# An Introduction to Quantum Computing

## Lecture 04

*Cats, No-Cloning, and Quantum Teleportation*

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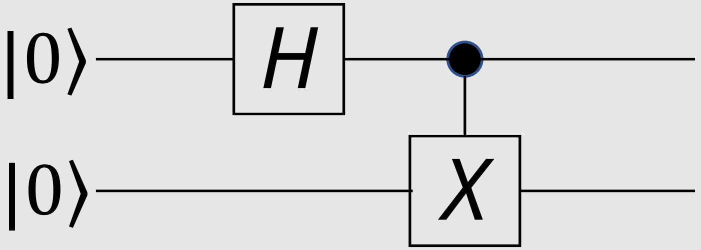
# Outline

- Half-dead cats?
- The No-cloning Theorem
- Quantum Teleportation

# Half-dead Cats

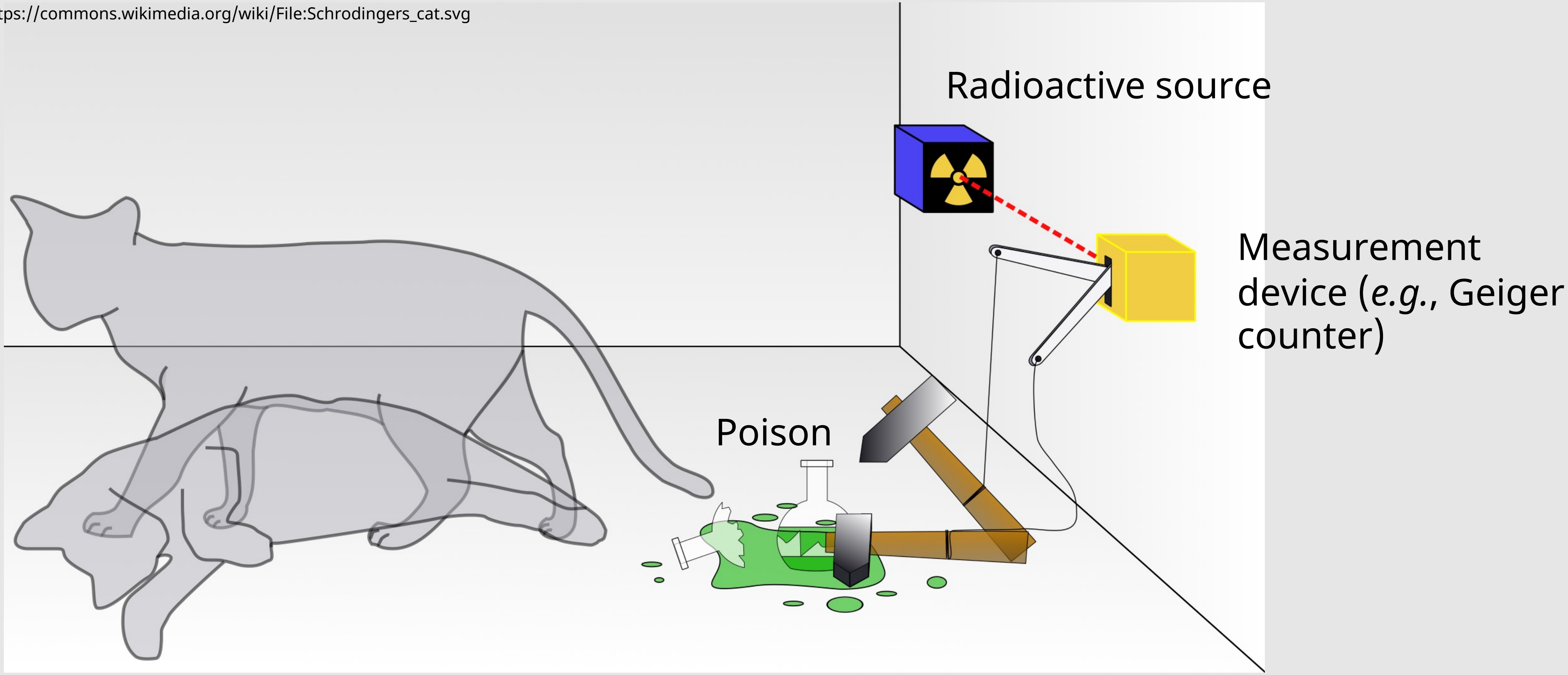
- Or: quantum mechanics cannot be (easily) applied to macroscopic objects
- Entangled states are (one of the) sources of the problem




