

Quantum Computing

Lecture $|04\rangle$

Cats, No-Cloning, and Quantum Teleportation

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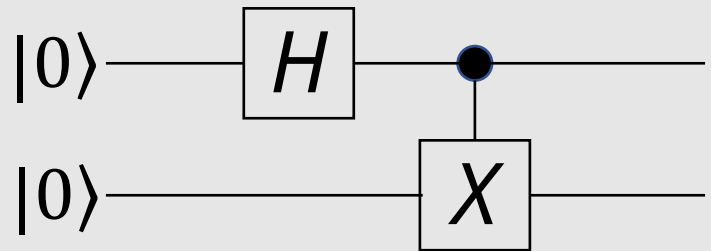
Outline

- Half-dead cats?
- The No-cloning Theorem
- Quantum Teleportation



Half-dead Cats

- Or: quantum mechanics cannot be (easily) applied to macroscopic objects
- Entangled states are (one of the) sources of the problem

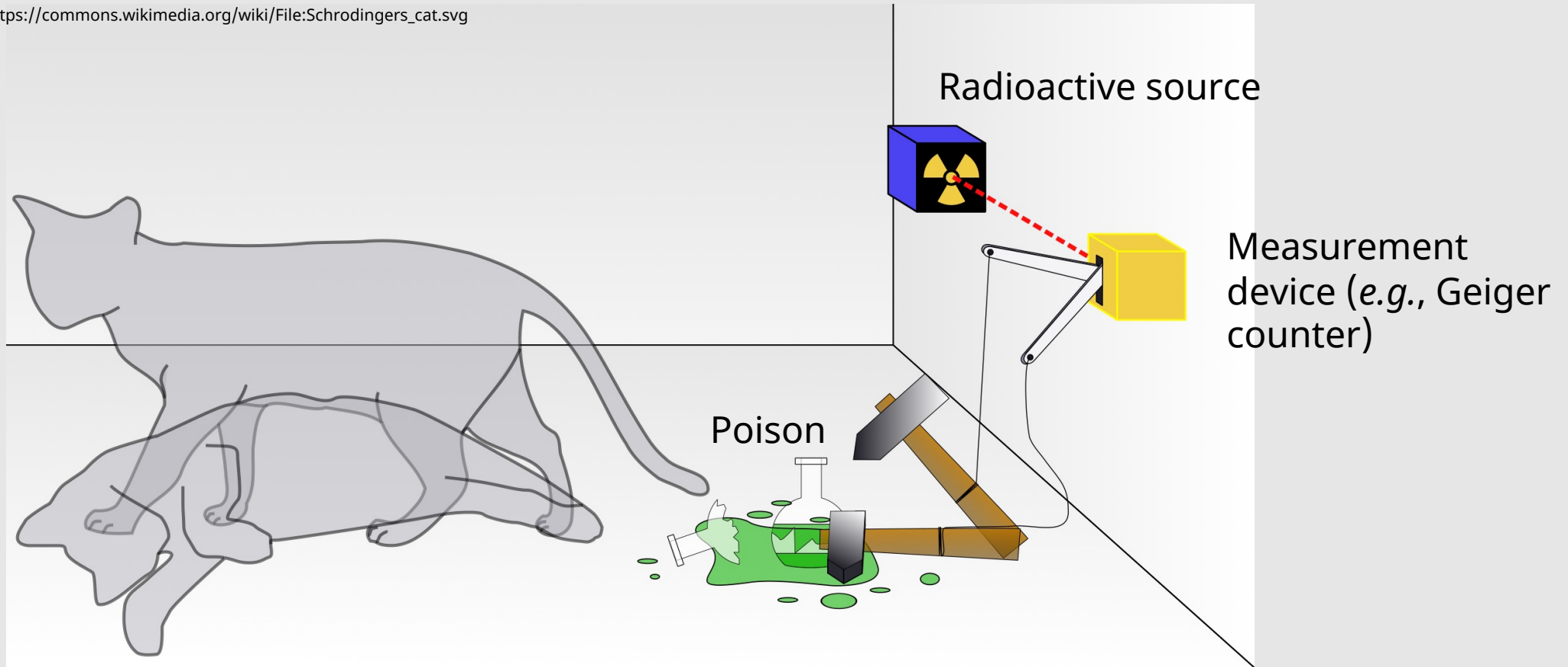


A quantum circuit diagram with two horizontal lines representing qubits. The top line starts with the state $|0\rangle$, followed by a box labeled H (Hadamard gate). The bottom line starts with the state $|0\rangle$, followed by a box labeled X (Pauli-X gate). A vertical line with a solid black dot at the top connects the two lines, representing a CNOT gate with the top line as control and the bottom line as target.

