

Quantum Computing

Lecture |03⟩

Systems with Multiple Qubits

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Outline

- Basis vectors
- Tensor products
- Entanglement
- Two-Qubit operations



Basis Vectors

Any vector can be “decomposed” as a sum of ‘basis’ vectors:

$$\begin{pmatrix} 3 \\ -2 \\ 1+i \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1+i) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This holds for complex vectors of ***any size*** (even infinite, with some caveats)!

Basis Vectors

In general, we write

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^n \alpha_i e_i$$

e_i is the vector that has 1 in the i -th position and 0 elsewhere.

The e_i 's are called **(diagonal) basis vectors**, and typically correspond to *classical* states.

Multiple Bits

- It is easy to describe *n classical* bits: just list them in a vector/array of *n* elements!
- Formally: *Cartesian* product (René Descartes, 1596-1650)

$$\mathcal{B} = \{0,1\}$$

$$\mathcal{B} \times \mathcal{B} = \{(a, b): a \in \mathcal{B}, b \in \mathcal{B}\} \quad (\text{similarly for } n \text{ copies of } \mathcal{B})$$

$$s = (b_1, b_2, \dots, b_n) \in \mathcal{B}^n$$

- **Note:**
 - The length of the vector grows linearly with *n*
 - The vector components are independent from each other (you can change a component without affecting the others)



Multiple Qubits

To describe n *quantum* bits we need the
tensor product



Tensor Products

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A qubit is $\alpha_0|0\rangle + \alpha_1|1\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$

$$\begin{aligned} \text{Two qubits: } \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} &= \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} \\ &= \alpha_0\beta_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_0\beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1\beta_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_1\beta_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \end{aligned}$$

Tensor Products

- Easily extended to any number of qubits
- The size of the resulting vector grows exponentially with the qubits number!

$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$ is a valid quantum state, *i.e.*, $\sum_{i,j \in \mathcal{B}} |\alpha_i \beta_j|^2 = 1$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Multiple Qubits

$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ is a superposition of all 2-bit states

- Built by juxtaposing two qubits in superposition
- Even though the state looks 'complicate', it can be decomposed in two valid qubit states!
- However, ...

Multiple Qubits

- Quantum mechanics tells us that *any*
 $\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$ with $\sum_i |\alpha_i|^2 = 1$
is a valid quantum state for a 2-qubit system!

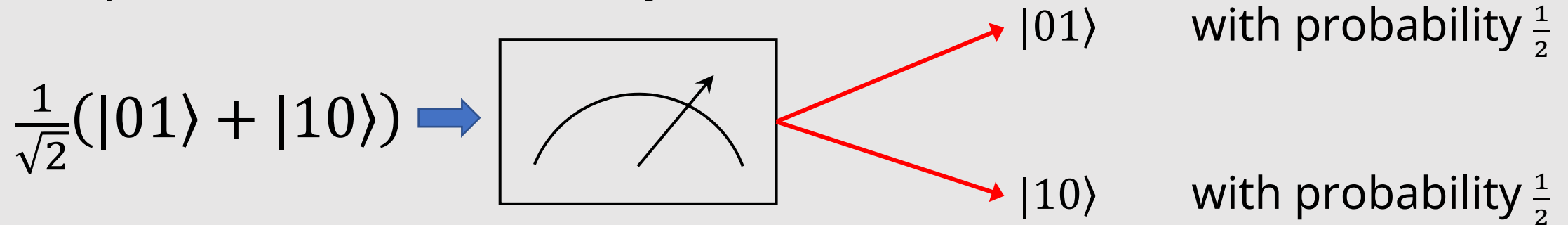
- *Example:*

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Can you separate this 2-qubit state into two, independent qubit states?

