An Introduction to Quantum Computing

Lecture 17:

Solving Linear Systems of Equations: Towards the HHL Algorithm (I)

Paolo Zuliani

Dipartimento di Informatica
Università di Roma "La Sapienza", Rome, Italy

SAPIENZA
UNIVERSITÀ DI ROMA

Agenda

- Quantum Phase Estimation
- Unitary Operators from Self-adjoint Operators
- Problem and Complexity
- General Idea of the Harrow-Hassidim-Lloyd (HHL) Algorithm

Phase Estimation

Fact: the eigenvalues of a unitary operator are complex numbers of **modulus 1**.

This means that any eigenvalue of a unitary operator can be written as $e^{2\pi i\varphi}$ for some real $\varphi \in [0, 1]$.

Phase Estimation

Fact: the eigenvalues of a unitary operator are complex numbers of **modulus 1**.

This means that any eigenvalue of a unitary operator can be written as $e^{2\pi i\varphi}$ for some real $\varphi \in [0, 1]$.

Definition (Phase Estimation Problem)

Let $\lambda = e^{2\pi i\varphi}$ be an eigenvalue of a unitary operator U. Find φ.

Phase Estimation

Fact: the eigenvalues of a unitary operator are complex numbers of **modulus 1**.

This means that any eigenvalue of a unitary operator can be written as $e^{2\pi i\varphi}$ for some real $\varphi \in [0, 1]$.

Definition (Phase Estimation Problem)

Let $\lambda = e^{2\pi i\varphi}$ be an eigenvalue of a unitary operator U. Find φ.

The quantum phase estimation algorithm returns φ with high precision and high probability.

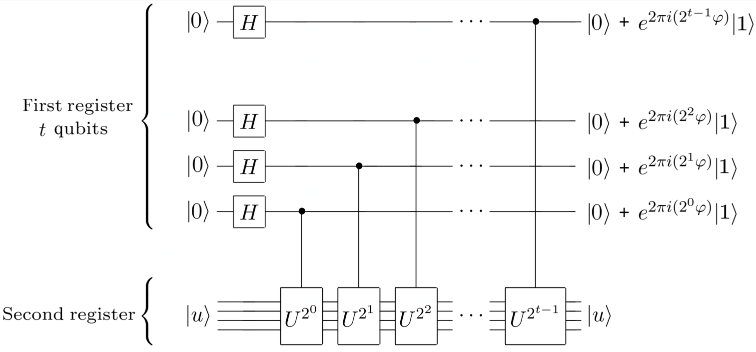
Quantum Phase Estimation Algorithm

Let u be an eigenvector associated to the unknown eigenvalue $e^{2\pi i\varphi}$ of a unitary operator U, *i.e.*, $U|u\rangle = e^{2\pi i\varphi}|u\rangle$.

Quantum Phase Estimation Algorithm

Let u be an eigenvector associated to the unknown eigenvalue $e^{2\pi i\varphi}$ of a unitary operator U, *i.e.*, $U|u\rangle = e^{2\pi i\varphi}|u\rangle$.

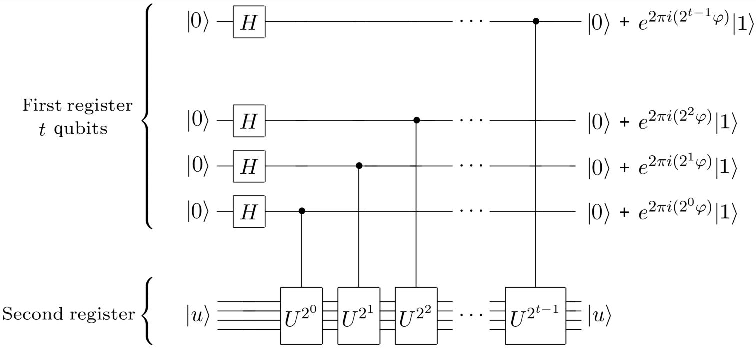
Consider the circuit below for some natural $t > 0$:



Quantum Phase Estimation Algorithm

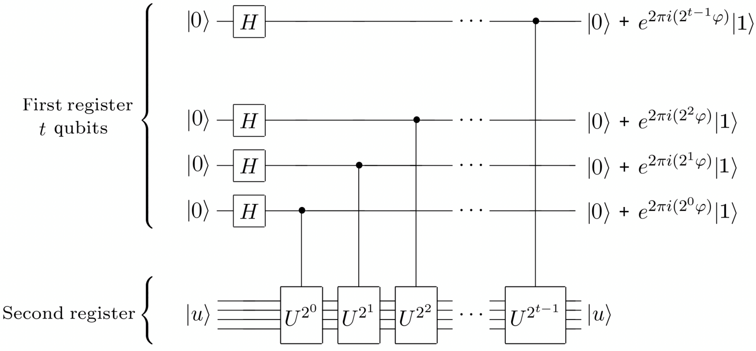
Let u be an eigenvector associated to the unknown eigenvalue $e^{2\pi i\varphi}$ of a unitary operator U, *i.e.*, $U|u\rangle = e^{2\pi i\varphi}|u\rangle$.