$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

# Schrödinger’s Cat

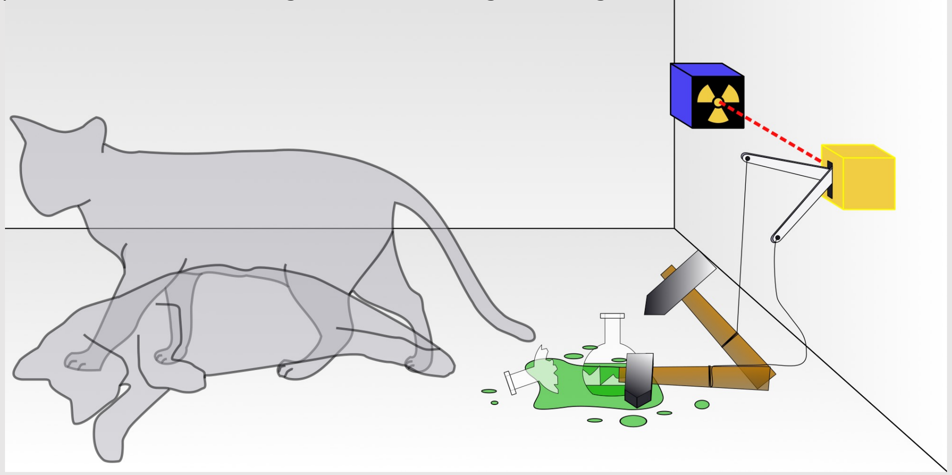
https://commons.wikimedia.org/wiki/File:Schrodingers_cat.svg



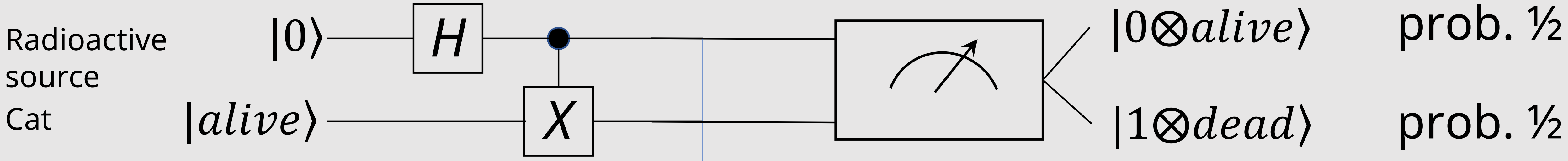

Is the cat dead or alive?

# Schrödinger's Cat

https://commons.wikimedia.org/wiki/File:Schrodingers_cat.svg



- The radioactive source emits particles described by a qubit
- The cat's state is a qubit, with basis states 'alive' ($|0\rangle$) and 'dead' ($|1\rangle$)