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Schrödinger's Cat

https://commons.wikimedia.org/wiki/File:Schrödingers_cat.svg

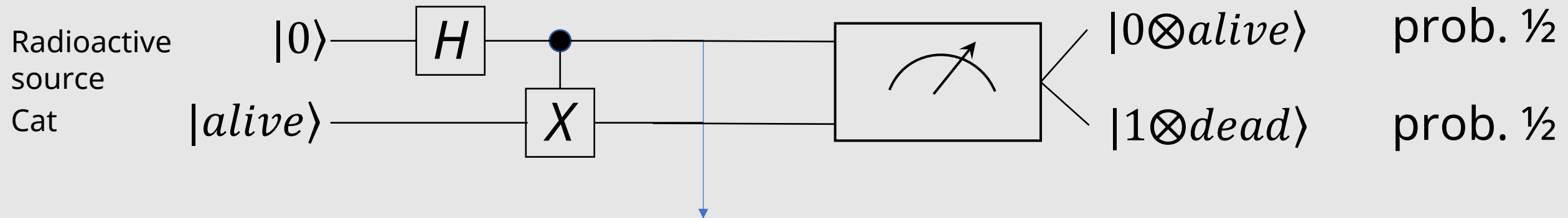
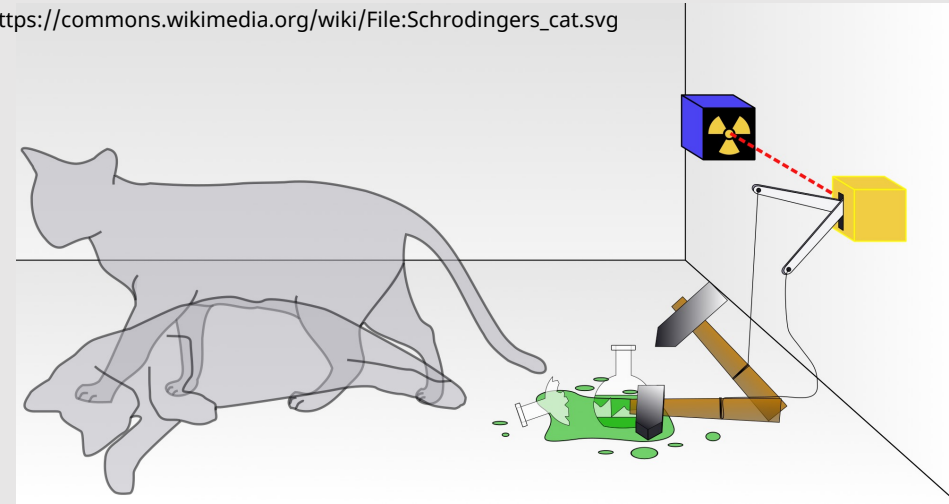


Is the cat dead or alive?

Schrödinger's Cat

- The radioactive source emits particles described by a qubit
- The cat's state is a qubit, with basis states 'alive' ($|0\rangle$) and 'dead' ($|1\rangle$)

https://commons.wikimedia.org/wiki/File:Schrödingers_cat.svg



The state of the system at this point is:

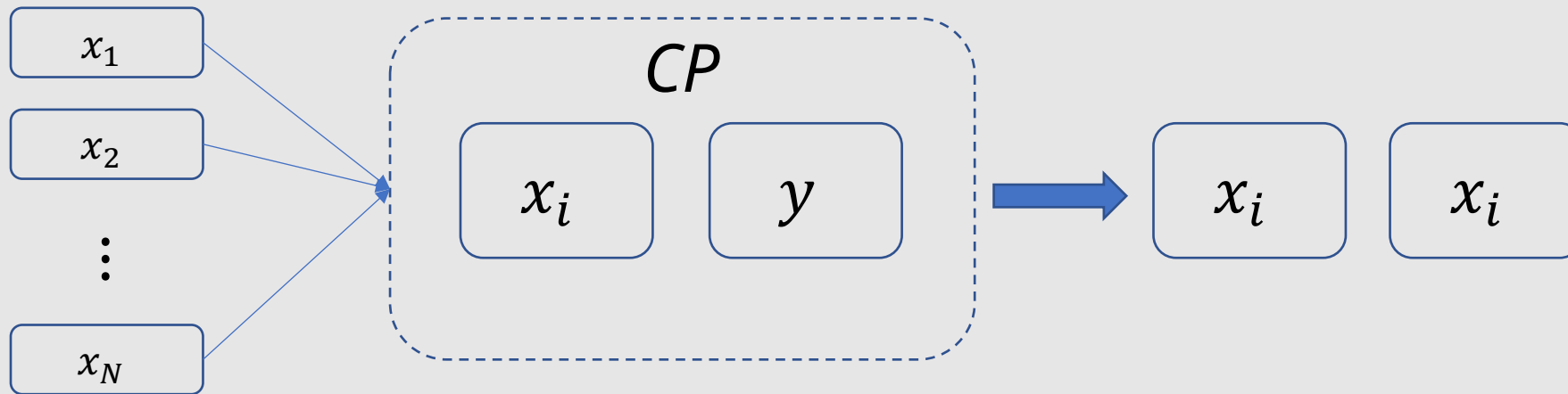
$$\frac{1}{\sqrt{2}}(|0 \otimes \text{alive}\rangle + |1 \otimes \text{dead}\rangle)$$

No Cloning (No Copy)

A quantum state cannot be perfectly copied (cloned)

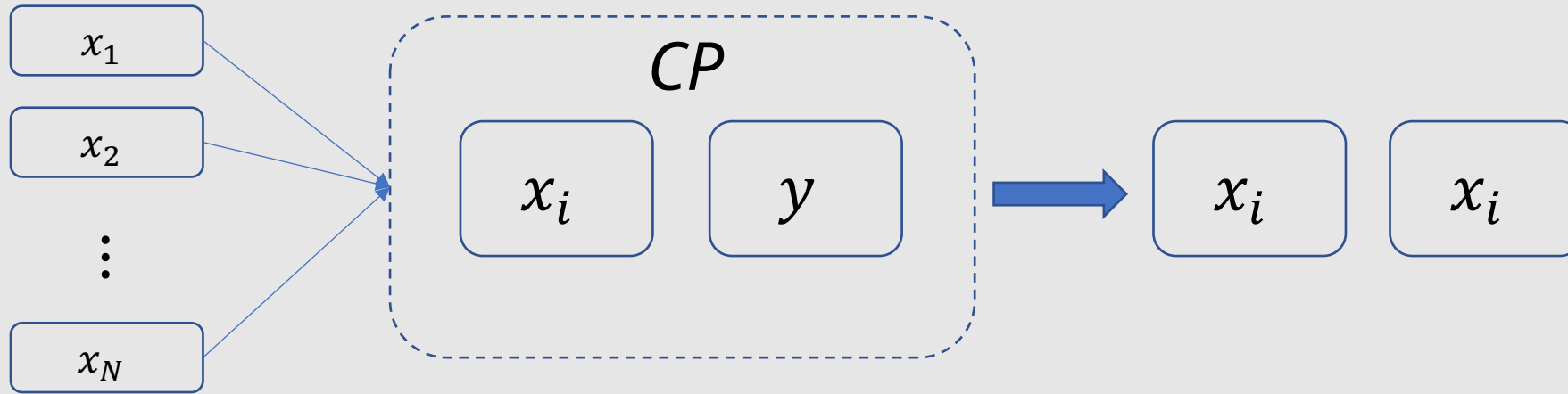
[Wooters and Zurek, 1982; Dieks, 1982]

A putative copy operation/transformation CP is something like:



CP takes a 'source' variable in input and copies its contents to the 'destination' variable, leaving the source untouched.

No Cloning (No Copy)



Formally, we write

$$\exists y \forall x CP(x, y) = (x, x)$$

“there exists a y such that for all x $CP(x, y) = (x, x)$ ”

No Cloning (No Copy)

In a quantum world, a putative copy CP should satisfy

$$\exists y \quad \forall x \quad CP(x \otimes y) = (x \otimes x)$$

However, this doesn't work!

Proof by contradiction.

