Quantum Computing

Exercises for Lecture 01-05

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Exercise 1

An operator T on a Hilbert space $\mathcal{H}$ is unitary iff $TT^\dagger = T^\dagger T = I$. Show that T is unitary iff:

1. T is bijective, and
2. $\forall u, v \in \mathcal{H}, \langle Tu|Tv\rangle = \langle u|v\rangle$.

Exercise 2

Given a vector space V and a basis set $S = \{e_1, \ldots, e_n\}$ such that $[S] = V$, prove that any $v \in V$ can be written as:

$$v = \sum_{i=1}^{n} \alpha_i e_i$$

where the coefficients α_i's are complex. Also, prove that the α_i's are unique (wrt a basis set). [Hint: start by noticing that the vectors $\{e_1, \ldots, e_n, v\}$ are linearly dependent.]

Exercise 3

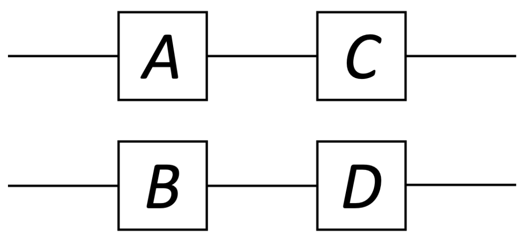
Prove the triangular inequality.

Exercise 4

Let H be the Hadamard matrix. Using the definition of unitary transformation via adjoint given in the lecture notes, verify that $H \otimes H \otimes H$ is unitary.
[Hints:

1. for any two matrices M_1, M_2 we have $(M_1 \otimes M_2)^\dagger = M_1^\dagger \otimes M_2^\dagger$. (This can be proved by writing down (the elements of) the matrix corresponding to $M_1 \otimes M_2$ and then apply the adjoint operation to it.)

2. if the dimensions of matrices A, B, C, and D are such that the products CA and BD are possible, then $(C \otimes D)(A \otimes B) = CA \otimes DB$. (It is easy to understand why this is true: if A, B, C, and D are single-qubit quantum gates applied in the circuit below then the action of the gates



A, B on the two qubits is described by $A \otimes B$. Next, the action of C, D on the circuit is described by $C \otimes D$. The action of the overall circuit can be then described as the sequential composition of $A \otimes B$ followed by $C \otimes D$, which is $(C \otimes D)(A \otimes B)$.)

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Exercise 5

Let V be a complex vector space. Show that for any $v, w \in V$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\langle \alpha u + \beta v, w \rangle = \alpha^* \langle u, w \rangle + \beta^* \langle v, w \rangle.$$

Exercise 6

If $\lambda \in \mathbb{C}$ is a non-degenerate eigenvalue of operator A, with notation abuse we write $A \left|\lambda\right\rangle = \lambda \left|\lambda\right\rangle$. Show that

$$\left\langle\lambda\right| A^\dagger = \lambda^* \left\langle\lambda\right| .$$

Exercise 7

Let W be a closed subspace an Hilbert space $\mathcal{H}$, and f_i be an orthonormal basis for W. The projection v_W of $v \in \mathcal{H}$ on W is defined as

$$v_W = \sum_i \langle f_i | v \rangle \, f_i.$$

Show that v_W does not depend on the choice of basis f_i.

Exercise 8

If P and Q are two orthogonal projectors, show that $P + Q$ is also a projector.

Exercise 9

Given a self-adjoint operator, show that

- Its eigenvalues are real numbers;
- Eigenvectors associated to different eigenvalues are orthogonal.

Exercise 10

Let A be a self-adjoint operator with eigenvalues $\lambda_1, \ldots, \lambda_m$. Show that

$$A = \sum_{i=1}^{m} \lambda_i P_i$$

where P_i is the projector on the eigenspace of λ_i.

Exercise 11

Let A be a self-adjoint operator and $f : \mathbb{R} \to \mathbb{R}$. Define the operator $f(A)$ as:

$$f(A) = \sum_{j=1}^{m} f(\lambda_j) P_j$$

where $\{\lambda_1, \ldots, \lambda_m\}$ are the eigenvalues of A and P_j is the projector on the j-th eigenspace. Prove that $f(A)$ is self-adjoint.

Exercise 12

Let A be a self-adjoint operator and $f : \mathbb{R} \to \mathbb{C}$. Define the operator $f(A)$ as:

$$f(A) = \sum_{j=1}^{m} f(\lambda_j) P_j$$

where $\{\lambda_1, \dots, \lambda_m\}$ are the eigenvalues of A and P_j is the projector on the j-th eigenspace. Prove that e^{iA} is unitary (and therefore any solution of the Schrödinger equation is unitary - see the notes for Lecture 05).