Elements of Linear Algebra

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Basic Definitions

Linear algebra for Neural Networks

In order to address the course topics, it is very important to be comfortable with linear algebra.

Working with vectors and matrices must *not* represent an obstacle.

You need to be able to mostly use matrix and vector product notation *quickly* and *easily*.

The best tip to learn linear algebra is to do a lot of practice problems.

Programming tools, like Python, can help you to learn.

Basic concepts

- **Linear algebra**: provides the main rules for manipulating vectors and matrices.
- **Matrix**: is defined as a table of numbers.
- **Vector**: is a particular case of the matrix.
- **Tensor**: is a generalization of a matrix.

Advantages of linear algebra

- Compact notation.
- Intuitive geometric representation.
- Convenient set of operations.
- Suitable data structures for signal manipulations.
- Widely used in modern textbooks and papers.
- Easy equation-to-code translation with many programming tools.

Linear algebra in algorithm development

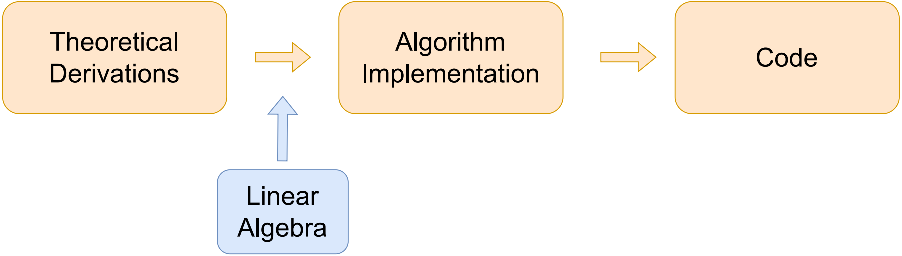


Figure 1: Linear algebra is fundamental to easily encode theory in an algorithm, whose form allows a direct implementation of math operations.

Scalars

A **scalar** is a physical quantity that is described by any real number.

It is usually denoted by a lowercase letter, e.g., x:

$$x_1 = 1, \quad x_2 = 0.3, \quad x_3 = -2, \; \ldots$$

With reference to signals, a scalar may represent for example:

- the amplitude value of a signal sample,
- the color for 1 pixel of an image.

A **vector** is an ordered linear arrangement of a set of scalars. It is usually denoted with a lowercase boldface letter, e.g., $\mathbf{x}$:

$$\mathbf{x} = [x_1 \quad x_2 \quad x_3] = [1 \quad 0.3 \quad -2] \,.$$

Vectors can be used to identify the path from origin to a location P in an N-dimensional space.

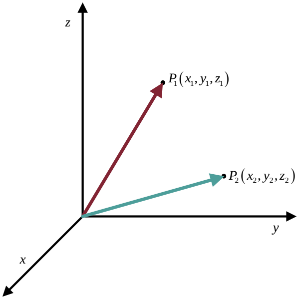


Figure 2: A geometrical representation of vectors.

Vectors II

With reference to signals, a vector may represent:

- 1-dimensional digital signals (e.g., audio signals, brain signals, time-series) which are composed of an ordered sequence of samples (i.e., a sequence of scalars).

As a common notation, 1D signals are usually stored in column vectors.

$\mathbf{x}^T =$

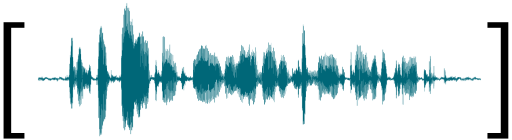


Figure 3: Vector is a typical data structure for 1D signals, like audio waveforms.

Matrices I

A **matrix** consists of set of ordered elements arranged in a number N of rows and M columns.

It is usually denoted with a bold capital letter, e.g., $\mathbf{X}$.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_M \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N1} & \cdots & x_{NM} \end{bmatrix}$$

A matrix can be seen as a vertical stacking of row vectors or as a horizontal arrangement of column vectors.

Matrices II

With reference to signals, matrices may represent:

- 2-dimensional digital signals (e.g., images),
- collections of 1D signals (e.g., multichannel audio, multisensor signals).

$\mathbf{X} =$

Figure 4: Matrix is a typical data structure for 2D signals.

Tensors I

A **tensor** is a linear arrangement of a set of K matrices.

While matrices are functions of two indices for rows and columns, tensors are functions of three or more indices.

It is usually denoted with a capital letter of a special font face, e.g., X.

$$\mathsf{X} = \left\{ \mathbf{X}_1 \quad \mathbf{X}_2 \quad \ldots \quad \mathbf{X}_K \right\}$$

A tensor is a generalization of a matrix.

It can be seen as a stacking of horizontal, vertical or lateral slices (e.g., matrices or higher-dimensional tensors).

Tensors II

With reference to signals, tensors may represent:

- 3-dimensional digital signals (e.g., video signals, time-frequency representation of EEG signals),
- or even higher-dimensional signals (e.g., MR brain imaging).

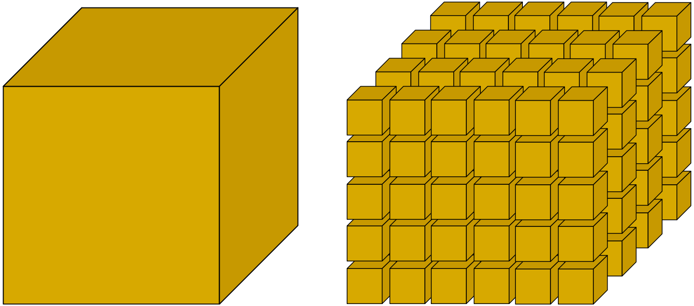


Figure 5: Typical examples of 3D tensor and higher-dimensional tensor.

