Null energy condition
and superluminal propagation

S. Dubovsky[a,b], T. Grégoire[a,c], A. Nicolis[d], R. Rattazzi[a]

[a] *Department of Physics, CERN Theory Division, CH-1211 Geneva 23, Switzerland*

[b] *Institute for Nuclear Research of the Russian Academy of Sciences,*
60th October Anniversary Prospect, 7a, 117312 Moscow, Russia

[c]*Physics Department, Boston University,*
590 Commonwealth Ave., Boston, MA 02215, USA

[d]*Jefferson Physical Laboratory, Harvard University, Cambridge MA 02138, USA*

Abstract

We study whether a violation of the null energy condition necessarily implies the presence of instabilities. We prove that this is the case in a large class of situations, including isotropic solids and fluids relevant for cosmology. On the other hand we present several counter-examples of consistent effective field theories possessing a stable background where the null energy condition is violated. Two necessary features of these counter-examples are the lack of isotropy of the background and the presence of superluminal modes. We argue that many of the properties of massive gravity can be understood by associating it to a solid at the edge of violating the null energy condition. We briefly analyze the difficulties of mimicking $\dot{H} > 0$ in scalar tensor theories of gravity.

1   Introduction

Energy conditions play an important role in general relativity and are a basis of singularity theorems [1] and entropy bounds [2]. These conditions reflect the microscopic properties of the medium sourcing the energy momentum tensor. In some cases they may be violated without making the system physically unacceptable. This is the case of the strong energy condition, whose violation in Nature is indeed suggested by the success of the theory of

inflation and also by the recent observation of cosmic acceleration. It is however widely believed that any well behaved physical systems should respect the weakest of these conditions stating that $T_{\mu\nu}n^\mu n^\nu \geq 0$ for any null vector n. This condition is often referred to as the null energy condition (NEC). For an isotropic fluid with positive energy density, NEC amounts to $w \equiv p/\rho \geq -1$. In some simple cases a connection between violation of NEC and the presence of instabilities in the system sourcing $T_{\mu\nu}$ was already established (see for instance [3]). It would however be desirable to investigate under what condition this connection becomes a theorem. This question is obviously theoretically interesting[1], but it may also turn out to be relevant to present day cosmology. Indeed, recent supernova data mildly favor a NEC-violating equation of state [5], $w < -1$. If this result were to be confirmed by more precise experiments in the future, it would become essential to understand under what conditions NEC can be violated in the absence of pathologies. A no-go theorem valid in ordinary 4D theories may point towards more exotic scenarii, like for instance the Dvali-Gabadadze-Porrati model [6] where an effective w slightly less than -1 for dark energy can be achieved [7, 8], without manifest pathologies [9]. Finally, as we will show, the study of NEC violating field theories opens up a new perspective on theories of massive gravity (see Refs. [10, 11, 12, 13]).

The purpose of the current paper is to study in detail the relation between instabilities and NEC violation. For most of the paper we will be dealing with a system of derivatively coupled scalar fields with coordinate dependent condensates. Although this system may seem somewhat artificial, its simplicity allows us to concentrate on the physically relevant features avoiding technicalities which arise when one includes higher spin fields (e.g. vectors). Moreover, as we will explain in section 5 our results are in fact significantly more general. In particular our arguments apply to arbitrary relativistic solids and fluids. The main reason for this generality is that NEC violation is related to instabilities in a particular sector of the theory. This sector consists of just the Goldstone degrees of freedom (sound waves) associated to the spontaneous breakdown of local space-time translations. The properties of the Goldstone modes are dictated by symmetry, and are therefore largely independent of the specific fields that gave rise to the background.

The paper is organized as follows. In section 2 we describe the basic setup. In section 3 we study the general conditions for stability. We concentrate on some unusual features of rotationally non-invariant backgrounds, which make it rather difficult to establish a relation between NEC violation and instabilities in general. In particular, we explain that the positive definiteness of the Hamiltonian is not necessary for stability. Another peculiarity is related to the fact that in an effective field theory there is no *a priori* reason to exclude the presence of superluminal excitations around a background which spontaneously violates Lorentz invariance. It turns out that in the presence of superluminal modes stability is not

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[1]For instance, NEC prevents the existence of Lorentzian wormholes [4] and of flat bouncing cosmologies.

a frame independent notion. More precisely a perfectly stable system may appear to have ghost or gradient instabilities in some Lorentzian reference frames.

In section 4 we establish a relation between NEC violation and instabilities. We start by analyzing two-dimensional systems. We show that if Goldstone modes have non-degenerate dispersion relations[2] and if the energy-momentum tensor can be diagonalized by a Lorentz boost, then NEC violation implies the presence of an instability. On the other hand, we construct a simple 2-dimensional model which is stable (the Hamiltonian is even positive definite), violates NEC but possesses a non-diagonalizable energy momentum tensor. We then present generalizations of these results to higher-dimensional space-times. We prove that a violation of NEC leads to instabilities in a large class of situations. These include systems that are either rotationally invariant or possess only subluminal excitations. Nevertheless we were able to construct an example of a stable low energy effective field theory that violates NEC. This system has a positive definite Hamiltonian. Unlike in the 2-dimensional example, the energy-momentum tensor of this system can be diagonalized by a Lorentz boost. A peculiar necessary feature of stable NEC violating systems is the presence of superluminal modes in the spectrum of perturbations.

In section 5 we provide a more general perspective on our results. We relate scalar field theories with coordinate dependent condensates to relativistic solids and fluids, and elaborate on this correspondence, rederiving in field theory language some of the known results of hydrodynamics (such as Kelvin theorem). Although most of the results of this section are well known, we find it useful to present this discussion as it provides an intuitive physical picture of the systems considered in the previous sections and in works on massive gravity. A pragmatic reader can skip this section and proceed directly to section 6. In the latter we study NEC violation in rotationally invariant systems with degenerate dispersion relations. We argue that, unlike in massive gravity [13], these degeneracies cannot be protected by symmetries.

In section 7 we speculate on the implication of our results were future more precise cosmological obervations to imply $w < -1$. We present simple arguments explaining why it is difficult to mimic such an expansion in scalar-tensor theories of gravity without introducing a NEC violating energy-momentum tensor. This allows us to propose a relatively simple candidate model which can do the job without obviously contradicting the precise tests of general relativity. We summarize our results in section 8.

It is worth noting that a recent paper [14] claims to have a proof that NEC violation implies instabilities in a broad class of systems including those considered in our work. However, we believe that proof is not correct as we have been able to provide explicit counterexamples that contradict it (see section 4).

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[2]Degenerate dispersion relations, where some modes have either vanishing gradient term or vanishing time derivative term (or both), always emerge in the consistent theories of massive gravity [10, 11, 12, 13].

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2 Setup

Let us consider a system of N scalar fields ϕ^I, $i = 1, \ldots, N$ interacting with gravity

$$S = S_{gr} + \int d^4x \, \sqrt{-g} \, F(\phi^I, B^{IJ}, \ldots) \,, \tag{1}$$

where S_{gr} is the gravitational part of the action,

$$B^{IJ} = g^{\mu\nu} \, \partial_\mu \phi^I \partial_\nu \phi^J \,, \tag{2}$$

and the dots stand for higher derivative terms. We work in the context of effective field theory and assume that the scalar action has a cut-off Λ smaller than the Planck scale M_{Pl}. More specifically we assume the function F in eq. (1) is of the form

$$F(\phi^I, B^{IJ}, \ldots) = \Lambda^4 \tilde{F}\left(\frac{\epsilon \phi^I}{\Lambda}, \frac{B^{IJ}}{\Lambda^4}, \frac{\partial^2 \phi^I \partial^2 \phi^J}{\Lambda^6}, \ldots\right) \,, \tag{3}$$

where $\epsilon \ll 1$ is a small parameter responsible for the "lightness" of the fields ϕ^I, while all the other dimensionless parameters in the function $\tilde{F}$ are of order one. Indeed for the very existence of a non trivial low energy effective field theory the mass of the scalars $\sim \epsilon \Lambda$ should be much less than the cut-off Λ. Notice also that in the limit $\epsilon \to 0$ we would be dealing with a set of naturally massless Goldstone bosons. We want now to consider some classical solution $g_{cl}^{\mu\nu}, \phi_{cl}^I$ and study perturbations around it. The validity of our effective field theory, or the control over quantum corrections, forces us to consider only solutions that are smooth on space-time scales of order Λ^{-1}

$$|\partial_\nu \phi^I| \gg \frac{|\partial_\mu \partial_\nu \phi^I|}{\Lambda}, \ldots \,. \tag{4}$$

In other words we assume that the fields do not vary appreciably over lengths of order $1/\Lambda$ or more formally $|\partial_\mu| \ll \Lambda$. Notice that this constraint does not necessarily imply[3] $\partial_\mu \phi^I \ll \Lambda^2$. Let us consider the relevant length scales. Since the energy momentum generated by the scalars is of order Λ^4, space-time will be curved over a length $r_G \sim M_{Pl}/\Lambda^2$. Under the assumptions $\Lambda \ll M_{Pl}$ and $\epsilon \ll 1$ there is a range of lengths

$$\frac{1}{\Lambda} \ll r \ll r_G, \frac{1}{\epsilon \Lambda} \tag{5}$$

where we can use effective theory, spacetime is approximately flat and the scalars can be treated as massless. In physically interesting situations where the scalars describe some sort

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[3]In the Goldstone limit, $\epsilon = 0$, the terms of the type $\left[B^{IJ}\right]^n$ represent the leading interaction among n-quanta in the derivative expansion. In this respect $|B^{IJ}/\Lambda^4|$ is controlling the size of classical non-linearities rather than the validity of the quantum expansion, which is instead more directly tight to the condition $|\partial_\mu| \ll \Lambda$ (see a related discussion in Ref. [15]).

of cosmic fluid or a generalized quintessence, the above range would encompass all relevant sub-horizon scales. We want to study perturbations around the solution in the above range of scales (cf., e.g., Ref. [16]), that is for space momenta k

$$\frac{\Lambda^2}{M_{Pl}}, \ \epsilon\Lambda \ll k \ll \Lambda \ . \tag{6}$$

As we focus on a patch of space-time where curvature and scalar masses are negligible, by a suitable change of coordinates we can approximate $g_{cl}^{\mu\nu}$ by the Minkowski metric $\eta^{\mu\nu}$ in the whole patch and the background scalar fields ϕ_{cl}^I by linear functions of the coordinates

$$\phi_{cl}^I = \phi_0^I + A_\mu^I x^\mu \ . \tag{7}$$

Since we can neglect both masses and higher derivative terms we can define our function F to be just $F \equiv \Lambda^4 \tilde{F}(B^{IJ}/\Lambda^4)$. Finally, at momenta larger than the curvature, $k \gg \Lambda^2/M_{Pl}$, we can neglect the mixing between scalar fields and metric perturbations. Indeed this mixing is a manifestation of the gravitational Higgs mechanism: the mixing scalar modes are the Goldstone bosons of spontaneously broken translations, and the graviton has effectively a mass of the order of the curvature Λ^2/M_{Pl}. By a suitable (unitary) choice of coordinates the Goldstones can be eliminated, in which case extra physical polarizations appear in the graviton field. However when working at $k \gg \Lambda^2/M_{Pl}$, the graviton mass can be neglected and, by an analogue of the equivalence theorem, the extra graviton polarizations can be accurately described by the Goldstone bosons. In the rest of the paper we will work in $\Lambda = 1$ units.

By defining the perturbed scalar field as $\phi^I = \phi_{cl}^I + \pi^I$ we can write the quadratic Lagrangian for scalar perturbations in Fourier space as

$$L = L_{IJ}(p)\pi^I(-p)\pi^J(p) \ . \tag{8}$$

Here p is the four-momentum and the matrix $L_{IJ}(p)$ is given by

$$L_{IJ} = 2F_{IK,JL} A_\mu^K p^\mu A_\nu^L p^\nu + F_{IJ} p^2 \ , \tag{9}$$

where

$$F_{IJ} = \frac{\partial F}{\partial B^{IJ}} \ , \quad F_{IJ,KL} = \frac{\partial^2 F}{\partial B^{IJ} \partial B^{KL}} \ . \tag{10}$$

It is straightforward to check that the background energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = 2F_{IJ}A_\mu^I A_\nu^J - F g_{\mu\nu} \ . \tag{11}$$

Our goal is now to study the stability of eq. (8) when the background violates NEC, that is when there exists some light-like vector n^μ such that

$$T_{\mu\nu}n^\mu n^\nu < 0 \ . \tag{12}$$

5

3 General criteria for stability

3.1 One flavor in two dimensions

Before addressing the stability of Lagrangian (8) in full generality, it is instructive to discuss a simple special case: a single scalar field in (1+1)-dimensional space-time with action ($\dot{\pi} \equiv \partial_t \pi$, $\pi' \equiv \partial_x \pi$)

$$S = \int dt dx \, \frac{1}{2} \left(K \dot{\pi}^2 + M \dot{\pi} \pi' - G \pi'^2 \right) \; . \tag{13}$$

The dispersion relation is

$$\lambda(\nu) \equiv K \nu^2 + M \sigma \nu - G = 0 \; , \tag{14}$$

where

$$\nu = \omega / |k| \tag{15}$$

is the ratio of the frequency and the absolute value of the spatial momentum k, and $\sigma = \pm 1$ depending on the direction of the spatial momentum.

In general, the system (13) can exhibit two different types of instability. First, there can be a gradient instability. This is present whenever equation (14) has complex roots. This is similar to the usual tachyonic instability which arises in a Lorentz invariant theory when some field has a negative mass squared, $m^2 < 0$. It is however different since in the latter case the time-scale τ with which the instability grows is set by the inverse mass, $\tau \sim |m|^{-1}$. This time scale may be very long compared to the other characteristic time scales if m^2 is small enough. Equivalently, conventional tachyonic instabilities are present only for long wavelength modes. On the other hand if eq. (14) has a complex root this implies the presence of instabilities at all wavelengths, so that their rate of growth is not bounded. Instabilities of this type are significantly more dangerous than the usual tachyonic ones. Nonetheless in the rest of the paper we will improperly refer to gradient instabilities as *tachyons*.

Another type of instability is the one associated with *ghosts*. In a Lorentz invariant theory a ghost is a field entering the action with a wrong sign kinetic term. At the classical level such a field has a negative definite free Hamiltonian. Therefore, neglecting non linear terms or interactions with other fields, no runaway behaviour occurs: there just happens to exist one decoupled sector using the opposite sign convention for time and energy. Runaway behaviour shows up when the ghost is set to interact with ordinary fields having positive definite free Hamiltonian. Notice that, being due to non-linearity, the time scale for growth becomes infinite in the limit of vanishing initial perturbation, *i.e.* in the classical vacuum. The situation is generalized, but gets worst, at the quantum level, just because the field fluctuates away from the origin already in the vacuum. The ghost quanta have negative energy[4], so that, once interactions are taken into account, the vacuum itself becomes unstable

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[4]We quantize on a physical Hilbert space with positive norm, so that all probabilities are non-negative.

with respect to spontaneous creation of positive and negative energy quanta with vanishing total energy. Again, ghost instabilities are entirely due to the non-trivial couplings between negative and positive energy fields, and strictly speaking require to go beyond the quadratic approximation. We also recall that in a Lorentz invariant theory, the rate of development of such an instability is uncalculable (formally infinite), even for arbitrary small couplings, due to the non-compact final state phase space volume. A way to solve this problem is to introduce a cutoff which explicitly breaks Lorentz invariance. In such a way the rate of growth of the instability can be at least estimated and phenomenological consistency satisfied if the coupling between the ghosts and the other fields is small enough (see for instance refs. [17, 18]). In what follows we will not be concerned with the latter possibility but instead require the stronger, more robust, condition that ghost instabilities be absent. As made clear from the discussion, whether or not negative energy states in the system (13) imply instability depends on the spectrum of the fields coupled to them. One coupling that surely exists is the one to the graviton. Moreover in the range we are focusing on, the graviton satisfies to a good approximation a Lorentz invariant dispersion relation $\omega^2 = k^2$. We will therefore require the system to be kinematically stable with respect to the associated production of π quanta and of some other massless particle (in particular the graviton)[5].

