Autonomous Networking

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Today’s plan

Formalization of sequential decision making

- Markov Processes
- Markov Reward Processes
- Markov Decision Processes

- The first step in applying reinforcement learning is to formulate the problem as an MDP
  - Markov process
  - We add rewards -> Markov Reward Processes
  - We add actions -> Markov Decision Processes

Why MDP?

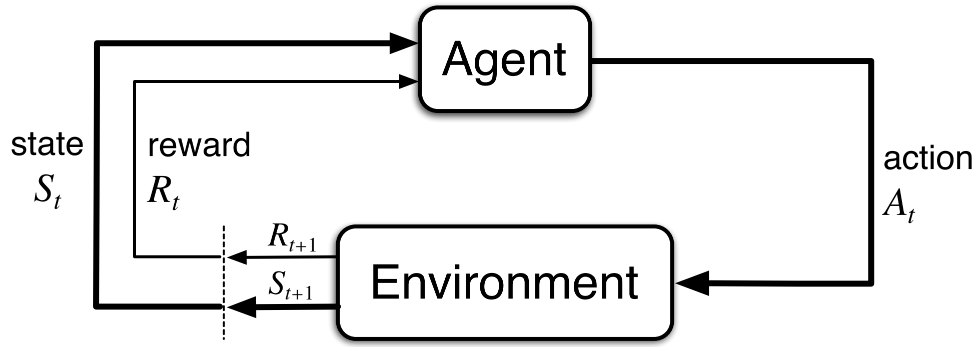
MDPs are a **classical formalization of sequential decision making**, where *actions influence* not just *immediate rewards*, but also *subsequent situations* (states) and through those future rewards

MDPs involve ***delayed*** rewards and the need to trade off immediate and delayed rewards

Whereas in bandit we estimated the q*(a) of each action a, in MDPs **we estimate the value q*(a,s) of each action a in each state s, or we estimate the value v*(s) of each state s given optimal action selection**

The agent-environment interface

- MDPs are meant to be a straightforward framing of the problem of learning from interaction to achieve a goal.



- The agent and environment interact at each of a sequence of discrete time steps, t = 0, 1, 2, 3, . . .
- At each timestep, the agent receives some representation of the environment state and on that basis selects an action
- One time step later, in part as a consequence of its action, the agent receives a numerical reward and finds itself in a new state

Introduction to MDP

- Markov decision processes formally **describe an environment for reinforcement learning**
- Where the environment is *fully observable*
- i.e. The current state completely characterises the process
- Almost all RL problems can be formalised as MDPs,
  - e.g. Bandits are MDPs with one state

Markov property

"The future is independent of the past given the present"

Definition

A state S_t is Markov if and only if $\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1,\ldots,S_t]$

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

State Transition Matrix

- For a Markov state *s* and successor state *s'*, the ***state transition probability*** is defined by

$$P_{ss'} = \mathbb{P}\left[S_{t+1}=s' \mid S_t=s \right]$$

- State transition matrix *P* defines transition probabilities from all states *s* to all successor states *s'*

$$\mathcal{P} = \begin{array}{cc} & \textit{to} \\ \textit{from} & \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \end{array}$$

- where each row of the matrix sums to 1.

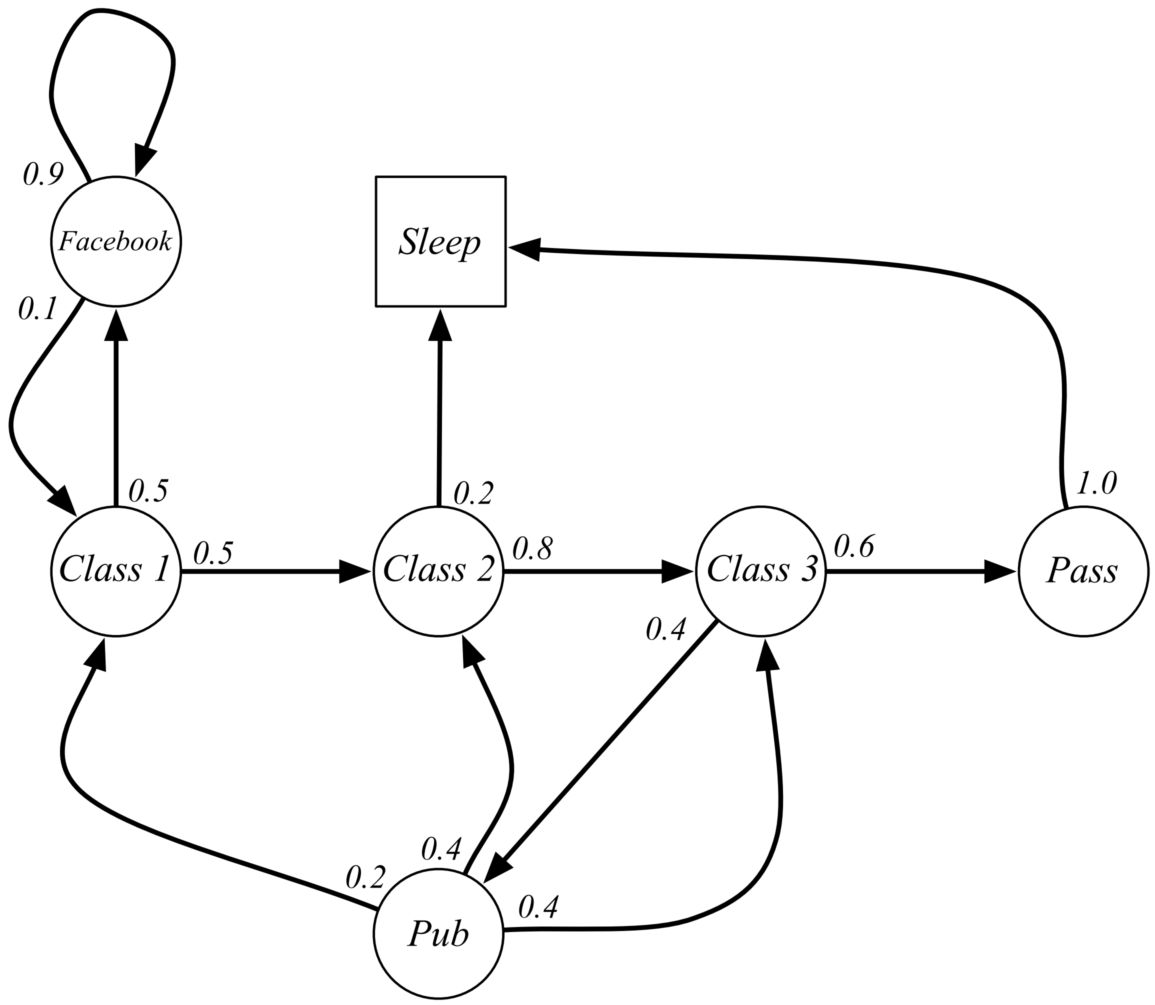
Markov process

- A Markov process is a **memoryless random process**, i.e. **a sequence of random states S1, S2, ... with the Markov property**.

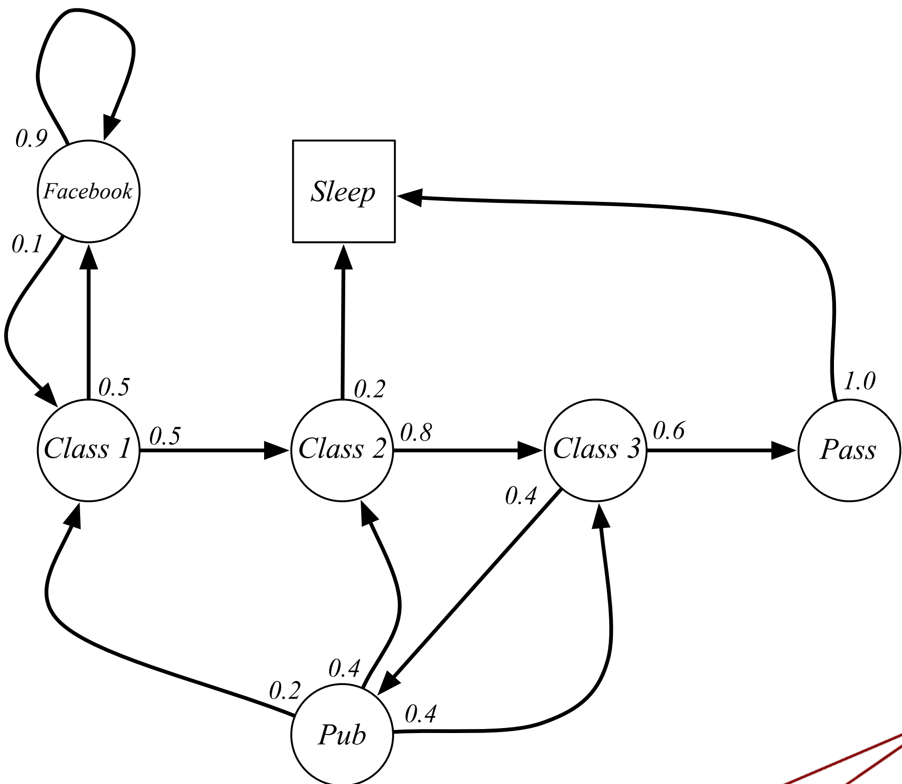
Definition

A Markov Process (or Markov Chain) is a tuple $\langle S, P \rangle$
- S is a (finite) set of states
- P is a state transition probability matrix, $P_{ss'} = \mathbb{P}\,[\,S_{t+1} = s' \mid S_t = s\,]$