Consider the circuit below for some natural $t > 0$:



A control-U^{2^k} gate conditionally applies $U^{2^k} = \underbrace{U \cdots U}_{2^k \textit{ times}}$ to the second qubit register.

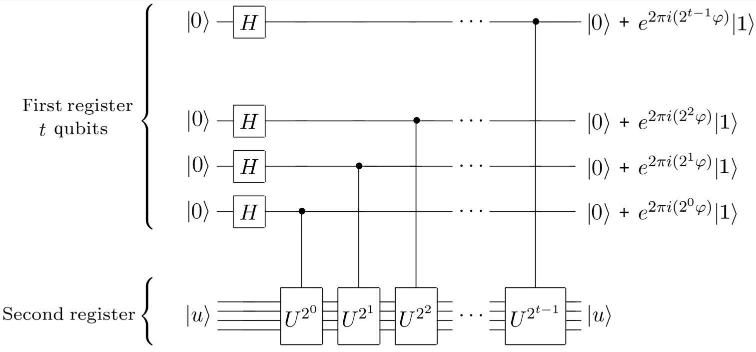
Quantum Phase Estimation Algorithm



The state of the t qubits at the end of the QPE circuit is:

$$\frac{1}{2^{t/2}}(|0\rangle+e^{2\pi i 2^{t-1}\varphi}|1\rangle)\otimes(|0\rangle+e^{2\pi i 2^{t-2}\varphi}|1\rangle)\otimes\cdots\otimes(|0\rangle+e^{2\pi i 2^{0}\varphi}|1\rangle)$$

Quantum Phase Estimation Algorithm



The state of the t qubits at the end of the QPE circuit is:

$$\frac{1}{2^{t/2}}(|0\rangle+e^{2\pi i 2^{t-1}\varphi}|1\rangle)\otimes(|0\rangle+e^{2\pi i 2^{t-2}\varphi}|1\rangle)\otimes\cdots\otimes(|0\rangle+e^{2\pi i 2^{0}\varphi}|1\rangle) = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{2\pi i k\varphi}|k\rangle$$

Quantum Phase Estimation Algorithm

Suppose now that φ can be written exactly with t bits:

$$\boxed{\varphi = 0.\varphi_1 \ldots \varphi_t}$$

Quantum Phase Estimation Algorithm

Suppose now that φ can be written <u>exactly</u> with t bits:

$$\boxed{\varphi = 0.\varphi_1 \dots \varphi_t}$$

Then, the state at the end of the QPE circuit is:

$$\frac{1}{2^{t/2}}(|0\rangle + e^{2\pi i 0.\varphi_t}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.\varphi_{t-1}\varphi_t}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i 0.\varphi_1\varphi_2\dots\varphi_t}|1\rangle)$$

which is *precisely* the final state of the QFT circuit (after the swap)!

Quantum Phase Estimation Algorithm

Suppose now that φ can be written <u>exactly</u> with t bits:

$$\boxed{\varphi = 0.\varphi_1 \ldots \varphi_t}$$

Then, the state at the end of the QPE circuit is:

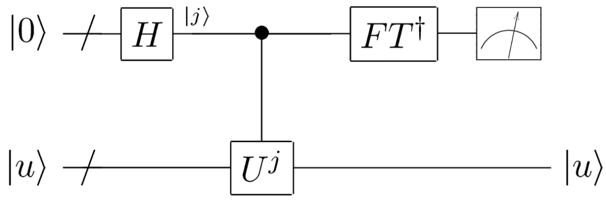
$$\frac{1}{2^{t/2}}(|0\rangle + e^{2\pi i 0.\varphi_t}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.\varphi_{t-1}\varphi_t}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i 0.\varphi_1\varphi_2\ldots\varphi_t}|1\rangle)$$

which is *precisely* the final state of the QFT circuit (after the swap)!

Therefore, we apply the inverse QFT circuit at the end of the QPE circuit and then measure to obtain the sought phase $|\varphi_1 \ldots \varphi_t\rangle$ with probability 1!

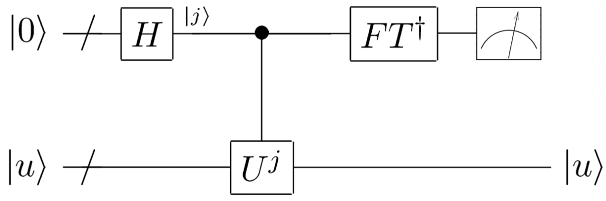
Quantum Phase Estimation Algorithm

The final quantum circuit for solving phase estimation for a unitary U is thus:



Quantum Phase Estimation Algorithm

The final quantum circuit for solving phase estimation for a unitary U is thus:



What if φ is not expressible in exactly t bits?