Let us now study what the stability conditions are. Absence of tachyons requires that both roots of the equation (14) be real, *i.e.*

$$M^2 + 4KG \geq 0 \ . \tag{16}$$

The issue of ghosts can be equivalently studied at the classical or quantum level. We find it convenient to work right away at the quantum level and use the perhaps more intuitive particle physics viewpoint. The general solution of the field equations following from the action (13) has the form

$$\pi(t,x) = \int dk \left(\alpha_1(k) \, e^{ik(\nu_1 t - x)} + \alpha_2(k) \, e^{ik(\nu_2 t - x)} \right) \ , \tag{17}$$

where $\nu_{1,2}$ are the two solutions of eq. (14) with $\sigma = 1$. Without loss of generality, we assume $\nu_1 > \nu_2$. Reality of the field π implies

$$\alpha_{1,2}(k)^\dagger = \alpha_{1,2}(-k) \ . \tag{18}$$

By eq. (13) the momentum Π canonically conjugated to π is given by

$$\Pi = K\dot{\pi} + \tfrac{1}{2}M\pi' \ . \tag{19}$$

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[5]Note, that in two dimensions the graviton does not have propagating degrees of freedom. This is not essential for our purposes as we will eventually be interested in four dimensional theories. In the 2d example we are therefore considering the coupling to some other massless field.

The canonical commutation relations in momentum space become

$$[\alpha_a(k), \alpha_b(k')] = \frac{2\delta_{ab}\sigma(K)(-1)^a}{\pi k\sqrt{M^2 + 4KG}}\delta(k + k') , \tag{20}$$

where $\sigma(K)$ stands for the sign of K. It is convenient to consider separately the two cases $K > 0$ and $K < 0$.

- $K > 0$.
In this case we define creation and annihilation operators $a^\dagger(k)$, $a(k)$ as

$$a^\dagger(k) = \alpha_1(k)\theta(k) + \alpha_2(k)\theta(-k) \tag{21}$$
$$a(k) = \alpha_1(-k)\theta(k) + \alpha_2(-k)\theta(-k) \tag{22}$$

Then, by defining the Fock vacuum as the state $|0\rangle$ satisfying

$$a(k)|0\rangle = 0 \qquad \forall k , \tag{23}$$

the Fock space generated by the action of the $a^\dagger(k)$'s on $|0\rangle$ has a positive definite inner product. Consequently, the energy spectrum of one particle states is

$$E(k) = \nu_1 k\,\theta(k) + \nu_2 k\,\theta(-k) . \tag{24}$$

Negative energy states are absent provided ν_1 is positive and ν_2 is negative. For $K > 0$ this condition is true if and only if $G > 0$. Notice that positivity of K and G coincides with positivity of the classical Hamiltonian H

$$H = \int dx\,\frac{1}{2}\left(K^{-1}(\Pi - \tfrac{1}{2}M\pi')^2 + G\pi'^2\right) > 0 . \tag{25}$$

However, $K, G > 0$ is not necessary for the stability of the system (cf. Ref. [15]). Indeed, consider the situation where ν_1 and ν_2 are both positive, in which eq. (24) is not positive definite. The ghost instability is still absent, provided the two-momentum of the multiparticle states of the π-system (13) never belongs to the interior of the lower light cone, that is if

$$E > -|k| . \tag{26}$$

In this case, in spite of the presence of negative energy states, vacuum production of negative energy π-particles and positive energy gravitons is not kinematically allowed. The basic reason for stability in this case is that in addition to energy there exists an extra conserved charge, the total momentum. Condition (26) holds even when both ν_1 and ν_2 are positive provided

$$\nu_2 < 1 , \tag{27}$$

or if both ν_1 and ν_2 are negative provided

$$\nu_1 > -1 \ . \tag{28}$$

In both cases the Hamiltonian is unbounded both from above and from below.

To better understand the physics of such systems let us consider a subluminal scalar field in the reference frame where its action is rotationally invariant[6],

$$S = \int dt dx \, \frac{1}{2} \left(\dot{\pi}^2 - v^2 \pi'^2 \right) \ , \tag{29}$$

with $0 < v^2 < 1$. The Hamiltonian associated to this action is positive definite and there is no doubt that this system is perfectly stable. Let us, however, consider the same system as seen in another Lorentzian reference frame, *i.e.* let us perform a change of time and space variables in the action (29) according to the rule[7]

$$\begin{pmatrix} t \\ x \end{pmatrix} \rightarrow \begin{pmatrix} ch & sh \\ sh & ch \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \ , \tag{30}$$

where $ch = \cosh \lambda$ and $sh = \sinh \lambda$ are the boost parameters. After this change of coordinates the action (29) takes the general form (13) with the coefficients K, M, and G equal to

$$\begin{aligned} K &= ch^2 - sh^2 v^2 \\ M &= \tfrac{1}{2} \, ch \, sh \, (v^2 - 1) \\ G &= ch^2 v^2 - sh^2 \end{aligned} \tag{31}$$

One observes that for $v^2 < 1$ the coefficient K remains positive in all reference frames, while G becomes negative for large enough boosts, $sh^2/ch^2 > v^2$. So, in the boosted frame, negative energy states appear and the Hamiltonian is unbounded both from above and from below. Dispersion relations for one-particle states in the original and in the boosted reference frames are shown in Fig. 1. It is clear from this figure that the conditions (27), (28) hold in both frames as it is natural to expect for a stable system.

Actually, the presence of negative energy states is easy to understand in the following way. Let us consider a massive particle with relativistic dispersion relation

$$E^2 = k^2 + m^2 \ , \tag{32}$$

moving with velocity V in the original reference frame. At $V > v$ the emission of Čerenkov radiation is kinematically allowed and the massive particle can emit one quantum of the π-field, thus losing part of its energy. On the other hand, from the point of view of an inertial

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[6]In two dimensions, rotations reduce to the $\mathbb{Z}_2$ reflection $x \rightarrow -x$.

[7]Recall that all our considerations are in the context specified in section 2: starting from a Lorentz invariant action, we study the dynamics of fluctuations around a generic Lorentz-breaking field configuration. Lorentz transformations are still a symmetry of the whole system.
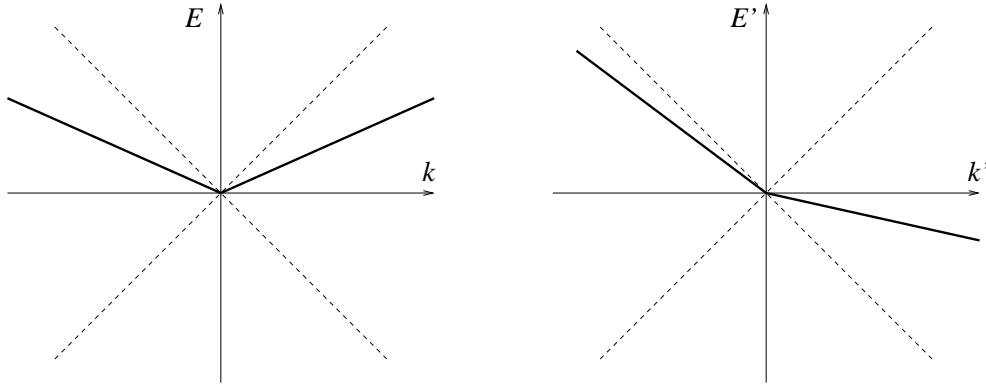
Figure 1: The dispersion relations (bold lines) of a subluminal scalar in the original frame in which eq. (29) holds (left), and in a highly boosted frame (right).

observer moving at velocity V, the massive particle is initially at rest and then starts moving by emitting a negative energy quantum.

- $K < 0$.

Let us now come back to the study of the general action (13) and consider the case $K < 0$. It is straightforward to check that the energy spectrum of one-particle states is now given by

$$E(k) = \nu_1 k \, \theta(-k) + \nu_2 k \, \theta(k), \tag{33}$$

so that a state with two particles of equal and opposite spatial momentum has negative total energy (recall that $\nu_1 > \nu_2$). At first sight, this seems to necessarily imply that theories with $K < 0$ suffer from a fatal ghost instability.

However, this naive conclusion is not quite correct. Indeed, let us again consider action in eq. (29) but for the case of a superluminal π ($v^2 > 1$). There is no doubt that this action describes a perfectly stable system in this case as well. Let us now look at this system in the boosted frame (30). eqs. (31) now imply that the kinetic coefficient K becomes negative for large enough boost parameters, $ch^2/sh^2 > v^2$ (see Fig. 2).

Naively, this result leads to a paradox where the vacuum of a stable system (29) seems unstable to a fast moving observer. Let us assume for the sake of the argument that there exists a vertex with two gravitons and one π. Then, compatibly with conservation of 2-momentum, the moving observer expects to see events of spontaneous creation of one negative energy π-particle and a pair of positive energy gravitons from the vacuum, see Fig. 3. He should then conclude that the vacuum is unstable.

The resolution of the paradox is just that the observer at rest and the moving one
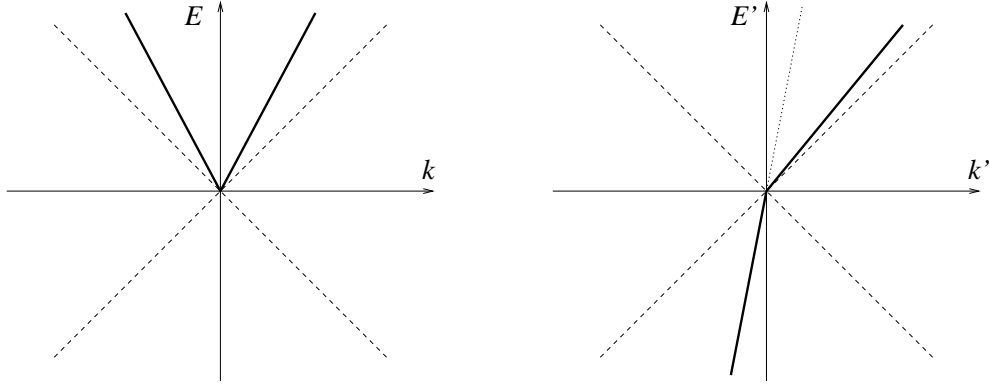
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Figure 2: The dispersion relations (bold lines) of a superluminal scalar in the original frame (left) and in a highly boosted frame (right).

have performed mutually inequivalent quantizations of the system. This is understood by considering the causal structure of the system and the initial conditions associated to the two different parametrizations of time. First notice that, for a large enough boost, part of the past causal cone is mapped to the future of the new space-time coordinates. Consequently, an incoming π-particle in the original (unboosted) reference frame, is mapped to an outgoing π-ghost in the boosted frame. Therefore the decay process $\pi \to 2g$ becomes the vacuum decay process $vacuum \to \pi + 2g$ in the boosted coordinates. Now the point is that the two observers have defined causality by two *inequivalent* Cauchy surfaces, corresponding to
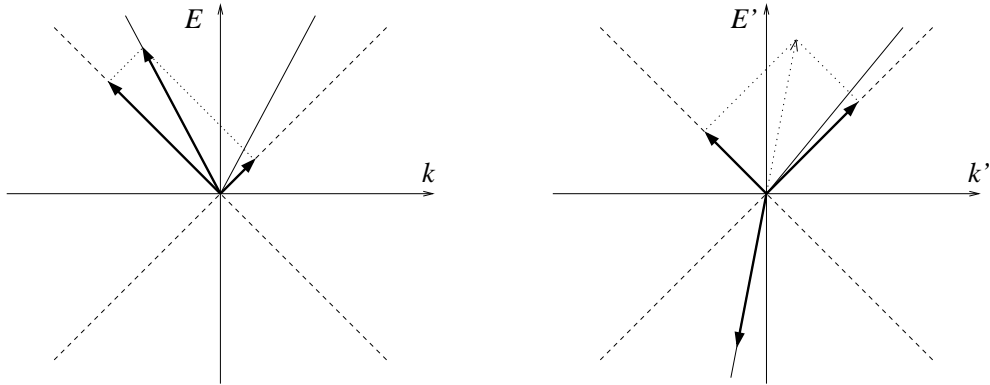

Figure 3: The creation of a negative energy particle and two gravitons out of vacuum in the boosted frame (right). The bold arrows are the particles momenta, which sum up to zero. The same process appears in the original frame as the decay of a superluminal particle into two gravitons (left).
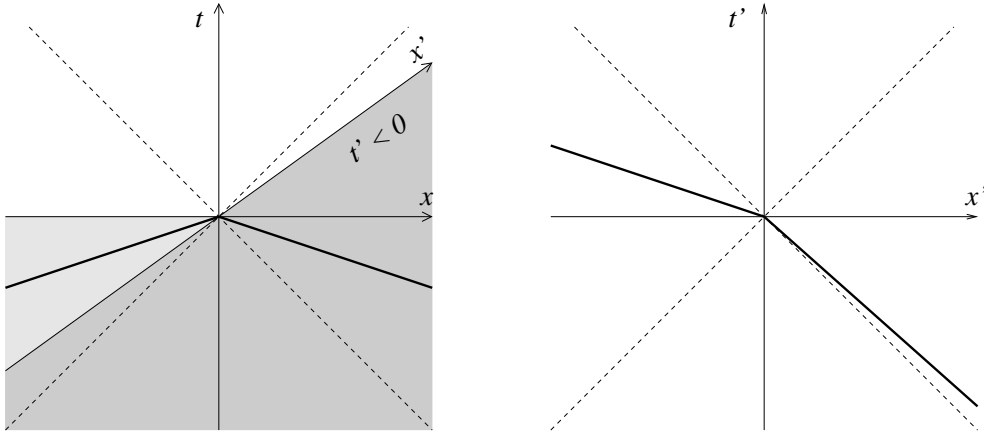
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Figure 4: The bold lines depict the past causal cones in the original reference frame (left) and in a highly boosted one (right). Left: The two distinct vacuum choices require no incoming wave in the light shaded region ($t < 0$) and in the dark shaded region ($t' < 0$), respectively. As clear from the picture these choices are inequivalent: an incoming particle moving along the left branch of the past causal cone is discarded in the former choice while it is accepted (as outgoing) in the latter.

surfaces of constant time in the two different coordinate systems. This is made evident in Fig. 4 where the two distinct vacuum choices correspond to the request of no-incoming wave in two distinct shaded regions. The two choices are inequivalent because there are incoming (outgoing) portions of particle trajectories that are allowed in one choice and not in the other. Another way to say it is that the time vector, which defines the hamiltonian flow (and its quantization), is inequivalently chosen in the two parametrizations. This inequivalence is reflected by the fact that the direction of time flow of one of the two particle trajectories that cross each spacetime point is mutually reversed in the two cases. As shown in Figure 4 this is precisely like comparing two parametrizations of spacetime where the role of time and space is interchanged[8].

