The University of Queensland
School of Earth and Environmental Sciences

Application of Finite Elements in Geophysical Modelling

Lutz Gross
E-mail : l.gross@uq.edu.au

Outline

I) Derive a generic PDE form → `PDE Template`

II) How does this work apply in some geophysical application?

III) Solution using the Finite Element Method (FEM)

PDE Template

- Generic from of a PDE with coefficient as parameters
  - framework to derive FEM
  - useful for development of software
- Problem: overhead & efficience vs. flexibility
- See also esys-escript for a more general form: https://github.com/esys-escript/esys-escript.github.io/blob/master/user.pdf

PDE Template

- Potential u
- Gravity force

$$\mathbf{g} = -\nabla u$$

- Gauss' Law

$$\nabla^T \mathbf{g} = -4\pi G\rho$$

- Unknown u
- Pseudo flux $\boldsymbol{F}$:

$$\mathbf{F} = -\mathbf{A}\nabla u + \mathbf{X}$$

- `Conservation` Law

$$\nabla^t \, \mathbf{F} + D \, u = Y$$

3D Case

- Assume $\boldsymbol{A}$ as a diagonal matrix:

$$\mathbf{A} = \begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}$$

$$\mathbf{X} = [X_0, X_1, X_2]^T$$

$$\mathbf{F} = -\mathbf{A}\nabla u + \mathbf{X}$$

$$\mathbf{F} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} -a_{00}\dfrac{\partial u}{\partial x_0} + X_0 \\ -a_{11}\dfrac{\partial u}{\partial x_1} + X_1 \\ -a_{22}\dfrac{\partial u}{\partial x_2} + X_2 \end{bmatrix}$$

3D Case (cont.)

- Conservation condition:

$$\nabla^t \mathbf{F} + D\, u = Y \qquad \frac{\partial F_0}{\partial x_0} + \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + Du = Y$$

PDE Template for Gravity

For the gravity case:

$$\mathbf{g} = -\nabla u \qquad \nabla^T \mathbf{g} = -4\pi G\rho$$

$$\mathbf{F} = -\mathbf{A}\nabla u + \mathbf{X} \qquad \nabla^t \mathbf{F} + D\, u = Y$$

$$\boldsymbol{A} = \boldsymbol{I} \qquad \boldsymbol{X} = \mathbf{0} \qquad D = 0 \qquad Y = -4\pi G\rho$$

$a_{00} = a_{11} = a_{22} = 1$ and $X_0 = X_1 = X_2 = 0$

$D = 0$ and $Y = -4\pi G\rho$

Formulation of PDE:

- Insert Flux into Conservation condition:

$$-\nabla^t \mathbf{A} \nabla u + D\, u = -\nabla^t \mathbf{X} + Y$$

- In 2D:

$$-\frac{\partial}{\partial x_0} a_{00} \frac{\partial u}{\partial x_0} - \frac{\partial}{\partial x_1} a_{11} \frac{\partial u}{\partial x_1} + D u = -\frac{\partial X_0}{\partial x_0} - \frac{\partial X_1}{\partial x_1} + Y$$

Boundary conditions

- Dirichlet-type boundary conditions on Γ_D

$$u = r$$

  - Γ_D = top of the domain Ω

- Neumann-type boundary conditions on Γ_N

$$\mathbf{n}^T\mathbf{F} = 0 \longrightarrow \mathbf{n}^T\mathbf{A}\nabla u = \mathbf{n}^T\mathbf{X}$$

  - With outer normal field $\boldsymbol{n}$
  - Γ_N = all other faces of the domain Ω

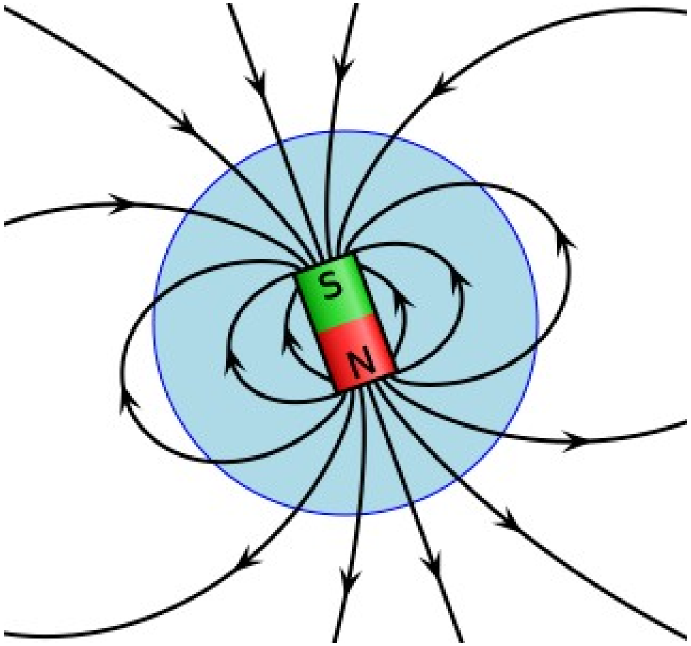
Magnetic Anomalies

- Magnetic Flux $\boldsymbol{B}_f$

$$\mathbf{B_f} = \mathbf{B}_t + \mathbf{M}$$

  - Total magnetic field $\boldsymbol{B}_t$
    - Background magnetic field $\boldsymbol{B}_b$
    - Magnetic field anomaly $\boldsymbol{B}_a$

$$\mathbf{B}_t = \mathbf{B}_a + \mathbf{B}_b$$

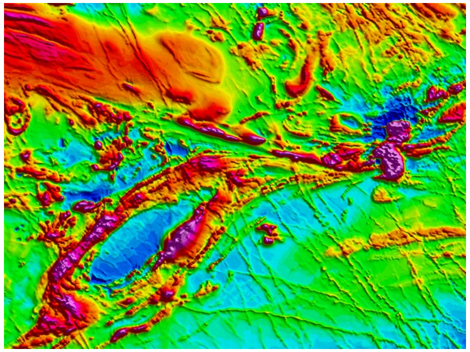


- Magnetization $\boldsymbol{M} = k\,\boldsymbol{B}_b$ with susceptibility k
  - Here: linearized model, also possible $\boldsymbol{M} = k\,\boldsymbol{B}_t$

Magnetic Anomalies (cont.)