Example: Student Markov Chain



Example: Student Markov Chain Episodes



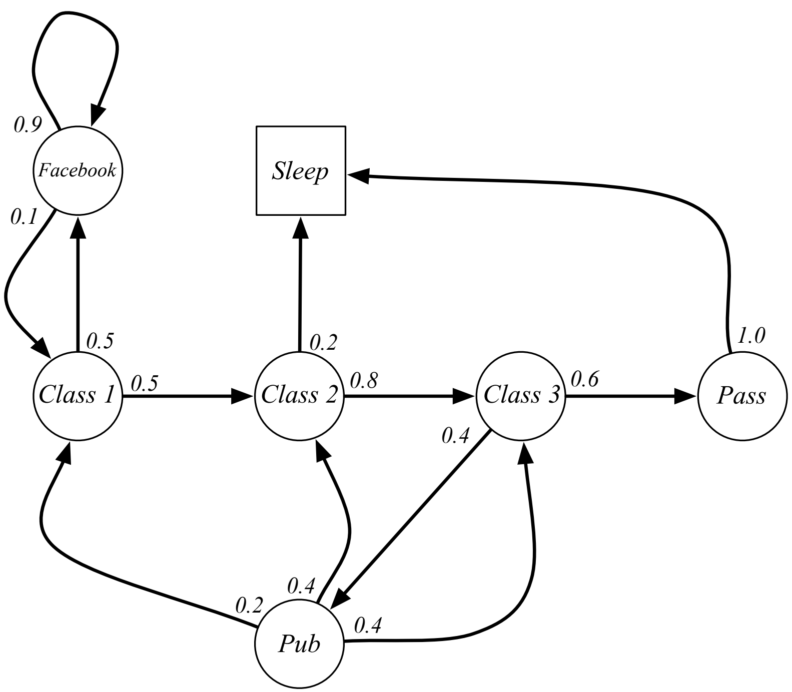
Random sequences drawn from probabilities

Sample episodes for Student Markov Chain starting from $S_1 = C_1$

$$S_1, S_2, \ldots, S_T$$

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

Example: Student Markov Chain Transition Matrix



$$\mathcal{P} = \begin{array}{cc} & \begin{array}{ccccccc} \mathit{C1} & \mathit{C2} & \mathit{C3} & \mathit{Pass} & \mathit{Pub} & \mathit{FB} & \mathit{Sleep} \end{array} \\ \begin{array}{c} \mathit{C1} \\ \mathit{C2} \\ \mathit{C3} \\ \mathit{Pass} \\ \mathit{Pub} \\ \mathit{FB} \\ \mathit{Sleep} \end{array} & \left[\begin{array}{ccccccc} & 0.5 & & & & 0.5 & \\ & & 0.8 & & & & 0.2 \\ & & & 0.6 & 0.4 & & \\ & & & & & & 1.0 \\ 0.2 & 0.4 & 0.4 & & & & \\ 0.1 & & & & & 0.9 & \\ & & & & & & 1 \end{array} \right] \end{array}$$

Outline

Formalization of sequential decision making

1. Markov Processes
   - We have seen the basics on Markov processes but we have not talked about RL

2. Markov Reward Processes
   - Let us add rewards to our process
   - How much reward do I accumulate across a particular sequence

3. Markov Decision Processes

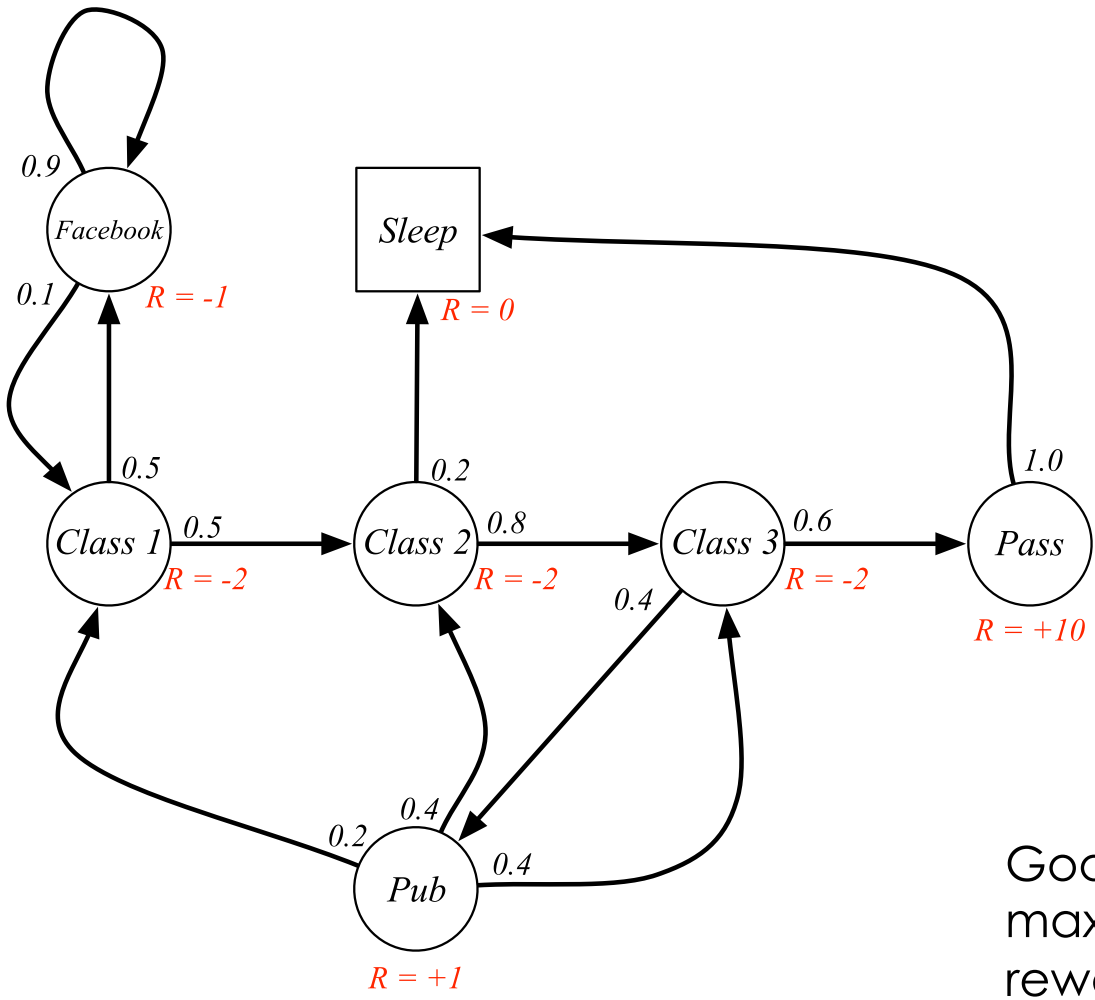
Markov Reward Process

- A *Markov reward process* is a Markov chain with values.

Definition

A Markov Reward Process is a tuple $\langle S, P, R, \gamma \rangle$
- S is a (finite) set of states
- P is a state transition probability matrix, $P_{ss'} = \mathbb{P} [S_{t+1}=s' \mid S_t=s]$
- R is a reward function, $R_s = \mathbb{E} [R_{t+1} \mid S_t = s]$
- γ is a discount factor, $\gamma \in [0, 1]$

Example: Student MRP



Goal: we want to maximize the rewards we obtain

Return

Definition

The return G_t is the total discounted reward from time-step t.

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The discount $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R, after k + 1 time-steps is $\gamma^k R$
- This values immediate reward above delayed reward
  - γ close to 0 leads to "short-sighted" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
- Uncertainty about the future may not be fully represented

- It is sometimes possible to use undiscounted Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate.

Value function

- The value function v(s) gives the long-term value of (being in) state s

Definition

The state value function v(s) of an MRP is the **expected return starting from state s**

$$V_s = \mathbb{E}\,[Gt\ |St\ = s]$$

Example: Student MRP Returns

- Sample returns for Student MRP:

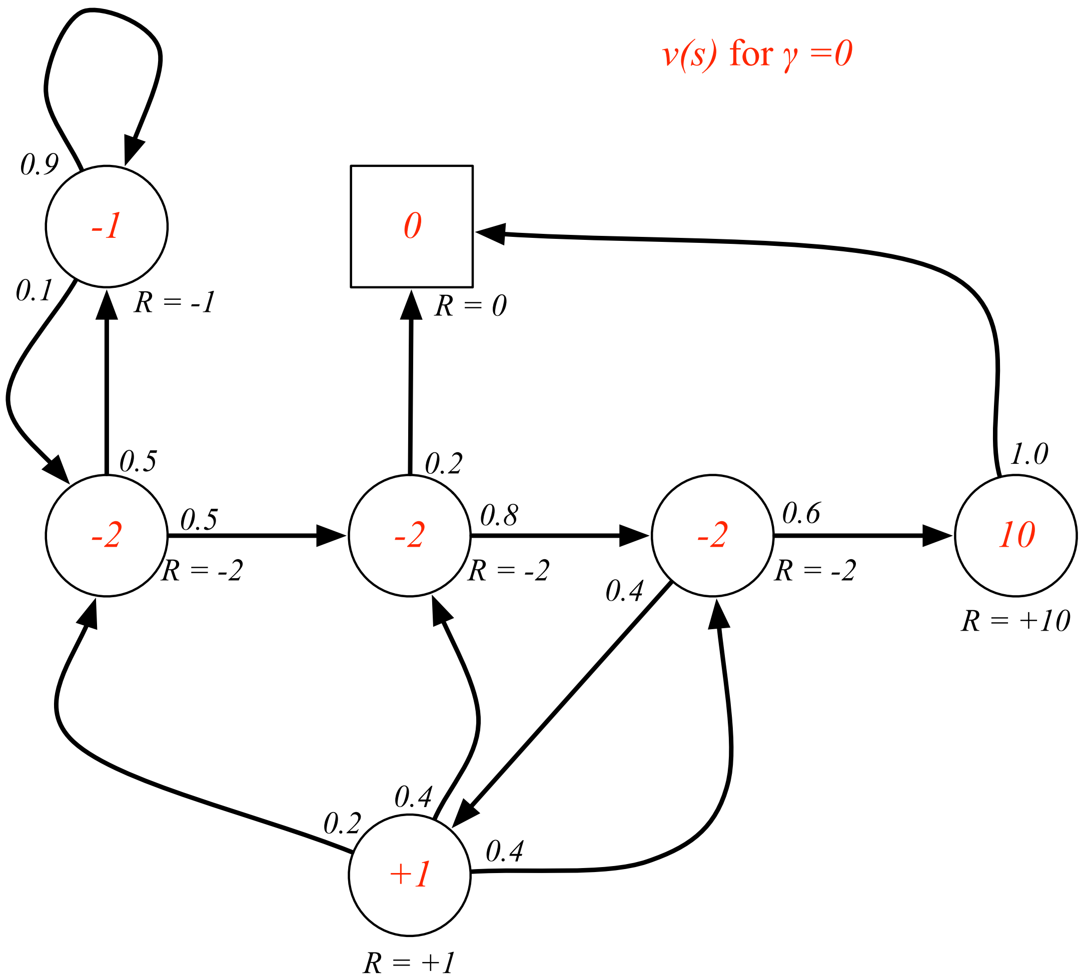
Starting from S_1 = C1 with $\gamma = \frac{1}{2}$