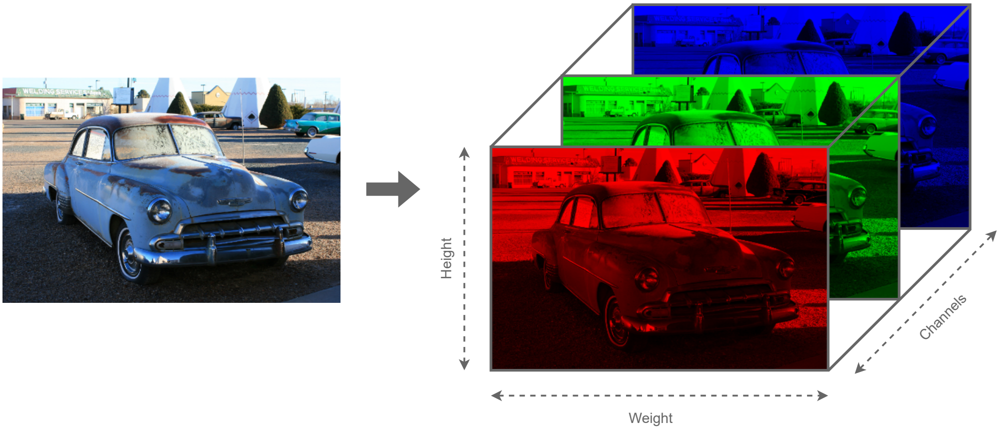


Figure 6: A color image can be stored in a 3D tensor X $= \{\ \mathbf{X}_{\text{R}}\quad \mathbf{X}_{\text{G}}\quad \mathbf{X}_{\text{B}}\ \}$, in which each matrix $\mathbf{X}_i$, with $i = \{\text{R, G, B}\}$, represents an image channel.

Additions and multiplications with matrices

Addition of matrices: Given two matrices $\mathbf{A}, \mathbf{B} \in (\mathbb{R}, \mathbb{C})^{N\times M}$ with the same size, their sum $\mathbf{C} \in (\mathbb{R}, \mathbb{C})^{N\times M} = \mathbf{A} + \mathbf{B}$ is defined by the sum of individual entries with the same index

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1,\ 2,\ \ldots,\ N;\ \ j = 1,\ 2,\ ...,\ M.$$

Matrices multiplication: Given two matrices $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N\times P}$ and $\mathbf{B} \in (\mathbb{R}, \mathbb{C})^{P\times M}$, their product $\mathbf{C} \in (\mathbb{R}, \mathbb{C})^{N\times M} = \mathbf{A} \cdot \mathbf{B}$, is defined as

$$c_{ij} = \sum_{k=1}^{P} a_{ik} b_{kj}, \quad i = 1,\ 2,\ \ldots,\ N;\ \ j = 1,\ 2,\ ...,\ M.$$

Multiplications in Python

Remark

In Python it is possible to perform multiplications and divisions using the following commands:

- `A@B`, performs multiplication of matrices;
- `A*B`, performs element-wise multiplication of matrices;
- `A/B`, performs element-wise division of matrices;
- `A**p`, performs element-wise exponentiation of order p.

Kroneker product

The **Kronecker product** between two matrices $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{P \times Q}$ and $\mathbf{B} \in (\mathbb{R}, \mathbb{C})^{N \times M}$, usually denoted as $\mathbf{A} \otimes \mathbf{B}$, is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1Q}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{P1}\mathbf{B} & \cdots & a_{PQ}\mathbf{B} \end{bmatrix} \in (\mathbb{R}, \mathbb{C})^{PN \times QM}.$$

Remark

The Kronecker product of matrices corresponds to the *abstract tensor product* of linear maps. Specifically, if matrices $\mathbf{A}$ and $\mathbf{B}$ represent the linear applications $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$, then the matrix $\mathbf{A} \otimes \mathbf{B}$ represents the tensor product between the two maps $X_1 \otimes X_2 \to Y_1 \otimes Y_2$.

In Python, $\mathbf{A} \otimes \mathbf{B}$ is achieved by the NumPy function `np.kron(A,B)`.

2 Matrices and Their Properties

Transpose matrix I

Given a matrix $\mathbf{A} \in \mathbb{R}^{N \times M}$ the **transpose matrix** $\mathbf{A}^{\mathsf{T}} \in \mathbb{R}^{M \times N}$ is obtained by interchanging the rows and columns of $\mathbf{A}$

$$\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{12} & a_{11} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1M} & a_{2M} & \cdots & a_{MN} \end{bmatrix}.$$

or

$$\mathbf{A}^{\mathsf{T}} = [a_{ji}], \qquad i = 1,\ 2,\ \ldots,\ N; \quad j = 1,\ 2,\ \ldots,\ M;$$

Hence, $\left(\mathbf{A}^{\mathsf{T}}\right)^{\mathsf{T}} = \mathbf{A}$.

Transposing a 2D signal is equivalent to swapping its dimensions.

$$\mathbf{X}^{\mathsf{T}} =$$

Figure 7: Matrix transpose of a 2D signal.

Hermitian matrix

Let us consider a matrix in the complex domain $\mathbf{A} \in \mathbb{K}^{N \times M}$, then it is possible to define the **Hermitian matrix** as a transpose and complex conjugate matrix

$$\mathbf{A}^{\mathsf{H}} = \left[a_{ji}^{*}\right], \qquad i = 1,\ 2,\ \ldots,\ N;\ \ j = 1,\ 2,\ \ldots,\ M.$$

In the real domain, $\mathbf{A}^{\mathsf{H}} = \mathbf{A}^{\mathsf{T}}$.

Remark

In Python, `A.conj().transpose()` or `A.conj().T` denotes a Hermitian transpose and `A.transpose()` or `A.T` a transpose.
In the real domain: `A.conj().T = A.T`.

Row of a matrix: Given a matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times M}$, its i-th *row vector*, with $i = 1, \dots, N$, is denoted as

$$\mathbf{a}_i \in (\mathbb{R}, \mathbb{C})^{1 \times M} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{iM} \end{bmatrix}$$

Column of a matrix: Given a matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times M}$, its j-th *column vector*, with $j = 1, \dots, M$, is denoted as

$$\mathbf{a}_j \in (\mathbb{R}, \mathbb{C})^{N \times 1} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{Nj} \end{bmatrix}.$$

Matrices as row or column vectors II

Remark

In Python you can extract entire columns or rows of a matrix with the following instructions:

- `A[(i-1)]` or `A[(i-1),:]`, extracts the entire i-th row in a row vector of dimension M;
- `A[(:,(j-1)]`, extracts the entire *j*-th column in column vector of size N;
- `A[0:K]` or `A[:K]` or `A[0:K,:]`, extracts the first K rows from A.
- `A[-K:]`, extracts the last K rows from A.