In the end, as it should, the behaviour of the system does not depend on the observer but on the choice of quantization procedure, *i.e.* on the vacuum state. Stability favors the obvious vacuum, the one associated to the time slicing operated by the observer at rest. Provided this is the vacuum, the moving observer will perhaps find it a little puzzling that he does not observe copious production of $\pi + 2g$ from nothing. However, with some thinking, he will realize that because of his unhappy choice of time coordinate, the correct causal past overlaps with a space time region with infinitely growing time coordinate.

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[8]Notice, by the way, that if we perform the exchange of time and space for an ordinary Lorentz invariant massive field in two dimensions, we find that it becomes a tachyon.

It is worth noting that there were suggestions [19] that boundary conditions at future infinity may help to solve the cosmological constant problem. It would be interesting to understand whether the situation just discussed may be relevant towards a realization of such a proposal, without invoking the anthropic principle and without acausalities.

To finish this discussion, we remark that in the presence of superluminal particles one can find examples where a stable system appears to have a tachyonic instability in some Lorentzian reference frame. It is straightforward to check that this happens, for instance, if one adds a mass term $-\frac{1}{2}(m^2\pi^2)$ to the action (29) (for $v^2 > 1$).

To summarize, since the goal of the present paper is to find a connection between a violation of the NEC and the presence of local instabilities, we will adopt a very weak criterion of stability. We will assume that if the system (13) possesses superluminal excitations one should not conclude that it is unstable after observing (apparent) instabilities in some specific Lorentzian frame. Instead one should check whether it is possible to find some frame where the system (13) is manifestly stable. If this frame exists, it should be used to quantize the system thus defining the notion of causality[9]. We leave aside the issue whether an effective theory with superluminal excitations can arise from a consistent microscopic theory.

To conclude let us stress that in the absence of superluminal excitations the above peculiarity does not arise and all Lorentzian frames are equally good for the analysis of stability.

3.2 Arbitrary number of flavors

Let us now discuss what the stability criterion is for a multiflavor system of general form (8) with $I, J = 1, \ldots, N$. We still focus, for the moment, on two-dimensional systems. We work in the limit discussed in section 2 in which the kinetic matrix $L_{IJ}(p)$ is a quadratic form of momentum,

$$L_{IJ}(p) = L_{IJ,\mu\nu}\, p^\mu p^\nu \,. \tag{34}$$

Dispersion relations for on-shell states are determined by the condition

$$\det L_{IJ}(p) = 0 \,. \tag{35}$$

This equation defines a set of rays in the (ω, k) plane. As before, we can rewrite it in terms of the slope (velocity) ν as defined by eq. (15),

$$\det \tilde{L}_{IJ}(\nu, \sigma) = 0 \,, \tag{36}$$

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[9]In the above discussion (and in the following sections) we have restricted to Lorentzian frames the search of coordinate where canonical quantization leads to stability. It is easy to see that this restriction is unimportant. As we explained, the choice of canonical quantization corresponds to the choice of time coordinate t. Assuming a good choice of equal time slicing exists, the invariant distance in that frame will in general be $ds^2 = adt^2 + c_i dt dx^i + g_{ij} dx^i dx^j$. It is clear that by a reparametrization of the form $t = ct'$, $x^i = c^i_j x'^j + c^i t'$ the metric can be made Lorentzian without changing the time slicing.

where $\tilde{L}_{IJ}(\nu) \equiv L_{IJ}(p)/k^2$ and $\sigma = \pm 1$ characterizes the direction of the spatial momentum k. Let us consider eq. (36) at $k > 0$ [10]. This equation has $2N$ roots. In analogy with the one flavor case, upon quantization half of these roots $\{\nu_{+I}\}$ will correspond to creation operators for particles with momentum k and energy

$$\omega = \nu_{+I}|k| . \tag{37}$$

The second half of the roots $\{\nu_{-I}\}$ corresponds to annihilation operators for particles with momentum $-k$ and energy

$$\omega = -\nu_{-I}|k| . \tag{38}$$

One necessary condition for stability is that all roots $\nu_{\pm I}$ be real.

To understand the issue of ghosts let us understand how the two sets, $\{\nu_{+I}\}$ and $\{\nu_{-I}\}$, are characterized. For this purpose, notice that (cfr. the analysis in Ref. [13]) at any value of ν, the real and symmetric matrix $\tilde{L}_{IJ}(\nu)$ has N real eigenvalues $\lambda_I(\nu)$. When ν takes one of the values in $\{\nu_{+I}\}$ or in $\{\nu_{-I}\}$ one of the eigenvalues λ_I is equal to zero. We will now prove that the two sets of zeros are characterized by the sign of the derivative of the corresponding eigenvalue at its zero,

$$\left.\frac{d\lambda_I}{d\nu}\right|_{\nu_{+J}} > 0 \text{ if } \lambda_I(\nu_{+J}) = 0 , \tag{39}$$

and

$$\left.\frac{d\lambda_I}{d\nu}\right|_{\nu_{-J}} < 0 \text{ if } \lambda_I(\nu_{-J}) = 0 . \tag{40}$$

It is straightforward to check that these conditions agree with the results of subsection 3.1 in the case of one flavor. Conditions (39), (40) are easily derived by using the Källen–Lehmann representation of the propagator. Consider indeed the Feynman correlator

$$G^{IJ}(t) = \langle 0| T \left(\pi^I(t)\pi^J(0)\right) |0\rangle$$

and rewrite it by inserting a complete set of states $|n\rangle$ between $\pi^I(t)$ and $\pi^J(0)$

$$G^{IJ}(t) = \sum_n \theta(t)e^{-iE_n t}M_{(n)}^{IJ} + \sum_n \theta(-t)e^{iE_n t}M_{(n)}^{JI} , \tag{41}$$

where E_n is the energy of the n-th state and

$$M_{(n)}^{IJ} = \langle 0|\pi^I(0)|n\rangle\langle n|\pi^J(0)|0\rangle . \tag{42}$$

Note that all the matrices $M_{(n)}^{IJ}$ are positive semi-definite. eq. (41) implies that the Fourier transform $G^{IJ}(E)$ of the propagator takes the form

$$G^{IJ}(E) = \sum_n \frac{i}{E - E_n + i\epsilon}M_{(n)}^{IJ} - \sum_n \frac{i}{E + E_n - i\epsilon}M_{(n)}^{JI} . \tag{43}$$

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[10]Note that $\tilde{L}_{IJ}(\nu, 1) = \tilde{L}_{IJ}(-\nu, -1)$, so that $\tilde{L}_{IJ}(\nu, 1)$ alone carries all relevant information about the spectrum.

14

Consequently, the poles of $-iG^{IJ}(E)$ with positive residue correspond to the energies of the physical states. On the other hand, from the Lagrangian we have

$$-iG^{IJ}(E) = L_{IJ}^{-1}(E) \ . \tag{44}$$

Consequently, using matrix notations we may write

$$-iG(E) = k^2 O^T(\nu) D(\nu) O(\nu) \ , \tag{45}$$

where $D(\nu)$ is a diagonal matrix with eigenvalues equal to $\lambda_I^{-1}(\nu)$ and $O(\nu)$ is an ortogonal matrix. Any pole of $G(E)$ corresponds to a zero of some $\lambda^I(\nu)$, while the sign of the residue is the same as the sign of the first derivative $d\lambda^I(\nu)/d\nu$. This implies eqs. (39, 40).

Using these conditions one can determine the spectrum of the system. According to our previous discussion, the system is stable provided we can so define time that upon quantization there are no multiparticle states in the lower light cone (see eq. (26)).

3.3   Arbitrary number of dimensions

Let us finally discuss how the above results generalize to field theories in arbitrary $(d + 1)$ dimensions. A straightforward generalization of the dispersion relation (36) is

$$\det \tilde{L}_{IJ}(\nu, \hat{\mathbf{n}}) = 0 \ , \tag{46}$$

where, as before, $\tilde{L}_{IJ}(\nu, \hat{\mathbf{n}}) \equiv L_{IJ}(p)/\mathbf{k}^2$ and $\hat{\mathbf{n}}$ is now a unit d-dimensional vector in the direction of the spatial momentum $\mathbf{k}$.

Again the system (8) is stable if there is a Lorentzian reference frame, where eq. (46) has $2N$ real roots for any $\hat{\mathbf{n}}$ and there are no states in the lower light cone. Note that our derivation of the conditions (39), (40) was not sensitive to the number of the dimensions. Consequently, these conditions are applicable for arbitrary d as well. The only difference is that now roots $\nu_{I\pm}$ are functions of the unit vector $\hat{\mathbf{n}}$, and $\nu_{I+}(\hat{\mathbf{n}})$-roots correspond to the particles with momentum along $\hat{\mathbf{n}}$ while $\nu_{I-}(\hat{\mathbf{n}})$-roots correspond to particles with momentum in the opposite direction.

4   NEC violation and instabilities

To understand the implications of the NEC violation for stability, let us first study how stability itself is encoded in the Lagrangian. We start with the analysis of two-dimensional systems. Consider a stable system of the type (1). According to the discussion of section 3 this means that a Lorentzian reference frame exists where the spectrum of states does not contain either tachyons or states with energy-momentum in the lower light cone. Let us

see in more detail how this restricts the properties of the matrix $L_{IJ}(p)$ in eq. (8). As we discussed in subsection 3.2 the properties of the spectrum can conveniently be read off from the properties of the eigenvalues $\lambda_I(\nu)$ of the matrix

$$\tilde{L}_{IJ}(\nu) = L_{IJ}(p)/k^2 \; , \tag{47}$$

where without loss of generality we assumed that $k > 0$. Absence of tachyons implies that these eigenvalues have $2N$ zeros in total. Let us assume that we are in a generic situation, where all these roots are simple. Multiple roots may emerge due to some coincidental degeneracy or as the result of a symmetry. We postpone the discussion of degeneracies to section 6.

Let us now understand what root configurations on the ν-axis are implied by stability. As before, we indicate by $\{\nu_+\}$ and $\{\nu_-\}$ the sets of roots with respectively positive and negative slope. In what follows we consider all the particles together, so that we include in the sets $\{\nu_\pm\}$ the roots $\nu = \pm 1$ associated to the "graviton", "photon", etc. By eqs. (37 – 40), if there exist $\nu_{1+} \in \{\nu_{I+}\}$ and $\nu_{1-} \in \{\nu_{I-}\}$ such that $\nu_{1+} < \nu_{1-}$ then we can construct a two-particle state with zero total momentum and negative energy

$$E = (\nu_{1+} - \nu_{1-})k \; . \tag{48}$$

By the discussion in the previous section we thus have a ghost instability. On the other hand it is straightforward to check that if

$$\nu_{I+} > \nu_{J-} \tag{49}$$

for any two members of the two sets (see Fig. 5), then the Hilbert space does not contain any state with zero energy-momentum other than the vacuum, so that there is no ghost instability. eq. (49) is just a generalization of the conditions (37), (38) obtained in the one flavor case.

Notice that eq. (49) implies that no eigenvalue $\lambda_I(\nu)$ has more than two zeroes on the ν line. Since we assume that there are $2N$ zeroes in total (we barr degeneracies), then each eigenvalue must have precisely two zeroes. The behaviour of the $\lambda_I(\nu)$ that follows is shown in Fig. 5. Since the signs of the eigenvalues define the sign properties of the matrix $L_{IJ}(\nu)$, we find that stability holds if and only if the following two conditions hold:

(i) There exists $\nu_{\max} > 1$, such that $\tilde{L}_{IJ}(\nu)$ is positive definite for $|\nu| > \nu_{\max}$.

(ii) There exists $-1 < \nu_{\min} < 1$, such that $\tilde{L}_{IJ}(\nu_{\min})$ is negative definite.

Condition (i) is just the usual requirement that the kinetic part of the Lagrangian (8) be positive definite. Moreover if condition (ii) is satisfied for the special value $\nu_{\min} = 0$, then the purely gradient part of the Lagrangian (8) is negative definite. Conditions (i), (ii) with

16

$\nu_{\min} = 0$ are equivalent to positivity of the Hamiltonian. In general, however, validity of *(i)* and *(ii)* is a weaker request than positivity of the Hamiltonian.

At this point it is convenient to go back to the matrix $L_{IJ}(p)$ and reformulate conditions *(i)* and *(ii)* in a manifestly Lorentz covariant way. Let us define the following scalar function of 2-vectors p and q

$$L(p,q) \equiv L_{IJ}(p) \, A_\mu^I q^\mu \, A_\nu^J q^\nu \ . \tag{50}$$

In practice we are evaluating the quadratic Lagrangian of a perturbation

$$\pi^I = q^\mu A_\mu^I \ . \tag{51}$$

Perturbations of this special form can be obtained by acting on the vacuum (7) with a space-time translation $x^\mu \to x^\mu + q^\mu$. Perturbations (51) are therefore just the Goldstones (sound waves) of broken space-time translations. That is why we do not lose any relevant information by focusing on these perturbations. By using the defining eqs. (9, 11), we obtain a simple relation

$$L(p,q) - L(q,p) = \tfrac{1}{2} \left(p^2 \, T_{\mu\nu} q^\mu q^\nu - q^2 \, T_{\mu\nu} p^\mu p^\nu \right) \equiv f(q,p,T_{\mu\nu}) \ , \tag{52}$$

which relates the kinetic matrix to the energy momentum tensor, and which will prove crucial in relating NEC to stability. As we shall see shortly, stability in the form of conditions *(i)*, *(ii)* forces the function f to be positive for properly chosen p^μ and q^μ. On the other hand, if NEC is violated then $T_{\mu\nu}$ is negative definite along some direction. Our strategy will be to prove that under generic conditions there exists a tension between these two facts.

Now, notice that $|\nu| > 1$ and $|\nu| < 1$ correspond respectively to time-like and space-like 2-momentum. Conditions *(i)*, *(ii)*, then imply the following two properties of $L(p,q)$. First,
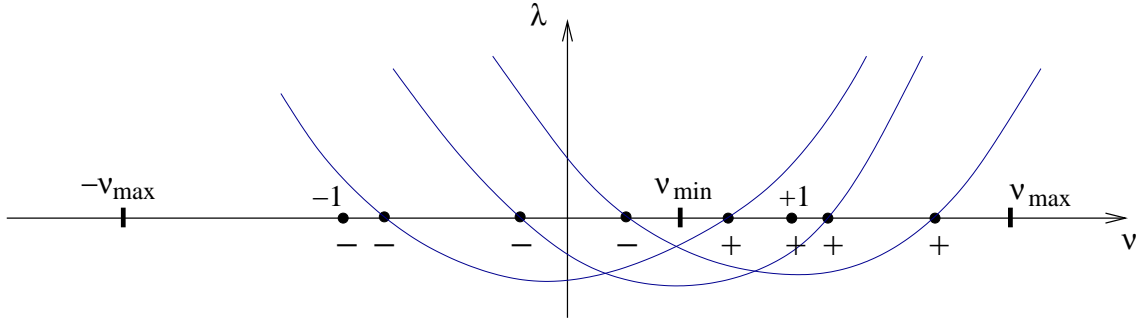

Figure 5: The eigenvalues λ_I of $\tilde{L}_{IJ}$ as functions of ν. The configuration shown corresponds to a stable system: all the positive derivative roots ν_{I+} are on the right of the negative derivative roots ν_{I-} (taking into account also the roots ± 1 corresponding to the dispersion relation of the graviton). As clear from the picture, this property is equivalent to conditions *(i)*, *(ii)*.

there exists a cone Γ_+ in the region of time-like momenta which includes the ($\omega \neq 0, k = 0$)-axis such that for any $t^\mu \in \Gamma_+$ and for arbitrary q^μ

$$L(t, q) > 0 \ . \tag{53}$$

Second, there exists a space-like line Γ_- such that for $s^\mu \in \Gamma_-$

$$L(s, q) < 0 \tag{54}$$

for any q^μ. By eqs. (53, 54) it then follows that

$$f(s, t, T_{\mu\nu}) > 0 \ , \tag{55}$$

for $t \in \Gamma_+$ and $s \in \Gamma_-$.

