# Quantum Computing

## Lecture $|11\rangle$: **Order Finding - Shor's Algorithm (II)**

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# Agenda

- Integer division
- Order-finding Problem
- Quantum Algorithm for Integer Factoring (Peter Shor, 1994)

# Integer (Euclidean) Division

**Proposition**

*Given two integers $n, p$ $(p \neq 0)$ there exist* **unique** *integers $q, r$ with $0 \leqslant r < |p|$ s.t.:*

$$n = p \times q + r$$

We say that $q$ is the **quotient** and $r$ is the **remainder (modulo)**.

Examples:

$$n = 31, p = 7 \qquad 31 = 4 \times 7 + 3$$

$$n = 73, p = 8 \qquad 73 = 9 \times 8 + 1$$

# Order-finding Problem

Let $x, N$ be two integers with $x < N$ and **coprime**, *i.e.*, $\gcd(x, N) = 1$.

**Definition**

The **order** of $x$ modulo $N$ is the **least** integer $r$ such that $x^r = 1 \bmod N$.

**Definition (Order-finding Problem)**

Given $x < N$ coprimes, find $r$.

Examples:

$x = 4, N = 7$ $\qquad$ $r = 3$ (because $4^3 = 64 = 9 \times 7 + 1$)

$x = 4, N = 11$ $\qquad$ $r = 5$ (because $4^5 = 1024 = 93 \times 11 + 1$)

# Order-finding Algorithms: Complexity

**Classical**: no algorithm (yet) with polynomial complexity in the input length ($\log N$).

**Quantum**: $poly(\log N)$ algorithm exists! [Quantum Phase Estimation.]

## Quantum Order-finding

**Problem**: Find **least** $r$ such that $x^r = 1 \bmod N$, with $x < N$ and **coprime.**

**Solution**: use QPE with

$$\boxed{U_x \lvert y \rangle = \lvert xy \bmod N \rangle}$$

for $y \in \{0,1\}^L$ and $L = \lceil \log N \rceil$. [If $y > N$, then $U_x$ does nothing, *i.e.*, it maps $y$ to $y$.]

**Proposition**

*$U_x \lvert y \rangle = \lvert xy \bmod N \rangle$ is **unitary**.*

We need to prove $U_x U_x^\dagger = U_x^\dagger U_x = I$, with:

$$U_x = \lvert xy \bmod N \rangle\langle y \rvert \qquad U_x^\dagger = \lvert y \rangle\langle xy \bmod N \rvert$$

Let us prove $U_x^\dagger U_x = I$. [Exercise: prove $U_x U_x^\dagger = I$.]

# Quantum Order-finding

$$
\begin{aligned}
U_x^\dagger U_x &= \sum_y |y\rangle\langle xy \bmod N| \sum_z |xz \bmod N\rangle\langle z| = \sum_{y,z} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| \\
&= \sum_{y=z} |y\rangle\langle z| + \sum_{y\neq z} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| \\
&= I + \sum_{y\neq z\geqslant N} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| + \sum_{y\neq z<N} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| \\
&= I + \sum_{y\neq z\geqslant N} |y\rangle \langle y|z\rangle \langle z| + \sum_{y\neq z<N} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| \qquad (\langle y|z\rangle = \delta_{yz}) \\
&= I + \sum_{y\neq z<N} |y\rangle \langle xy \bmod N|xz \bmod N\rangle \langle z| \\
&= I \qquad (\text{if } x \text{ is coprime with } N \text{ then } xy \equiv xz \bmod N \text{ iff } y \equiv z \bmod N, \text{and } y, z < N)
\end{aligned}
$$

# Quantum Order-finding

What are $U_x$'s eigenvectors and eigenvalues?

**Proposition**

*For any* $0 \leqslant s \leqslant r-1$ *(*$r$ *is the order of* $x \bmod N$*) the vector*

$$|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} \left| x^k \bmod N \right\rangle$$

*is an* **eigenvector** *of* $U_x$*.*

Let's prove it.

We need to find $\lambda \in \mathbb{C}$ such that $U_x |u_s\rangle = \lambda |u_s\rangle$.

