Quantum Computing: Exercises for Lectures 10-11

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Exercise 1

The quantum Fourier transform on an N-dimensional Hilbert space is:

$$QFT = \sum_{j=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle \right) \langle j| \,.$$

Show that $QFT\, QFT^{\dagger} = I$.

Exercise 2

Show that an equivalent definition of the quantum Fourier transform (assuming $N = 2^n$, hence n qubits) is the following:

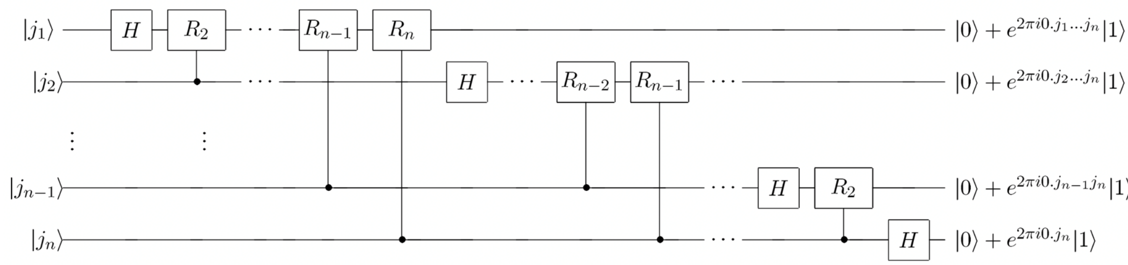
$$|j_1 \ldots j_n\rangle \xrightarrow{QFT} \frac{(|0\rangle + e^{2\pi i 0.j_n} |1\rangle) \otimes (|0\rangle + e^{2\pi i 0.j_{n-1}j_n} |1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i 0.j_1 \cdots j_n} |1\rangle)}{2^{n/2}}$$

where $j_1, \ldots, j_n$ are bits, and the binary fraction

$$0.j_l j_{l+1} j_m = \frac{j_l}{2} + \frac{j_{l+1}}{4} + \cdots + \frac{j_m}{2^{m-l+1}}.$$

Exercise 3

Compute the final state of the quantum Fourier transform circuit:

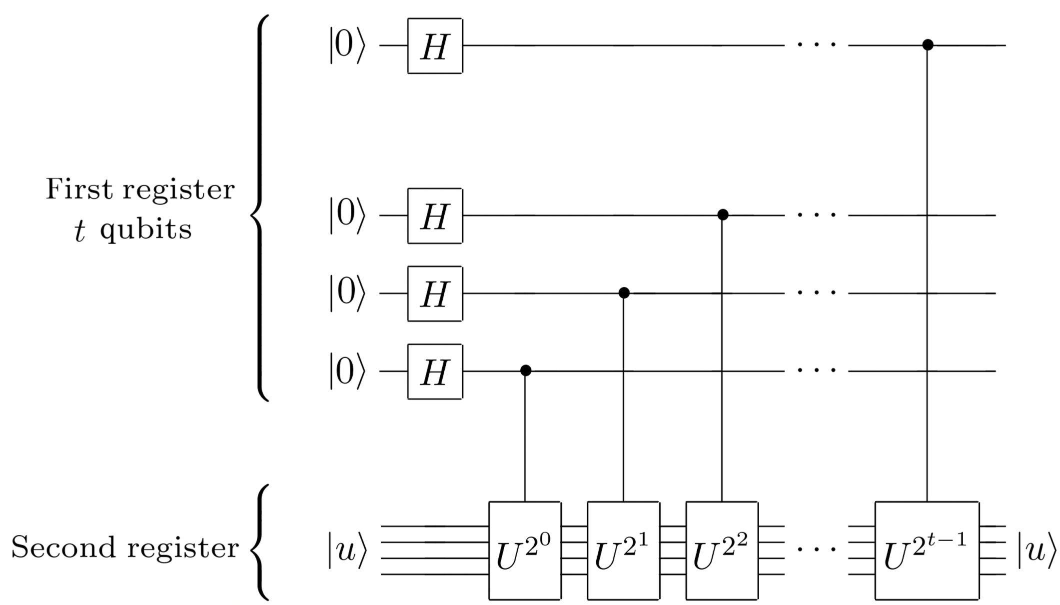


that is, show that the state of each qubit is what is reported on the right-hand side of the circuit above. (Note that the normalization constant is not shown.)

The controlled-R_k gate is defined in Lecture 10.

Exercise 4

Let U be a unitary operator with eigenvalue $e^{2\pi i \varphi}$ (for $\varphi \in \mathbb{R}$) and associated eigenvector $|u\rangle$. The quantum phase estimation circuit is:



Show that the state of the t qubits at the end of the circuit is:

$$\frac{1}{2^{t/2}}(|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle) \otimes (|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i 2^{0}\varphi}|1\rangle)$$

which is in turn:

$$\frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{2\pi i k\varphi}|k\rangle .$$

Exercise 5

Let x be coprime with N. Show that $xy \equiv xz \bmod N$ iff $y \equiv z \bmod N$.

Exercise 6

Given naturals $x < N$, let U_x be the modular multiplication operator

$$U_x|y\rangle = |xy \bmod N\rangle$$

for $y \in \{0,1\}^L$ and $L = \lceil \log N \rceil$. If $y > N$, then U_x does nothing, *i.e.*, it maps y to y. Show that $U_x U_x^\dagger = I$.