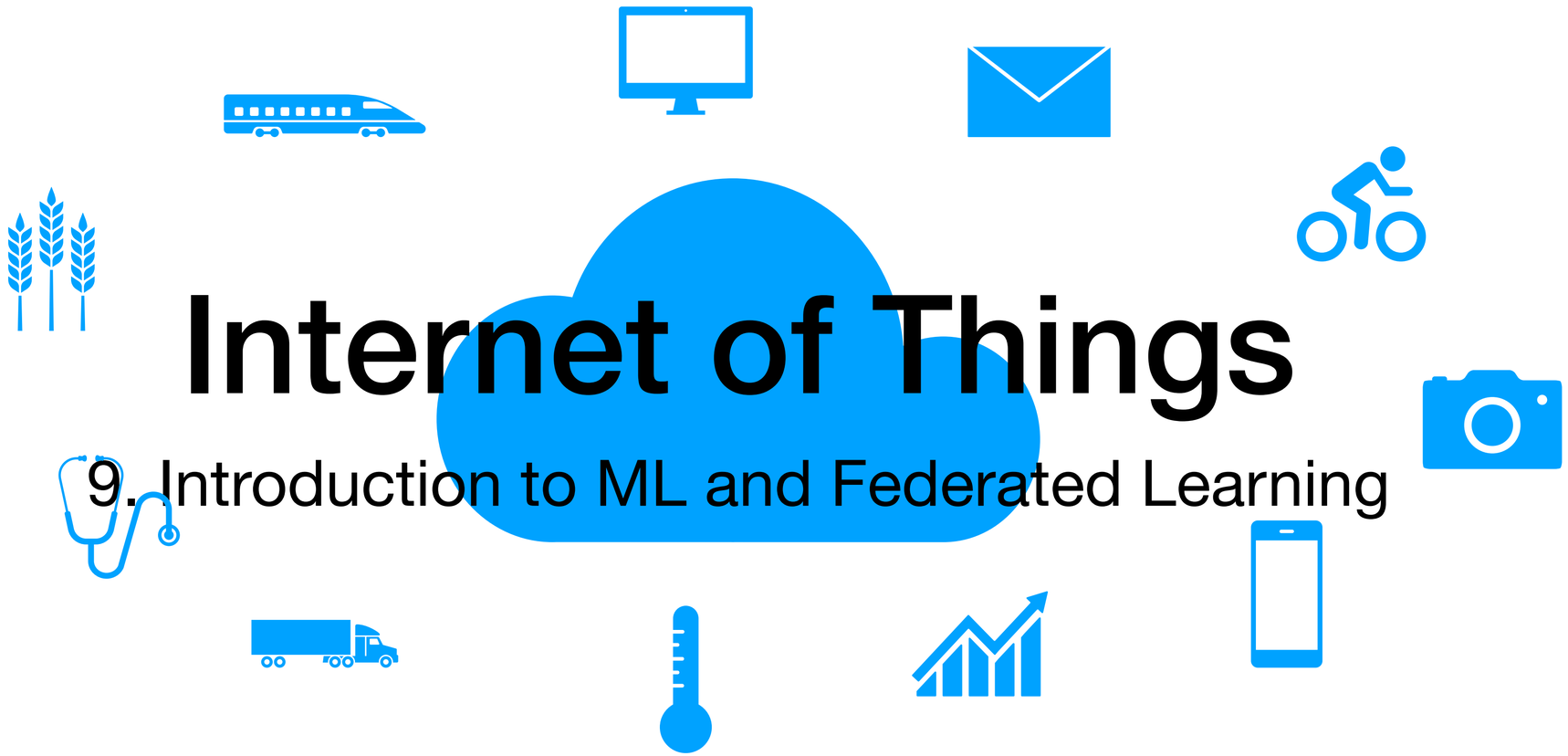


Internet of Things

9. Introduction to ML and Federated Learning

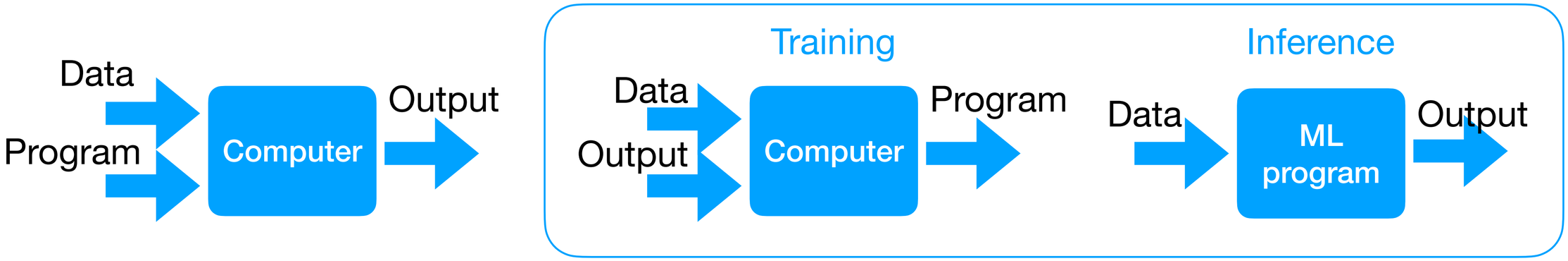
M.Sc. Computer Science 2024-2025

Viviana Arrigoni

ML/AI

- Artificial Intelligence is the field of developing computers and robots that are capable of behaving in ways that mimic human capabilities without human interference.

- Machine Learning is a sub-field of AI that uses algorithms to automatically learn insights and recognize patterns from **data**.

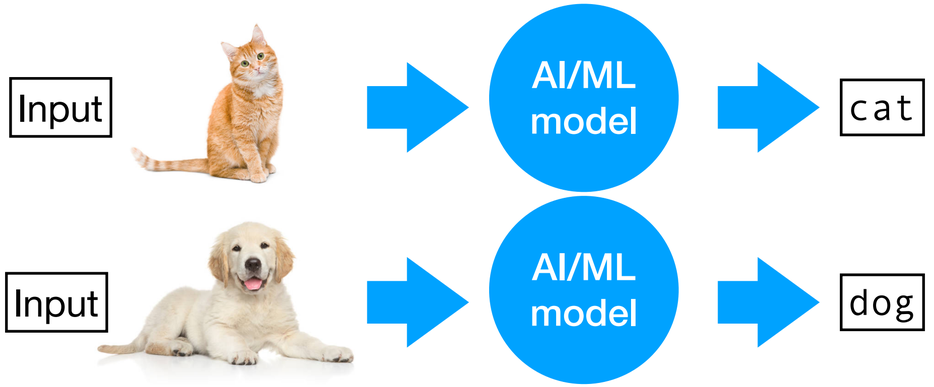


Traditional programming

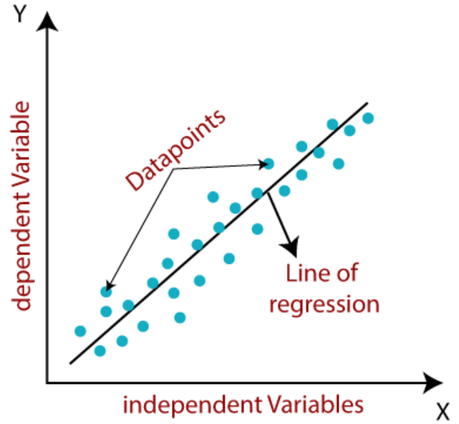
ML programming

Classification and regression

- In machine learning, there are two main types of task:
  - **Classification**- involves predicting a category or **class label**. The output is discrete, meaning the model tries to classify data into predefined labels or groups.

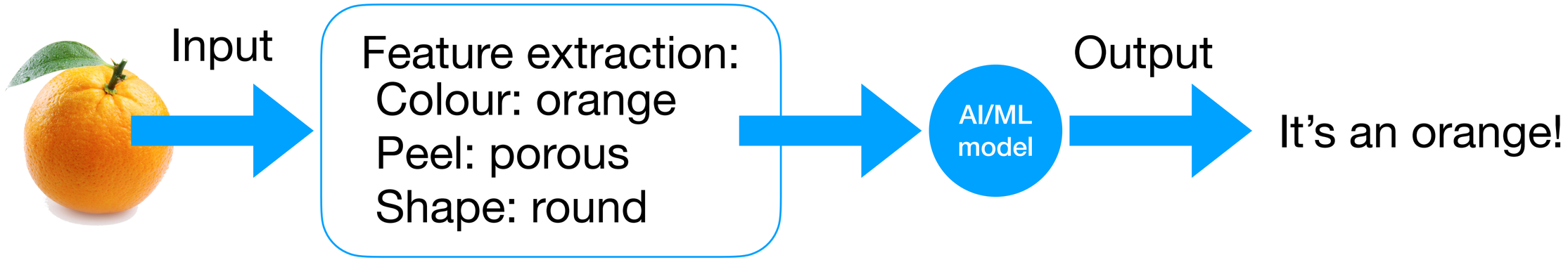


  - **Regression**- involves predicting a (possibly continuous) value or quantity (dependent variable) on a numerical scale by observing variables (features - independent variables) that influence prediction.



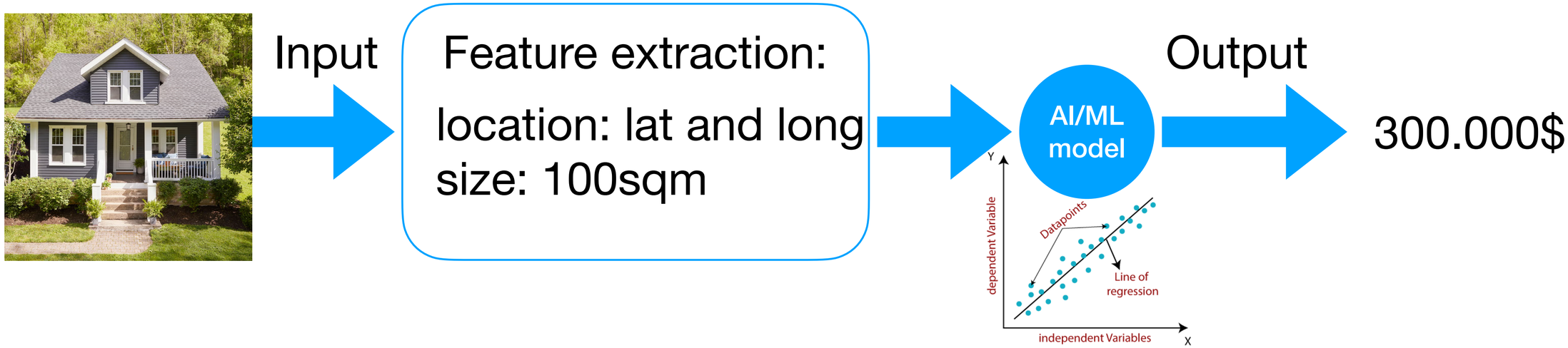
Features

- Data is preprocessed to identify **features**, i.e., characteristics of the input which help in describing it.
- High level classification example:
  task: identify the fruit
  three classes: apples, oranges, pears.
  - Features = colour, peel, shape.



Features

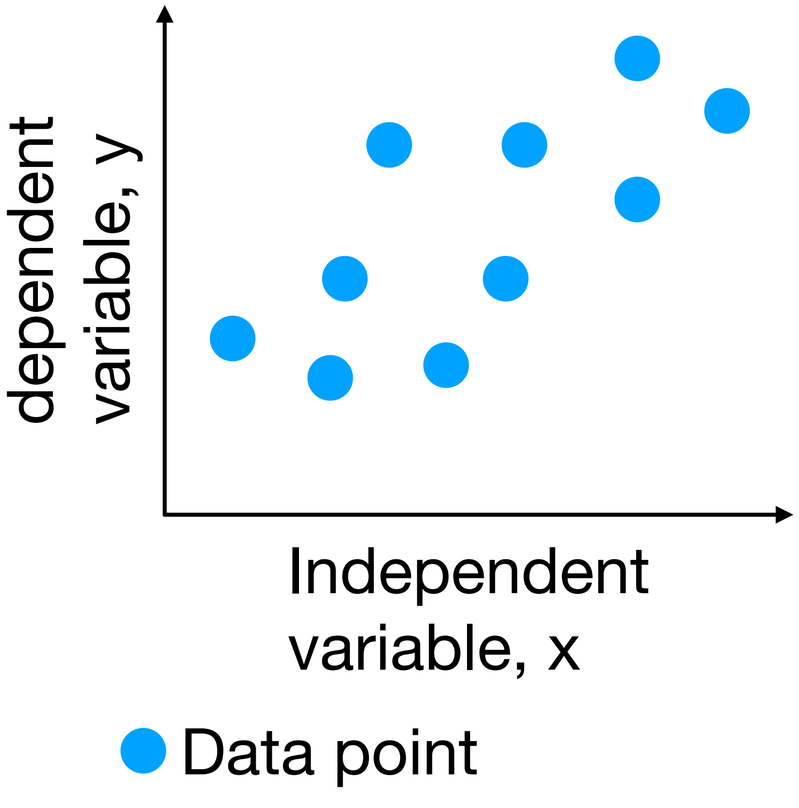
- Data is preprocessed to identify **features**, i.e., characteristics of the input which help in describing it.
- High level regression example:
  task: predict house price
  output: a value in $\mathbb{R}$
  Features = location, size.



9.1 Basics of Regression

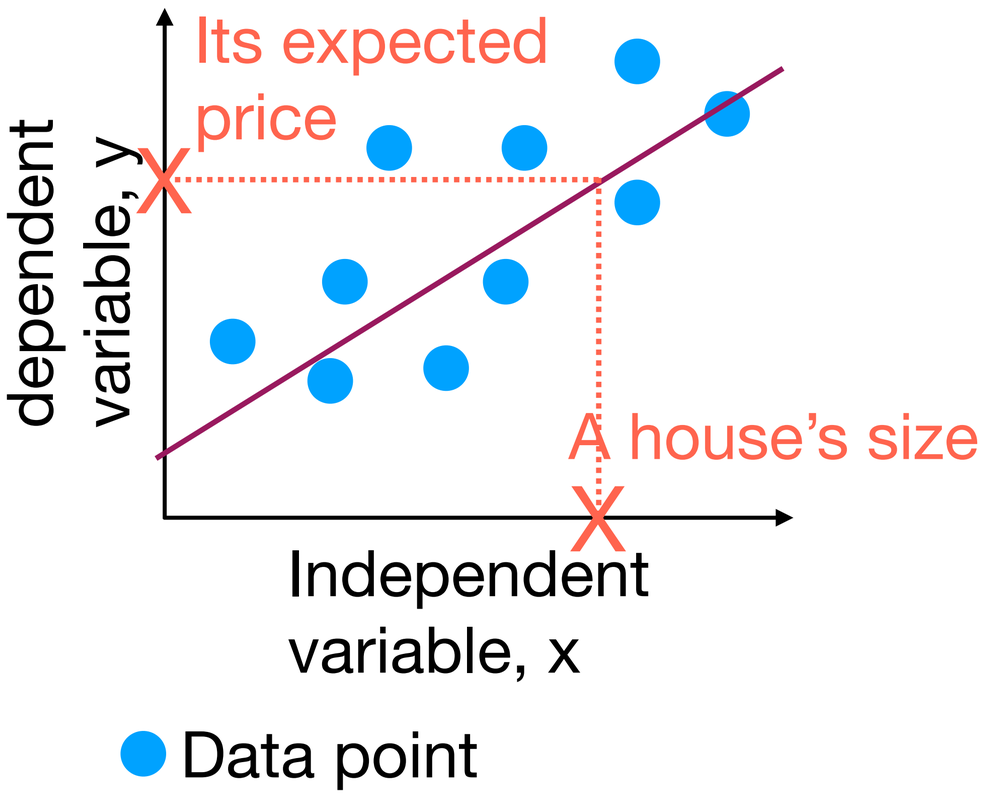
Linear Regression

- Consider a dataset as a set of data point, characterised by an independent variable (feature) and a dependent variable (to predict), for instance:
  - Independent variable = size
  - Dependent variable = cost
- Each data point is a house.
- By observing the distribution of the data points, we can notice a linear relationship between the two variables, e.g., the cost of the house grows linearly with its size.
- Objective: find the line that better “fits” the data point.



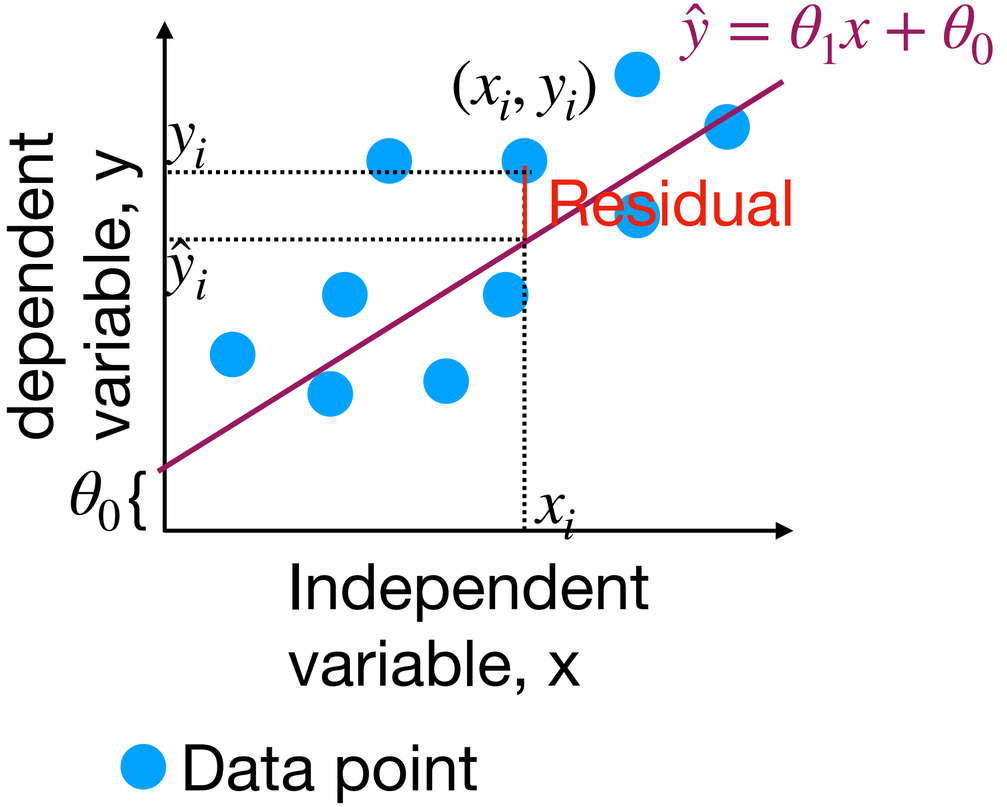
Linear Regression

- The line that better “fits” the data point is the line that better represents the relationship between the two variables, given a set of data points.
  - By finding such a line, given the size of a house, we can predict how much it costs with some confidence.



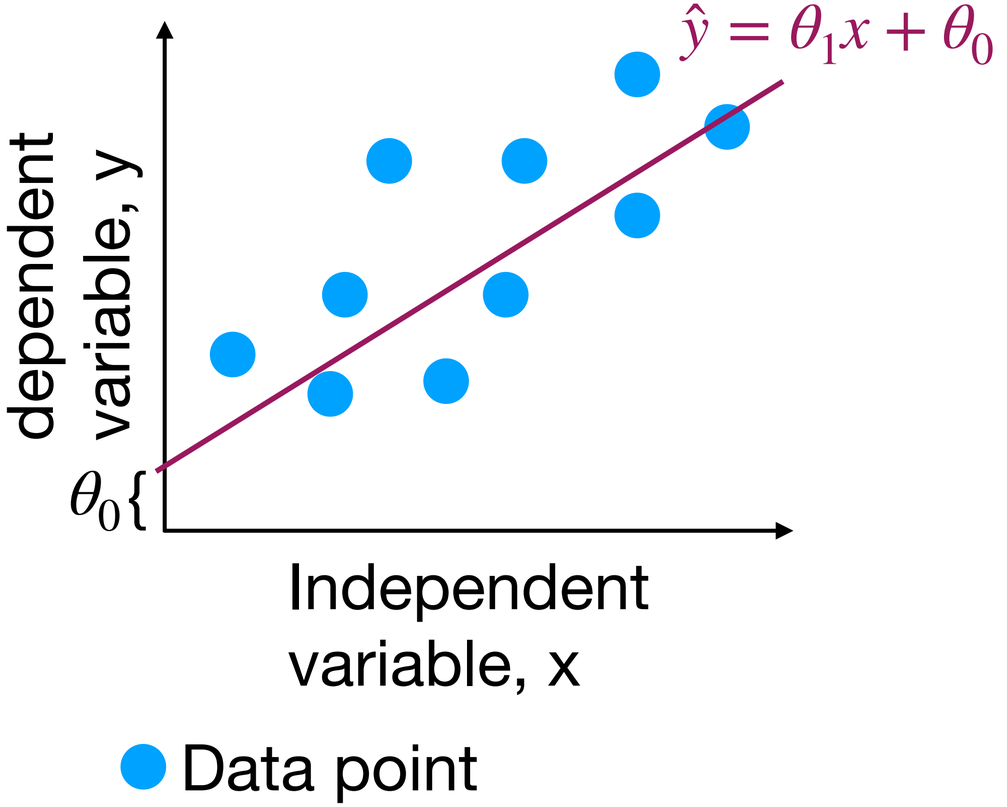
Linear Regression

- We can fit many lines, none of them is “perfect”, since data points are not aligned.
- How to we find the “best” one? And how do we decide if a line is “better” than the other?
- Recall that the function of a line is:
  $\hat{y} = \theta_1 x + \theta_0$,
  where θ_1 is the slope and θ_0 is the intercept.
- Given the function of a line, for each point (x_i, y_i), we can compute the **residual**, i.e., the difference between y_i (real value) and $\hat{y}_i = \theta_1 x_i + \theta_0$ (predicted value).



Linear Regression

- To define how good a line fits the data, we can define different *loss functions,* that are functions of the residuals of the data points in the dataset.
  - L1 loss = $\sum_{i=1}^{n} |\hat{y}_i - y_i|$
  - Mean Absolute Error (MAE) = $\frac{1}{n}\sum_{i=1}^{n} |\hat{y}_i - y_i|$
  - L2 loss = $\sum_{i=1}^{n} (\hat{y}_i - y_i)^2$
  - Mean Square Error (MSE) = $\frac{1}{n}\sum_{i=1}^{n} (\hat{y}_i - y_i)^2$
- Where n is the number of data points.



Which loss function to use? Usually…

Linear Regression

- To define how good a line fits the data, we can define different *loss functions,* that are functions of the residuals of the data points in the dataset.
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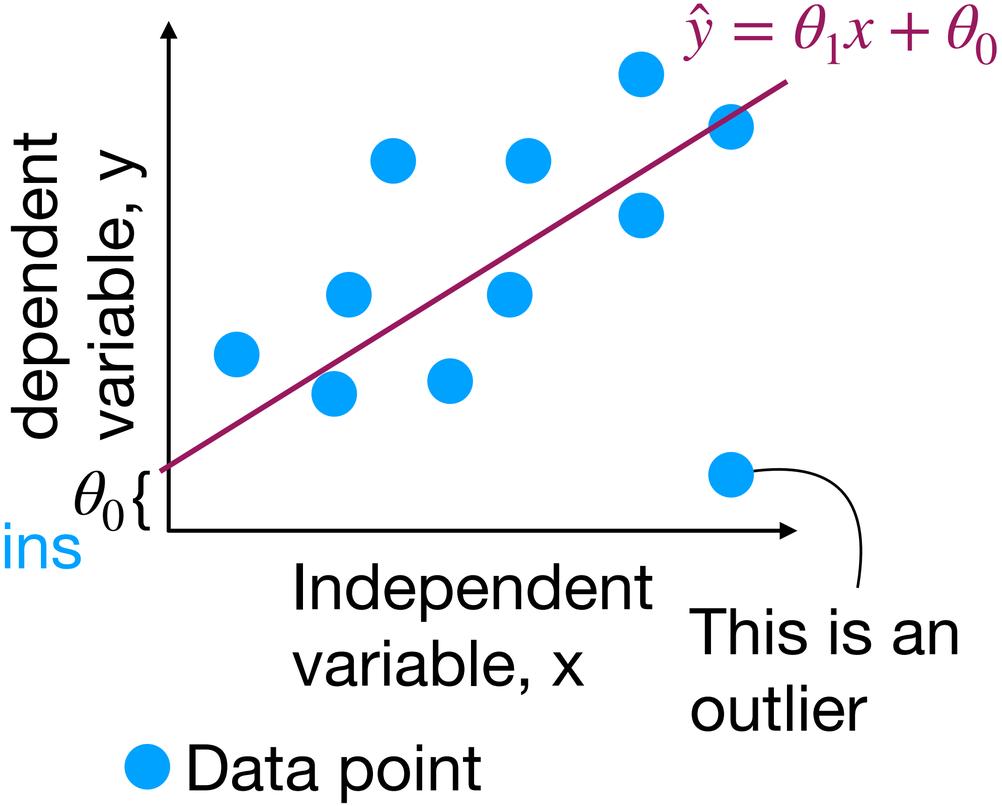
- Where n is the number of data points.

When data set contains many outliers

When data set does not contain many outliers

Which loss function to use? Usually…



Linear Regression

- To define how good a line fits the data, we can define different *loss functions,* that are functions of the residuals of the data points in the dataset.
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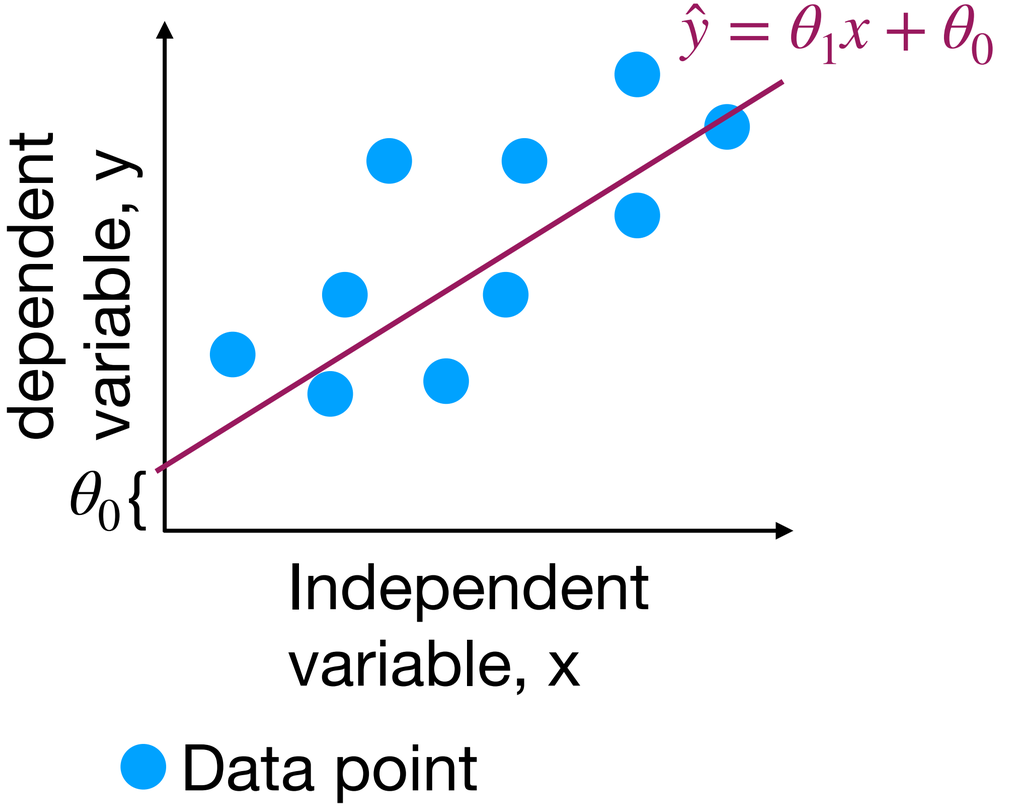
- Where n is the number of data points.



Objective: choose one of these loss functions, and find the line that minimises it

Linear Regression

- For example, if we want to find the line that minimises the MSE, we need to solve the following problem:
  - $\min_{\theta_0,\theta_1} f_{MSE}(\theta_0, \theta_1) = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i - y_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(\theta_i x_i + \theta_0 - y_i)^2$
- Analytically:
  - Find the zeros of the partial derivatives: $\frac{\partial f_{MSE}}{\partial \theta_0} = 0$, $\frac{\partial f_{MSE}}{\partial \theta_1} = 0$ ((θ_0^c, θ_1^c) "critical point", each is either a point of minimum, of maximum or a saddle point).
  - Compute the determinant of the Hessian matrix, $D = \frac{\partial^2 f_{MSE}}{\partial \theta_0^2} \cdot \frac{\partial^2 f_{MSE}}{\partial \theta_1^2} - \left(\frac{\partial^2 f_{MSE}}{\partial \theta_0 \partial \theta_1}\right)^2$, evaluate it in the critical points. If $D(\theta_0^c, \theta_1^c) > 0$ and $\frac{\partial^2 f_{MSE}(\theta_0^c, \theta_1^c)}{\partial \theta_0^2} > 0$, then (θ_0^c, θ_1^c) is a point of minimum, so the line that minimises the MSE has equation $\hat{y} = \theta_1^c x + \theta_0^c$

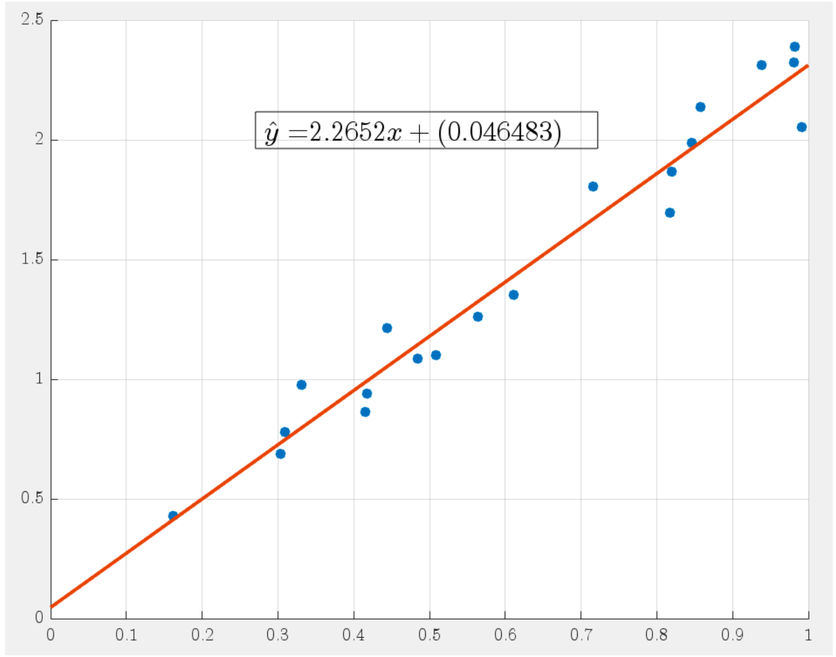
Linear Regression

- For example, if we want to find the line that minimises the MSE, we need to solve the following problem:
  - $\min_{\theta_0,\theta_1} f_{MSE}(\theta_0, \theta_1) = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i - y_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(\theta_i x_i + \theta_0 - y_i)^2$
- Geometrically:
  - Let $X = [x_1, \ldots, x_n]^T$, $Y = [y_1, \ldots, y_n]^T$, and $X_+ = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$. Let $\theta = [\theta_0, \theta_1]^T$. Then:
    - $\theta = (X_+^T \cdot X_+)^{-1} \cdot (X_+^T \cdot Y)$
  - $X_+^T \cdot X_+\theta = X_+^T Y$ is called **normal equation.**

Linear Regression

- For example, if we want to find the line that minimises the MSE, we need to solve the following problem:
  - $$\min_{\theta_0,\theta_1} f_{MSE}(\theta_0,\theta_1) = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i - y_i)^2 = \frac{1}{n}\sum_{i=1}^{n}(\theta_i x_i + \theta_0 - y_i)^2$$
- Geometrically:

```
X = rand(20,1);
X = sort(X);
Y = 2 * X + 0.5*rand(20,1);
Xp = [ones(20,1),X];
theta = inv(Xp'*Xp)*(Xp'*Y);
```

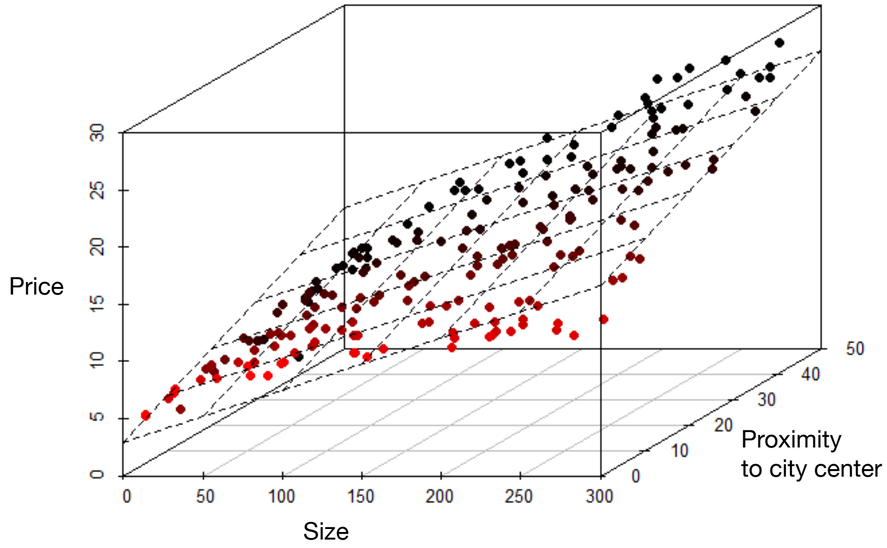


Linear Regression

- If the matrix $X_+^T \cdot X_+$ is not invertible, linear regression can still be used by computing its pseudo-inverse.
- The geometrical approach is very powerful because it gives a closed-form formula to perform linear regression for MSE minimisation!
- Very often, we do not have just one independent variable, but multiple ones.
  - For instance, the price of a house does not only depend on its size, but also on other factors, e.g., on the proximity to the city center.
  - Assume that also the relationship between the proximity to the city center and the price is also linear.