$$G_1 = R_2 + \gamma R_3 + \cdots + \gamma^{T-2} R_T$$

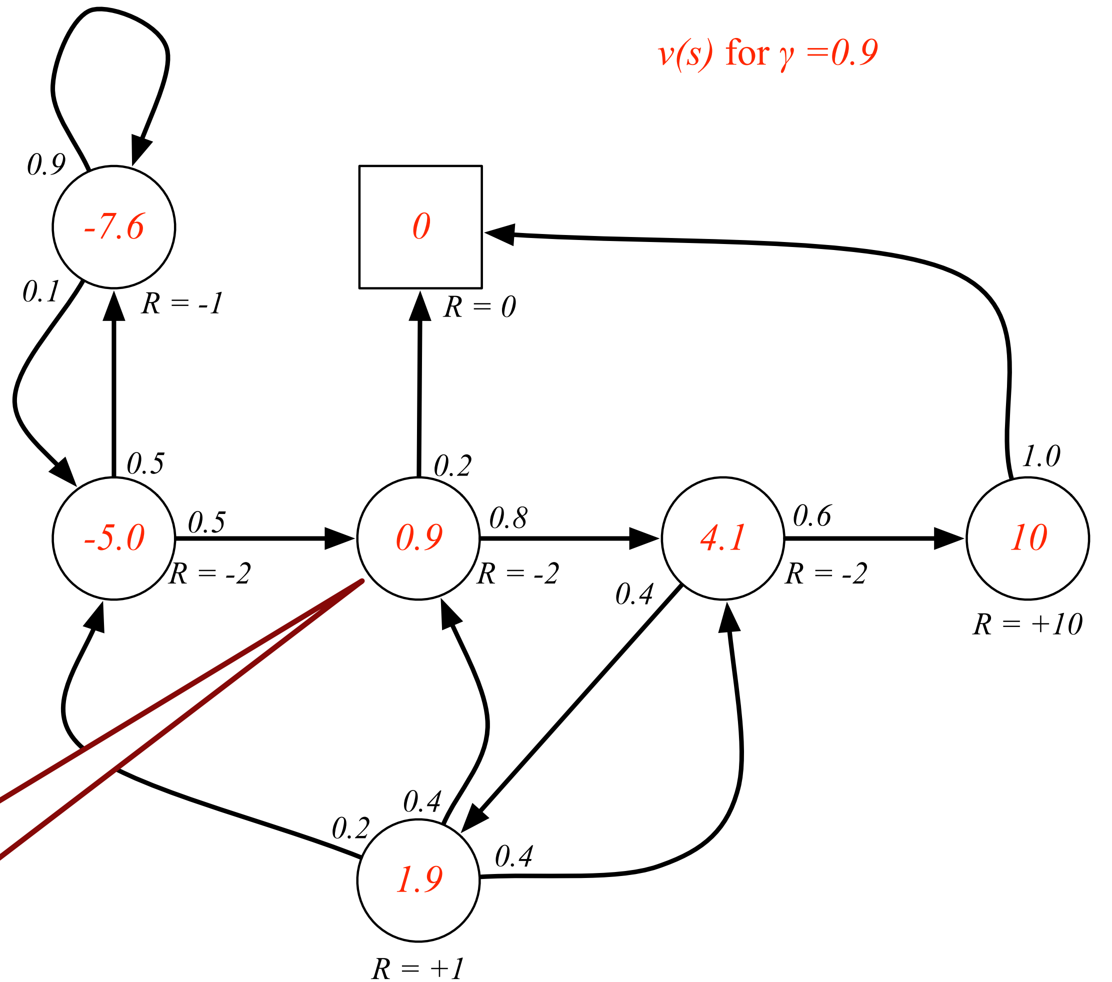
γ

C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.20
FB FB FB C1 C2 C3 Pub C2 Sleep			

Example: State-Value Function for Student MRP (γ=0)

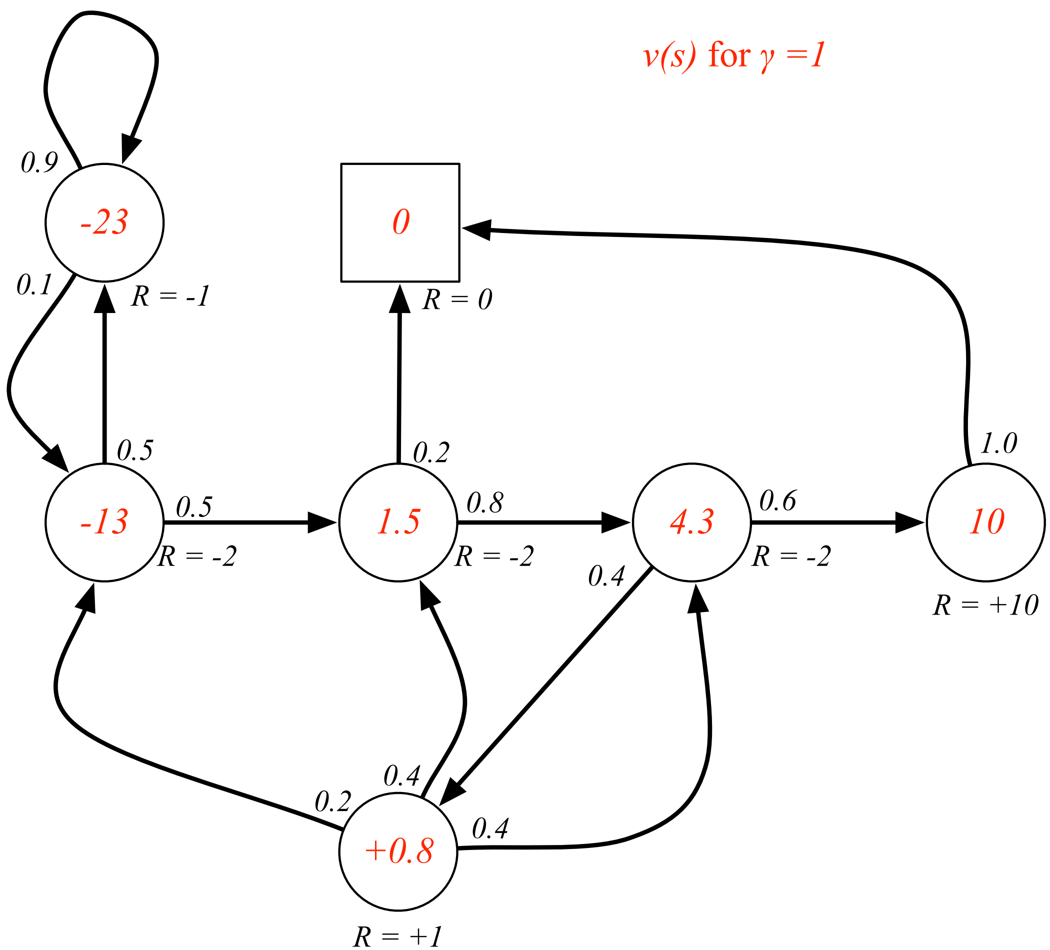


Example: State-Value Function for Student MRP (γ=0.9)



Averages over the probability of falling asleep and continue classes and also considers future rewards

Example: State-Value Function for Student MRP (γ=1)



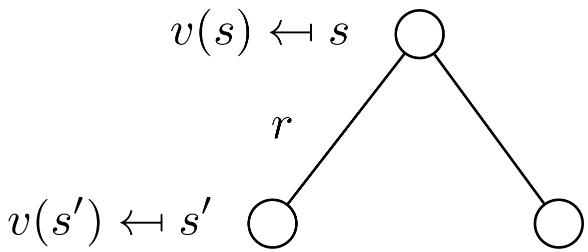
Bellman Equation for MRPs

- The value function can be decomposed into two parts:
  - immediate reward R_{t+1}
  - discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}\left[G_t \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma\left(R_{t+2} + \gamma R_{t+3} + \ldots\right) \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma G_{t+1} \mid S_t = s\right] \\
&= \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]
\end{aligned}
$$

Bellman Equation for MRPs (2)

$$v(s) = \mathbb{E}\left[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s\right]$$



- Bellman equations expresses a relationship between the value of a state and the values of its **successor states**

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

- Bellman equation **averages over all the possibilities, weighting each by its probability of occurring**
- The value of the start state must be equal the (discounted) value of the expected next state, plus the reward expected along the way

