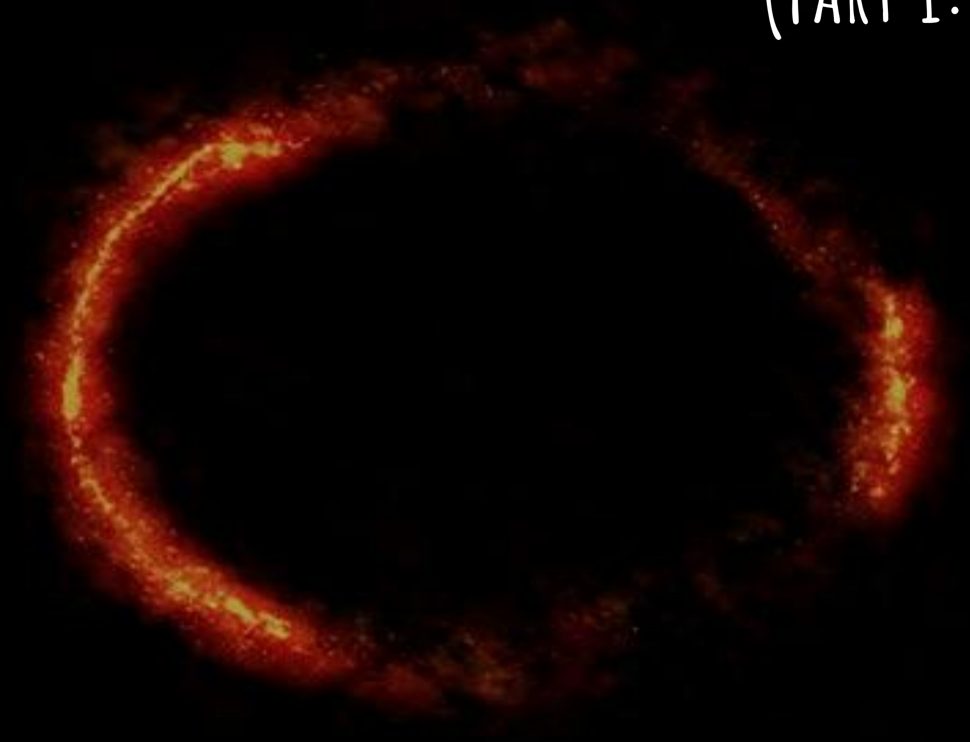
Gravitational Lensing
(part 1: lensing theory)



Observational cosmology Course, Master's Degree in Astrophysics, A.A 2023-2024

Theory of Gravitational Lensing

$$\hat{\vec{\alpha}}(\xi) = \frac{4GM}{c^2 \xi} \vec{e}_\xi = \frac{4GM}{c^2 \xi^2} \vec{\xi}$$

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\rm LS}}{D_{\rm L} D_{\rm S}} \frac{2}{c^2} \int \Phi(D_{\rm L}\vec{\theta}, z)\mathrm{d}z$$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

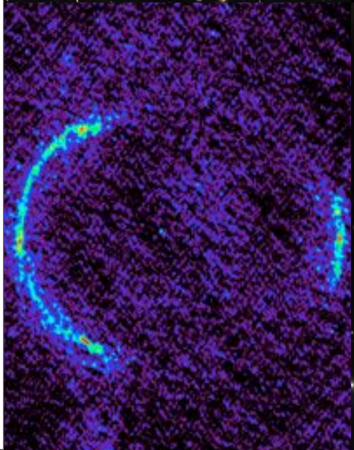
$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} \mathrm{d}^2\xi'$$

$$t(\vec{\theta}) = \frac{(1 + z_{\rm L})}{c} \frac{D_{\rm L} D_{\rm S}}{D_{\rm LS}} \left[\frac{1}{2}(\vec{\theta} - \vec{\beta})^2 - \hat{\Psi}(\vec{\theta})\right]$$

$$\hat{\alpha}(\theta) = \frac{4GM}{c^2 D_{\rm L}\theta}$$

$$\theta_E \approx (10^{-3})'' \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{D}{10\mathrm{kpc}}\right)^{-1/2}$$

Lensing, applied



Outline

Notes/References

Section 2

2. DEVIATIONS FROM HOMOGENEITY: THE GRAVITATIONAL *LENSING*

1. THE GRAVITATIONAL *LENSING*

We have seen during the third year course how to treat the space-time geometry of an expanding universe on the basis of very general considerations referring to the Cosmological Principle, and essentially making use of the symmetry properties of the space-time inherent the Principle. It has thus been possible to define a metrics that has allowed us to set up in a rigorous way the concept of distances and, based on this, all cosmological observables.