- Observed is total magnetic field anomaly b_a is the difference of the intensity of the total magnetic field $\boldsymbol{B}_t$ and of the background field $\boldsymbol{B}_b$:

$$b_a = |\mathbf{B}_t| - |\mathbf{B}_b|$$



- With Linearization

$$b_a = \frac{\mathbf{B}_b^T}{|\mathbf{B}_b|}\mathbf{B}_a \ .$$

- Task: calculate magnetic field anomaly $\boldsymbol{B}_a$ from subsurface susceptibility k

Towards a PDE

- magnetic flux $\boldsymbol{B}_f$ is divergence free:

$$\nabla^t \mathbf{B}_f = 0$$

- With definition of $\boldsymbol{B}_f$:

$$\mathbf{B_f} = \mathbf{B}_t + \mathbf{M} \quad \nabla^t \mathbf{B}_b = 0 \quad \longrightarrow \quad \nabla^t(\mathbf{B}_a + k\mathbf{B}_b) = 0$$

- Analogously to gravity: scalar potential u with

$$\mathbf{B}_a = -\nabla u$$

- Leading to PDE

$$-\nabla^t \nabla u = -\nabla^t k \mathbf{B}_b$$

Using the PDE template

- pseudo flux $\boldsymbol{F} = \boldsymbol{B}_f - \boldsymbol{B}_b$

$$\mathbf{F} = -\nabla u + k\mathbf{B}_b$$

$$\boldsymbol{A} = \boldsymbol{I} \qquad \boldsymbol{X} = k\,B_b$$

$$\mathbf{F} = -\mathbf{A}\nabla u + \mathbf{X}$$

- Also: $D=0$, $Y=0$:

$$\nabla^t(\mathbf{B}_a + k\mathbf{B}_b) = 0 \longrightarrow \nabla^t\,\mathbf{F} + D\,u = Y$$

- Neumann Boundary conditions:

$$\mathbf{n}^T\mathbf{F} = 0 \longrightarrow \mathbf{n}^T\mathbf{B}_f = \mathbf{n}^T\mathbf{B}_b$$

Magnetotellurics

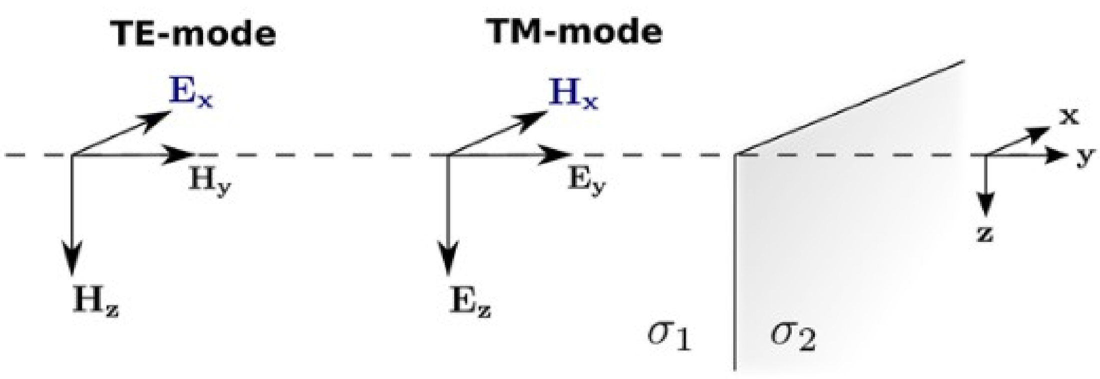
- Maxwell's equations for
  - magnetic field H
  - electric field E
  - Conductivity σ , resistivity ρ ($\sigma\rho=1$)
    - Conductivity in air layer $\sigma=0$
  - Angular frequency ω
  - magnetic permeability μ_0

$$\nabla \times \rho \times \nabla \mathbf{H} + \mathbf{j}\mu_0 \omega \mathbf{H} = 0$$

$$\nabla \times \nabla \times \mathbf{E} + \mathbf{j}\mu_0 \sigma \omega \mathbf{E} = 0$$

2D Model

- no variation along the strike = x-direction
- incoming wave fronts:
  - transverse electric (TE) along Ex
    - $E_y = E_z = 0, H_x = 0$
  - transverse magnetic (TM) along Hx
    - $H_x = H_z = 0, E_x = 0$



TE mode

- Maxwell equation simplifies to

$$-\nabla^T \nabla E_x + \mathbf{j}\omega\sigma\mu_0 E_x = 0$$

- Recover magnetic field component:

$$H_z = -\frac{\mathbf{j}}{\omega\mu_0}\frac{\partial E_x}{\partial x_0} \qquad H_y = \frac{\mathbf{j}}{\omega\mu_0}\frac{\partial E_x}{\partial x_1}$$

- Dirichlet boundary conditions
  - Set as E_x from ‘air plus homogeneous subsurface‘

PDE Template: TE mode

$$u = E_x$$

$$-\nabla^T \nabla E_x + \mathbf{j}\omega\sigma\mu_0 E_x = 0$$

$$a_{00} = a_{11} = 1 \text{ and } D = \mathbf{j}\omega\sigma\mu_0$$

$$-\frac{\partial}{\partial x_0} a_{00} \frac{\partial u}{\partial x_0} - \frac{\partial}{\partial x_1} a_{11} \frac{\partial u}{\partial x_1} + Du = \cancel{-\frac{\partial X_0}{\partial x_0} - \frac{\partial X_1}{\partial x_1}} + Y$$

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

TM mode

- Maxwell equation simplifies to

$$-\nabla^T \rho \nabla H_x + \mathbf{j}\omega\mu H_x = 0$$

  - No air-layer required as $\rho=\infty \rightarrow H_x=const$

- Recover magnetic field component:

$$E_z = \rho\frac{\partial H_x}{\partial x_0} \qquad E_y = \rho\frac{\partial H_x}{\partial x_1}$$