Example: Bellman Equation for Student MRP

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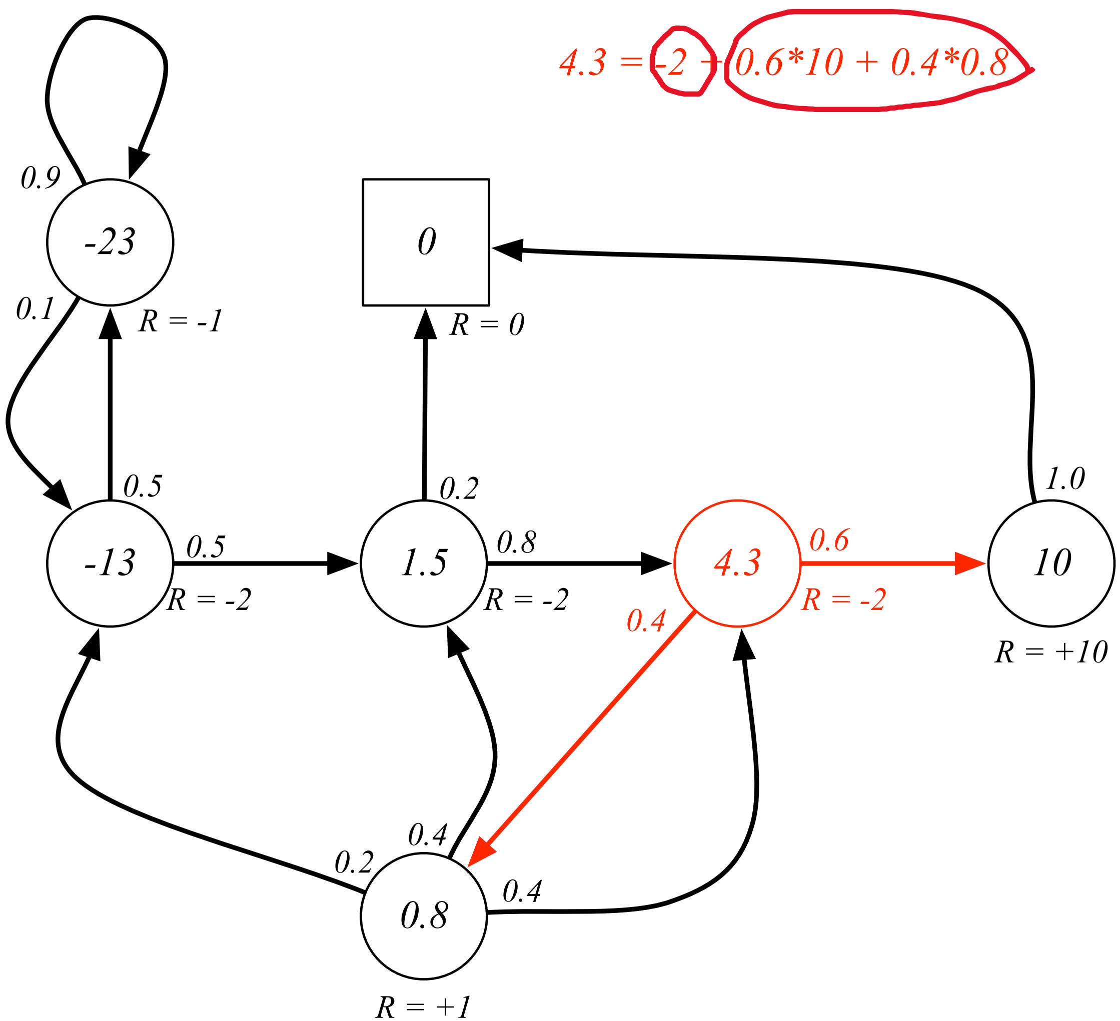
$\gamma=1$



Bellman Equation in Matrix Form

- The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

- where v is a column vector with one entry per state, R is the vector of immediate reward, P is transition probability matrix

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \dots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \dots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$
\begin{aligned}
v &= \mathcal{R} + \gamma \mathcal{P} v \\
(I - \gamma \mathcal{P})\, v &= \mathcal{R} \\
v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Outline

Formalization of sequential decision making

1. Markov Processes
2. Markov Reward Processes
3. Markov Decision Processes

Markov Decision Process

- A Markov decision process (MDP) is a **Markov reward process with decisions.** It is an *environment* in which all states are Markov

Definition

A Markov Reward Process is a tuple ⟨S, A, P, R, γ⟩

- S is a (finite) set of states
- A is a finite set of actions
- P is a state transition probability matrix,

  $$P^{a}_{ss'} = \mathbb{P}\,[\,S_{t+1}=s' \mid S_t=s, A_t=a\,]$$

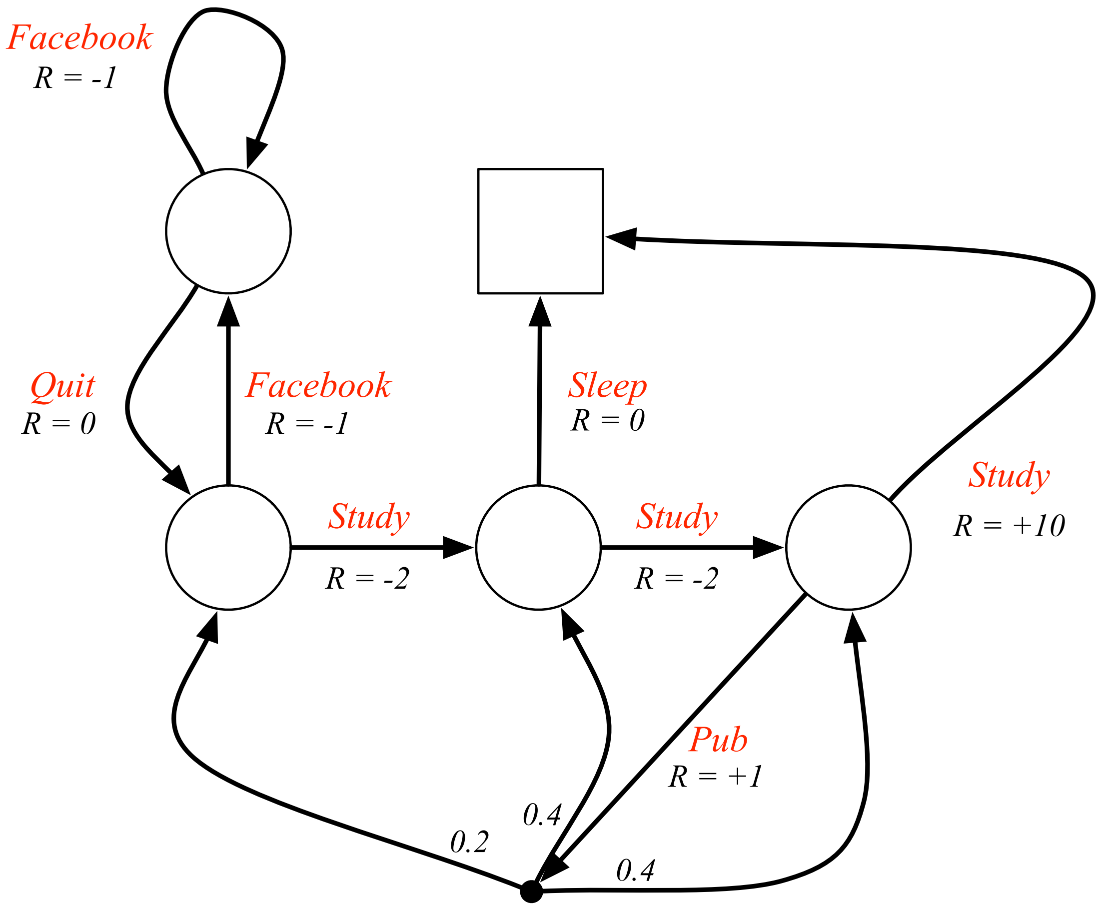
- R is a reward function,

  $$R^{a}_{s} = \mathbb{E}\,[\,R_{t+1} \mid S_t = s, A_t=a\,]$$

- γ is a discount factor, $\gamma \in [0, 1]$

One matrix for each action

Example: Student MDP



- **Actions** in red
- Now **I choose the action**, e.g. study or go to facebook
- The goal is to find the best path to maximize rewards
- How do we make decisions?

Policies (1)

Definition

A policy is a distribution over actions given states

$$\pi(a \mid s) = \mathbb{P} [A_t = a \mid S_t = s]$$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are stationary (time-independent, do not depends on the time step, but only on the state)

Value Function

Definition

The *state-value function* $v_\pi(s)$ of an MDP is the expected return starting from state *s*, and then **following policy** $\boldsymbol{\pi}$

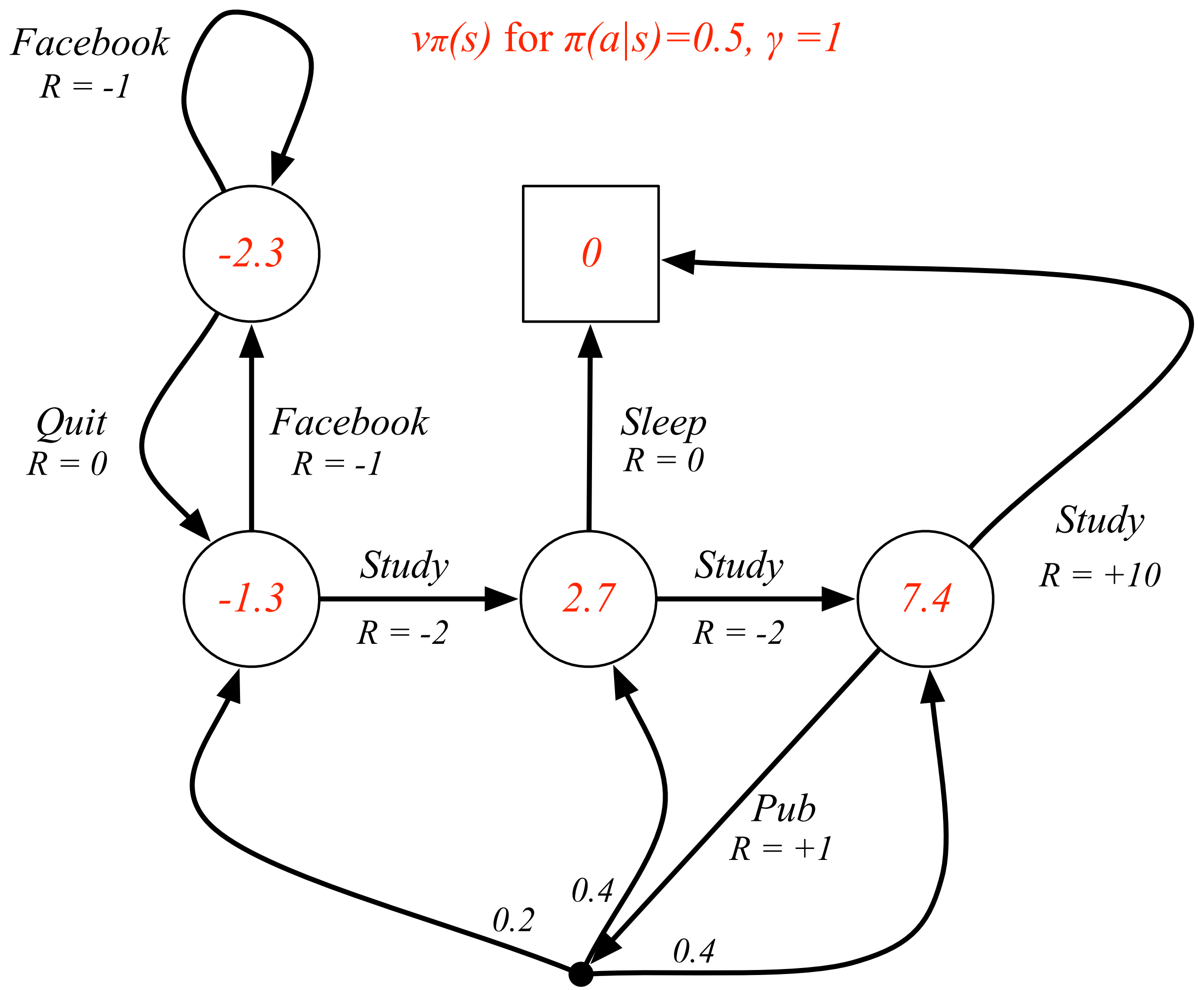
$$v_\pi(s) = \mathbb{E}_\pi [\, G_t \mid S_t{=}s \,]$$