Entanglement

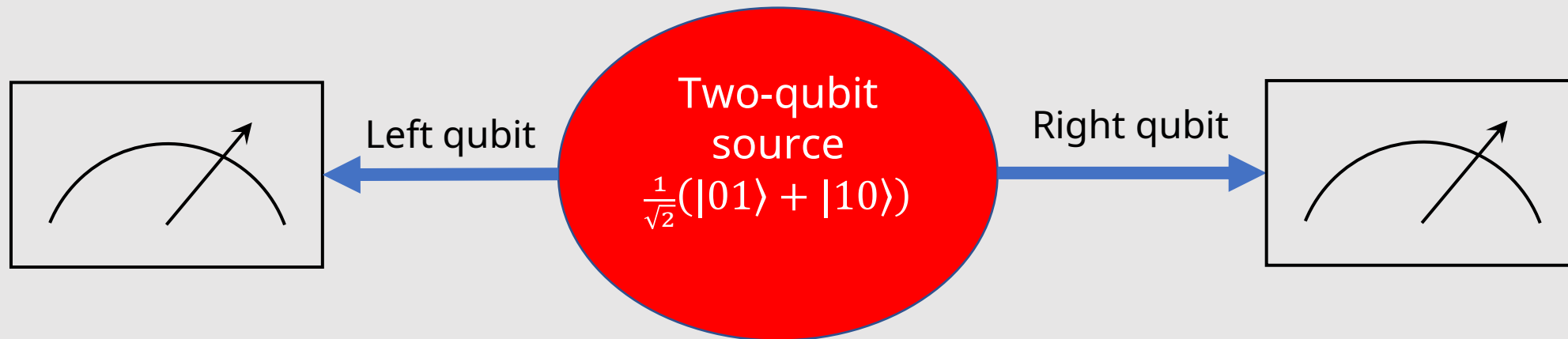
- NO! This state is **entangled**.
- There is no way to describe it as two 'local' qubit states: it describes the system *as a whole*.
- *EPR* pair (Einstein, Podolsky, Rosen. 1935)



- Measuring either qubit will tell us about the other!!
- [That won't be the case with, *e.g.*, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|0\rangle)$]

Entanglement: EPR 'Paradox'

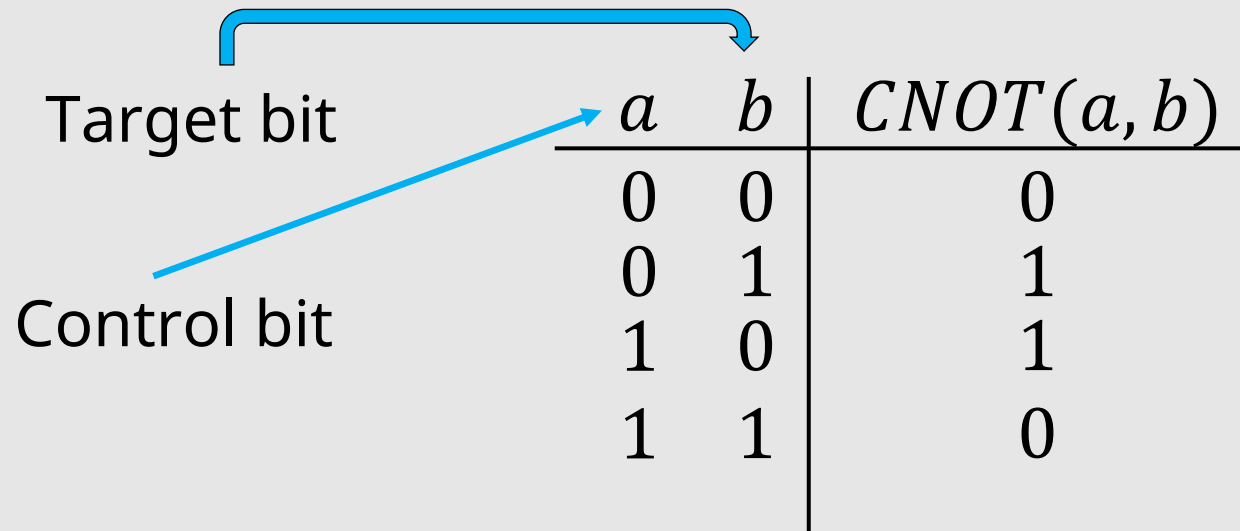
- But wait! Suppose now the two qubits 'fly' apart



- The two qubits could end up being light-years apart and still maintain their anticorrelation!
 - Einstein: "spooky actions at distance"
 - *Experimentally validated* on distances of $\sim 1,000$ km
 - 2022 Nobel Prize in Physics to Aspect, Clauser, Zeilinger ([talk](#) by Aspect)

Multiple Qubits: Operations

- The Controlled-NOT (*CNOT*) operates on two qubits
- Classically: $CNOT(a,b) = \text{if } a \text{ then } b := \text{NOT}(b) \text{ else skip}$



The diagram illustrates the CNOT operation. A blue arrow points from the 'Control bit' label to the 'a' column of the truth table. Another blue arrow points from the 'Target bit' label to the 'b' column. A third blue arrow points from the 'Control bit' label to the 'a' column. The truth table shows the output of the CNOT operation for all combinations of input bits a and b.