The Cosmological Principle and the consequent Robertson-Walker metrics assume a rigorously homogeneous and isotropic universe. As we have seen in the previous Section, this is valid on the large scales (essentially on scales larger than at least a hundred Mpc), but obviously breaks down on smaller scales where we have seen a high degree of inhomogeneity. In this Section we discuss the phenomenon of ***gravitational lensing***, originating from deformations of the space-time metrics due to the inhomogeneities of the gravitational field induced by cosmic structures.

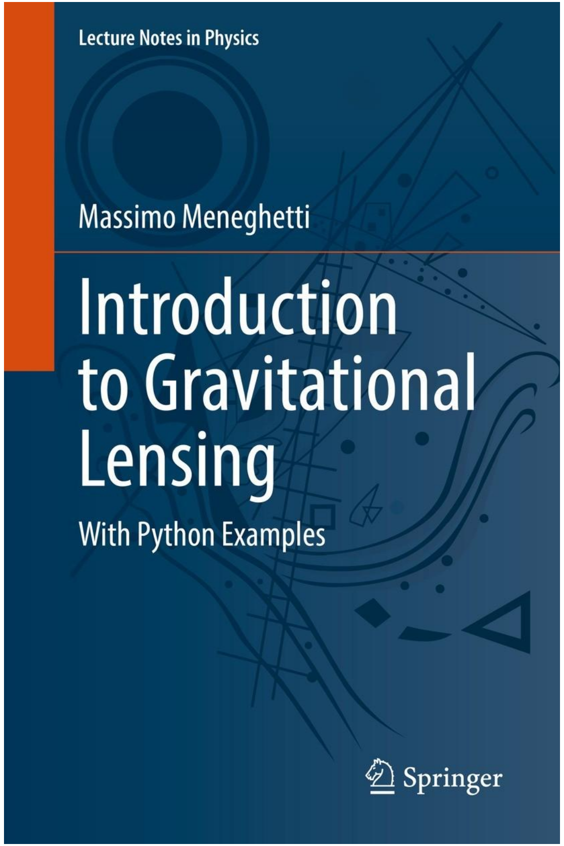
The gravitational lensing offers not only a solid verification of the General Relativity theory (indeed it made its first experimental test), but even a unique instrument for studying the gravitational matter (baryons + dark matter) distribution in the Universe (hence potentially bypassing the very serious problem of cosmological *bias* mentioned in Sect. 1.4) and, finally, a method for constraining the cosmological parameters, in particular the Ω_Λ parameter that so far we have found difficult to quantify in a direct way. However, gravitational lensing has countless applications in astrophysics and cosmology.

For our cosmological applications, gravitational lensing offers a unique, model independent way to probe directly the dark matter distribution, and addressing then the bias problem discussed in Sect. 2.5 below.

Introduction

In agreement with the Einstein theory of General Relativity, an object of a given mass produces a curvature in the space-time with its gravitational field, and imposes a

Gravitational *Lensing* 2. 1



https://link.springer.com/book/10.1007/978-3-030-73582-1

These slides are available in a GitHub repository here:
https://github.com/AndreaEnia/PadovaGravLens
and on Moodle

for anything, contact me at **andrea.enia@unibo.it**

Outline

Notes/References

Brief History of GL

Outline

Notes/References

Brief History of GL

The idea that gravity can deflect light is way older than you might think.

Michell (1784) & Laplace (1796) independently assumed that the mass of a star could be measured from how much the light slows down in its proximity (pre-Einstein…)

Cavendish (late XVII century) and von Soldner (1802) actually did the math:

$$\hat{\alpha}(R) = \frac{2GM}{c^2 R}$$

~ 0.875" for the Sun, which is surprisingly close to the correct value for the deflection angle.