Definition

The *action-value function* $q_\pi(s,a)$ is the expected return starting from state *s*, **taking action a, and then following policy** $\boldsymbol{\pi}$

$$q_\pi(a \mid s) = \mathbb{E}_\pi [\, G_t \mid S_t{=}s, A_t{=}a \,]$$

Example: State-Value Function for Student MDP



Bellman Expectation Equation (with policy)

- The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s \right]$$

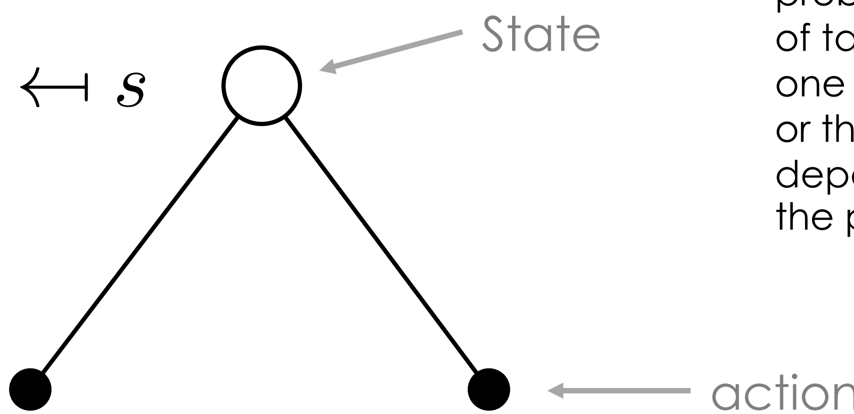
- The action-value function can similarly be decomposed,

$$q_\pi(s, a) = \mathbb{E}_\pi \left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a \right]$$

Bellman Expectation Equation for V^{π}

The probability of taking one action or the other depends on the policy

$$v_{\pi}(s) \leftarrow s$$

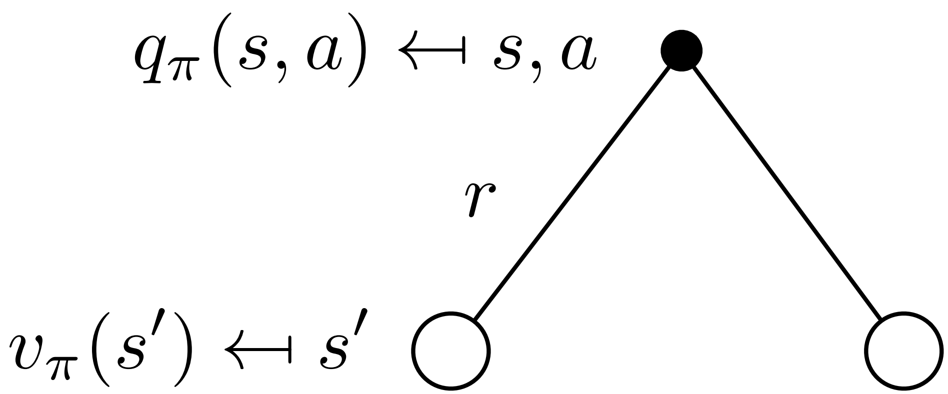


For each action I might take there is a q value

$$q_{\pi}(s,a) \leftarrow a$$

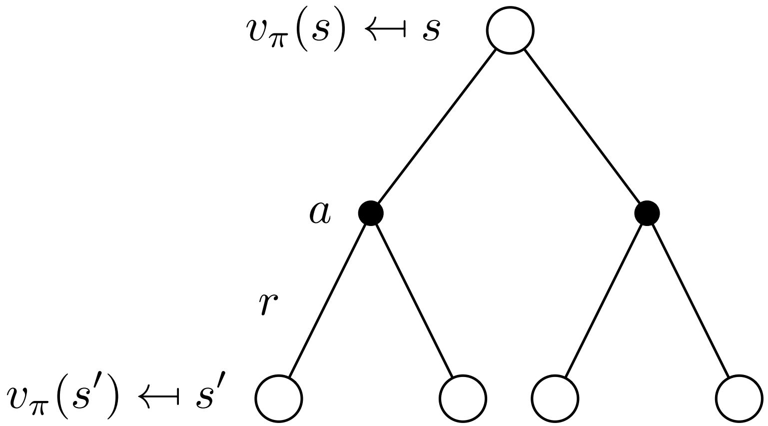
$$v_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_{\pi}(s,a)$$

Bellman Expectation Equation for Q^π



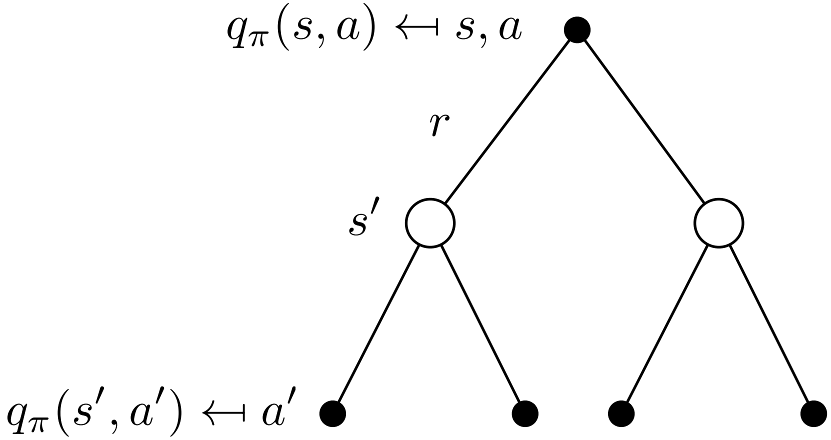
$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

Bellman Expectation Equation for v_π (2)



$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Bellman Expectation Equation for q_π (2)