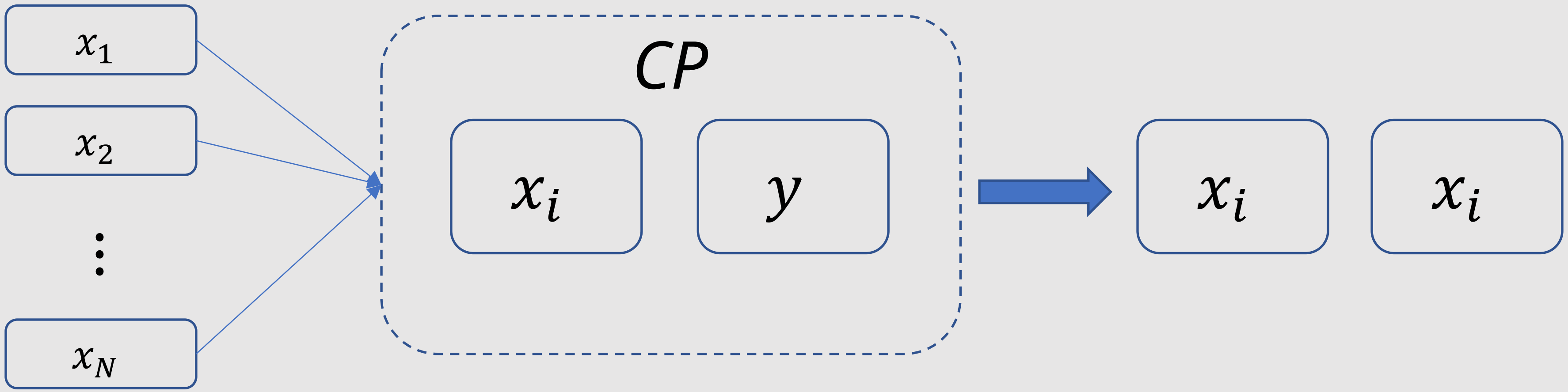

The state of the system at this point is:

$$\frac{1}{\sqrt{2}}(|0\otimes alive\rangle + |1\otimes dead\rangle)$$

# No Cloning (No Copy)

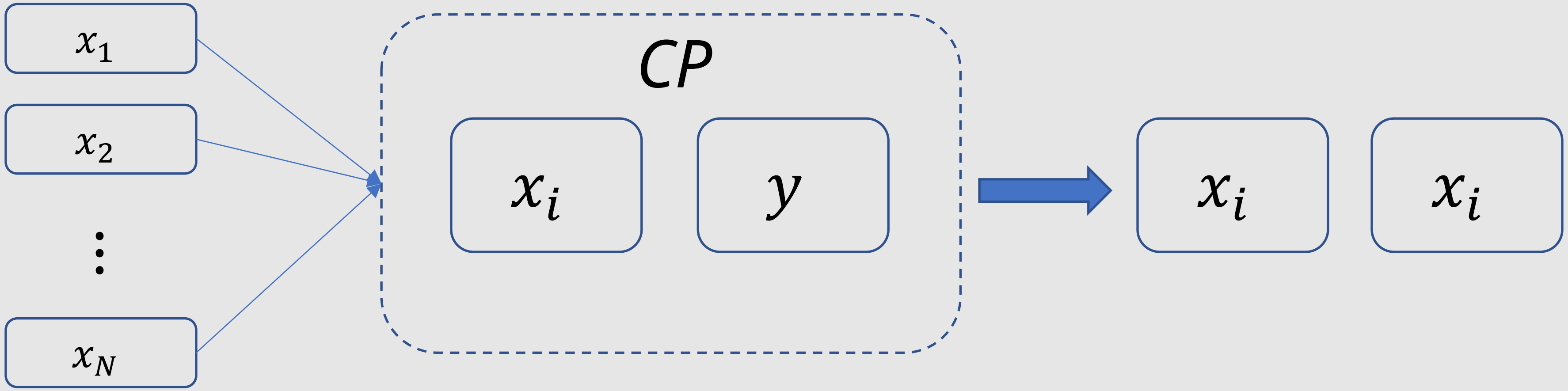
*A quantum state cannot be perfectly copied (cloned)*
[Wooters and Zurek, 1982; Dieks, 1982]

A putative copy operation/transformation *CP* is something like:




*CP* takes a 'source' variable in input and copies its contents to the 'destination' variable, leaving the source untouched.

# No Cloning (No Copy)




Formally, we write

$$\exists y\ \forall x\ CP(x, y) = (x, x)$$

“there exists a $y$ such that for all $x$ $CP(x, y) = (x, x)$”

# No Cloning (No Copy)

In a quantum world, a putative copy *CP* should satisfy

$$\exists y \quad \forall x \quad CP(x \otimes y) = (x \otimes x)$$

However, this doesn't work!

Proof by contradiction.

