Quantum Computing

Lecture $|07\rangle$

The Deutsch-Jozsa Algorithm

Paolo Zuliani

zuliani@di.uniroma1.it

Outline

- The Deutsch-Jozsa Problem
- The quantum algorithm

The Deutsch-Jozsa Problem

- For $n \in \mathbb{N}$, consider $f: \mathcal{B}^n \rightarrow \mathcal{B}$

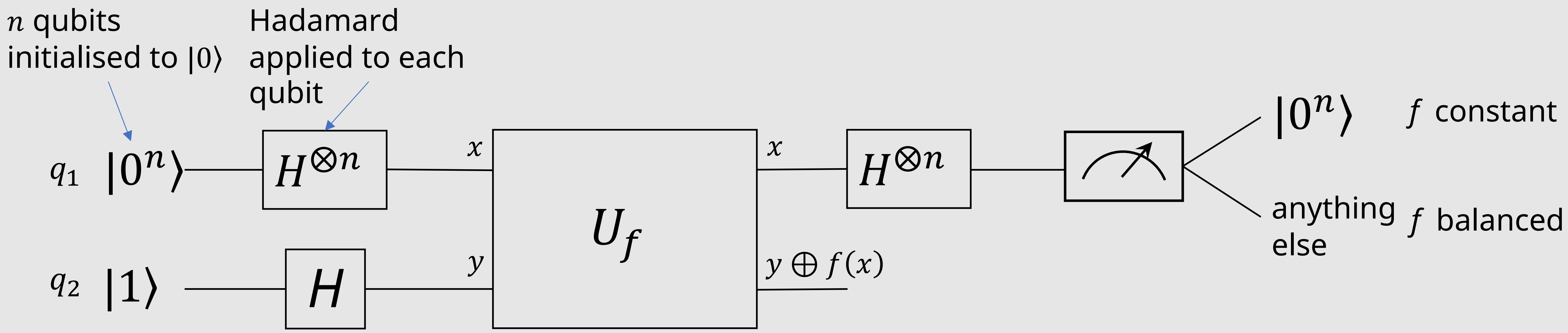
$\boxed{\mathcal{B} = \{0,1\}}$

- f is ***constant***: $\forall x \in \mathcal{B}^n \quad f(x) = 0 \qquad (\text{or } 1)$
- f is ***balanced***: $\sum_{x \in \mathcal{B}^n} f(x) = \frac{2^n}{2} = 2^{n-1}$
- A constant function cannot be balanced, and a balanced function cannot be constant
- Most Boolean functions are neither constant nor balanced

The Deutsch-Jozsa Problem

- $f\colon \mathcal{B}^n \rightarrow \mathcal{B}$ is either ***constant*** or ***balanced*** (we don't know which one)
- **D-J Problem**: decide whether f is constant or f is balanced
- Complexity (*classical*): in the worst case, we need $(2^{n-1}+1)$ evaluations of f to make a correct decision
- Complexity (*quantum*): ***one*** evaluation of f suffices!!

The Deutsch-Jozsa Algorithm



Using a "programming" notation:

$q_1, q_2 = |0^n 1\rangle;$
$q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2);$
$q_1, q_2 = U_f(q_1, q_2);$
$q_1 = H^{\otimes n}(q_1);$
$b = Measure(q_1)$

// q_1 is a quantum register of *n* qubits!

Preliminary: The Hadamard Transform

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\Rightarrow \quad H|a\rangle = \frac{1}{\sqrt{2}}\sum_{b\in\mathcal{B}}(-1)^{a\cdot b}|b\rangle$$

Now two qubits:

$$H\otimes H|x\rangle\otimes|y\rangle = H\otimes H|xy\rangle$$

$$= \Big(\frac{1}{\sqrt{2}}\sum_{a\in\mathcal{B}}(-1)^{x\cdot a}|a\rangle\Big)\otimes\Big(\frac{1}{\sqrt{2}}\sum_{b\in\mathcal{B}}(-1)^{y\cdot b}|b\rangle\Big)$$

$$= \frac{1}{2}\sum_{a,b\in\mathcal{B}}(-1)^{x\cdot a+y\cdot b}|a\rangle|b\rangle = \frac{1}{2}\sum_{a\in\mathcal{B}^2}(-1)^{xy\cdot a}|a\rangle$$

