Quantum Computing: Solutions to (some) Exercises

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0) Verify that the tensor product is *not* commutative by computing, for example, $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |0\rangle$.

Solution: We have that

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

while

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

thus $|01\rangle \neq |10\rangle$.

1) Let q be the quantum bit state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Compute the tensor product $q \otimes q \otimes q$ and its norm (it should be 1).

Solution:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) =$$
$$\frac{1}{2\sqrt{2}}[(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)] =$$
$$\frac{1}{2\sqrt{2}}[(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes (|0\rangle + |1\rangle)] =$$
$$\frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle) + |110\rangle + |111\rangle)$$

where as usual we have omitted the tensor sign $\otimes$ in the basis states (*e.g.*, $|010\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle$).

The norm of the product vector is $\sqrt{8 \cdot (\frac{1}{2\sqrt{2}})^2} = \sqrt{8 \cdot \frac{1}{4 \cdot 2}} = 1.$

2) Consider measuring a qubit in the state $\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$: what are the probabilities of obtaining $|0\rangle$ and $|1\rangle$? Do they sum to 1?

Solution:

Prob("measure $|0\rangle$") $= (\frac{1}{3})^2 = \frac{1}{9}$
Prob("measure $|1\rangle$") $= (\frac{\sqrt{8}}{3})^2 = \frac{8}{9}$
Their sum is 1.

3) Consider two qubits in the state $\left[\frac{1}{4}(|00\rangle + |01\rangle) + i\frac{\sqrt{28}}{8}(|10\rangle + |11\rangle)\right]$. They are now measured. Compute:

a. the probability of each of the four 2-qubit basis states and verify that their sum is 1;

b. the probability that the first qubit (starting from left) is $|0\rangle$;

c. the probability that the two qubits are anticorrelated (*e.g.*, one $|0\rangle$ and the other $|1\rangle$).

Solution:

a. Prob("measure $|00\rangle$") = Prob("measure $|01\rangle$") $= (\frac{1}{4})^2 = \frac{1}{16}$, while Prob("measure $|10\rangle$") = Prob("measure $|11\rangle$") $= (\frac{\sqrt{28}}{8})^2 = \frac{28}{64}$. Their sum is 1.

b. Prob("measure $|00\rangle$" or "measure $|01\rangle$") $= 2 \cdot \frac{1}{16} = \frac{1}{8}$

c. Prob("measure $|01\rangle$" or "measure $|10\rangle$") $= \frac{1}{16} + \frac{28}{64} = \frac{1}{2}$

4) Consider two qubits in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (this is an entangled state). Which basis states are measurable with non-zero probability? What is it?

Solution: Only $|00\rangle$ and $|11\rangle$ are measurable with non-zero probability, which is $\frac{1}{2}$ for both.

5) Apply the CNOT gate to two qubits in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$: what is the resulting state? Is it entangled?

Solution:

$$
\begin{aligned}
&\mathsf{CNOT}(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)) = \frac{1}{\sqrt{2}}\mathsf{CNOT}(|00\rangle + |11\rangle) = \\
&\frac{1}{\sqrt{2}}[\mathsf{CNOT}(|00\rangle) + \mathsf{CNOT}(|11\rangle)] = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \\
&\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle) \otimes |0\rangle]
\end{aligned}
$$

This is not entangled since it is the tensor product of two single-qubit states.

6) Compute $\mathsf{CNOT}(\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle))$.

Solution:

$$
\begin{aligned}
&\mathsf{CNOT}(\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)) = \\
&\frac{1}{2}[\mathsf{CNOT}(|00\rangle) + \mathsf{CNOT}(|01\rangle) + \mathsf{CNOT}(|10\rangle) + \mathsf{CNOT}(|11\rangle)] = \\
&\frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}
$$

7) Let I_n be the $n \times n$ identity matrix. Verify that $I_2 \otimes I_2 = I_4$, that is, the tensor product of the 2×2 identity matrices is the 4×4 identity matrix.

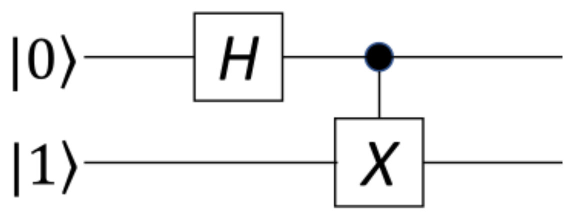
Solution: Recall that:

$$
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

Now:

$$I_2 \otimes I_2 = \begin{pmatrix} 1 \cdot I_2 & 0 \cdot I_2 \\ 0 \cdot I_2 & 1 \cdot I_2 \end{pmatrix} = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4$$

8) Verify that the circuit

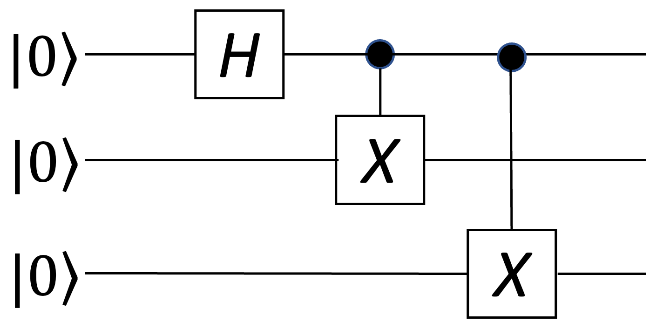


results in the state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

Solution: Let us denote by q_1 and q_2 the two qubits so that the initial state is $q_1 \otimes q_2 = |01\rangle$. We now proceed:

$$\begin{aligned}
&|0 \otimes 1\rangle && \text{apply } H \text{ on } q_1 \\
&\Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle && \text{refactoring} \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) && \text{apply CNOT}(q_1, q_2) \\
&\Rightarrow \text{CNOT}(\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)) && \text{linearity of CNOT} \\
&= \frac{1}{\sqrt{2}}(\text{CNOT}(|01\rangle) + \text{CNOT}(|11\rangle)) && \text{definition of CNOT} \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)
\end{aligned}$$

9) Compute the output of the following three-qubit circuit:



Solution: Let us denote by q_1, q_2 and q_3 the three qubits, from top to bottom. Note that the rightmost CNOT gate in the circuit acts on qubits q_1 and q_3.

$$\begin{aligned}
& q_1 \otimes q_2 \otimes q_3 = |0 \otimes 0 \otimes 0\rangle && \text{apply } H \text{ on } q_1 \\
& \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \otimes |0\rangle && \text{refactoring} \\
& = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \otimes |0\rangle && \text{apply CNOT}(q_1, q_2) \\
& \Rightarrow \text{CNOT}\Big(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\Big) \otimes |0\rangle && \text{linearity of CNOT} \\
& = \frac{1}{\sqrt{2}}(\text{CNOT}(|00\rangle) + \text{CNOT}(|10\rangle)) \otimes |0\rangle && \text{definition of CNOT} \\
& = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle && \text{refactoring} \\
& = \frac{1}{\sqrt{2}}(|\underline{0}0\underline{0}\rangle + |\underline{1}1\underline{0}\rangle) && \text{apply CNOT}(q_1, q_3)\text{; } q_2 \text{ is left untouched} \\
& \Rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)
\end{aligned}$$

This is the Greenberger–Horne–Zeilinger (GHZ) state. It is clearly entangled.