A matrix $\mathbf{A} \in (\mathbb{R},\mathbb{C})^{N\times M}$ can be represented by its N row vectors or by its M column vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{\mathsf{H}} \\ \mathbf{a}_2^{\mathsf{H}} \\ \vdots \\ \mathbf{a}_N^{\mathsf{H}} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_N \end{bmatrix}^{\mathsf{H}}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_M \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{\mathsf{H}} \\ \mathbf{a}_2^{\mathsf{H}} \\ \vdots \\ \mathbf{a}_M^{\mathsf{H}} \end{bmatrix}^{\mathsf{H}}.$$

Then, given a matrix $\mathbf{A} \in (\mathbb{R},\mathbb{C})^{N\times M}$, we can associate a vector $\mathrm{vec}\,(\mathbf{A}) \in (\mathbb{R},\mathbb{C})^{NM\times 1}$ containing, all the stacked column vectors of $\mathbf{A}$

$$\begin{aligned} \mathrm{vec}\,(\mathbf{A}) &= \begin{bmatrix} \mathbf{a}_1^{\mathsf{H}} & \mathbf{a}_2^{\mathsf{H}} & \cdots & \mathbf{a}_M^{\mathsf{H}} \end{bmatrix}^{\mathsf{H}}_{M(N)\times 1} \\ &= [a_{11}, ..., a_{N1},\; a_{12}, ..., a_{N2}, \cdots\cdots, a_{1M}, ..., a_{NM}]^{\mathsf{H}}_{NM\times 1}. \end{aligned}$$

In Python, reshaping is performed by the NumPy function `np.reshape(A,(N,M))`.

Diagonal matrix

A given matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times N}$ is called **diagonal** if $a_{ji} = 0$ for $i \neq j$.

In Python, the `diag` operator yields both a diagonal matrix from a vector, e.g., `A = diag(a);`

$$\mathbf{A} = \operatorname{diag}(\mathbf{a}) = \operatorname{diag}\begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_N \end{bmatrix}$$

and a vector from a matrix, e.g., `a = diag(A);`

$$\mathbf{a} = \operatorname{diag}^{-1}(\mathbf{A}) = \operatorname{diag}^{-1}\begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{NN} \end{bmatrix}$$

Trace and symmetry

The **trace** of a matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times N}$ is given by the sum of its diagonal elements

$$\text{tr}\left(\mathbf{A}\right) = \text{tr}\begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} = \sum_{i=1}^{N} a_{ii}$$

In Python, this can be expressed as `A.trace`.

A matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times N}$ is **symmetric** if $a_{ji} = a_{ij}$ or $a_{ji} = a_{ij}^*$ in the complex-valued domain, whereby $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$ for real domain and $\mathbf{A}^{\mathsf{H}} = \mathbf{A}$ for the complex-valued domain.

Toeplitz matrix

$\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times N}$ is a **Toeplitz matrix** if $[a_{i,j}] = [a_{i+1,j+1}] = [a_{i-j}]$, i.e., each descending diagonal of $\mathbf{A}$, from left to right, is constant.

$$\mathbf{A}_T = \begin{bmatrix} a_i & a_{i-1} & a_{i-2} & a_{i-3} & \cdots \\ a_{i+1} & a_i & a_{i-1} & a_{i-2} & \ddots \\ a_{i+2} & a_{i+1} & a_i & a_{i-1} & \ddots \\ a_{i+3} & a_{i+2} & a_{i+1} & a_i & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

In Python, from scipy.linalg, `A = toeplitz(c,r)` returns a *nonsymmetric* Toeplitz matrix with `c` as its first column and `r` as its first row.

A *symmetric* Toeplitz matrix is simply created by `A = toeplitz(c)`.

Positive semidefinite matrix

A matrix $\mathbf{A} \in \mathbb{K}^{N \times N}$ is **positive semidefinite**, or *nonnegative*, if

$$\forall \mathbf{w} \in \mathbb{K}^{N} \Rightarrow \Re\left(\mathbf{w}^{\mathsf{H}} \mathbf{A} \mathbf{w}\right) \geq 0.$$

A positive semidefinite matrix can be also denoted as $\mathbf{A} \succeq 0$.
The symbol $\succeq$ represents inequality between matrices, i.e., given two matrices $(\mathbf{A}, \mathbf{B}) \in (\mathbb{R}, \mathbb{C})^{N \times N}$, $\mathbf{A} \succeq \mathbf{B}$ means that

$$\mathbf{z}^{\mathsf{T}} \mathbf{A} \mathbf{z} \geq \mathbf{z}^{\mathsf{T}} \mathbf{B} \mathbf{z}, \quad \forall \mathbf{z} \in (\mathbb{R}, \mathbb{C})^{N \times 1}.$$

If $\mathbf{w}^{\mathsf{H}} \mathbf{A} \mathbf{w} > 0$, then the matrix is referred to as positive definite $\mathbf{A} \succ 0$.

Inverse matrix

A square matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N\times N}$ is **invertible**, or *nonsingular*, if there exists a matrix $\mathbf{B} \in (\mathbb{R}, \mathbb{C})^{N\times N}$ such that

$$\mathbf{B}\mathbf{A} = \mathbf{I},$$

where $\mathbf{I}_{N\times N}$ is an *identity matrix* defined as

$$\mathbf{I} = \text{diag}\,(1,\ 1, \ldots\ , 1)\,.$$

In such a case, the matrix $\mathbf{B}$ is uniquely determined from $\mathbf{A}$ and defined as the **inverse** of $\mathbf{A}$, or also denoted as $\mathbf{A}^{-1}$. This implies:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

In Python, $\mathbf{A}^{-1}$ is performed by `linalg.inv(A)`.