No Cloning (No Copy)

$$\exists y \forall x \quad CP(x \otimes y) = (x \otimes x)$$

thus, since CP should work on superposed states, too

$$\exists y \forall x, a \quad CP((x + a) \otimes y) = (x + a) \otimes (x + a)$$

$$\exists y \forall x, a \quad CP(x \otimes y + a \otimes y) = (x + a) \otimes (x + a)$$

thus, since CP must be unitary hence linear

$$\exists y \forall x, a \quad CP(x \otimes y) + CP(a \otimes y) = (x + a) \otimes (x + a)$$

thus, since CP copies its input

$$\forall x, a \quad (x \otimes x) + (a \otimes a) = (x + a) \otimes (x + a)$$

$$\forall x, a \quad (x \otimes x) + (a \otimes a) = (x \otimes x) + (x \otimes a) + (a \otimes x) + (a \otimes a)$$

FALSE (try for example $x = |0\rangle$ $a = |1\rangle$)

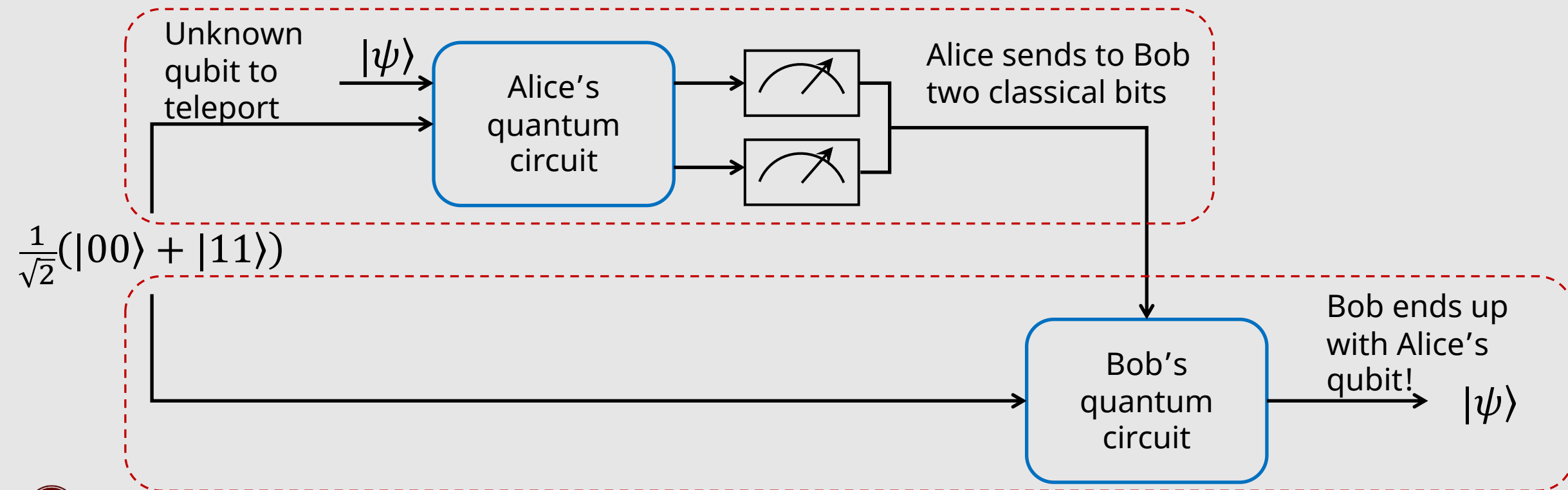
Summing Up

- While we cannot clone perfectly, it has been shown that *imperfect cloning* (i.e., with high probability) is possible!
 - The quantum internet is being built!
- Quantum mechanics cannot be straightforwardly applied to 'big' (macroscopic) objects
 - Physicists are very busy working out a *quantum theory of gravity*!
- Next: We can *teleport* a quantum state *perfectly*!