Linear Regression

- Each data point has 3 coordinates, (x, y, z), that are (size, proximity, price).
- We can still perform linear regression! In this case, we are searching for the **plane** that minimises a loss function. The equation of the plane is:
  - $\hat{z} = \theta_2 y + \theta_1 x + \theta_0$



- In general, if data points have coordinates: $(x_1, \ldots, x_d, y)$, then we want to find the **hyperplane**, which has general equation

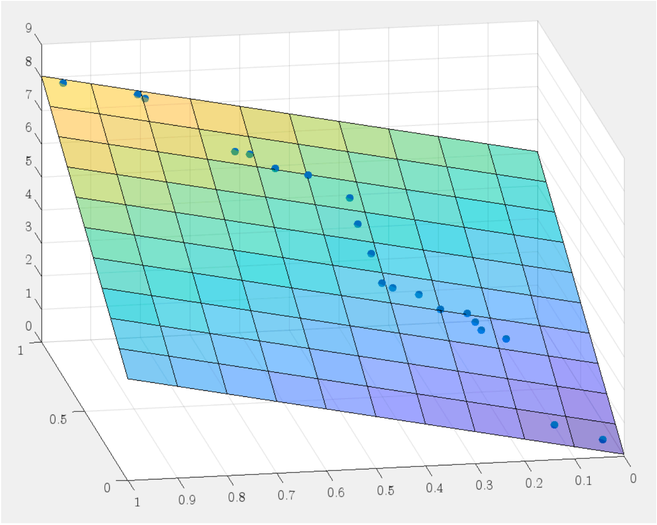
$$\hat{y} = \theta_0 + \theta_1 x_1 + \ldots + \theta_d x_d$$

that minimises a loss function.

Linear Regression

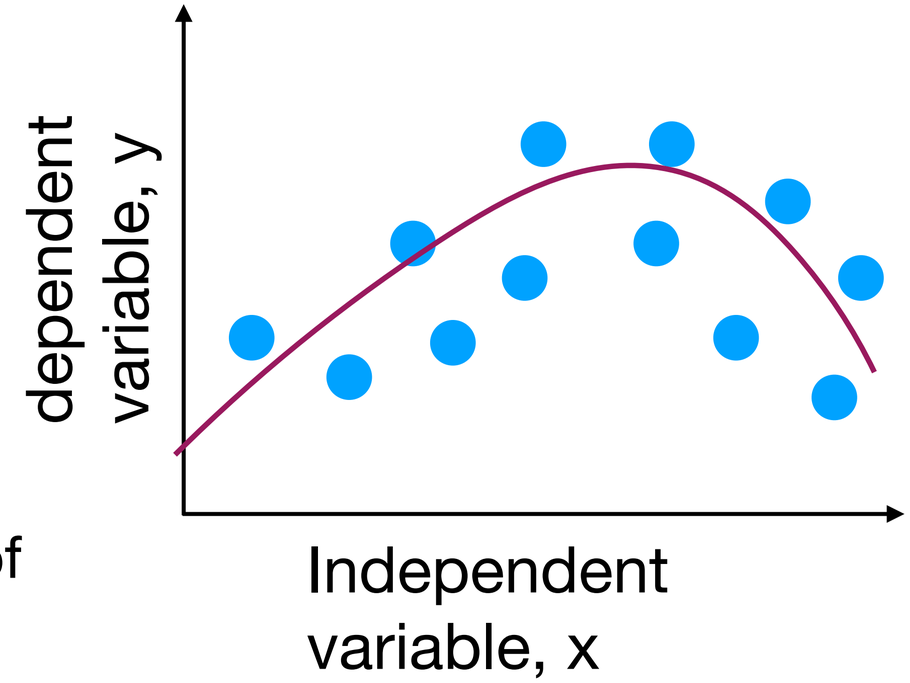
- Can we still use the normal equations approach to solve this?
  - Yes! It is enough to define X_+ as the matrix of size $n \times (d+1)$, where n is the number of data points, and d is the number of features (that are the independent variables - in the previous example, the house size and proximity to the center). The first column of X_+ is all ones. Y is still the vector $Y = [y_1, \ldots, y_n]^T$, and $\theta = [\theta_0, \theta_1, \ldots, \theta_d]^T$

```
Z = 5*X + 3*Y + rand(20,1)*0.1;
Xp = [ones(20,1), X, Y];
theta = inv(Xp'*Xp)*(Xp'*Z);
```



Polynomial Regression

- What if the data points do not exhibit a linear behaviour, but rather a polynomial one?
- In this case, we can search for the best polynomial that fits the data.
- $\hat{y} = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_m x^m$
- As before, we can do that by searching for the coefficients $(\theta_0, \theta_1, \ldots, \theta_m)$ that minimise a loss of function.

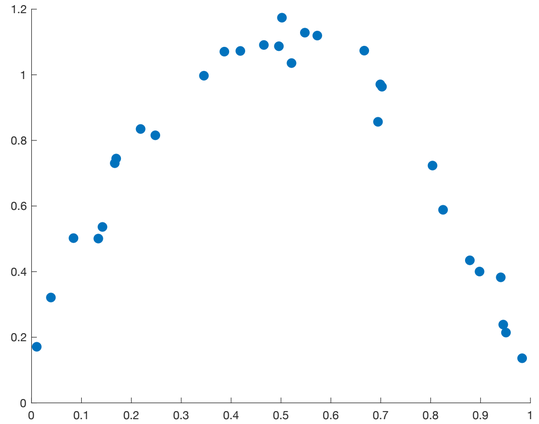


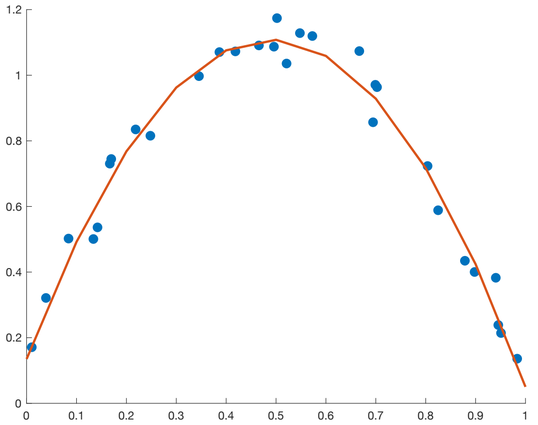
- Notice that, in the scenario that we are considering now, the dependent variable only depends on one independent variable, but it depends also on its powers.

Polynomial Regression

- If we want to find the polynomial that minimises the Mean Square Error, can we still use the normal equation?
- Yes! Given n data points (x_i, y_i), we define
- $$X_+ = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ (known)}$$

  and $\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_m \end{bmatrix}$ (unknown)
- Then we can find $\theta = (X_+^T \cdot X_+)^{-1} \cdot (X_+^T \cdot Y)$





Polynomial Regression

- What if the polynomial behaviour occurs over multiple variables?
- We can still use normal equations. Given n data points $(x_{1,i}, \ldots, x_{d,i}, y_i)$, we define
- $$X_+ = \begin{bmatrix} 1 & x_{1,1} & x_{1,1}^2 & \cdots & x_{1,1}^m & \cdots & x_{1,d} & x_{1,d}^2 & \cdots & x_{1,d}^m \\ 1 & x_{2,1} & x_{2,1}^2 & \ldots & x_{2,1}^m & \cdots & x_{2,d} & x_{2,d}^2 & \ldots & x_{2,d}^m \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n,1} & x_{n,1}^2 & \ldots & x_{n,1}^m & \cdots & x_{n,d} & x_{n,d}^2 & \ldots & x_{n,d}^m \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$ (known)

  and $\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{d \cdot m} \end{bmatrix}$ (unknown)

- Then we can find $\theta = (X_+^T \cdot X_+)^{-1} \cdot (X_+^T \cdot Y)$

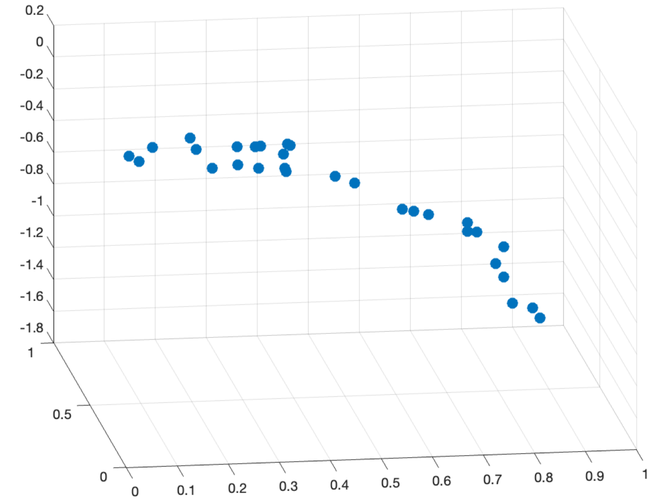
n = number of data points

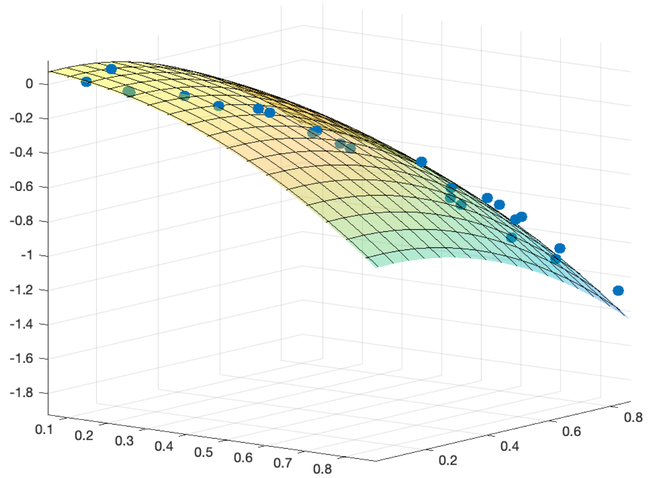
d = number of features

m = degree of the polynomials

Polynomial Regression

- Example with 30 datapoints, 2 features, degree = 2
- $$X_+ = \begin{bmatrix} 1 & x_{1,1} & x_{1,1}^2 & x_{1,2} & x_{1,2}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{30,1} & x_{30,1}^2 & x_{30,2} & x_{30,2}^2 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{30} \end{bmatrix}$$
- $f = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_2 + \theta_4 x_2^2$





- The problems with this approach are:
  - Computing the (pseudo) inverse of $X_+^T \cdot X_+$ is computationally expensive if the matrix is big
  - Works only for MSE, and not for other loss functions as Mean Absolute Error (MAE) = $\frac{1}{n}\sum_{i=1}^{n} |\hat{y}_i - y_i|$
- Alternative: use a **numerical** approach.
- Recall the **chain rule** for derivatives of composition of function:

$$\frac{d[f(g(x))]}{dx} = g'(x) \cdot f'(g(x))$$

Chain Rule

$$\frac{d[f(g(x))]}{dx} = g'(x) \cdot f'(g(x))$$

- **Example 1**
  - $g(x) = 2x + 1, f(x) = \frac{1}{3}x - 2.$
  - The composition of f and g is $f(g(x)) = \frac{1}{3}(2x + 1) - 2.$
  - Its derivative is $\frac{d[f(g(x))]}{dx} = \frac{2}{3}$, which is equal to:

    $$\frac{d[f(g(x))]}{dx} = g'(x) \cdot f'(g(x)) = 2 \cdot \frac{1}{3}$$

Chain Rule

$$\frac{d[f(g(x))]}{dx} = g'(x) \cdot f'(g(x))$$

- **Example 2**
  - $g(x) = 2x^2 + 1, f(x) = 3\log(x).$
  - The composition of f and g is $f(g(x)) = 3\log(2x^2 + 1).$
  - $g'(x) = 4x, f'(x) = \frac{3}{|x|}$, then

$$\frac{d[f(g(x))]}{dx} = g'(x) \cdot f'(g(x)) = 4x \cdot \frac{3}{|g(x)|} = 4x \cdot \frac{3}{2x^2 + 1}$$

9.2 Gradient Descent

Gradient

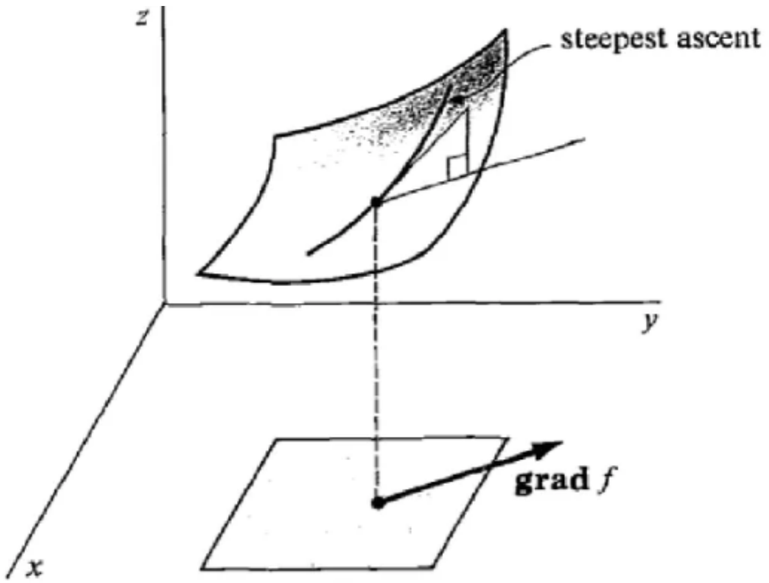
- The **gradient** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the vector of its partial derivatives: $\nabla f(x_1, \ldots, x_n) = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$.

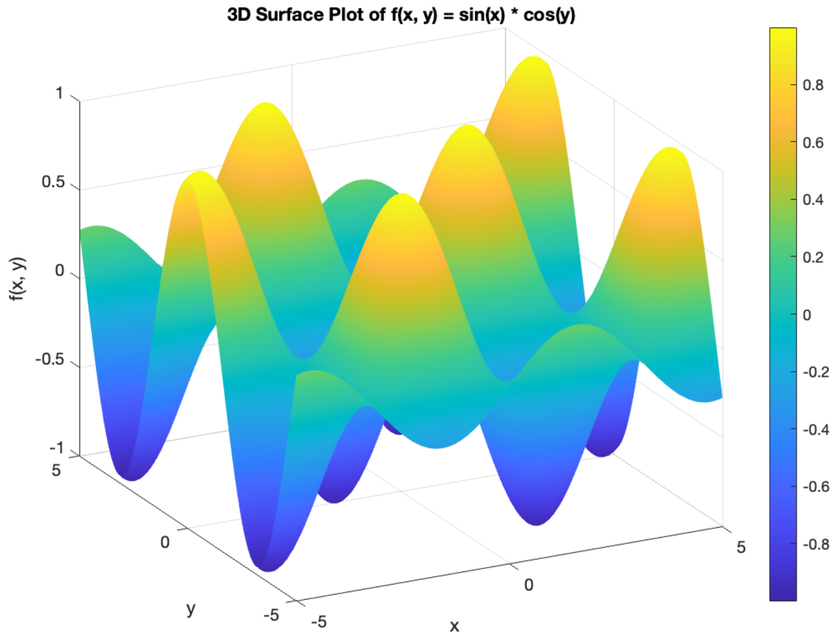
- If $n = 1$, the gradient is simply the derivative of f.

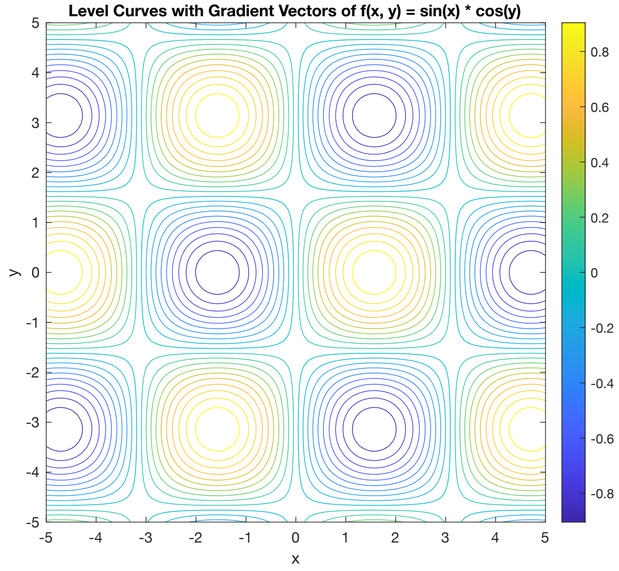
- The gradient generalises the concept of derivative in a multi-dimensional space.

- In each point of the domain, $(x_1, \ldots, x_n)$, it presents the **slope** of the surface $f(x_1, \ldots, x_n)$ in the **direction** of the **steepest ascent**.

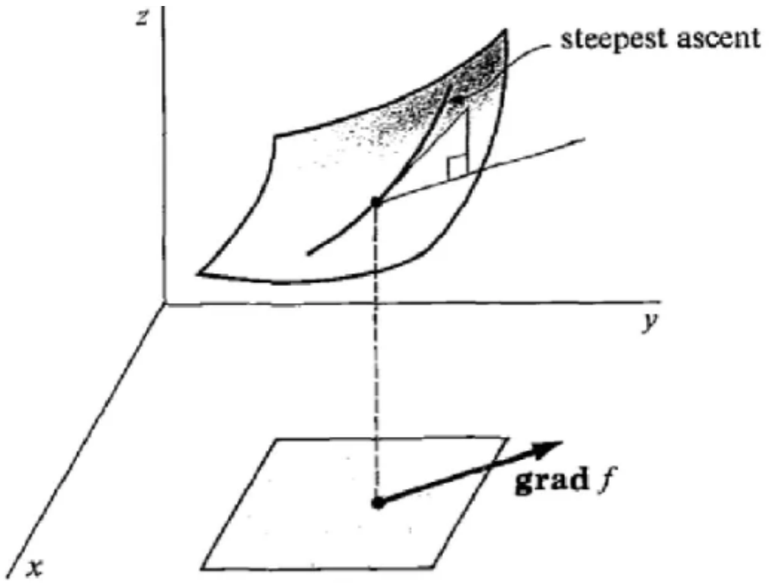
Gradient

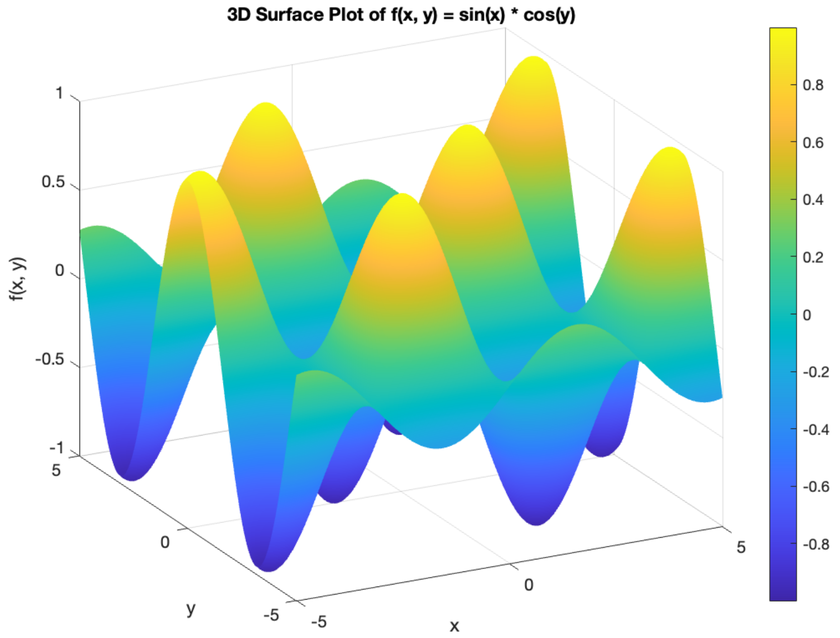

z
steepest ascent
y
grad f
x

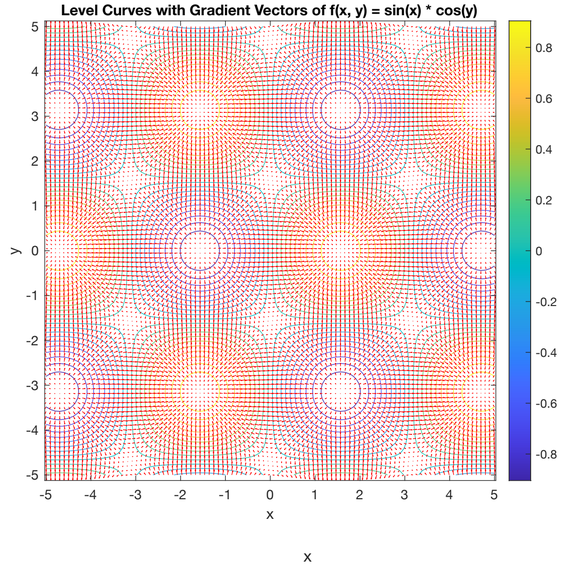

3D Surface Plot of f(x, y) = sin(x) * cos(y)
f(x, y)
1
0.5
0
-0.5
-1
5
0
y
-5
-5
0
5
x
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8


Level Curves with Gradient Vectors of f(x, y) = sin(x) * cos(y)
y
x
5
4
3
2
1
0
-1
-2
-3
-4
-5
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

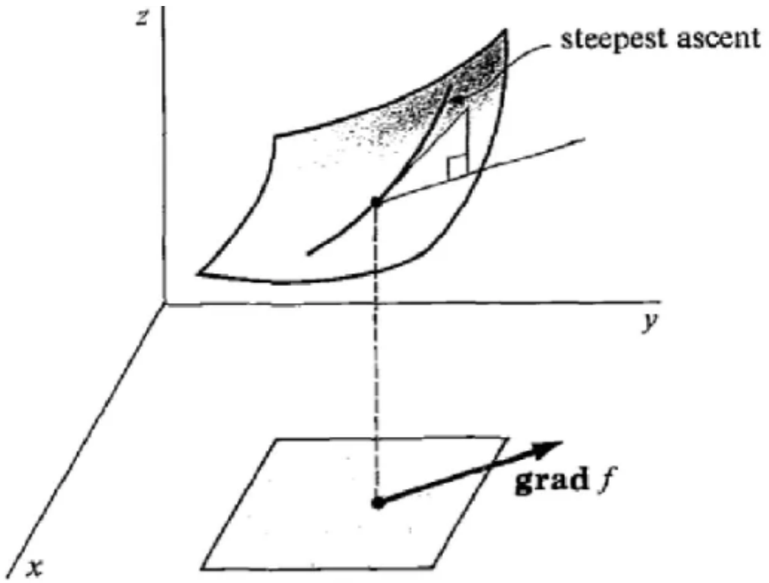
Gradient

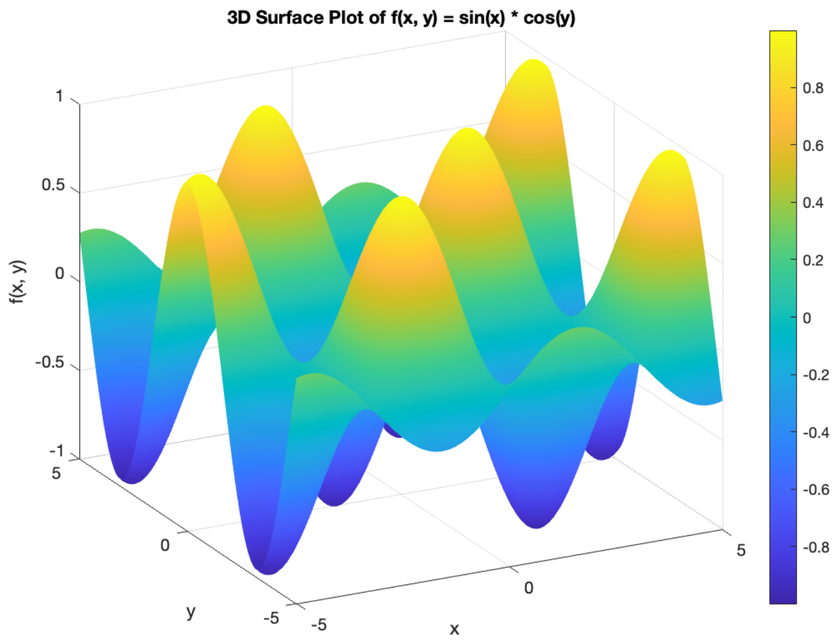

z
steepest ascent
y
grad f
x


3D Surface Plot of f(x, y) = sin(x) * cos(y)
f(x, y)
1
0.5
0
-0.5
-1
5
0
-5
y
-5
0
5
x
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

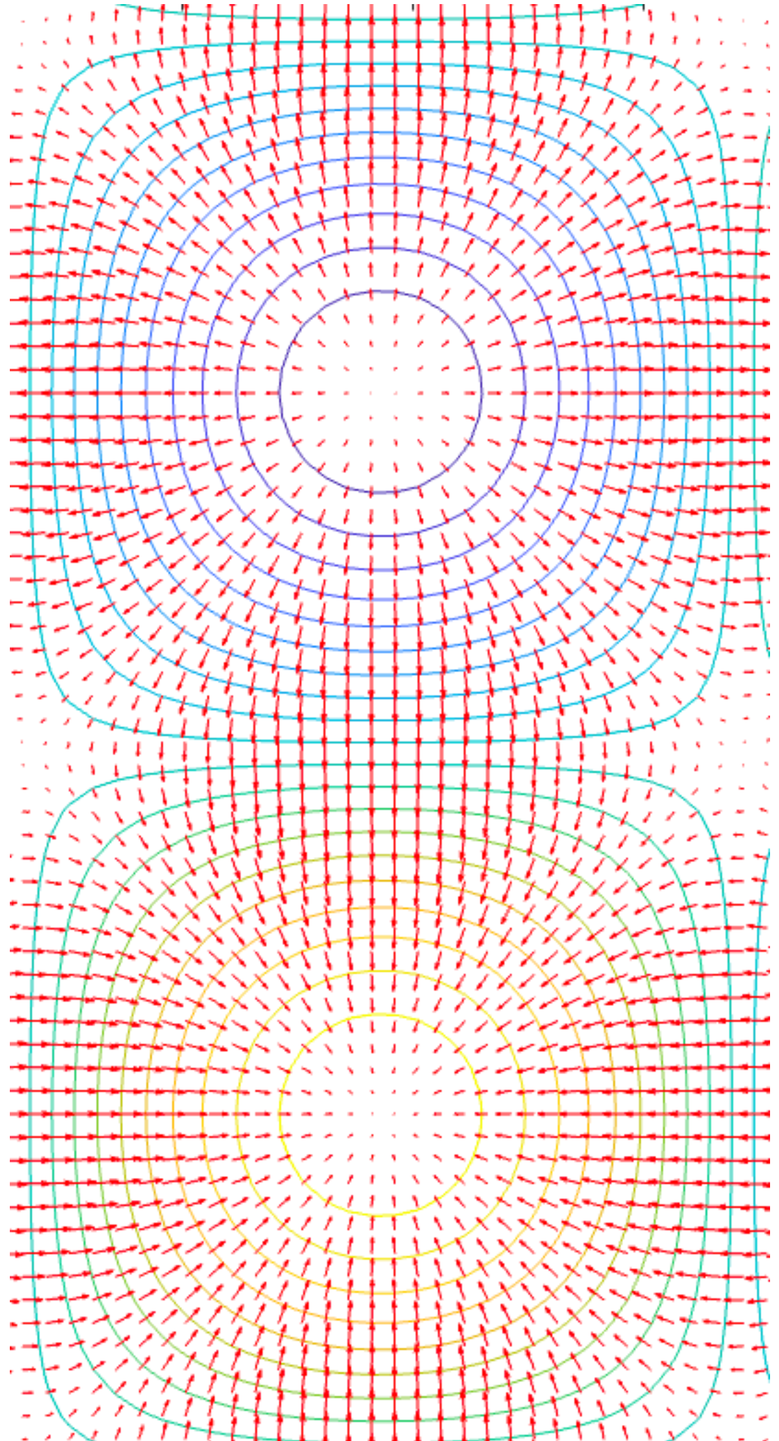

Level Curves with Gradient Vectors of f(x, y) = sin(x) * cos(y)
y
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5 -4 -3 -2 -1 0 1 2 3 4 5
x
x
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8

Gradient


z
steepest ascent
y
grad f
x


3D Surface Plot of f(x, y) = sin(x) * cos(y)
f(x, y)
1
0.5
0
-0.5
-1
5
0
-5
y
-5
0
5
x

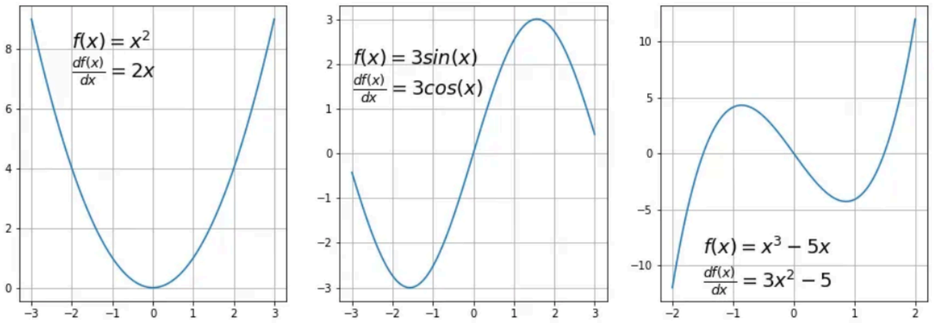
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8



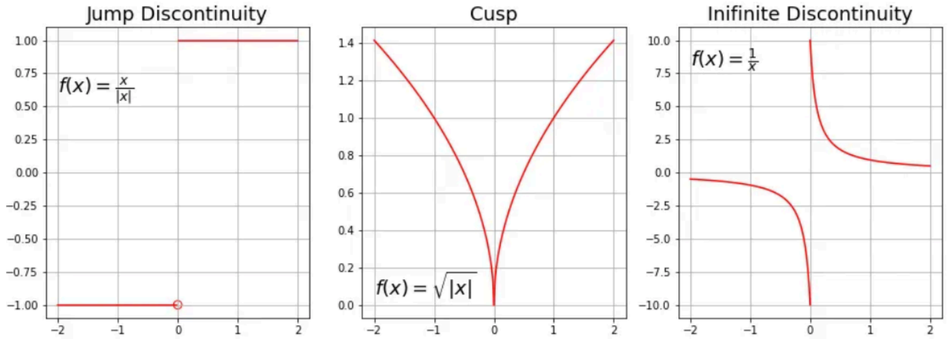
Gradient Descent

- The gradient descent is an iterative algorithm to find the minimum of a function f. It can be applied to f if:
  - f is differentiable.
    - If $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$, then it is differentiable if its derivative is defined in each point of D.

Differentiable



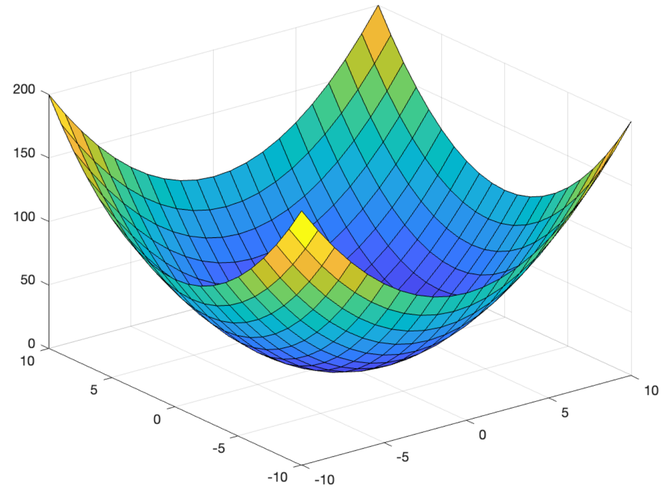
Non-differentiable



Gradient Descent

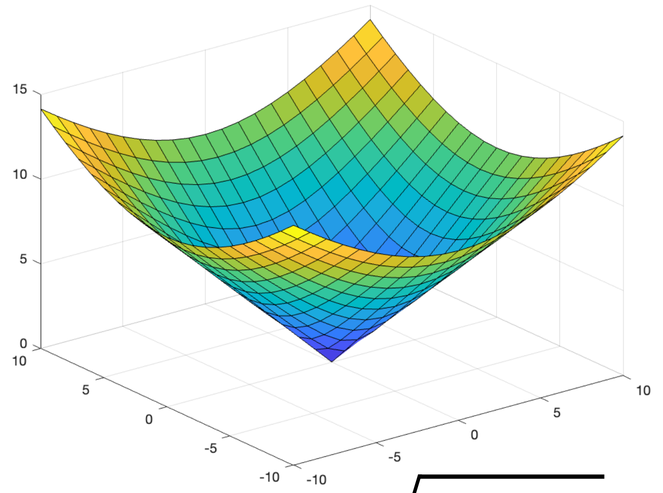
- If $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, saying whether it is differentiable or not is more complicated.
  - Sufficient condition: if for each point $a \in D$, all partial derivatives exist and are continuous in a neighbourhood of a, then the function is differentiable.

Differentiable



$$f(x_1, x_2) = x_1^2 + x_2^2$$

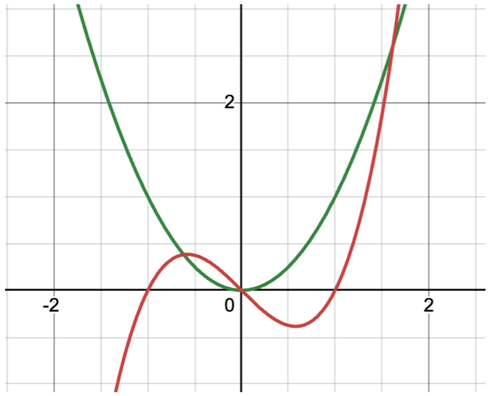
Non-differentiable



$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

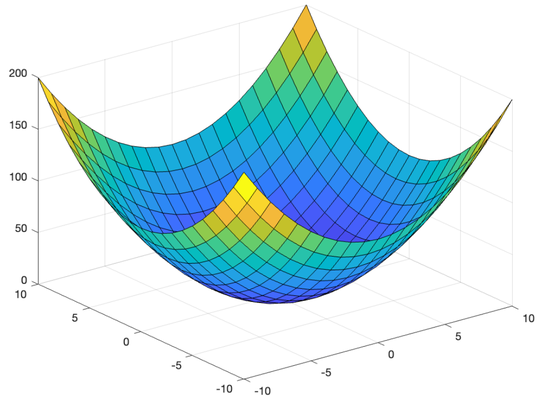
Gradient Descent

- The gradient descent is an iterative algorithm to find the minimum of a function f. It always finds the point of minimum if:
  - f is convex.