- Dirichlet boundary conditions
  - Set as H_x from ‘homogeneous subsurface‘

PDE Template: TM mode

$$u = H_x \qquad -\nabla^T \rho \nabla H_x + \mathbf{j}\omega\mu H_x = 0$$

$$a_{00} = a_{11} = \rho \text{ and } D = \mathbf{j}\omega\mu_0$$

$$-\frac{\partial}{\partial x_0} a_{00} \frac{\partial u}{\partial x_0} - \frac{\partial}{\partial x_1} a_{11} \frac{\partial u}{\partial x_1} + Du = \cancel{-\frac{\partial X_0}{\partial x_0} - \frac{\partial X_1}{\partial x_1}} + Y$$

Impedance

- Defined as

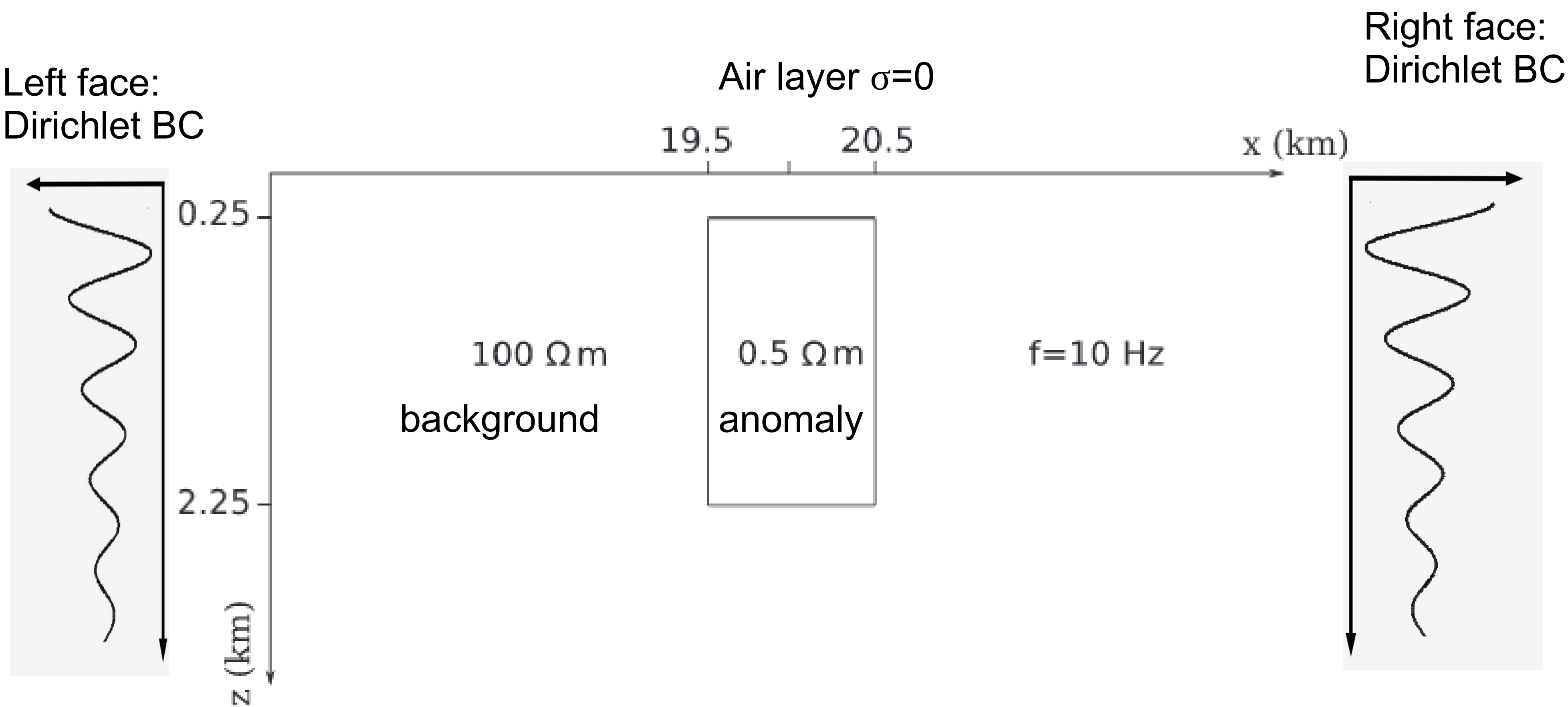
$$Z_{xy}(\omega) = \frac{E_x}{H_y} \qquad Z_{yx}(\omega) = \frac{E_y}{H_x}$$

- Expressed on term of apparent resistivity and phase:

$$\rho_a(\omega) = \frac{1}{\omega\mu_0}|Z_{xy}(\omega)|^2 \qquad \phi(\omega) = \arctan\left(\frac{Z_{xy}(\omega).imag}{Z_{xy}(\omega).real}\right)$$