# No Cloning (No Copy)

$\exists y\ \ \forall x\ \ \ CP(x \otimes y) = (x \otimes x)$

thus, since *CP* should work on superposed states, too

$\exists y\ \ \forall x, a\ \ \ CP\big((x + a) \otimes y\big) = (x + a) \otimes (x + a)$

$\exists y\ \ \forall x, a\ \ \ CP(x \otimes y + a \otimes y) = (x + a) \otimes (x + a)$

thus, since *CP* must be unitary hence linear

$\exists y\ \ \forall x, a\ \ \ CP(x \otimes y) + CP(a \otimes y) = (x + a) \otimes (x + a)$

thus, since *CP* copies its input

$\forall x, a\ \ \ (x \otimes x) + (a \otimes a) = (x + a) \otimes (x + a)$

$\forall x, a\ \ \ (x \otimes x) + (a \otimes a) = (x \otimes x) + (x \otimes a) + (a \otimes x) + (a \otimes a)$

**FALSE** (try for example $x = |0\rangle\ \ a = |1\rangle$)

# Summing Up

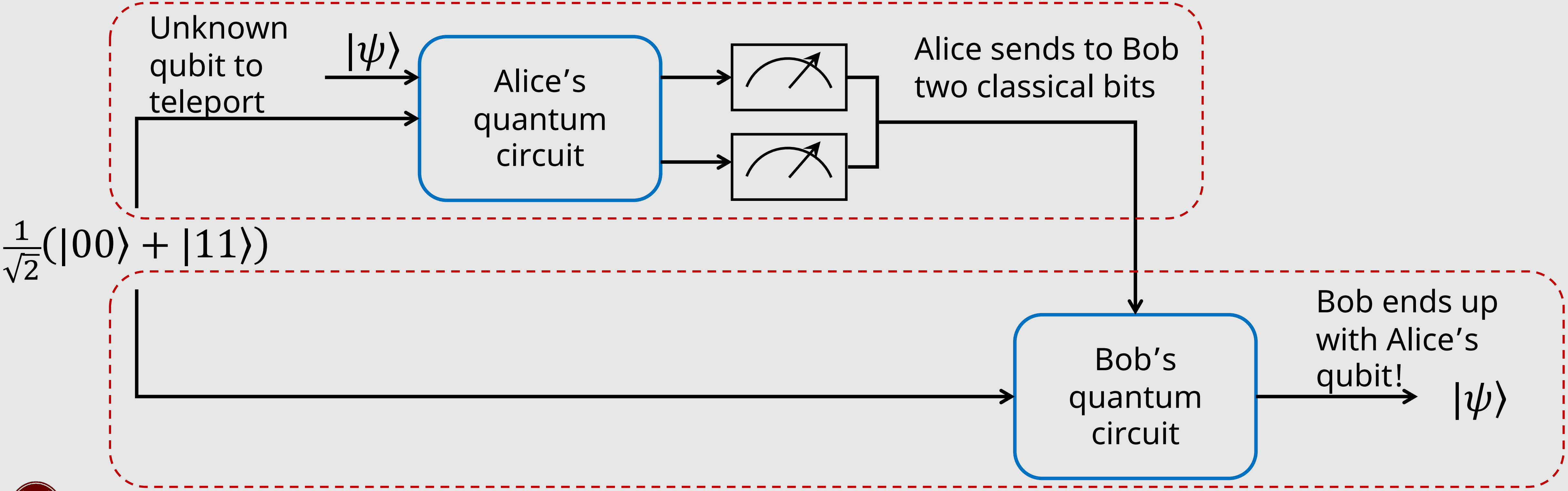
- While we cannot clone perfectly, it has been shown that *imperfect cloning* (*i.e.*, with high probability) is possible!
  - The quantum internet is being built!

- Quantum mechanics cannot be straightforwardly applied to ‘big’ (macroscopic) objects
  - Physicists are very busy working out a *quantum theory of gravity*!
- Next: We can *teleport* a quantum state *perfectly*!