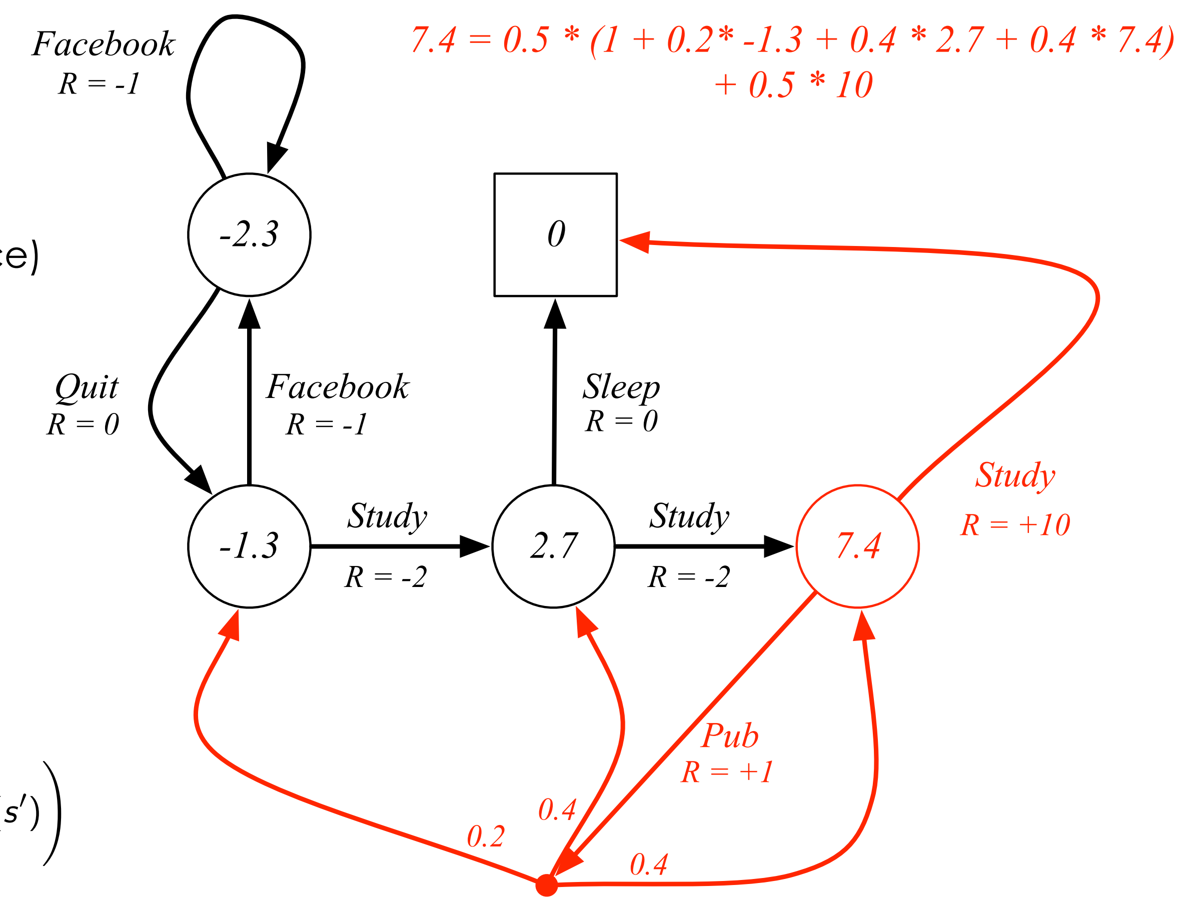


$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a')$$

Example: Bellman Expectation Equation in Student MDP

Let us verify the value of the red state

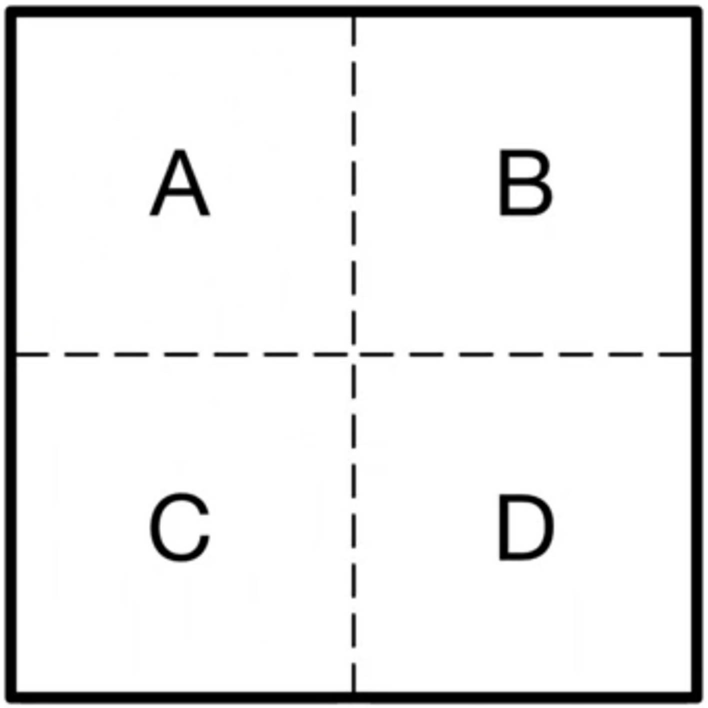
Policy is random: fifty-fifty
(equal probability for each choice)
γ=1



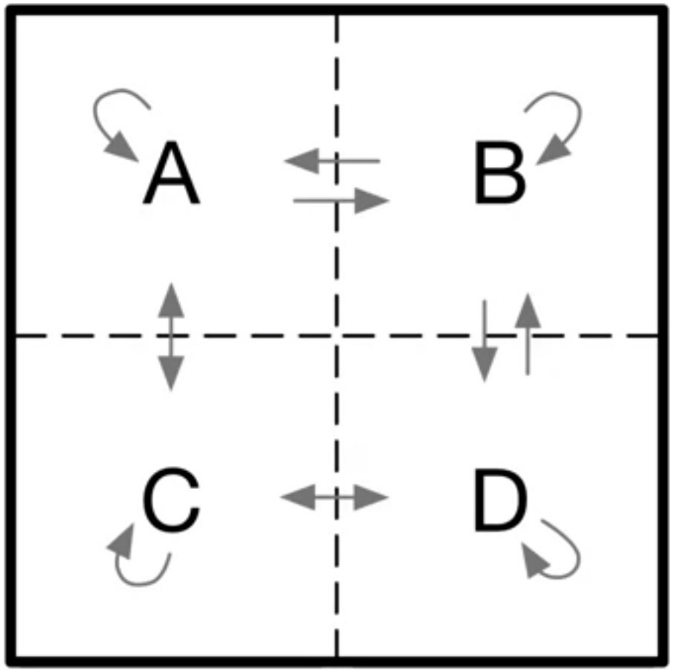
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

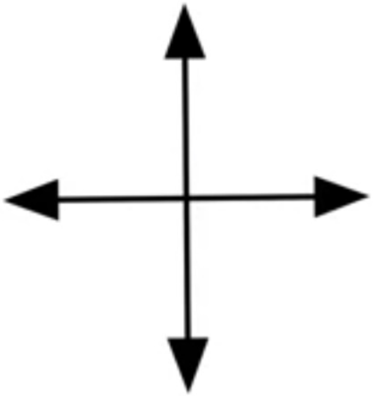
Example of value function calculation

Example: Gridworld



Example: Gridworld





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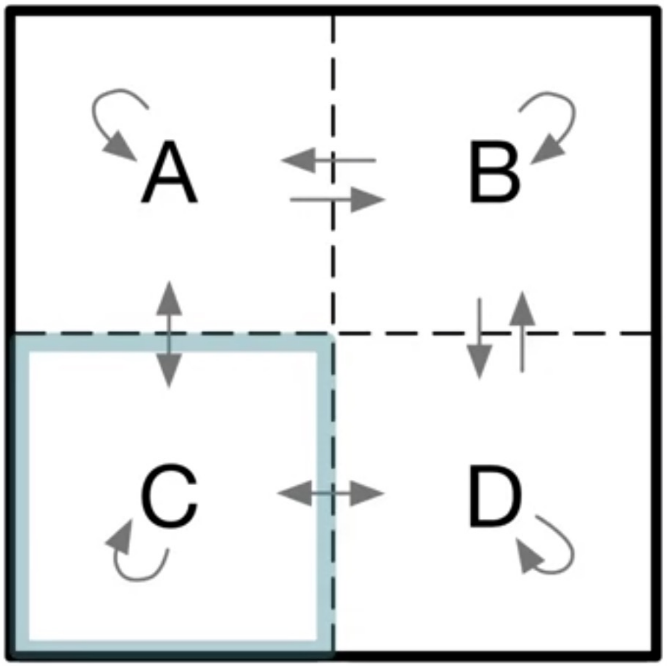

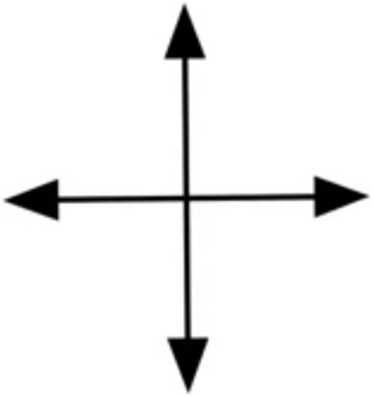

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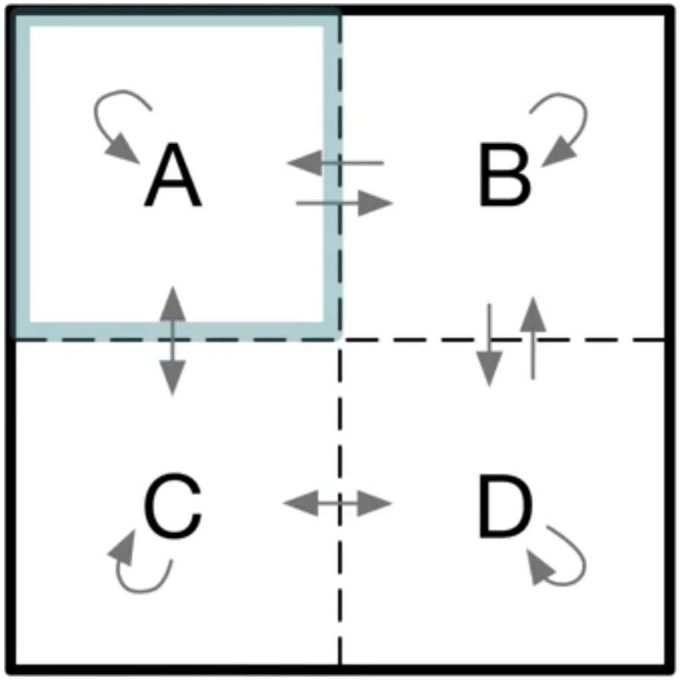