	<i>a</i>	<i>b</i>	$CNOT(a, b)$
	0	0	0
	0	1	1
	1	0	1
	1	1	0

Quantum CNOT

inputs		outputs	
a	b	a	$CNOT(a, b)$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

$$CNOT(|00\rangle) = |00\rangle$$

$$CNOT(|01\rangle) = |01\rangle$$

$$CNOT(|10\rangle) = |11\rangle$$

$$CNOT(|11\rangle) = |10\rangle$$

Xor

$$CNOT(|a, b\rangle) = |a, a \oplus b\rangle$$

$$CNOT \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum CNOT

The quantum CNOT is a linear operator, thus

$$CNOT(\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)) = \frac{1}{\sqrt{2}}[CNOT(|00\rangle) + CNOT(|01\rangle)] = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

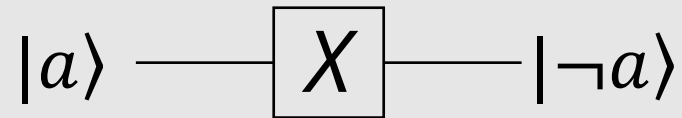
$$CNOT(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle) = CNOT(\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

EPR pair!!!

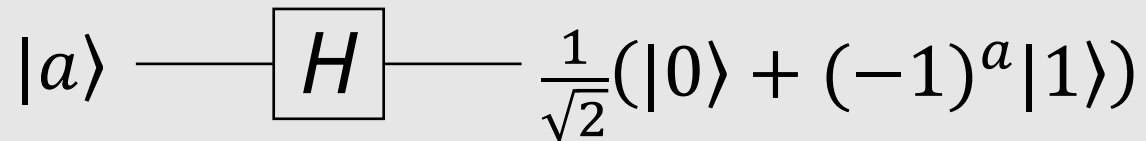
$$CNOT\left(\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\right) = \dots$$

Notation

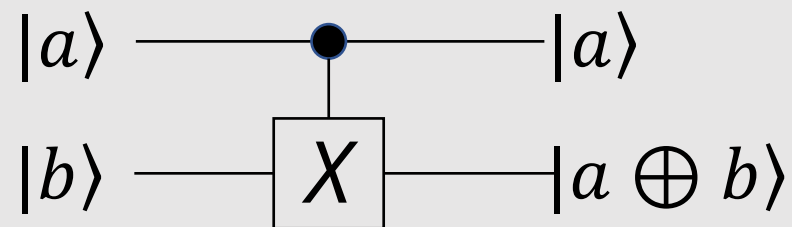
We 'program' using gates and circuits!!