Proposition

To estimate φ *with* n bits of precision *and* success probability *at least* $1-\epsilon$, *it is sufficient to use the QPE circuit with* r *qubits*

$$r = n + \left\lceil \log\left(2 + \frac{1}{2\epsilon}\right)\right\rceil$$

Unitary Operators and Self-adjoint Operators

Remember:

1. Both unitary and self-adjoint operators are *normal*, *i.e.*, $A^\dagger A = AA^\dagger$;

Unitary Operators and Self-adjoint Operators

Remember:

1. Both unitary and self-adjoint operators are *normal*, *i.e.*, $A^\dagger A = AA^\dagger$;
2. *Spectral Theorem*: the eigenvectors $\{|u_i\rangle\}$ of a normal operator $A : \mathcal{H} \to \mathcal{H}$ form an orthonormal basis for the n-dimensional $\mathcal{H}$:

$$\forall\, |\psi\rangle \in \mathcal{H} \quad |\psi\rangle = \sum_{i=1}^{n} \alpha_i\, |u_i\rangle \qquad \text{where } \alpha_i = \langle u_i|\psi\rangle$$

Unitary Operators and Self-adjoint Operators

Remember:

1. Both unitary and self-adjoint operators are *normal*, *i.e.*, $A^\dagger A = AA^\dagger$;
2. *Spectral Theorem*: the eigenvectors $\{|u_i\rangle\}$ of a normal operator $A : \mathcal{H} \to \mathcal{H}$ form an orthonormal basis for the n-dimensional $\mathcal{H}$:

$$\forall\, |\psi\rangle \in \mathcal{H} \quad |\psi\rangle = \sum_{i=1}^{n} \alpha_i \, |u_i\rangle \qquad \text{where } \alpha_i = \langle u_i|\psi\rangle$$

3. *Schrödinger's Equation*:

$$i\hbar \frac{d\,|\psi(t)\rangle}{dt} = H\,|\psi(t)\rangle \qquad \text{with } \ |\psi(0)\rangle = |\psi_0\rangle \ \text{ (initial condition)}$$

where $|\psi(t)\rangle$ is a complex function of time and H is the so-called *Hamiltonian* of the system (a self-adjoint operator).

Unitary Operators and Self-adjoint Operators

It can be shown that the solution of Schrödinger's Equation is

$$\forall t \in \mathbb{R}^+ \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}t} |\psi_0\rangle$$

Unitary Operators and Self-adjoint Operators

It can be shown that the solution of Schrödinger's Equation is

$$\forall t \in \mathbb{R}^{+} \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}t}\,|\psi_0\rangle$$

and that the operator

$$e^{-\frac{i}{\hbar}Ht} = \sum_{i=1}^{n} e^{-\frac{i}{\hbar}\lambda_i t}\,|u_i\rangle\langle u_i|$$

is *unitary*, where the λ_i's and the $|u_i\rangle' s$ are the eigenvalues and eigenvectors of the Hamiltonian H, respectively.

Unitary Operators and Self-adjoint Operators

It can be shown that the solution of Schrödinger's Equation is

$$\forall t \in \mathbb{R}^+ \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}t} |\psi_0\rangle$$

and that the operator

$$e^{-\frac{i}{\hbar}Ht} = \sum_{i=1}^{n} e^{-\frac{i}{\hbar}\lambda_i t} |u_i\rangle\langle u_i|$$

is *unitary*, where the λ_i's and the $|u_i\rangle' s$ are the eigenvalues and eigenvectors of the Hamiltonian H, respectively.

[We have just defined what we mean by the (real) function of a normal operator.]

Solving Linear Systems of Equations: The Problem

Definition

Given a $N \times N$ Hermitian (self-adjoint) matrix A and unit vector b, find the vector x such that $Ax = b$.

Solving Linear Systems of Equations: The Problem

Definition

Given a $N \times N$ Hermitian (self-adjoint) matrix A and unit vector b, find the vector x such that $Ax = b$.

Equivalently, compute $A^{-1}b$, since the solution x must satisfy $A^{-1}b = A^{-1}Ax = x$.

Solving Linear Systems of Equations: The Problem

Definition

Given a $N \times N$ Hermitian (self-adjoint) matrix A and unit vector b, find the vector x such that $Ax = b$.

Equivalently, compute $A^{-1}b$, since the solution x must satisfy $A^{-1}b = A^{-1}Ax = x$.