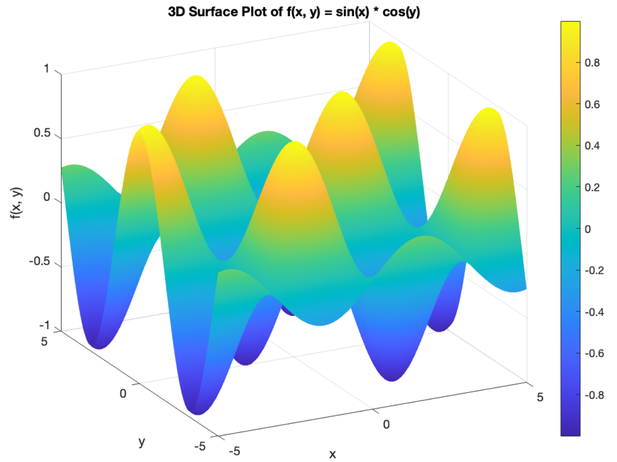
$f(x) = x^2$ Convex

$f(x) = x^3 - x$ Non convex



$f(x_1, x_2) = x_1^2 + x_2^2$

Convex



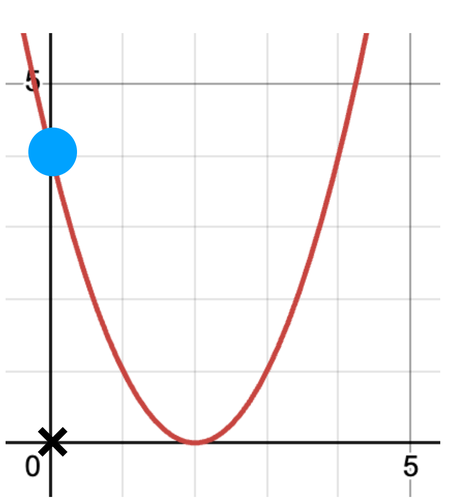
$f(x_1, x_2) = \sin(x_1) \cdot \cos(x_2)$

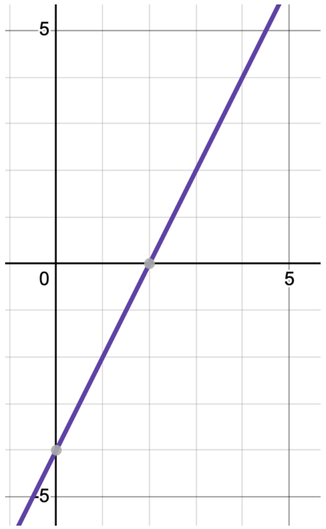
Non convex

Gradient Descent

- Idea: given a function, choose an initial point.
- Compute the gradient of the function, and see what is its value at the initial point.
- Move of one step towards the opposite direction. You will land in another point.
- Repeat for a maximum number iterations, or until the improvement is smaller than the tolerance.

$f(x) = (x-2)^2 \quad f'(x) = 2(x-2)$



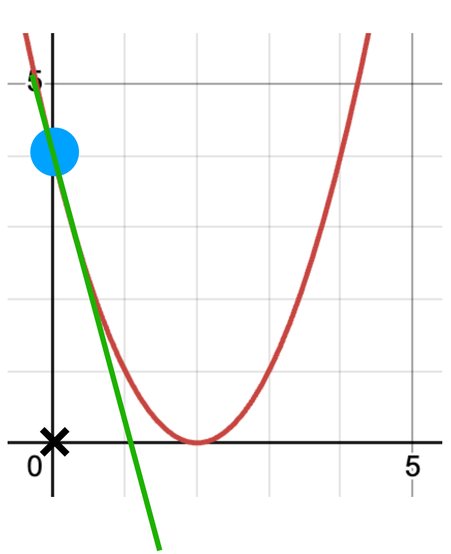


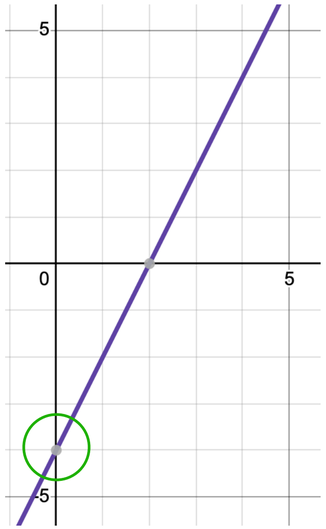
$x = 0$

Gradient Descent

- Idea: given a function, choose an initial point.
- Compute the gradient of the function, and see what is its value at the initial point.
- Move of one step towards the opposite direction. You will land in another point.
- Repeat for a maximum number iterations, or until the improvement is smaller than the tolerance.

$f(x) = (x-2)^2$ $f'(x) = 2(x-2)$





$x = 0$ $f'(0) = -4$

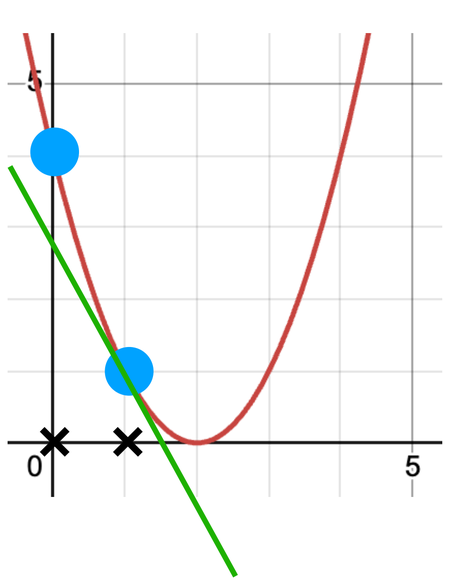
Line tangent to f in 0 has slope -4.

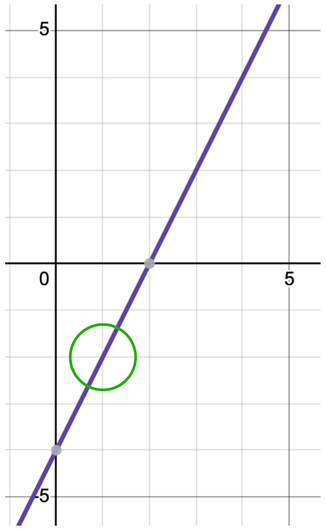
Means that f grows when "moving left". Since we want to move towards where the function decreases, we move right

Gradient Descent

- Idea: given a function, choose an initial point.
- Compute the gradient of the function, and see what is its value at the initial point.
- Move of one step towards the opposite direction. You will land in another point.
- Repeat for a maximum number iterations, or until the improvement is smaller than the tolerance.

$f(x) = (x-2)^2 \quad f'(x) = 2(x-2)$





$x = 1$ $\qquad f'(1) = -2$

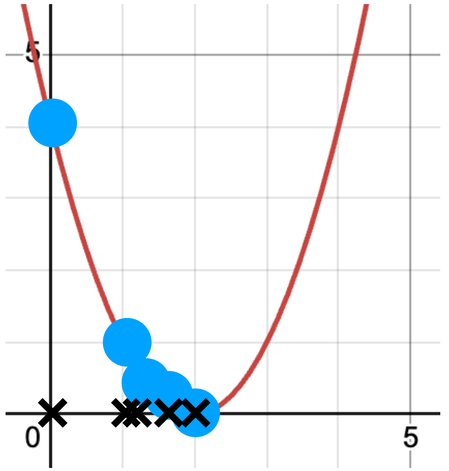
Line tangent to f in 1 has slope -2.

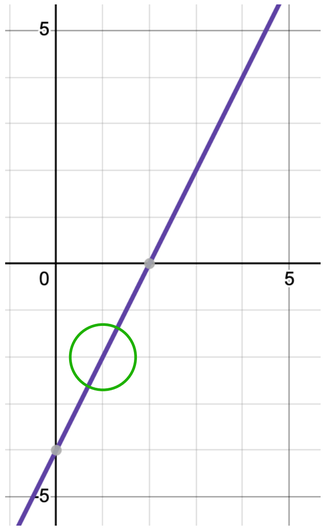
Means that f grows when "moving left". Since we want to move towards where the function decreases, we move right

Gradient Descent

- Idea: given a function, choose an initial point.
- Compute the gradient of the function, and see what is its value at the initial point.
- Move of one step towards the opposite direction. You will land in another point.
- Repeat for a maximum number iterations, or until the improvement is smaller than the tolerance.

$f(x) = (x-2)^2 \quad f'(x) = 2(x-2)$



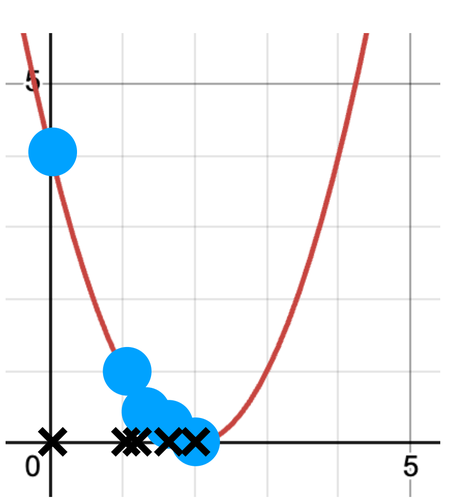


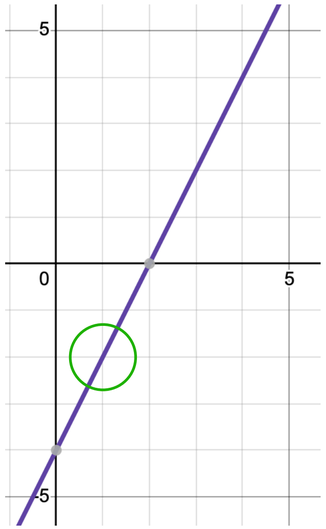
Continue until convergence

Gradient Descent

- Idea: given a function, choose an initial point.
- Compute the gradient of the function, and see what is its value at the initial point.
- Move of one step towards the opposite direction. You will land in another point.
- Repeat for a maximum number iterations, or until the improvement is smaller than the tolerance.

$f(x) = (x-2)^2$ $f'(x) = 2(x-2)$





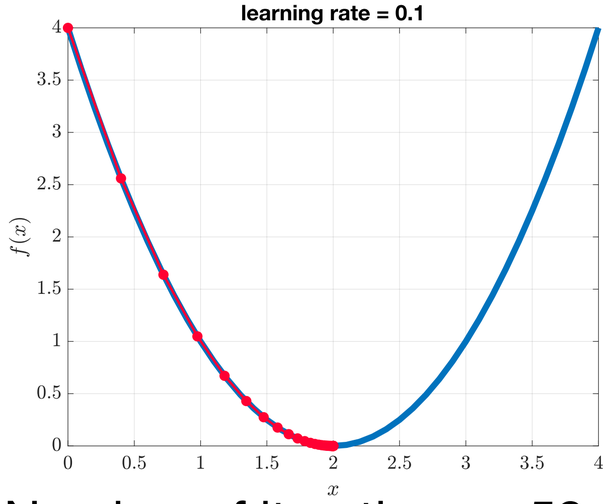
The step is known as “learning rate”

Large learning rate: move fast towards the opposite direction of the gradient
Small learning rate: move slowly towards the opposite direction of the gradient

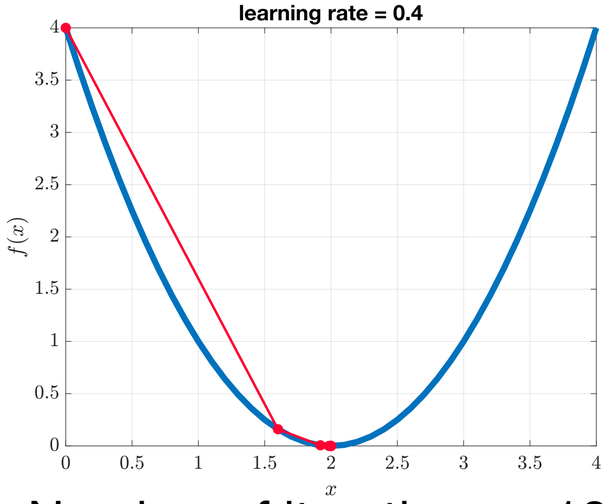
Gradient Descent

GradientDescent$(f(x), x_0, \eta, I_{max}, \epsilon)$

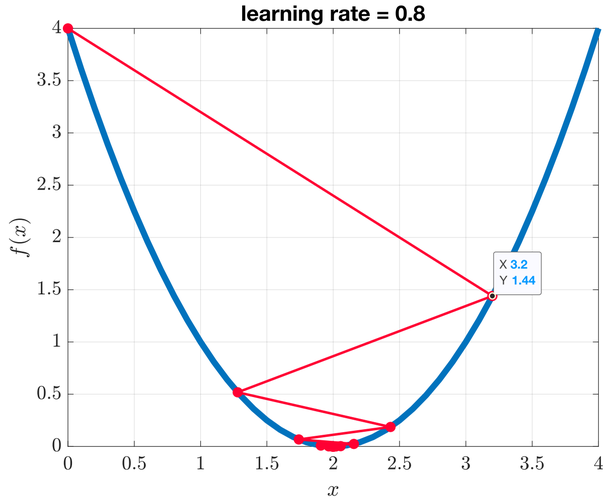
- Compute $\nabla f(x)$
- $x \leftarrow x_0$ (random)
- Initialise d
- For $i = 1, \ldots, I_{max}$
  - $d' \leftarrow \nabla f(x) \cdot \eta$
  - If $|d - d'| < \epsilon$
    - return x
  - $d \leftarrow d'$
  - $x \leftarrow x - d$



Number of iterations = 59



Number of iterations = 10



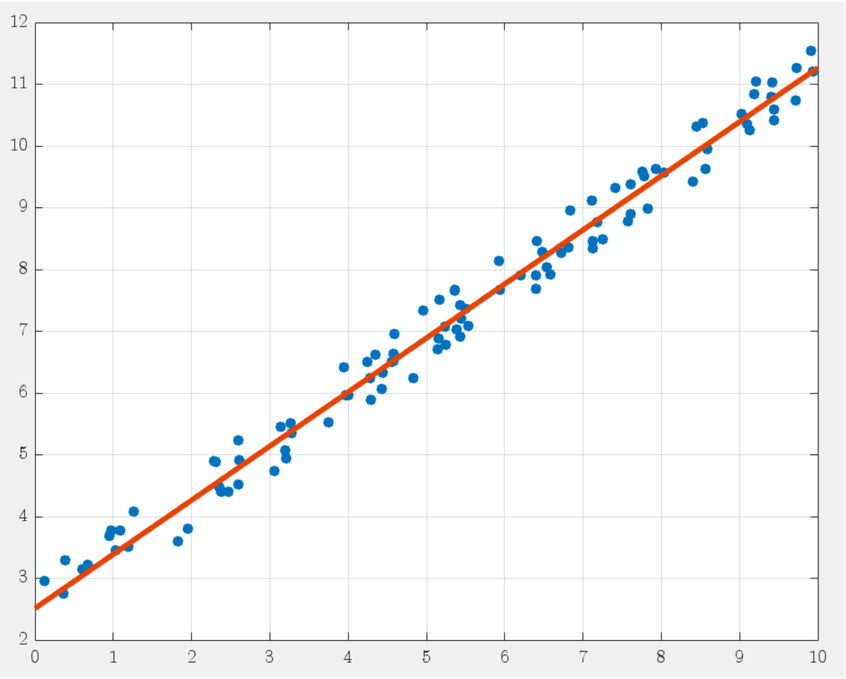
Number of iterations = 31

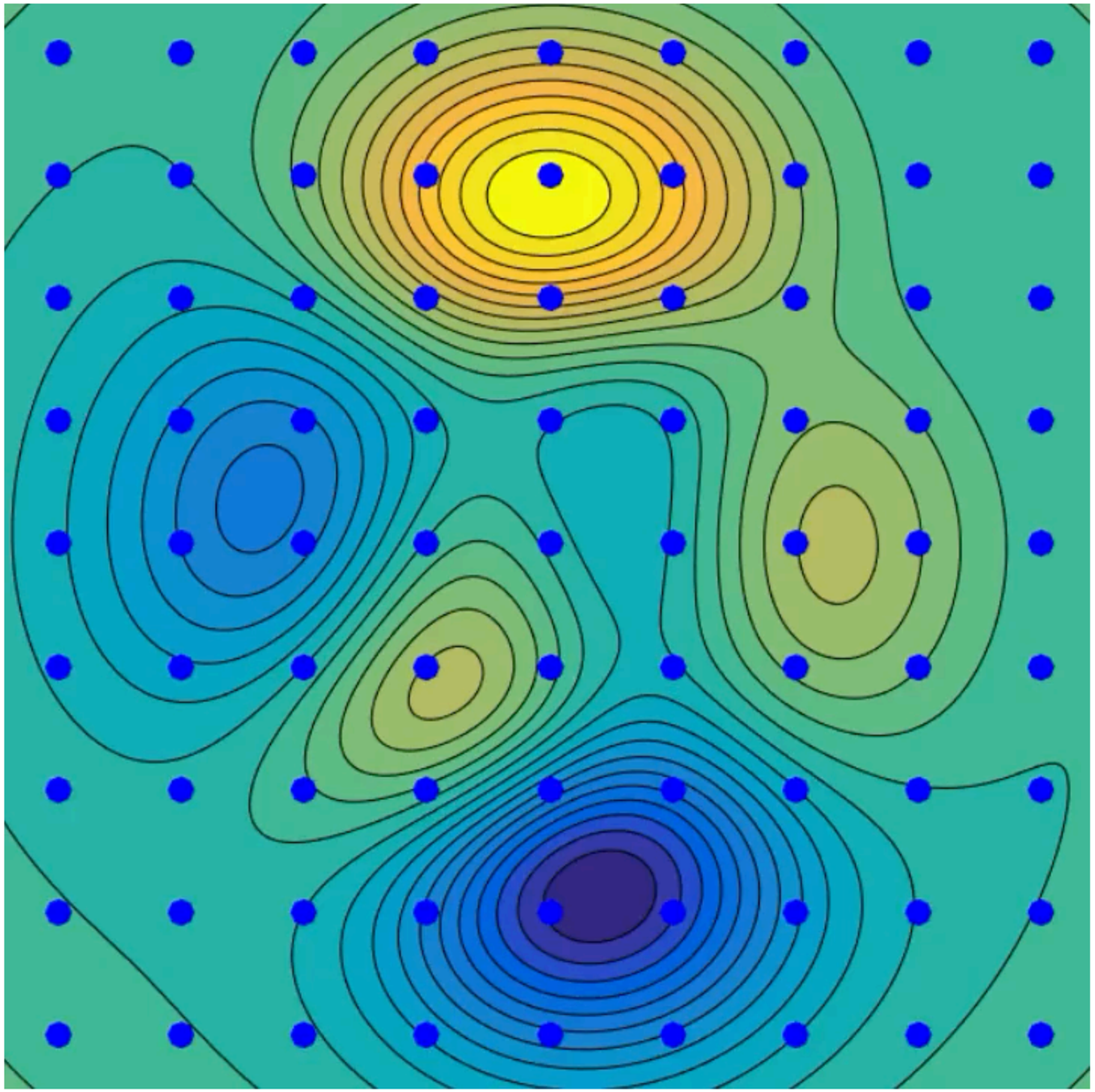
Number of iterations = I_{max}

Gradient Descent - example with MSE

GradientDescent$(f(\theta_0, \theta_1) = \frac{1}{n}\sum_{i=1}^{n}(\theta_0 + \theta_1 x_i - y_i)^2, \theta_{in}, \eta, I_{max}, \epsilon)$

- Compute $\nabla f(x) = \left[\frac{1}{n}\sum_{i=1}^{n} 2(\theta_0 + \theta_1 x_i - y_i), \frac{1}{n}\sum_{i=1}^{n} 2x_i(\theta_0 + \theta_1 x_i - y_i)\right]$
- $\theta \leftarrow \theta_{in}$
- Initialise d
- For $i = 1, \ldots, I_{max}$
  - $d' \leftarrow \nabla f(x) \cdot \eta$
  - If $||d - d'|| < \epsilon$
    - return θ
  - $d \leftarrow d'$
  - $\theta \leftarrow \theta - d$



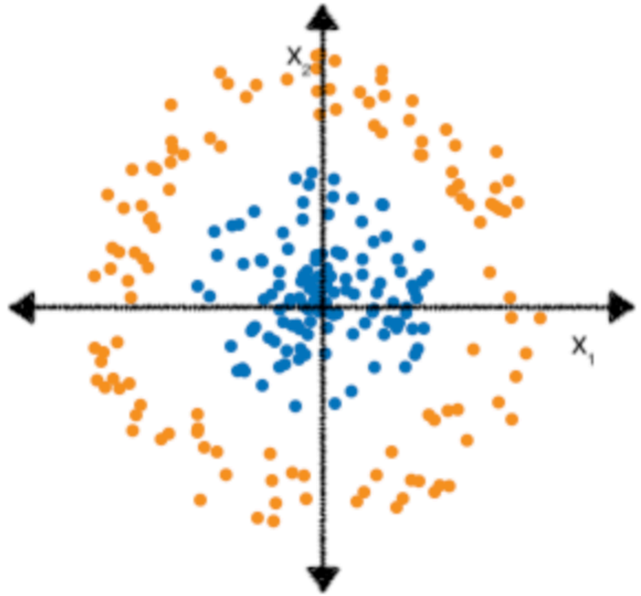


9.3 Introduction to Neural Networks (MLP and back propagation)

- Very often, data is represented with many features and exhibit complex non-linear patterns which are hard to detect with regression.

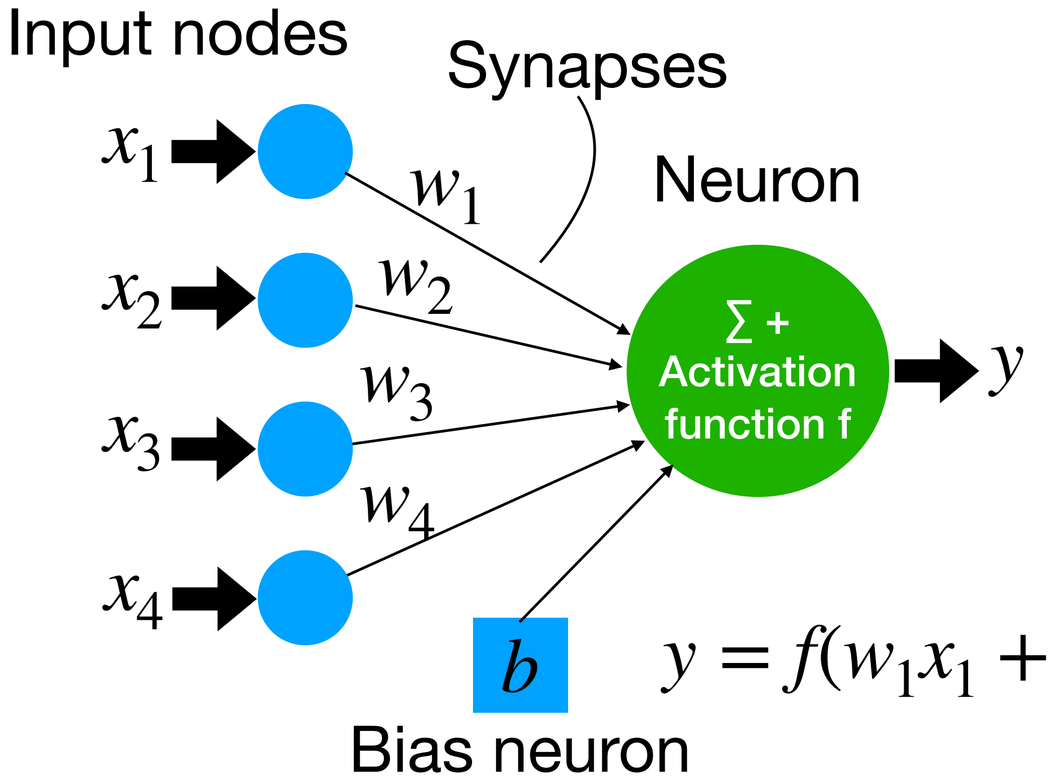
- Neural networks are a family of model architectures designed to find nonlinear patterns in data.

- Their structure and components are inspired by the biological structure of the humans' neuron.



Perceptron

- The perception is the building block of a neural network. The number of input nodes is given by the number of features of the data points.

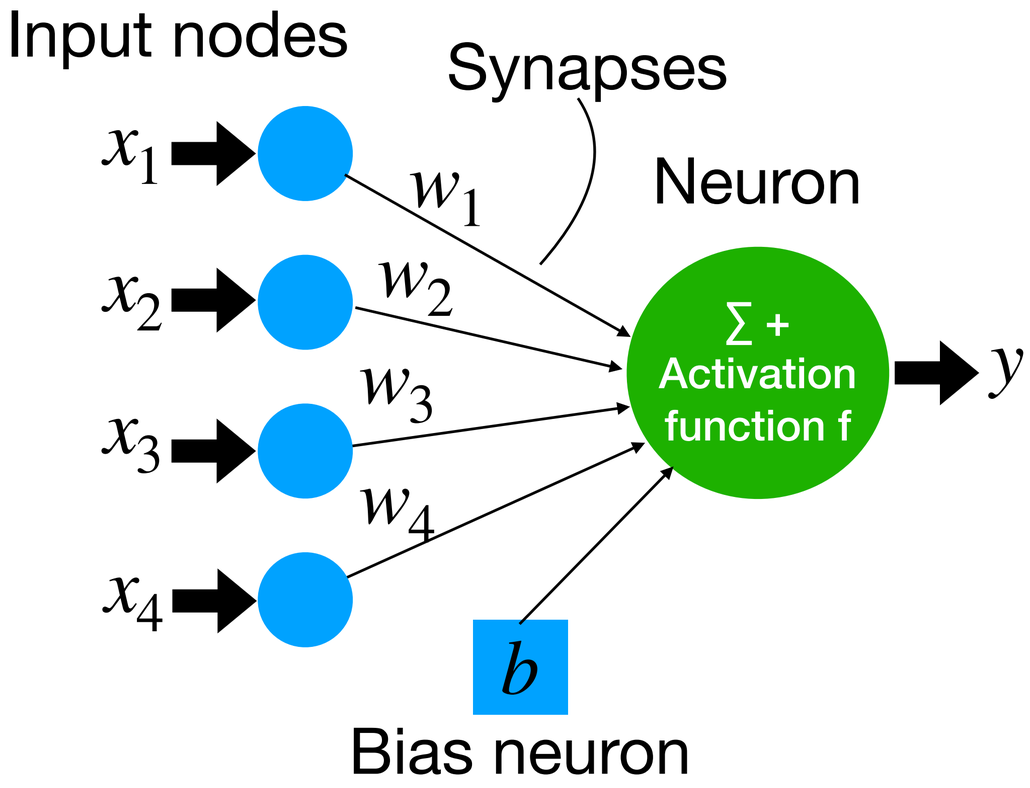


- If the activation function f is the identity function, this is a simple linear regression model, where b is the intercept.

$$y = f(w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + b)$$

Perceptron

- The perception is the building block of a neural network.



- Given an input data point with d features $x_1, \ldots, x_d$, the neuron:
- Computes a linear combination of the input and the synapses weights, possibly adding the bias:

  $b + \sum_{i=1}^{d} w_i x_i$, or, in matrix form: $b + \mathbf{w}^T \cdot \mathbf{x}$

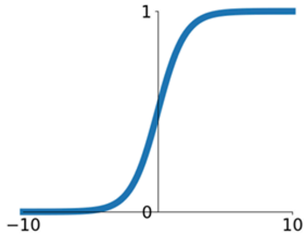
  where $\mathbf{w} = [w_1, \ldots, w_d]^T$ and $\mathbf{x} = [x_1, \ldots, x_d]^T$
- Applies the **activation function** f to the combination.

Activation functions

- Activation functions enables a perceptron to learn nonlinear and complex relationships between features and label.

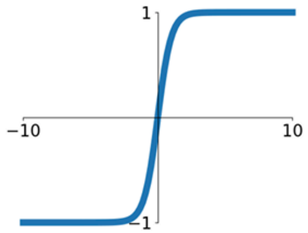
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



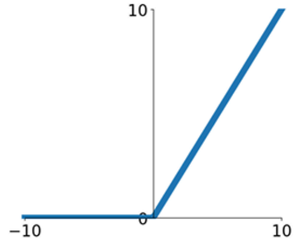
tanh

$$\tanh(x)$$



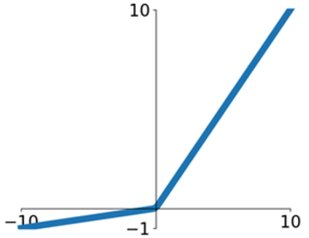
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$



Maxout

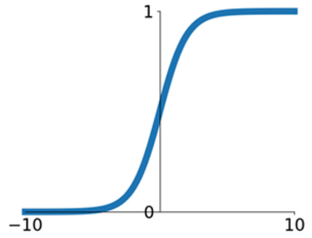
$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

Activation functions

- Activation functions enables a perceptron to learn nonlinear and complex relationships between features and label.

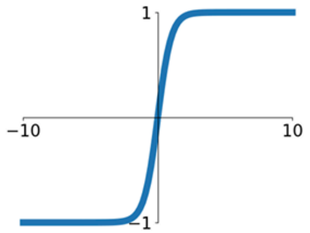
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



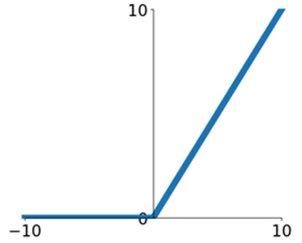
tanh

$$\tanh(x)$$



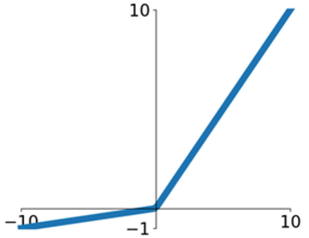
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$



Maxout

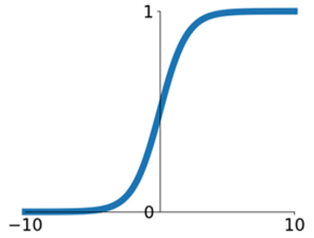
$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

Activation functions

- Activation functions enables a perceptron to learn nonlinear and complex relationships between features and label.

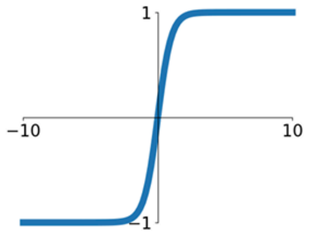
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



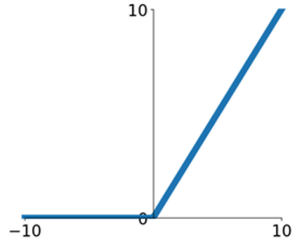
tanh

$$\tanh(x)$$



ReLU

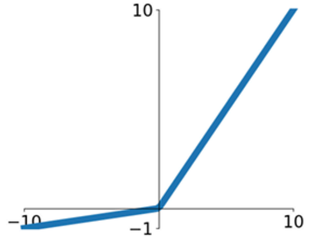
$$\max(0, x)$$



Notice that this is not differentiable

Leaky ReLU

$$\max(0.1x, x)$$

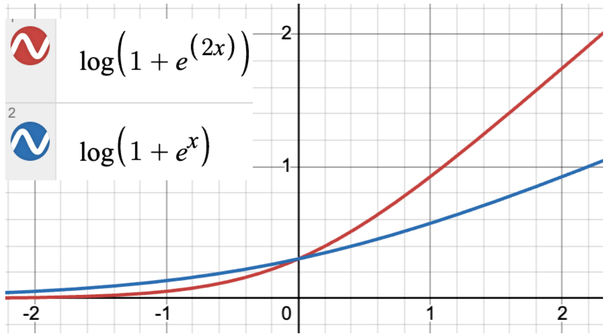


Maxout

$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

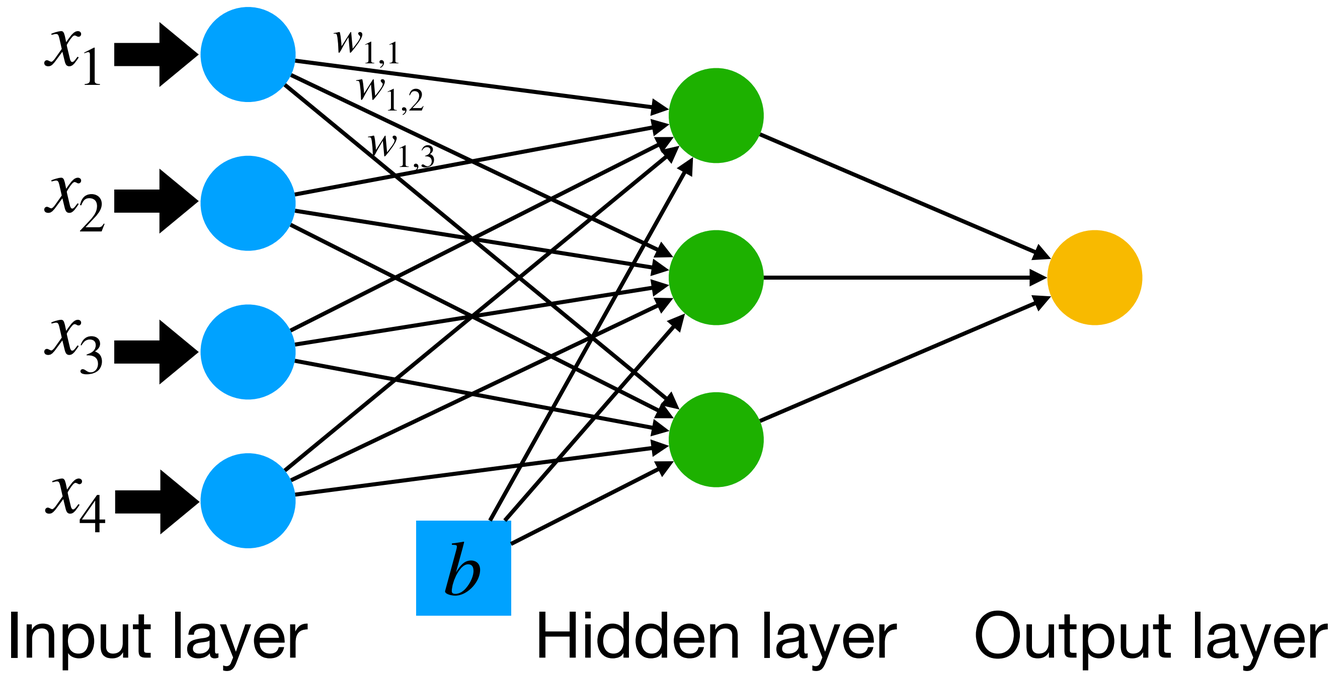
Softplus

$$\log(1 + e^x)$$



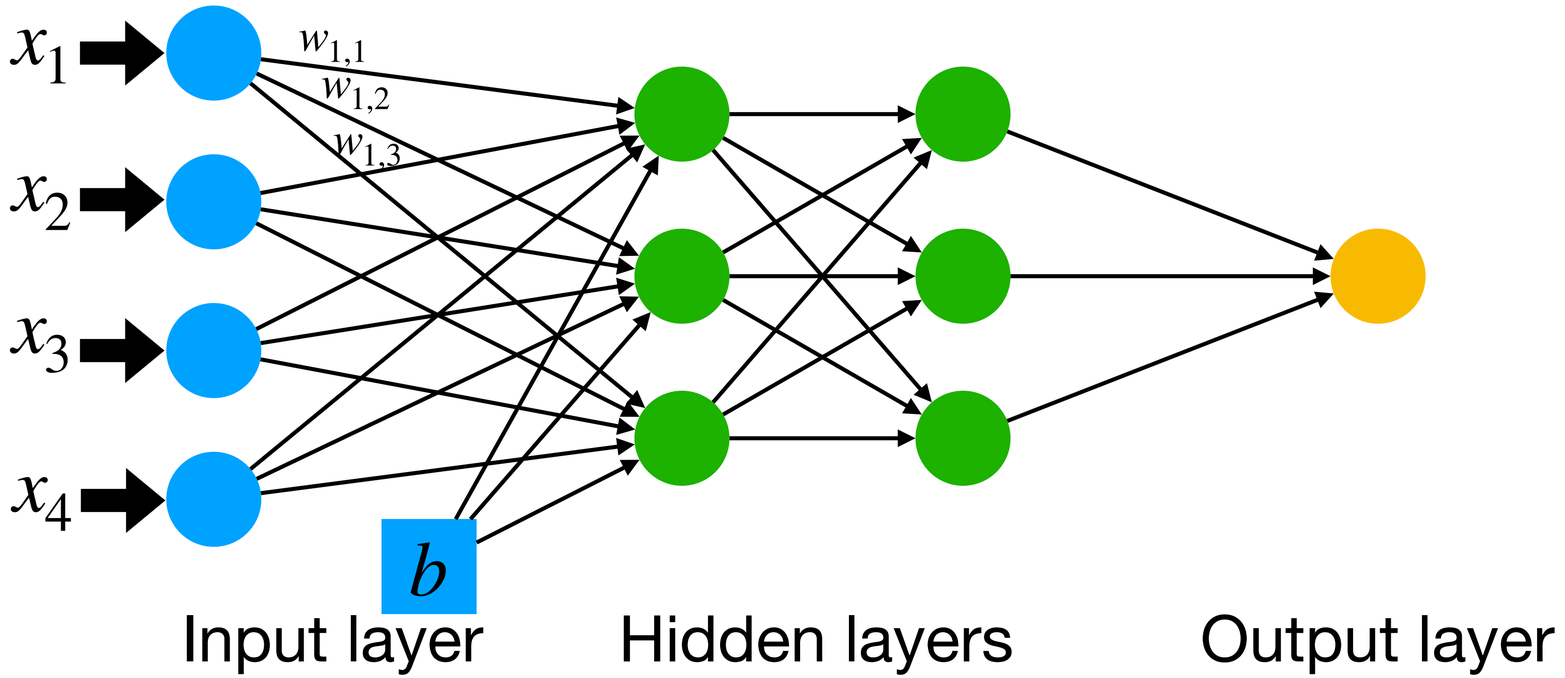
Single layer Perceptron

- Neurons and Synopsis are organised in layers: the input layer, one or more **hidden layers**, and the output layer.

