# An Introduction to Quantum Computing

## Lecture 06
*The Deutsch Algorithm*

Paolo Zuliani

zuliani@di.uniroma1.it

SAPIENZA
UNIVERSITÀ DI ROMA

# Outline

- Quantum gates
- Reversible computing
- Deutsch’s problem
- Quantum parallelism
- Deutsch’s algorithm

# Quantum Gates

- Quantum gates are so-called ***unitary*** transforms that preserve the vector norm. $U$ is unitary if and only if

$$\|Uv\| = \|v\| \qquad \text{for all vectors } v$$

- Equivalently:

$$UU^\dagger = U^\dagger U = I$$

where $U^\dagger$ is the *adjoint* of $U$, given by $U^\dagger_{i,j} = U^*_{j,i}$ for $i \neq j$ and $U^\dagger_{i,j} = U_{i,j}$ otherwise.

---

The adjoint can be defined for *any* complex matrix, *e.g.*:

$$A = \begin{pmatrix} 1+2i & 1+i \\ -3i & -1 \end{pmatrix} \qquad A^\dagger = \begin{pmatrix} 1+2i & \mathbf{3}\boldsymbol{i} \\ \mathbf{1} - \boldsymbol{i} & -1 \end{pmatrix}$$

# Why Is Quantum Evolution Unitary?

Schrödinger's equation is a *linear* differential equation ($H$ is a self-adjoint operator and $v(t)$ is a complex function of time)

$$i\hbar \frac{dv(t)}{dt} = Hv(t)$$

It can be shown that, for $t_2 \geq t_1$:

$$v(t_2) = e^{-\frac{i}{\hbar}H(t_2 - t_1)} v(t_1)$$

where the (linear) operator

$$U(t_2, t_1) = e^{-\frac{i}{\hbar}H(t_2 - t_1)} = \sum_{j=1}^{m} e^{-\frac{i}{\hbar}\lambda_j (t_2 - t_1)} P_j$$

is **unitary**.

$H$ has $m$ eigenvalues $\lambda_1 \ldots \lambda_m$

SAPIENZA UNIVERSITÀ DI ROMA

# Quantum Gates

- Any quantum gate *G*, being unitary, thus admits an *inverse* gate that 'uncomputes' *G*!

$$GG^\dagger = G^\dagger G = I = \text{“do nothing”}$$

- Therefore, quantum computing (<u>except measurement</u>) is ***reversible***.
- Most classical computations are NOT reversible!
- Does this mean that quantum computers cannot run classical algorithms?

# Reversible Classical Computing

- 1963: Lecerf defined a *reversible,* deterministic Turing machine that is as efficient as a regular Turing machine
- 1973: Bennett independently did the same thing
- 2000: Z showed how to make imperative (*probabilistic* and *nondeterministic*) programs reversible

- Thus, reversibility does not pose an issue for computing!
- Essentially, one only needs more memory/larger circuits to store the 'history' of the computation.

# Reversible Classical Computing

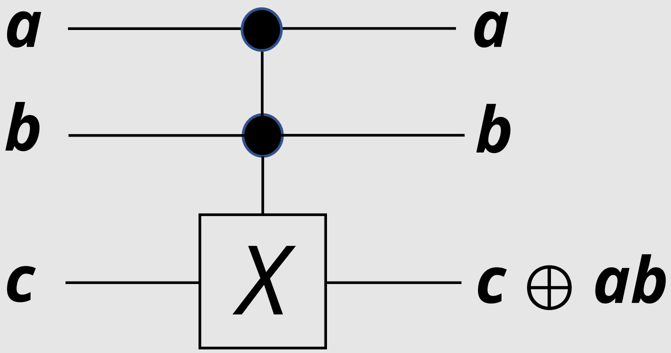
- The Toffoli gate (1980) is ***reversible***

**Inputs**

| *a* | *b* | *c* |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |
| 1 | 1 | 1 |

**Outputs**

| *a'* | *b'* | *c'* |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |
| 1 | 1 | 0 |