John Michell

Pierre Simon de Laplace

Henry Cavendish

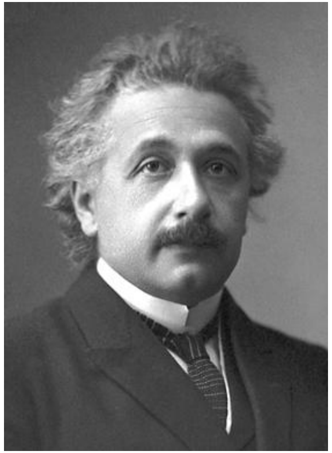
Johann von Soldner

Outline

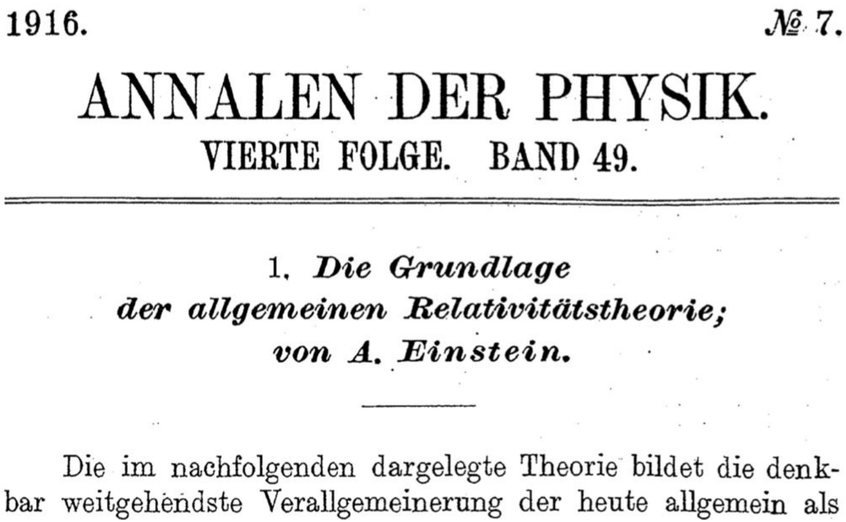
Notes/References

Brief History of GL

Of course, everything changed after 1915, with the publication of Einstein's Theory of Gravity (but you may also know it as General Relativity)



Albert Einstein

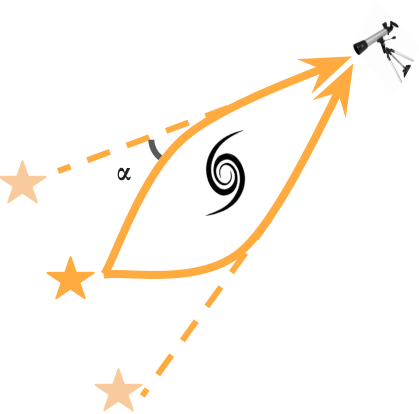

1916. № 7.

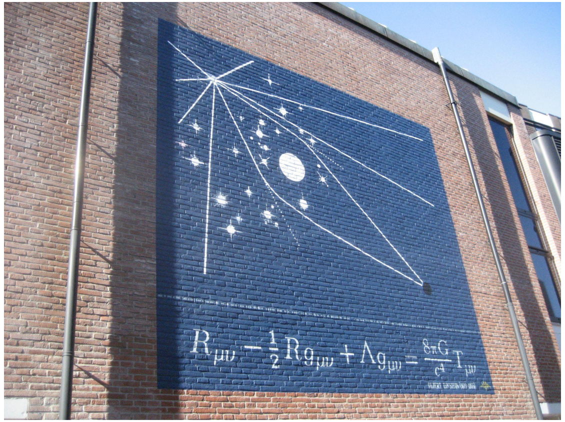
ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 49.

1. *Die Grundlage der allgemeinen Relativitätstheorie;*
von A. Einstein.

Die im nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der heute allgemein als





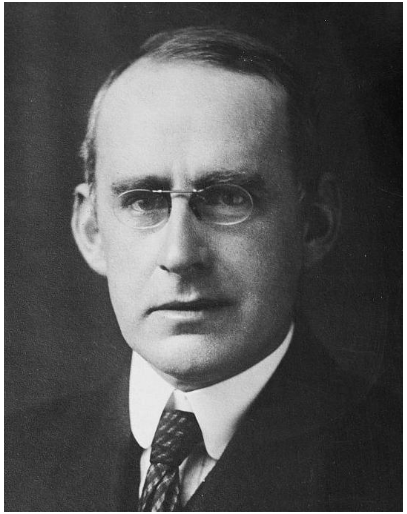
With the new formalism, Einstein predicted that a point source with radius R and mass M would deflect the light of a background object with a deflection angle of:

$$\hat{\alpha}(R) = \frac{4GM}{c^2 R}$$

~ 1.75" for the Sun.

