An Introduction to Quantum Computing

Lecture 03

Systems with Multiple Qubits

Paolo Zuliani

zuliani@di.uniroma1.it

SAPIENZA UNIVERSITÀ DI ROMA

Outline

- Basis vectors
- Tensor products
- Entanglement
- Two-Qubit operations

Basis Vectors

Any vector can be ``decomposed'' as a sum of 'basis' vectors:

$$\begin{pmatrix} 3 \\ -2 \\ 1+i \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (1+i)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This holds for complex vectors of ***any size*** (even infinite, with some caveats)!

zuliani@di.uniroma1.it

Basis Vectors

In general, we write

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} \alpha_i e_i$$

e_i is the vector that has 1 in the i-th position and 0 elsewhere.

The e_i's are called (**diagonal**) **basis vectors**, and typically correspond to *classical* states.

zuliani@di.uniroma1.it

Multiple Bits

- It is easy to describe *n classical* bits: just list them in a vector/array of *n* elements!
- Formally: *Cartesian* product (René Descartes, 1596-1650)

$\mathcal{B} = \{0,1\}$

$\mathcal{B} \times \mathcal{B} = \{(a, b): a \in \mathcal{B},\ b \in \mathcal{B}\}$ (similarly for *n* copies of $\mathcal{B}$)

$s = (b_1, b_2, \dots, b_n) \in \mathcal{B}^n$

- **Note**:
  - The length of the vector grows *linearly* with *n*
  - The vector components are *independent* from each other (you can change a component without affecting the others)

Multiple Qubits

To describe *n quantum* bits we need the

tensor product

Tensor Products

$|0\rangle = \binom{1}{0} \qquad |1\rangle = \binom{0}{1}$

A qubit is $\alpha_0|0\rangle + \alpha_1|1\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$

Two qubits: $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$

$$= \alpha_0\beta_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_0\beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1\beta_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_1\beta_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Tensor Products

- Easily extended to any number of qubits
- The size of the resulting vector grows *exponentially* with the qubits number!

$\begin{pmatrix}\alpha_0\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}$ is a valid quantum state, *i.e.*, $\sum_{i,j\in\mathcal{B}}\left|\alpha_i\beta_j\right|^2 = 1$

$$|0\rangle\otimes|1\rangle=\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=|01\rangle$$

$$\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\tfrac{1}{\sqrt{2}}(|0\rangle+|1\rangle)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)$$

Multiple Qubits

$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ is a superposition of all 2-bit states

- Built by juxtaposing two qubits in superposition
- Even though the state looks 'complicate', it can be decomposed in two valid qubit states!
- However, …

zuliani@di.uniroma1.it

Multiple Qubits

- Quantum mechanics tells us that *any*

$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$ with $\sum_i |\alpha_i|^2 = 1$

is a valid quantum state for a 2-qubit system!

- *Example*:

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

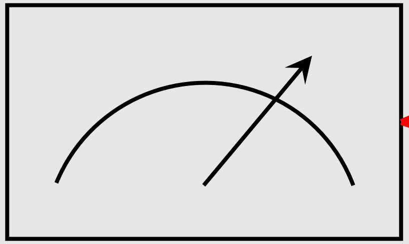
Can you separate this 2-qubit state into two, independent qubit states?

Entanglement

- NO! This state is ***entangled***.
- There is no way to describe it as two ‘local’ qubit states: it describes the system *as a whole*.

- *EPR* pair (Einstein, Podolsky, Rosen. 1935)

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ ➡



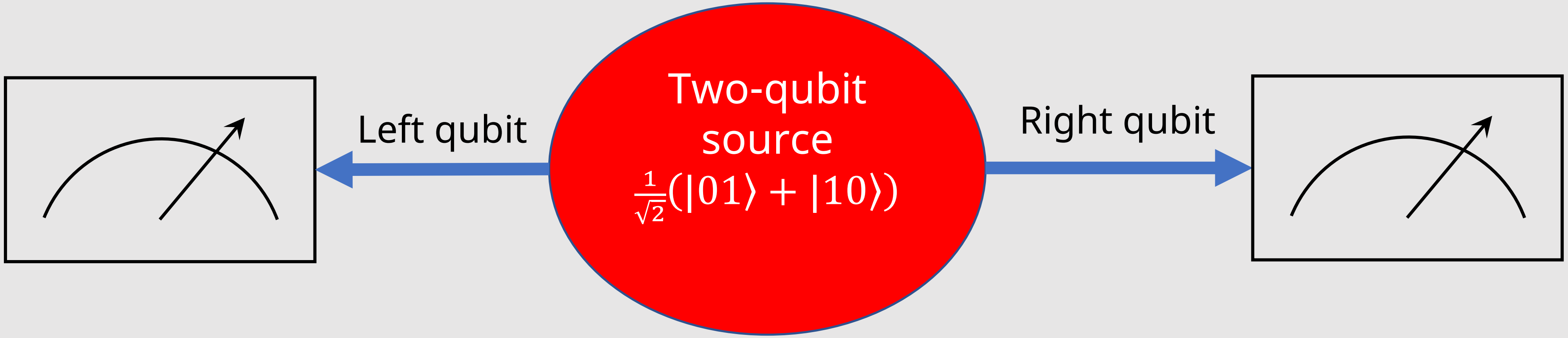
$|01\rangle$ with probability $\frac{1}{2}$