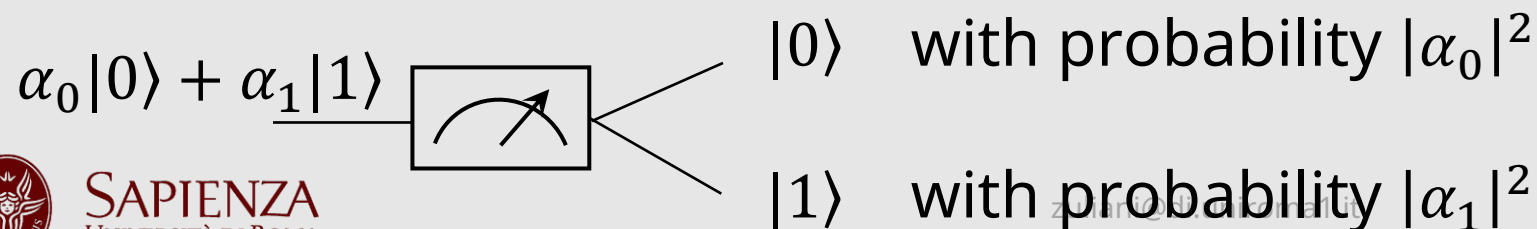
NOT gate



Hadamard gate

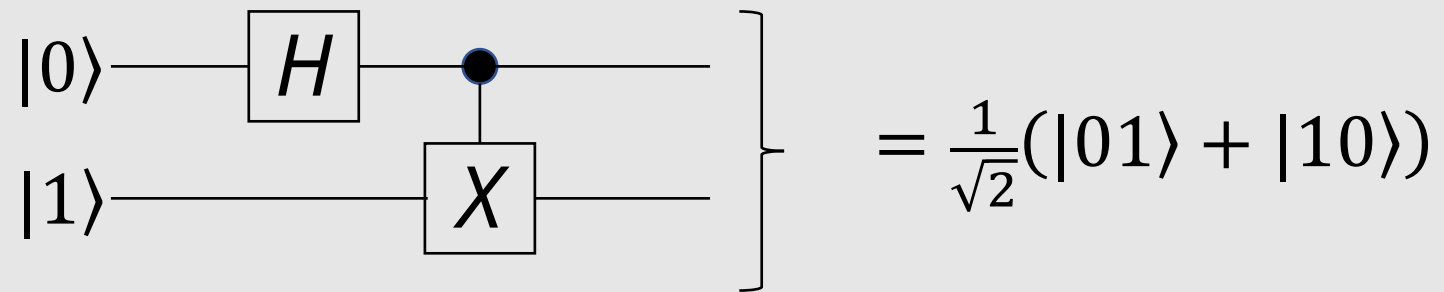


CNOT gate

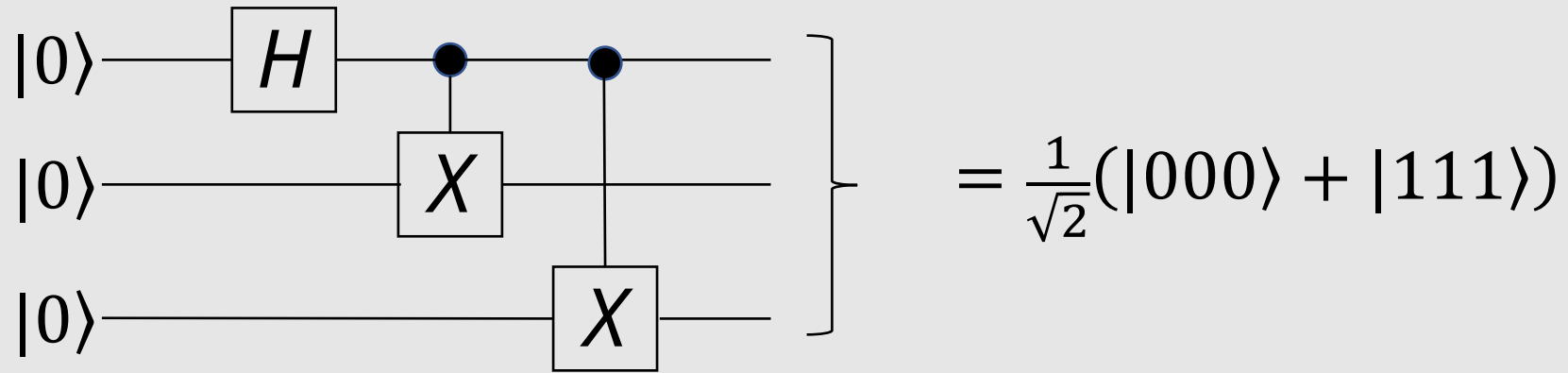


Measurement gate

How to Create EPR Pairs



The GHZ State



GHZ = Greenberger, Horne, Zeilinger

Tensor Product of (Unitary) Gates

$$\left. \begin{array}{l} |0\rangle \text{---} [H] \text{---} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |0\rangle \text{---} [H] \text{---} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \end{array} \right\} \otimes = \frac{1}{2}(|00\rangle + \dots)$$

Take two 2x2 matrices A, B

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B \\ a_{2,1}B & a_{2,2}B \end{pmatrix} \quad \text{is a 4x4 matrix}$$

Tensor Product of (Unitary) Gates

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1H & 1H \\ 1H & -1H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ (check this!)}$$