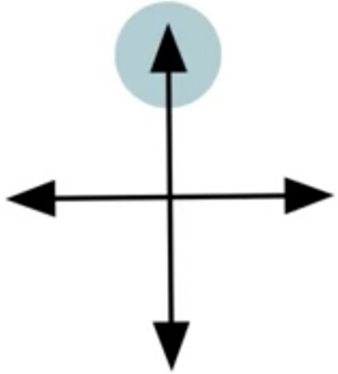

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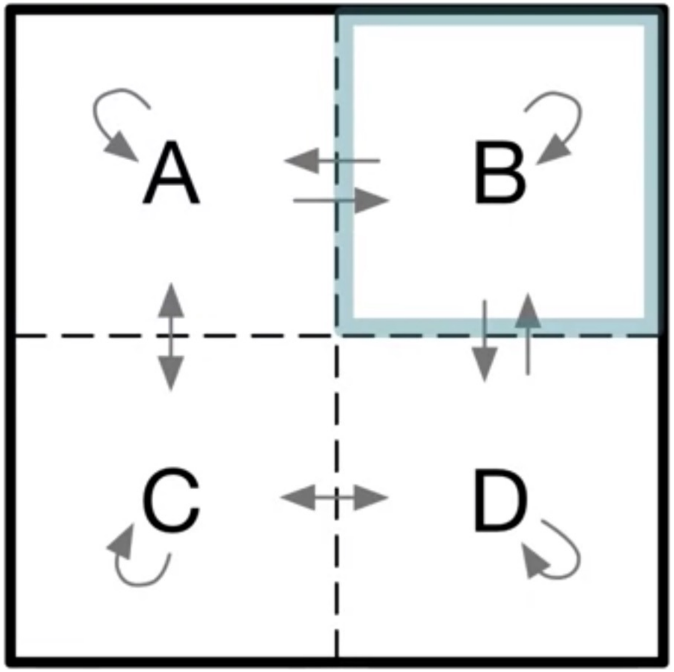
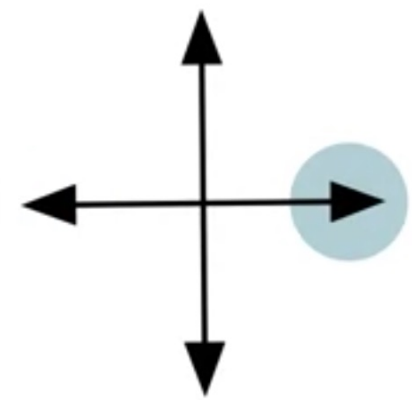
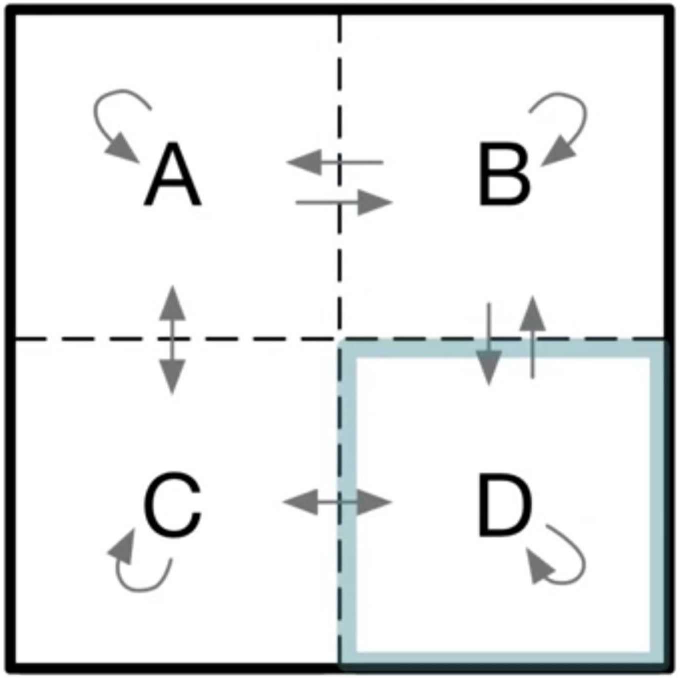
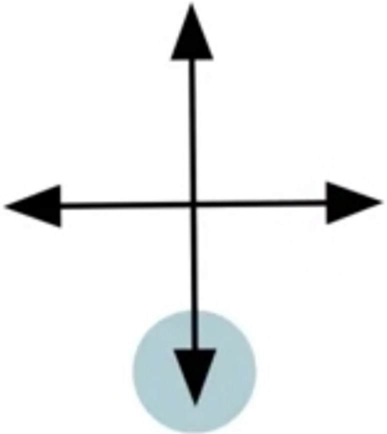