# Quantum Order-finding

$$
\begin{aligned}
U_x \left| u_s \right\rangle &= U_x \left( \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} \left| x^k \bmod N \right\rangle \right) \\
&= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} U_x \left| x^k \bmod N \right\rangle \\
&= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} \left| x (x^k \bmod N) \bmod N \right\rangle \\
&= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} \left| x^{k+1} \bmod N \bmod N \right\rangle \\
&= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k / r} \left| x^{k+1} \bmod N \right\rangle
\end{aligned}
$$

# Quantum Order-finding

$$= \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} \left| x^{k+1} \bmod N \right\rangle$$

$$= \frac{1}{\sqrt{r}} e^{2\pi i s/r} e^{-2\pi i s/r} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} \left| x^{k+1} \bmod N \right\rangle$$

$$= \frac{1}{\sqrt{r}} e^{2\pi i s/r} \sum_{k=0}^{r-1} e^{-2\pi i s(k+1)/r} \left| x^{k+1} \bmod N \right\rangle$$

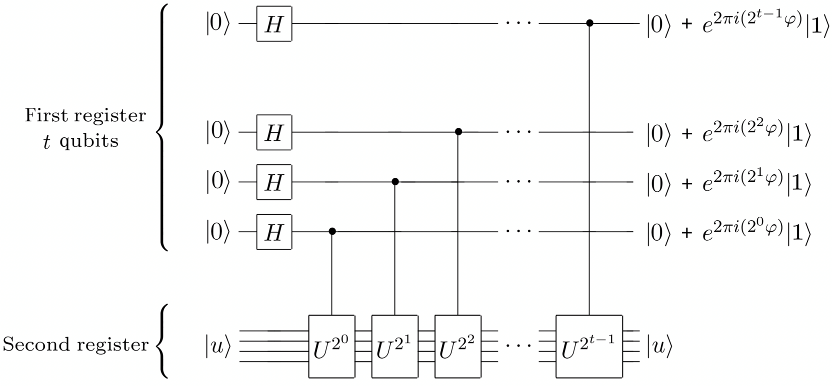
$$= \frac{1}{\sqrt{r}} e^{2\pi i s/r} \sum_{k=0}^{r-1} e^{-2\pi i s k/r} \left| x^{k} \bmod N \right\rangle \qquad \text{(previous sum “wraps” around last term)}$$

$$= e^{2\pi i s/r} \left| u_s \right\rangle$$

Therefore, $|u_s\rangle$ is an eigenvector of $U_x$ with eigenvalue $e^{2\pi i s/r}$.

# Quantum Order-finding

Using QPE we can compute with **high accuracy** the phase of $e^{2\pi is/r}$, *i.e.*, $s/r$.



## Quantum Order-finding: Quantum Circuit

Two problems with QPE:

1. We need controlled-$U$ operations (**modular exponentiation** – non-trivial, but can be done with $O(L^3)$ gates)
2. We must prepare $|u_s\rangle$ in the lower quantum register of the QPE circuit. However, it can be shown that:

$$|1\rangle = \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle$$

   where $|1\rangle$ is an $L$-qubit state.

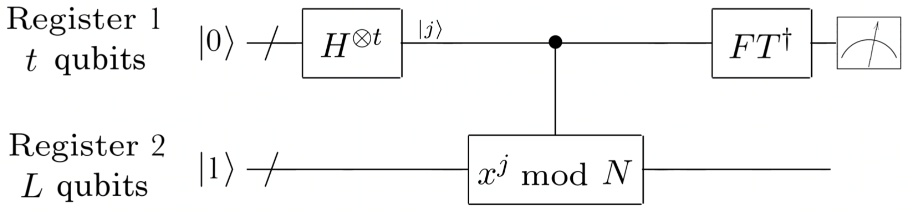
Let us prove problem 2.

# Quantum Order-finding: Quantum Circuit

$$
\begin{aligned}
\frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|u_s\rangle &= \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}\frac{1}{\sqrt{r}}\sum_{k=0}^{r-1} e^{-2\pi isk/r}\,|x^k \bmod N\rangle = \frac{1}{r}\sum_{s,k=0}^{r-1} e^{-2\pi isk/r}\,|x^k \bmod N\rangle \\
&= \frac{1}{r}\sum_{s=0}^{r-1}|1\rangle + \frac{1}{r}\sum_{s=0,k=1}^{r-1} e^{-2\pi isk/r}\,|x^k \bmod N\rangle \\
&= |1\rangle + \frac{1}{r}\sum_{k=1}^{r-1}|x^k \bmod N\rangle \sum_{s=0}^{r-1} e^{-2\pi isk/r} \\
&= |1\rangle + \frac{1}{r}\sum_{k=1}^{r-1}|x^k \bmod N\rangle \sum_{s=0}^{r-1}(e^{-2\pi ik/r})^s \qquad \text{(geometric sum)} \\
&= |1\rangle + \frac{1}{r}\sum_{k=1}^{r-1}|x^k \bmod N\rangle \frac{1-(e^{-2\pi ik/r})^r}{1-e^{-2\pi ik/r}} = |1\rangle \qquad (e^{-2\pi ik} = 1)
\end{aligned}
$$

# Quantum Order-finding: Quantum Circuit

Thus, by using QPE we can get an estimate of $s/r$ for any $s$.



