# An Introduction to Quantum Computing

## Lecture 18:

## Solving Linear Systems of Equations: The HHL Algorithm (II)

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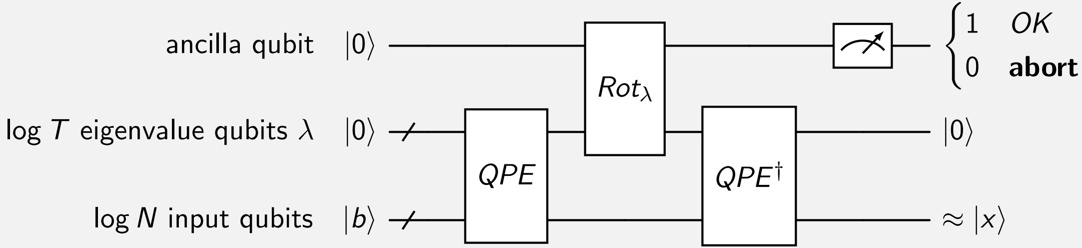
# The HHL Circuit (Algorithm)




If we measure 1 in the ancilla qubit, then the bottom quantum register will be in a state proportial to $x$, the solution vector we are looking for.

The 'rotation' gate $Rot_\lambda$ will be defined later.

# The HHL Circuit (QPE part)

Define $t_0 = O(\frac{\kappa}{\epsilon})$.

The QPE circuit is applied to the unitary $U$ defined as

$$U = e^{iA\frac{t_0}{T}} = \sum_{j=0}^{N-1} e^{i\lambda_j\frac{t_0}{T}} \left|u_j\middle\rangle\middle\langle u_j\right|$$

where $T$ is large enough to fit $A$'s eigenvalues (the $\lambda_j$'s).

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where $T$ is large enough to fit $A$'s eigenvalues (the $\lambda_j$'s). What happens to $U|b\rangle$?

$$U|b\rangle = U\left(\sum_{k=0}^{N-1} \beta_k |u_k\rangle\right) = \sum_{k=0}^{N-1} \beta_k U |u_k\rangle =$$

$$\sum_{k=0}^{N-1} \beta_k \left(\sum_{j=0}^{N-1} e^{i\lambda_j \frac{t_0}{T}} |u_j\rangle \langle u_j|u_k\rangle\right) = \sum_{k=0}^{N-1} \beta_k e^{i\lambda_k \frac{t_0}{T}} |u_k\rangle$$

# The HHL Circuit (QPE part)

Now, for $\tau = 0, 1, \ldots, T-1$ we have that

$$U^{\tau} |b\rangle = \sum_{k=0}^{N-1} \beta_k e^{i\lambda_k t_0 \frac{\tau}{T}} |u_k\rangle$$

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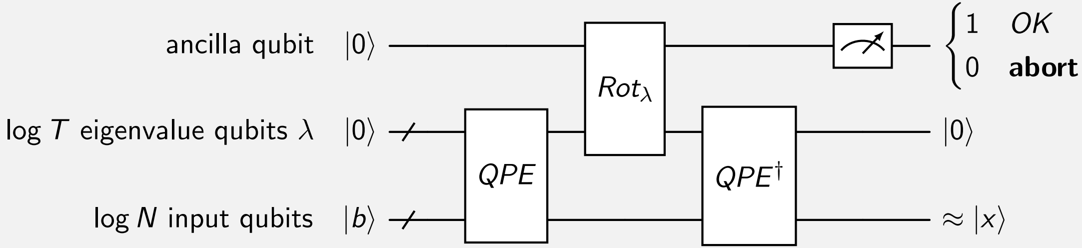
Recall the HHL circuit:



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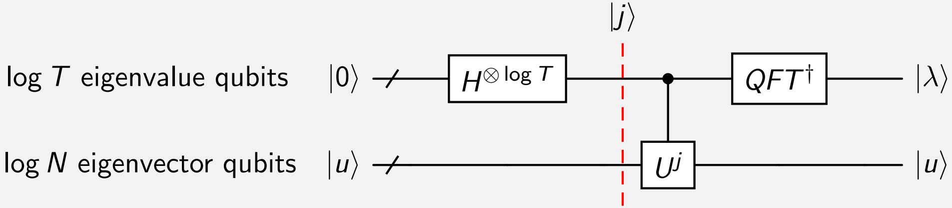
Recall the QPE circuit:



# The HHL Circuit (QPE part)

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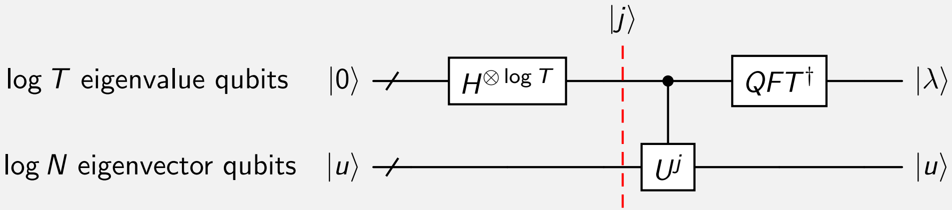


Instead of using Hadamards, we prepare the eigenvalue qubits to the state

$$|\psi\rangle = \sqrt{\frac{2}{T}} \sum_{j=0}^{T-1} \sin \frac{\pi(j+\frac{1}{2})}{T} |j\rangle$$

