Quantum Computing

Lecture $|09\rangle$: **Fixed-Point Quantum Search**

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Agenda

- The "soufflé problem"
- Grover's fixed-point quantum search algorithm (2005)

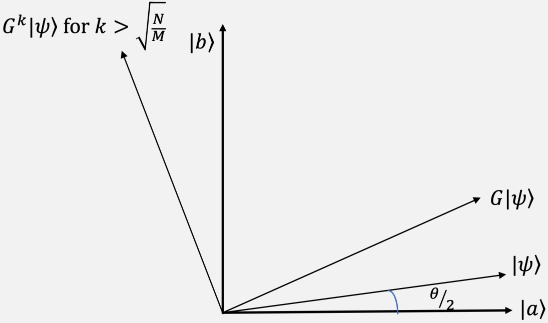
The “souffle problem”

Quantum searching is like cooking a soufflé – one needs to open the oven (stop the Grover iterations) at the right time, otherwise one risks:

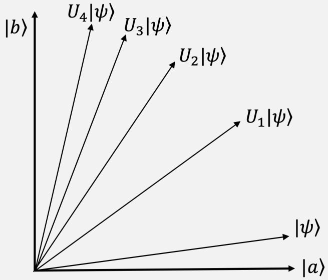
- burning the soufflé (overshooting the target states); or
- undercooking the soufflé (undershooting the targets states)

To know the right time (number of iterations), one needs to know the number of solution elements in the array.

Fixed-Point Quantum Search vs. Grover's Algorithm



Grover's algorithm overshoots the target



Grover's **fixed-point** algorithm converges to the target

The fixed-point algorithm **monotonically** moves towards the target!

Fixed-Point Quantum Search: Operators

Define the **phase shift** operators:

$$R_s = I - (1 - e^{i\theta})\,|s\rangle\langle s|$$

$$R_t = I - (1 - e^{i\theta})\,|t\rangle\langle t|$$

where $\theta = \frac{\pi}{3}$, $\quad |s\rangle =$ starting state, $\quad |t\rangle =$ target state.

[Exercise: show that both operators are unitary.]

Fixed-Point Quantum Search: Operators

Let U be **any** unitary operator such that, for some (small) $\epsilon > 0$:

$$|\langle t|Us\rangle|^2 = 1 - \epsilon.$$

Informally, "U drives $|s\rangle$ towards $|t\rangle$ with a probability of $(1 - \epsilon)$."

Define the (unitary) operator:

$$UR_sU^\dagger R_tU.$$

Now, it can be shown that $UR_sU^\dagger R_tU|s\rangle$ gets to $|t\rangle$ with a probability of $(1 - \epsilon^3)$.

Fixed-Point Quantum Search: Towards the Algorithm

More formally, by denoting $U_{ts} = \langle t|Us\rangle$ it can be shown that

$$UR_s U^\dagger R_t U |s\rangle = U |s\rangle [e^{i\theta} + |U_{ts}|^2 (e^{i\theta} - 1)^2] + |t\rangle U_{ts}(e^{i\theta} - 1). \tag{1}$$

Now, to know how close the state above is to $|t\rangle$, we compute the squared norm of:

$$\begin{aligned} &|t\rangle\langle t| \left[U |s\rangle [e^{i\theta} + |U_{ts}|^2 (e^{i\theta} - 1)^2] + |t\rangle U_{ts}(e^{i\theta} - 1)\right] \\ =\, &|t\rangle \left[U_{ts}[e^{i\theta} + |U_{ts}|^2 (e^{i\theta} - 1)^2] + U_{ts}(e^{i\theta} - 1)\right] \\ =\, &|t\rangle U_{ts}[2e^{i\theta} - 1 + |U_{ts}|^2 (e^{i\theta} - 1)^2] \end{aligned}$$

which is simply $\left|U_{ts}[2e^{i\theta} - 1 + |U_{ts}|^2 (e^{i\theta} - 1)^2]\right|^2$ since $\||t\rangle\| = 1$. A bit more work ...