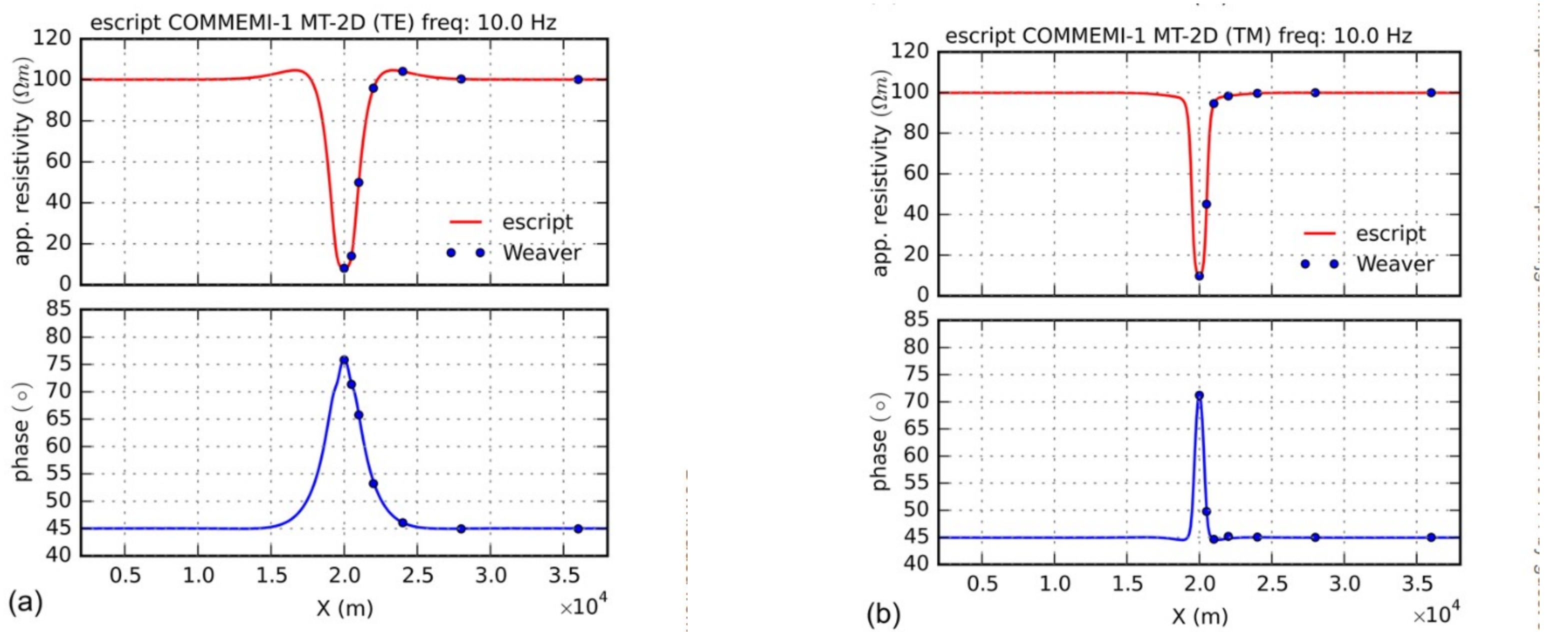
FEM solution

- Test case: See https://doi.org/10.1088/1742-2132/13/2/S59



Result TE mode

- FEM with 60000 nodes
  - solved with esys-escript in python
- Weaver – reference solution



Seismic

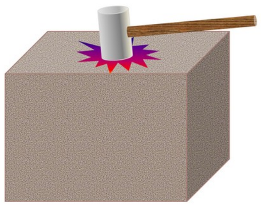
- acoustic wave equation for pressure $p=p(t,\mathbf{x})$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^T \nabla p = f(t) \cdot \delta_{\mathbf{x}_s}$$

Propagation speed c

Source amplitude speed $f(t)$

Source location $\boldsymbol{x}_s$

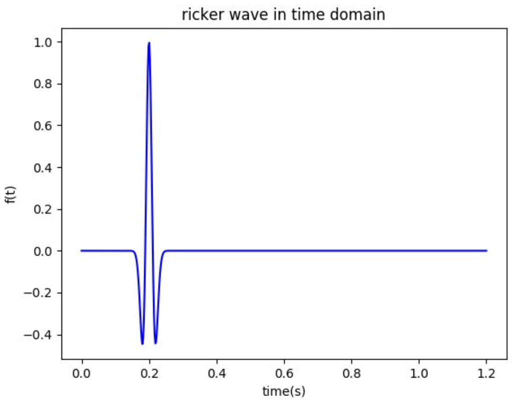


In Frequency Domain

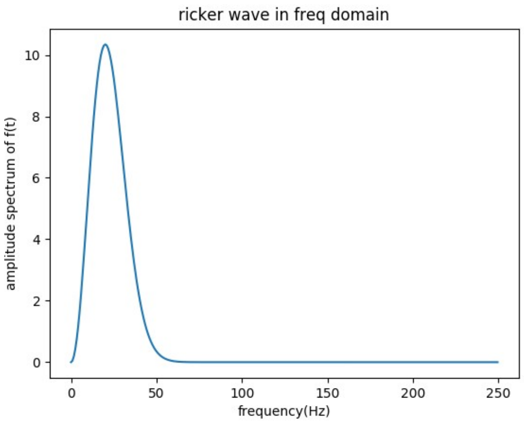
- Apply Fourier Transformation
  - For any angular frequency ω :

$$-\nabla^T \nabla u - k^2 u = f_\omega \delta_{\mathbf{x}_s} \qquad k = \frac{\omega}{c}$$

Source amplitude speed *f(t)*

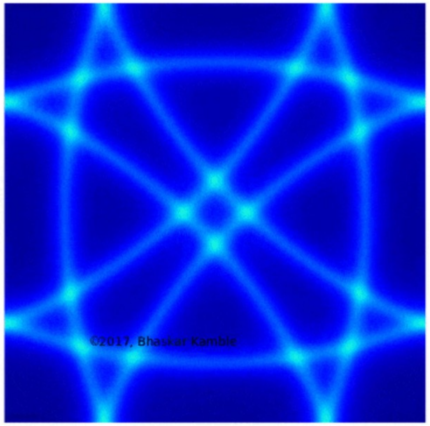


Source Spectrum speed f_ω



Where Are We?

- Boundary conditions?
  - For instance: u=0 on boundary
- Wave equation in frequency fits PDE template: $\boldsymbol{A}=\boldsymbol{I}$, $D=-k^2$
- But: reflection from boundary
  - As domain is bounded
- Introduce absorption near boundary → eg. PML



PML

- PML = Perfect Matching Layer
- Apply a coordinate transformation:

$$x_i \rightarrow \gamma_i x_i$$

- With

$$\gamma_i = 1 - \mathbf{j}\frac{c\,\alpha}{\omega} Q_i$$

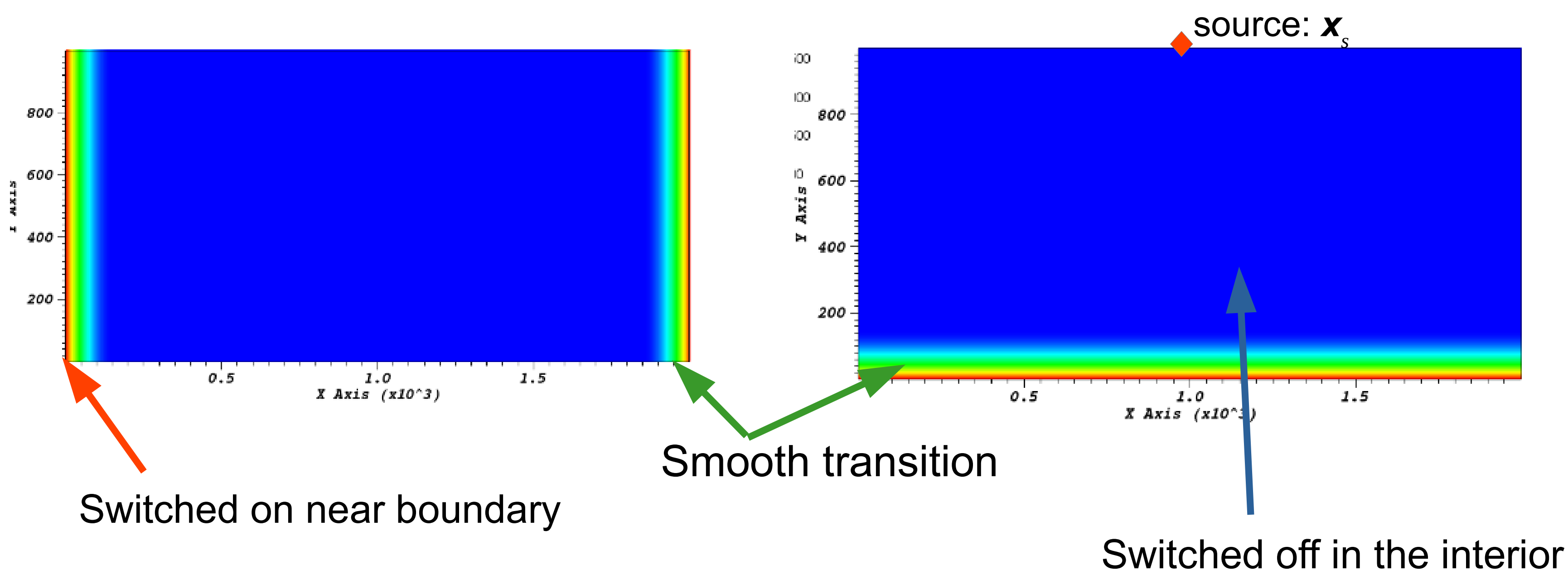
  - absorption strength parameter α
  - absorbing functions Q_i : switches on/off absorption

Absorbing Function

Red: $Q_i=1$, blue: $Q_i=0$

Horizontally: Q_0

Vertically: Q_1



Wave Equation With PML

$$-\nabla^T \nabla u - k^2 u = f_\omega \delta_{\mathbf{x}_s}$$

$$\Big\downarrow \; x_i \rightarrow \gamma_i x_i$$

$$-\frac{1}{\gamma_0}\frac{\partial}{\partial x_0}\frac{1}{\gamma_0}\frac{\partial u}{\partial x_0} - \frac{1}{\gamma_1}\frac{\partial}{\partial x_1}\frac{1}{\gamma_1}\frac{\partial u}{\partial x_1} - k^2 u = f_\omega \delta_{\mathbf{x}_s}$$

Does not fit PDE Template!!!!

A Little Trick

- Multiply by $\gamma_0\gamma_1$:

$$-\gamma_1 \frac{\partial}{\partial x_0} \frac{1}{\gamma_0} \frac{\partial u}{\partial x_0} - \gamma_0 \frac{\partial}{\partial x_1} \frac{1}{\gamma_1} \frac{\partial u}{\partial x_1} - \gamma_0\gamma_1 k^2 u = f_\omega \delta_{\mathbf{x}_s}$$

- Notice: $\gamma_1=\gamma_1(x_1)$ and $\gamma_0=\gamma_0(x_0)$

$$-\frac{\partial}{\partial x_0} \frac{\gamma_1}{\gamma_0} \frac{\partial u}{\partial x_0} - \frac{\partial}{\partial x_1} \frac{\gamma_0}{\gamma_1} \frac{\partial u}{\partial x_1} - \gamma_0\gamma_1 k^2 u = f_\omega \delta_{\mathbf{x}_s}$$

- Works also in 3D

PDE Template

$$-\frac{\partial}{\partial x_0}\frac{\gamma_1}{\gamma_0}\frac{\partial u}{\partial x_0} - \frac{\partial}{\partial x_1}\frac{\gamma_0}{\gamma_1}\frac{\partial u}{\partial x_1} - \gamma_0\gamma_1 k^2 u = f_\omega \delta_{\mathbf{x}_s}$$

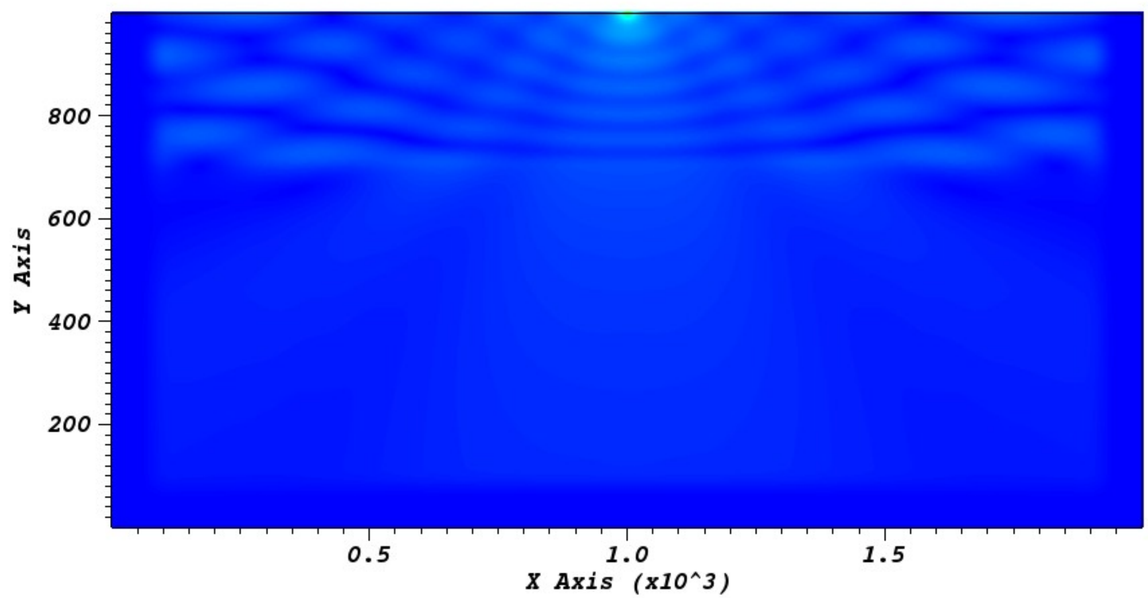
$$a_{00} = \frac{\gamma_1}{\gamma_0}, a_{11} = \frac{\gamma_0}{\gamma_1}, D = -\gamma_0\gamma_1 k^2 \text{ and } Y = f_\omega \delta_{\mathbf{x}_s}$$