Some properties of inverse matrices

It is worth noting that if $\mathbf{A}$ is nonsingular, the system equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has a unique solution given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. In Python: `linalg.solve(A,B)` if `A` is square; `linalg.lstsq(A,B)` otherwise.

Property

$$\begin{aligned}
(\mathbf{A}\mathbf{B}\mathbf{C}\cdots)^{-1} &= \cdots\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\
\left(\mathbf{A}^{H}\right)^{-1} &= \left(\mathbf{A}^{-1}\right)^{H} \\
(\mathbf{A}+\mathbf{B})^{H} &= \mathbf{A}^{H}+\mathbf{B}^{H} \\
(\mathbf{A}\mathbf{B})^{H} &= \mathbf{B}^{H}\mathbf{A}^{H} \\
(\mathbf{A}\mathbf{B}\mathbf{C}\cdots)^{H} &= \cdots\mathbf{C}^{H}\mathbf{B}^{H}\mathbf{A}^{H}.
\end{aligned}$$

Pseudoinverse matrix

The *generalized inverse* matrix, or **Moore-Penrose pseudoinverse**, represents a general way to determine of the solution of a linear real or complex system equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, when $\mathbf{A}$ is a rectangular matrix, i.e., $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N\times M}$, $\mathbf{x} \in (\mathbb{R}, \mathbb{C})^{M\times 1}$, $\mathbf{b} \in (\mathbb{R}, \mathbb{C})^{N\times 1}$.

The pseudoinverse of a generic rectangular matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N\times M}$ is denoted as $\mathbf{A}^{\#}$ and it is characterized by the following properties.

Property

$$
\begin{aligned}
\mathbf{A}\mathbf{A}^{\#}\mathbf{A} &= \mathbf{A} \\
\mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} &= \mathbf{A}^{\#} \\
\mathbf{A}\mathbf{A}^{\#} &= \left(\mathbf{A}\mathbf{A}^{\#}\right)^{H} \\
\mathbf{A}^{\#}\mathbf{A} &= (\mathbf{A}^{\#}\mathbf{A})^{H}.
\end{aligned}
$$

Solution by a pseudoinverse matrix

The solution of a general linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ depends on the size of the matrix $\mathbf{A}$:

$$\mathbf{A}^{\#} = \begin{cases} \mathbf{A}^{-1} & N = M & \text{square matrix} \\ \mathbf{A}^{\mathsf{H}} \left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)^{-1} & N < M & \text{“fat” matrix} \\ \left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathsf{H}} & N > M & \text{“tall” matrix.} \end{cases}$$

Anyway, the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ may always be expressed as

$$\mathbf{x} = \mathbf{A}^{\#}\mathbf{b}.$$

In Python, $\mathbf{A}^{\#}$ is performed by `linalg.pinv(A)`, while the solution can be obtained by `x = linalg.pinv(A)@b = linalg.solve(A,B)` for $\mathbf{A}$ square, or `x = linalg.pinv(A)@b = linalg.lstsq(A,B)` otherwise.

Different methods for calculating the pseudoinverse refer to possible decompositions of the matrix $\mathbf{A}$.

Matrix inversion lemma

The **matrix inversion lemma** (MIL) is a very useful property in the development of online machine learning algorithms.

The MIL states that if $\mathbf{A}^{-1}$ and $\mathbf{C}^{-1}$ exist, the following equation is true (as well as several equivalent variants):

$$[\mathbf{A}+\mathbf{BCD}]^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\left[\mathbf{C}^{-1}+\mathbf{DA}^{-1}\mathbf{B}\right]^{-1}\mathbf{DA}^{-1}$$

where $\mathbf{A}\in\mathbb{K}^{M\times M}$, $\mathbf{B}\in\mathbb{K}^{M\times N}$, $\mathbf{C}\in\mathbb{K}^{N\times N}$ and $\mathbf{D}\in\mathbb{K}^{N\times M}$.

The **rank** r of a matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ represents the number of independent columns of $\mathbf{A}$.

It is useful in linear system equations to understand if a system admits a solution (i.e., *the rank of the coefficient matrix must be the same of the complete matrix including the constant terms*).

An algebraic tool to easily determine the rank of any matrix is the determinant.

The **determinant** of a matrix $\mathbf{A}$ is denoted as $\det(\mathbf{A})$ or $\Delta_{\mathbf{A}}$, and it can be computed by using several algorithms according to the size of the matrix.

Rank and determinant of a matrix II

The determinant of a square matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ is defined in terms of the determinant of order $N-1$ with the following recursive expression

$$\det(\mathbf{A}) = \sum_{j=1}^{N} a_{ij} \left[(-1)^{j+i} \det(\mathbf{A}_{ij}) \right]$$

where $\mathbf{A}_{ij} \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix obtained by eliminating the i-th row and the j-th column of $\mathbf{A}$.

A matrix $\mathbf{A}$ with $\det(\mathbf{A}) \neq 0$ is called *nonsingular* and it is always invertible.

Note that the determinant of a diagonal or triangular matrix is the product of the values on the diagonal.

Low-rank approximation

In several machine learning problems, low-rank approximation methods are used to describe a large matrix $\mathbf{Y} \in \mathbb{R}^{M \times N}$ with a smaller one $\mathbf{X} \in \mathbb{R}^{R \times N}$, with $R < M$:

$$\mathbf{Y} \approx \mathbf{A}\mathbf{X}$$

being $\mathbf{A} \in \mathbb{R}^{M \times R}$.

Note

In problems of data compression, very often the minimization of a cost function that measures the fit between a given data matrix and an approximating matrix is formulated under a low-rank approximation constraint, i.e., the approximating matrix is constrained to have a reduced rank. This kind of constraint is equivalent to reducing the complexity of a model that fits the data.