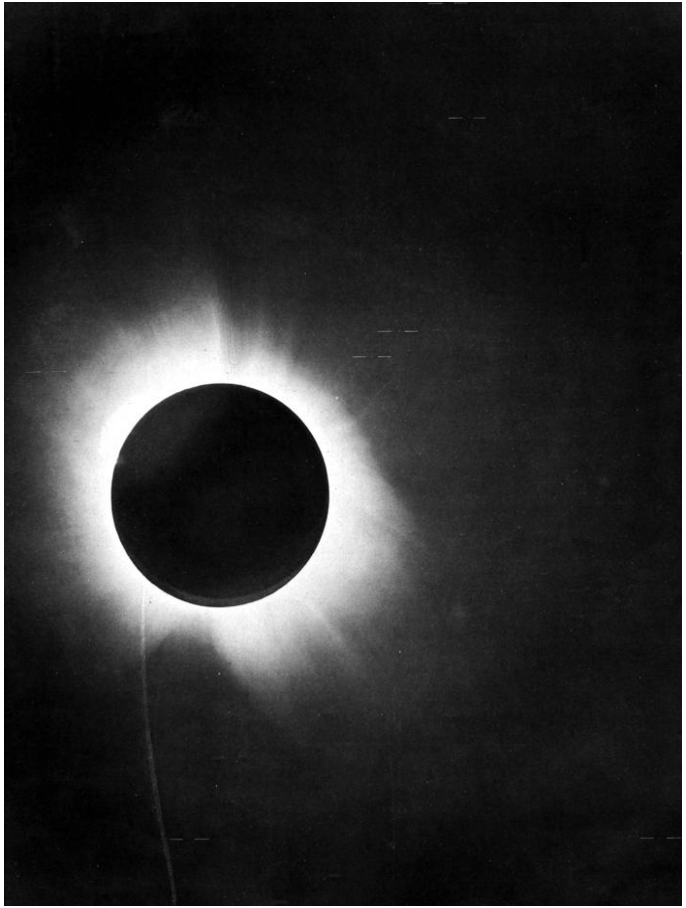
Gravitational lensing played a fundamental role in showing that Einstein's General Relativity is true.

(well, Popper would argue that it just probed Newton's Gravity as false, since you can't *really* probe something as true, but whatever, the concept is clear)



Arthur Eddington

After a series of failed attempts (bad weather, Crimean war, WWI, lousy instrumentation, any other business), Eddington's expedition in Principe Island found a deflection angle of 1.60±0.31 arcsec, in spectacular accord with what Einstein predicted.



ESO/Landerssternwarte Heidelberg-Konigstuhl
F. W. Dyson, A. S. Eddington & C. Davidson

Outline

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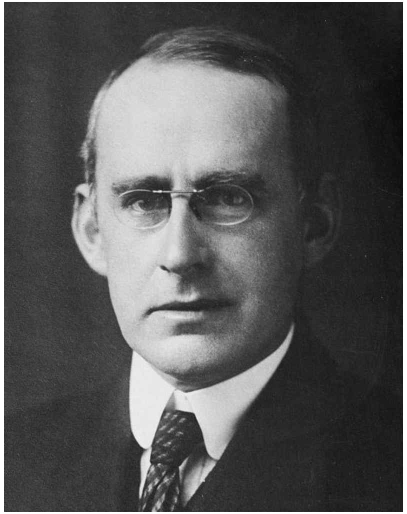