It is now straightforward to extend the discussion to systems in $d > 2$. Consider indeed the subspace of the states where all particles move collinearly (*i.e* the spatial momentum is along some given direction $\hat{\mathbf{n}}$). Then one can literally repeat the arguments above and obtain that the conditions *(i)*, *(ii)* should hold for any direction $\hat{\mathbf{n}}$. (In analogy with subsection 3.3 the matrix $\tilde{L}_{IJ}(\nu)$ now depends on $\hat{\mathbf{n}}$ as well). We stress however that these conditions, while necessary, are no longer sufficient to ensure the absence of ghost instabilities. Indeed, for anisotropic systems one may imagine a situation where states with total momentum in the lower light cone are absent if one considers only states with collinear particles, but are present in general. Nonetheless, these necessary conditions allow us to establish a relation between NEC and stability in a rather broad class of theories. On the other hand we will be able to construct examples of stable systems that violate NEC. Therefore we do not see much motivation in trying to formulate stronger conditions.

Focusing just on states of collinearly moving particles the conditions *(i)*, *(ii)* are easily generalized. As a consequence condition (53) is unchanged: there must exist a cone Γ_+ in momentum space, which contains the ω-axis and such that the matrix $L_{IJ}(p)$ is positive definite for any $p \in \Gamma_+$. Condition (54) is similarly generalized: as before, for any given space direction $\hat{\mathbf{n}}$ there exists a line of space-like momenta along which the matrix $L_{IJ}(p)$ is negative definite. By varying $\hat{\mathbf{n}}$ we find a skirt-shaped cone Γ_- (see fig. 6). As before stability implies that eq. (55) holds for $t \in \Gamma_+$ and $s \in \Gamma_-$.

Now that we have characterized the implications of stability, let us assume that the NEC condition is violated along some null vector n^μ: $T_{\mu\nu} n^\mu n^\nu < 0$. In $d > 2$ by continuity there must exists a whole neighborhood of null directions along which NEC is violated. The two-dimensional case is singular in this respect, since there the light-cone is made up of two disconnected directions; we will analyze the implications of this fact later. Let us therefore concentrate on $d > 2$ for the moment. Any pair of such linearly independent NEC-violating null vectors n_1 and n_2 defines a plane $\mathcal{N}$. Now, both the NEC and the scalar quantity f are
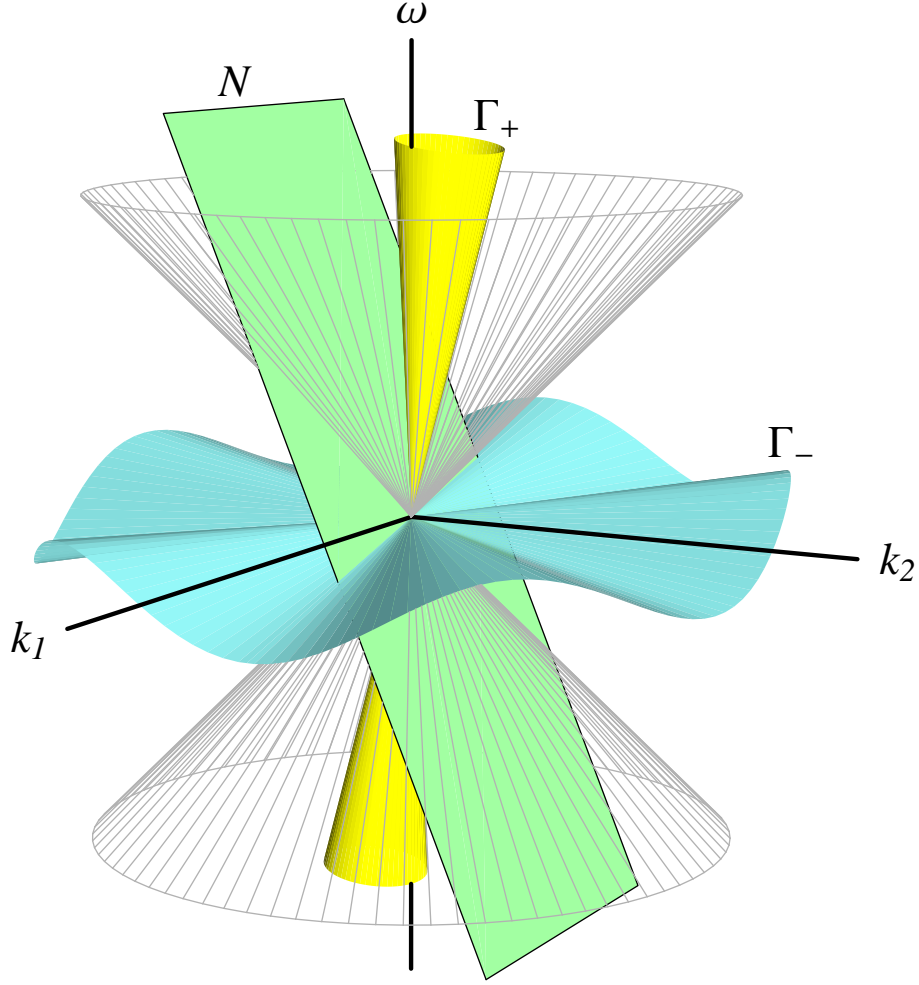
Figure 6: The various regions in the $(\omega, \mathbf{k})$ space discussed in the text. L_{IJ} is positive definite inside the yellow narrow cone Γ_+, while it is negative definite on the skirt-shaped blue cone Γ_-. The green plane $\mathcal{N}$ is one of the planes on which the stress-energy tensor can be made negative definite. The grid-like cones are the light-cones.

invariant under the shift symmetry

$$T_{\mu\nu} \to \tilde{T}_{\mu\nu} = T_{\mu\nu} + \alpha \, \eta_{\mu\nu} \, , \tag{56}$$

for arbitrary α. By choosing $\alpha = -T_{\mu\nu} \, n_1^\mu n_2^\nu / (n_1 \cdot n_2)$ we can make $T_{\mu\nu}$ strictly negative definite on the whole plane $\mathcal{N}$, *i.e.*, on all linear combinations of n_1 and n_2. Notice that $\mathcal{N}$ is a time-like plane: it intersects the light-cone along two lines (defined by n_1 and n_2 themselves), while a space-like plane would only intersect the light-cone at the origin. By varying n_1 and n_2 inside the 'NEC-violating neighborhood' we can construct a whole family

of such time-like planes $\mathcal{N}$ on which $T_{\mu\nu}$ can be made strictly negative by means of the suitable plane-dependent shift. Then if we can prove that at least one of these $\mathcal{N}$'s must intersect both Γ_+ and Γ_- we have a contradiction. Indeed, on the one hand stability forces $f(s,t,T_{\mu\nu})$ to be positive for all $s \in \Gamma_-$ and $t \in \Gamma_+$. On the other hand, if NEC is violated and one of the $\mathcal{N}$'s intersects both Γ_+ and Γ_- we can pick t^μ and s^μ along these intersections. In this case $T_{\mu\nu}$ is negative definite both on t^μ and on s^μ, and $f(s,t,T_{\mu\nu})$ is thus negative, by its very definition eq. (55).

It is evident that all $\mathcal{N}$'s must intersect Γ_-, since the former are time-like planes while the latter is a space-like hypersurface. On the other hand they may not intersect Γ_+. This clearly happens when, in the frame where stability is manifest, all $\mathcal{N}$'s are almost tangent to the light cone (which is the case if the neighborhood of NEC-violating null vectors is very small) and Γ_+ is a tiny cone around the energy axis. This suggests that in order to establish a connection between NEC and stability we need extra assumptions. These are indeed offered in at least two physically relevant situations:

- The first is the case where all excitations are subluminal, for which the region Γ_+ necessarily contains the light cone. Then all planes $\mathcal{N}$ intersects Γ_+ and we obtain that the assumption of stability and NEC violation contradict each other.

- The other example consists of systems in which superluminal modes may be present but there are two NEC violating null vectors with opposite spatial components $n_{1,2}^\mu = (1, \pm\hat{\mathbf{n}})$. In this case the plane $\mathcal{N}$ generated by n_1 and n_2 contains the ω-axis and necessarily intersect the Γ_+-region. This property trivially holds for systems, where the NEC is violated for all null vectors, in particular for isotropic systems. This is the case which is relevant for cosmological applications.

The fact that NEC follows from stability and subluminality, is perhaps not completely surprising. As discussed for instance if ref. [20] the dominant energy condition, which implies NEC, corresponds to a background with positive energy density and subluminal energy flow. However the energy density and flow of the background do not precisely correspond to the Hamiltonian and velocity of the fluctuations around the background itself. Mathematically this is seen by comparing eq. (9) and eq. (11). The matrix describing the propagation of perturbations involves also the second derivative tensor F_{IJKL} while the energy momentum does not. Conversely $T_{\mu\nu}$ involves the cosmological constant term $\propto g_{\mu\nu}F$ which does not affect the propagation of the modes. These differences suggest that a quick conclusion that stability and subluminality of the modes implies NEC would be too naive. However, the quantities we have used in eq. (52) somewhat cleverly eliminate the contributions of F_{IJKL} and F in such a way that the naive conclusion is correct.

Let us now come back to the two-dimensional case. In $d = 2$ the light-cone is composed of the two null directions $n_\pm^\mu = (1, \pm 1)$. If NEC is violated along both null directions, then,

as before, the stress-energy tensor can be made negative definite on the plane $\mathcal{N}$ generated by n_+ and n_-, which in this case coincides with the whole (ω, k) plane! This plane obviously intersects (in fact contains) both Γ_+ and Γ_-. We are therefore led to the above contradiction. If instead NEC is only violated along, say, n_+^μ, then the region $\mathcal{N}$ on which the stress-energy tensor can be made negative definite is just the line generated by n_+^μ. In this case $\mathcal{N}$ cannot intersect both Γ_+ and Γ_-, and there is no evident contradiction. A peculiar feature of two-dimensional systems is that the condition that NEC be violated along one and only one null direction is equivalent to the requirement that the energy-momentum tensor cannot be diagonalized by a Lorentz-boost. Indeed they are both equivalent to the inequality

$$|T_{00} + T_{11}| < 2|T_{01}| \,. \tag{57}$$

We can therefore summarize the $d = 2$ case as follows. If the stress-energy tensor is diagonalizable by a Lorentz boost and the NEC condition is violated, then the system is unstable. On the other hand we will now show that by giving up the assumption that $T_{\mu\nu}$ be diagonalizable we can construct a system that violates NEC while being perfectly stable, indeed with a positive Hamiltonian. We stress again that such an assumption is only crucial in $d = 2$, given its accidental relation to the 'extent' at which the NEC is violated. Indeed in the next subsection we will also show an example in $d > 2$ of a stable NEC violating system with diagonalizable energy-momentum tensor.

4.1   Examples of stable, NEC-violating systems

In $d = 2$, consider two scalar fields ϕ^I, $I = 1, 2$ with a purely derivative action and choose the following classical background

$$\phi^0 = t, \qquad \phi^1 = x \,. \tag{58}$$

By defining

$$\tilde{B}^{IJ} \equiv B^{IJ} - \delta^{IJ} \tag{59}$$

(with the "meson" field B^{IJ} defined in eq. (2)) we now present an action written in term of $\tilde{B}^{IJ}$ such that $T_{\mu\nu}$ satisfies eq. (57) and such that the quadratic Hamiltonian of small perturbations is positive definite. The advantage of the definition (59) is that $\tilde{B}^{IJ} = 0$ on the background (58). This simplifies some of the algebra. The Lagrangian is of the form

$$\mathcal{L} = \left(\tilde{B}^{00} + \tilde{B}^{11} + 2(1 + a^2)\, \tilde{B}^{01} - (a^2 + b^2)\, \tilde{B}^{00} \tilde{B}^{01} + (a^2 + c^2)\, \tilde{B}^{11} \tilde{B}^{01} \right) \,. \tag{60}$$

with

$$b^2 < 2, \qquad c^2 < 2 \,.$$

21

The energy-momentum tensor for this system can be calculated using equation (11),

$$T_{\mu\nu} = \begin{pmatrix} 1 & 1+a^2 \\ 1+a^2 & 1 \end{pmatrix} \tag{61}$$

which satisfies (57) and violates the NEC. To study the stability of the system we need to write the Lagrangian for the fluctuations and this is done in the appendix.

To conclude this example, let us note that this construction can be readily extended to provide an example of a stable system with non-diagonalizable (and, consequently, violating NEC) energy-momentum tensor for an arbitrary number of dimensions d. In addition to ϕ^0, ϕ^1 one introduces more scalar fields ϕ^i, $i = 2, \ldots, d-1$ with background values

$$\phi_{cl}^i = x^i \ .$$

As before we will denote small perturbations of the ϕ^i-fields about this background by π^i. We also add the following term to the Lagrangian (60)

$$\Delta\mathcal{L} = \left(N^2 \tilde{B}^{ii} - f^2 \big[(\tilde{B}^{0i})^2 + (\tilde{B}^{1i})^2 \big] \right) \tag{62}$$

where we generalized the definition of the $\tilde{B}_{ij}$ in the obvious way. For large enough f^2 the last two terms in eq. (62) provide a dominant contribution of the correct sign to the gradient terms of the $\pi^{0,1}$-fields in the x^i-direction. The first term in eq. (62) provides healthy kinetic and gradient terms for the π^i-perturbations. In the limit $N^2 \gg a^2$, b^2, c^2, 1 one may neglect extra mixings involving π^i-fields and coming from the last two terms.

As anticipated, we now want to show that the requirement that $T_{\mu\nu}$ cannot be diagonalized by a Lorentz boost is not crucial at $d > 2$. Indeed we construct an example of a stable NEC violating system with diagonalizable energy-momentum tensor. We work in $d = 3$, the generalization to $d > 3$ being straightforward. As a first step, we take the stress tensor, in the frame where it is diagonal, to be of the form

$$T_{\mu\nu}^{(\text{diag})} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & -b^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{63}$$

with $(a^2 - b^2) < 0$. By the discussion in the previous section the reference frame where a system with such $T_{\mu\nu}$ has a chance of being manifestly stable, must be boosted in such a way that all the null NEC-violating vectors are close to each other and the $\mathcal{N}$-planes they define are thus almost tangent to the light cone. The above stress-energy tensor violates NEC for null vectors whose spatial component mainly extends in the x^1-direction. We thus consider a large boost in the x^2-direction. In the boosted frame the energy-momentum tensor becomes

$$T_{\mu\nu} = \begin{pmatrix} ch^2 a^2 + sh^2 & 0 & ch\,sh\,(1+a^2) \\ 0 & -b^2 & 0 \\ ch\,sh\,(1+a^2) & 0 & ch^2 + sh^2 a^2 \end{pmatrix} \tag{64}$$

where $sh = \sinh\lambda$ and $ch = \cosh\lambda \gg a^2$, b^2, 1, are the boost parameters. To construct a stable system with stress tensor given by eq. (64), we introduce three scalar field ϕ^I with vev's

$$\phi^I_{cl} = x^I \ . \tag{65}$$

and write the Lagrangian in term of the mesons $\tilde{B}^{IJ}$ defined previously

$$\mathcal{L} = T_{IJ}\,\tilde{B}^{IJ} + \Big(\alpha^2\big[(\tilde{B}^{01})^2 - (\tilde{B}^{11})^2 - (\tilde{B}^{12})^2\big] +$$
$$ch\,sh\,(1+a^2)\big[\tilde{B}^{02}(-\tilde{B}^{11}+\tilde{B}^{22}) + 2\tilde{B}^{01}\tilde{B}^{12}\big]\Big) \tag{66}$$

with $a^2 \sim b^2 \sim \alpha^2 \sim 1$. The stability of this system is discussed in the appendix. In agreement with our general expectations it contains a highly superluminal mode.