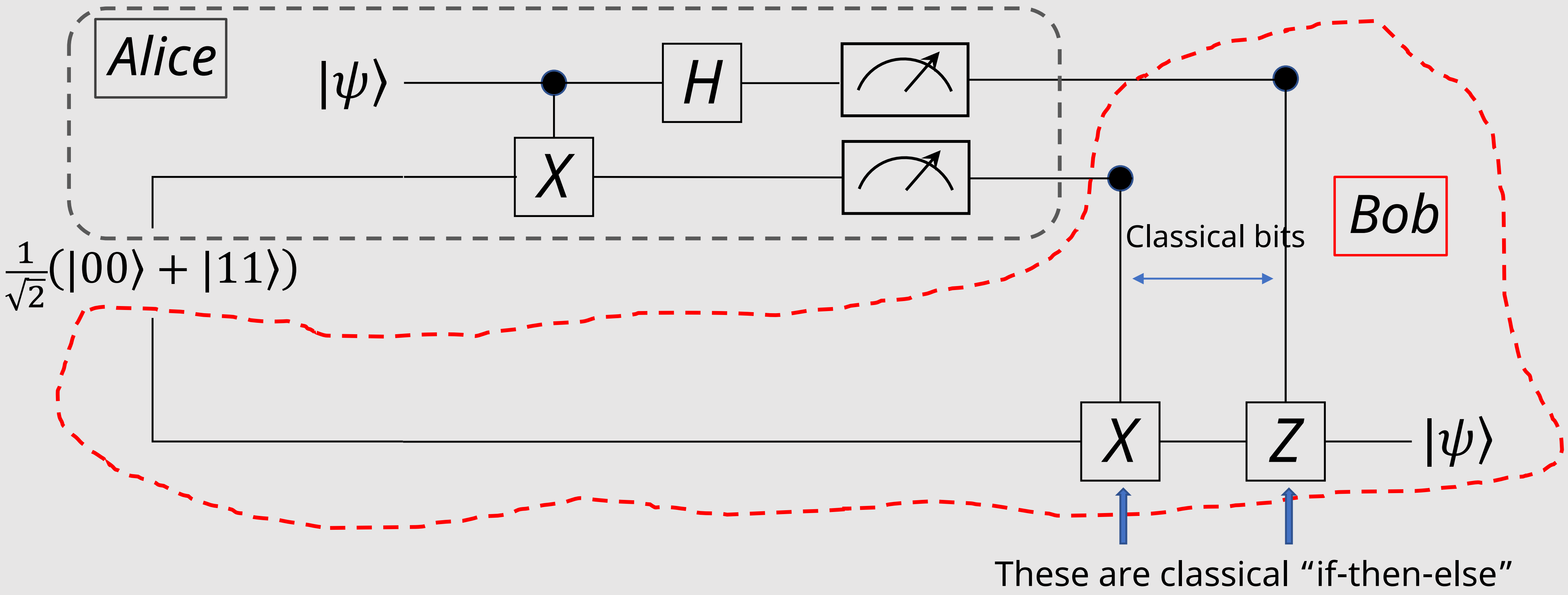
# Quantum Teleportation

[Bennett, Brassard, Crépeau, Jozsa, Peres, Wootters. 1993]

- Two parties, Alice and Bob, are separated by a large distance
- Alice has a qubit she would like to send to Bob
- Teleportation achieve this by using **two qubits** and **two classical bits**