3 Geometrical Properties of Vectors

Inner and Outer Products

Projection Operators

Inner and outer products

Given two vectors $\mathbf{u} \in (\mathbb{R}, \mathbb{C})^{N\times 1}$ and $\mathbf{v} \in (\mathbb{R}, \mathbb{C})^{N\times 1}$, the *inner product*, or *scalar product* or **dot product**, denoted as $\langle \mathbf{u}, \mathbf{v} \rangle$, is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathsf{H}}\mathbf{v} = \left(\mathbf{u}^{\mathsf{H}}\mathbf{v}\right)^* = \sum_{i=1}^{N} u_i v_i^* .$$

The **outer product** between two vectors $\mathbf{u} \in (\mathbb{R}, \mathbb{C})^{M\times 1}$ and $\mathbf{v} \in (\mathbb{R}, \mathbb{C})^{N\times 1}$, denoted as $\rangle \, \mathbf{u}, \mathbf{v} \, \langle \, \in (\mathbb{R}, \mathbb{C})^{M\times N}$, is a matrix defined by the product

$$\rangle \, \mathbf{u}, \mathbf{v} \, \langle = \mathbf{u}\mathbf{v}^{\mathsf{H}} = \begin{bmatrix} u_1 v_1^* & \cdots & u_1 v_N^* \\ \vdots & \ddots & \vdots \\ u_M v_1^* & \cdots & u_M v_N^* \end{bmatrix}_{M\times N}$$

The inner product represents a basic operation in many NNs models.

A neuron of a NN receives inputs x_i, $i = 1 \ldots, N$, from the N nodes of the previous layer; each input x_i is weighted with its own coefficient w_i, thus:

$$s = w_1 x_1 + w_2 x_2 + \ldots + w_N x_N = \sum_{i=1}^{N} w_i x_i = \mathbf{x}^\mathsf{T}\mathbf{w} = \mathbf{w}^\mathsf{T}\mathbf{x}$$

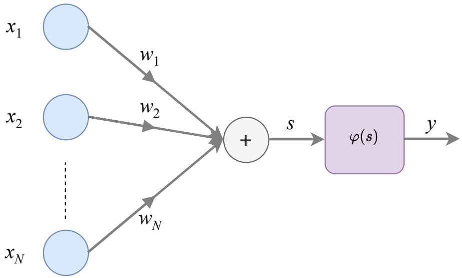


Figure 8: Graphical representation of a neuron receiving N inputs frm the previous layer.

Geometrical interpretation I

The inner product of a vector for itself $\mathbf{u}^{\mathsf{H}}\mathbf{u}$, can be written as

$$\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^{\mathsf{H}}\mathbf{u} \triangleq \|\mathbf{u}\|_2^2$$

which defines the square of its length in a Euclidean space.

In Euclidean geometry, the inner product of vectors expressed in an orthonormal basis is related to their *length* and *angle*.

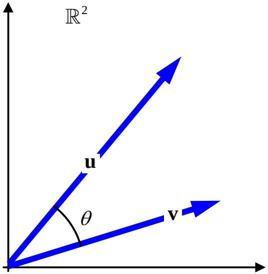
Considering two vectors $\mathbf{u}$ and $\mathbf{v}$, their lengths can be expressed as

$$\|\mathbf{u}\| \triangleq \sqrt{\|\mathbf{u}\|_2^2}, \qquad \|\mathbf{v}\| \triangleq \sqrt{\|\mathbf{v}\|_2^2}$$

Therefore, the inner product can be also expressed as

$$\mathbf{u}^{\mathsf{H}}\mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.



Inner product

provides X with a structure can be viewed as a '*similarity*'

- $\langle \mathbf{u}, \mathbf{v} \rangle > 0$, $\mathbf{u}$ and $\mathbf{v}$ point to the 'same direction'
- $\langle \mathbf{u}, \mathbf{v} \rangle < 0$, $\mathbf{u}$ and $\mathbf{v}$ point to the 'opposite direction'
- $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\mathbf{u} \perp \mathbf{v}$.

Figure 9: Graphical representation of the inner product.

Orthogonality

Whenever dealing with signals, the conditions of **orthogonality** (but also orthonormality and bi-orthogonality) represent a fundamental tool to understand some characterizing properties of signals, e.g., correlations between signals, convergence properties, and much more.

Two vectors (e.g., two signals) $\mathbf{x}, \mathbf{w} \in (\mathbb{R}, \mathbb{C})^N$ are orthogonal if their inner product is zero $\langle \mathbf{x}, \mathbf{w} \rangle = 0$.
This is sometimes also referred to as $\mathbf{x} \perp \mathbf{w}$.

This concept can be extended to a set of vectors $\{\mathbf{a}_i\} \triangleq \{\mathbf{a}_i \in (\mathbb{R}, \mathbb{C})^N, \ \ \forall i, \ i = 1, ..., N\}$, which is called *orthogonal* if

$$\mathbf{a}_i^{\mathsf{H}} \mathbf{a}_j = 0, \qquad i \neq j.$$

Orthonormality

A set of vectors $\{\mathbf{a}_i\}$, is *orthonormal* if

$$\mathbf{a}_i^{\mathsf{H}}\mathbf{a}_j = \delta_{ij} = \delta\left[i - j\right]$$

where δ_{ij} is the Kronecker symbol defined as: $\delta_{ij} = 1$ for $i = j$; $\delta_{ij} = 0$ for $i \neq j$.

A matrix $\mathbf{A} \in \mathbb{K}^{N\times N}$ is orthonormal if its columns are an orthonormal set of vectors.

Formally, a matrix $\mathbf{A} \in \mathbb{K}^{N\times N}$ is an **orthogonal matrix** if

$$\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{I},$$

where $\mathbf{A}^{\mathsf{T}}$ is the transpose of $\mathbf{A}$ and $\mathbf{I}$ is the *identity* matrix.

Properties of orthonormal matrices

An orthogonal matrix is always *invertible*.

Moreover, $\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}}$, i.e., in component form, $a_{ij}^{-1} = a_{ji}$.

These properties make orthogonal matrices particularly easy to compute, since the transpose operation is much simpler than computing a matrix inversion.