with Neumann boundary conditions

Example

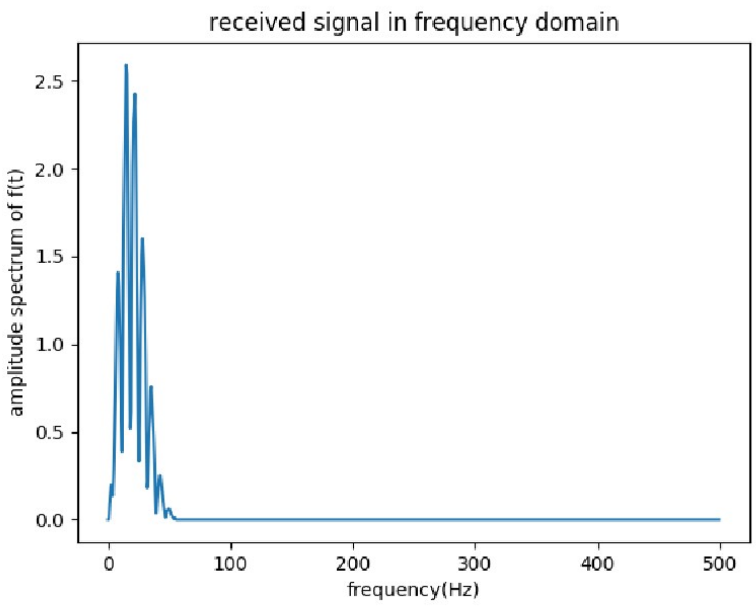
- Single reflector 2000m/s vs 3000m/s @ 300m
- 2000x1000 elements with edge length h=1
- PML: $\alpha=0.5$, layer thickness = $150m$

Wave field u for 20Hz

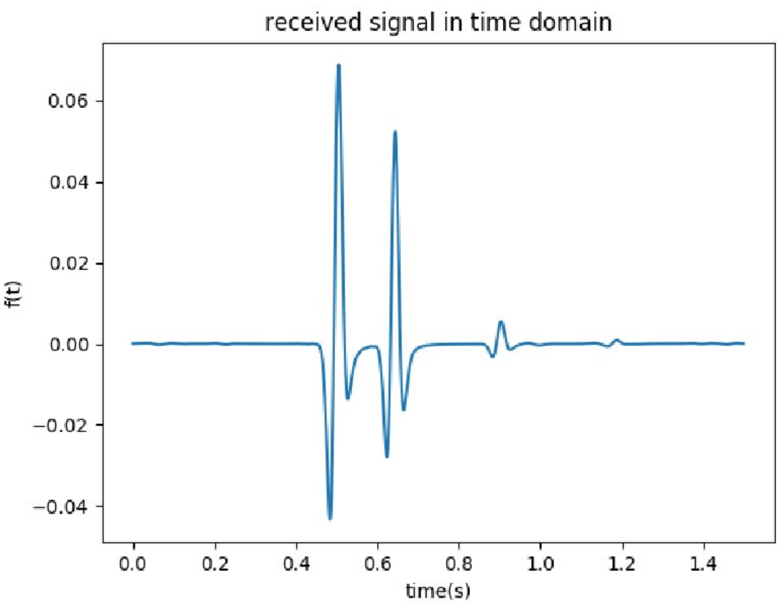


Example (cont.)

- Recover signal at 500m off source
  - Sampled at 1ms



(a) Power spectrum



(b) Time signal

  - By Ao Chang using esys-escript

FEM solver for the PDE template

- Follow what we did for gravity
  - Define node and elements
    - Element = connect vertices with nodes
  - Define the local basis function N_i

    $\rightarrow$ linked to FEM basis ϕ_p^h
  - Get Local Element matrices
    - Evaluate PDE with local basis functions N_i

      Here comes in the PDE template
  - Assemble local element matrices to linear system
  - Insert Dirichlet conditions
  - Solve the linear system of equations

$\rightarrow$ Done

PDE To Linear System

`strong` form of PDE:

$$-\nabla^T \nabla u = -4\pi G \rho \ .$$

For PDE template ?

`Weak` form of PDE:

$$\int_\Omega \nabla^T v^h \, \nabla u^h \, d\mathbf{x} = \int_\Omega (-4\pi G)\rho \, v^h \, d\mathbf{x}$$

Local element matrices:

$$S_{ij}^E = \int_E \nabla^T N_i \, \nabla N_j \, d\mathbf{x} \text{ for } i,j = 0,\ldots 3 \qquad b_i^E = \int_E (-4\pi G)\rho \, N_i \, d\mathbf{x} \text{ for } i = 0,\ldots 3$$

Global system of linear equations:

$$\mathbf{S}^h \mathbf{U}^h = \mathbf{b}^h$$

Weak Form PDE Template

- Conservation law

$$\nabla^t \mathbf{F} + D\,u = Y$$

- Multiply by v and integrate:

$$\int_\Omega v\left(\nabla^t \mathbf{F} + D\,u - Y\right) d\mathbf{x} = 0$$

- Again: apply generalized Green's first identity to pseudo-flux:

$$\int_\Omega v \nabla^T \mathbf{F} d\mathbf{x} = -\int_\Omega \nabla^T v\,\mathbf{F} d\mathbf{x} + \int_{\partial\Omega} v\,\mathbf{n}^T \mathbf{F} ds$$

- Boundary integral is zero as
  - Neumann condition: $\mathbf{n}^T \mathbf{F} = 0$
  - Dirichlet boundary condition: where $u=r \rightarrow v=0$

Weak Form PDE Template (cont.)

- So we get:

$$\int_{\Omega} \left(v \; (D \; u - Y) - \nabla^T v \; \mathbf{F} \right) \mathbf{x} = 0$$

- Recall: pseudo-flux $\mathbf{F} = -\mathbf{A}\nabla u + \mathbf{X}$
- Finally: the ‘weak’ form of the PDE template

$$\int_{\Omega} (\nabla^T v \; \mathbf{A} \; \nabla u + \; D \; vu) \, d\mathbf{x} = \int_{\Omega} (\mathbf{X}^T \nabla v + Y \; v) \, d\mathbf{x}$$

Weak Form PDE Template (cont.)