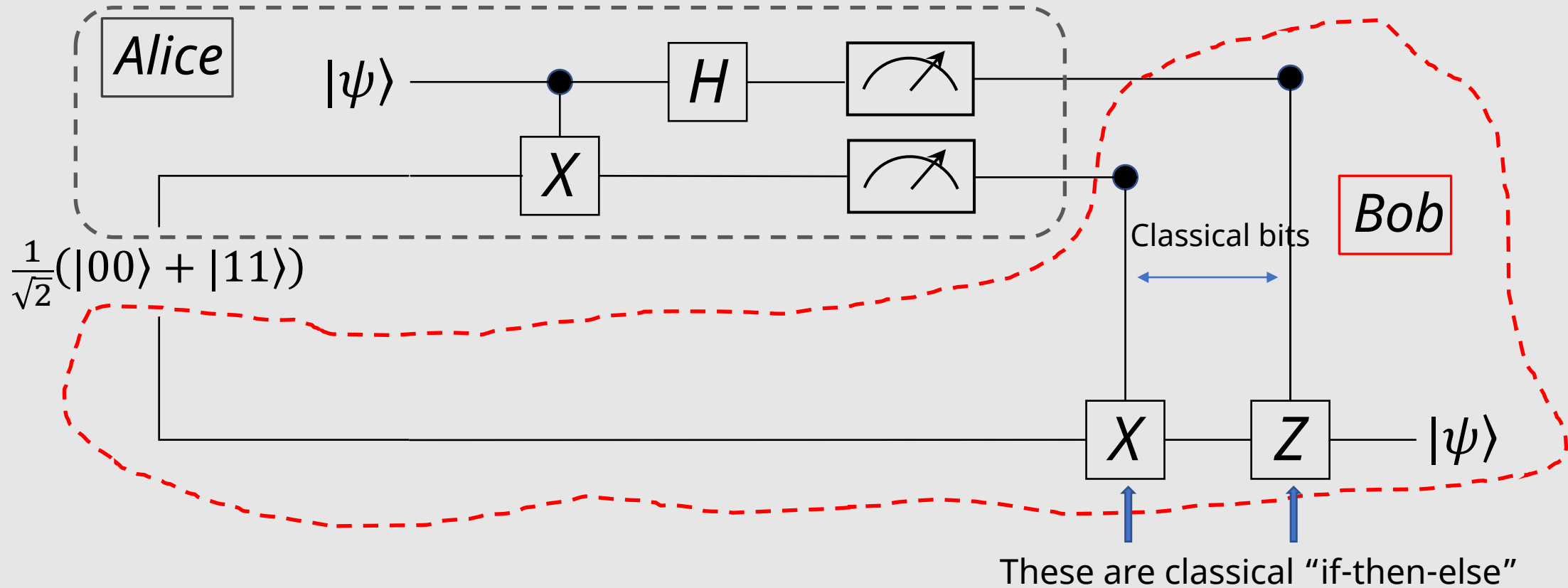
Quantum Teleportation

[Bennett, Brassard, Crépeau, Jozsa, Peres, Wootters. 1993]

- Two parties, Alice and Bob, are separated by a large distance
- Alice has a qubit she would like to send to Bob
- Teleportation achieve this by using **two qubits** and **two classical bits**

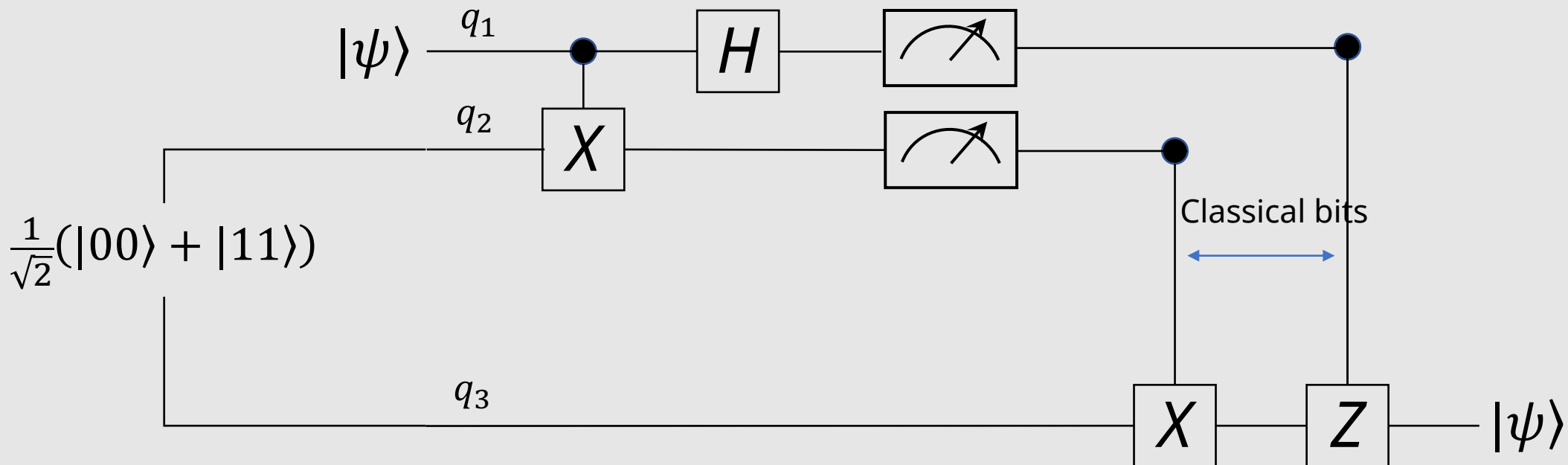


Quantum Teleportation Circuit



Hence, to teleport one qubit Alice needs to send one qubit and two classical bits

