Quantum Computing

Exercises for Lectures 01-04

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Recall that a complex number z in the *standard form* is $z = a + ib$, for reals a, b.

1 Lectures 01 and 02

Exercise 1

Compute $(4 + \frac{i}{2})(1 + i)$. Show that for any complex z, w the following is true

$$zw = wz$$

that is, complex multiplication is *commutative*. (Answer: $\frac{7}{2} + \frac{9i}{2}$.)

Exercise 2

Express the following fraction in standard form:

$$\frac{3 + 7i}{2 + 5i}$$

(Answer: $\frac{1}{29}(41 - i)$.)

Exercise 3

Compute $\text{Re}(3 + 7i)$ and $\text{Im}(2 + 5i)$. Show that for any complex z the following is true:

$$z + \overline{z} = 2\,\text{Re}(z) \quad \text{and} \quad z - \overline{z} = 2i\,\text{Im}(z)$$

Exercise 4

Compute $|3+7i|$ and $|\sqrt{2}+i\sqrt{7}|$. Show that for any complex z, w the following is true

$$|zw| = |z||w| \quad \text{and} \quad |z| = \sqrt{z\overline{z}}$$

Exercise 5

Show that

$$e^{i\frac{\pi}{2}} = i \quad \text{and} \quad e^{i\pi} = -1$$

Exercise 6

Show that

$$e^{(\log\left(\frac{2}{\sqrt{2}}\right)+i\frac{\pi}{4})} = 1+i$$

(Hint: use the laws of powers.)

Exercise 7

Compute $|e^{i\frac{\pi}{2}}|$ and $|e^{i\pi}|$.

Exercise 8

Show that for any real r, we have $|e^{ir}| = 1$.

Exercise 9

Let n be any natural number and z any complex number. Using polar coordinates for z, show that $|z^n| = |z|^n$.

Exercise 10

Let $w = se^{i\phi}$ for $s \geqslant 0$ and $\phi \in \mathbb{R}$. Solve the equation $z^n = w$ in $\mathbb{C}$ where n is a natural number. Without using the Fundamental Theorem of Algebra, how many solutions are there? [Hint: use polar coordinates.]

Exercise 11

Show that $\|(-i, e^{i\frac{\pi}{2}}, i, 1)\| = 2$.

Exercise 12

Given the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the vectors $|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, verify that

$$\begin{aligned} \sigma_y|0\rangle &= i|1\rangle & \qquad \sigma_y|1\rangle &= -i|0\rangle \\ \sigma_z|0\rangle &= |0\rangle & \qquad \sigma_z|1\rangle &= -|1\rangle \ . \end{aligned}$$

Compute $\sigma_x^2, \sigma_y^2, \sigma_z^2$.

Exercise 13

Let the Hadamard matrix H be

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

Verify that $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute $H|1\rangle$. Give a *single* expression that describes the action of H on a basis state $|a\rangle$, where $a \in \{0, 1\}$.

Exercise 14

Recall that the norm of a vector $v = (\alpha|0\rangle + \beta|1\rangle) = \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ is $\|v\| = \sqrt{|\alpha|^2 + |\beta|^2}$, a linear transformation (*i.e.*, a matrix) T is unitary if and only if $\|Tv\| = \|v\|$ and that a matrix is self-adjoint if $T = T^\dagger$. Verify that the Pauli and Hadamard matrices are unitary and self-adjoint.

Exercise 15

Picture in the Bloch sphere the vectors $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$.

Exercise 16

Compute the eigenvalues of the Pauli and the Hadamard matrices.

Exercise 17

Show that any eigenvalue of a unitary operator is a complex number of modulus 1.

2 Lecture 03 and 04

Exercise 18

Verify that the tensor product of vectors is *not* commutative by computing, for example, $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |0\rangle$.

Exercise 19

Let $q = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute the tensor product $q \otimes q \otimes q$ and its norm (it should be 1).

Exercise 20

Consider measuring a qubit in the state $\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$: what are the probabilities of obtaining $|0\rangle$ and $|1\rangle$? Do they sum to 1?