Orthonormality has no effect on inner product, i.e.

$$\langle \mathbf{Ax}, \mathbf{Ay} \rangle = (\mathbf{Ax})^{\mathsf{H}} \mathbf{Ay} = \mathbf{x}^{\mathsf{H}} \mathbf{A}^{\mathsf{H}} \mathbf{Ay} = \langle \mathbf{x}, \mathbf{y} \rangle$$

Furthermore, if multiplied to a vector, it does not change its length

$$\|\mathbf{Ax}\|_2^2 = (\mathbf{Ax})^{\mathsf{H}} \mathbf{Ax} = \mathbf{x}^{\mathsf{H}} \mathbf{A}^{\mathsf{H}} \mathbf{Ax} = \|\mathbf{x}\|_2^2 .$$

Projection matrix I

A square matrix $\mathbf{P} \in (\mathbb{R}, \mathbb{C})^{N \times N}$ is a **projection operator** iff $\mathbf{P}^2 = \mathbf{P}$, i.e., it is *idempotent*.

If $\mathbf{P}$ is symmetric, then the projection is orthogonal.

If $\mathbf{P}$ is a projection matrix, $(\mathbf{I} - \mathbf{P})$ is also a projection matrix.

Examples of orthogonal projection matrices are matrices associated with the pseudoinverse $\mathbf{A}^{\#}$ in the over- and under-determined cases.

Projection matrix II

In the overdetermined case, $N > M$ and $\mathbf{A}^{\#} = \left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}$, so the orthogonal projection operator can be defined as

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}$$

On the other hand, the *orthogonal complement projection operator* is defined as $\mathbf{P}^{\perp} = \mathbf{I} - \mathbf{A}\left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}$, so that $\mathbf{P} + \mathbf{P}^{\perp} = \mathbf{I}$.

Remark

The operator $\mathbf{P}$ projects a vector on the subspace $\Psi = \mathcal{R}\left(\mathbf{A}\right)$, while $\mathbf{P}^{\perp}$ on its orthogonal complement $\Psi^{\perp} = \mathcal{R}^{\perp}\left(\mathbf{A}\right)$ or $\sum = \mathcal{N}\left(\mathbf{A}^{\mathsf{H}}\right)$.

Indeed, given $\mathbf{x} \in (\mathbb{R}, \mathbb{C})^{M\times 1}$ and $\mathbf{y} \in (\mathbb{R}, \mathbb{C})^{N\times 1}$, such that $\mathbf{Ax} = \mathbf{y}$, we have that $\mathbf{Py} = \mathbf{u}$ and $\mathbf{P}^{\perp}\mathbf{y} = \mathbf{v}$ such that $\mathbf{u} \in \mathcal{R}\left(\mathbf{A}\right)$ and $\mathbf{v} \in \mathcal{N}\left(\mathbf{A}^{\mathsf{H}}\right)$.
In the underdetermined case, where $N < M$ and $\mathbf{A}^{\#} = \mathbf{A}^{\mathsf{H}}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)^{-1}$, we have

$$\mathbf{P} = \mathbf{A}^{\mathsf{H}}\left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A} \qquad \text{with} \quad \mathbf{P}^{\perp} = \mathbf{I} - \mathbf{A}^{\mathsf{H}}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)^{-1}\mathbf{A}.$$

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

$$\mathbf{P}^{\perp} = \mathbf{I} - \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$$

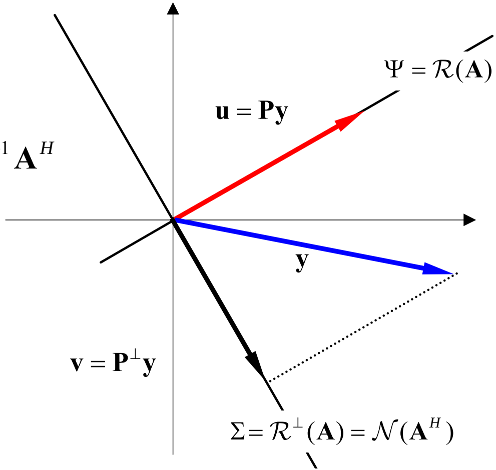


Figure 10: Representation of the orthogonal projection operator.

We consider an audio signal, in this case a gong, and a musical note A. We want to know "how much" of the note A is contained in the gong.

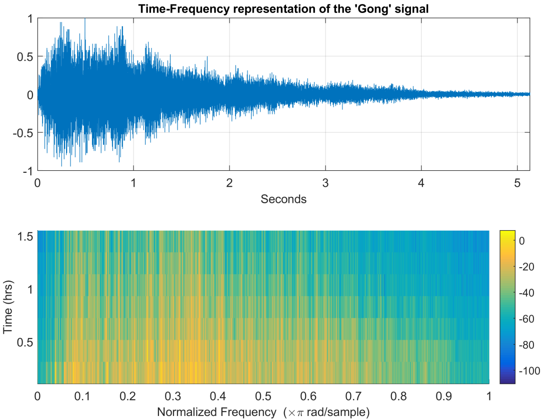


Figure 11: Representation of the orthogonal projection operator.

Example: projection of audio signals II

To this end we project the original signal into the musical note A.

We represent the gong signal $\mathbf{x}$ and the note signal $\mathbf{a}$ in the time-frequency domain by applying a short-time Fourier transform (SFTF):

$$\mathbf{X} = \text{STFT}\left\{\mathbf{x}\right\}, \qquad \mathbf{A} = \text{STFT}\left\{\mathbf{a}\right\}.$$

Then, we compute the projection matrix and the projected signal in its time-frequency domain:

$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}$$
$$\mathbf{U} = \mathbf{P}\mathbf{X}$$

Example: projection of audio signals III

By anti-transforming $\mathbf{U}$ in the time domain, we achieve the resulting signal containing the information of the musical note A in the gong signal, i.e., $\mathbf{u} = \text{STFT}^{-1}\{\mathbf{U}\}$.