- This is how it looks in the 2D case:

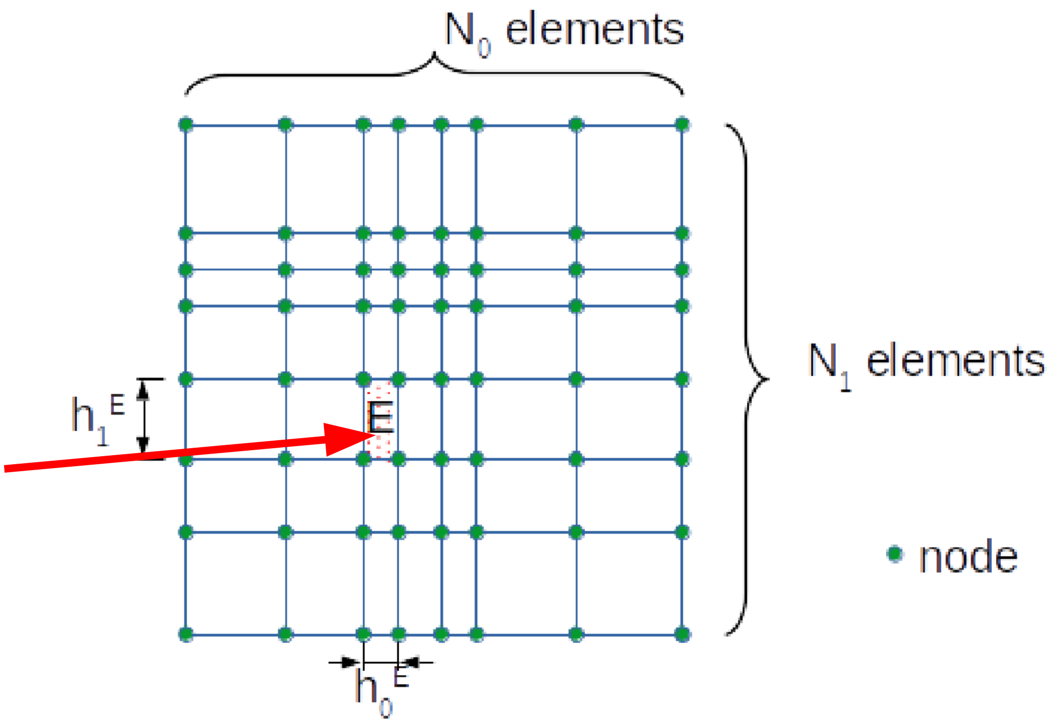
$$\int_\Omega \left(a_{00} \frac{\partial v}{\partial x_0} \frac{\partial u}{\partial x_0} + a_{11} \frac{\partial v}{\partial x_1} \frac{\partial u}{\partial x_0} + D\ vu \right) d\mathbf{x} = \int_\Omega \left(X_0 \frac{\partial v}{\partial x_0} + X_1 \frac{\partial v}{\partial x_1} + Y\ v \right) d\mathbf{x}$$

- Next: break integration down to elements

A More General Grid

- Allow for rectangular shaped elements
- Nodes are not are **not equally** spaced anymore

Element E is shape $h_0^E \times h_1^E$.
and area $A^E = h_0^E h_1^E$



Modified local basis functions

Modifications relative to square case

$$N_0(x_0, x_1) = \frac{1}{A_E}(c_0 + h_0 - x_0) \cdot (c_1 + h_1 - x_1)$$

$$N_2(x_0, x_1) = -\frac{1}{A_E}(c_0 + h_0 - x_0) \cdot (c_1 - x_1)$$

$$N_1(x_0, x_1) = -\frac{1}{A_E}(c_0 - x_0) \cdot (c_1 + h_1 - x_1)$$

$$N_3(x_0, x_1) = \frac{1}{A_E}(c_0 - x_0) \cdot (c_1 - x_1)$$

Local Element Matrices

- This gives the local element matrices:

$$\int_\Omega \left(\nabla^T v \,\mathbf{A}\, \nabla u + \; D\, vu\right) d\mathbf{x} = \int_\Omega \left(\mathbf{X}^T \nabla v + Y\; v\right) d\mathbf{x}$$

$$S_{ij}^E = \int_\Omega \left(\nabla^T N_i \,\mathbf{A}\, \nabla N_j + \; D\; N_i N_j\right) d\mathbf{x}$$

$$b_i^E = \int_\Omega \left(\mathbf{X}^T \nabla N_i + Y\; N_i\right) d\mathbf{x}$$

The 2D case

- Assume: coefficients are constant at each element

$$S_{ij}^E = a_{00}^E \cdot \int_E \frac{\partial N_i}{\partial x_0}\frac{\partial N_j}{\partial x_0} d\mathbf{x} + a_{11}^E \cdot \int_E \frac{\partial N_i}{\partial x_1}\frac{\partial N_j}{\partial x_1} d\mathbf{x} + D^E \cdot \int_E N_i N_j d\mathbf{x}$$

$$b_i^E = X_0^E \cdot \int_E \frac{\partial N_i}{\partial x_0} d\mathbf{x} + X_1^E \cdot \int_E \frac{\partial N_i}{\partial x_1} d\mathbf{x} + Y^E \cdot \int_E N_i d\mathbf{x}$$

Evaluation

- For instance:

$$\int_E \frac{\partial N_i}{\partial x_0}\frac{\partial N_j}{\partial x_0} d\mathbf{x} = \frac{A^E}{(h_0^E)^2}\hat{S}_{ij}^{1,00}$$

For edge length h=1

$$\hat{\mathbf{S}}^{1,00} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Element Matrix Calculation

$$\mathbf{S}^E = \frac{a_{00}^E \, A^E}{(h_0^E)^2} \, \hat{\mathbf{S}}^{1,00} + \frac{a_{11}^E \, A^E}{(h_1^E)^2} \, \hat{\mathbf{S}}^{1,11} + D^E A^E \, \hat{\mathbf{S}}^0$$

$$\mathbf{b}^E = \frac{X_0^E A^E}{h_0^E} \, \hat{\mathbf{b}}^{1,0} + \frac{X_1^E A^E}{h_1^E} \, \hat{\mathbf{b}}^{1,1} + Y^E A^E \, \hat{\mathbf{b}}^0$$