$|10\rangle$ with probability $\frac{1}{2}$

- Measuring either qubit will tell us about the other!!
- [That won’t be the case with, *e.g.*, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|0\rangle)$]

Entanglement: EPR ‘Paradox’

- But wait! Suppose now the two qubits ‘fly’ apart



- The two qubits could end up being light-years apart and still maintain their anticorrelation!
  - Einstein: “spooky actions at distance”
  - *Experimentally validated* on distances of ~1,000 km
  - 2022 Nobel Prize in Physics to Aspect, Clauser, Zeilinger (talk by Aspect)

Multiple Qubits: Operations

- The Controlled-NOT (*CNOT*) operates on *two* qubits
- Classically: *CNOT*(*a*,*b*) = if *a* then *b*:=NOT(*b*) else *skip*

Target bit

Control bit

a	b	$CNOT(a,b)$
0	0	0
0	1	1
1	0	1
1	1	0

Quantum CNOT

inputs		outputs	
a	b	a	$CNOT(a,b)$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

$$CNOT(|00\rangle) = |00\rangle$$
$$CNOT(|01\rangle) = |01\rangle$$
$$CNOT(|10\rangle) = |11\rangle$$
$$CNOT(|11\rangle) = |10\rangle$$

Xor

$$CNOT(|a,b\rangle) = |a, a \oplus b\rangle$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{CNOT} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{|10\rangle} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum CNOT

The quantum CNOT is a linear operator, thus

$$CNOT(\tfrac{1}{\sqrt{2}}(|00\rangle + |01\rangle)) = \tfrac{1}{\sqrt{2}}[(CNOT(|00\rangle) + CNOT(|01\rangle)] = \tfrac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

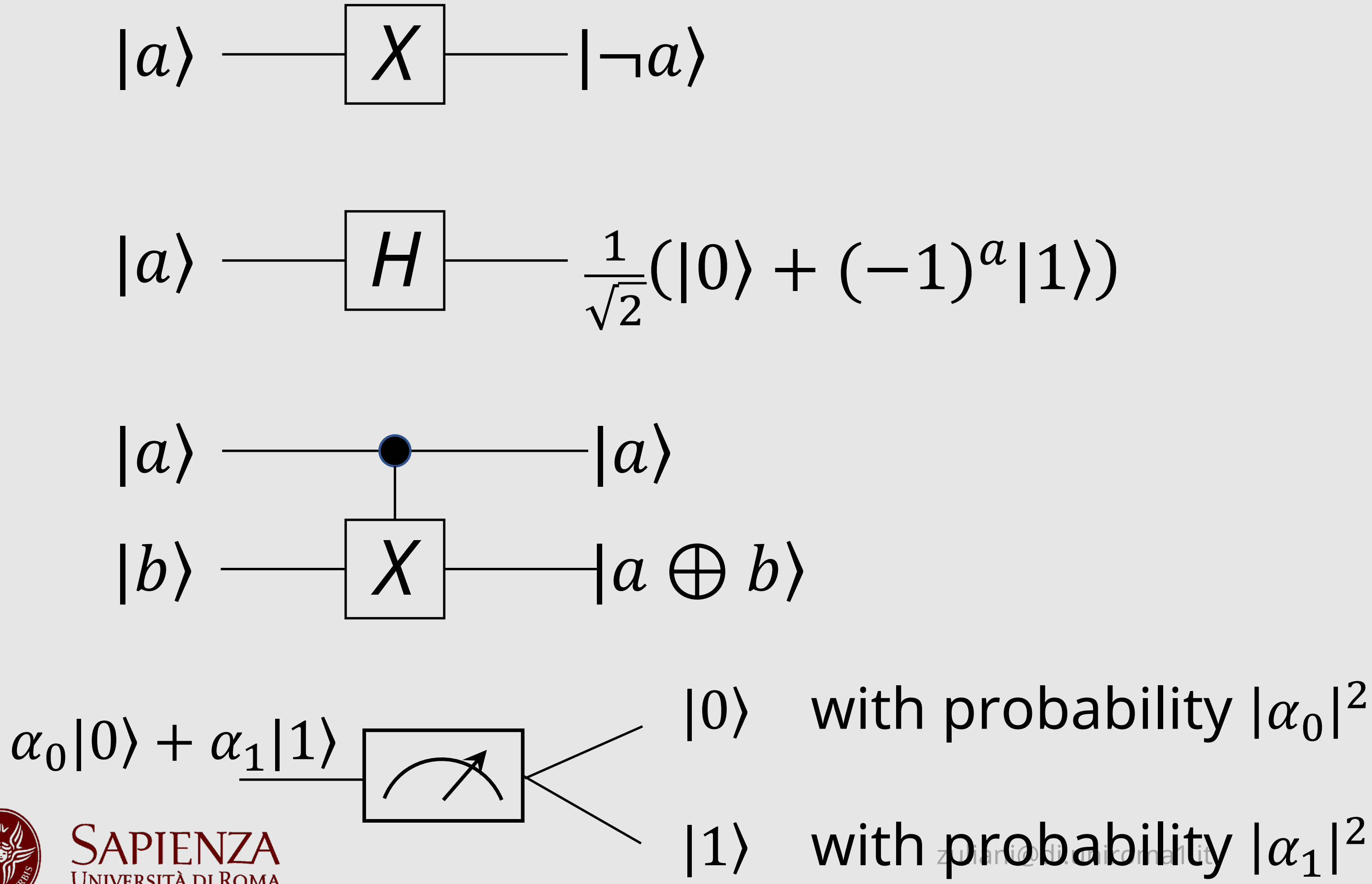
$$CNOT(\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle) = CNOT(\tfrac{1}{\sqrt{2}}(|01\rangle + |11\rangle)) = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