5 Effective field theory for solids and fluids

In the previous sections we focused our attention on systems involving derivatively coupled scalar fields. It is not obvious a priori whether our results can be exported to different physical systems. In particular, can we apply them to fluids? After all, most of the cosmic history is dominated by a mixture of fluids, their physical relevance in cosmology certainly overwhelms that of scalar fields. In this section we show that our results are straightforwardly applied to universe filling continua, solids, and in particular to perfect fluids. Indeed the dynamics of these systems is described by an effective theory of three derivatively coupled scalars, as we will now discuss. Although most of the results we will discuss are well known, we believe our presentation to be still useful as it provides a direct connection to the subject of the previous sections and to the investigations on modifications of gravity. For a recent thorough review on the Lagrangian formulation of fluid dynamics we refer to Ref. [21]. Our treatment is closer to the effective (non-relativistic) theory for supersolids developed in Ref. [22], but we work from the beginning with the fully relativistic formulation.

5.1 Generalities

Let us consider a mechanical continuum filling all of (3-dimensional) flat space and such that at each space point there resides just one (and only one) dynamical entity. There thus exists an invertible mapping ϕ between coordinate space and the internal space $\mathcal{M}$ labeling the dynamical variables (for instance the positions of the atoms in a crystal). This is made explicit by working with a field theory of three scalar fields $\phi^I(x,t)$, $I = 1,2,3$, which describe both the mapping and the dynamics. Since $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^3$ we can always choose field coordinates ϕ^I such that in some given stationary state, for instance the ground state itself, the mapping is just $\phi^I(x^i,t) = x^I$. In what follows we shall indeed

stationary configuration $\mathcal{M} \xrightarrow{\phi_t} \mathbb{R}^3$

x $\quad\quad$ $t_*(x)$

23

(margin note: now $\phi^2 = \phi^2(x,t)$)

assume $\phi^I(x^i, t) = x^I$ on the ground state. Notice also that the ϕ^I's correspond to a choice of space coordinates comoving with the solid. This is to say that, given an arbitrary time dependent configuration $\phi^I(x^i, t)$, we can choose space-time coordinates by

$$x'^i = \phi^i(x, t) \qquad t' = t \tag{67}$$

for which the scalar fields take constant value

(margin note: in the sense that each constant choice of $\phi(x,t) = \phi(x)$ is)

$$\phi'^I(x', t') = \phi^I(x(x'), t') = x'^I \ . \tag{68}$$

Of course in the primed coordinates all the information about the state of the solid is contained in the metric, which is no longer Minkowskian.

Let us work in Minkowskian coordinates around the ground state $\phi^I(x^i, t) = x^I$, and let us consider the symmetries of the system. Ordinary space translations $T_x : \phi^I(x, t) \to \phi^I(x + a, t)$ are obviously a (spontaneously) broken symmetry. On the other hand we would like to focus on *homogeneous solids*, for which there still exists an unbroken translational invariance. It is evident that the only way to achieve that, is to further assume the additional internal shift symmetry

$$T_\phi : \quad \phi^I(x, t) \to \phi^I(x, t) + c^I \ . \tag{69}$$

(margin note: condition)

This way the ground state is invariant under a combination of ordinary translations and T_ϕ with $a^I = -c^I$. Therefore $T_x - T_\phi$ represents the conserved three momentum for processes corresponding to small fluctuations around the vacuum (for instance in the scattering of phonons). The broken generators $T_x + T_\phi$ are instead in one to one correspondence, by Goldstone theorem, with 3 gapless excitations, the phonons. At the lowest derivative level, the relativistic invariant Lagrangian for such a system will clearly be written as a function of the 'meson' fields $B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$. It thus falls precisely into the class of field theories we have been discussing throughout the paper! If the Lagrangian is a generic function of the B^{IJ}, the system will describe an homogeneous but otherwise anisotropic solid. More symmetric systems can be constructed by assuming additional internal symmetries acting on ϕ^I. For instance we can consider an invariance

$$\phi^I \to D^I{}_J \phi^J \tag{70}$$

where $D^I{}_J$ belong to some discrete subgroup of $O(3)$. In this case our system will have the same macroscopic properties as a crystal with the corresponding symmetry. We can also consider the limiting case in which $D^I{}_J$ is extended to the full rotation group $SO(3)$. In which case our system will essentially describe a jelly.

The jelly system does not correspond yet to the most symmetric possibility. Indeed one can consider a system for which this symmetry is extended to the group of volume preserving diffeomorphisms

$$\phi^I \to \hat{\phi}^I(\phi), \qquad \det \frac{\partial \hat{\phi}^I}{\partial \phi^J} = 1 \ . \tag{71}$$

While this is a huge internal symmetry it still allows a non trivial invariant quantity: the determinant of B^{IJ}. The most general Lagrangian for such a system is then

$$\mathcal{L} = F(B) , \qquad B \equiv \det B^{IJ} , \tag{72}$$

where F is a generic function. It is pretty obvious that, with three fields, we cannot further extend the volume preserving diffeomorphism to a larger symmetry that still allows for a non-trivial action. What is the physical interpretation of a system of this type? The answer is intuitively obvious: our Lagrangian describes the dynamics of a (relativistic) fluid. Consider indeed the ground state of the system $\phi^I(x) = x^I$. We can imagine, starting from this configuration, to build another one by moving around the individual points: $x \to f(x)$. In the new configuration the field profile will be given by

$$\phi^I(x) = (f^{-1})^I(x) . \tag{73}$$

Now, the intuitive notion of fluid suggests that its energy does not change if we do not compress it, no matter what different form its individual parts assume. Therefore we expect that for a coordinate transformation f that preserves the volume in physical space, the energy of the system should remain the same. This is obviously true in our case because eq. (73) can be compensated by an internal symmetry transformation.

5.2   Fluids

In this subsection we want to work out in some detail the properties of the relativistic fluid that follow from the Lagrangian in eq. (72). We want first to make contact with the standard description that can be found, for instance, in Ref. [23]. In particular we want to find the field theoretic description of quantities like the fluid 4-velocity, the current, the pressure and energy density.

The definition of the ϕ's as comoving coordinates makes the characterization of the fluid velocity field $u^\mu(x)$ immediate: u^μ is a vector field along which all the ϕ^I's stay constant,

$$\frac{d}{d\tau}\phi^I(x(\tau)) \equiv u^\mu \, \partial_\mu \phi^I = 0 \qquad I = 1, 2, 3 . \tag{74}$$

These orthogonality constraints together with the normalization constraint $u^\mu u_\mu = -1$ completely determine u^μ (except for the discrete choice of the direction, which is set by the request that u^μ be future-directed). Indeed the orthogonality constraints forces u^μ to be proportional to the 'vector product' $\epsilon^{\mu\alpha\beta\gamma} \partial_\alpha \phi^1 \partial_\beta \phi^2 \partial_\gamma \phi^3 \propto \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K$. The proper normalization is

$$u^\mu = \frac{1}{6\sqrt{B}} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{IJK} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^K . \tag{75}$$

$\mathcal{L} = \det B$

$$\delta dS = \left(\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\partial_\mu\phi + \frac{\partial \mathcal{L}}{\partial x^\mu}\,\delta x^\mu\right)dx + \mathcal{L}\,\partial_\mu \delta x^\mu dx^\mu$$

$$= \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi + \mathcal{L}\,\delta x^\mu\right]$$

sym is
$\phi \rightarrow \phi(x^\mu - \epsilon\delta^\mu_\alpha)$
$x \mapsto x$

Note that this expression is invariant under generic (not necessarily volume-preserving) diffeomorphisms acting on the ϕ's. This is correct, since what matters for characterizing u^μ is only that the ϕ's are comoving coordinates. By the above equation, u^μ is associated to the volume form Ω over the fluid space
$\delta\phi = \phi(x^\mu - \epsilon\delta^\mu_\alpha)$ $\quad \delta\phi = -\epsilon\partial_\alpha\phi$ $\quad \delta x^\mu = \epsilon\delta^\mu_\alpha$

$$u_\mu dx^\mu = \frac{1}{\sqrt{B}} \star \left(\tfrac{1}{6}\epsilon_{IJK}d\phi^I \wedge d\phi^J \wedge d\phi^K\right) \equiv \frac{1}{\sqrt{B}} \star \Omega \tag{76}$$

and the one form $\star\Omega \equiv J_\mu dx^\mu$ is naturally interpreted as the current of fluid points. Consistent with this interpretation, the current is conserved

$$d \star (J_\mu dx^\mu) = d(\tfrac{1}{6}\epsilon_{IJK}d\phi^I \wedge d\phi^J \wedge d\phi^K) = 0 . \tag{77}$$

Notice that the conservation of J_μ, rather than arising on-shell from a symmetry of the Lagrangian via Noether's theorem, is already valid off-shell since the number of degrees of freedom, that is the volume of $\mathcal{M}$, is fixed by construction. From $J_\mu = n \cdot u_\mu$ it also follows that the density of particles n is $\quad \int$

$$n = \sqrt{B} \tag{78}$$

What about the equation of state? The stress-energy tensor is given by eq. (11), with $F_{IJ} = \frac{\partial F}{\partial B^{IJ}} = F' B B_{IJ}^{-1}$,

$$T_{\mu\nu} = 2F' B B_{IJ}^{-1} \partial_\mu\phi^I \partial_\nu\phi^J - g_{\mu\nu}F . \tag{79}$$

On the other hand for a perfect fluid the general expression is $T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu + p\,g_{\mu\nu}$. The energy density and pressure therefore can be extracted by contracting $T_{\mu\nu}$ with $g^{\mu\nu}$ and with $u^\mu u^\nu$. By making use of the constraints eq. (74) we get ρ and p as functions of B,

$$\rho = -F(B) , \qquad p = -2F'(B)\,B + F(B) . \tag{80}$$

Note that although the stress-energy tensor eq. (79) for a generic configuration depends on the full structure of $\partial_\mu\phi^I$, the energy density and pressure depend on B only. This is physically correct: like the Lagrangian they must be scalar quantities only sensitive to compressional modes.

By eqs. (78, 80), the three quantities n, p, ρ are determined by just one parameter. On the other hand, the equation of state of a fluid corresponds to just one constraint $f(n, p, \rho) = 0$. The additional local constraint arises, in our description, from the homogeneity of the initial entropy density together with the isoentropicity of the fluid transformations, as we now explain. In the ordinary treatment, see [23], the current and energy momentum conservation equations $(\partial_\mu J^\mu = \partial_\mu T^{\mu\nu} = 0)$ can be combined to give an equation that establishes the conservation of the entropy per comoving volume σ along the flux

$$0 = u^\mu\partial_\mu\sigma = \frac{1}{kT}\left\{p\,\partial_\mu(1/n) + \partial_\mu(\rho/n)\right\}u^\mu \tag{81}$$

26

$\frac{dP}{dV}\big|_{\sigma} = -$

$P = -\frac{dV}{dV}\big|_{\sigma}$

$T\;F\;\;v$
$G\;\;\;\;U$
$P\;\;h\;S$

$-P = \frac{dU}{dV}$

$U = \int V = \int \frac{N}{2} = \int \frac{N}{\sqrt{B}}$

$\frac{dP}{dV}\big|_{\sigma}$

$= -\frac{1}{1V}\{s^v\}_{1_J}$

$= -J - \frac{1}{dt_L v}\,s = -\left(1 + \frac{1}{J_L v}\right)J$

In other words the motion of each fluid portion is isoentropic, although the entropy density σ varies in general through space. Therefore the above equation introduces an additional, non-algebraic—i.e., differential—constraint between n, p, ρ. Now, the fluid we are describing by our field theory is homogeneous from all stand points, in the sense that any given configuration can always be brought to the homogeneous ground state by means of *mechanical* transformations. In other words, in our formalism there is no room for genuine thermodynamic degrees of freedom: to increase the temperature we have to compress the fluid, we cannot 'heat it up'. In particular our fluid is homoentropic (the entropy per comoving volume is the same for all fluid elements), in which case the last equation is replaced by the stronger one $\partial_\mu \sigma = 0$ which corresponds precisely to the additional algebraic constraint we have between n, p, ρ.

We can specify our analysis of quadratic fluctuation to the fluid. Consider a background $\phi^I = \alpha x^I$, where α parameterizes the level of compression $B = \alpha^6$, and let us parameterize the perturbations by $\phi^I = \alpha(x^I + \pi^I)$. By expanding eq. (72), integrating by parts and neglecting total derivatives we get the Lagrangian for the Goldstones,

$$\mathcal{L} = -F'(B)B \cdot (\dot\pi^I)^2 + (F'(B)B + 2F''(B)B^2) \cdot (\partial_J \pi^J)^2 \ . \tag{82}$$

Only the longitudinal Goldstone has a gradient energy term; as expected it is the only propagating degree of freedom: sound waves are longitudinal. The dispersion relation corresponds to a speed of sound c_S^2

$$c_S^2 = \frac{F' + 2F''B\,B}{F'} \equiv \frac{p'(B)}{\rho'(B)} = \frac{dp}{d\rho}\bigg|_{\sigma=\text{const}} \ , \tag{83}$$

where in the last equality we have made use of the equations of state eq. (80) and have thus made contact with the usual expression for the speed of adiabatic sound waves in a fluid. On the other hand, the transverse phonons, $\pi^i \propto \epsilon^{ijk}\partial_j f_k$ do not propagate. Their general solution at the infinitesimal level is linear in time

$$\pi_T^i = \epsilon^{ijk}\partial_j \left(f_k(x) + g_k(x)t\right) \tag{84}$$

and corresponds to vortices in constant rotation. In the linearized approximation the fluid 4-velocity is indeed $u^\mu = (1, -\dot\pi^i)$, so that we have[11] $v^i = \epsilon^{ijk}\partial_j g_k(x)$. The trivial time evolution of infinitesimal vortex perturbations follows directly from the internal volume preserving diffeomorphism invariance of the fluid, eq. (71). As a matter of fact, (also at the full nonlinear level) this symmetry implies the existence of an infinite set of conserved currents and

---

[11]For large t the linearized approximation for this solution breaks down. Notice however that the bulk of the value of π_T^i at large t can be eliminated by a volume preserving diffeomorphism around the stationary solution. This means that what matters physically is just the time derivative $\dot\pi_T^i$, which is just the fluid local velocity.

of the associated charges

$$J_\mu^f = F'(B)B\,B_{IJ}^{-1}\,\partial_\mu\phi^J f^I(\phi) \tag{85}$$

where f^I is any vector function satisfying $\partial f^I/\partial\phi^I = 0$. The conservation of the infinite set of associated charges basically corresponds to the trivial dynamical evolution of the vortices.