Quantum Teleportation Circuit



Using a “programming” notation:

$$q_1, q_2, q_3 = |\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$q_1, q_2 = CNOT(q_1, q_2)$$

$$q_1 = H(q_1)$$

$$b_1, b_2 = Measure(q_1, q_2)$$

$$b_2, q_3 = CNOT(b_2, q_3)$$

$$b_1, q_3 = CZ(b_1, q_3)$$

$$// \text{ this means } |\psi\rangle \otimes \frac{1}{\sqrt{2}}(|0\otimes 0\rangle + |1\otimes 1\rangle)$$

$$// \text{ if } b_2 \text{ then } q_3 = NOT(q_3)$$

$$// \text{ if } b_1 \text{ then } q_3 = \sigma_z(q_3)$$

Quantum Teleportation

$$\begin{aligned} |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &= (\alpha|0\rangle + \beta|1\rangle) \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= 1/\sqrt{2} [\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)] \end{aligned}$$

Apply $q_1, q_2 = CNOT(q_1, q_2)$

$$= 1/\sqrt{2} [\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)]$$

Apply $q_1 = H(q_1)$

$$\begin{aligned} &= 1/2 [\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)] \\ &= 1/2 [\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)] \end{aligned}$$

$$q_1, q_2, q_3 = |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$q_1, q_2 = CNOT(q_1, q_2)$$

$$q_1 = H(q_1)$$

$$b_1, b_2 = \text{Measure}(q_1, q_2)$$

$$b_2, q_3 = CNOT(b_2, q_3)$$

$$b_1, q_3 = CZ(b_1, q_3)$$



Quantum Teleportation

$$\begin{aligned} q_1, q_2, q_3 &= |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ q_1, q_2 &= CNOT(q_1, q_2) \\ q_1 &= H(q_1) \\ b_1, b_2 &= Measure(q_1, q_2) \\ b_2, q_3 &= CNOT(b_2, q_3) \\ b_1, q_3 &= CZ(b_1, q_3) \end{aligned}$$

$$\begin{aligned} &1/2[\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)] \\ &= 1/2[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)] \end{aligned}$$

Apply $b_1, b_2 = Measure(q_1, q_2)$

From the above state, we have four possible outcomes:

Alice's measured qubits Bob's qubit (which has NOT been measured)

$ 00\rangle$	$(\alpha 0\rangle + \beta 1\rangle)$
$ 01\rangle$	$(\alpha 1\rangle + \beta 0\rangle)$
$ 10\rangle$	$(\alpha 0\rangle - \beta 1\rangle)$
$ 11\rangle$	$(\alpha 1\rangle - \beta 0\rangle)$



Quantum Teleportation

$$\begin{aligned}
 q_1, q_2, q_3 &= |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
 q_1, q_2 &= CNOT(q_1, q_2) \\
 q_1 &= H(q_1) \\
 b_1, b_2 &= Measure(q_1, q_2) \\
 b_2, q_3 &= CNOT(b_2, q_3) \\
 b_1, q_3 &= CZ(b_1, q_3)
 \end{aligned}$$

Alice's measured qubits Bob's qubit (which has NOT been measured)

$ 00\rangle$	$(\alpha 0\rangle + \beta 1\rangle)$
$ 01\rangle$	$(\alpha 1\rangle + \beta 0\rangle)$
$ 10\rangle$	$(\alpha 0\rangle - \beta 1\rangle)$
$ 11\rangle$	$(\alpha 1\rangle - \beta 0\rangle)$

Alice sends two classical bits (the output of her measurements) to Bob, who will then retrieve the original state $\alpha|0\rangle + \beta|1\rangle$!

Alice sends	Bob
00	Nothing to do
01	Applies NOT to qubit
10	Applies Z to qubit
11	Applies NOT, then Z to qubit



Quantum Teleportation

- Note that Alice's qubit is “destroyed” by her measurement (recall the no-cloning theorem!)
- Bob only has to apply unitary gates at his end
- Quantum teleportation is used in quantum error-correcting codes and in quantum chips