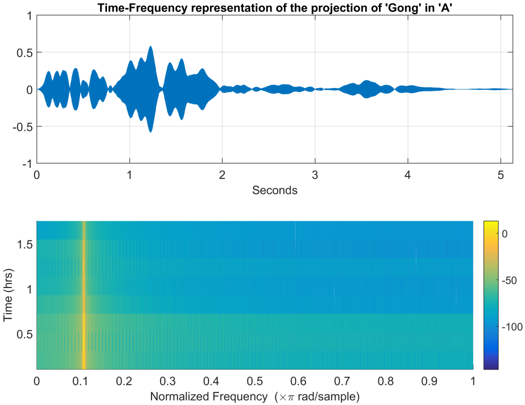


Figure 12: Representation of the orthogonal projection of the gong signal in the musical note A.

4 Metrics and Norms

Distance Metrics

Norm of Vectors

Notable Inequalities

Norm of Matrices

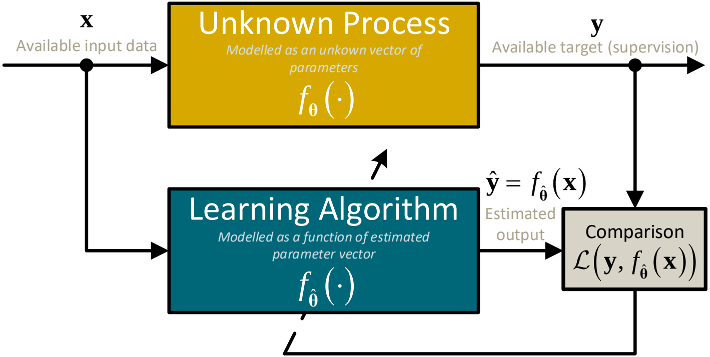


Figure 13: The unknown system is modelled as a function of estimated parameters. Learning is performed after comparing the target signal $\mathbf{y}$ (for *supervised learning*) and the estimated output $\hat{\mathbf{y}} = f_{\hat{\boldsymbol{\theta}}}(\cdot)$. Now we focus on how to measure the distance between the target signal (or also denoted as *ground truth*) and the estimated output.

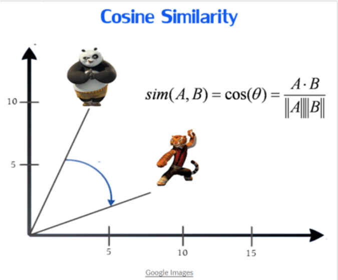
Distance metrics in learning algorithms

Why Distance Metrics are important in learning algorithms?

- In machine learning, many supervised and unsupervised algorithms use Distance Metrics to analyze input data patterns.
- Distance Metrics are used to recognize similarities among patterns.
- Distance Metrics are used to define the “rewarding function” of learning algorithm (usually called *loss functions*, e.g., the mean squared error.

The choice of a “good” metric is fundamental to achieve optimal performance in classification, regression and clustering algorithms.

Word2Vec (W2V) is a neural network that preprocesses text.



Word	Cosine distance
spain	0.678515
belgium	0.665923
netherlands	0.652428
italy	0.633130
switzerland	0.622323
luxembourg	0.610033
portugal	0.577154
russia	0.571507
germany	0.563291
catalonia	0.534176

Figure 14: Example of distances from the word “france”.

Vector spaces

A **Vector Space** $(X, \mathbb{K})$ is a collection of objects called vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots \in X$, (e.g., arrays, continuous or discrete functions, polynomials,...) that can be added together and multiplied by scalars $c \in \mathbb{K}$.

A **Metric Vector Space**, denoted as $(X, \mathbb{K}, d)$, or as (X, d), is a vector space equipped with a strictly positive function $d : X \times X \to R$, denoted **distance**, that assigns a *length* (i.e., a measure) to each vector in a vector space.

A **Normed Vector Space** $(X, \|\cdot\|)$ is a vector space over the real or complex numbers on which the distance is a *norm*.

Examples of vector spaces

- **Banach spaces** $(B, \|\cdot\|)$ is a complete *vector space* where the metric is the norm $d = \|\mathbf{x} - \mathbf{y}|$.

- **Hilbert spaces** $(H, < \cdot, \cdot >)$ is a complete *inner product space* where the norm is induced by the inner product $d =< \mathbf{x}, \mathbf{y} >$.

- **Reproducing kernel Hilbert space** (RKHS) is a Hilbert space where functionals with particular characteristics of *smoothness* are defined, which make them suitable for functions approximation/interpolation problems.

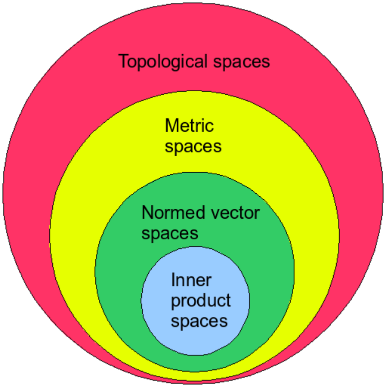


Figure 15: Hierarchy of mathematical spaces. Normed vector spaces are a superset of inner product spaces and a subset of metric spaces, which in turn is a subset of topological spaces. Hilbert spaces are subsets of iner product spaces as well as the Euclidean vetor space. Source: Wikipedia.

Norm of vectors

Together with the scalar product, the **norm** operator is one the most natural ways to define or measure the distance in a *metric space*.

Given a vector $\mathbf{x} \in (\mathbb{R}, \mathbb{C})^N$, its *norm* refers to its length with respect to a vector space.

In the case of a normed-space of order p, said ℓ_p-*space*, the norm is denoted as $\|\mathbf{x}\|^{\ell_p}$ or $\|\mathbf{x}\|_p$, and it is defined as:

Definition (ℓ_p Vector Norm)

A generic norm of a vector is defined as

$$\|\mathbf{x}\|_p \triangleq \left[\sum_{i=1}^{N} |x_i|^p\right]^{1/p}, \quad p \geq 1.$$

ℓ_0 vector "norm"

The previous definition of the ℓ_p vector norm is valid even when $0 < p < 1$; however, the results have a different meaning.

Particular attention is paid to ℓ_0, ℓ_1, ℓ_2 and ℓ_∞ norms.