This can be more explicitly seen by working in the comoving coordinate frame, *i.e.* by choosing the ϕ^I as space coordinates and by taking $x^i(\phi, t)$ as the dynamical variable. From the basic relation $\phi^I(x(\phi, t), t) = \text{const}$ we obtain by differentiation

$$\dot\phi^I = -\partial_k\phi^I \dot x^k \equiv -W^I{}_k \dot x^k \,. \tag{86}$$

By noticing that $\det(\delta^{ij} - \dot x^i\dot x^j) = (1 - \dot x^2)$ we have

$$B = (\det W)^2(1 - \dot x^2)\,, \tag{87}$$

so that the action now reads

$$S = \int dt\, d^3\phi\, \frac{F\left((\det W)^2(1 - \dot x^2)\right)}{|\det W|}\,. \tag{88}$$

This action is manifestly invariant under volume preserving diffeomorphisms acting on the comoving coordinates ϕ^I. On the under hand Lorentz symmetry acts now as a combined transformation of the time variable t and the dynamical fields $x^i(\phi, t)$, and invariance of the action is not explicitly manifest. The volume preserving diffs are realized on the dynamical variables as $\delta x^i = f^I\partial_I x^i$, with $\partial_I f^I = 0$, so that the set of conserved charges is given by

$$C^f = \int d^3\phi\, \frac{\delta L}{\delta\dot x^i} f^I\partial_I x^i \tag{89}$$

For $f^I = a^I = \text{const}$ the 3 conserved charges just correspond to spatial momentum. The genuine fluid invariants are obtained by choosing $f^I = \epsilon^{IJK}\partial_J\left(a_K\delta^3(\phi - \phi_0)\right)$, for arbitrary constant a_K and ϕ_0. By eqs. (88, 89) we obtain, after integration by parts, that the local 2-form

$$\partial_J\left[F'|\det W|\,\dot x_i\partial_K x^i\right]d\phi^J \wedge d\phi^K = d(F'\sqrt{B}\,u_i dx^i) \tag{90}$$

is time independent, where in the last equality we have used the relation $\dot x_i = \sqrt{1 - \dot x^2}\,u_i$ for the space components of the 4-velocity. Considering any 2-surface Σ bounded by a closed circuit $C = \partial\Sigma$ we have then

$$\frac{d}{dt}\oint_C F'\sqrt{B}\,u_i dx^i = \frac{d}{dt}\int_\Sigma d\left[F'\sqrt{B}\,u_i dx^i\right] = 0\,. \tag{91}$$

From the point of view of coordinate space the circuit C depends on time as it moves with the fluid flux, so the statement is that the line integral of $F'\sqrt{B}u_i$ is conserved along the

flux. This quantity computed for all closed paths C measures indeed the vorticity of the fluid's configuration. Our result states that vorticity, thus defined, does not evolve with time. This result corresponds to the relativistic generalization of Kelvin's theorem. In the non-relativistic limit, we have $p \ll \rho$, which from eq. (80) corresponds to $F(B) \simeq \sqrt{B}$. In this case the conserved quantity becomes simply the circuit integral of the velocity v_i, as stated in Kelvin's theorem.

In the last discussion Lorentz invariance was not explicitly manifest. We can however easily express the result in eq. (91) in a manifestly Lorentz covariant manner. From the definition of the 4-velocity and from eq. (91) the relevant one-form we should focus on is

$$F'\sqrt{B}\,u_\mu dx^\mu \equiv V_\mu dx^\mu\,. \tag{92}$$

Using V the equations of motion

$$\partial_\mu \left[F'B\,B_{IJ}^{-1}\,\partial^\mu \phi^J\right] = 0 \tag{93}$$

are written conveniently as

$$(dV) \wedge d\phi^J \wedge d\phi^K \epsilon_{IJK} = 0 \tag{94}$$

The geometric implication of this formula is that $(dV) = 0$ on any two surface Σ tangent at all points to the four velocity u^μ. This is easily seen by working in comoving coordinates, where $u^\mu = (1, 0, 0, 0)$ and eq. (94) reads $(dV)_{0I} = 0$ for $I = 1, 2, 3$. We can create such a surface Σ by choosing a spacelike closed cycle C_0 and by considering the world-surface it describes along the fluid motion up to some final cycle C_1. We have $\partial \Sigma = C_0 - C_1$ so that by Stokes theorem we find

$$0 = \oint_\Sigma dV = \oint_{C_0} V - \oint_{C_1} V \tag{95}$$

As expected, the quantity $\oint F'\sqrt{B}\,u_\mu dx^\mu$ is conserved along the flow. Notice however that neither C_0 nor C_1 has to be necessarily chosen at fixed time. To summarize, our result states that given a two-surface Σ with the topology of a cylinder and tangent to the local 4-velocity at any point, the integral of $V = F'\sqrt{B}\,u_\mu dx^\mu$ around Σ is independent of the cycle chosen to compute the integral.

5.3   From Fluid to Superfluid and to Ghost Condensate

We have seen that vorticity is conserved through the time evolution of the system, corresponding to the 'trivial' dynamics of the vortices. In particular if vorticity vanishes on any initial time surface it will vanish throughout the time evolution. For this subclass of the fluids configuration it is natural to expect our system to be described by just one scalar field instead of three. Let us see how this works.

The most direct, although at first sight unintuitive, way to proceed is to introduce an auxiliary 1-form V_μ and start with the following action

$$\mathcal{S}_{V,\Omega} = \int G(X) - V \wedge \Omega \tag{96}$$

where, $X = -V^\mu V_\mu$ and Ω is the fluid volume form defined before. The V equations of motion

$$-2G'(X)V^\mu = (\star\Omega)^\mu \equiv J^\mu \tag{97}$$

allows us to algebraically solve for V_μ as a function of J^μ and to write eq. (96) as a functional of the fluid variables

$$\int G(X) - V \wedge \Omega \Big|_{\delta S/\delta V = 0} = \int F(B) \,. \tag{98}$$

The functions $G(X = -V^2)$ and $F(B = -J^2)$ are the Legendre transforms of one another, and V_μ and J^μ are the conjugated variables. By the usual properties of Legendre transforms, eqs. (96, 98) also imply the conjugate to eq. (97)

$$2F'(B)J_\mu = V_\mu \,. \tag{99}$$

Up to a factor of 2, the form V coincides on-shell with the form V we defined in the previous section, hence our notation. (This is also evident by varying eq. (96) with respect to ϕ_I and comparing to eq. (94)). Notice that eqs. (97, 99) imply the scalar relation

$$2F'(B) = \frac{-1}{2G'(X)} \tag{100}$$

as well as

$$B = X(G'(X))^2 \qquad X = B(F'(B))^2 \,. \tag{101}$$

Moreover eq. (98) can be also written as

$$G - F = -2BF' = 2XG' \,. \tag{102}$$

We have now all the ingredients to discuss fluid motion at zero vorticity. From the discussion at the end of last section, this situation is realized when

$$V = d\psi \tag{103}$$

where ψ is a scalar field. In this case $u_\mu \propto \partial_\mu \psi$, corresponding to the Lorentz covariant generalization of potential flow in a fluid [24]. When eq. (103) holds, the equation of motion for ϕ are trivially satisfied. However the dynamics of ψ is non-trivial and is determined by taking the divergence of eq. (97)

$$\partial_\mu(G'(X)\partial^\mu \psi) = 0 \,. \tag{104}$$

30

We have therefore established that the solutions in the absence of vorticity are into one-to-one correspondence with the solutions for the dynamics of a derivatively coupled scalar with Lagrangian $L = G(-\partial_\mu \psi \partial^\mu \psi)$. Moreover since $\partial_\mu \psi \propto u_\mu$ is a time-like directed vector, the scalar ψ depends linearly on time in the ground state $\psi = ct$. This is precisely the dependence of the charged field phase in a superfluid. We have thus established that, in the absence of vortices, the classical dynamics of a fluid is equivalent to that of a superfluid with suitably chosen Lagrangian (cf. [25]). As a further check of this equivalence, notice that the energy momentum tensors in the two descriptions coincide. This is manifest by eq. (98), since the fluid and superfluid action differ by the integral of the four form $V \wedge \Omega$ which does not depend on the metric. It can also be checked directly by using eq. (102) [12].

The relation between fluid and superfluid is a close relative of the duality between a 2- and a 0-form. Consider indeed eq. (96), with $\Omega = dA_2$. The 2-form A_2 acts like a Lagrange multiplier setting $dV = 0$ thus reducing to a 0-form $V = d\psi$ action. On the other hand, by solving the V equations first we would get a fully equivalent local action for A_2. The fluid is distinguished from the two form because $\Omega = d\phi^1 \wedge d\phi^2 \wedge d\phi^3$ is not the most general field strength a two form can have (it is a function of 3 field variables, instead of 6). However although the off-shell field space of the fluid is of smaller dimensionality that for the 2-form theory, the space of solutions is nonetheless bigger. This is perhaps counterintuitive, but it becames less so if one recalls that in a gauge theory some of the field variables act just as Lagrange multipliers. We do not want to explore these issues further here, but just make a few extra remarks elucidating the duality character of the fluid/superfluid relation. Notice that dynamical equations and structural constraints are interchanged. So the conservation of J_μ is structural for the fluid while it follows from the equations of motion in the supefluid. Conversely vorticity is dynamically conserved in the fluid and structurally zero in the superfluid. More importantly the fluid/superfluid relation is only valid for the classical action on-shell, and breaks down when quantum corrections are taken into account. In fact in a duality relationship the quantum perturbation series are not mapped into one another, and when one description is weakly coupled that other is not.

This last property and others are made clear by focussing on a particular superfluid, the so called ghost-condensate model, which also provides an interesting and consistent modification of gravity [11]. The model is defined by assuming that the function G is positive and has a minimum (where $G' = 0$) at some finite point $X = c_*^2$ in field space. This is the ghost-condensate point. At this point, although the field is time dependent $\psi = c_* t$, the energy momentum tensor is still proportional to the metric. Thus the universe is in a de Sitter phase with an effective cosmological constant $\equiv -G(c_*^2)$, which differs from the

---

[12]Indeed we originally derived our relation between fluid and supefluid by equating their energy-momentum and number current. This leads precisely to the relation described at the beginning of this section. We believe however the Lagrangian approach we followed is analitycally more transparent.

genuine cosmological constant defined at zero field $-G(0)$. What does this point correspond to in the fluid description? By the first of eqs. (101) the answer is $B = 0$, i.e the point of infinite dilution, while, not surprisingly, eq. (100) tells us that this is a singular point. More precisely, by solving the second of eqs. (101) we find two solutions, corresponding to a branch singularity

$$F_\pm(B) = -\sqrt{B} f_\pm(B) + F_0 \tag{105}$$

where $f_\pm$ are smooth functions satisfying

$$\lim_{B \to 0} f_\pm(B) = \pm c_* \,. \tag{106}$$

F_+ and F_- respectively corresponds to the superfluid branches $X > c_*^2$ ($G' > 0$) and $X < c_*^2$ ($G' < 0$). As $B \to 0$ the F_+ branch becomes a diluted dust with positive energy density $\rho \sim c_* \sqrt{B}$ and positive pressure[13] superposed to a cosmological constant $F_+(0) = F_0$. Notice that by eq. (102) $F_0 = G(c_*^2)$, which propels de Sitter expansion, is now truly the cosmological constant. On the ther hand, the branch F_- corresponds to a diluted fluid with negative energy density $\rho \sim -c_* \sqrt{B}$ but positive pressure $p = B^{3/2} f'_-(B)$. The two branches $X > c_*^2$ and $X < c_*^2$ thus correspond to two distinct fluids. These two fluids cannot mix as they are always separated by a region of infinite dilution, where $B = 0$. Moreover the regions corresponding to F_+ will tend to expand as they correspond to regions of higher density and pressure in an ordinary fluid. The regions corresponding to F_- will instead contract [26, 27] as they correspond to a fluid with negative energy density but whose pressure grows with compression[14]. Notice also that in the region with $G' < 0$ the superfluid theory features a gradient instability and this is consistent with the imaginary speed of sound implied by eq. (82). However in the fluid description, unlike for the superfluid, the corresponding mode is also a ghost, in the sense that its time derivative term has the wrong sign in the action, see eqs. (82, 100). This makes sense since duality corresponds to a canonical transformation that exchanges $p \to -q$ and $q \to p$ for some of the variables[15], so that if in one description the potential is negative in the dual description it is the kinetic term to be negative. As a final remark, the quantum cut-off Λ of the fluid theory goes to zero like $\rho^{1/4} \sim B^{1/8}$ as $B \to 0$. Therefore the fluid is formally infinitely strongly coupled at the ghost-condensate point[16] while the superfluid has a finite UV cut-off, and thus represents a well to do effective field theory. This, and the presence of two distinct phases with opposite energy density, represent the novelty of the ghost condensate model with respect to ordinary fluids.

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[13] By eq. (102) we have $p = B^{3/2} f'_+ \equiv G(X) - G(c_*^2) \geq 0$.

[14] This follows from Euler equations where acceleration along fluid moton is proportional to $\nabla p / \rho$.

[15] Example B and E in electric-magnetic duality.

[16] For ordinary fluids associated to weakly coupled theories, the physical cut-off Λ_{phys} where the hydrodynamic description breaks down is always smaller than $\rho^{1/4}$ and is determined by the inverse mean free path. For instance in a hot plasma we have $\Lambda_{\text{phys}} \sim g^2 T \ll T \sim \rho^{1/4}$.

6 Degenerate cosmic supersolids

Throughout the paper we have been focussing on the scalar field excitations, neglecting the backreaction of gravity. As we explained in the beginning, this is consistent as long as we limit ourselves to distances $r \ll r_G = M_{Pl}/\Lambda^2$, where Λ is the characteristic energy scale of the scalar Lagrangian. The gravitational length r_G normally controls both the mixing between the Goldstone boson and the graviton, basically the graviton mass, and the curvature of space time[17]. Because of the coincidence of these two length scales, the mixing between graviton and Goldstones cannot normally be discussed around flat space, and is thus qualitatively different from the Higgs phenomenon in ordinary gauge theories. One can however think of peculiar situations where $|T_{\mu\nu}| \ll \Lambda^4$ can be practically neglected, while the graviton mass is still of order $1/r_G$. In this situation, the propagation of gravity is modified at distances r_G where curvature can be neglected, thus defining infrared modifications of gravity that are the analogue of the ordinary Higgs mechanism. These theories include the Fierz-Pauli model, the ghost condensate model [11] and the general class of massive gravities of [13] and are indeed particular cases of the class of scalar theories that we have been focussing on. In these models, the difficulties and peculiarities associated to the dispersion relation of the Goldstone fields are also directly connected to the difficulty in violating NEC in general. Indeed when $T_{\mu\nu} = 0$ eq. (52) simply reduces to $L(p,q) - L(q,p) = 0$. Since the matrix A_μ^I is non-zero, it immediately follows that either Γ_- or Γ_+, or both, do not exist. Therefore, these theories can only be acceptable in limiting situations in which some dispersion relation is degenerate, *i.e.* either $\omega = 0$ or $k = 0$, and the corresponding mode does not propagate. This is in agreement with the results of Ref. [13] where the conditions for stability were studied in detail. Because of these degenerate dispersion relations, the stability of these models is threatened both by higher order effects, ex. higher derivative terms, and by slight deformations of the background. However in Ref. [13] it was also shown that in some cases the singular dispersion relations can be protected by internal symmetries of the scalar manifold, thus ensuring in a natural way the stability of the system.