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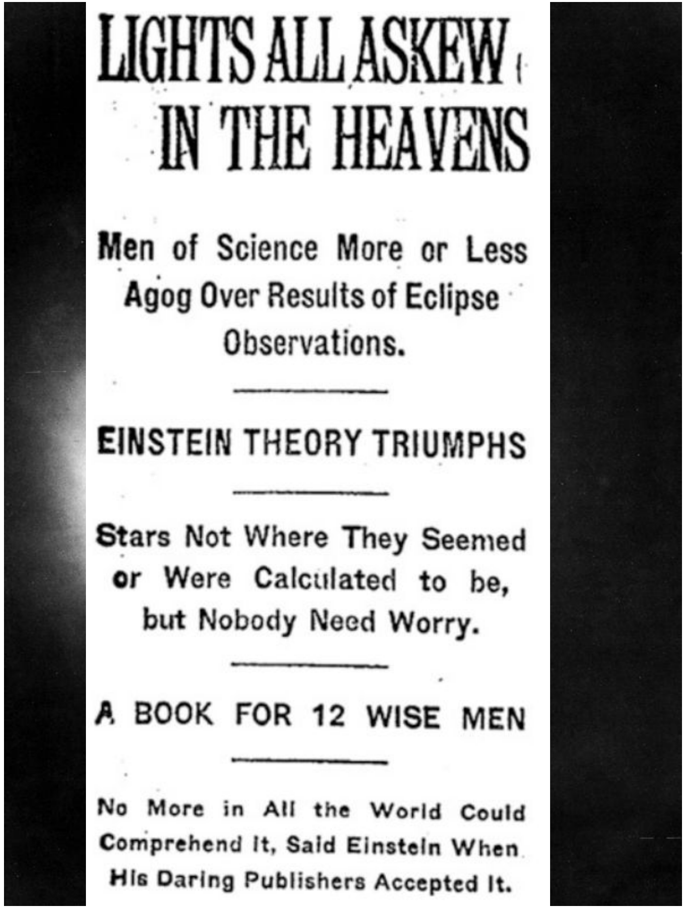
All hell broke loose, Einstein became a celebrity, GR one of the two pillars of modern physics…



LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

New York Times, 10 Nov 1919

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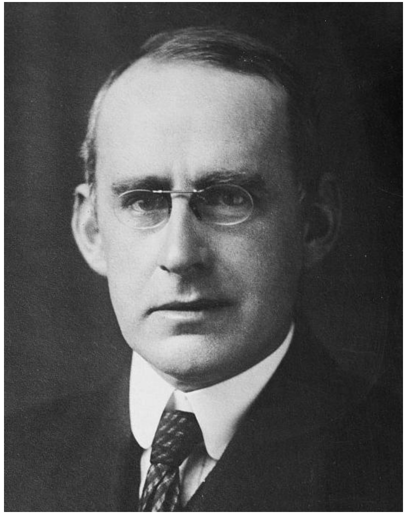

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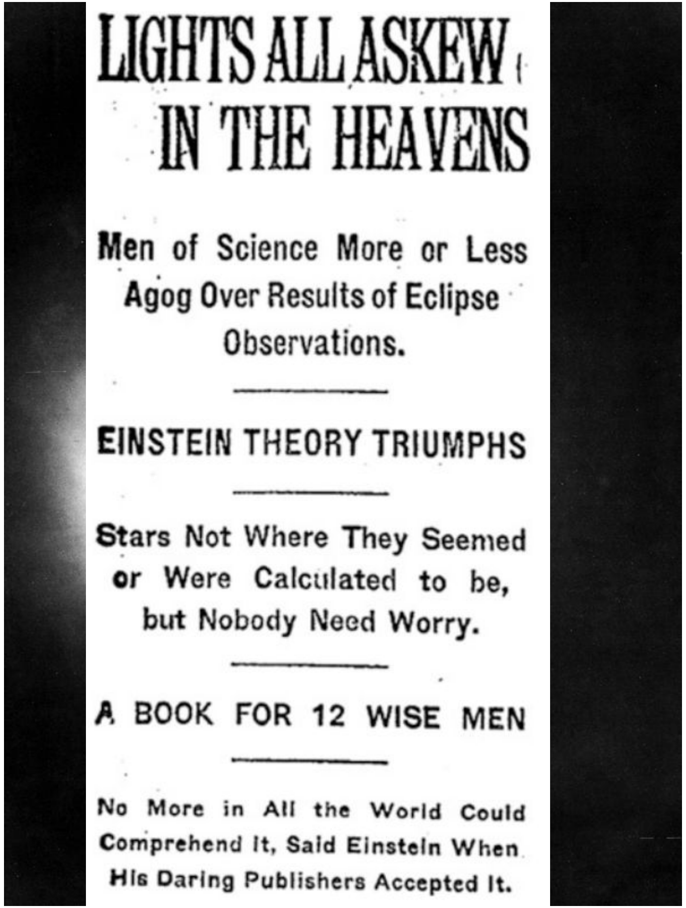
All hell broke loose, Einstein became a celebrity, GR one of the two pillars of modern physics…

… and as for lensing, nothing new happened for 50 years.