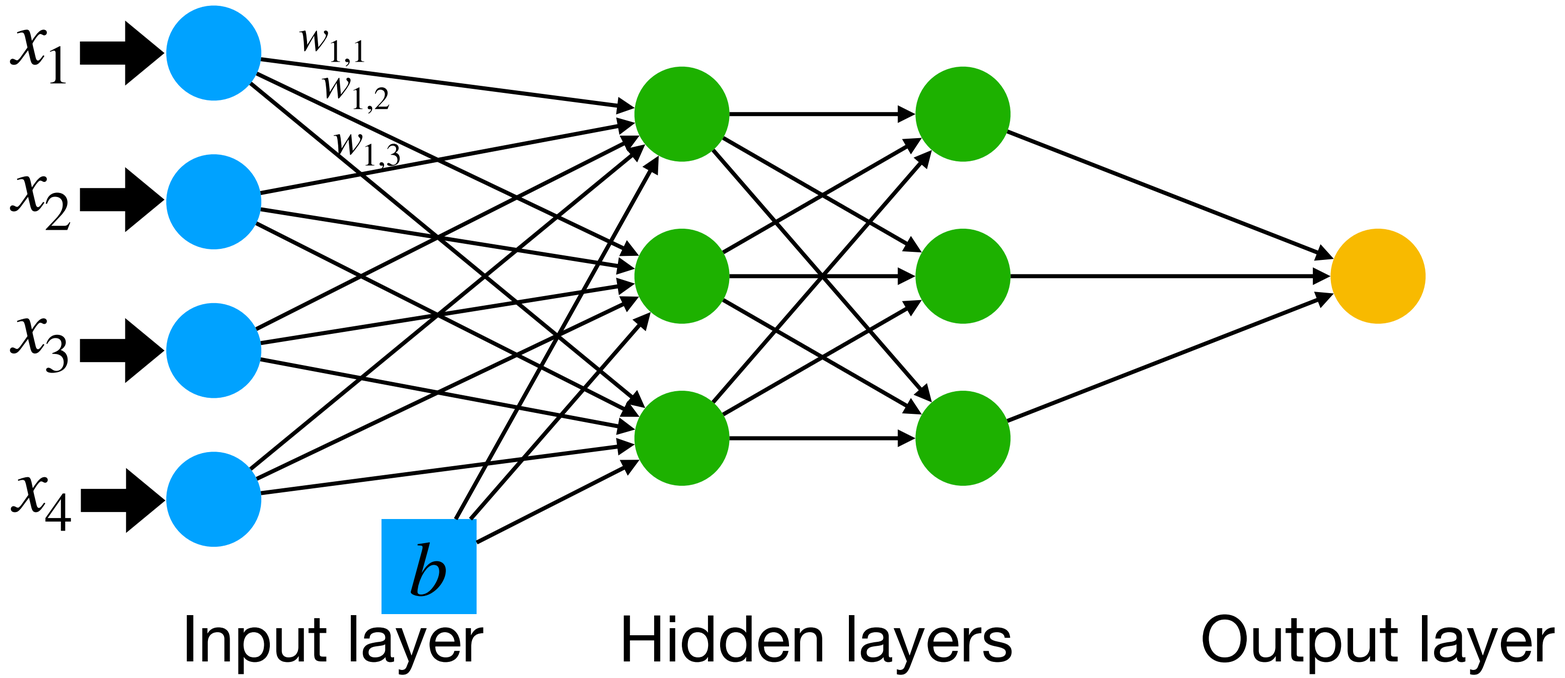


Multi layer Perceptron (MLP)

- Neurons and Synopsis are organised in layers: the input layer, one or more hidden layers, and the output layer.



- **Deep learning**: is a subfield of machine learning based on multilayered neural networks (at least one hidden layer) to perform regression and classification
- Multilayer perceptron is an example of **feed forward neural network**, i.e. the directed links in the network do not form cycles.
- This is in contrast to **recurrent neural network**, where the output of a node at hidden layer j can be fed to a node in a hidden layer < j.



- Weights and biases are the **model parameters,** and learnt during the **training** process.

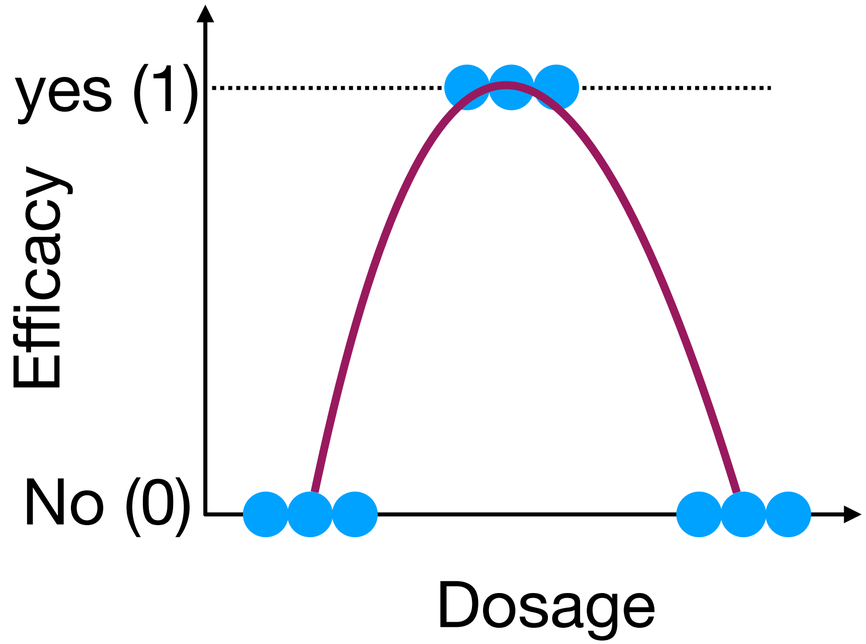
- The structure of the neural network, the number of layers, the activation functions, the number of nodes per hidden layers, are **hyperparameters**, which are defined before training and influence the performance of a neural network.

  - They are not learnt during the training process

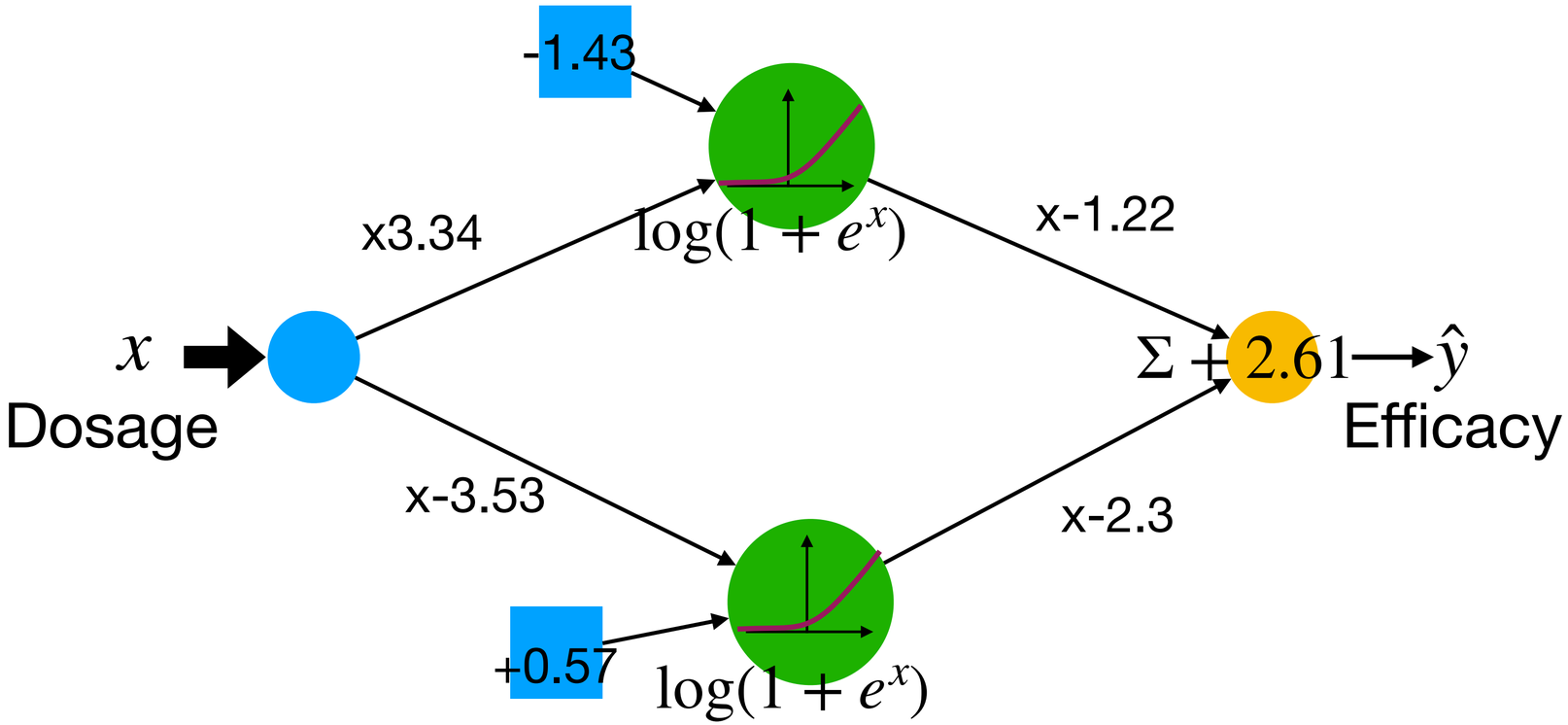
Example

- Assume we are testing a drug to treat an illness. We gave the drug to three different groups of people with three different dosages: low, medium and high.

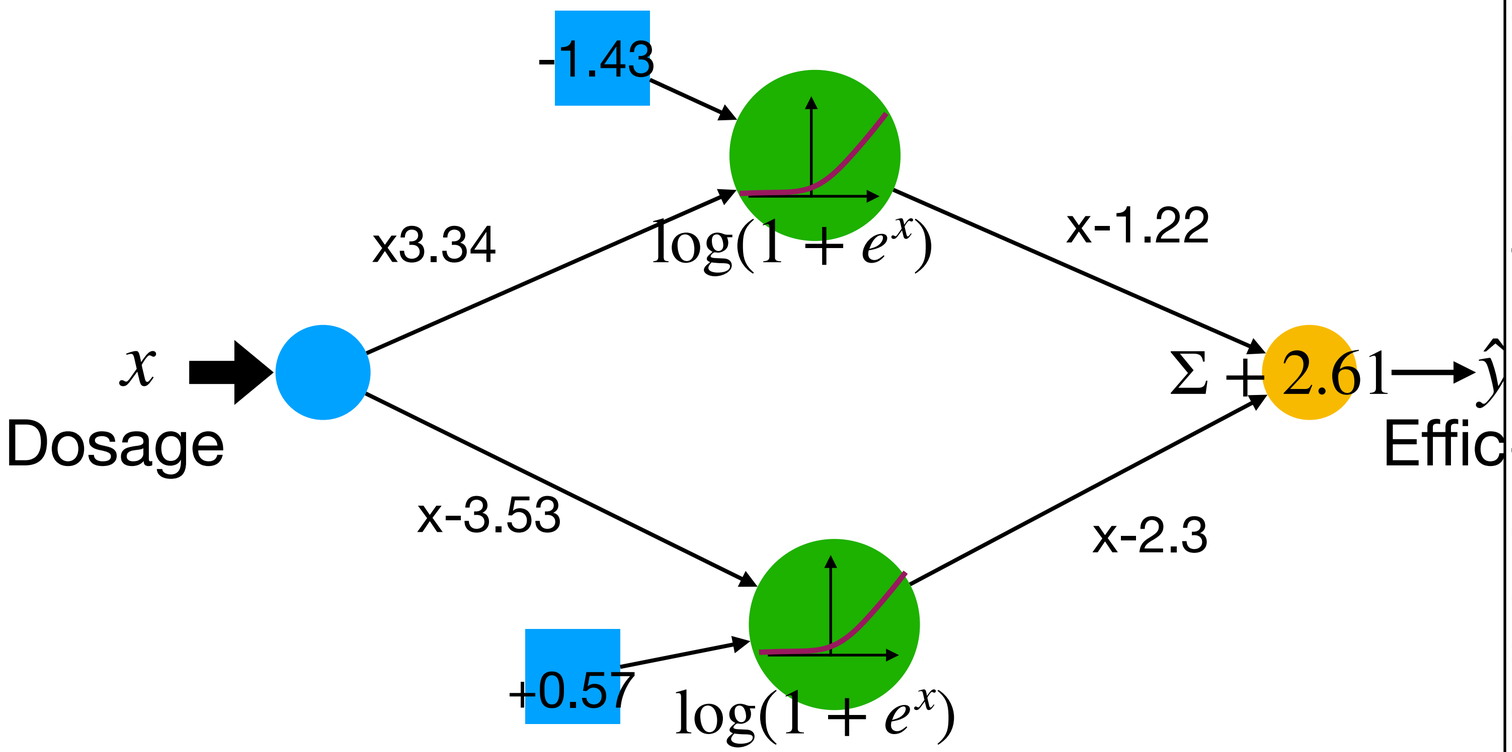
- Low and high dosages were not effective, while medium dosages were effective.



Example


-1.43
x3.34
$\log(1+e^x)$
x-1.22
x
Dosage
$\Sigma + 2.61 \rightarrow \hat{y}$
Efficacy
x-3.53
x-2.3
+0.57
$\log(1+e^x)$

Example



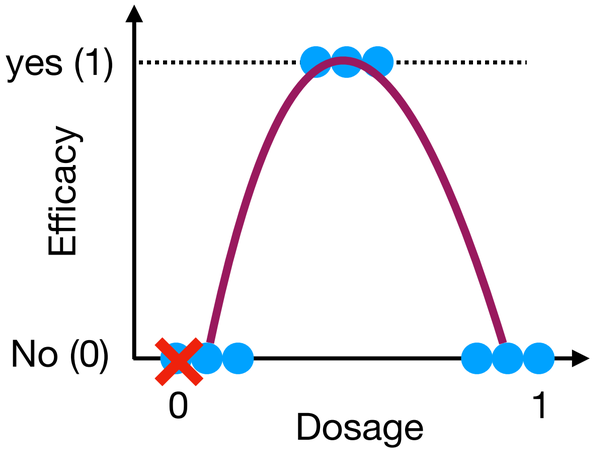
Notice that the parameters (weights and bias) of this neural network are already defined.

The neural network was already trained.

The parameter estimates are analogous to the slope and intercept estimates that we saw in linear regression.

Example

Inference: given a trained NN, feed it with an input to get a prediction/ classification



$\log(1 + e^{0 \cdot 3.34 - 1.43}) = 0.21$

-1.43

$0.21 \cdot (-1.22) + 1.02 \cdot (-2.3) + 2.61 = 0.08$

x3.34

$\log(1 + e^{x})$

x-1.22

$x = 0$

Dosage

$\Sigma + 2.61 \rightarrow \hat{y}$

Efficacy

x-3.53

x-2.3

+0.57

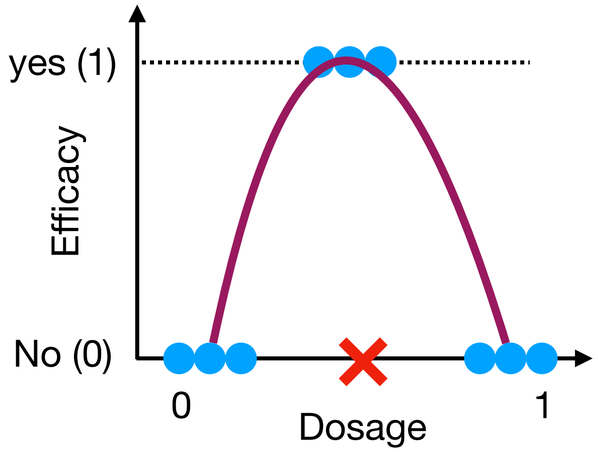
$\log(1 + e^{x})$

$\log(1 + e^{0 \cdot (-3.53) + 0.57}) = 1.02$

Since the output is close to 0, we infer that the dosage is not effective.

Example

Inference: given a trained NN, feed it with an input to get a prediction/ classification



$\log(1 + e^{0.5 \cdot 3.34 - 1.43}) = 0.82$

$0.82 \cdot (-1.22) + 0.26 \cdot (-2.3) + 2.61 = 1.001$

-1.43

x3.34

$\log(1 + e^x)$

x-1.22

$x = 0.5$
Dosage

$\Sigma + 2.61 \rightarrow \hat{y}$
Efficacy

x-3.53

x-2.3

+0.57

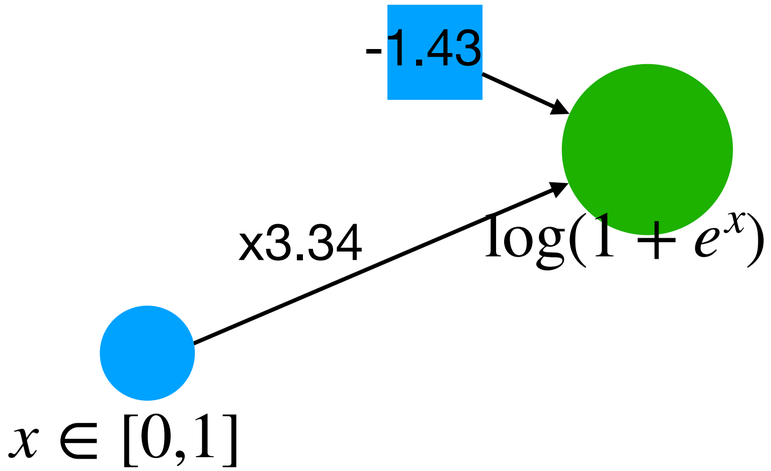
$\log(1 + e^x)$

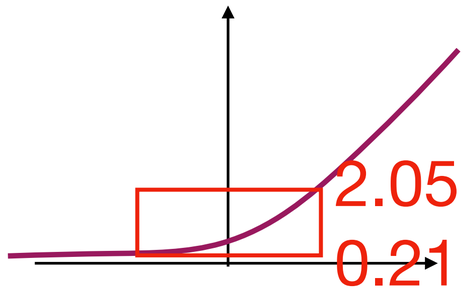
$\log(1 + e^{0.5 \cdot (-3.53) + 0.57}) = 0.26$

Since the output is close to 1, we infer that the dosage is effective.

Example

The output of this node are always in this range:





The input to the soft plus function ranges in

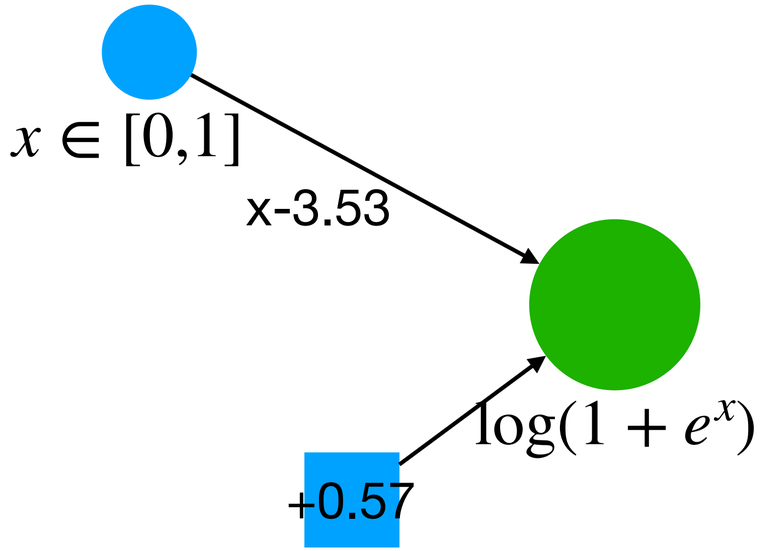
$$[0 \cdot 3.34 - 1.43, 1 \cdot 3.34 - 1.43] =$$

$$[-1.43, 1.91]$$

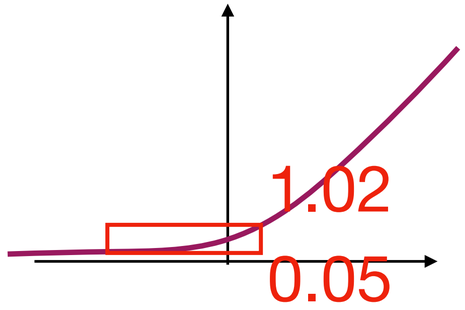
Example

The input to the soft plus function ranges in

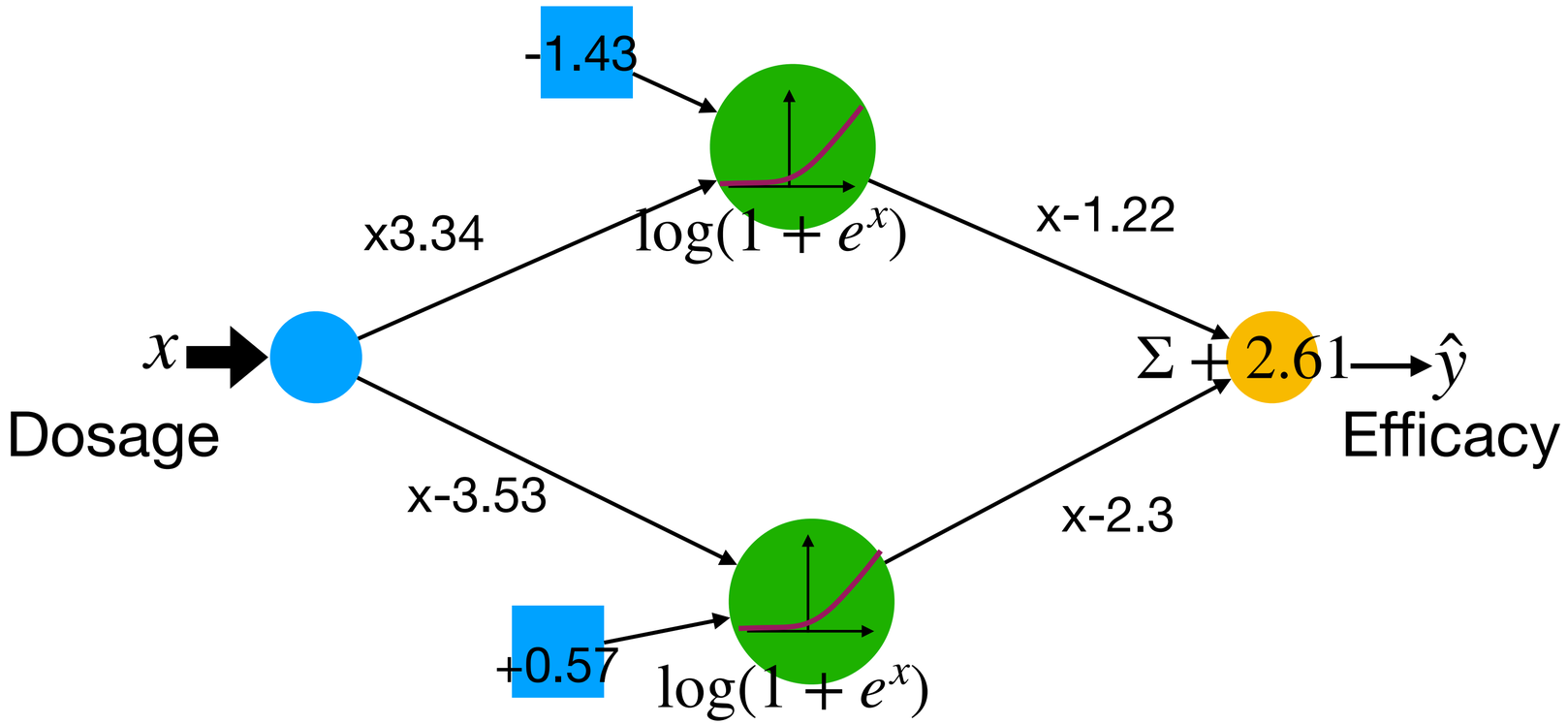
$$[1 \cdot (-3.53) + 0.57, 0 \cdot (-3.53) + 0.57] = [-2.96, 0.57]$$



The output of this node are always in this range:



Example



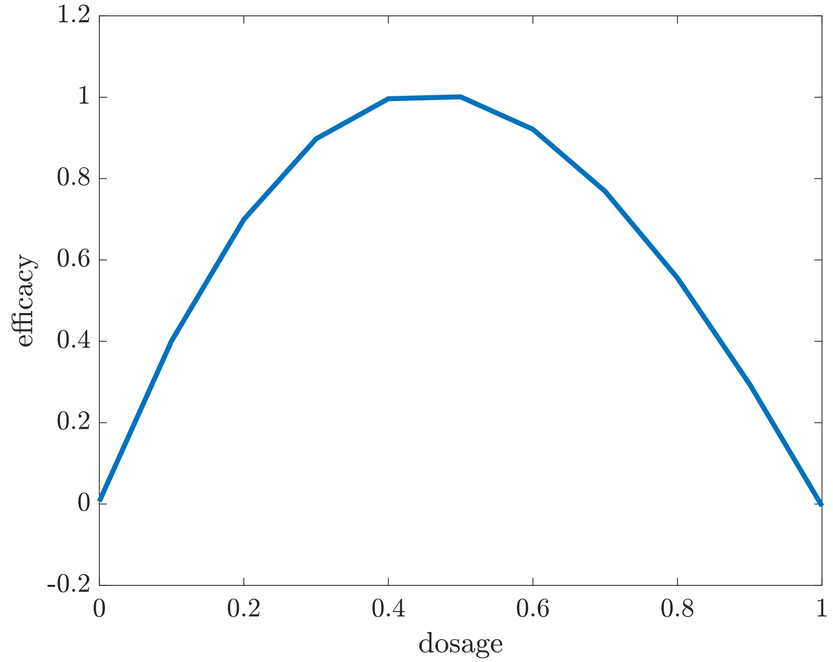
- At the end of the day, this neural network is just this big function of x:

$$-1.22\log(1+\exp(3.34x-1.43))-2.3\log(1+\exp(-3.53x+0.57))+2.61$$

1

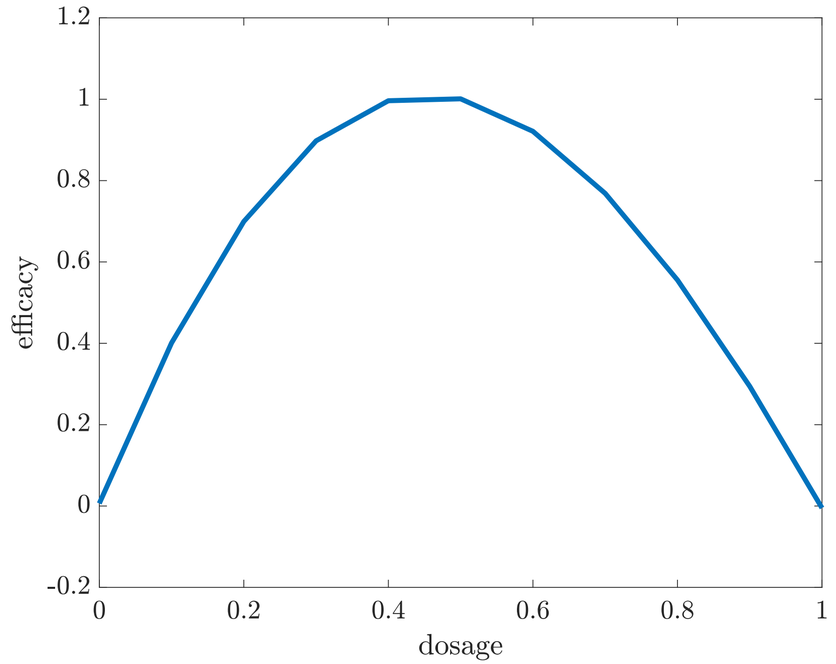
Example

- By plotting the function $-1.22\log(1+\exp(3.34x-1.43))-2.3\log(1+\exp(-3.53x+0.57))+2.61$ for $x \in [0,1]$, we get the following graph.



Example

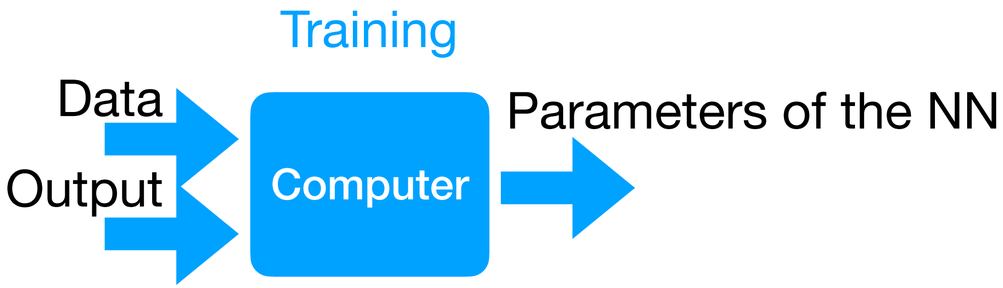
- By plotting the function $-1.22\log(1+\exp(3.34x-1.43))-2.3\log(1+\exp(-3.53x+0.57))+2.61$ for $x \in [0,1]$, we get the following graph.



These are unknown

Backpropagation

- Backpropagation is an algorithm to **train** feedforward neural networks.
- Training a neural network means to find the optimal **parameters** (weights and bias) that model the behaviour of a phenomenon (e.g., how the dosage of a drug influence its efficacy).
- The phenomenon is not described by a mathematical model (e.g., differential equations modelling the body's response to the chemicals in the drug), but only by **observed data** (**data points**, e.g., couples (dosage, efficacy)).
  - Hence, to find the optimal parameters means to find the parameters that minimise the error with respect to the input data.
- To train a neural network, we need to feed it with many data points or samples, that are the **training set**- supervised learning.

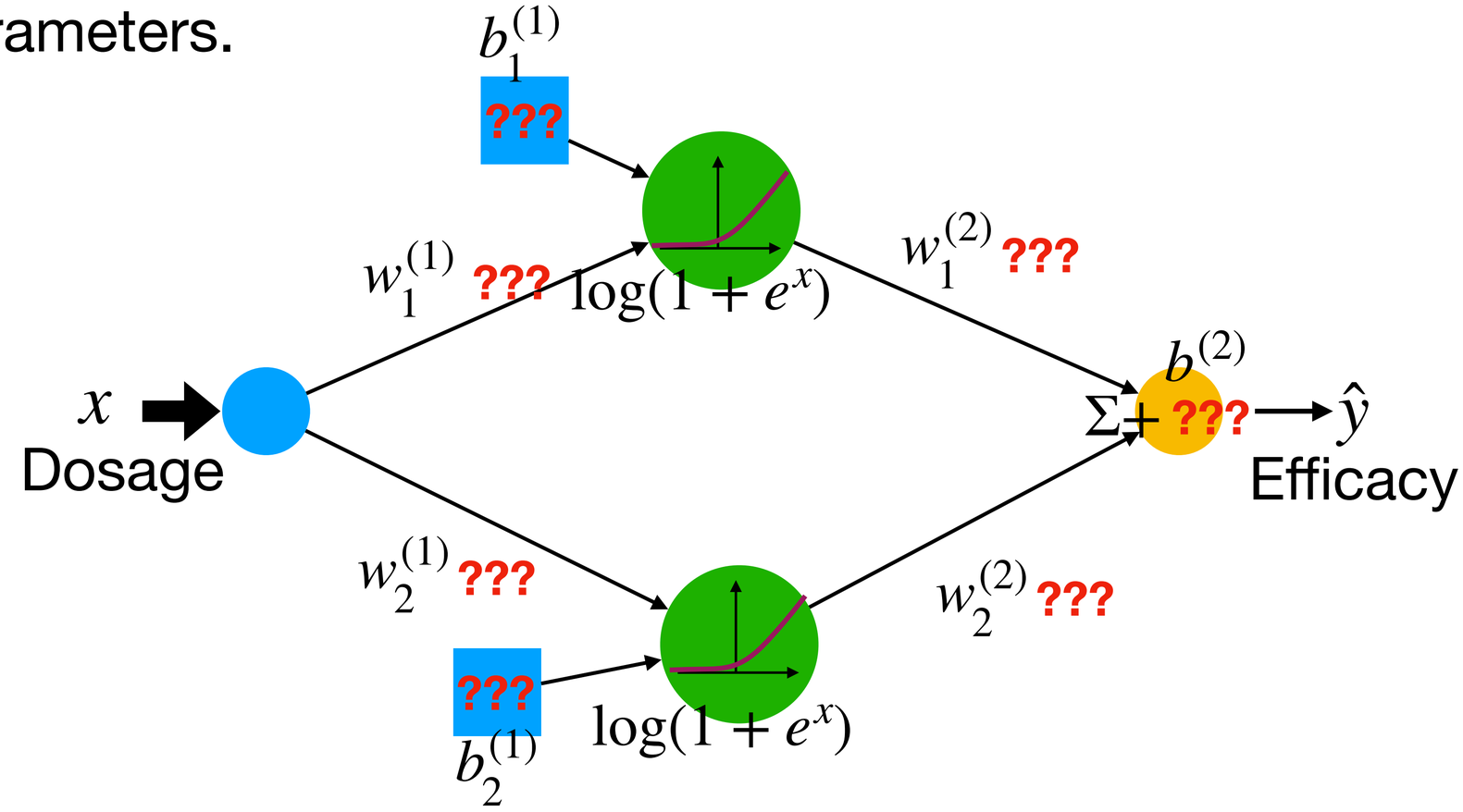


Backpropagation - high level idea

- **Initialisation:** initialise weights and biases at random.
- **Forward step:** feed the network with a data point $(x_1, \ldots, x_d, y)$ and get the expressions of the parameters layer by layer, by writing each parameter as a function of the parameters in the previous layers.
- **Backward step:** Compute the gradients of a loss function $\mathscr{L}$ with respect to **each parameter**, using the **chain rule** from the rightmost layer.
  - Update the parameters with the **gradient descent** step:
    $w_{i,j}^{(l)} \leftarrow w_{i,j}^{(l)} - \eta \partial_{w_{i,j}^{(l)}} \mathscr{L}$
    $b_{j}^{(l)} \leftarrow b_{j}^{(l)} - \eta \partial_{b_{j}^{(l)}} \mathscr{L}$
  - Repeat the updates as in the gradient descent until stopping criteria.