Fixed-Point Quantum Search: Towards the Algorithm

Recalling that $\theta = \frac{\pi}{3}$, we get that the sought after deviation from $|t\rangle$ is:

$$\left| U_{ts}[2e^{i\theta} - 1 + |U_{ts}|^2(e^{i\theta} - 1)^2] \right|^2 = \left| U_{ts} \left(-\frac{1}{2}|U_{ts}|^2 + i\frac{\sqrt{3}}{2}(2 - |U_{ts}|^2) \right) \right|^2$$

Finally, recalling that $|U_{ts}|^2 = (1 - \epsilon)$, we get

$$\left| U_{ts} \left(-\frac{1}{2}|U_{ts}|^2 + i\frac{\sqrt{3}}{2}(2 - |U_{ts}|^2) \right) \right|^2 = 1 - \epsilon^3$$

and therefore the deviation from $|t\rangle$ has been **reduced** from ϵ to ϵ^3!

Fixed-Point Quantum Search: the Algorithm

Given U satisfying $|U_{ts}|^2 = 1 - \epsilon$, we define the **recursive** sequence of operators:

$$\begin{aligned} U_0 &= U \\ U_m &= U_{m-1} R_s U_{m-1}^{\dagger} R_t U_{m-1} \qquad \text{for } m \geqslant 1 \end{aligned}$$

It can be shown that:

$$|\langle t|U_m s\rangle|^2 = 1 - \epsilon^{2q_m+1} \qquad\qquad q_m = \#\text{ of queries to the oracle}$$

This is (unfortunately) similar to a classical probabilistic algorithm where the failure probability drops as ϵ^{q+1} after q oracle queries. The quantum advantage is thus lost.

Better Fixed-Point Quantum Search

- A 2014 paper proposed a fixed-point quantum algorithm that monotonically converges to the target state while **retaining quadratic advantage** over classical algorithms.
- The algorithm involves phase-shift operators that are parameterized with angles different from $\frac{\pi}{3}$ and again involves building a sequence of operators using said phase shifts, the oracle, and the unitary which prepares the starting state.
- Reference: Fixed-Point Quantum Search with an Optimal Number of Queries. Physical Review Letters 113, 210501 (2014).

Proof of Claim (1)

Need to show that $UR_sU^\dagger R_tU\,|s\rangle = U\,|s\rangle\,[e^{i\theta} + |U_{ts}|^2(e^{i\theta}-1)^2] + |t\rangle\,U_{ts}(e^{i\theta}-1)$.

First, let us massage the operator form (and define $P_s = |s\rangle\langle s|$ and $P_t = |t\rangle\langle t|$):

$$\begin{aligned}
UR_sU^\dagger R_tU =& U(I-(1-e^{i\theta})P_s)U^\dagger(I-(1-e^{i\theta})P_t)U \qquad \text{[expand } R_s \text{ and } R_t\text{]}\\
=&(U-(1-e^{i\theta})UP_s)U^\dagger(U-(1-e^{i\theta})P_tU)\\
=&(I-(1-e^{i\theta})UP_sU^\dagger)(U-(1-e^{i\theta})P_tU)\\
=&U-(1-e^{i\theta})P_tU-(1-e^{i\theta})UP_s+(1-e^{i\theta})^2UP_sU^\dagger P_tU
\end{aligned}$$

Proof of Claim (1)

Now, we apply the previous (massaged) operator to $|s\rangle$:

$$UR_sU^\dagger R_tU|s\rangle = U|s\rangle - (1-e^{i\theta})P_tU|s\rangle - (1-e^{i\theta})UP_s|s\rangle + (1-e^{i\theta})^2UP_sU^\dagger P_tU|s\rangle$$

$$= \qquad \text{[apply projectors; } U_{ts} = \langle t|Us\rangle\text{]}$$

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2UP_sU^\dagger|t\rangle U_{ts} =$$

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2U|s\rangle\langle s|U^\dagger|t\rangle U_{ts} =$$

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2U|s\rangle\left\langle s\middle|U^\dagger t\right\rangle U_{ts}$$

$$= \qquad \text{[adjoint property]}$$

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2U|s\rangle\langle Us|t\rangle U_{ts} =$$

Proof of Claim (1)

Resuming from the last equality:

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2 U|s\rangle \langle Us|t\rangle U_{ts}$$

$$= \qquad [U^*_{ts} = \langle Us|t\rangle\ ]$$

$$U|s\rangle - (1-e^{i\theta})|t\rangle U_{ts} - (1-e^{i\theta})U|s\rangle + (1-e^{i\theta})^2 U|s\rangle |U_{ts}|^2 =$$

$$= \qquad [\text{algebra}]$$

$$U|s\rangle [e^{i\theta} + |U_{ts}|^2 (e^{i\theta}-1)^2] + |t\rangle U_{ts}(e^{i\theta}-1).$$