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Observing a gravitational lensing phenomena has been impossibile for decades, for two reasons: insufficient instrumentation and (at the time) the need of pure serendipitous observation.

However, theoretical studies on gravitational lensing kept going:

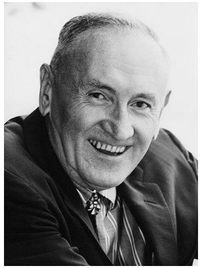
- Chwolson, 1924 and Einstein, 1936 both theorized Einstein's rings

- Zwicky, 1937a, b "*Nebulae* as gravitational lenses*"

- Klimov 1963, galaxy-galaxy strong lensing

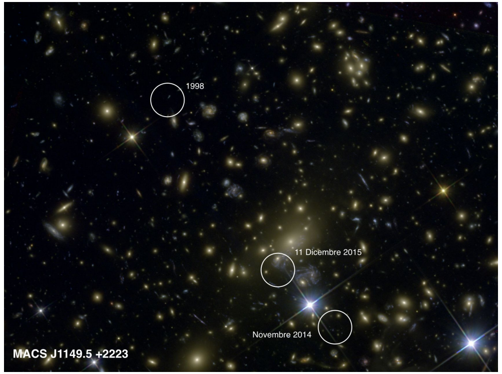
&

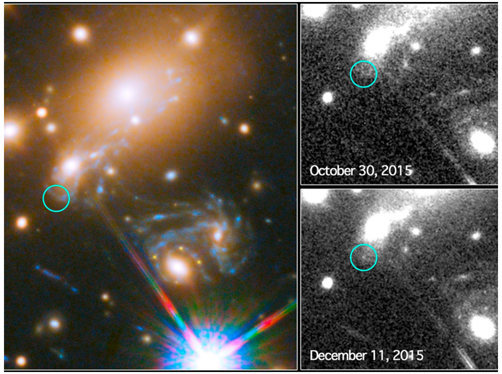
- Liebes 1964, stars and globular clusters as lenses