Backpropagation - example

- Consider the same scenario as before, but we have no clue about any optimal parameters.

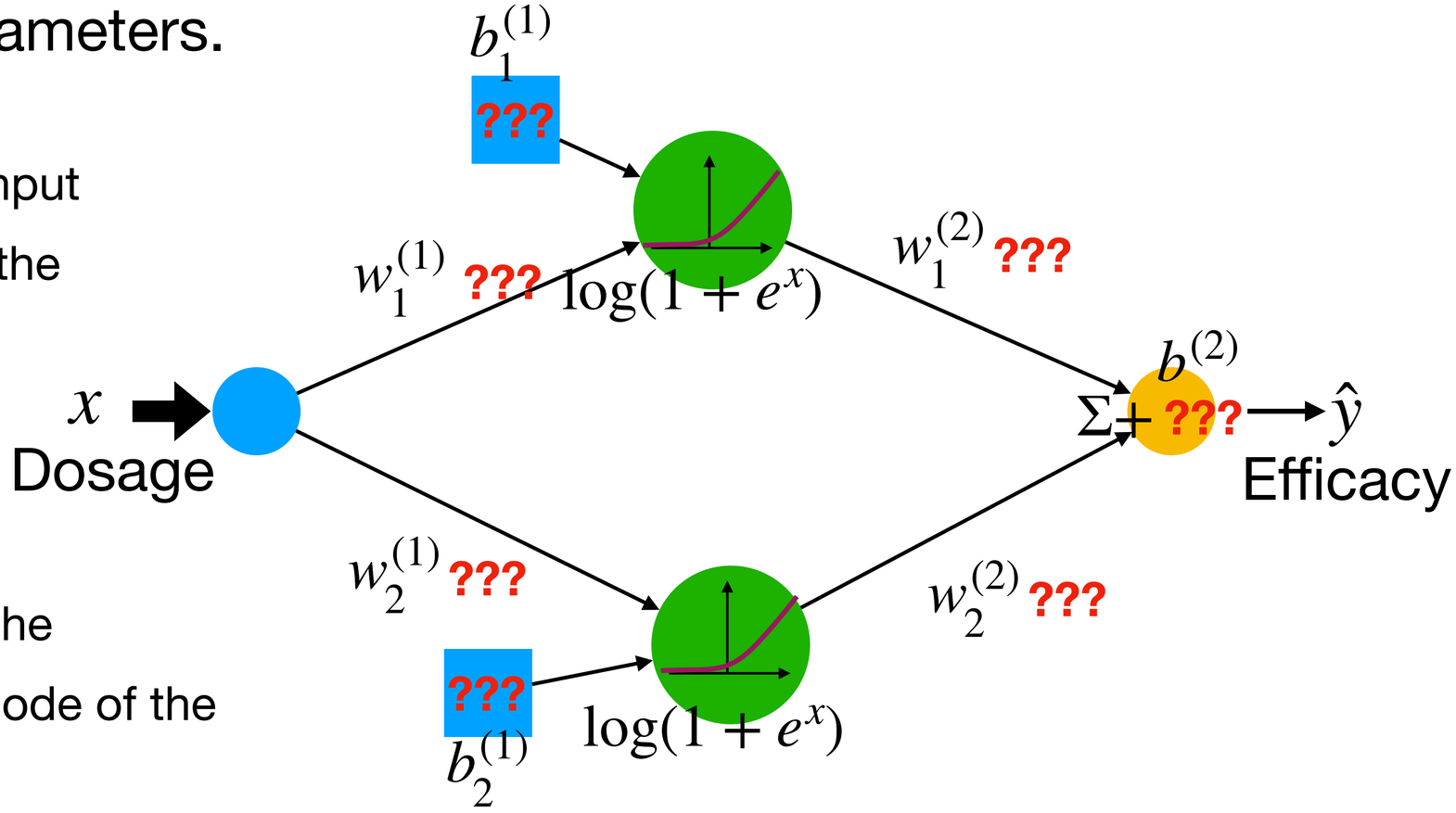


Backpropagation - example

- Consider the same scenario as before, but we have no clue about any optimal parameters.

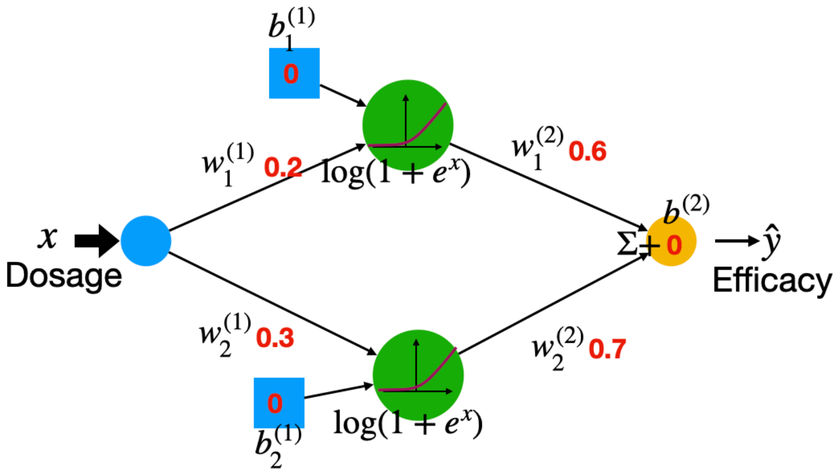
Let's call $net_i^{(l)}$ the input to the $i-$th node of the $l-$th layer.

And let's call $out_i^{(l)}$ the output of the $i-$th node of the $l-$th layer.



Backpropagation - example

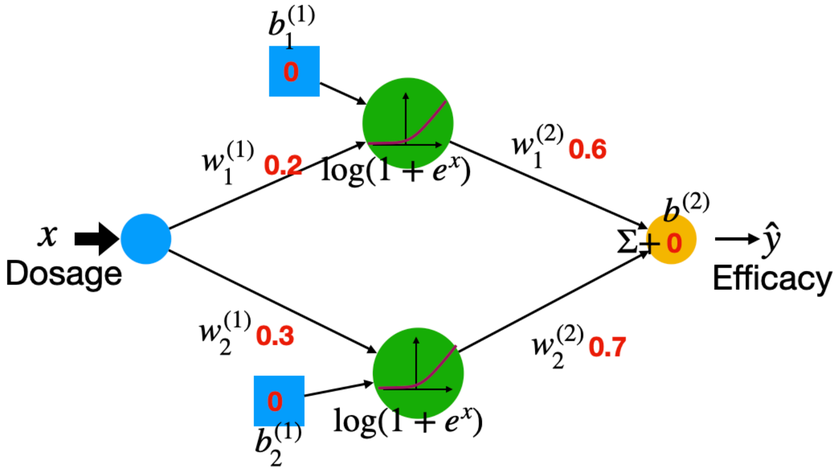
- Random initialisation
- Forward step on training point (0,0)

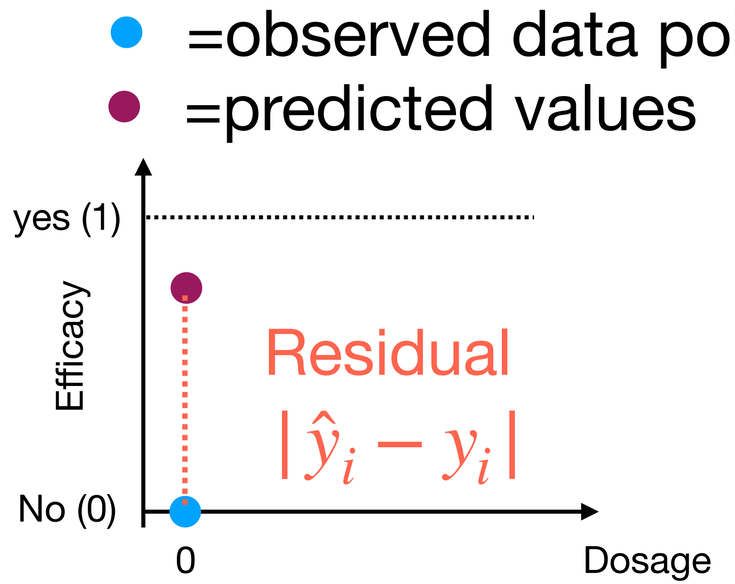


- $net_1^{(1)} = w_1^{(1)}x + b_1^{(1)} = 0$
- $net_2^{(1)} = w_2^{(1)}x + b_2^{(1)} = 0$
- $out_1^{(1)} = \log(1 + \exp(net_1^{(1)}) = 0.69$
- $out_2^{(1)} = \log(1 + \exp(net_2^{(1)}) = 0.69$
- $net^{(2)} = w_1^{(2)}out_1^{(1)} + w_2^{(2)}out_2^{(1)} = 0.9$
- $\hat{y} = net^{(2)} + b^{(2)} = 0.9$

Backpropagation - example

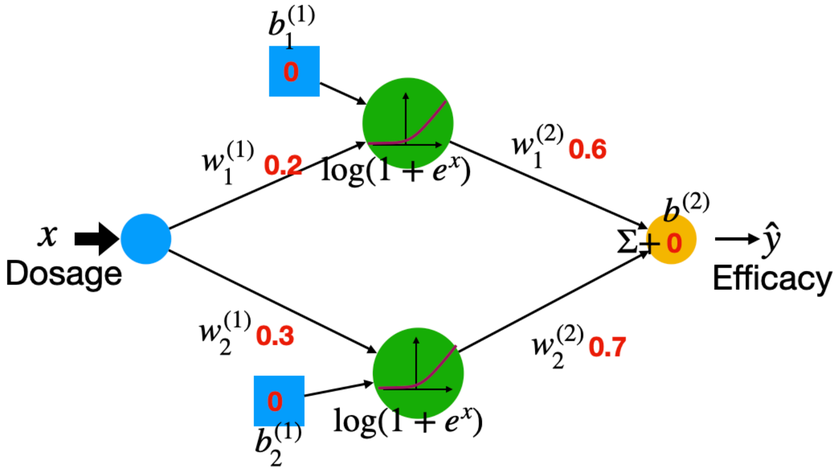
- Calculate the value of the loss function $\mathscr{L} = \frac{1}{2}(\hat{y} - y)^2 = \frac{1}{2}(0.9 - 0)^2 = 0.41$





Backpropagation - example

- Backward step:
  - Let's begin by computing the gradients of the loss function wrt all parameters, starting from right-most layer and using the chain rule.



$$net_1^{(1)} = w_1^{(1)}x + b_1^{(1)} = 0 \qquad \mathscr{L} = \frac{1}{2}(\hat{y} - y)^2 = \frac{1}{2}(0.9 - 0)^2 = 0.41$$

$$net_2^{(1)} = w_2^{(1)}x + b_2^{(1)} = 0$$

$$out_1^{(1)} = \log(1 + \exp(net_1^{(1)}) = 0.69$$

$$out_2^{(1)} = \log(1 + \exp(net_2^{(1)}) = 0.69$$

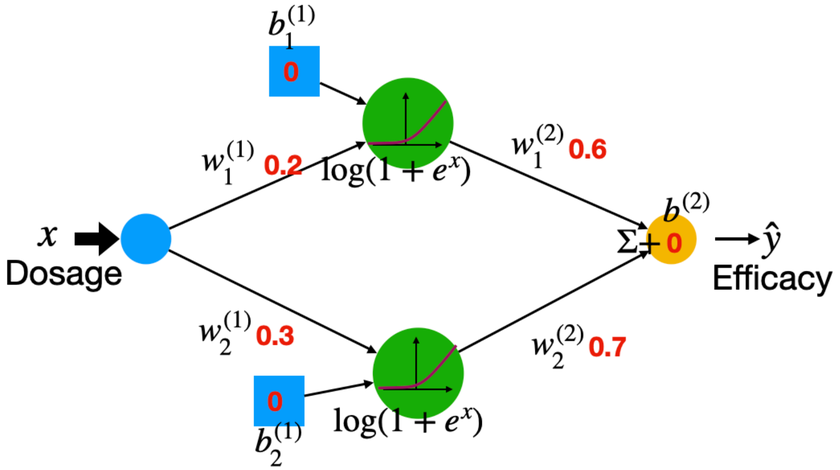
$$net^{(2)} = w_1^{(2)}out_1^{(1)} + w_2^{(2)}out_2^{(1)} = 0.9$$

$$\hat{y} = net^{(2)} + b^{(2)} = 0.9$$

- $\dfrac{\partial \mathscr{L}}{\partial \hat{y}} = \hat{y} - y = 0.9$
- $\dfrac{\partial \mathscr{L}}{\partial b^{(2)}} = \dfrac{\partial \mathscr{L}}{\partial \hat{y}}\dfrac{\partial \hat{y}}{\partial b^{(2)}} = (\hat{y} - y) \cdot 1 = 0.9$
- $\dfrac{\partial \mathscr{L}}{\partial w_1^{(2)}} = \dfrac{\partial \mathscr{L}}{\partial \hat{y}}\dfrac{\partial \hat{y}}{\partial net^{(2)}}\dfrac{\partial net^{(2)}}{\partial w_1^{(2)}} = (\hat{y} - y) \cdot 1 \cdot out_1^{(1)} = 0.9 \cdot 0.69 = 0.62$
- $\dfrac{\partial \mathscr{L}}{\partial w_2^{(2)}} = \dfrac{\partial \mathscr{L}}{\partial \hat{y}}\dfrac{\partial \hat{y}}{\partial net^{(2)}}\dfrac{\partial net^{(2)}}{\partial w_2^{(2)}} = (\hat{y} - y) \cdot 1 \cdot out_2^{(1)} = 0.9 \cdot 0.69 = 0.62$

Backpropagation - example

- Backward step:
  - Let's begin by computing the gradients of the loss function wrt all parameters, starting from right-most layer and using the chain rule.



$$net_1^{(1)} = w_1^{(1)}x + b_1^{(1)} = 0 \qquad \mathscr{L} = \frac{1}{2}(\hat{y}-y)^2 = \frac{1}{2}(0.9-0)^2 = 0.41$$

$$net_2^{(1)} = w_2^{(1)}x + b_2^{(1)} = 0$$

$$out_1^{(1)} = \log(1+\exp(net_1^{(1)}) = 0.69$$

$$out_2^{(1)} = \log(1+\exp(net_2^{(1)}) = 0.69$$

$$net^{(2)} = w_1^{(2)}out_1^{(1)} + w_2^{(2)}out_2^{(1)} = 0.9$$

$$\hat{y} = net^{(2)} + b^{(2)} = 0.9$$

- $$\frac{\partial \mathscr{L}}{\partial b_1^{(1)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial net^{(2)}}\frac{\partial net^{(2)}}{\partial out_1^{(1)}}\frac{\partial out_1^{(1)}}{\partial net_1^{(1)}}\frac{\partial net_1^{(1)}}{\partial b_1^{(1)}} = (\hat{y}-y)\cdot 1\cdot w_1^{(2)}\frac{\exp(net_1^{(1)})}{(1+\exp(net_1^{(1)}))}\cdot 1 = 0.315$$
- $$\frac{\partial \mathscr{L}}{\partial b_2^{(1)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial net^{(2)}}\frac{\partial net^{(2)}}{\partial out_2^{(1)}}\frac{\partial out_2^{(1)}}{\partial net_2^{(1)}}\frac{\partial net_2^{(1)}}{\partial b_2^{(1)}} = (\hat{y}-y)\cdot 1\cdot w_2^{(2)}\frac{\exp(net_2^{(1)})}{(1+\exp(net_2^{(1)}))}\cdot 1 = 0.315$$
- $$\frac{\partial \mathscr{L}}{\partial w_1^{(1)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial net^{(2)}}\frac{\partial net^{(2)}}{\partial out_1^{(1)}}\frac{\partial out_1^{(1)}}{\partial net_1^{(1)}}\frac{\partial net_1^{(1)}}{\partial w_1^{(1)}} = (\hat{y}-y)\cdot 1\cdot w_1^{(2)}\frac{\exp(net_1^{(1)})}{(1+\exp(net_1^{(1)}))}\cdot x = 0$$
- $$\frac{\partial \mathscr{L}}{\partial w_2^{(1)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial net^{(2)}}\frac{\partial net^{(2)}}{\partial out_2^{(1)}}\frac{\partial out_2^{(1)}}{\partial net_2^{(1)}}\frac{\partial net_2^{(1)}}{\partial w_2^{(1)}} = (\hat{y}-y)\cdot 1\cdot w_2^{(2)}\frac{\exp(net_2^{(1)})}{(1+\exp(net_2^{(1)}))}\cdot x = 0$$

Backpropagation - example

$\delta^{(2)}$
- $\frac{\partial \mathcal{L}}{\partial \hat{y}} = \hat{y} - y$
- $\frac{\partial \mathcal{L}}{\partial b^{(2)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial b^{(2)}} = (\hat{y} - y) \cdot 1 = \delta^{(2)} \frac{\partial \hat{y}}{\partial b^{(2)}}$
- $\frac{\partial \mathcal{L}}{\partial w_1^{(2)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial w_1^{(2)}} = (\hat{y} - y) \cdot 1 \cdot out_1^{(1)} = \delta^{(2)} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial w_1^{(2)}}$
- $\frac{\partial \mathcal{L}}{\partial w_2^{(2)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial w_2^{(2)}} = (\hat{y} - y) \cdot 1 \cdot out_2^{(1)} = \delta^{(2)} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial w_2^{(2)}}$

$\delta_1^{(1)}$
- $\frac{\partial \mathcal{L}}{\partial b_1^{(1)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_1^{(1)}} \frac{\partial out_1^{(1)}}{\partial net_1^{(1)}} \frac{\partial net_1^{(1)}}{\partial b_1^{(1)}} = (\hat{y} - y) \cdot 1 \cdot w_1^{(2)} \frac{\exp(net_1^{(1)})}{(1 + \exp(net_1^{(1)}))} \cdot 1 = \delta_1^{(1)} \frac{\partial net_1^{(1)}}{\partial b_1^{(1)}}$

$\delta_2^{(1)}$
- $\frac{\partial \mathcal{L}}{\partial b_2^{(1)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_2^{(1)}} \frac{\partial out_2^{(1)}}{\partial net_2^{(1)}} \frac{\partial net_2^{(1)}}{\partial b_2^{(1)}} = (\hat{y} - y) \cdot 1 \cdot w_2^{(2)} \frac{\exp(net_2^{(1)})}{(1 + \exp(net_2^{(1)}))} \cdot 1 = \delta_2^{(1)} \frac{\partial net_2^{(1)}}{\partial b_2^{(1)}}$
- $\frac{\partial \mathcal{L}}{\partial w_1^{(1)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_1^{(1)}} \frac{\partial out_1^{(1)}}{\partial net_1^{(1)}} \frac{\partial net_1^{(1)}}{\partial w_1^{(1)}} = (\hat{y} - y) \cdot 1 \cdot w_1^{(2)} \frac{\exp(net_1^{(1)})}{(1 + \exp(net_1^{(1)}))} \cdot x = \delta_1^{(1)} \frac{\partial net_1^{(1)}}{\partial w_1^{(1)}}$
- $\frac{\partial \mathcal{L}}{\partial w_2^{(1)}} = \frac{\partial \mathcal{L}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_2^{(1)}} \frac{\partial out_2^{(1)}}{\partial net_2^{(1)}} \frac{\partial net_2^{(1)}}{\partial w_2^{(1)}} = (\hat{y} - y) \cdot 1 \cdot w_2^{(2)} \frac{\exp(net_2^{(1)})}{(1 + \exp(net_2^{(1)}))} \cdot x = \delta_2^{(1)} \frac{\partial net_2^{(1)}}{\partial w_2^{(1)}}$

$$\delta_1^{(1)} = \delta^{(2)} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_1^{(1)}} \frac{\partial out_1^{(1)}}{\partial net_1^{(1)}}$$

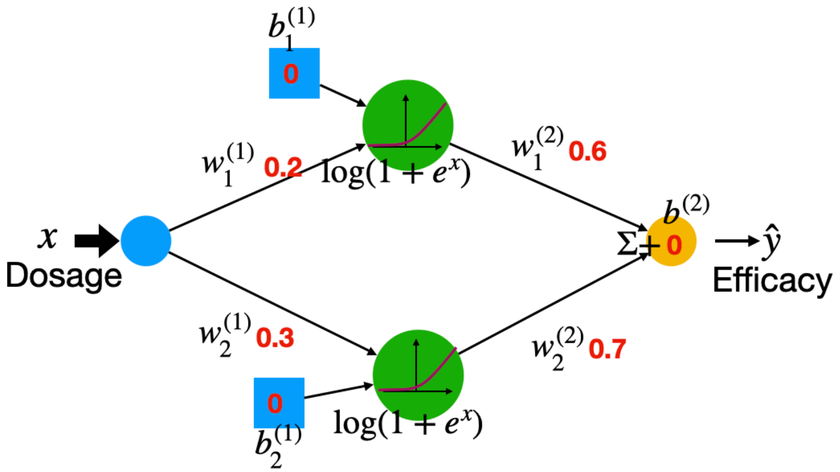
$$\delta_2^{(1)} = \delta^{(2)} \frac{\partial \hat{y}}{\partial net^{(2)}} \frac{\partial net^{(2)}}{\partial out_2^{(1)}} \frac{\partial out_2^{(1)}}{\partial net_2^{(1)}}$$

$$\delta_v^{(l)} = f'\left(net_v^{(l)}\right) \cdot \sum_{i=1}^{n} w_{v,i}^{(l+1)} \cdot \delta_i^{(l+1)}$$

General formulation for a node v at a hidden layer l, v having n outgoing edges

Backpropagation - example

- Backward step:
  - Updates all weights and biases and iterate until convergence.



$$w_{i,j}^{(l)} \leftarrow w_{i,j}^{(l)} - \eta \frac{\partial \mathscr{L}}{\partial w_{i,j}^{(l)}}$$

$$b_j^{(l)} \leftarrow b_j^{(l)} - \eta \frac{\partial \mathscr{L}}{\partial b_j^{(l)}}$$

$$net_1^{(1)} = w_1^{(1)} x + b_1^{(1)} = 0 \qquad \mathscr{L} = \frac{1}{2}(\hat{y} - y)^2 = \frac{1}{2}(0.9 - 0)^2 = 0.41$$

$$net_2^{(1)} = w_2^{(1)} x + b_2^{(1)} = 0$$

$$out_1^{(1)} = \log(1 + \exp(net_1^{(1)}) = 0.69$$

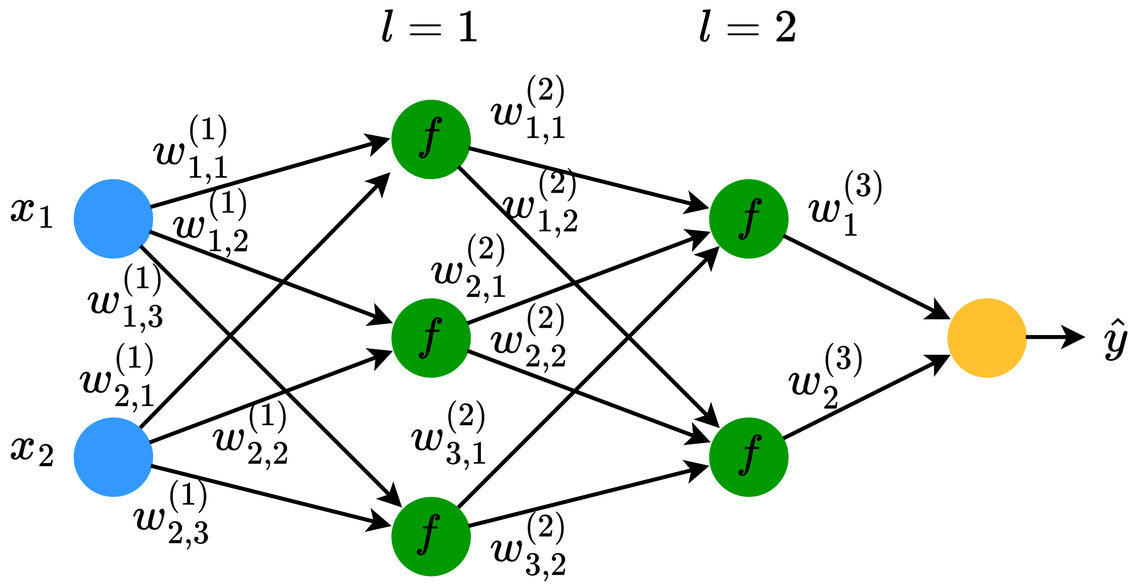
$$out_2^{(1)} = \log(1 + \exp(net_2^{(1)}) = 0.69$$

$$net^{(2)} = w_1^{(2)} out_1^{(1)} + w_2^{(2)} out_2^{(1)} = 0.9$$

$$\hat{y} = net^{(2)} + b^{(2)} = 0.9$$

Backpropagation - example 2

- Forward step

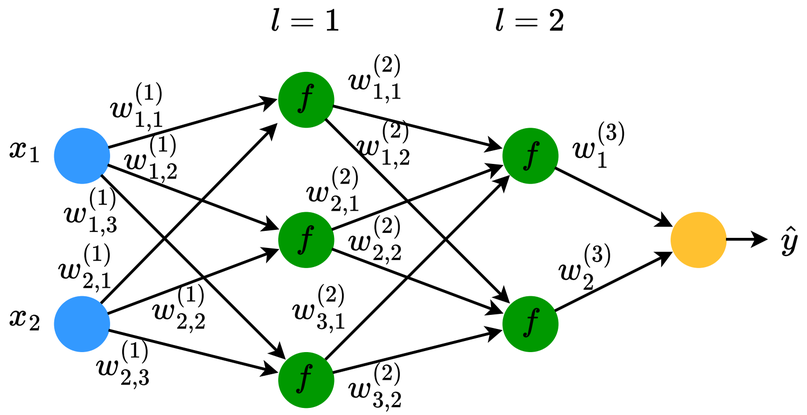


$$f(x) = \log(1 + e^x)$$

- $net_1^{(1)} = w_{1,1}^{(1)} x_1 + w_{2,1}^{(1)} x_2$
- $net_2^{(1)} = w_{1,2}^{(1)} x_1 + w_{2,2}^{(1)} x_2$
- $net_3^{(1)} = w_{1,3}^{(1)} x_1 + w_{2,3}^{(1)} x_2$
- $out_1^{(1)} = \log(1 + \exp(net_1^{(1)}))$
- $out_2^{(1)} = \log(1 + \exp(net_2^{(1)}))$
- $out_3^{(1)} = \log(1 + \exp(net_3^{(1)}))$
- $net_1^{(2)} = w_{1,1}^{(2)} out_1^{(1)} + w_{2,1}^{(2)} out_2^{(1)} + w_{3,1}^{(2)} out_3^{(1)}$
- $net_2^{(2)} = w_{1,2}^{(2)} out_1^{(1)} + w_{2,2}^{(2)} out_2^{(1)} + w_{3,2}^{(2)} out_3^{(1)}$
- $out_1^{(2)} = \log(1 + \exp(net_1^{(2)}))$
- $out_2^{(2)} = \log(1 + \exp(net_2^{(2)}))$
- $\hat{y} = out_1^{(2)} w_1^{(3)} + out_2^{(2)} w_2^{(3)}$

Backpropagation - example 2

- Define loss function as $\mathscr{L} = \frac{1}{2}(\hat{y} - y)^2$ and compute the gradients backwards



$$f(x) = \log(1 + e^x)$$

$$net_1^{(1)} = w_{1,1}^{(1)}x_1 + w_{2,1}^{(1)}x_2$$
$$net_2^{(1)} = w_{1,2}^{(1)}x_1 + w_{2,2}^{(1)}x_2$$
$$net_3^{(1)} = w_{1,3}^{(1)}x_1 + w_{2,3}^{(1)}x_2$$
$$out_1^{(1)} = \log(1 + \exp(net_1^{(1)}))$$
$$out_2^{(1)} = \log(1 + \exp(net_2^{(1)}))$$
$$out_3^{(1)} = \log(1 + \exp(net_3^{(1)}))$$

$$net_1^{(2)} = w_{1,1}^{(2)}out_1^{(1)} + w_{2,1}^{(2)}out_2^{(1)} + w_{3,1}^{(2)}out_3^{(1)}$$
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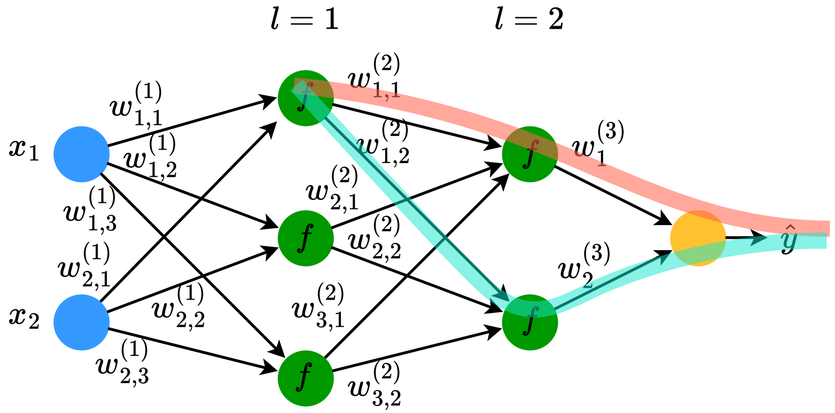
$$out_1^{(2)} = \log(1 + \exp(net_1^{(2)}))$$
$$out_2^{(2)} = \log(1 + \exp(net_2^{(2)}))$$
$$\hat{y} = out_1^{(2)}w_1^{(3)} + out_2^{(2)}w_2^{(3)}$$

- $\frac{\partial \mathscr{L}}{\partial \hat{y}} = (\hat{y} - y)$
- $\frac{\partial \mathscr{L}}{\partial w_1^{(3)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial w_1^{(3)}} = (\hat{y} - y) \cdot out_1^{(2)}$
- $\frac{\partial \mathscr{L}}{\partial w_{1,1}^{(2)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial out_1^{(2)}}\frac{\partial out_1^{(2)}}{\partial net_1^{(2)}}\frac{\partial net_1^{(2)}}{\partial w_{1,1}^{(2)}} =$

$$= (\hat{y} - y) \cdot w_1^{(3)}\frac{\exp(net_1(2))}{(1 + \exp(net_1^{(2)}))}out_1^{(1)}$$

Backpropagation - example 2

- Define loss function as $\mathscr{L} = \frac{1}{2}(\hat{y} - y)^2$ and compute the gradients backwards



$$f(x) = \log(1 + e^x)$$

$$net_1^{(1)} = w_{1,1}^{(1)} x_1 + w_{2,1}^{(1)} x_2$$

$$net_2^{(1)} = w_{1,2}^{(1)} x_1 + w_{2,2}^{(1)} x_2$$

$$net_3^{(1)} = w_{1,3}^{(1)} x_1 + w_{2,3}^{(1)} x_2$$

$$out_1^{(1)} = \log(1 + \exp(net_1^{(1)}))$$

$$out_2^{(1)} = \log(1 + \exp(net_2^{(1)}))$$

$$out_3^{(1)} = \log(1 + \exp(net_3^{(1)}))$$

$$net_1^{(2)} = w_{1,1}^{(2)} out_1^{(1)} + w_{2,1}^{(2)} out_2^{(1)} + w_{3,1}^{(2)} out_3^{(1)}$$

$$net_2^{(2)} = w_{1,2}^{(2)} out_1^{(1)} + w_{2,2}^{(2)} out_2^{(1)} + w_{3,2}^{(2)} out_3^{(1)}$$

$$out_1^{(2)} = \log(1 + \exp(net_1^{(2)}))$$

$$out_2^{(2)} = \log(1 + \exp(net_2^{(2)}))$$

$$\hat{y} = out_1^{(2)} w_1^{(3)} + out_2^{(2)} w_2^{(3)}$$

$$\frac{\partial \mathscr{L}}{\partial w_{1,1}^{(1)}} = \frac{\partial \mathscr{L}}{\partial \hat{y}} \left(\frac{\partial \hat{y}}{\partial out_1^{(2)}} \frac{\partial out_1^{(2)}}{\partial net_1^{(2)}} \frac{\partial net_1^{(2)}}{\partial out_1^{(1)}} + \frac{\partial \hat{y}}{\partial out_2^{(2)}} \frac{\partial out_2^{(2)}}{\partial net_2^{(2)}} \frac{\partial net_2^{(2)}}{\partial out_1^{(1)}} \right) \frac{\partial out_1^{(1)}}{\partial net_1^{(1)}} \frac{net_1^{(1)}}{\partial w_{1,1}^{(1)}}$$

The partial derivatives with respect to the other weights in the first layer are computed similarly.