Let’s take a closer look

Element geometry factors

$$\mathbf{S}^E = \frac{\boxed{a_{00}^E} A^E}{(h_0^E)^2}\,\hat{\mathbf{S}}^{1,00} + \frac{\boxed{a_{11}^E} A^E}{(h_1^E)^2}\,\hat{\mathbf{S}}^{1,11} + \boxed{D^E} A^E\,\hat{\mathbf{S}}^0$$

PDE coefficient

For square element edge length h=1

$$\mathbf{b}^E = \frac{\boxed{X_0^E} A^E}{h_0^E}\,\hat{\mathbf{b}}^{1,0} + \frac{\boxed{X_1^E} A^E}{h_1^E}\,\hat{\mathbf{b}}^{1,1} + \boxed{Y^E} A^E\,\hat{\mathbf{b}}^0$$

Element geometry factors

Dirichlet-Type Boundary condition

- When node q has a Dirichlet condition:
  - overwrite row p with equation:

$$S^h_{qp} = \delta_{qp}$$

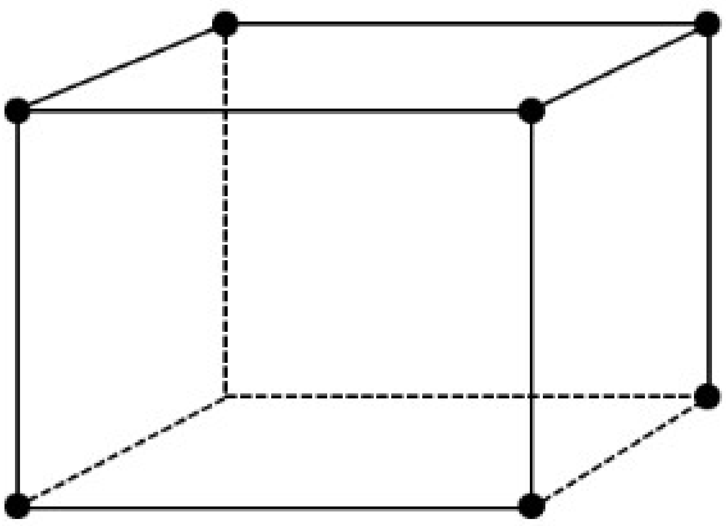
  - And in right hand side
    -

$$b^h_q = r(\check{\mathbf{x}}_q)$$

Extension into 3D

- Now an element has 8 vertices = 8 FEM nodes
  - Define link: element vertices → FEM nodes
  - Define local basis $N_0,\ldots N_7$
  - Calculate

    $\hat{\mathbf{S}}^{1,00}$, $\hat{\mathbf{S}}^{1,11}$, $\hat{\mathbf{S}}^{1,22}$ and $\hat{\mathbf{S}}^{0}$



Anisotropy

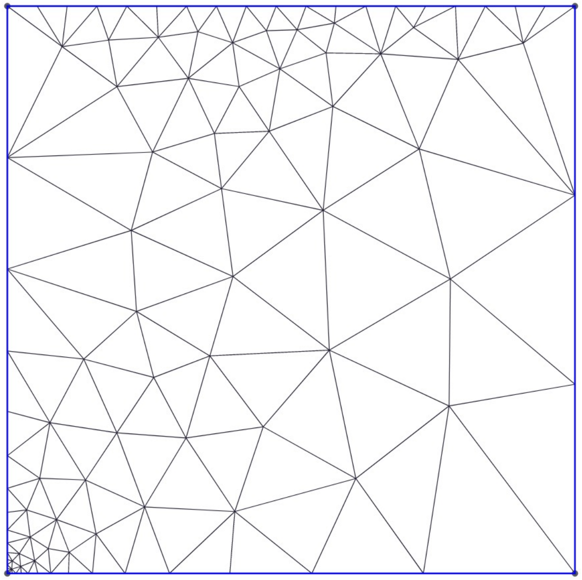
- PDE template with full coefficient matrix $\mathbf{A}$:

$$\boldsymbol{A} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \text{ with } a_{01} \neq 0 \text{ or } a_{10} \neq 0$$

  - for instance modelling anisotropic conductivity
- Introduces mixed derivatives of v and u in week form
- An element level requires $\int_E \frac{\partial N_i}{\partial x_0} \frac{\partial N_j}{\partial x_1} d\mathbf{x}$

Other Element Shapes

- For instance triangles or tetrahedrons
  - Allows for local refinement
- Modification analogously to 3D
  - Construct local basis N_0, N_1, N_2
  - Link Vertex → FEM node
  - Element matrix calculation
    - Not only stretching + **rotation**

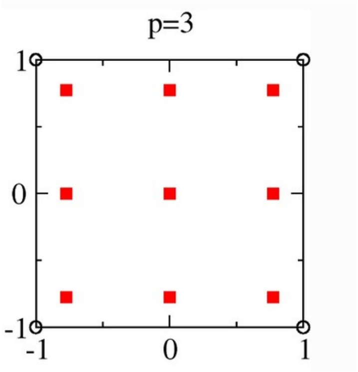


Numerical Integration

- Assumption: PDE coefficient constant across element.
- Better resolution of PDE coefficient on element?
- Apply numerical integration on each element E:

$$\int_E f(\mathbf{x})d\mathbf{x} \approx \sum_k \omega_k^E \, f(\mathbf{q}_k^E)$$

  - Integration nodes q_k^E
  - Integration weights ω_k^E



Numerical Integration (Cont.)

- Evaluate element matrices using numerical integration:

PDE coefficient at integration points q_k^E

$$\int_E DN_iN_j d\mathbf{x} \approx \sum_k D_k^E \omega_k^E N_i(\mathbf{q}_k^E) N_j(\mathbf{q}_k^E) = A^E \sum_k D_k^E \ \hat{S}_{kij}^0$$

Element geometry factor

$$\omega_k^E N_i(\mathbf{q}_k^E) N_j(\mathbf{q}_k^E)$$

For edge length 1

- Warning: This requires more memory and compute time.