Definition (ℓ_0 Vector Norm)

The ℓ_p vector norm, for $p = 0$, can be expressed as

$$\|\mathbf{x}\|_0 \triangleq \lim_{p\to 0} \|\mathbf{x}\|_p = \sum_{i=1}^{N} |x_i|^0$$

that measures the *numerosity* of non-zero elements, i.e., the number of non-zero entries of the vector $\mathbf{x}$.

ℓ_1 and ℓ_∞ vector norms

Definition (ℓ_1 Vector Norm)

For $p = 1$, the vector norm becomes

$$\|\mathbf{x}\|_1 \triangleq \sum_{i=1}^{N} |x_i|,$$

which represents the sum of the absolute values of the elements of $\mathbf{x}$.

Definition (ℓ_∞ Vector Norm)

For $p \to \infty$, the vector norm becomes

$$\|\mathbf{x}\|_\infty \triangleq \max_{i=1,N} \{|x_i|\},$$

which is also known as *uniform norm*, *supremum norm* (or sup norm), *Chebyshev norm*, or *infinity norm*.

Definition (ℓ_2 Vector Norm)

The norm is defined for $p = 2$, also known as **Euclidean norm**, expresses the standard length of the vector

$$\|\mathbf{x}\|_2 \triangleq \sqrt{\sum_{i=1}^{N} |x|_i^2} = \sqrt{\mathbf{x}^{\mathsf{H}}\mathbf{x}}$$

$$\|\mathbf{x}\|_2^2 \triangleq \mathbf{x}^{\mathsf{H}}\mathbf{x}$$

Let $\mathbf{G}$ be a suited diagonal matrix, said weighing matrix, the expression:

$$\|\mathbf{x}\|_{\mathbf{G}}^2 \triangleq \left\|\mathbf{x}^{\mathsf{H}}\mathbf{G}\mathbf{x}\right\|$$

represents the so called weighted norm.

Distance measures

Property

The distance between two vectors $\mathbf{x}$ *and* $\mathbf{y}$ *is defined as*

$$\|\mathbf{x}-\mathbf{y}\|_p \triangleq \left[\sum_{i=1}^{N} |x_i - y_i|^p\right]^{1/p}, \quad p > 0$$

and represents a similarity measure in the Minkowsky metric.

Definition (Manhattan Distance)

The Manhattan (or *taxicab*) distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ is:

$$\|\mathbf{x}-\mathbf{y}\|_1 = \sum_{i=1}^{N} |x_i - y_i|.$$

In a Cartesian system, it represents the sum of the lengths of the projections of the segments between the points onto the coordinate axes.

Hölder's inequality

The triangular and Cauchy-Schwarz inequalities can be generalized to the case of norm of order $p \in [1, \infty]$.

The **Hölder's inequality**, which is extension of the Cauchy-Schwarz's inequality, can be defined by the following theorem.

Theorem (Hölder's Inequality)

Given two vector $\mathbf{x}, \mathbf{y} \in (\mathbb{R}, \mathbb{C})^N$ with possible infinite length, let $p, q \in [1, \infty)$ with $1/p + 1/q = 1$, the following identity holds

$$\left| \sum_{i=1}^{\infty} x_i^\star y_i \right| \leq \left[\sum_{i=1}^{\infty} |x_i|^p \right]^{1/p} \cdot \left[\sum_{i=1}^{\infty} |y_i|^q \right]^{1/q}$$

$$\text{i.e.,} \qquad \left| \mathbf{x}^H \mathbf{y} \right| \leq \|\mathbf{x}\|_p \cdot \|\mathbf{y}\|_q$$

with the equality that applies only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent (i.e., lie on the same line).

Minkowski's inequality

Using the Hölder's inequality, we can establish the triangle inequality for ℓ_p norms denoted as **Minkowski's inequality** and defined as follows.

Theorem (Minkowski's Inequality)

Given two vector $\mathbf{x}, \mathbf{y} \in (\mathbb{R}, \mathbb{C})^N$ *with possible infinite length, let* $p \in [1, \infty)$*, we have that*

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$$

The norm of a matrix can be defined similarly to the vector norms. Let us consider a matrix $\mathbf{A} \in (\mathbb{R}, \mathbb{C})^{N \times M}$, the matrix norm can be defined as follows.

Definition (ℓ_1 Matrix Norm)

$$\|\mathbf{A}\|_1 \triangleq \max_{j \in [1, M]} \sum_{i=1}^{N} |a_{ij}|$$

i.e., $\|\mathbf{A}\|_1$ represents the maximum absolute column sum of $\mathbf{A}$.

Definition (ℓ_2 Matrix Norm)

The **Euclidean norm** is defined for the space $p = 2$ and it can be expresses as

$$\|\mathbf{A}\|_2 \triangleq \sqrt{\lambda_{\max}} \quad \Rightarrow \quad \max_{\lambda_i} \left[\text{eig}\left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)\right] \subseteq \max_{\lambda_i} \left[\text{eig}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)\right]$$

Definition (Frobenius Matrix Norm)

$$\|\mathbf{A}\|_{\mathsf{F}} \triangleq \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{M} |a_{ij}|^2} = \sqrt{\text{tr}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)} = \sqrt{\sum_{i=1}^{\min(N,M)} \sigma_i^2}$$

where $\text{tr}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)$ is the trace of $\mathbf{A}\mathbf{A}^{\mathsf{H}}$, and σ_i are the singular values of the matrix $\mathbf{A}$.

Note that the Frobenius norm is nothing but the Euclidean norm applied to $\text{vec}\,(\mathbf{A})$.

Definition (ℓ_∞ Matrix Norm)

Given a $\mathbf{A}_{N\times M}$ matrix the ℓ_∞ matrix norm is defined as

$$\|\mathbf{A}\|_\infty \triangleq \max_{i\in[1,N]} \sum_{j=1}^{M} |a_{ij}|$$

i.e. $\|\mathbf{A}\|_\infty$ represents the maximum absolute row sum of $\mathbf{A}$.

Next lecture

- We introduce elements of unconstrained convex optimization.
- We focus on one of the most popular cost functions in learning algorithms that is the least-square cost function.
- We introduce the gradient operator that is strictly related to the concept of *differentiability* in neural networks.

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Elements of Linear Algebra

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