Backpropagation

- We can train the neural network over a dataset of N samples $(x_{i,1}, \ldots, x_{i,d}, y_i)$.
- The process works as follows:
  - Initialise weights and biases at random.
  - For each batch of n elements:
    - Feed the network with each sample (data point), obtain the prediction $\hat{y}_i$ and compute the corresponding values of the loss function $\mathscr{L}(x_{i,1}, \ldots, x_{i,d}, y_i) = \mathscr{L}_i = \frac{1}{2}(\hat{y}_i - y)^2$
    - Consider the average of the values of the loss functions: $\mathscr{L} = \frac{1}{n}\sum_{i=1}^{n} \mathscr{L}_i$
    - Compute gradients of $\mathscr{L} = \frac{1}{n}\sum_{i=1}^{n} \mathscr{L}_i$ wrt each parameter, and proceed until stopping

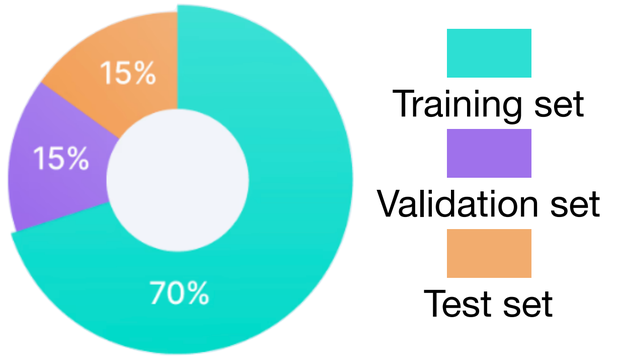
This is one epoch

Backpropagation

- This procedure can be repeated on multiple **epochs**.
  - Divide the training set into a number of batches.
  - **First epoch:** Initialise weights and biases at random. Train the network on the **all the batches**, one at a time (forward and backward).
  - **k-th epoch**: start the training using the latest parameters computed in the previous epoch. Train the network on all the batches.
  - Repeat until the maximum number of epochs has been reached.

- Notice:
  - When we do not apply the gradient descent to the entire dataset, but to just one sample or a batch of samples, we refer to it as to "**stochastic gradient descent**" (**SDG**). Batches are usually chosen uniformly at random.

- The entire dataset is divided into a **training set**, a **validation set**, and a **test set.**
- At each epoch, evaluate the performance of the obtained NN on an independent **validation set**.

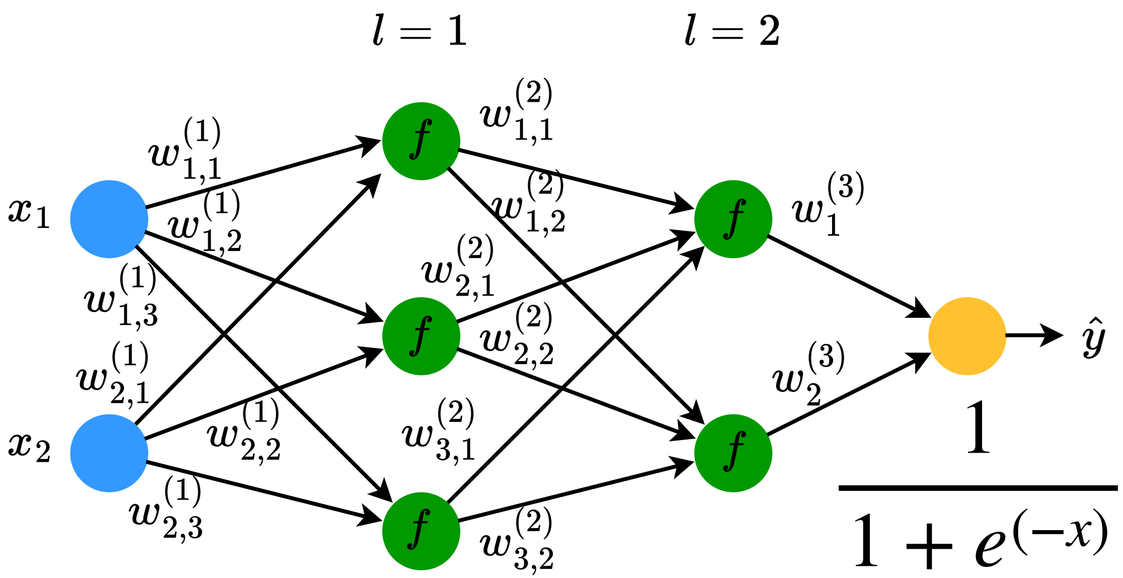


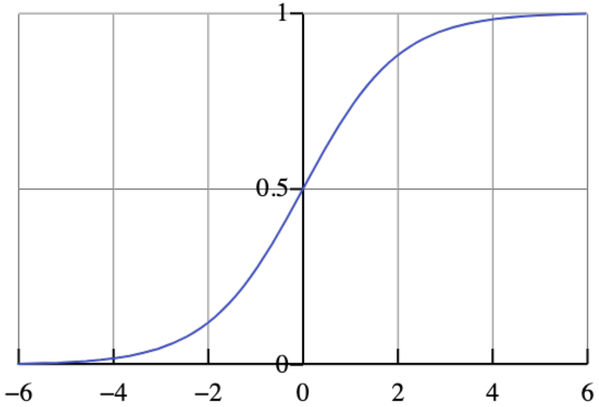
- After the last epoch, select the parameters that achieved the best performance on the validation set.
- Evaluate the generalisability of the NN on an independent **test set.**
- How to evaluate it?
  - If the NN's task is **regression**, then use MAE, MSE.
  - If the NN's task is **classification**, then use accuracy/precision/recall/F1-score.
- Learning rate, batch size, number of epochs are additional hyper parameters to be set before the training phase.

NN for classification

- Neural network for **binary classification**, where the samples belong to one of two possible classes (e.g., spam detection, disease detection).

- One output node with sigmoid activation function.



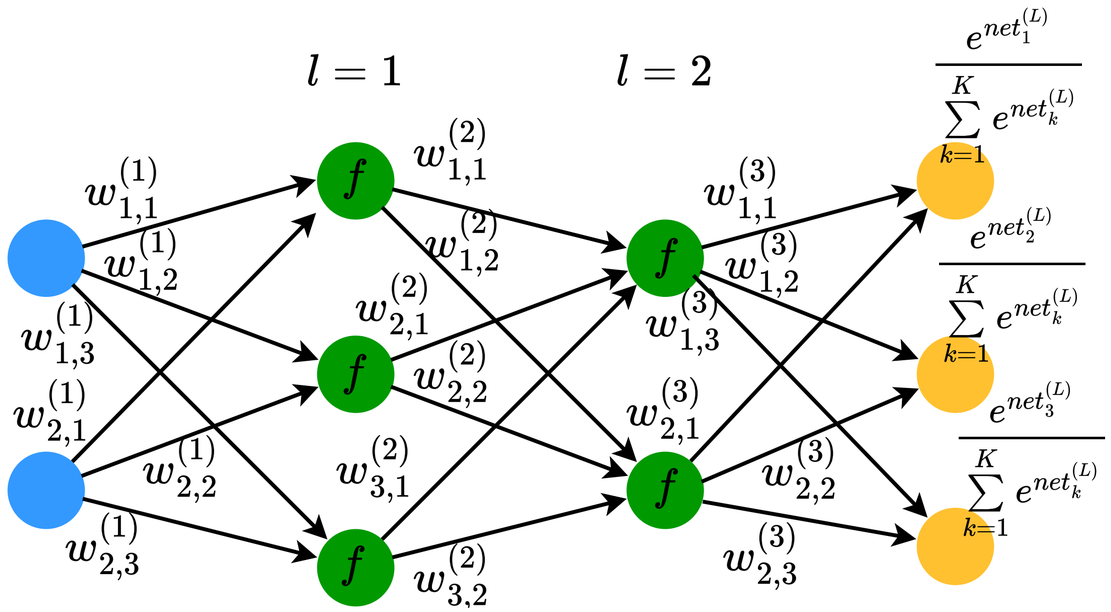


The output is a number between 0 and 1, that represents the "probability" that the sample belongs to one of the two classes

NN for classification

- Neural network for classification, where the samples belong to one of K >2 possible classes (e.g., image classification, sentiment analysis, etc).

- Number of output nodes = number of classes, each with **softmax activation** function.



Each output node outputs a number between 0 and 1, representing the probability of belonging to the class associated with the node.

NN for classification

- If classes are not numerical (e.g., animal recognition, presence or absence of a disease), we perform **label encoding**, meaning that we assign a numerical value to each class (e.g., cats=0, dogs = 1, mice=2, etc).
  - Some similar techniques are used also for representing non numerical features
- What loss function to use for classification tasks?

Binary classification:
binary cross entropy

One sample: $-y\log_2(p) - (1-y)\log(1-p)$

n samples: $-\frac{1}{n}\sum_{i=1}^{n}[y_i\log_2(p_i) + (1-y_1)\log(1-p_i)]$

Multi-class classification:
Categorical Cross-Entropy

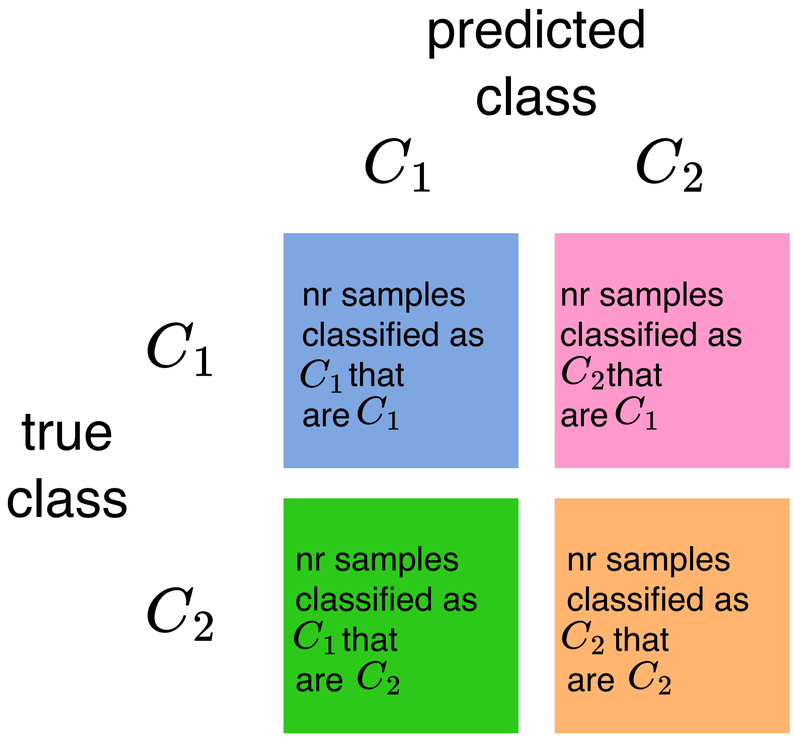
One sample: $-\sum_{k=1}^{K} y_k \log_2(p_k)$

n samples: $-\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{K} y_i \log_2(p_i)$

p is the inferred probability and equivalent to $\hat{y}$

Metrics for classification

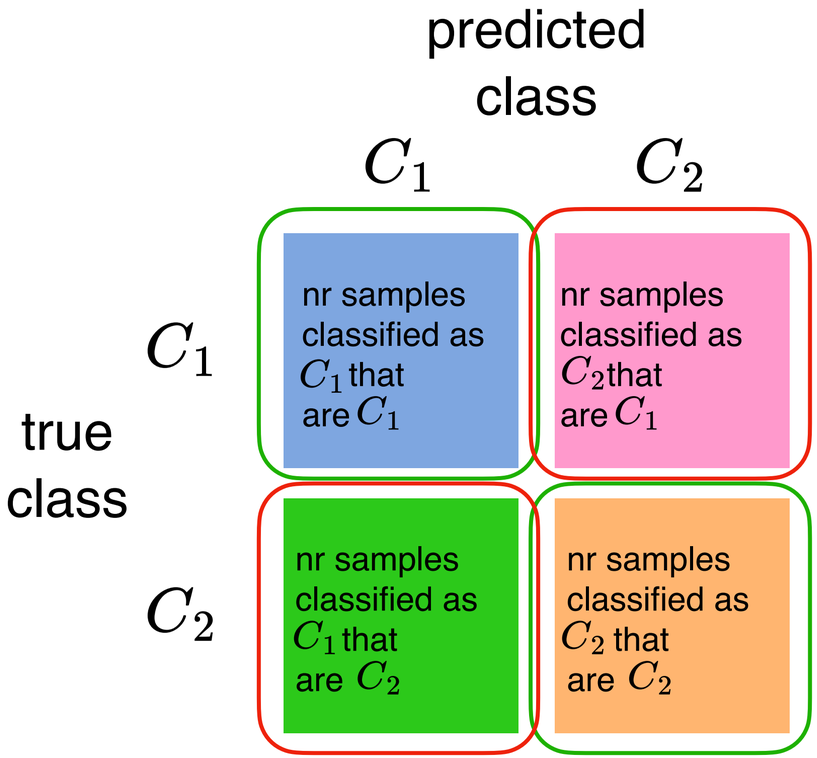
- Binary classification, two classes, C_1 and C_2.



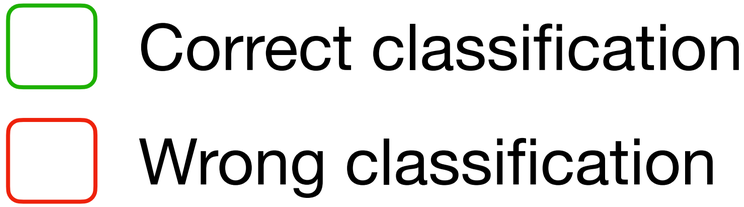
Confusion matrix

Metrics for classification

- Binary classification, two classes, C_1 and C_2.

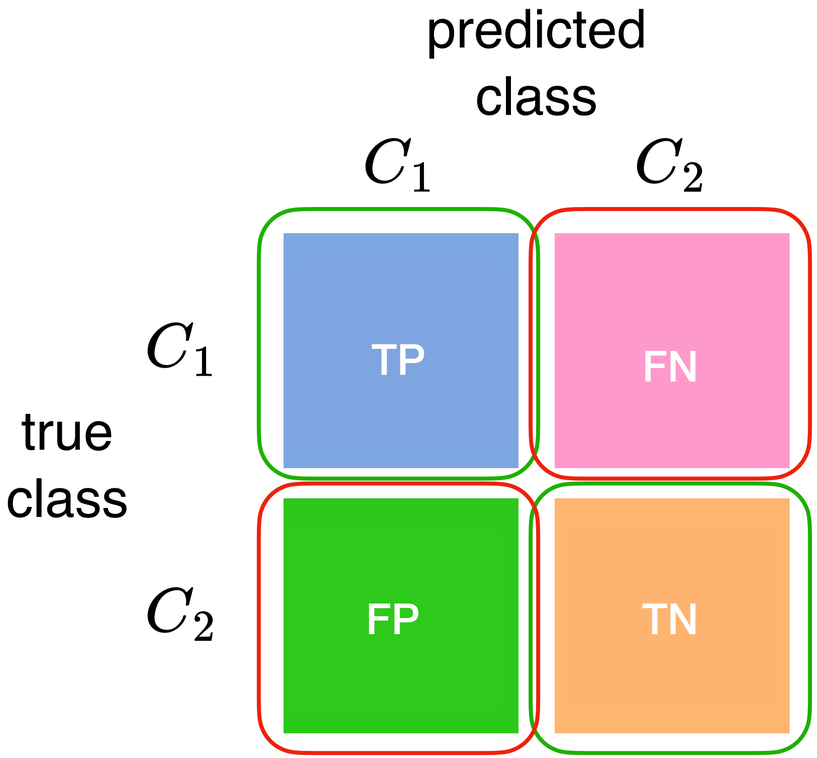


Confusion matrix



Metrics for classification

- Binary classification, two classes, C_1 and C_2.



Confusion matrix

Correct classification

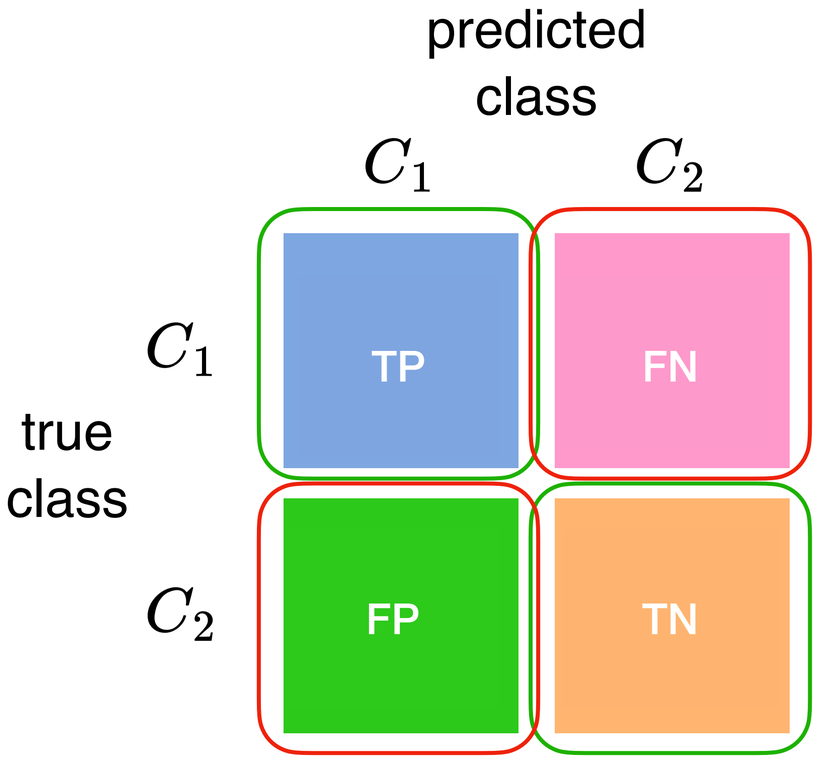
Wrong classification

- Assume C_1 is the "positive" class and C_2 is the negative class.
- TP = true positive
- TN = true negative
- FP = false positive
- FN = false negative

By dividing them by the total number of samples, we get their rates.

Metrics for classification

- Binary classification, two classes, C_1 and C_2.



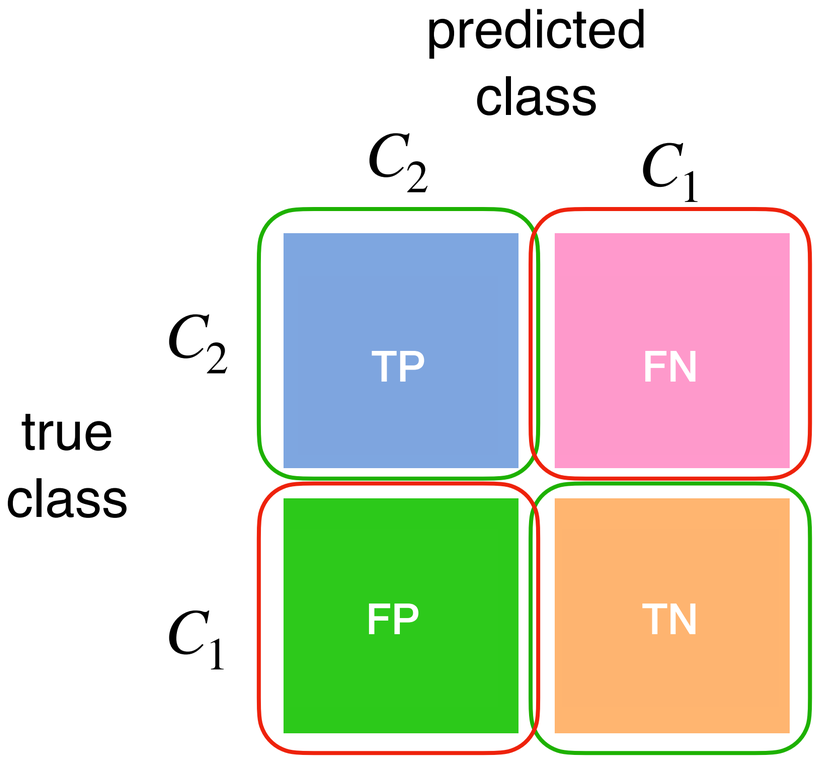
Confusion matrix

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN}$ Symmetric
- Recall $_{C_1}$: $\frac{TP}{P} = \frac{TP}{TP + FN}$ Non Symmetric
- Precision $_{C_1}$: $\frac{TP}{TP + FP}$ Non Symmetric

$$F1_{C_1} = \frac{1}{2} \frac{\text{Precision}_{C_1} \cdot \text{Recall}_{C_1}}{\text{Precision}_{C_1} + \text{Recall}_{C_1}}$$

Metrics for classification

- Binary classification, two classes, C_1 and C_2.



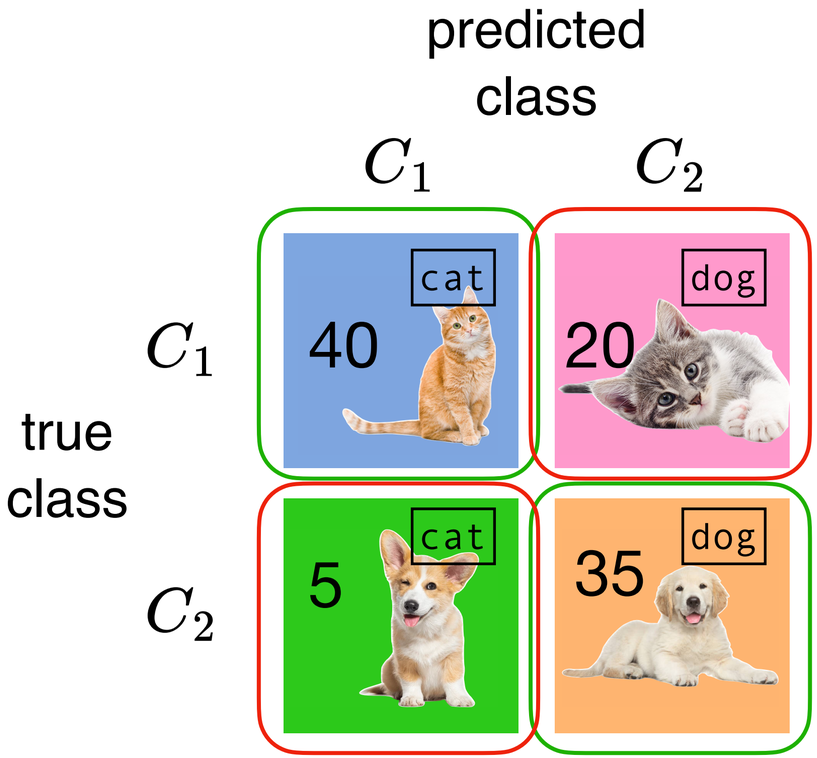
Confusion matrix

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN}$
- Recall $_{C_2}$: $\frac{TP}{P} = \frac{TP}{TP + FN}$
- Precision $_{C_2}$: $\frac{TP}{TP + FP}$

Invert classes in the confusion matrix or invert positives and negatives

Metrics for classification - example

- Binary classification, two classes, C_1, cats, and C_2, dogs.
- Training set: 100 samples, 60 cats and 40 dogs.

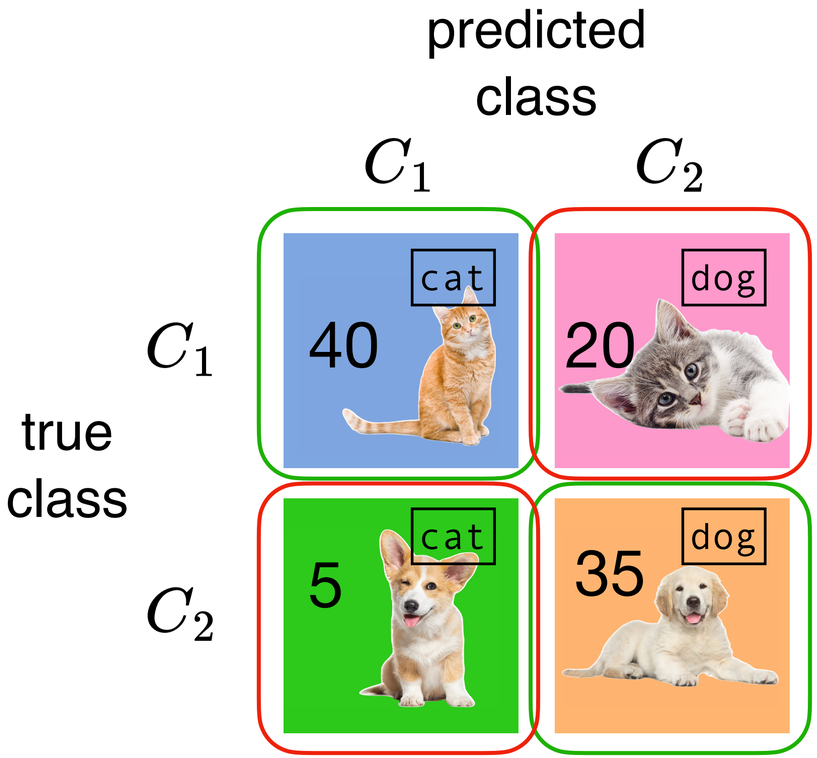


TP=40, TN=35, FP = 5, FN = 20

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN} = \frac{75}{100}$
- Recall$_{C_1}$: $\frac{TP}{P} = \frac{TP}{TP + FN} = \frac{40}{60}$
- Precision$_{C_1}$: $\frac{TP}{TP + FP} = \frac{40}{45}$

Metrics for classification - example

- Binary classification, two classes, C_1, cats, and C_2, dogs.
- Training set: 100 samples, 60 cats and 40 dogs.

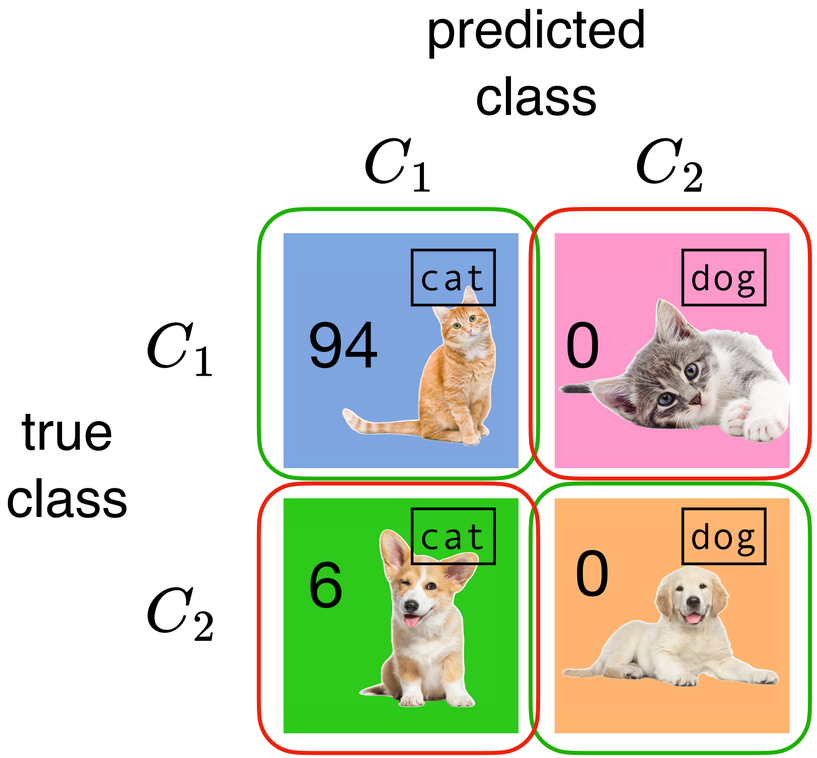


TP=40, TN=35, FP = 5, FN = 20

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN} = \frac{75}{100}$
- Recall$_{C_2}$: $\frac{TP}{P} = \frac{TP}{TP + FN} = \frac{35}{40}$
- Precision$_{C_2}$: $\frac{TP}{TP + FP} = \frac{35}{55}$

Metrics for classification - example

- Accuracy can be misleading for unbalanced datasets.
- Training set: 100 samples, 94 cats and 6 dogs. NN just outputs “cats”.

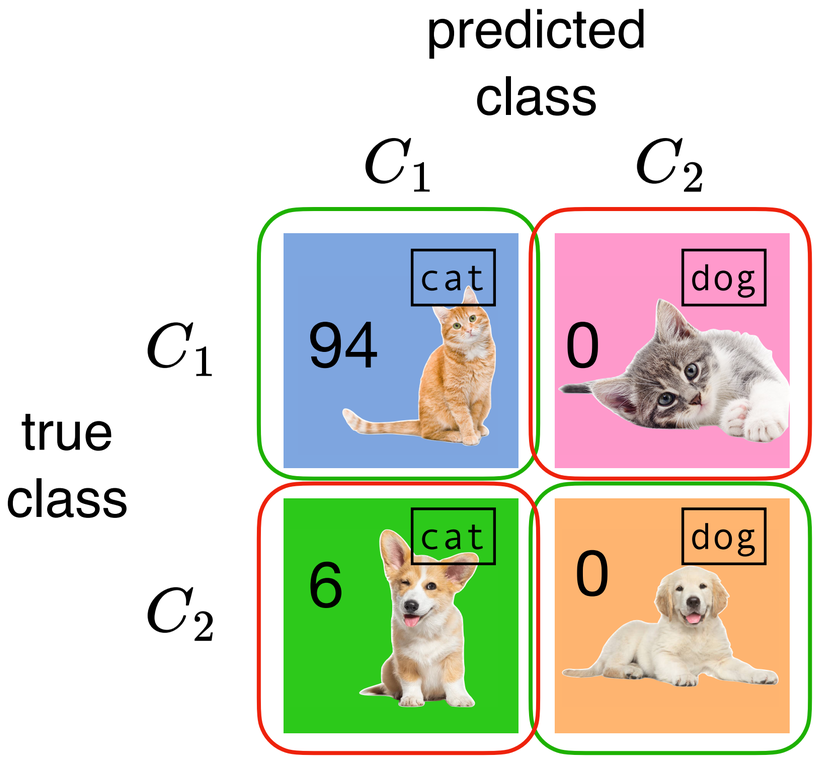


TP=94, TN=0, FP = 6, FN = 0

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN} = \frac{94}{100}$
- Recall$_{C_1}$: $\frac{TP}{P} = \frac{TP}{TP + FN} = \frac{94}{94}$
- Precision$_{C_1}$: $\frac{TP}{TP + FP} = \frac{94}{100}$

Metrics for classification - example

- Accuracy can be misleading for unbalanced datasets.
- Training set: 100 samples, 94 cats and 6 dogs. NN just outputs “cats”.

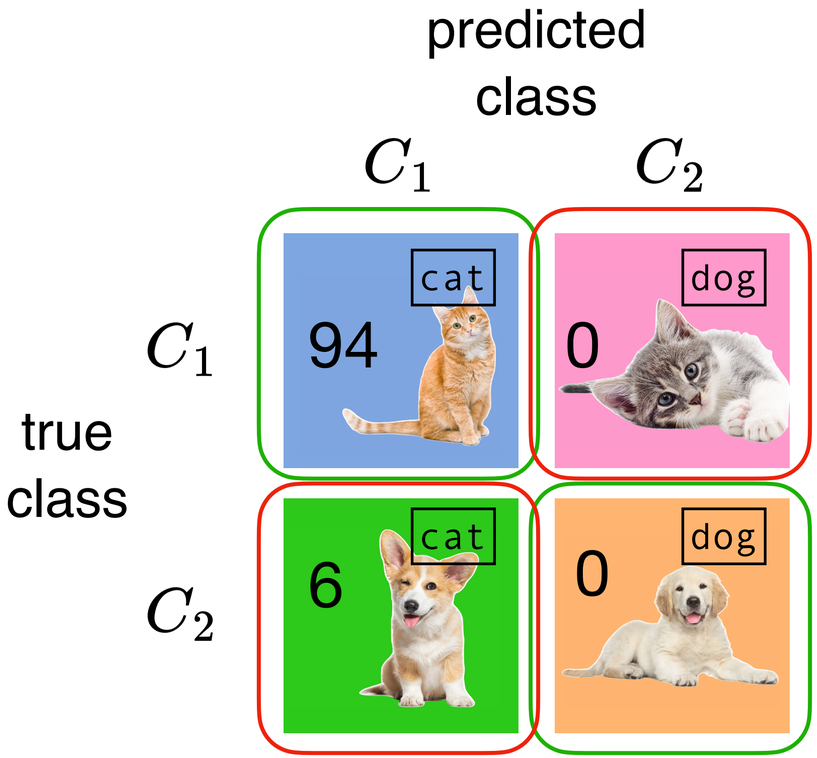


TP=94, TN=0, FP = 6, FN = 0

- Accuracy: $\frac{TP + TN}{TP + TN + FP + FN} = \frac{94}{100}$
- Recall$_{C_2}$: $\frac{TP}{P} = \frac{TP}{TP + FN} = \frac{0}{6}$
- Precision$_{C_2}$: $\frac{TP}{TP + FP} = \frac{0}{0}$

Metrics for classification - example

- Training set: 100 samples, 94 cats and 6 dogs. NN just outputs “cats”.



TP=94, TN=0, FP = 6, FN = 0

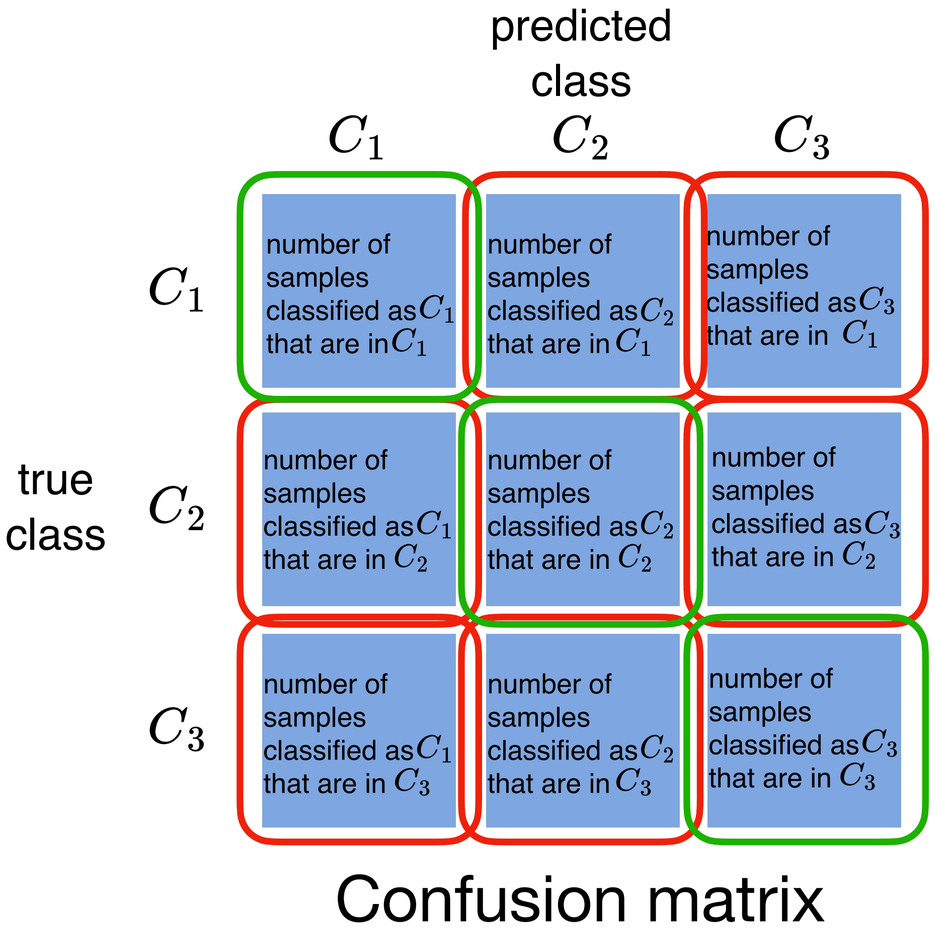
- Possible metric for unbalanced datasets:
- Balanced accuracy

$$\frac{1}{2}\left(\frac{TP}{TP+FN}+\frac{TN}{TN+FP}\right)=0.5$$

Balanced accuracy equal to 0.5 means either that the model is just guessing, or that it can predict well just one class.