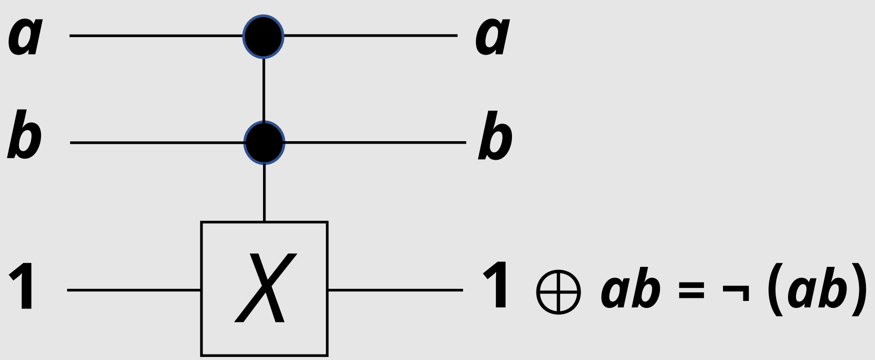

Recall the XOR

| *a* | *b* | $a \oplus b$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

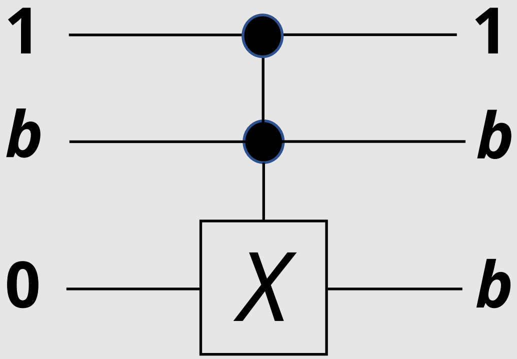
The Toffoli gate $T$ is self-inverse: $T \cdot T = I$

# Reversible Classical Computing

The NAND gate




The FANOUT gate




Any Boolean circuit can then be implemented using only Toffoli gates. Hence, the Toffoli gate is ***universal*** for classical computing.

# Reversible Classical Computing

## Reversibility in programs (Z, 2000):

| $pGCL$ atomic statement $S$ | reversible statement $S_r$ | inverse statement $S_i$ |
|---|---|---|
| $v := e$ | $\mathbf{push}\ v \fatsemi\ v := e$ | $\mathbf{pop}\ v$ |
| $\mathbf{skip}$ | $\mathbf{skip}$ | $\mathbf{skip}$ |
| $pGCL$ constructor $C$ | reversible constructor $C_r$ | inverse constructor $C_i$ |
| $R \fatsemi S$ | $R_r \fatsemi S_r$ | $S_i \fatsemi R_i$ |
| $\mathbf{while}\ \ c\ \ \mathbf{do}\ S\ \mathbf{od}$ | $\mathbf{push}\ b \fatsemi \mathbf{push}\ F \fatsemi$<br>$\mathbf{while}\ \ c\ \ \mathbf{do}$<br>$\quad S_r \fatsemi \mathbf{push}\ T$<br>$\mathbf{od}$ | $\mathbf{pop}\ b \fatsemi$<br>$\mathbf{while}\ \ b\ \ \mathbf{do}$<br>$\quad S_i \fatsemi \mathbf{pop}\ b$<br>$\mathbf{od} \fatsemi$<br>$\mathbf{pop}\ b$ |
| $R \triangleleft c \triangleright S$ | $\mathbf{push}\ b \fatsemi$<br>$(R_r \fatsemi \mathbf{push}\ T\ ) \triangleleft c \triangleright (S_r \fatsemi \mathbf{push}\ F)$ | $\mathbf{pop}\ b \fatsemi$<br>$(R_i \triangleleft b\ \ \triangleright S_i) \fatsemi$<br>$\mathbf{pop}\ b$ |
| $R \ \Box\ S$ | $\mathbf{push}\ b \fatsemi$<br>$(R_r \fatsemi \mathbf{push}\ T) \ \Box\ (S_r \fatsemi \mathbf{push}\ F)$ | $\mathbf{pop}\ b \fatsemi$<br>$(R_i \triangleleft b\ \ \triangleright S_i) \fatsemi$<br>$\mathbf{pop}\ b$ |
| $R\ {}_p\oplus\ S$ | $\mathbf{push}\ b \fatsemi$<br>$(R_r \fatsemi \mathbf{push}\ T)\ {}_p\oplus\ (S_r \fatsemi \mathbf{push}\ F)$ | $\mathbf{pop}\ b \fatsemi$<br>$(R_i \triangleleft b\ \ \triangleright S_i) \fatsemi$<br>$\mathbf{pop}\ b$ |
| $\mathbf{proc}\ \ Q(param) := body$ | $\mathbf{proc}\ \ Q_r(param) := body_r$ | $\mathbf{proc}\ \ Q_i(param) := body_i$ |

Formally, for any *pGCL* program *P*:

$$(P_r; P_c; P_i) \equiv (P_c; P)$$

reversible

copy input

inverse

# Quantum Parallelism (Linearity)

Given <u>any</u> Boolean function $f$, one can show that the function

$$U_f : (x, y) \to (x, y \oplus f(x))$$

is ***reversible*** ⇒ ***unitary*** (⇒ a valid quantum gate!)