* at that time *nebula* was still used in lieu of galaxy

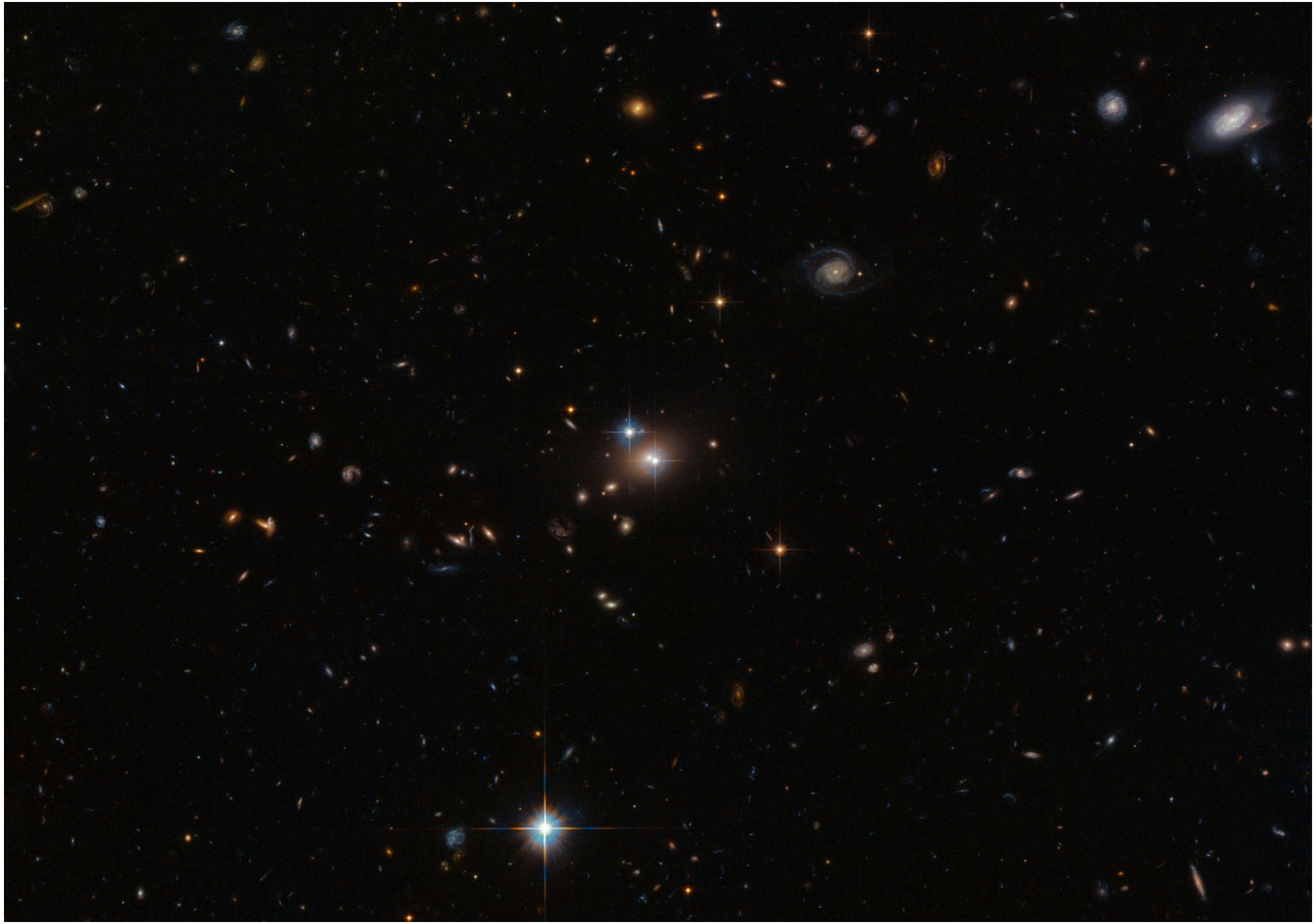
- Refsdal, 1964, discussed on multiple images, magnifications, time delays, and firstly theorized lensed SNae





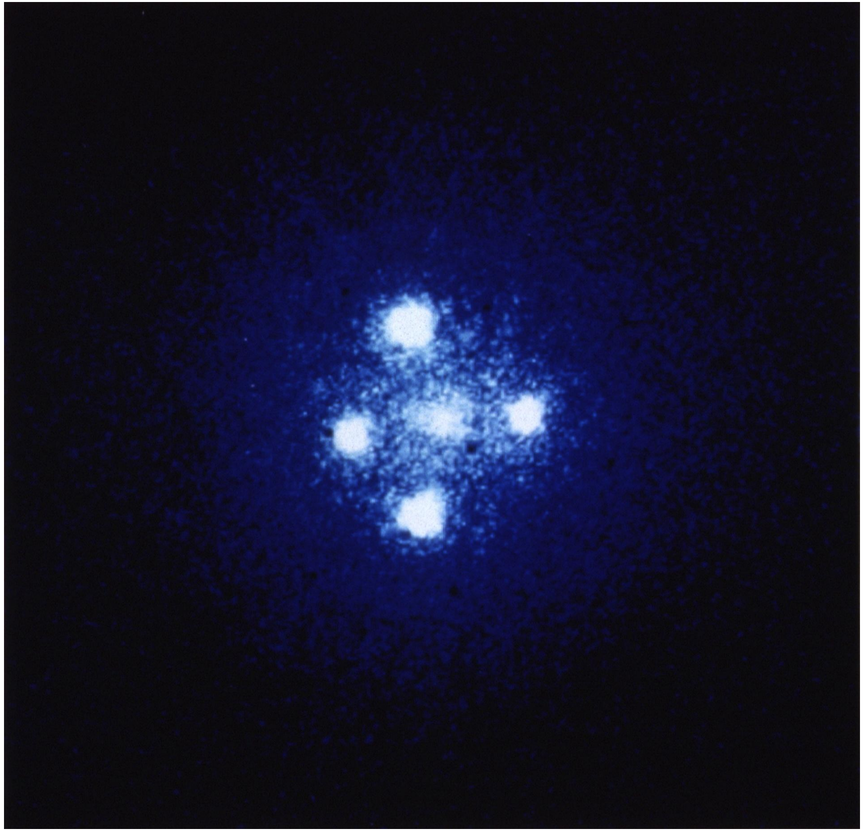
QSO 0957+561: the first ever* detected gravitational lensing event (Walsh et al., 1979)

