An Introduction to Quantum Computing

Lecture 12: On Measurements and Quantum Key Distribution

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Agenda

- Principle of Deferred Measurement
- Measurements in Non-Diagonal Bases
- Bell's Inequalities
- The E91 Quantum Key Distribution Protocol

Principle of Deferred Measurement

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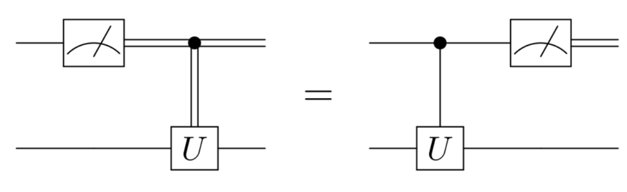
Pictorially:



https://commons.wikimedia.org/wiki/File:Qcircuit_measurement-commute.svg

Principle of Deferred Measurement

In our "programming notation" (assuming two qubits q_0, q_1):

$$\begin{array}{l} b = \text{Measure}(q_0); \\ \text{if } b \text{ then } q_1 = U(q_1) \text{ else skip} \end{array} \quad \equiv \quad \begin{array}{l} \text{ControlledU}(q_0, q_1); \\ b = \text{Measure}(q_0) \end{array}$$

Let's why this is true. We start from the LHS:

Principle of Deferred Measurement

The initial state of our program/circuit is an arbitrary state of two qubits q_0, q_1:

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$$\begin{cases} b = 0 & |0\rangle (|0\rangle + |1\rangle) \qquad \text{with probability } |\alpha_0|^2 + |\alpha_1|^2 \\ b = 1 & |1\rangle (|0\rangle + |1\rangle) \qquad \text{with probability } |\alpha_1|^2 + |\alpha_3|^2 \end{cases}$$

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Compute the RHS as an exercise!

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So far we have seen *diagonal* measurements (the $|0\rangle, |1\rangle$ basis) and the *non-diagonal* measurements in the $|+\rangle, |-\rangle$ basis in the BB84 protocol.

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The set of all eigenvectors of a self-adjoint operator acting on a Hilbert space $\mathcal{H}$ is an orthonormal basis for $\mathcal{H}$.

Therefore:

1. *unitarily* change from a non-diagonal basis to the diagonal basis;
2. measure diagonally.

Measurements in Non-Diagonal Bases

Let $\{e_i\}$ and $\{f_i\}$ be two orthonormal bases.

To change basis from, say $\{f_i\}$ to $\{e_i\}$ we need an operator that satisfies:

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This can be obtained by the unitary operator (exercise):

$$\sum_i |e_i\rangle \langle f_i|$$

Measurements in Non-Diagonal Bases

Example. Measure a qubit in the basis $\{|+\rangle, |-\rangle\}$:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad\qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

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$$U = |0\rangle\langle +| \ + \ |1\rangle\langle -|$$

U is unitary since $U^\dagger = (|0\rangle\langle +| \ + \ |1\rangle\langle -|)^\dagger = |+\rangle\langle 0| + |-\rangle\langle 1|$ and thus

$$\begin{aligned} UU^\dagger =& (|0\rangle\langle +| \ + \ |1\rangle\langle -|)(|+\rangle\langle 0| + |-\rangle\langle 1|) \\ =& |0\rangle\langle +|\,|+\rangle\langle 0| + |0\rangle\langle +|\,|-\rangle\langle 1| + |1\rangle\langle -|\,|+\rangle\langle 0| + |1\rangle\langle -|\,|-\rangle\langle 1| \\ =& |0\rangle\langle 0| + |1\rangle\langle 1| = I \end{aligned}$$

Similary for $U^\dagger U$.

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Hence, we have that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, and thus $U = H$. In fact, let us write the matrix representation of U:

$$\begin{aligned} U &= |0\rangle\langle +| \;+\; |1\rangle\langle -| \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}^T + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}^T \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = H \end{aligned}$$

Bell's Inequalities

Also known as Bell's Theorem, after physicist John Bell (1928-1990).
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The EPR thought experiment:

$$\text{Alice} \xleftarrow{\textit{one particle}} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \xrightarrow{\textit{one particle}} \text{Bob}$$

EPR's "paradox": Alice's and Bob's measurement results (in the *diagonal* basis) are always <u>anticorrelated</u>!

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Suppose now that Alice measures one of two different observables (say Q and R), and Bob one of another pair of observables (say S and T, not necessarily the same as Alice's). What happens?

Bell's Inequalities

Suppose the four observables (Alice's + Bob's) always return ± 1, *i.e.*, the eigenvalues of the four self-adjoint operators are ± 1.

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Hence, we can say that Alice's particle has two "properties" that describe the result of measuring either of the two possible measurements, say, $Q = \pm 1$ and $R = \pm 1$. Similarly for Bob's particle with, say, $S = \pm 1$ and $T = \pm 1$.

Bell's Inequalities

Graphically:

Alice's particle values:		Bob's particle values:
$Q = \pm 1$ $R = \pm 1$	$\xleftarrow{\textit{one particle}}\xrightarrow{\textit{one particle}}$	$S = \pm 1$ $T = \pm 1$

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Let us consider the quantity: $QS + RS + RT - QT$. It is easy to see that:

$$QS + RS + RT - QT \leqslant 2.$$

Bell's Inequalities

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We compute the *expected value* $\mathcal{A}$ (or average) of $QS + RS + RT - QT$:

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Let us test this inequality in quantum mechanics!

Bell's Inequalities

Try this "experimental setting":

Alice measures one of $\begin{cases} Q = Z \\ R = X \end{cases}$ Bob measures one of $\begin{cases} S = \frac{-Z-X}{\sqrt{2}} \\ T = \frac{Z-X}{\sqrt{2}} \end{cases}$

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One can show (Rule 3 of Quantum Mechanics) that

$$E[QS] = \langle QS \rangle = \frac{1}{\sqrt{2}}; \quad \langle RS \rangle = \frac{1}{\sqrt{2}}; \quad \langle RT \rangle = \frac{1}{\sqrt{2}}; \quad \langle QT \rangle = -\frac{1}{\sqrt{2}}$$

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and thus

$$\mathcal{A} = \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$

The Bell inequality is **violated** by quantum mechanics!!!!

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Does this actually happen?

YES!!

The 2022 Nobel Prize in Physics was given to Aspect, Clauser, and Zeilinger for "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science".

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Say, Alice and Bob take a random portion of their qubits and use them for computing the value of $\mathcal{A} = \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle$:

- if $\mathcal{A} \approx 2\sqrt{2}$ then all fine (distil and generate key as BB84);
- else `abort` (too much eavesdropping or noise).

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Research: can you find other uses of the Bell inequalities?