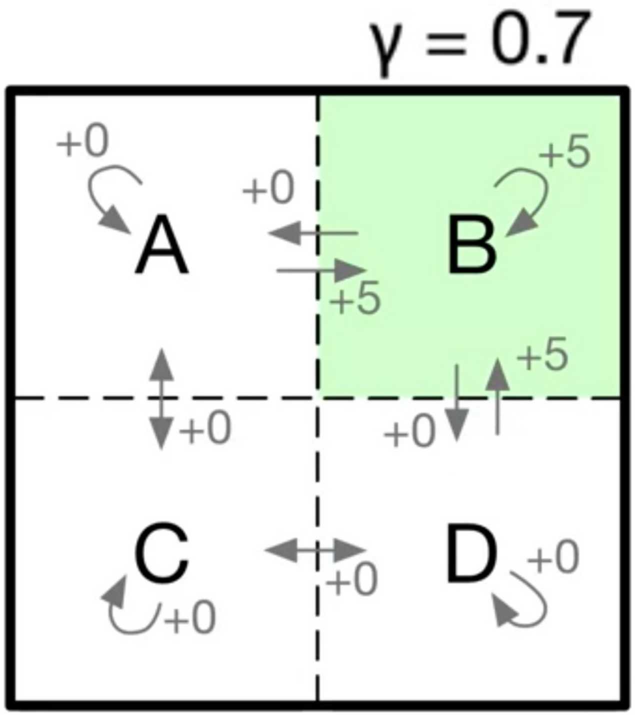


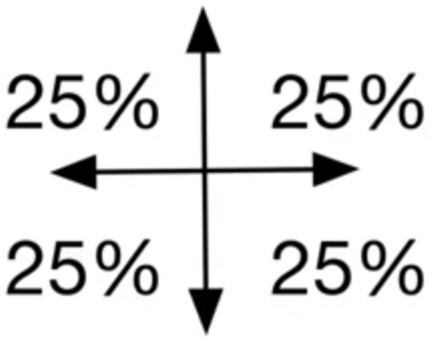
Example: Gridworld



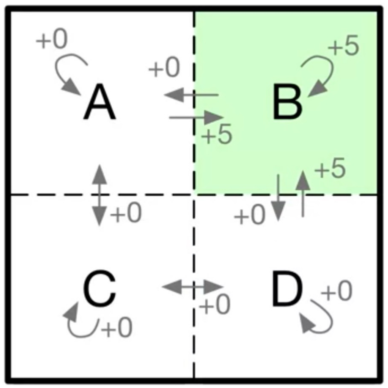


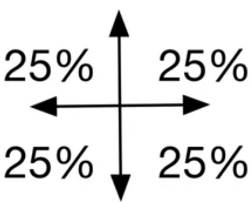
Example: Gridworld

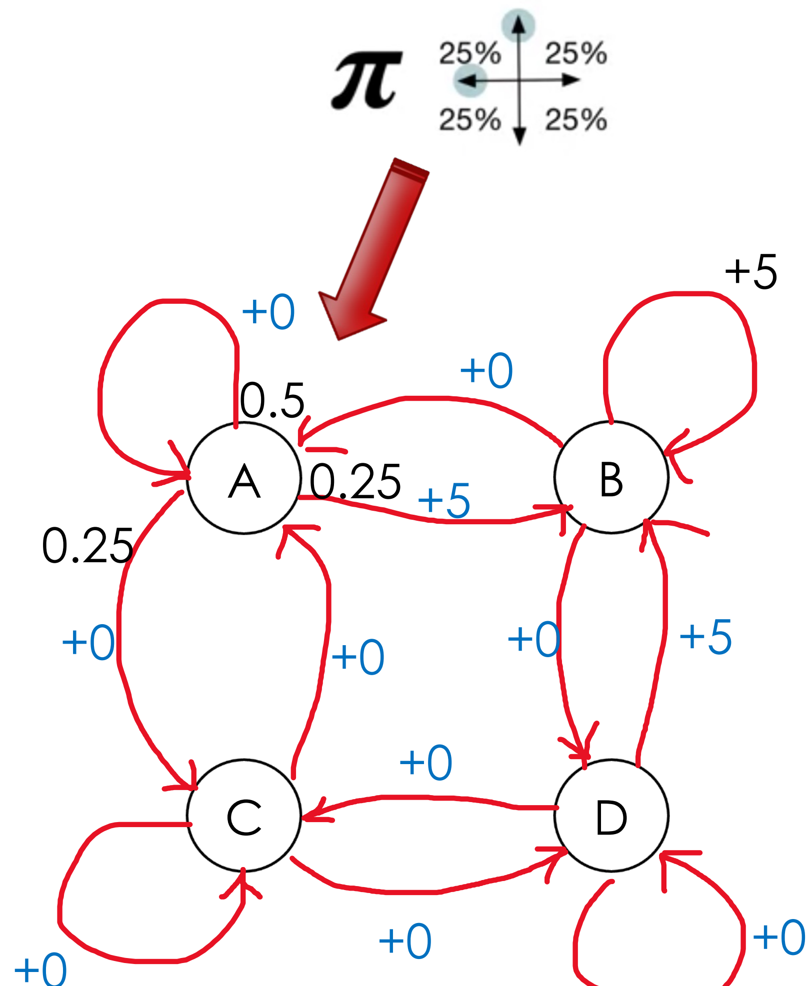




Example: Gridworld

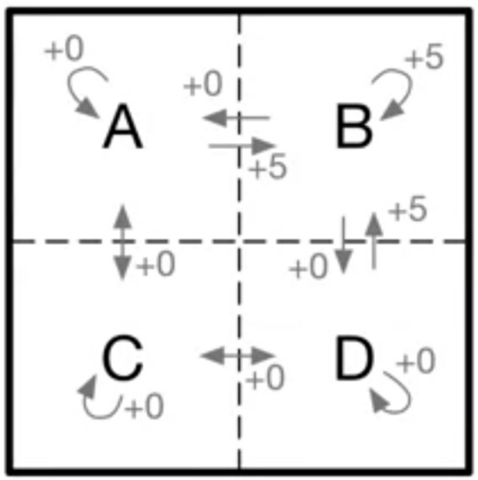




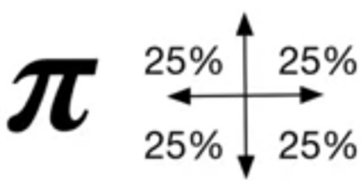


Example: Gridworld

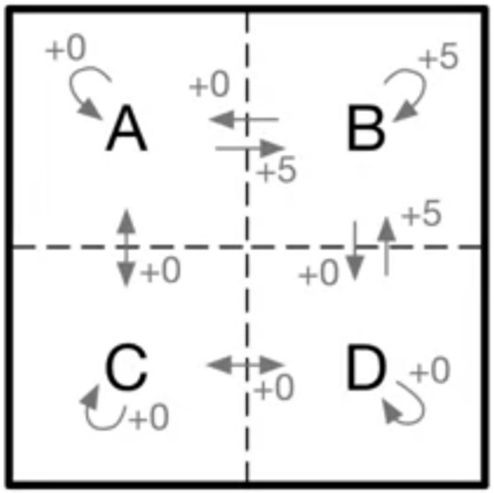
γ=0.7

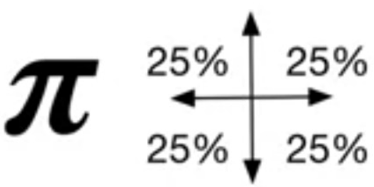


$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$



Example: Gridworld

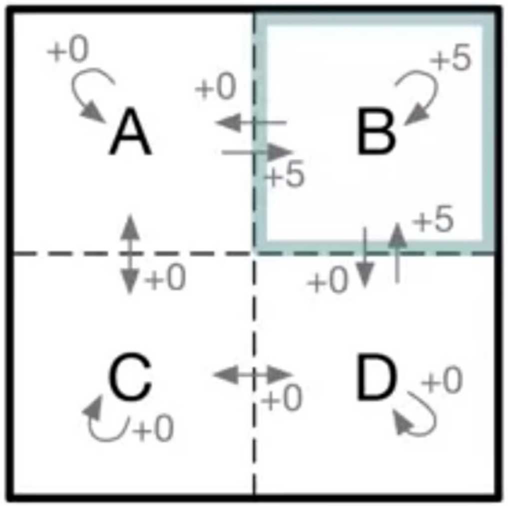


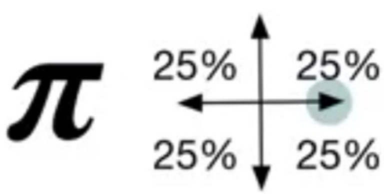


$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

$$V_\pi(A) = \sum_a \pi(a \mid A)\left(r + 0.7 V_\pi(s')\right)$$

Example: Gridworld



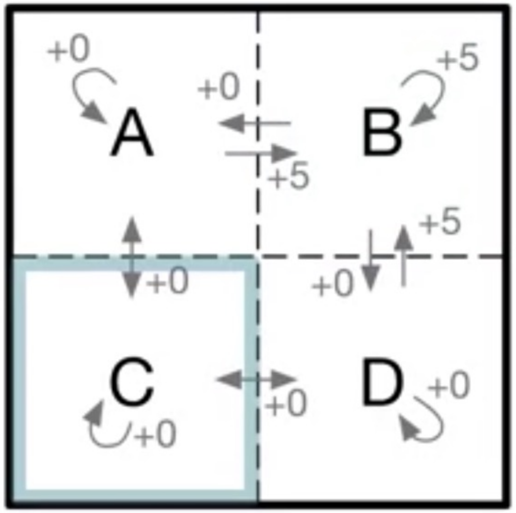


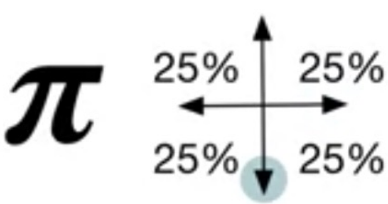
$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

$$V_\pi(A) = \sum_a \pi(a \mid A)\left(r + 0.7V_\pi(s')\right)$$

$$V_\pi(A) = \frac{1}{4}(5 + 0.7V_\pi(B))$$

Example: Gridworld



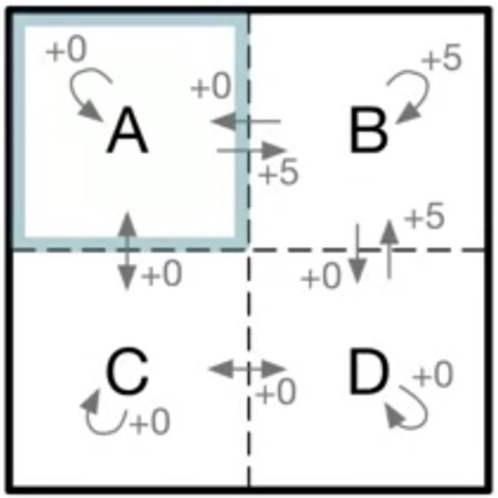


$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

$$V_\pi(A) = \sum_a \pi(a \mid A)\big(r + 0.7 V_\pi(s')\big)$$

$$V_\pi(A) = \frac{1}{4}(5 + 0.7 V_\pi(B)) + \frac{1}{4} 0.7 V_\pi(C)$$

Example: Gridworld



$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

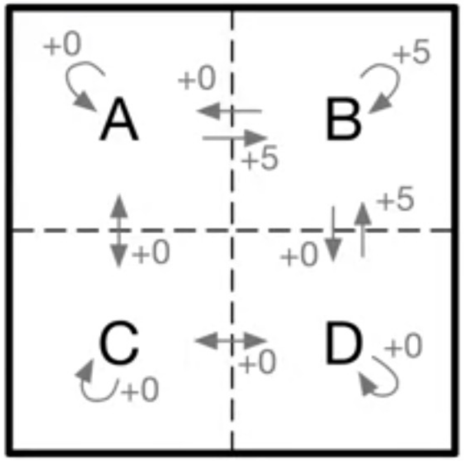
$$V_\pi(A) = \sum_a \pi(a \mid A)\left(r + 0.7V_\pi(s')\right)$$

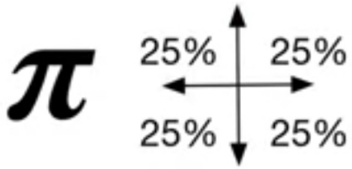
$$V_\pi(A) = \frac{1}{4}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(C) + \frac{1}{2}0.7V_\pi(A)$$

Example: Gridworld

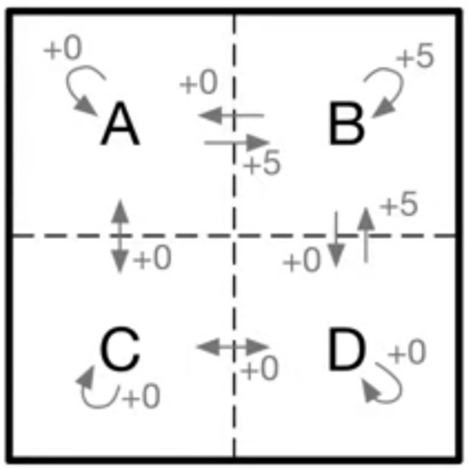
$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

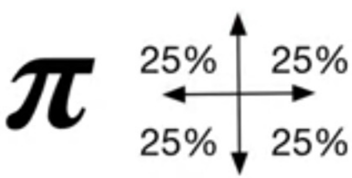
$$V_\pi(A) = \frac{1}{4}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(C) + \frac{1}{2}0.7V_\pi(A)$$





Example: Gridworld



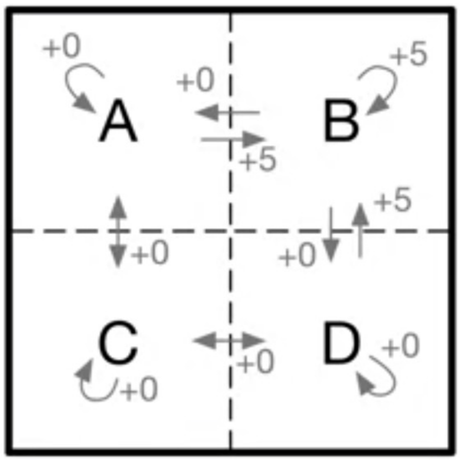


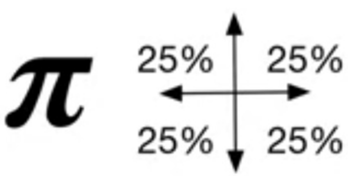
$$V_\pi(s) = \sum_a \pi(a|s) \sum_r \sum_{s'} p(s', r | s, a)[r + \gamma V_\pi(s')]$$

$$V_\pi(A) = \frac{1}{4}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(C) + \frac{1}{2}0.7V_\pi(A)$$

$$V_\pi(B) = \frac{1}{2}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(A) + \frac{1}{4}0.7V_\pi(D)$$

Example: Gridworld





$$V_\pi(s) = \sum_a \pi(a \mid s) \sum_r \sum_{s'} p(s', r \mid s, a)[r + \gamma V_\pi(s')]$$

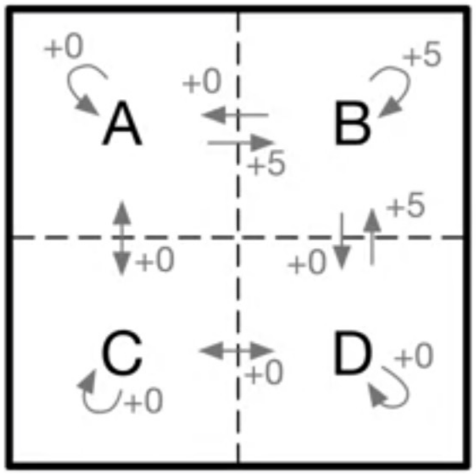
$$V_\pi(A) = \frac{1}{4}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(C) + \frac{1}{2}0.7V_\pi(A)$$

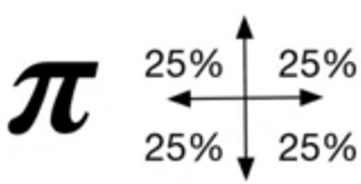
$$V_\pi(B) = \frac{1}{2}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(A) + \frac{1}{4}0.7V_\pi(D)$$

$$V_\pi(C) = \frac{1}{4}0.7V_\pi(A) + \frac{1}{4}0.7V_\pi(D) + \frac{1}{2}0.7V_\pi(C)$$

$$V_\pi(D) = \frac{1}{4}(5 + 0.7V_\pi(B)) + \frac{1}{4}0.7V_\pi(C) + \frac{1}{2}0.7V_\pi(D)$$

Example: Gridworld





$$V_\pi(s) = \sum_a \pi(a|s) \sum_r \sum_{s'} p(s', r | s, a)[r + \gamma V_\pi(s')]$$

$$V_\pi(A) = 4.2$$

$$V_\pi(B) = 6.1$$

$$V_\pi(C) = 2.2$$

$$V_\pi(D) = 4.2$$