Definition

Given matrix A, the condition number κ of A is

$$\kappa = \frac{|\text{largest eigenvalue of } A|}{|\text{smallest eigenvalue of } A|}$$

Solving Linear Systems of Equations: The Problem

Definition

Given a $N \times N$ Hermitian (self-adjoint) matrix A and unit vector b, find the vector x such that $Ax = b$.

Equivalently, compute $A^{-1}b$, since the solution x must satisfy $A^{-1}b = A^{-1}Ax = x$.

Definition

Given matrix A, the condition number κ of A is

$$\kappa = \frac{|\text{largest eigenvalue of } A|}{|\text{smallest eigenvalue of } A|}$$

The HHL algorithm assumes that the *singular values* of A (the square root of the eigenvalues of $A^\dagger A$) lie between $\frac{1}{\kappa}$ and 1.

Solving Linear Systems of Equations: Computational Complexity

Define $\epsilon = \|x - \hat{x}\|$ (error of the computed solution $\hat{x}$ wrt the correct solution x)

Solving Linear Systems of Equations: Computational Complexity

Define $\epsilon = \|x - \hat{x}\|$ (error of the computed solution $\hat{x}$ wrt the correct solution x)

The HHL algorithm also assumes that A is s-sparse and efficiently computable, that is:

1. A has at most s non-zero entries per row; and
2. entries can be computed in $O(s)$ time.

Solving Linear Systems of Equations: Computational Complexity

Define $\epsilon = \|x - \hat{x}\|$ (error of the computed solution $\hat{x}$ wrt the correct solution x)

The HHL algorithm also assumes that A is s-sparse and efficiently computable, that is:

1. A has at most s non-zero entries per row; and
2. entries can be computed in $O(s)$ time.

Computational complexity of solving linear systems:

- Classical: $O(N\, s\, \kappa \log \frac{1}{\epsilon})$
- **Quantum** (HHL): $O((\log N)\, s^2\, \kappa^2\, \frac{1}{\epsilon})$

Solving Linear Systems of Equations: Computational Complexity

Define $\epsilon = \|x - \hat{x}\|$ (error of the computed solution $\hat{x}$ wrt the correct solution x)

The HHL algorithm also assumes that A is s-sparse and efficiently computable, that is:

1. A has at most s non-zero entries per row; and
2. entries can be computed in $O(s)$ time.

Computational complexity of solving linear systems:

- Classical: $O(N\, s\, \kappa\, \log \frac{1}{\epsilon})$
- **Quantum** (HHL): $O((\log N)\, s^2\, \kappa^2\, \frac{1}{\epsilon})$

Childs, Kothari & Somma in 2017 improved the quantum complexity to $\log \frac{1}{\epsilon}$.

In practice, whether you should go quantum or classical depends on your use case.

The Harrow-Hassidim-Lloyd (HHL) Algorithm: General Idea

The input matrix A is Hermitian, so

$$A = \sum_{i=1}^{N} \lambda_i \left|u_i\right\rangle\!\left\langle u_i\right|$$

The Harrow-Hassidim-Lloyd (HHL) Algorithm: General Idea

The input matrix A is Hermitian, so

$$A = \sum_{i=1}^{N} \lambda_i \left|u_i\right\rangle\!\left\langle u_i\right|$$

and its eigenvectors $|u_i\rangle$'s are an orthonormal basis, thus:

$$|b\rangle = \sum_{i=1}^{N} b_i \left|e_i\right\rangle = \sum_{i=1}^{N} \beta_i \left|u_i\right\rangle$$

The Harrow-Hassidim-Lloyd (HHL) Algorithm: General Idea

The input matrix A is Hermitian, so

$$A = \sum_{i=1}^{N} \lambda_i \left|u_i\right\rangle\!\left\langle u_i\right|$$

and its eigenvectors $\left|u_i\right\rangle$'s are an orthonormal basis, thus:

$$\left|b\right\rangle = \sum_{i=1}^{N} b_i \left|e_i\right\rangle = \sum_{i=1}^{N} \beta_i \left|u_i\right\rangle$$

Since $A^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_i} \left|u_i\right\rangle\!\left\langle u_i\right|$, we have that

$$\left|x\right\rangle = A^{-1}\left|b\right\rangle = \sum_{i=1}^{N} \frac{1}{\lambda_i} \left|u_i\right\rangle \left\langle u_i\middle|b\right\rangle = \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i} \left|u_i\right\rangle$$

The Harrow-Hassidim-Lloyd (HHL) Algorithm: General Idea

Note that if A is not Hermitian, then the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}$$

is Hermitian, and we solve $\mathcal{A}y = \begin{pmatrix} b \\ 0 \end{pmatrix}$ to obtain $y = \begin{pmatrix} 0 \\ x \end{pmatrix}$.