Exercise 21

Given two qubits in the state $\left[\frac{1}{4}(|00\rangle + |01\rangle) + i\frac{\sqrt{28}}{8}(|10\rangle + |11\rangle)\right]$ compute:

- the probability of each of the four 2-qubit basis states and verify that their sum is 1;
- the probability that the first qubit (starting from left) is $|0\rangle$;
- the probability that the two qubits are anticorrelated (*e.g.*, one $|0\rangle$ and the other $|1\rangle$).

Exercise 22

Given two qubits in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, which basis states are measurable with non-zero probability? What is it?

Exercise 23

Apply the CNOT gate to two qubits in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$: what is the resulting state? Is it entangled?

Exercise 24

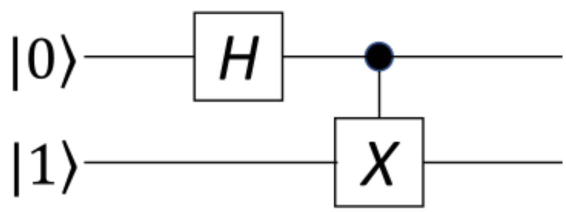
Compute CNOT$(\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle))$.

Exercise 25

Let I_n be the $n \times n$ identity matrix. Verify that $I_2 \otimes I_2 = I_4$, that is, the tensor product of the 2×2 identity matrices is the 4×4 identity matrix.

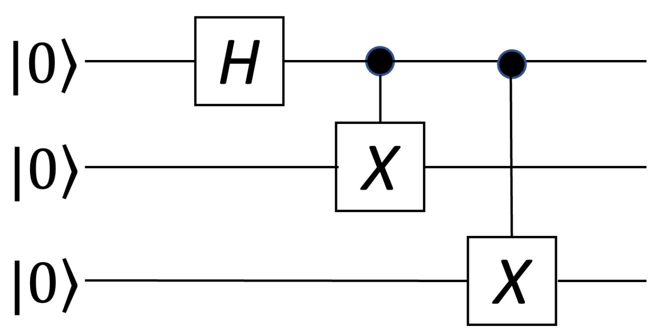
Exercise 26

Verify that the circuit below produces the state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$:



Exercise 27

Compute the output of the following three-qubit circuit:



Exercise 28

Compute the state $(H \otimes H \otimes H)|000\rangle$, where H is the Hadamard matrix. (Recall that $|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$.)

Exercise 29

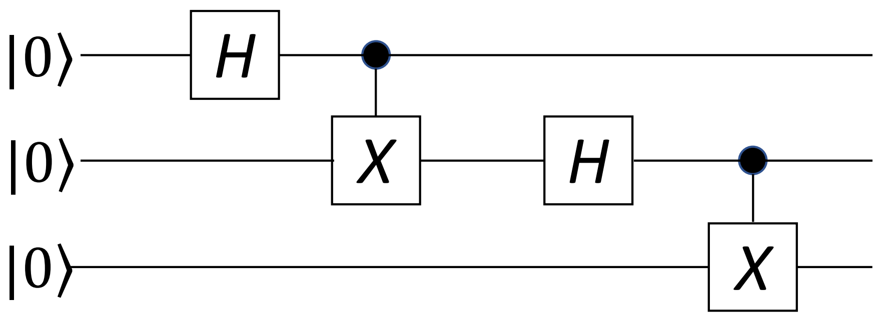
Let $\{q_1, \ldots, q_n\}$ be qubits, *i.e.*, $\|q_i\| = 1$ for all i. Show that

$$\left\| \bigotimes_{i=1}^{n} q_i \right\| = 1$$

i.e., their tensor product is a vector of norm 1. How many complex numbers are needed to describe the product?

Exercise 30

Compute the output of the following three-qubit circuit:



[Recall that for $a, b \in \{0, 1\}$, $H(|a\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^a|1\rangle)$ and CNOT$(|a \otimes b\rangle) = |a \otimes (a \oplus b)\rangle$ (*i.e.*, CNOT flips the left-hand side qubit if the first-hand side is 1.]