Example: 1-bit function:

**Inputs**

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 0 | 1 |
| 1 | 0 |
| 1 | 1 |

**Outputs**

| $x$ | $y \oplus f(x)$ $[f(x) = 0]$ | $y \oplus f(x)$ $[f(x) = 1]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

Thus: $U_f(x, 0) = (x, f(x))$ and $U_f(x, 1) = (x, \neg f(x))$

# Quantum Parallelism (Linearity)

$U_f: (x, y) \to (x, y \oplus f(x))$ can be implemented unitarily, thus can be applied to qubits:

$$U_f: |x \otimes y\rangle \to |x \otimes (y \oplus f(x))\rangle$$

$$U_f|x \otimes 0\rangle = |x \otimes f(x)\rangle \quad U_f|x \otimes 1\rangle = |x \otimes \neg f(x)\rangle$$

Thus:

$$U_f\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(U_f|0 \otimes 0\rangle + U_f|1 \otimes 0\rangle)$$
$$= \frac{1}{\sqrt{2}}(|0 \otimes f(0)\rangle + |1 \otimes f(1)\rangle)$$

$f(0)$ and $f(1)$ are **computed in "parallel"**!!

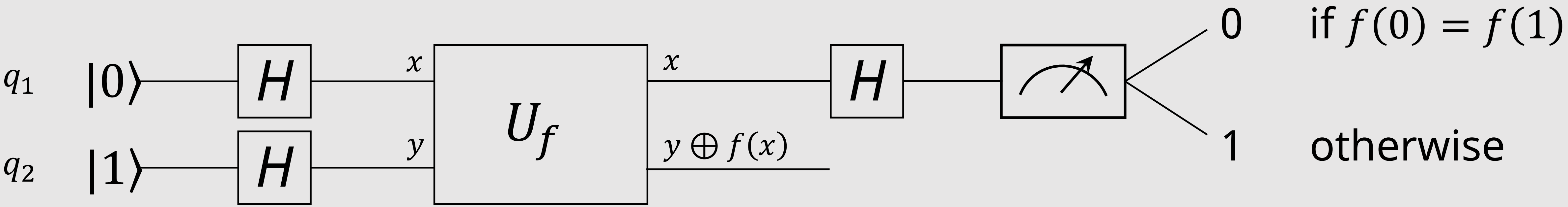
# Deutsch’s Problem

- A Boolean function $f : \mathcal{B} \to \mathcal{B}$
- We want to compute $f(0) \oplus f(1)$

| $f(0)$ | $f(1)$ | $f(0) \oplus f(1)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

- Classically: we need to evaluate $f$ twice
- Quantum: one evaluation of $f$ suffices!

# Deutsch's Algorithm (1985)




Using a "programming" notation:

$$
\begin{aligned}
&q_1, q_2 = |01\rangle; \\
&q_1, q_2 = H \otimes H(q_1, q_2); \\
&q_1, q_2 = U_f(q_1, q_2); \\
&q_1 = H(q_1); \\
&b = Measure(q_1);
\end{aligned}
$$

# Deutsch's Algorithm

$$q_1, q_2 = |01\rangle;$$
$$q_1, q_2 = H \otimes H(q_1, q_2);$$
$$q_1, q_2 = U_f(q_1, q_2);$$
$$q_1 = H(q_1);$$
$$b = Measure(q_1);$$

$$|01\rangle$$

Apply $q_1, q_2 = H \otimes H(q_1, q_2)$

$$= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}} |0\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} + \frac{1}{\sqrt{2}} |1\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

For the next step: note (and verify!) that

$$U_f(|a\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}) = (-1)^{f(a)} |a\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

# Deutsch's Algorithm

$q_1, q_2 = |01\rangle;$
$q_1, q_2 = H \otimes H(q_1, q_2);$
$q_1, q_2 = U_f(q_1, q_2);$
$q_1 = H(q_1);$
$b = Measure(q_1);$

$$\frac{1}{\sqrt{2}}|0\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} + \frac{1}{\sqrt{2}}|1\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

Apply $q_1, q_2 = U_f(q_1, q_2)$

$$U_f\left(|a\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}\right) = (-1)^{f(a)}|a\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$\begin{cases} \pm \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) = f(1) \\ \pm \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) \neq f(1) \end{cases}$$

Apply $q_1 = H(q_1)$

Note that $HH = I$

$$\begin{cases} \pm |0\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) = f(1) \\ \pm |1\rangle \otimes \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) \neq f(1) \end{cases}$$

# Deutsch's Algorithm

$q_1, q_2 = |01\rangle;$
$q_1, q_2 = H \otimes H(q_1, q_2);$
$q_1, q_2 = U_f(q_1, q_2);$
$q_1 = H(q_1);$
$b = Measure(q_1);$

$$\begin{cases} \pm|0\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) = f(1) \\ \pm|1\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}} & \text{if} \quad f(0) \neq f(1) \end{cases}$$

Apply $b = Measure(q_1)$

If we measure 0 on $q_1$ we know **for sure** that $f(0) = f(1)$ (with only one evaluation of $f$)!!