Preliminary: The Hadamard Transform

So, for two qubits:

$$H \otimes H |xy\rangle = \frac{1}{2} \sum_{a \in \mathcal{B}^2} (-1)^{xy \cdot a} |a\rangle$$

In general, for $x \in \mathcal{B}^n$ (*i.e.*, a string of n bits)

$$H^{\otimes n} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{a \in \mathcal{B}^n} (-1)^{x \cdot a} |a\rangle$$

where $x \cdot a = \left(\sum_{i=0}^{n-1} x_i \cdot a_i\right) \bmod 2$

The Deutsch-Jozsa Algorithm

$q_1, q_2 = |0^n 1\rangle;$
$q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2);$
$q_1, q_2 = U_f(q_1, q_2);$
$q_1 = H^{\otimes n}(q_1);$
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$|0^n 1\rangle$

Apply $q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2)$

$$= \sum_{x \in \mathcal{B}^n} \frac{|x\rangle}{\sqrt{2^n}} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Apply $q_1, q_2 = U_f(q_1, q_2)$

$$= \sum_{x \in \mathcal{B}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Recall that, for $a \in \mathcal{B}^n$

$$U_f(|a\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}) = (-1)^{f(a)}|a\rangle \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

The Deutsch-Jozsa Algorithm

$\cancel{q_1, q_2 = |0^n 1\rangle;}$
$\cancel{q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2);}$
$\cancel{q_1, q_2 = U_f(q_1, q_2);}$
$q_1 = H^{\otimes n}(q_1);$
$b = Measure(q_1)$

$$= \sum_{x \in \mathcal{B}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Apply $q_1 = H^{\otimes n}(q_1)$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{a \in \mathcal{B}^n} (-1)^{x \cdot a}|a\rangle$$

$$= \sum_{x \in \mathcal{B}^n} \sum_{a \in \mathcal{B}^n} \frac{(-1)^{x \cdot a + f(x)}|a\rangle}{2^n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The Deutsch-Jozsa Algorithm

$~~q_1, q_2 = |0^n 1\rangle;~~$
$~~q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2);~~$
$~~q_1, q_2 = U_f(q_1, q_2);~~$
$~~q_1 = H^{\otimes n}(q_1);~~$
$~~b = Measure(q_1)~~$

$$= \sum_{x \in \mathcal{B}^n} \sum_{a \in \mathcal{B}^n} \frac{(-1)^{x \cdot a + f(x)} |a\rangle}{2^n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \sum_{a \in \mathcal{B}^n} \sum_{x \in \mathcal{B}^n} \frac{(-1)^{x \cdot a + f(x)} |a\rangle}{2^n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \sum_{a \in \mathcal{B}^n} |a\rangle \sum_{x \in \mathcal{B}^n} \frac{(-1)^{x \cdot a + f(x)}}{2^n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Apply $b = Measure(q_1)$

The amplitude for $|0^n\rangle$ is computed from the sum above by setting $a = 0^n$

$$\Rightarrow \quad \frac{1}{2^n} \sum_{x \in \mathcal{B}^n} (-1)^{x \cdot a + f(x)} = \frac{1}{2^n} \sum_{x \in \mathcal{B}^n} (-1)^{f(x)}$$

The Deutsch-Jozsa Algorithm

$\cancel{q_1, q_2 = |0^n 1\rangle;}$
$\cancel{q_1, q_2 = H^{\otimes n} \otimes H(q_1, q_2);}$
$\cancel{q_1, q_2 = U_f(q_1, q_2);}$
$\cancel{q_1 = H^{\otimes n}(q_1);}$
$\cancel{b = Measure(q_1)}$

The amplitude for $|0^n\rangle$ is thus

$$\frac{1}{2^n} \sum_{x \in \mathcal{B}^n} (-1)^{f(x)} = \begin{cases} \pm 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ is balanced} \end{cases}$$

If we measure 0^n we know <u>precisely</u> that f is constant.
If we measure any other state, then f <u>must be</u> balanced.

Only <u>one</u> evaluation of f is needed!