# The HHL Circuit (QPE part)

The application of the controlled-$U$ in the QPE gate can be described by:

$$\sum_{\tau=0}^{T-1}(|\tau\rangle\langle\tau| \otimes U^{\tau})(|\psi\rangle \otimes |b\rangle) = \sqrt{\frac{2}{T}}\sum_{j=0}^{N-1}\beta_j\sum_{\tau=0}^{T-1} e^{i\lambda_j t_0 \frac{\tau}{T}} \sin\frac{\pi(\tau+\frac{1}{2})}{T}\,|\tau\rangle\,|u_j\rangle$$

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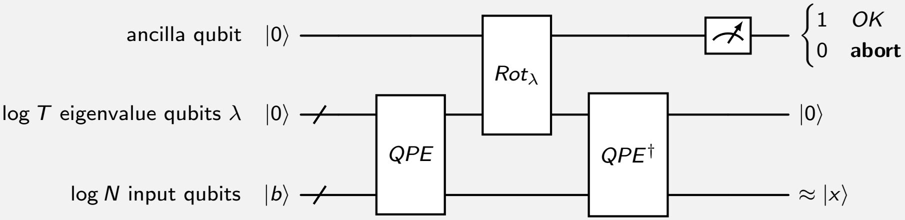
We now apply $QFT^{\dagger}$ and get

$$\frac{\sqrt{2}}{T}\sum_{j=0}^{N-1}\beta_j \sum_{\tau=0}^{T-1}\sum_{k=0}^{T-1} e^{i\frac{k}{T}(\lambda_j t_0 - 2\pi\tau)} \sin\frac{\pi(k+\frac{1}{2})}{T}\, |\tau\rangle\, |u_j\rangle = \sum_{j=0}^{N-1}\beta_j \sum_{\tau=0}^{T-1}\alpha_{j\tau}\, |\tau\rangle\, |u_j\rangle$$

where $\alpha_{j\tau} = \frac{\sqrt{2}}{T}\sum_{k=0}^{T-1} e^{i\frac{k}{T}(\lambda_j t_0 - 2\pi\tau)} \sin\frac{\pi(k+\frac{1}{2})}{T}$.

# The HHL Circuit (Rotation part)




For a constant $C = O(1/\kappa)$ we rotate the ancilla qubit controlled by the eigenvalue qubits – this is the $Rot_\lambda$ gate.

The state of the full system is then:

$$\sum_{j=0}^{N-1} \sum_{\tau=0}^{T-1} \beta_j \alpha_{j\tau} \left( \sqrt{1 - \frac{C^2}{\lambda_\tau^2}} |0\rangle + \frac{C}{\lambda_\tau} |1\rangle \right) |\lambda_\tau\rangle |u_j\rangle$$

where $|\lambda_\tau\rangle = |2\pi\tau/t_0\rangle$.

# The HHL Circuit (Undo QPE)

$$\sum_{j=0}^{N-1}\sum_{\tau=0}^{T-1}\beta_j\alpha_{j\tau}\left(\sqrt{1-\frac{C^2}{\lambda_\tau^2}}\left|0\right\rangle+\frac{C}{\lambda_\tau}\left|1\right\rangle\right)\left|\lambda_\tau\right\rangle\left|u_j\right\rangle$$

Next, we apply $QPE^\dagger$, *i.e.*, we uncompute the $|\lambda_t\rangle$'s, and obtain

$$\sum_{j=0}^{N-1}\beta_j\left(\sqrt{1-\frac{C^2}{\lambda_j^2}}\left|0\right\rangle+\frac{C}{\lambda_j}\left|1\right\rangle\right)\left|\psi\right\rangle\left|u_j\right\rangle$$

# The HHL Circuit (Undo QPE)

$$\sum_{j=0}^{N-1}\sum_{\tau=0}^{T-1}\beta_j\alpha_{j\tau}\left(\sqrt{1-\frac{C^2}{\lambda_\tau^2}}\,|0\rangle+\frac{C}{\lambda_\tau}\,|1\rangle\right)|\lambda_\tau\rangle\,|u_j\rangle$$

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$$\sum_{j=0}^{N-1}\beta_j\left(\sqrt{1-\frac{C^2}{\lambda_j^2}}\,|0\rangle+\frac{C}{\lambda_j}\,|1\rangle\right)|\psi\rangle\,|u_j\rangle$$

Finally, we measure the ancilla qubit: if we obtain 1 then the state is

$$\sqrt{\frac{1}{\sum_{k=0}^{N-1}C^2|\beta_k|^2/|\lambda_k|^2}}\,|1\rangle\,|\psi\rangle\sum_{j=0}^{N-1}\beta_j\frac{C}{\lambda_j}\,|u_j\rangle$$

# The HHL Circuit

Recall from the previous lecture that our sought after solution $x$ can be written as

$$|x\rangle = A^{-1}|b\rangle = \sum_{j=0}^{N-1} \frac{1}{\lambda_j} |u_j\rangle \langle u_j|b\rangle = \sum_{j=0}^{N-1} \frac{\beta_j}{\lambda_i} |u_j\rangle$$

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Therefore, from the previous derivation we have that

$$\sqrt{\frac{1}{\sum_{k=0}^{N-1} C^2|\beta_k|^2/|\lambda_k|^2}} |1\rangle |\psi\rangle \sum_{j=0}^{N-1} \beta_j \frac{C}{\lambda_j} |u_j\rangle \approx |1\rangle |\psi\rangle |x\rangle$$

and the normalization constant in front of the sum can be estimated by sampling.