Metrics for classification

- Multi-class classification, three classes, C_1, C_2 and C_3



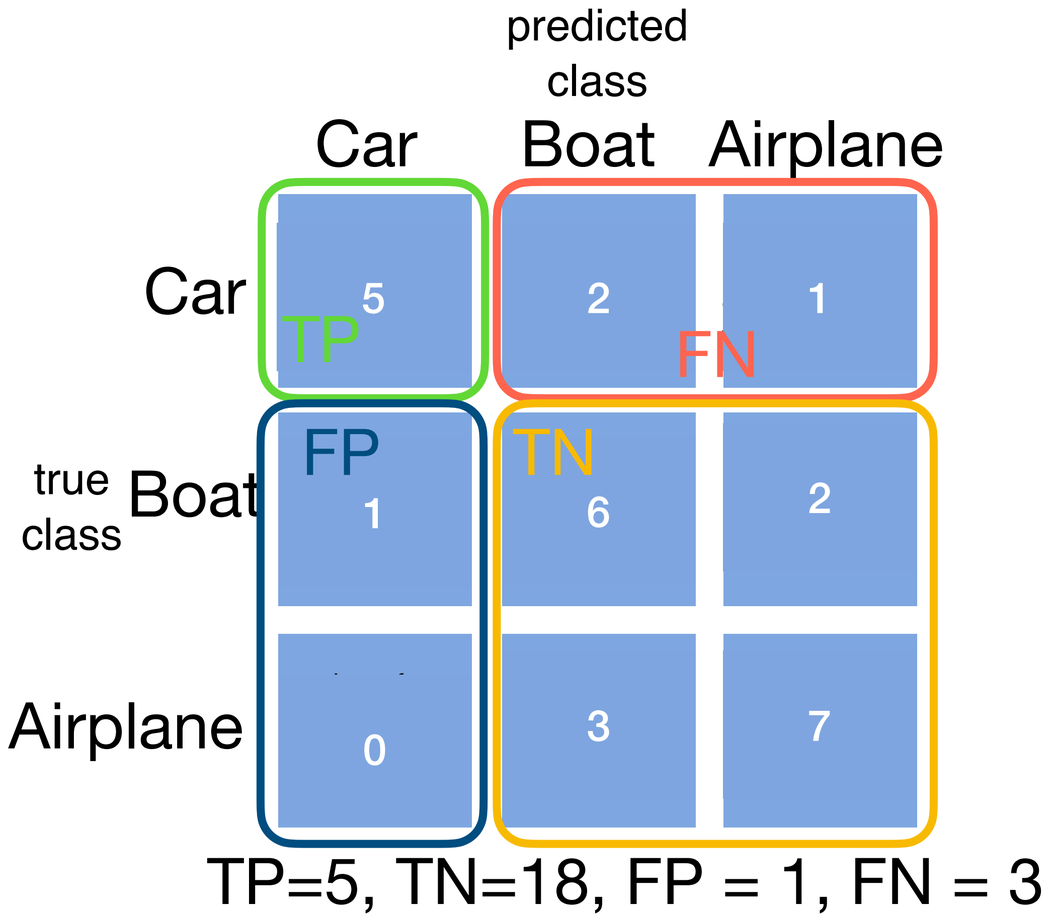
Confusion matrix

Correct classification

Wrong classification

Metrics for classification

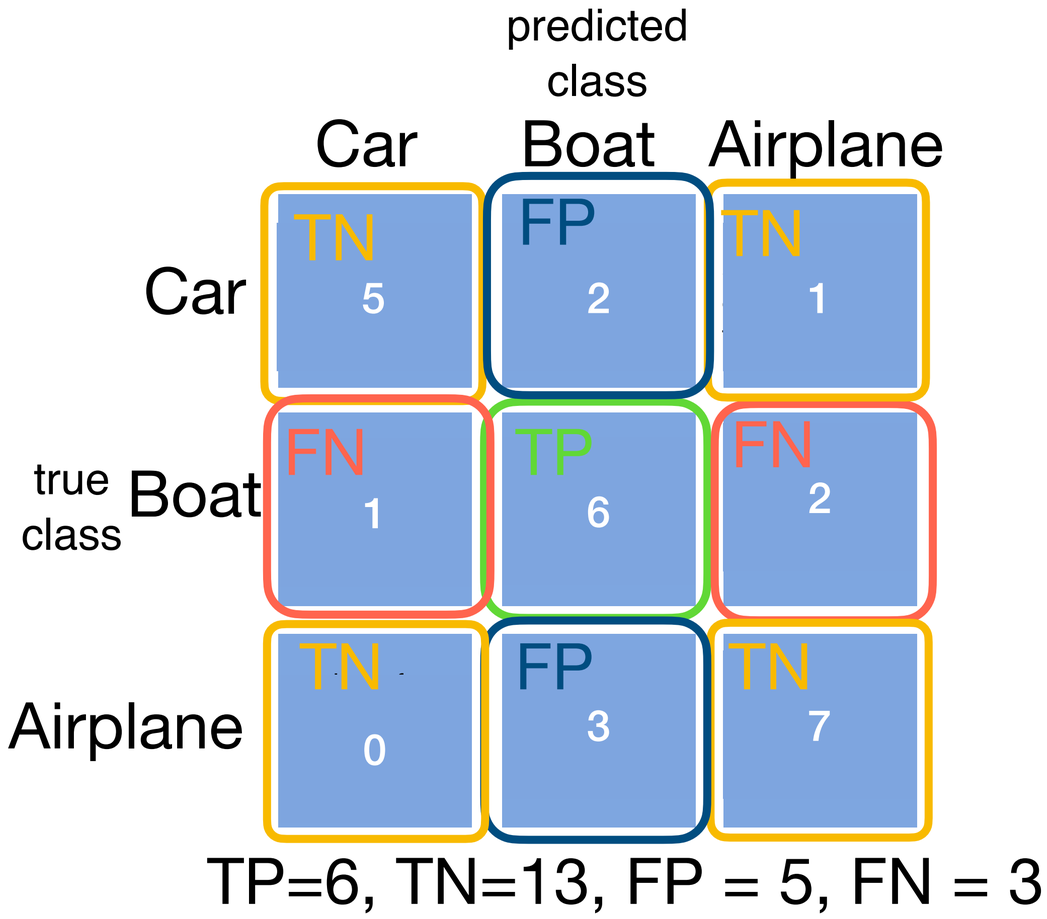
- Classes are: Car, boat, airplane. For each class, we can define the TP, FP, TN, FN, and accuracy, precision, recall and F1 score accordingly



TP, FP, TN, FN for class “car”.

Metrics for classification

- Classes are: Car, boat, airplane. For each class, we can define the TP, FP, TN, FN, and accuracy, precision, recall and F1 score accordingly

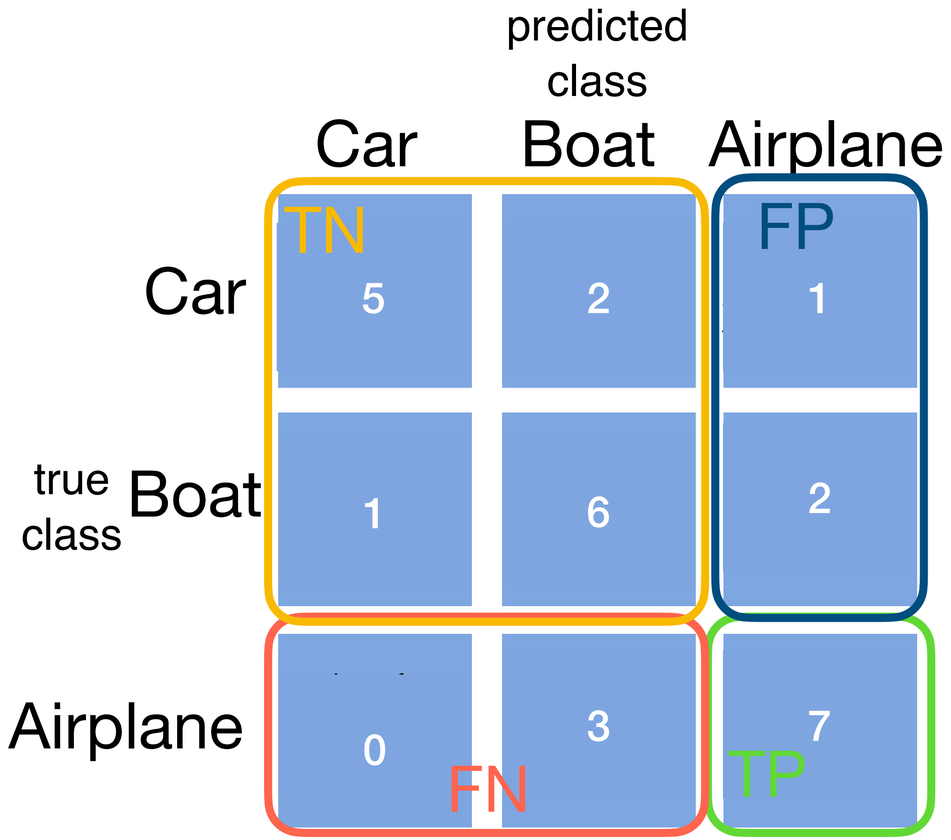


TP=6, TN=13, FP = 5, FN = 3

TP, FP, TN, FN for class “boat”.

Metrics for classification

- Classes are: Car, boat, airplane. For each class, we can define the TP, FP, TN, FN, and accuracy, precision, recall and F1 score accordingly

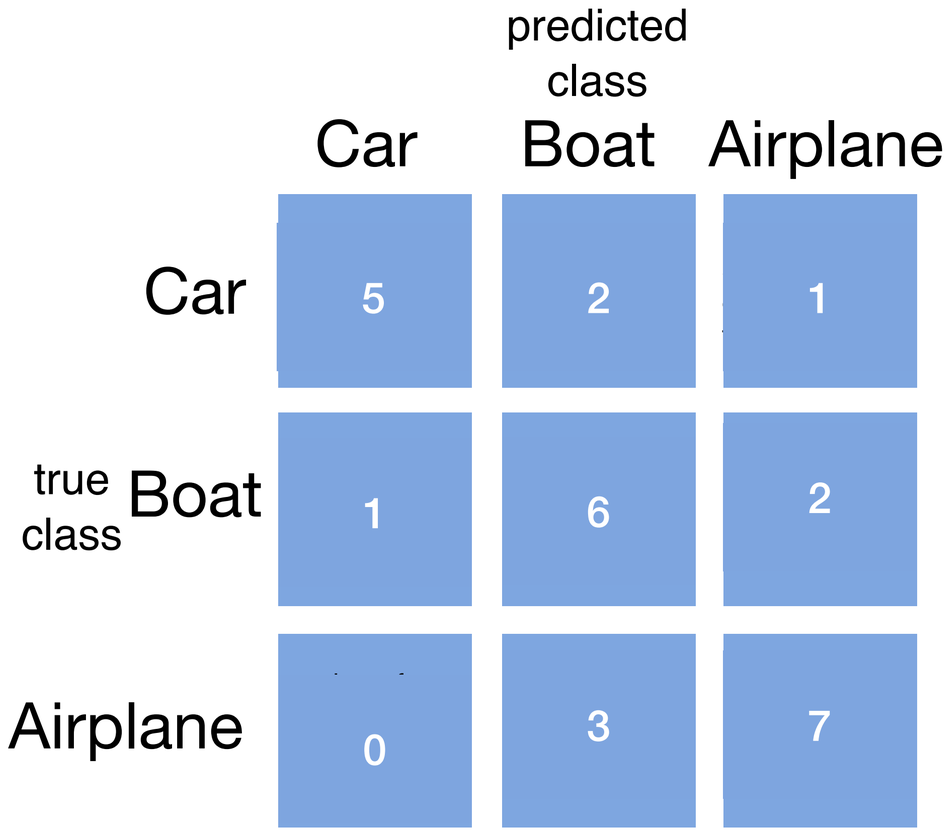


TP=7, TN=14, FP = 3, FN = 3

TP, FP, TN, FN for class “airplane”.

Metrics for classification

- Classes are: Car, boat, airplane. For each class, we can define the TP, FP, TN, FN, and accuracy, precision, recall and F1 score accordingly



$$\text{Accuracy} = \frac{\text{correct positives}}{\text{all samples}} = \frac{18}{27}$$

	Precision	Recall	F1-score
Car	0.83	0.62	0.18
Boat	0.55	0.67	0.15
Airplane	0.7	0.7	0.17

Metrics for classification

- Classes are: Car, boat, airplane. For each class, we can define the TP, FP, TN, FN, and accuracy, precision, recall and F1 score accordingly
- We then need to take the average of precision, recall, and F1-score for each class. There are two ways for doing that.

	Precision	Recall	F1-score
Car	0.83	0.62	0.18
Boat	0.55	0.67	0.15
Airplane	0.7	0.7	0.17

Micro average

$$P_\mu = \frac{TP_{\text{car}} + TP_{\text{boat}} + TP_{\text{airplaine}}}{TP_{\text{car}} + TP_{\text{boat}} + TP_{\text{airplaine}} + FP_{\text{car}} + FP_{\text{boat}} + FP_{\text{airplaine}}} = \frac{18}{27}$$

$$R_\mu = \frac{TP_{\text{car}} + TP_{\text{boat}} + TP_{\text{airplaine}}}{TP_{\text{car}} + TP_{\text{boat}} + TP_{\text{airplaine}} + FN_{\text{car}} + FN_{\text{boat}} + FN_{\text{airplaine}}} = \frac{18}{27}$$

$$P_M = \frac{P_{\text{car}} + P_{\text{boat}} + P_{\text{airplaine}}}{\text{number of classes}} = \frac{2.08}{3} = 0.69$$

$$R_M = \frac{R_{\text{car}} + R_{\text{boat}} + R_{\text{airplaine}}}{\text{number of classes}} = \frac{1.99}{3} = 0.66$$

Macro average

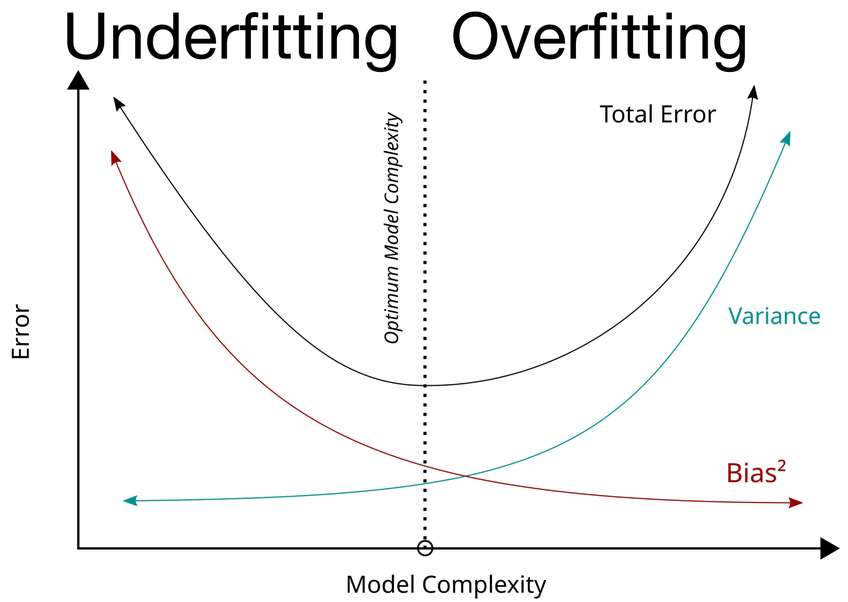
Bias (dataset)

- Machine learning (ML) models are not inherently objective.
- Selection bias occur when the sample selection of the training set does not accurately reflect the target population.
- Historical bias occur when the data distribution changed during time, and the model is not adapted to cope with the new distribution.
    - Example: an IoT air quality sensor is installed outdoors and trains a local model to predict pollution spikes. The model is trained in the summer, when most of the town activities are shut. When in winter time activities open again, the quality of air has a different distribution, but the model cannot cope with this.

Bias-Variance Tradeoff

- The bias–variance tradeoff describes the relationship between a model's complexity (i.e., number of hidden layers, number of nodes), the accuracy of its predictions, and how well it can make predictions on previously unseen data.

Underfitting: the model is not complex enough, it makes many prediction errors (biases) as it cannot model the complexity of relevant relations between features and target outputs



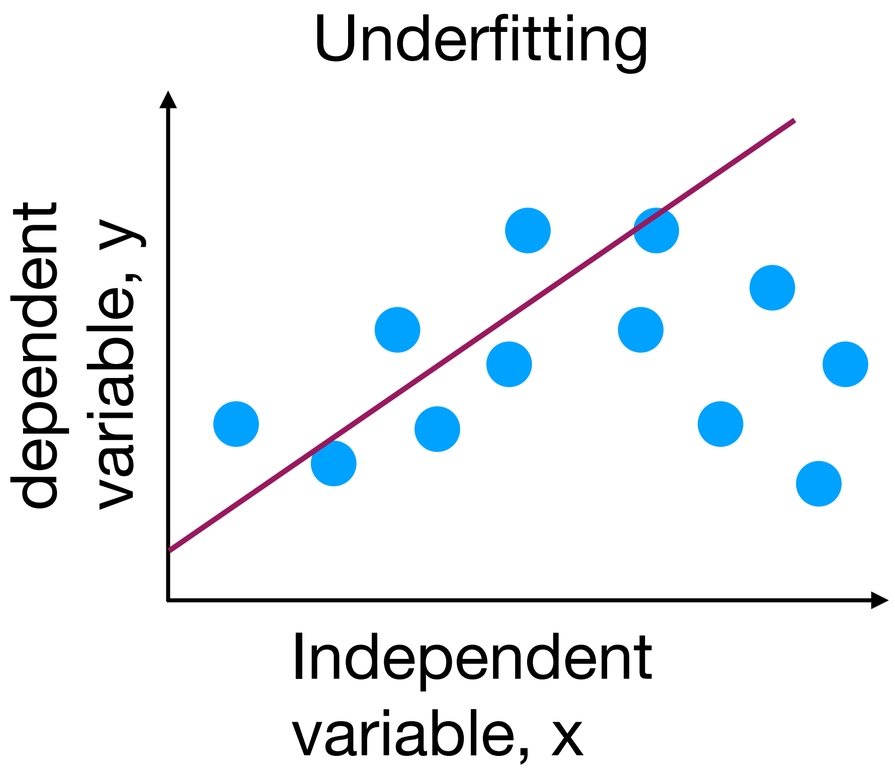
Overfitting: the model is too complex, it fitted the training data perfectly, missing the necessary generalisation capabilities. It makes many prediction mistakes as it sees different data than its training data (variance errors).

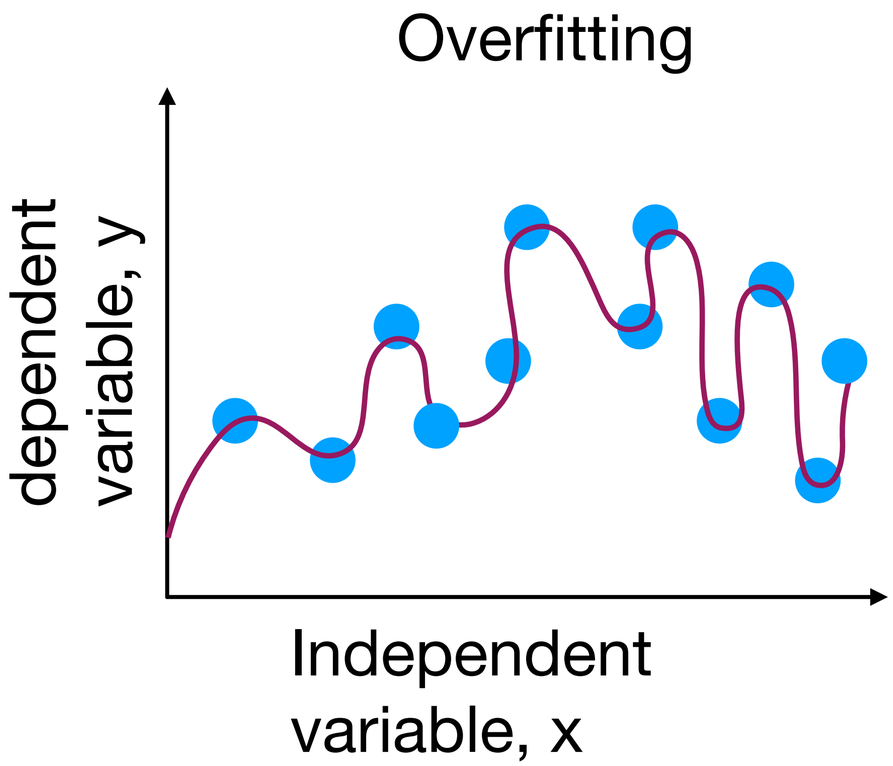
Bias errors are due to the model being too simple and unable to represent the complexity of the data patterns.

Variance errors are due to a model being too complex, and therefore too sensitive to the training data.

Bias-Variance Tradeoff

- The bias–variance tradeoff describes the relationship between a model's complexity (i.e., number of hidden layers, number of nodes), the accuracy of its predictions, and how well it can make predictions on previously unseen data.





9.4 Federated Learning

Can data live at the edge?

- Billions of phones and IoT devices constantly generate data.
- Data processing is moving on the device:
  - Improved latency
  - Work offline
  - Privacy advantages

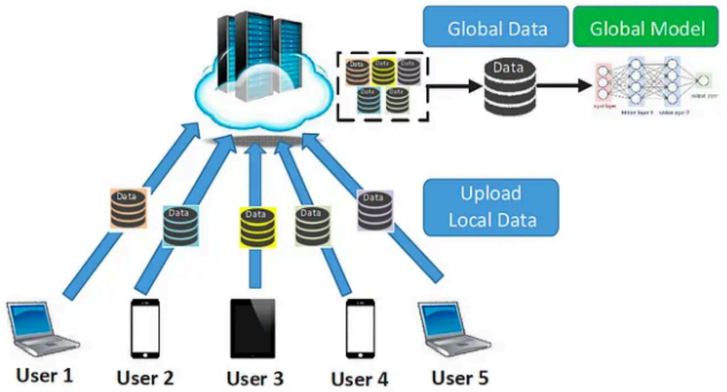
- Example: On-device inference for next word/emoji prediction in mobile phones.

Non local/Local Learning

Non local learning

High accuracy

Operators have access to sensitive data



- In non-local learning, end points send data to a central server/multiple central servers that train a NN on the data collected by the end points.
  - Users can access the model to query inference tasks (MLaaS).
- In local learning, each device stores, trains and maintains its own ML model and trains it on its local data.

Problems with non local Learning in IoT

- **Sending data may be too costly.**
- Take as an example a self driving car (very delay-sensitive application). It must make real-time decisions based on several factors:
  - How fast is the car moving? What is each person/car around doing? Is it raining? Is the floor slippery? How wide is the road? What are the speed limits? Etc.
  - To make decisions like speed, breaking, steering, etc, a complex NN continuously observing the environment and collecting different types of data must be queried for inference, resulting in the correct set of actions to take.
  - If the NN resides in the cloud, the car might respond too slowly, or might fail to respond if there is no connection.
  - Data can be video streams, requiring large bandwidth.

Problems with non local Learning in IoT

- **Data to be sent might be private or sensitive.**
  - Consider an healthcare device, collecting data as blood pressure, heart rate, medication intake, etc.
  - If the data is sent to a cloud server for ML inference, at a certain point of the long path between the device and the server a hacker could intercept the data.
  - Consequences can be very critical when data is so sensitive, e.g., identity theft, insurance discrimination.
- Devices produce a huge amount of data that could **overwhelm the server**.
- They might be energy-constrained or be part of a lossy network, **limiting the amount of data** that they can send.

Problems with local Learning in IoT

- IoT endpoints or the coordinator of an IoT system implement some ML model (e.g., a NN), train it on its own dataset locally, and use it for inference in decision making.

- **Local dataset may be too small**, the ML model can be biased.

  - Agriculture environment: sensors deployed in a farm, send data to a local server in a private network that collects data, train a local ML model to make autonomous decisions (irrigate, alert for fungal diseases, recommend pesticide application, harvest).

  - The field might never experienced the presence of a particular fungus, a particular weed. The ML model trained on local data cannot recognise these events and make correct decisions.

Federated Learning

- **Federated Learning (FL)** represents a valuable alternative to combine the advantages of non-local training (i.e., more data storage availability, less computation at the edge, more accurate NN) and local training (data privacy, low latency inference).
- For these reasons, FL is widely used for IoT applications.
- Main idea:
  - Several devices share the same NN architecture.
  - Each device trains the NN on its own data.
  - Devices share some **parameters** of the NN (e.g., weights and biases, or gradients)
  - Each device **aggregates** the parameters of the other devices, advantaging from the training and the data of other devices, but without its local data ever being shared.

Devices form a “federation” and collaborate while keeping data decentralised

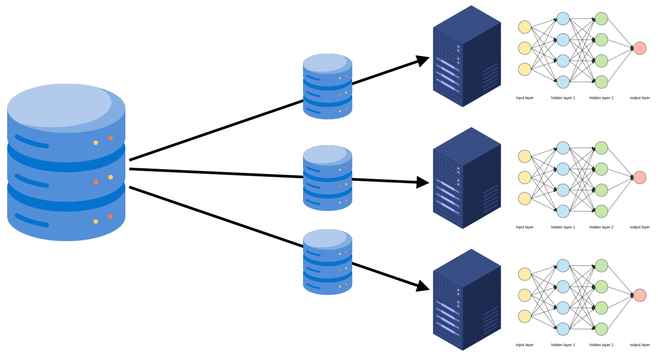
Federated Learning

- Ideal scenario: the model of each device performs as good as a centralised model trained on the entire union of each local dataset would.
- Realistic objective: the ML model on each device in the federation performs better than it would if trained only on the local data.

Federated Learning vs Distributed Learning

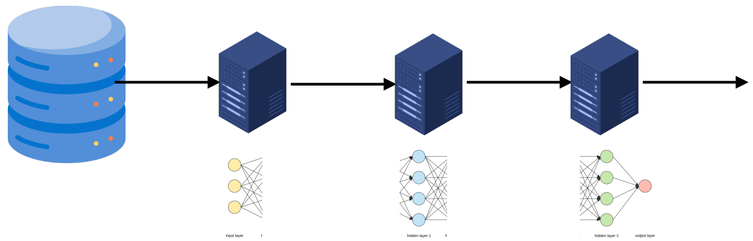
- In distributed learning, data is usually centrally stored.
  - Multiple devices (usually powerful servers) train the model **in parallel.**
  - Different kinds of parallelism can be implemented.

Data parallelism



Each server trains a ML model on a subset of the training set. Results are then aggregated.

Model parallelism



Each server trains a portion of the ML model on the entire training set

Pipeline model

- Objective: train faster large and complex models over huge datasets.

Federated Learning vs Distributed Learning

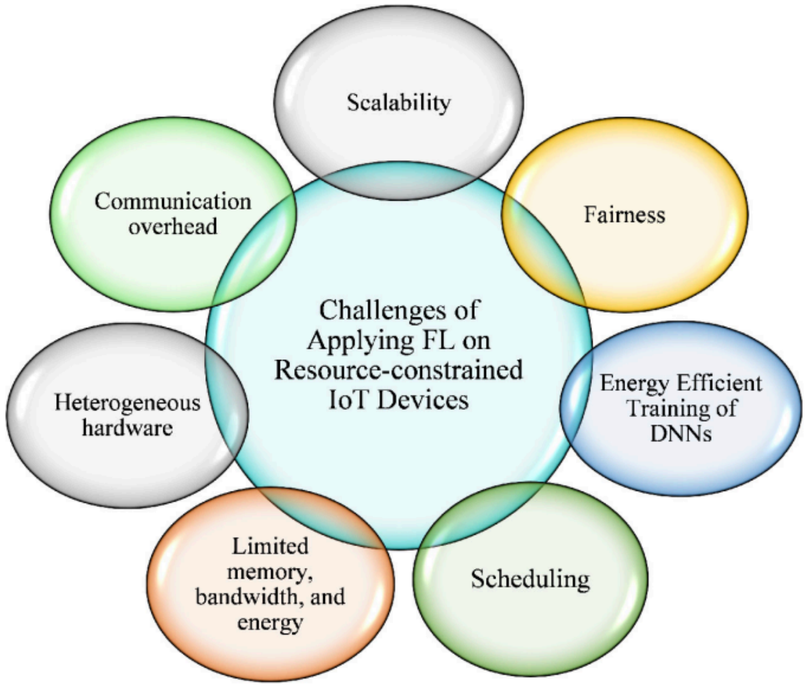
- In federated learning, data is stored on each device (usually non powerful end nodes). Devices train the model **in parallel**, each on its own chunk of data.
- Objective: train a more accurate and generalised model than each device would on its own data while keeping data privacy, preserving computing resources.
- Challenges:
  - local datasets are not iid (Independent and identically distributed - different datasets exhibit different probability distributions), and may be imbalanced.
    -> need for intelligent model aggregation solution.
  - Vulnerable to attacks: some attacks show that the original raw data can be retrieved by observing the model's parameters exchanged between parties. Some attackers can participate in the federation and share wrong parameters, preventing the other parties' models to converge.
  - Node availability: some nodes are not always available for training, since they might be power-constrained.

FL applications in IoT

- FL finds many applications in IoT:
- Smart Robotics (Deep reinforcement learning for mobile robots)
- Smart Object Detection (Self driving cars, smart surveillance)
- Smart homes (Wake-Up-Word speech recognition and automatic speech recognition)
- Smart cities (Smart Traffic Management, Smart Waste Management, Energy Demand Forecasting etc)
- Healthcare (Wearable Health Monitoring, Personalised Drug Dosage Prediction)
- And many others

FL in IoT - challenges

- IoT systems are characterised by low power, low computational capabilities of possibly hundreds of nodes. It is hard to train complex models on them, and communication must be optimised.
- Nodes are not reliable, they want to be in idle mode as much as possible.
- Nodes are heterogeneous in hardware, geographically distributed. They do not share a common standard to represent processed data. The data they collect is not iid (independent and identically distributed).



- Nodes live in low bandwidth networks.
- Nodes want to participate in the federation, and client selection can favour more powerful nodes, leaving the less powerful ones in starvation.

Taxonomy

Horizontal FL

datasets across different participants share the same feature space but differ in samples.

Vertical FL

participants have datasets that share the same sample space but differ in features. Each participant holds different attributes about the same set of entities.

Cross-device FL

large number of devices (smartphones, IoT devices, edge devices) that are typically not reliable and have varying computational capabilities and intermittent network connectivity.

Cross-silo FL

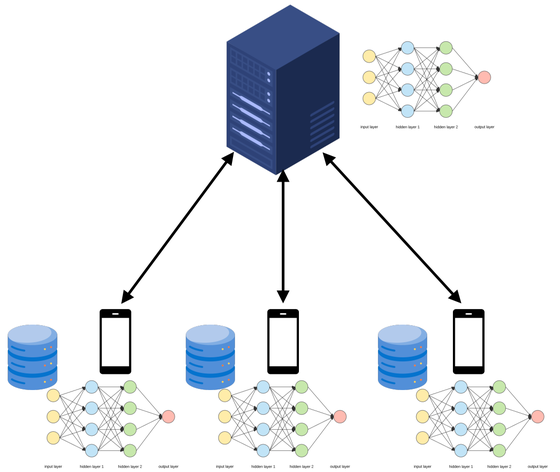
small number of reliable and stable participants (silos) (organizations or institutions) with significant computational resources and stable network connections

Centralised FL

Decentralised FL

Centralised FL

- A central server orchestrates the training process. Participants train local models on their data.
- Participants' send the updated model parameters to the central server.
- The server aggregates these parameters to update the global model, which is then distributed back to the clients.
- The central server coordinates the training, aggregation, and distribution processes.
- Model updates are aggregated at a single point, simplifying the update process.

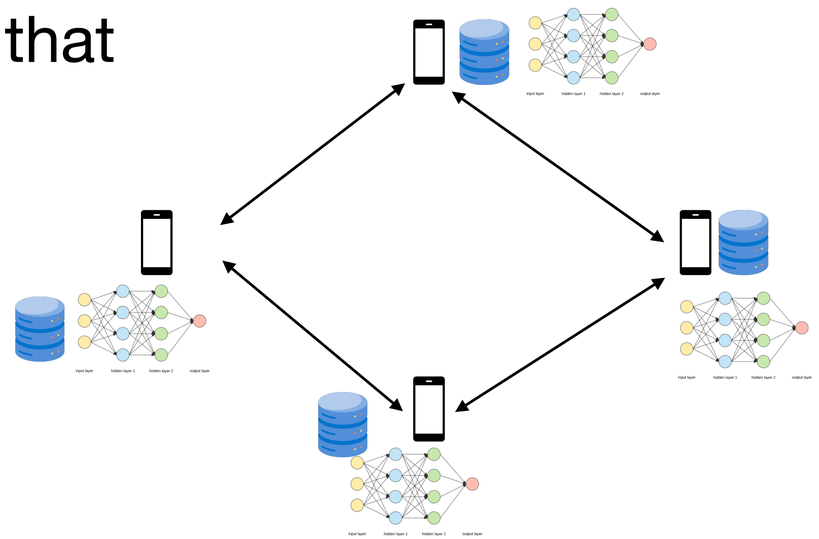


Decentralised FL

- In decentralized Federated Learning participants communicate directly with each other to share and aggregate model updates (peer-to-peer).

- Each party keeps a **mixing matrix**, that weighs the contribution of each other party of the aggregation. Can have 0 entries, can vary over time.

- This peer-to-peer communication ensures that there is no single point of failure and enhances privacy by distributing the aggregation process.



FedAvg

Centralised FL, first to share models' parameters, weighted average aggregation

- Fundamental algorithm for centralised FL [McMahan 2017].
- Consider a set of K parties (devices).
- Each party k store a local dataset D_k of n_k points, on which it trains a neural network.
- Let $D = D_1 \cup \ldots \cup D_K$ be the joint dataset and $n = \sum_k n_k$ the total number of points.
- Goal: minimising the average loss function $\mathscr{L}(\theta, D)$, i.e., solve the problem:

$$\min_{\theta \in \mathbb{R}^p} \mathscr{L}(\theta, D)$$

where

- $\theta = (\theta_1, \ldots, \theta_p)$ is the set of parameters of the NN,
- $\mathscr{L}(\theta, D) = \sum_{k=1}^{K} \frac{n_k}{n} \mathscr{L}_k(\theta_k, D_k)$ is the average of the loss function of the parties, weighted by their training set sizes.
- $\mathscr{L}_k(\theta_k, D_k) = \sum_{d \in D_k} \mathscr{L}(\theta, d)$ is the average loss function achieved by party k on all points of its training set.