EPR pair!!!

$$CNOT\left(\tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\right) = \ldots$$

Notation

We ‘program’ using gates and circuits!!



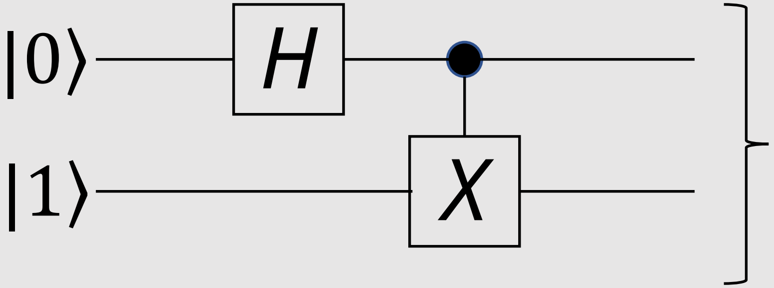
NOT gate

Hadamard gate

CNOT gate

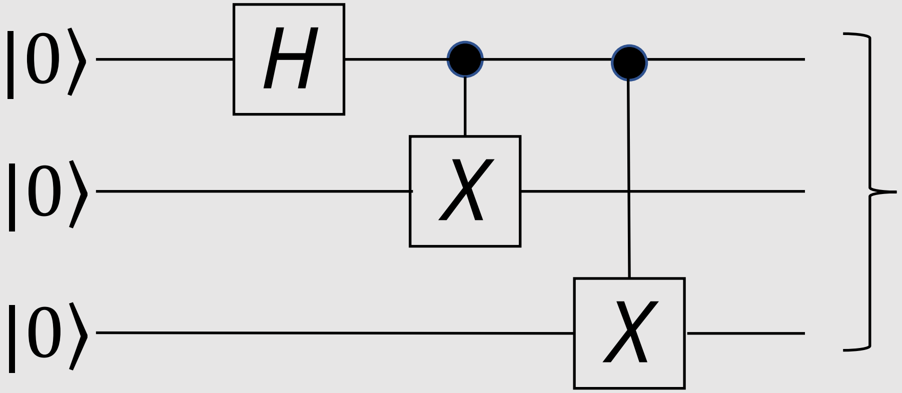
Measurement gate

How to Create EPR Pairs



$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

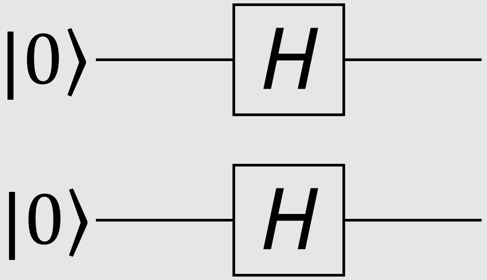
The GHZ State



$$= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

GHZ = Greenberger, Horne, Zeilinger

Tensor Product of (Unitary) Gates



$$\left.\begin{array}{c}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\\ \otimes \\ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\end{array}\right\} = \frac{1}{2}(|00\rangle + \cdots)$$

Take two 2x2 matrices *A, B*

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B \\ a_{2,1}B & a_{2,2}B \end{pmatrix}$$ is a 4x4 matrix

Tensor Product of (Unitary) Gates

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1H & 1H \\ 1H & -1H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ (check this!)}$$