Now, in the presence of these non-propagating modes, the general discussion of the stability conditions in section 4 does not apply directly, since there we had assumed that the determinant of the kinetic matrix $L_{IJ}(p)/k^2$ had $2N$ single roots. Therefore, as in [13], we must take a case-by-case approach, and do a very similar analysis to what was done there, to look at what happen if we relax the assumption that the energy-momentum tensor vanish. In particular we would like to know if there are stable cases when $w < -1$.

The case we study is not as general as the one considered in section 4 as we consider only 4 fields ϕ_I with $I = 0, 1, 2, 3$ in a rotationally invariant situation. However, the assumption of rotational symmetry is not a limitation when thinking of cosmological applications, while the

---

[17]We are treating the function $\tilde{F}$ (see eq. (3)), as well as its derivatives and $\partial\phi/\Lambda^2$ as $O(1)$ quantities.

4-fields corresponds to the simplest case where the full translation group is broken. Setting as before $\Lambda = 1$ the background values for ϕ_I are taken to be

$$
\begin{aligned}
\phi_0 &= \alpha t \\
\phi_i &= \beta x^i \quad i = 1, 2, 3.
\end{aligned}
\tag{107}
$$

Here, α and β are arbitrary. By the discussion in section 5 this system consists of a superfluid ($\phi_0 \propto t$) interacting with a solid ($\phi_i \propto x^i$), hence the denomination supersolid.

The Lagrangian for the fluctuations π_0, π_i around the background (107) can be written, in the notation we have been using so far, as

$$
\begin{aligned}
\left(F_{00} + 2F_{00,00}\alpha^2 \right) (\partial_t \pi_0)^2 + \left(-F_{00} + F_{0i,0i}\frac{\beta^2}{2} \right) (\partial_i \pi^0)^2 + \left(\frac{F_{0i,0i}}{2}\alpha^2 + \frac{F_{ii}}{3} \right) (\partial_t \pi_i)^2 \\
- \frac{1}{3} \left(4F_{00,ii} + F_{0i,0i} \right) \alpha\beta\pi_0 \partial_j \partial_t \pi_j + \left(-\frac{1}{3}F_{ii} + \beta^2 \frac{3F_{ii,jj} - F_{ij,ij}}{48} \right) (\partial_i \pi_j)^2 \\
+ \beta^2 \left(\frac{3F_{ij,ij} - F_{ii,jj}}{48} \right) (\partial_i \pi_i)^2 \quad (108)
\end{aligned}
$$

We can also calculate the energy-momentum tensor in this background,

$$
T_{\mu\nu} = c\eta_{\mu\nu} + \left(\alpha^2 F_{00}\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{3}F_{ii}\beta^2 \delta_{\mu j}\delta_{\nu j} \right)
\tag{109}
$$

The null energy condition is violated if

$$
p + \rho = \alpha^2 F_{00} + \frac{1}{3}F_{ii}\beta^2 < 0
\tag{110}
$$

To analyze this system, it is convenient to rescale π_0 and π_1 and rewrite the Lagrangian as in [12, 13],

$$
\begin{aligned}
2m_0^2 (\partial_t \pi_0)^2 + m_1^2 (\partial_t \pi_i)^2 + \left[m_1^2 - (p + \rho) \right] (\partial_i \pi_0)^2 + \left(4m_4^2 - 2m_1^2 \right) \pi_0 \partial_t \partial_i \pi_i \\
- m_2^2 (\partial_i \pi_j)^2 - \left(m_2^2 - 2m_3^2 \right) (\partial_i \pi_i)^2 \quad (111)
\end{aligned}
$$

The only difference with the Lagrangian of [13] is the appearance of $p + \rho$ in the coefficient of $(\partial_i \pi_0)^2$. As we explained before, for rotational invariant systems the positivity of the Hamiltonian is a necessary and sufficient condition for stability. In eq. (111) for $p + \rho < 0$, the gradient energy is made positive by having m_1^2 sufficiently large and negative. In this case, however, the coefficient of $(\partial_t \pi_i)^2$ would have the wrong sign, and the Hamiltonian would still be not positive definite. We thus recover our general result that, in the rotational invariant case, we cannot achieve stability with all the modes propagating. We have thus to

34

consider the degenerate cases. The analysis is immediate in the vector sector (the modes π_i^T that satisfy $\partial^i \pi_i^T = 0$) where the Lagrangian is simply

$$\mathcal{L}_{\text{vector}} = m_1^2 \left(\partial_0 \pi_i^T\right)^2 - m_2^2 \left(\partial_i \pi_j^T\right)^2 \tag{112}$$

In the scalar sector, consisting of the modes π_0 and π_L defined by $\pi_i = 1/\sqrt{\partial_i^2} \partial_i \pi_L$, the Lagrangian can be written in momentum space as

$$k^2 \begin{pmatrix} \pi_0 & \pi_L \end{pmatrix} \begin{pmatrix} m_0^2 \nu^2 + m_1^2 - (p+\rho) & -i\mu\nu \\ i\mu\nu & m_1^2 \nu^2 - \eta \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_L \end{pmatrix} , \tag{113}$$

with $\nu = \omega/k$, $\mu = (2m_4^2 - m_1^2)$ and $\eta = 2(m_2^2 - m_3^2)$. The analysis of the stability of this theory is very similar to the analysis done in [13]. The absence of ghosts and gradient instabilities constrains the sign of the coefficients in the diagonal entries of the kinetic matrix

$$m_0^2 > 0 \qquad m_1^2 > 0 , \qquad -m_1^2 + (p+\rho) > 0 , \qquad \eta > 0 \tag{114}$$

This condition agrees with our general result: for $p + \rho < 0$ there cannot be stability if all modes are normal. We must however analyze with care the situation where some of the modes become degenerate, corresponding to some root ν going to either 0 or ∞. If the unstable modes can be forced to these limiting dispersion relation by consistent symmetry requirements, then the system has a chance of becoming acceptable. The limiting situations arise when one or more of the coefficients in eq. (114) equal 0. For instance for $m_0^2 = 0$ one of the roots is $\nu = \infty$ and the corresponding field has a pure gradient quadratic action. By the arguments in refs. [12, 13], we may thus forget about this mode, provided it does not reappear when higher derivative terms are considered or the solution is slightly perturbed. The other mode has a normal dispersion relation and is not a ghost as long as $m_1^2, \eta > 0$, irrespective of $p + \rho < 0$. However we know of no symmetry that can naturally enforce $m_0^2 = 0$ without introducing extra unprotected degenerate modes, so this possibility does not seem viable. Similarly, the case $\eta = 2(m_2^2 - m_3^2) = 0$ allows to send the unwanted root at $\omega = 0$ while keeping $p + \rho < 0$. But also this second choice cannot be enforced by symmetries.

In Ref. [13] several internal symmetries of the scalar field manifold which protect the required degeneracies at $T_{\mu\nu} = 0$ were presented. These symmetries can be thought of as generalizations of the volume preserving diffeomorphism invariance which is relevant for ordinary fluids. In what follows we would like to survey the various symmetries to see whether they can make the $p + \rho < 0$ case acceptable.

First we consider the symmetry

$$\phi^i \to \phi^i + \xi^i(\phi^0) , \tag{115}$$

which forces $m_1 = 0$. This symmetry implies the presence of three instantaneous sound waves with dispersion relation $k = 0$, so we may call this system a rigid supersolid. Notice that

while for a fluid the internal symmetry (see eq. (71)), is making the solid totally deformable, eq. (115) is making the solid infinitely rigid. Cosmological and phenomenological properties of rigid supersolids with zero energy-momentum were studied in [16, 28]. In general symmetry (115) implies that the vector modes have a dispersion relation of the form $k = 0$. The Lagrangian for the scalar modes can be written as

$$
k^2 \begin{pmatrix} \pi_0 & \pi_L \end{pmatrix} \begin{pmatrix} 2m_0^2 - (p + \rho) & -2im_4^2\nu \\ 2im_4^2\nu & 2(m_3^2 - m_2^2) \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_L \end{pmatrix} , \tag{116}
$$

According to the previous discussion one mode has dispersion relation $k = 0$ and the second one

$$
\omega^2 = \frac{p + \rho}{\left(m_0^2 - \frac{m_4^4}{m_3^2 - m_2^2} \right)} k^2 . \tag{117}
$$

For real ω the sign of the corresponding residue is the same of $p + \rho$ so that this mode is a ghost when NEC is violated.

Another interesting symmetry that works in case of massive gravity is

$$
\phi^i \to \phi^i(\phi^j) \tag{118}
$$

that is the full group of diffeomorphisms of the ϕ^i submanifold. Notice that, although this is a very big symmetry, one can still write a non-trivial action thanks to the presence of ϕ^0. This symmetry implies $m_2 = m_3 = m_4 = 0$, and the presence of three sound waves with zero propagation velocity, $\omega = 0$. Consequently, a system possessing such a symmetry may be called a superdust. Both vector modes in the superdust have dispersion relation $\omega = 0$. The scalar sector is characterized by the following kinetic term,

$$
k^2 \begin{pmatrix} \pi_0 & \pi_L \end{pmatrix} \begin{pmatrix} 2m_0^2\nu^2 + m_1^2 - (p + \rho) & im_1^2\nu \\ -im_1^2\nu & m_1^2\nu^2 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_L \end{pmatrix} , \tag{119}
$$

The eigenvalues of the kinetic matrix have zeros at $\omega = 0$ and, $\omega^2 = k^2 (p + \rho)/2m_0^2$. Again, in the case of $p + \rho < 0$, gradient instabilities can be avoided by taking $m_0^2 < 0$, but then the propagating mode is a ghost. Similarly, the symmetries $\phi^0 \to \phi^0 + \xi^0(\phi^0)$ and $\phi^0 \to \phi^0 + \xi^0(\phi^i)$ do not lead to stable situations.

We conclude that for $p + \rho < 0$ not even the extra symmetries that helped make sense of modifications of gravity can help avoid instability. $p + \rho = 0$ emerges once again as a truly limiting case even for this class of generalized cosmic solids.

7   Towards $w < -1$

One of the motivations to study NEC violations in effective field theories is to understand whether the deceleration parameter of the cosmological expansion can be strongly negative,

$w < -1$. It is then natural to ask what are in this respect the implications of our results. Is there still room for a cosmology with increasing[18] Hubble parameter?

Our results make it very unlikely that such a cosmology may be driven by some NEC violating field theory. However one should keep in mind that our considerations, strictly speaking, apply only to scalar theories in the leading derivative approximation. For instance we did not consider situations where higher derivative terms in the effective action are important. An example of a non-trivial higher derivative theory from which to start this study could be offered by the dynamics of the brane bending mode in the DGP model [6, 15]. Moreover our analysis of the symmetries which could protect the degenerate dispersion relations did not cover the most general case, even for scalar theories. Also it may be possible to construct models where instabilities are present, but slow enough to be benign. On the other hand, we expect our results to be quite robust, especially because they relate NEC violation to the instabilitites in a sector of the theory, the Goldstones of spontaneously broken space-time translations, that exists whenever there is a non-trivial field background.

Here we would like to discuss yet another possibility for having $w < -1$. The basic idea is that the accelerated expansion of the Universe is due to a genuine "modification of gravity", by which the observed metric depends on new degrees of freedom and thus does not satisfy the Friedmann equation. For instance, one may imagine a probe 3-brane moving in a curved bulk space. This brane may be accelerated by forces of non-gravitational origin (e.g., it may be charged under some form fields). Then the induced metric on the brane will evolve in time even in the absence of an energy-momentum tensor localized on the brane. Clearly, the challenge for proposals of this type is to explain why gravity is described by the 4-dimensional Einstein theory at short length scales. Scalar tensor theories of gravity are also simple enough arena to toy with this idea. This was indeed done in Ref. [30] (for earlier work see [31]) with the conclusion that "only highly contrived models would lead observers to measure $w < -1$". So our first purpose here is to understand in simple terms the basic reason for contrivedness in such models. This will allow us to propose a relatively simple candidate model where $w < -1$ can be achieved without fine-tuning[19]. A minimal 4d setup possessing all the ingredients needed to implement this idea is described by the following action

$$S[g, \psi, \phi] = M_{Pl}^2 \int d^4x \sqrt{-g} R + S_m[\psi, f(\phi/M)g_{\mu\nu}] + S_\phi[\phi, g_{\mu\nu}] \ . \tag{120}$$

[18]Notice that there are proposals [8, 29] that mimic the presence of a dark energy component with $w < -1$ while the Hubble parameter still always decreases with time. In other words in these models the total $(p+\rho)$, dark energy plus dark matter, is still positive.

[19]More precisely, in the model we shall present there is no extra fine-tuning associated to $w < -1$. Still there is a necessity to fine-tune the potential of certain scalar field to make it very smooth. This is a consequence of the presence of a dynamical "dark energy" with $w \neq -1$. This type of fine-tuning is inherent in most "quintessence" scenarios.

Here ψ collectively denotes the Standard Model fields, while ϕ is an extra scalar field. The goal is to construct an action of the form (120) allowing for a solution with the following properties.

(A) The field ϕ changes in time and the physical metric $f(\phi)g_{\mu\nu}$ has the FRW form. The main source of visible FRW expansion is the time dependence of the factor $f(\phi/M)$. In particular, this implies that the energy density of the field ϕ itself should be small enough not to cause significant backreaction.

(B) Deviations from Einstein theory at short length scales due to coupling of the field ϕ to the matter energy-momentum tensor are small enough.

In other words, the field ϕ is a non-gravitating cosmic clock. Let us see that a conventional scalar field with the action

$$S_\phi = \int d^4x \sqrt{-g}(\partial_\mu\phi)^2 - V(\phi)$$

is very unlikely to play such a role. In order to proceed, let us first assume that the interaction of ϕ with matter is weaker than gravity, $M \gg M_{Pl}$. In this case, condition **(B)** is satisfied and the time evolution of the $f(\phi/M)$ factor in the physical metric gives a contribution to the observed expansion rate of the Universe

$$H_f^2 = \left(\frac{\dot{f}}{f^2}\right)^2 \sim \frac{\dot{\phi}^2}{M^2} , \tag{121}$$

where we assumed that $f(x)$ is a generic smooth function of order one. On the other hand there is a contribution to the expansion rate of the Universe from the energy-momentum tensor of the field ϕ itself, which is at least of order

$$H_\phi^2 \sim \frac{\dot{\phi}^2}{M_{Pl}^2} . \tag{122}$$

The latter contribution gives $w > -1$ and dominates the first one in the considered regime, $M \gg M_{Pl}$.

Let us now discuss the case in which the clock field ϕ is coupled to matter more strongly than gravity, $M \lesssim M_{Pl}$. In this case measurements of the deflection of light by the Sun combined with tests of the Newton law at sub-millimeter scales imply that ϕ should have rather big mass,

$$m_\phi \gtrsim 0.1 \text{ mm}^{-1} . \tag{123}$$

Now, during a Hubble time ϕ varies by

$$\Delta\phi \sim M \tag{124}$$

so that the potential $V(\phi)$ gives a contribution to the energy density of order

$$\rho_\phi \sim m_\phi^2 \Delta \phi^2 \sim m_\phi^2 M^2 \ . \tag{125}$$

In order for this energy to be subdominant in today's Universe one needs

$$M \lesssim m_\phi \cdot (0.1 \text{ mm})^{-2} \ . \tag{126}$$

In other words, the coupling of the clock field to matter becomes non-perturbative below 0.1 mm. It is hard to imagine how such a theory could make sense.