FedAvg

Algorithm FedAvg (server-side)

Parameters: client sampling rate ρ

initialize θ

for each round $t = 0, 1, \ldots$ **do**

$\mathcal{S}_t \leftarrow$ random set of $m = \lceil \rho K \rceil$ clients

for each client $k \in \mathcal{S}_t$ in parallel **do**

$\theta_k \leftarrow \text{ClientUpdate}(k, \theta)$

$\theta \leftarrow \sum_{k \in \mathcal{S}_t} \frac{n_k}{n} \theta_k$

Send θ to each client

Aggregates clients' parameters

Back propagation with SGD

Algorithm ClientUpdate(k, θ)

Parameters: batch size B, number of local steps L, learning rate η

for each local step $1, \ldots, L$ **do**

$\mathcal{B} \leftarrow$ mini-batch of B examples from $\mathcal{D}_k$

$\theta \leftarrow \theta - \frac{n_k}{B} \eta \sum_{d \in \mathcal{B}} \nabla f(\theta; d)$

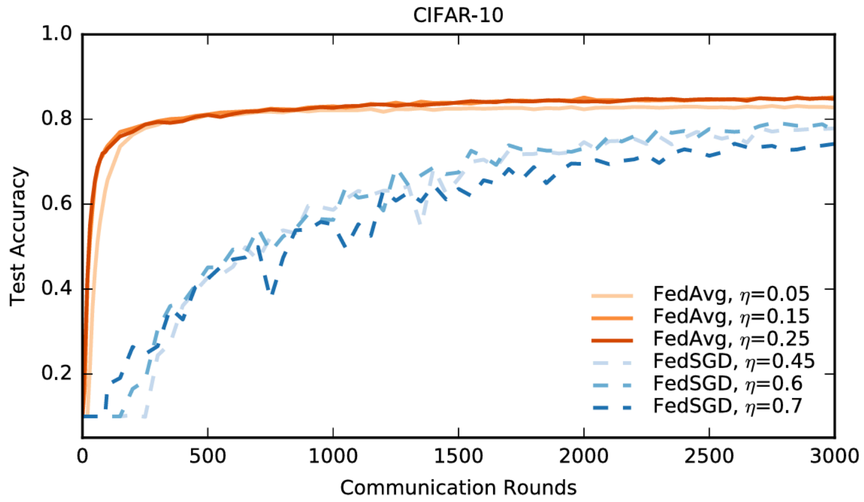
send θ to server

Performs local updates and receives aggregated parameters from the server.

- ρ specifies the percentage of clients selected for aggregation at each round.
- L says the number of local steps (batches) in training the NN that the client performs before sending the parameters θ_k to the server.
- For $L = 1$, $\rho = 1$, it is equivalent to classical parallel SDG.
- For $L > 1$, each client performs multiple local SGD steps before communicating.

FedAvg

Comparisons with FedSDG



This is what the parties in FedAVG send (θ_k)

$$w_{i,j}^{(l)} \leftarrow w_{i,j}^{(l)} - \eta \frac{\partial \mathscr{L}}{\partial w_{i,j}^{(l)}}$$

$$b_{j}^{(l)} \leftarrow b_{j}^{(l)} - \eta \frac{\partial \mathscr{L}}{\partial b_{j}^{(l)}}$$

This is what the parties in FedSDG send $(\nabla(\theta_k))$

- FedAvg with $L > 1$ allows to reduce the number of communication rounds, which is often the bottleneck in FL.
- Convergence to the optimal model can be guaranteed for i.i.d. data [Stich,2019] [Woodworthetal.,2020] but issues arise in strongly non-i.i.d. cases.

Decentralised FedAvg

Like FedAvg, but decentralised

- We can derive an algorithm based on FedAvg but for fully decentralised settings, where parties act as peers and do not interact with a central server for aggregation.
- Let $G = (V = \{1, \ldots, K\}, E)$ be a undirected graph where nodes are parties and an edge (k, l) exist if k and l can exchange messages.
- Let $W \in [0,1]^{K \times K}$ be a symmetric, bistochastic matrix such that if $\nexists (k, l)$ then $W_{k,l} = W_{l.k} = 0$ (mixing matrix).
- Given parties' parameters $\Theta = [\theta_1, \ldots, \theta_K]$, $W\Theta$ is a weighted aggregation between neighbouring nodes in G

$$[W\Theta]_k = \sum_{l \in \mathcal{N}_k} W_{k,l}\theta_l$$

each party k only cares about the k-th row of the matrix, which says how much weight it must give to the parameters of each of its neighbours in $l \in \mathcal{N}_k$. Each θ_k is the vector of the parameters of party k.

Decentralised FedAvg

Algorithm Distributed FedAvg (run by party k)

Parameters: batch size B, learning rate η, sequence of matrices $W^{(t)}$

initialize $\theta_k^{(0)}$

for each round $t = 0, 1, \ldots$ **do**

$\mathcal{B} \leftarrow$ mini-batch of B examples from $\mathcal{D}_k$

$\theta_k^{(t+\frac{1}{2})} \leftarrow \theta_k^{(t)} - \frac{n_k}{B}\eta \sum_{d \in \mathcal{B}} \nabla f(\theta_k^{(t)}; d)$ ← Local update

Send $\theta_k^{(t+\frac{1}{2})}$ and receive $\theta_k^{(t+\frac{1}{2})}$ from all $l \in \mathcal{N}_k$ ← Aggregation update

$\theta_k^{(t+1)} \leftarrow \sum_{l \in \mathcal{N}_k^{(t)}} W_{k,l}^{(t)} \theta_l^{(t+\frac{1}{2})}$

- Alternates between local updates and aggregation updates.
- Notice that the mixing matrix is actually time-dependent, $W^{(t)}$.
- Convergence rate depends on the topology: more connected means faster convergence, but also more resource usage.
- Can decide to do aggregation updates every τ local updates by selecting $W^{(t)} = I$ for τ consecutive steps.

Decentralised FedAvg

Algorithm Distributed FedAvg (run by party k)

Parameters: batch size B, learning rate η, sequence of matrices $W^{(t)}$

initialize $\theta_k^{(0)}$

for each round $t = 0, 1, \ldots$ **do**

$\mathcal{B} \leftarrow$ mini-batch of B examples from $\mathcal{D}_k$

$\theta_k^{(t+\frac{1}{2})} \leftarrow \theta_k^{(t)} - \frac{n_k}{B}\eta \sum_{d \in \mathcal{B}} \nabla f(\theta_k^{(t)}; d)$ ← Local update

Send $\theta_k^{(t+\frac{1}{2})}$ and receive $\theta_k^{(t+\frac{1}{2})}$ from all $l \in \mathcal{N}_k$ ← Aggregation update

$\theta_k^{(t+1)} \leftarrow \sum_{l \in \mathcal{N}_k^{(t)}} W_{k,l}^{(t)} \theta_l^{(t+\frac{1}{2})}$

- Alternates between local updates and aggregation updates.
- Notice that the mixing matrix is actually time-dependent, $W^{(t)}$.
- Convergence rate depends on the topology: more connected means faster convergence, but also more resource usage.
- Can decide to do aggregation updates every τ local updates by selecting $W^{(t)} = I$ for τ consecutive steps.

How do we set τ?

Resource optimisation in decentralised FedAvg

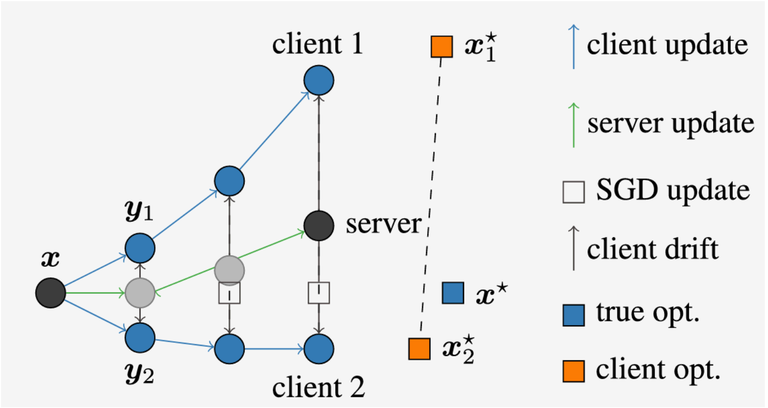
- τ: number of local updates in each round
- T : total number of local updates performed by each party, $T = \tau \cdot I_{max}$, I_{max}= maximum number of rounds.
- Each party has M different types of resources (time, energy, bandwidth, etc.).
- For each party and for each resource $m \in M$, a local update consumes c_m resources, and a global (aggregation) update consumes b_m resources. For each $m \in M$, each party has a maximum resource budget R_m.
- Goal: decide how to set τ, I_{max} to minimise the average loss function while respecting the resource budget constraints.

$$\min_{\tau, K \in \{1,2,3,\ldots\}} F(\mathbf{w}^{\mathrm{f}})$$
$$\text{s.t. } (T+1)c_m + (K+1)\, b_m \leq R_m, \quad \forall m \in \{1, \ldots, M\}$$
$$T = K\tau. \qquad (7)$$

Wang, Shiqiang, et al. "Adaptive federated learning in resource constrained edge computing systems." IEEE journal on selected areas in communications 37.6 (2019): 1205-1221.

Non i.i.d. data

- One of the greatest challenges in FL comes from the fact that the local datasets of the parties might follow very different probability distributions, causing **client drifts**.
  - Local model updates toward local optimal solutions, resulting in performance degradation of the global model.

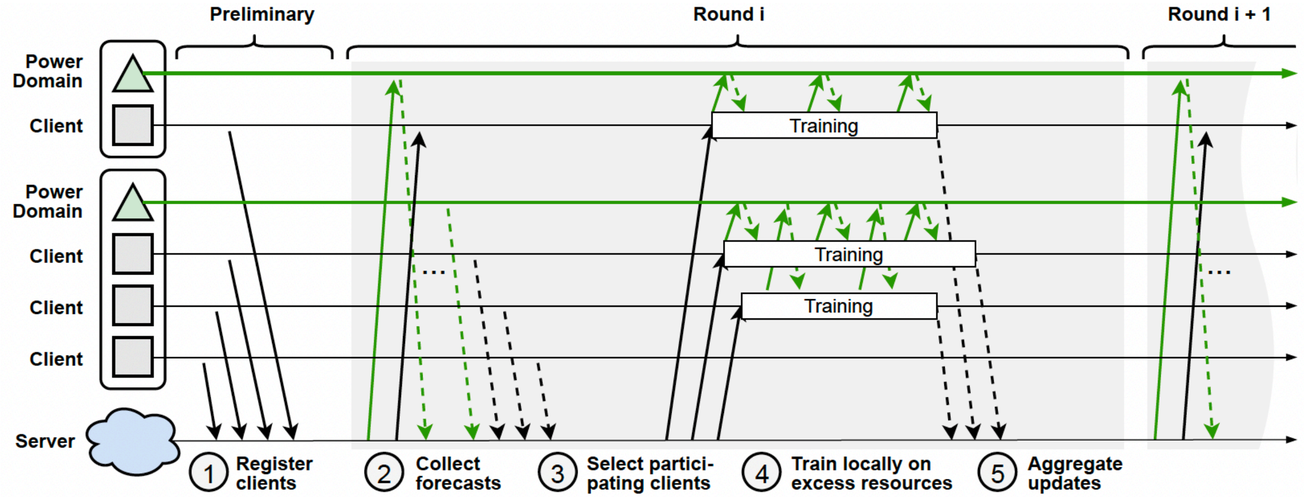


- Averaging the parameters in these cases does not guarantee the convergence to the optimal model.
- More sophisticated ways to aggregate models are needed.

FedZero

Centralised FL with client selection based on energy resources and fairness

- FedZero: Centralised FL framework that minimises the models' error while guaranteeing zero carbon emissions and fair training under energy and resource constraints.



- Clients c register to the central server and provide the following information:
  - Maximum computing capacity m_c expressed in batches per time step.
  - Energy efficiency δ_c, expressed as energy required per batch.
  - Power domain. Clients are divided into power domains. Clients grouped in one power domain have access to the same renewable energy source.

FedZero

- Some power domains might have abundance of energy. The clients in such domains are more likely to be selected for local updates.
- $r_{p,t}$=forecast energy excess of power domain p at time t.
- $m_{c,t}^{spare} \in [0, m_c]$ is the forecast spare computing capacity at time t of client c (batches/time step).
- To avoid picking clients with access to little or no resources during a round, FedZero relies on **multistep-ahead forecasts** of excess energy at power domains ($r_{p,t}$) and spare capacity at clients ($m_{c,t}^{spare}$) to decide what clients to include in the aggregation at each round.

FedZero

Algorithm 1 Determine clients and round duration

$C \leftarrow$ set of clients
$P \leftarrow$ set of power domains
# search for shortest possible round duration
for $d \leftarrow 1$ **to** $d^{\max}$ **do**
  # filter out power domains without excess energy
  $\bar{P} \leftarrow \{\forall p \in P, \forall t = 1, ..., d : r_{p,t} > 0\}$
  # filter out clients that over-participated in the past
  $\bar{C} \leftarrow \{\forall c \in C : \sigma_c > 0\}$
  # filter out clients without sufficient computing capacity or energy
  for $p \in \bar{P}$ **do**
    $\bar{C} \leftarrow \bar{C} \setminus \{\forall c \in C_p : \sum_{t=0}^{d} \min(m_{c,t}^{\text{spare}}, \frac{r_{p,t}}{\delta_c}) < m_c^{min}\}$
  # increase duration if there are not at least n valid clients
  if $|\bar{C}| < n$ **then**
    continue
  # select optimal clients
  $b \leftarrow \text{findOptimalClients}(\bar{C}, \bar{P}, d)$
  if b is valid solution **then**
    return b, d
wait, if no solution is found for $d = d^{max}$

Goal: Find n clients at each round for training and select the minimum possible training round duration to spare energy and computing resources.

->Avoids consuming all the energy resources of the clients

->Fairness step

->Include a client c that is in the power domain p only if it has enough spare computing capacity and enough energy to train with the minimum number of batches required m_c^{min}

With a larger round duration, more clients might become eligible for training, as their forecast energy excess can only increase over time

FedZero

Find Optimal Clients:

$$\max_{b_c, m_{c,t}^{\exp}} \quad \sum_{c \in C} b_c \cdot \sigma_c \sum_{t=0}^{d} m_{c,t}^{\exp}$$

Maximise the number of expected batches executed by the client c, weighted for its **statistical utility.**

$$\text{s.t.} \quad b_c = 1 \implies m_c^{\min} \leq \sum_{t=0}^{d} m_{c,t}^{\exp} \leq m_c^{\max} \quad \forall c \in C$$

limits each selected client to compute between m_c^{min} and m_c^{max} batches.

$$\sum_{c \in C_e} m_{c,t}^{\exp} \cdot \delta_c \leq r_{e,t} \qquad \forall e \in E, t = 0, \ldots, d$$

All selected clients should not use more energy than available

$$\sum_{c \in C} b_c = n$$

Exactly n clients are selected per round

- $b_c \in \{0,1\}$, =1 if client c participates in the round.
- $m_{c,t}^{exp} \in [0, m_{c,t}^{spare}]$ is the expected number of batches client c compute at time t

FedZero

- The statistical utility of a client c is defined as:

$$\sigma_c = \begin{cases} |B_c| \sqrt{\frac{1}{|B_c|} \sum_{k \in B_c} loss(k)^2}, & \text{if } p(c) \geq 1 \\ 1, & \text{otherwise} \end{cases}$$

- Where $p(c)$ describes the number of rounds a client previously participated in, B_c is the number of available samples at client c,

- A client has high statistical relevance if it has a high loss and a large data set.

FedZero

- FedZero aims to prevent the same clients to be selected many times in order to enhance the fairness of the model.
  - If a client has large computing resources and energy availability, it would be selected many times, gaining the privilege of having the global model specialised for its own data.
  - To address this, after each round, all clients which participated in the round are put in a black list, and their statistical utility σ_c is set to 0.
  - At each round, it is released from the black list with probability $P(c)$:

$$P(c) = \begin{cases} (p(c) - \omega)^{-\alpha}, & \text{if } p(c) - \omega > 0 \\ 1, & \text{otherwise} \end{cases}$$

- The parameter ω is updated at each round as $\omega \leftarrow \text{mean}\{p(c), \forall c \in C\}$. If $p(c) > \omega$ it means that client c participated in more rounds than the average.
- α is a parameter to control the speed at which clients get released from the blacklist. High α, clients remain longer in the blacklist.

ProxyFL

Decentralised FL with privacy preserving

- Some attacks have been crafted, showing that it is sometimes possible to retrieve the original data points used for training from parameters and gradient updates (i.e., the data shared between FL parties during training).
- **ProxyFL**: Decentralized FL setting involving a set of clients K, each with a local data distribution D_k.
- In ProxyFL, each client k trains **two models:**
  - A private model f_{ϕ_k}, whose parameters ϕ_k are not shared, used for inference. Parameters will be updated based on the parameter updates of the proxy model.
  - A **proxy model** h_{θ_k}, whose parameters θ_k are shared with the other peers.

ProxyFL - differential privacy

- Let D be a set of datapoints, and M a probabilistic function, or “mechanism” of D.

- We say that the function M is (ε, δ)-differentially private if for all subsets of its possible outputs $S \subset Range(M)$, and for all pairs of databases D and D' that differ by one element, it holds that:

$$Pr[M(D) \in S] \leq \exp(\varepsilon) \cdot Pr[M(D') \in S] + \delta, \text{ where } \varepsilon, \delta > 0.$$

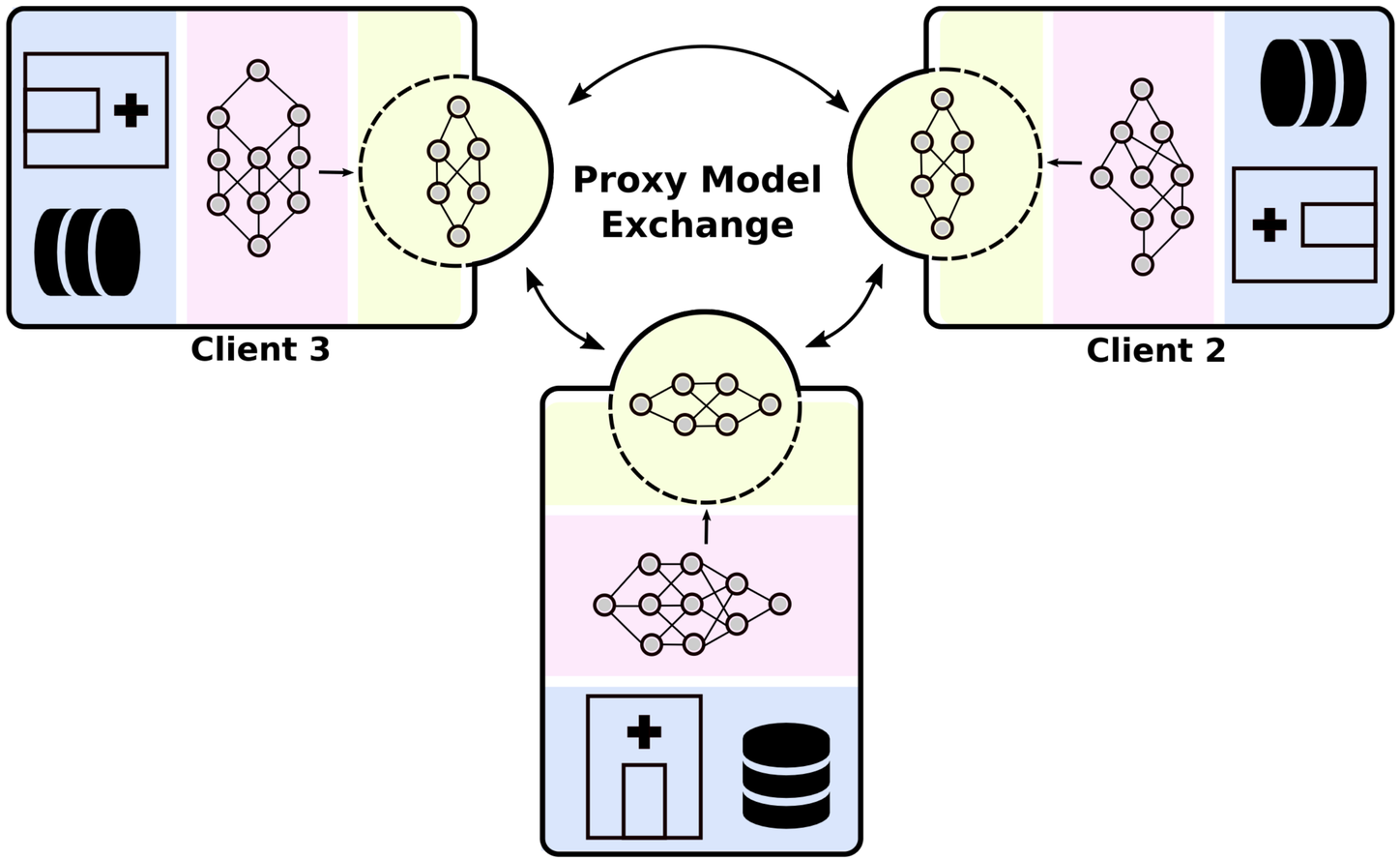
- Meaning: when a single data is added or removed from the database, the outcomes of M should be unchanged in distribution.

- **IDEA**: if the proxy models (whose parameters are exchanged) are differentially private, an adversary intercepting the parameters would not be able to learn about the individual’s data, preserving privacy.

Dwork, C., McSherry, F., Nissim, K., and Smith, A. Calibrating noise to sensitivity in private data analysis. In: Theory of Cryptography, 265–284, Berlin, Heidelberg (2006).

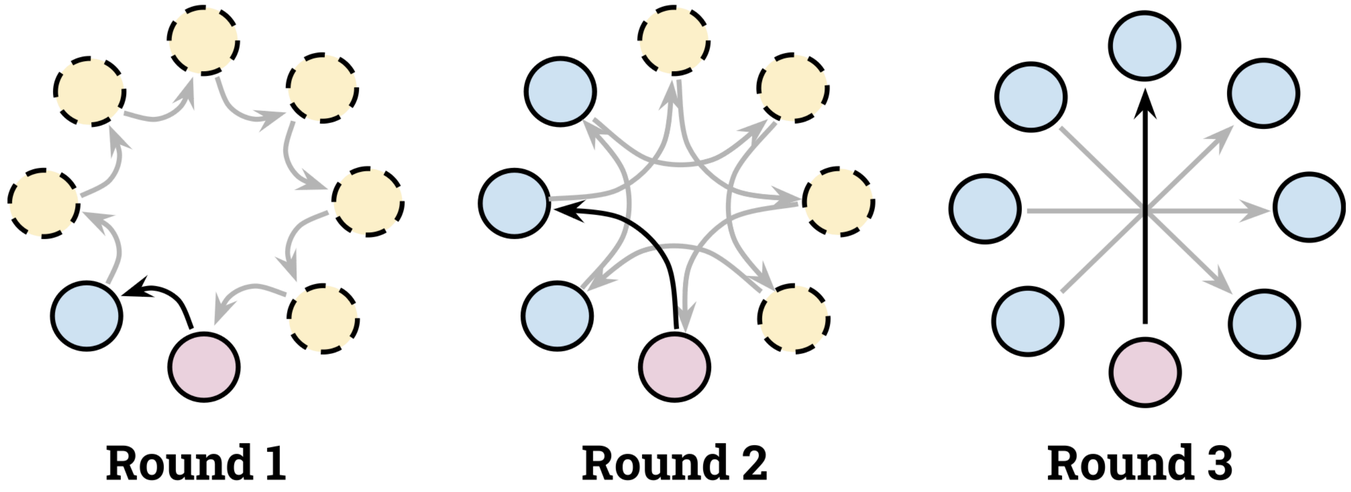
ProxyFL - methodology

- In every round of ProxyFL, each client trains its private and proxy models.
- The proxy model is trained with **differentially private training**, and is smaller than the private model.
- All parties agree on the same proxy architecture, while private architectures can be different.
- Each client sends its proxy parameters to its out-neighbours and receives new proxies from its in-neighbours according to a communication graph (mixing matrix).
- Finally, each client aggregates the proxies they received, and replaces their current proxy.



Communication Protocol[1]

- Each client communicates with its peer that is $2^0, 2^1, \ldots, 2^{\lfloor \log_2 |K| - 1 \rfloor}$ steps away periodically.
- “Exponential communication protocol”.



Assran, M., Loizou, N., Ballas, N., and Rabbat, M. Stochastic gradient push for distributed deep learning. In: International Conference on Machine Learning, 344–353. PMLR (2019).

ProxyFL - Algorithm

Require: Proxy parameters $\boldsymbol{\theta}_k^{(0)}$, private parameters $\boldsymbol{\phi}_k^{(0)}$, de-biasing weight $w_k^{(0)}$ for client k,
DML weights $\alpha, \beta \in (0,1)$, learning rate $\eta > 0$, adjacency matrix $P^{(t)}$

for *each round* $t = 0, \ldots, T-1$ *at client* $k \in \mathcal{K}$ **do**
  for *each local optimization step* **do**
    Sample mini-batch $\mathcal{B}_k = \{(\mathbf{x}_i, y_i)\}_{i=1}^{B}$ from $\mathcal{D}_k$;
    Update local proxy and private models:

$$\boldsymbol{\theta}_k^{(t)} \leftarrow \boldsymbol{\theta}_k^{(t)} - \eta \widetilde{\nabla} \widehat{\mathcal{L}}_{\boldsymbol{\theta}_k}(\mathcal{B}_k) \quad \# \text{ DP update}$$

$$\boldsymbol{\phi}_k^{(t)} \leftarrow \boldsymbol{\phi}_k^{(t)} - \eta \nabla \widehat{\mathcal{L}}_{\boldsymbol{\phi}_k}(\mathcal{B}_k) \quad \# \text{ non-DP update}$$

→ proxy model $h_{\boldsymbol{\theta}_k} : X \to Y$ with parameters $\boldsymbol{\theta}_k$

→ private model $f_{\boldsymbol{\phi}_k} : X \to Y$ with parameters $\boldsymbol{\phi}_k$

  end
  $\boldsymbol{\phi}_k^{(t+1)} \leftarrow \boldsymbol{\phi}_k^{(t)}$;
  Send $\left(P_{k',k}^{(t)} \boldsymbol{\theta}_k^{(t)}, P_{k',k}^{(t)} w_k^{(t)}\right)$ to out-neighbors;
  receive $\left(P_{k,k'}^{(t)} \boldsymbol{\theta}_{k'}^{(t)}, P_{k,k'}^{(t)} w_{k'}^{(t)}\right)$ from in-neighbors;
  Update local proxy $\boldsymbol{\theta}_k^{(t+1)} \leftarrow \sum_{k'} P_{k,k'}^{(t)} \boldsymbol{\theta}_{k'}^{(t)}$;
  Update de-biasing weight $w_k^{(t+1)} \leftarrow \sum_{k'} P_{k,k'}^{(t)} w_{k'}^{(t)}$;
  De-bias $\boldsymbol{\theta}_k^{(t+1)} \leftarrow \boldsymbol{\theta}_k^{(t+1)} / w_k^{(t+1)}$;
end
return $\boldsymbol{\theta}_k^{(T)}$, $\boldsymbol{\phi}_k^{(T)}$;

→ Nodes exchange their private model's parameters with their neighbours and aggregate them by taking a weighted average

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Q1 How do the private and the proxy model of each client interact?

Q2 How does the proxy model implement differential privacy?

Q1: How do the private and the proxy model of each client interact?

- We have seen in the algorithm that the two models update as follows:

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  where, unsurprisingly, η is the learning rate, and $\hat{\nabla}\hat{\mathscr{L}}_{\phi_k}(\mathscr{B}_k)$, $\nabla\hat{\mathscr{L}}_{\theta_k}(\mathscr{B}_k)$ are the gradients of two loss functions $\hat{\mathscr{L}}_{\phi_k}(\mathscr{B}_k)$, $\hat{\mathscr{L}}_{\theta_k}(\mathscr{B}_k)$.

- The loss function for the private model $\hat{\mathscr{L}}_{\phi_k}(\mathscr{B}_k)$ is a linear combination of two components:
  $\mathscr{L}_{\mathrm{CE}}(f_{\phi_k}) := \mathbb{E}_{(\mathbf{x},y)\sim D_k}\mathrm{CE}[f_{\phi_k}(\mathbf{x}) \,\|\, y]$: the expected value of the Cross Entropy loss
  $\mathscr{L}_{\mathrm{KL}}(f_{\phi_k}; h_{\theta_k}) := \mathbb{E}_{(\mathbf{x},y)\sim D_k}\mathrm{KL}[f_{\phi_k}(\mathbf{x}) \,\|\, h_{\theta_k}(\mathbf{x})]$: the expected value of the KL divergence

  $\mathscr{L}_{\phi_k} = (1-\alpha)\mathscr{L}_{\mathrm{CE}} + \alpha\mathscr{L}_{\mathrm{KL}}$, $\alpha \in (0,1)$. $\hat{\mathscr{L}}_{\phi_k}(\mathscr{B}_k)$ is the average on the batch $\mathscr{B}_k$

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  In this way the private model can retrieve the aggregated proxy parameters

  $\mathscr{L}_{\phi_k} = (1-\alpha)\mathscr{L}_{\mathrm{CE}} + \alpha\mathscr{L}_{\mathrm{KL}}$, $\alpha \in (0,1)$. $\hat{\mathscr{L}}_{\phi_k}(\mathscr{B}_k)$ is the average on the batch $\mathscr{B}_k$

Q1: How do the private and the proxy model of each client interact?

- Similarly, the loss function for the private model $\hat{\mathscr{L}}_{\theta_k}(\mathscr{B}_k)$ is a linear combination of two components:
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  $$\mathscr{L}_{\phi_k} = (1-\beta)\mathscr{L}_{\mathrm{CE}} + \beta\mathscr{L}_{\mathrm{KL}}, \beta \in (0,1)$$

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$$\mathscr{L}_{\phi_k} = (1-\beta)\mathscr{L}_{\mathrm{CE}} + \beta\mathscr{L}_{\mathrm{KL}}, \beta \in (0,1)$$

In this way the proxy model can encapsulate the training done at the private model

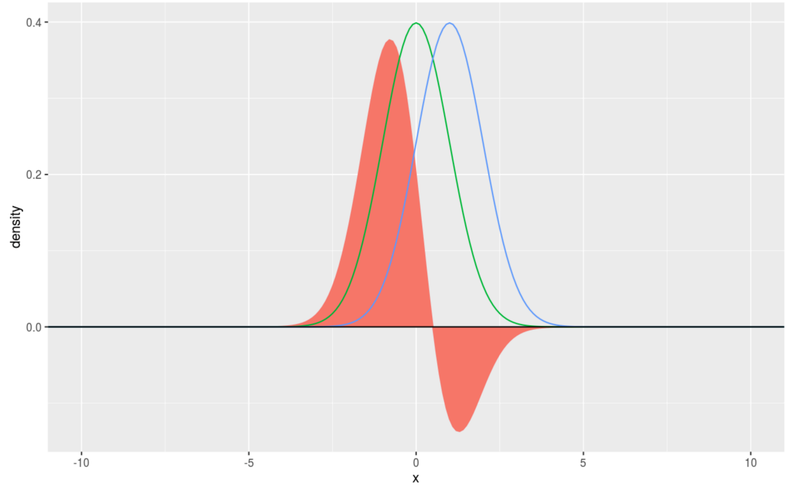
KL Divergence

- Given two distributions P and Q, the Kullback-Leibler divergence is defined as:

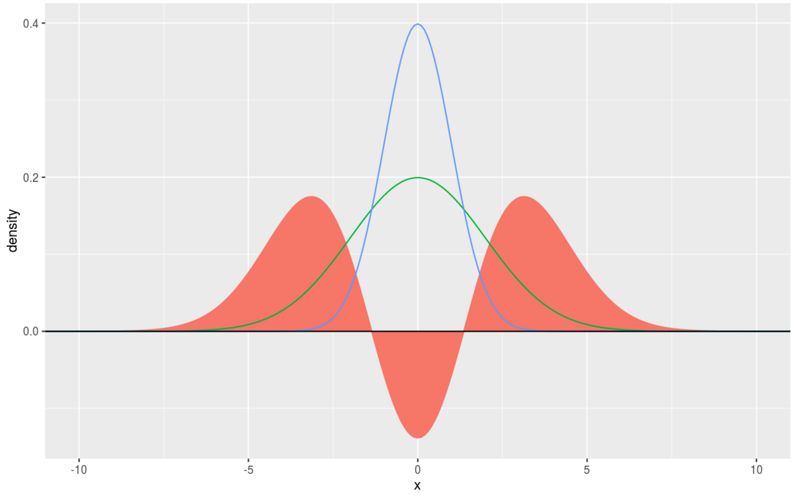
$$KL[P||Q] = \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}$$

- It is an asymmetrical measure of how different two distributions are.

- $KL[P||Q]$ says how badly the distribution Q fails to model the distribution P.



KL = 0.5



KL = 0.8069

Q2: How does the proxy model implement differential privacy?

- We have defined the loss at the proxy model $h_{\boldsymbol{\theta}_k} : X \to Y$ as:
  $\mathscr{L}^{(i)}_{\phi_k} = (1-\beta)\mathscr{L}^{(i)}_{\text{CE}} + \beta\mathscr{L}^{(i)}_{\text{KL}}$, for each data point i.
- Let's call its gradient wrt parameters θ_k $\mathbf{g}^{(i)}_{\boldsymbol{\theta}_k} := (1-\beta)\,\nabla_{\boldsymbol{\theta}_k}\text{CE}[h_{\boldsymbol{\theta}_k}(\mathbf{x}_i)\|y_i] + \beta\,\nabla_{\boldsymbol{\theta}_k}\text{KL}[f_{\boldsymbol{\phi}_k}(\mathbf{x}_i)\|h_{\boldsymbol{\theta}_k}(\mathbf{x}_i)]$
- To perform DP training for the proxy, apply:
  - Scaling: $\bar{\mathbf{g}}^{(i)}_{\boldsymbol{\theta}_k} := \dfrac{\mathbf{g}^{(i)}_{\boldsymbol{\theta}_k}}{\max\left\{1, \dfrac{\|\mathbf{g}^{(i)}_{\boldsymbol{\theta}_k}\|_2}{C}\right\}}$, where C is a scaling factor
  - Add Gaussian noise and average on the batch obtaining: :
    $\tilde{\nabla}\hat{L}_{\theta_k}(B_k) := \dfrac{1}{B}\left(\sum_{i=1}^{B}\bar{\mathbf{g}}^{(i)}_{\theta_k} + \mathcal{N}(0,\sigma^2C^2I)\right)$

Differential privacy stochastic gradient descent (DP-SGD)

- FedSL: Federated Split Learning for Collaborative Healthcare Analytics on Resource-Constrained Wearable IoMT Devices

- https://github.com/niwanli/FedSL-IoMT.git

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