# Quantum Order-finding

Hold on! We can get an accurate estimate for $s/r$, but we actually want $r$.

$r$ can be extracted by the **continued fractions** algorithm $[O(L^3)]$:

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \frac{1}{a_M}}}}$$

where $a_0, a_1, \dots, a_M$ are positive integers. [$r$ can be recovered from the $a_0, a_1, \dots, a_M$.]

## Brief Recap

1. Eigenvalues of unitary operators can be written as $e^{2\pi i\varphi}$, where $\varphi$ is the **phase** (a real number).
2. One can (efficiently) find $\varphi$ using the Quantum Phase Estimation algorithm, which in turn exploits the QFT.
3. The order-finding problem: Find least integer $r$ such that $x^r = 1 \bmod N$, with integers $x < N$ and **coprime** (no common factors).
4. Solving order-finding "quantumly": define a suitable unitary operator that encodes the sought order $r$ in the phase of an eigenvalue.
5. Use QPE to compute the phase and the continued fractions algorithm to extract the order $r$ from the phase.

# Integer Factoring

**Theorem (Fundamental Theorem of Arithmetic (Euclid, 300BC (!)))**

*Any integer $N$ can be written* **uniquely** *as:*

$$N = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots p_m^{\alpha_m}$$

*where $p_1, p_2, \ldots, p_m$ are* **primes** *and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are positive integers.*

**Definition (Integer Factoring Problem)**

Given $N$, find the factors $p_1, p_2, \ldots, p_m$ (and the powers $\alpha_1, \alpha_2, \ldots, \alpha_m$).

Next, we reduce factoring to order-finding.

# Factoring via Order-Finding

Two key theorems:

**Theorem (1)**

*Suppose $N$ is an $L$-bit composite number, and $x$ is a non-trivial solution to the equation $x^2 = 1 \bmod N$ for $1 \leqslant x \leqslant N$ (i.e., neither $x = 1 \bmod N$ nor $x = N - 1 = -1 \bmod N$). Then* **at least one of** $\gcd(x-1, N)$ *and* $\gcd(x+1, N)$ *is a* **non-trivial factor** *of $N$ that can be computed using $O(L^3)$ operations.*

"a non-trivial solution to $x^2 = 1 \bmod N$ can be (efficiently) turned into a factor of $N$"

# Factoring via Order-Finding

**Theorem (2)**

*Suppose* $N = p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots p_m^{\alpha_m}$ *is the prime factorization of an odd composite positive integer* $N$*. Let* $x$ *be an integer chosen uniformly at random between 1 and* $N-1$*, and coprime to* $N$*. Let* $r$ *be the order of* $x \bmod N$*. Then*

$$\mathit{Prob}(r \text{ is even and } x^{r/2} \neq -1 \bmod N) \geqslant 1 - 2^{-m}$$

"with probability at least 50% the order $r$ of $x$ is even and $x^{r/2}$ is not a trivial solution of $x^2 = 1 \bmod N$"

# Quantum Factoring: Shor's Algorithm

**Algorithm 1:** Reduction of factoring to order-finding

**Input:** A composite number $N$

**Output:** A non-trivial factor of $N$

**if** $N$ *is even* **then**
    **return** *2*;

`// there is an efficient classical algorithm for this`

**if** $N = a^b$ *for* $a \geqslant 1$ *and* $b \geqslant 2$ **then**
    **return** $a$;

$x \leftarrow \text{rand}(1 \ldots N-1)$;

**if** $\gcd(x, N) > 1$ **then**
    **return** $\gcd(x, N)$;

$r \leftarrow$ order of $x \bmod N$ ; `// use quantum order-finding algorithm`

**if** $r$ *is even and* $x^{r/2} \neq -1 \bmod N$ **then**
    compute $\gcd(x^{r/2} - 1, N)$ and $\gcd(x^{r/2} + 1, N)$ and return the one that is a non-trivial factor

**else**
    **abort**