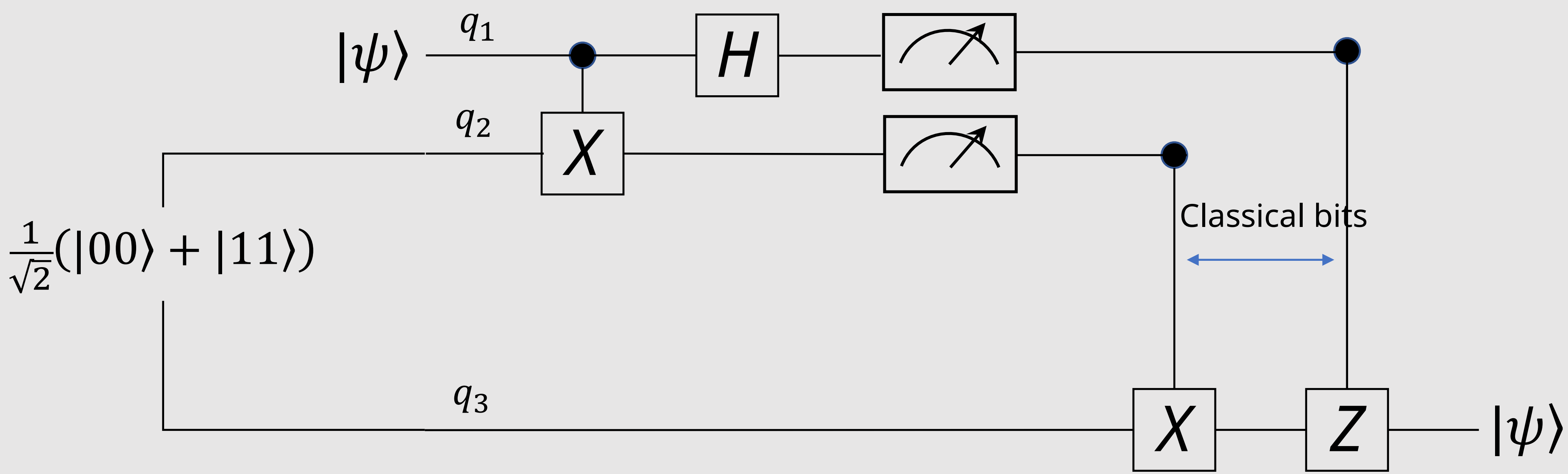


# Quantum Teleportation Circuit




Hence, to teleport one qubit Alice needs to send <u>one qubit</u> and <u>two classical bits</u>

# Quantum Teleportation Circuit




Using a “programming” notation:

$q_1, q_2, q_3 = |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ // this means $|\psi\rangle \otimes |\frac{1}{\sqrt{2}}(|0 \otimes 0\rangle + |1 \otimes 1\rangle)$

$q_1, q_2 = CNOT(q_1, q_2)$

$q_1 = H(q_1)$

$b_1, b_2 = Measure(q_1, q_2)$

$b_2, q_3 = CNOT(b_2, q_3)$ // if $b_2$ then $q_3 = NOT(q_3)$

$b_1, q_3 = CZ(b_1, q_3)$ // if $b_1$ then $q_3 = \sigma_z(q_3)$

# Quantum Teleportation

$q_1, q_2, q_3 = |\psi\rangle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
$q_1, q_2 = CNOT(q_1, q_2)$
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$b_2, q_3 = CNOT(b_2, q_3)$
$b_1, q_3 = CZ(b_1, q_3)$

$$|\psi\rangle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= 1/\sqrt{2}\,[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)]$$

Apply $q_1, q_2 = CNOT(q_1, q_2)$

$$= 1/\sqrt{2}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)]$$

Apply $q_1 = H(q_1)$

$$= 1/2[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)]$$

$$= 1/2[\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)]$$

# Quantum Teleportation

$q_1, q_2, q_3 = |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
$q_1, q_2 = CNOT(q_1, q_2)$
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$b_2, q_3 = CNOT(b_2, q_3)$
$b_1, q_3 = CZ(b_1, q_3)$

$$1/2[\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)]$$

$$= 1/2[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)]$$

Apply $b_1, b_2 = Measure(q_1, q_2)$

From the above state, we have four possible outcomes:

| Alice's measured qubits | Bob's qubit (which has NOT been measured) |
|---|---|
| $\|00\rangle$ | $(\alpha\|0\rangle + \beta\|1\rangle)$ |
| $\|01\rangle$ | $(\alpha\|1\rangle + \beta\|0\rangle)$ |
| $\|10\rangle$ | $(\alpha\|0\rangle - \beta\|1\rangle)$ |
| $\|01\rangle$ | $(\alpha\|1\rangle - \beta\|0\rangle)$ |

# Quantum Teleportation

$q_1, q_2, q_3 = |\psi\rangle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
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| Alice's measured qubits | Bob's qubit (which has NOT been measured) |
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| $|00\rangle$ | $(\alpha|0\rangle + \beta|1\rangle)$ |
| $|01\rangle$ | $(\alpha|1\rangle + \beta|0\rangle)$ |
| $|10\rangle$ | $(\alpha|0\rangle - \beta|1\rangle)$ |
| $|01\rangle$ | $(\alpha|1\rangle - \beta|0\rangle)$ |

Alice sends two classical bits (the output of her measurements) to Bob, who will then retrieve the original state $\alpha|0\rangle + \beta|1\rangle$!

| Alice sends | Bob |
|---|---|
| 00 | Nothing to do |
| 01 | Applies NOT to qubit |
| 10 | Applies Z to qubit |
| 11 | Applies NOT, then Z to qubit |

# Quantum Teleportation

- Note that Alice’s qubit is “destroyed” by her measurement (recall the no-cloning theorem!)

- Bob only has to apply unitary gates at his end

- Quantum teleportation is used in quantum error-correcting codes and in quantum chips