To summarize, we see that in both cases the essence of the problem is that the gravitational backreaction of the cosmic clock field ϕ tends to give a larger contribution to the expansion rate than the direct coupling $f(\phi)$.

One way to avoid this problem might perhaps be to consider the case $M \lesssim M_{Pl}$ and try to hide the field ϕ through the chameleon mechanism [32], instead of introducing the large mass m_ϕ.

Another possibility, instead, is to consider a clock field coupled to matter more weakly than gravity, $M \gg M_{Pl}$, and to try to minimize its gravitational backreaction. A natural candidate to the role of non-gravitating clock is the ghost condensate

$$S_\phi[\phi, g_{\mu\nu}] = \int d^4x \sqrt{-g} \Lambda^4 G(X) \tag{127}$$

where $X = (\partial_\mu \phi)^2/\Lambda^4$ as in sect. 5.3 but we also assume

$$G'(X_0) = G(X_0) = 0 \ . \tag{128}$$

for some X_0. Without loss of generality we may set $X_0 = 1$. Note that the direct coupling to matter breaks the shift symmetry of the ghost condensate. Then, in general, matter loops will generate a potential for the field ϕ, which is too large for the construction below to work. The fine-tuning needed to cancel this potential is not related to the requirement $w < -1$ and is similar to one present in most quintessence scenarios. Then one finds the following "cosmological" solution,

$$g_{\mu\nu} = \eta_{\mu\nu}, \ \phi = \Lambda^2 t \tag{129}$$

with matter fields in their vacuum state. This solution provides an extreme implementation of the original idea, in which the dependence on the field ϕ is the only source of time evolution for the physical metric. The Hubble parameter is given by (121) and is of order

$$H_f \sim \Lambda^2/M \ . \tag{130}$$

Its time evolution can be fairly arbitrary, in particular it can grow with time. To check that this model may present interest for cosmological applications let us check that values of H of

the order of the current Hubble rate H_0 can be achieved without contradicting to the tests of general relativity. Two type of effects should be suppressed in this model. First there is a modification of gravity due to the mixing between the metric and the ghost condensate mode. Second there is an extra scalar force arising from the direct coupling of the trace of the energy-momentum tensor to the ghost condensate field. Mixing with gravity gives rise to a Jeans-like instability with characteristic time-scale

$$t_c \sim M_{Pl}^2/\Lambda^3 \ .$$

There are no sizable effects due to this mixing as long as t_c is larger than the current age of the Universe (and even for somewhat shorter values of t_c). This implies that as large values as $\Lambda \sim 100$ MeV are allowed. Then eq. (130) implies that the parameter M which determines the strength of the direct coupling of the ghost condensate field to matter can be well above M_{Pl} still providing a realistic value of H. The only subtlety here is that in the limit in which gravity is decoupled the ghost condensate has the unusual dispersion relation

$$\omega^2 - \frac{k^4}{\Lambda^2} = 0 \ , \tag{131}$$

with the k^4-term coming from higher-derivative terms. Naively, this dispersion relation leads to the puzzling conclusion that the exchange of the ghost condensate field leads to a static potential which grows proportionally to the distance from the source. This would imply the breakdown of perturbation theory far from the source even for very large values of M. A resolution to this problem is that the dispersion relation (131) implies that the velocity of the scalar quanta

$$v = \frac{k}{\Lambda} \tag{132}$$

is very low for small values of k. As a result a very long time $\sim L^2\Lambda$ is needed for this potential to be established at a large distance scale L. More precisely accounting for the mixing with gravity the dispersion relation (131) is modified to

$$\omega^2 - \frac{k^4}{\Lambda^2} + \frac{k^2\Lambda^2}{M_{Pl}^2} = 0 \ . \tag{133}$$

The last term gives rise to the Jeans-like instability discussed above and dominates over the second one for

$$k < m \equiv \frac{\Lambda^2}{M_{Pl}} \ . \tag{134}$$

Consequently, for time-scales shorter than t_c the linear potential is established only for distances shorter than m^{-1}. At larger distances mixing with gravity trades this growing-with-distance-potential with an instability that develops in time. For small enough $\Lambda \lesssim 100$ MeV and large enough $M \gg M_{Pl}$ both effects can be neglected.

Note, that in the presence of massive matter sources non-linear effects dominate the dynamics of the ghost condensate [27]. These effects may introduce local spatial gradients of the field ϕ of order $\partial_i \phi \sim \Lambda^2$. Then direct coupling of the ghost condensate field to matter may result in local anomalous accelerations of order $\Lambda^2/M \sim H_0$. This may be an interesting or deadly feature of this model, depending on whether such effects are present inside the Solar system. The detailed study of these effects is beyond the scope of this paper.

8   Summary

We have studied the connection between the null energy condition and the microscopic stability of the system that sources the energy momentum tensor. We have not considered extra pathologies that may arise through the gravitational backreaction. Our attention has been limited to a general system of derivatively coupled scalar fields, but we expect that our arguments can be generalized to include vector fields. However, we have explained that the class of theories we consider includes relativistic solids and fluids, and is thus relevant to cosmology.

We have first worked under the assumptions that all gapless scalar modes have a non-degenerate dispersion relation $\omega = vk$ with $v \neq 0, \infty$. Our main result then is that a violation of the null energy condition does not strictly imply instabilities (ghosts or imaginary frequencies) in the system. We have proven that by producing counter-examples, that is acceptable effective field theories with a positive definite Hamiltonian for quadratic perturbations around a background whose energy momentum violates the null energy condition. A necessary feature of all the counter-examples is the anisotropy of the background and, perhaps more importantly, the presence of superluminal modes. In fact we proved that for systems that are either isotropic or do not feature superluminality, a violation of the null energy condition always implies an unescapable instability. From a purely effective field theoretic viewpoint, in the presence of a Lorentz violating background, there does not seem to exist any limitation on the presence of superluminal modes. Indeed many (most?) of the models that have been considered in recent years in connection to cosmic acceleration [33, 6, 11] feature superluminal excitations around some background. Our study suggests that effective field theories that allow superluminality are subtly distinct from those which do not. The apparent paradoxes that we discussed in connection to causality are another sign of this distinction and a simple, but deep, characterization of it will be given in Ref. [34].

We have then considered isotropic systems in which some mode may have a degenerate dispersion relation, either $\omega = 0$ or $k = 0$. We expect such systems to be acceptable only when the degenerate dispersion relations are protected by symmetries. We have studied various internal symmetries of the scalar field manifold. These symmetries nicely generalize to relativistic solids the internal symmetry that distinguishes ordinary liquids from ordinary

solids (see paragraph below). Although we have not been able to perform a completely general analysis, we believe we have given fairly convincing evidence that a violation of the null energy condition leads to instability even in the presence of such symmetries. It is interesting, in this respect, that the so called theories of massive gravity, broadly defined by having a non-trivial field background while the energy momentum tensor vanishes exactly, lie right at the boundary between theories that violate and theories that respect the null energy condition. This feature largely explains the difficulty to make such theories viable.

Aside the above results, two other directions were partially investigated. In sect.5 we considered the subclass of scalar field theories involving three scalar fields ϕ^I, $I = 1, 2, 3$, with space coordinate dependent expectation values $\langle \phi^I(x) \rangle = x^I$. We explained that these systems are naturally interpreted as solids filling the universe. We characterized the limit where the solid becomes a fluid by the appearance of a volume preserving reparametrization invariance of the scalar field manifold. We then generalized to the relativistic case in a Lagrangian formulation some of the well known results of fluid mechanics, like Kelvin's theorem. We also discussed a sort of duality relation between the recently proposed ghost condensate model and an ordinary diluted fluid. In this respect the novelty of the ghost condensate models partially lies in its being a weakly coupled theory, which is not the case for the corresponding diluted fluid model. In our opinion, the results in sect. 5, while not completely original and not fully clear cut, are still useful as they put in a wider perspective the classes of scalar theories we have been working on.

Finally in section 7 we briefly stepped into another class of theories, where one scalar with non-trivial time dependent background is directly coupled to the matter energy momentum and effectively mimics a cosmic acceleration with $\dot{H} > 0$. We have simply explained the two difficulties that have made past attempts of this type very clumsy. These are due to the tests of general relativity and to the gravitational backreaction of the metric on the scalar field energy momentum. The two difficulties point to a simple solution, where the scalar field is a ghost condensate field, for which the energy momentum can be exactly zero thus causing no backreaction. The possible difficulties of such a model lie in the non-linear dynamics of the ghost condensate in the presence of massive bodies, though it has reasonably been argued that such non-linearities may not necessarily lead to trouble. Our simple proposal potentially opens the way to genuine modifications of gravity that make $w < -1$ consistent, but further study is needed.

Acknowledgments

A Stability of the NEC-violating systems of section 4.1

In section 4.1, we presented two Lagrangians which were claimed to be stable despite violating the NEC. Here we sketch the stability analysis of these systems.

A.1 Two dimensional example

The Lagrangian for the two dimensional example is given by

$$\mathcal{L} = \left(\tilde{B}^{00} + \tilde{B}^{11} + 2(1 + a^2)\, \tilde{B}^{01} - (a^2 + b^2)\, \tilde{B}^{00}\tilde{B}^{01} + (a^2 + c^2)\, \tilde{B}^{11}\tilde{B}^{01} \right) \ . \tag{135}$$

with

$$b^2 < 2, \qquad c^2 < 2 \ .$$

To study the stability of the fluctuations, it is useful to know the expansion of the meson field $\tilde{B}^{IJ}$,

$$\tilde{B}^{00} = 2\, \partial_0 \pi^0 + \partial_\mu \pi^0 \partial^\mu \pi^0$$
$$\tilde{B}^{11} = -2\, \partial_1 \pi^1 + \partial_\mu \pi^1 \partial^\mu \pi^1$$
$$\tilde{B}^{01} = (\partial_0 \pi^1 - \partial_1 \pi^0) + \partial_\mu \pi^0 \partial^\mu \pi^1$$

The Lagrangian for the fluctuations can then be written as

$$\left(\partial_\mu \pi^0 \right)^2 + \left(\partial_\mu \pi^1 \right)^2 + 2(1 + a^2) \left(\partial_\mu \pi^0 \right) \left(\partial^\mu \pi^1 \right)$$
$$- (a^2 + b^2) 2 \partial_0 \pi^0 \partial_0 \pi^1 + (a^2 + c^2) \partial_1 \pi^1 \partial_1 \pi^0 + (a^2 + c^2) 2 \partial_0 \pi^0 \partial_1 \pi^0 - (a^2 + c^2) \partial_1 \pi^1 \partial_0 \pi^1 \ . \tag{136}$$

The first line of (136) comes from the first three terms of (135) which are linear in $\tilde{B}^{IJ}$. Those are the only term of (135) that contribute to the background energy-momentum tensor and are responsible for the NEC violation. We can see that taken alone, they would form an unstable system as both the kinetic and gradient matrices of the first line of (136) have negative eigenvalues, due to the mixing between π^0 and π^1. But this defect is cured by the terms $\tilde{B}^{00}\tilde{B}^{01}$ and $\tilde{B}^{11}\tilde{B}^{01}$ that provide terms in the fluctuation Lagrangian that suppress the kinetic and gradient mixing respectively. They also give terms with one space and one time derivative which do not affect the positivity of the Hamiltonian (cf. eq. (25)).

A.2 Three dimensional example

The Lagrangian written in terms of $\tilde{B}^{IJ}$ has form

$$\mathcal{L} = T_{IJ}\, \tilde{B}^{IJ} + \left(\alpha^2 \left[(\tilde{B}^{01})^2 - (\tilde{B}^{11})^2 - (\tilde{B}^{12})^2 \right] + ch\, sh\, (1 + a^2) \left[\tilde{B}^{02}(-\tilde{B}^{11} + \tilde{B}^{22}) + 2\tilde{B}^{01}\tilde{B}^{12} \right] \right) \tag{137}$$

43

with $a^2 \sim b^2 \sim \alpha^2 \sim 1$ and T_{IJ} as given by eq. (64). As in the two dimensional example, terms linear in $\tilde{B}^{IJ}$ give a Lagrangian for the fluctuation of the form $T_{IJ}\partial_\mu \pi^I \partial^\mu \pi^J$ that is Lorentz invariant and unstable, due to the $-b^2(\partial_\mu \pi^1)^2$ that it contains. And as before, this can be cured by adding the terms that are quadratic in $\tilde{B}^{IJ}$. To understand what these terms do, it is useful to keep in mind the expansions of the $\tilde{B}^{IJ}$ to linear order in π^I,

$$\tilde{B}^{00} = 2\,\partial_0 \pi^0$$
$$\tilde{B}^{11} = -2\,\partial_1 \pi^1$$
$$\tilde{B}^{22} = -2\,\partial_2 \pi^2$$
$$\tilde{B}^{01} = \partial_0 \pi^1 - \partial_1 \pi^0$$
$$\tilde{B}^{12} = -\partial_1 \pi^2 - \partial_2 \pi^1$$
$$\tilde{B}^{02} = \partial_0 \pi^2 - \partial_2 \pi^0$$

The kinetic part of the Lagrangian for the fluctuations is given by

$$K = (ch^2 a^2 + sh^2)\left(\partial_0 \pi^0\right)^2 + 2chsh(1+a^2)\partial_0\pi^0 \partial_0\pi^2 + (ch^2 + a^2 sh^2)\left(\partial_0\pi^2\right)^2 + (\alpha^2 - b^2)\left(\partial_0\pi^1\right)^2 \ , \tag{138}$$

while the gradient matrix is

$$G = (ch^2 a^2 + sh^2)\left(\partial_i \pi^0\right)^2 + (sh^2 a^2 + ch^2)\left(\partial_i \pi^2\right)^2 + (\alpha^2 - b^2)\left(\partial_i \pi^1\right)^2$$
$$- \alpha^2 \left[\left(\partial_1 \pi^0\right)^2 + \partial_1 \pi^1 \partial_2 \pi^2 - \left(\partial_1 \pi^2\right)^2\right] \tag{139}$$

The role of the term $\alpha^2(\tilde{B}^{01})^2$ in (137) is to make the kinetic matrix positive definite, which is the case if

$$\alpha^2 - b^2 > 0 \ . \tag{140}$$

The term $ch\,sh\,(1+a^2)\left[\tilde{B}^{02}(-\tilde{B}^{11}+\tilde{B}^{22})+2\tilde{B}^{01}\tilde{B}^{12}\right]$ removes mixing between π^2 and π^0 in the gradient matrix. This is needed because with the mixing the gradient matrix, despite having entries of order $ch\,sh$, has an eigenvalue that is small and the term $-\alpha^2(\partial_1 \pi^0)^2$ contained in $\alpha^2(\tilde{B}^{01})^2$ could make this matrix non-positive definite. Note, that mixing between π^2 and π^0 is still present in the kinetic matrix. As a result this matrix has a small eigenvalue leading to the highly superluminal mode in the spectrum in agreement with our results.

Finally, the terms $-\alpha^2\left[(\tilde{B}^{11})^2 - (\tilde{B}^{12})^2\right]$ are there to change the sign of the gradient term for $(\partial_i \pi^1)^2$. The terms on the last line of (139) are of order 1 and subdominant compared to the terms of order ch, sh and do not affect the positivity of the gradient matrix.

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