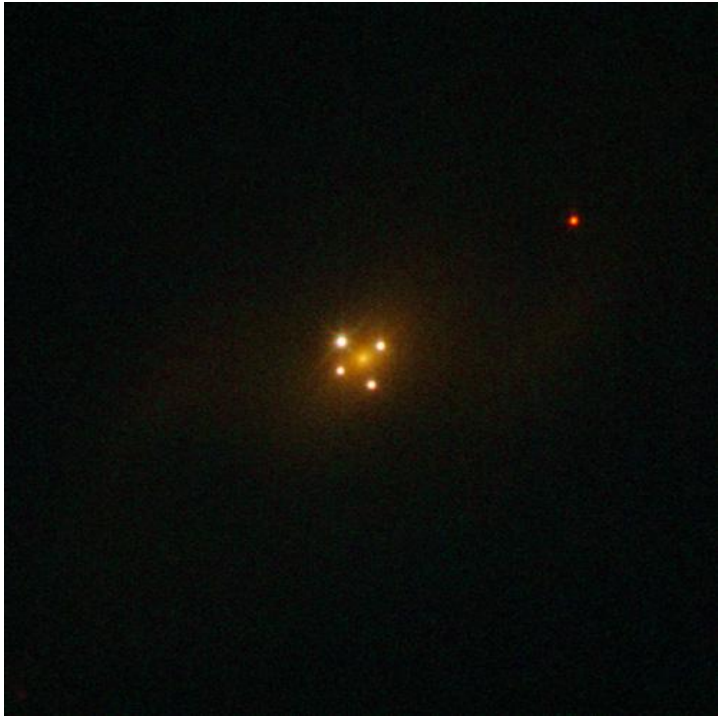
* well, technically Eddington's measurements were the first one ever



ESA/Hubble & NASA

QSO-Q2237+0305: the first Einstein cross (Huchra et al., 1985)





ESA/Hubble & NASA

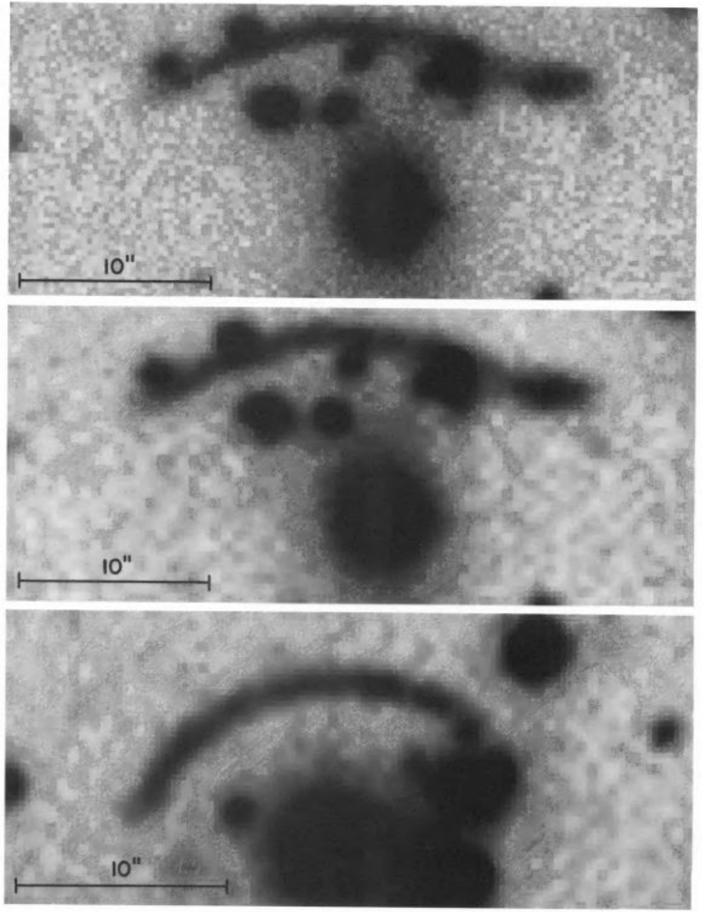
Outline

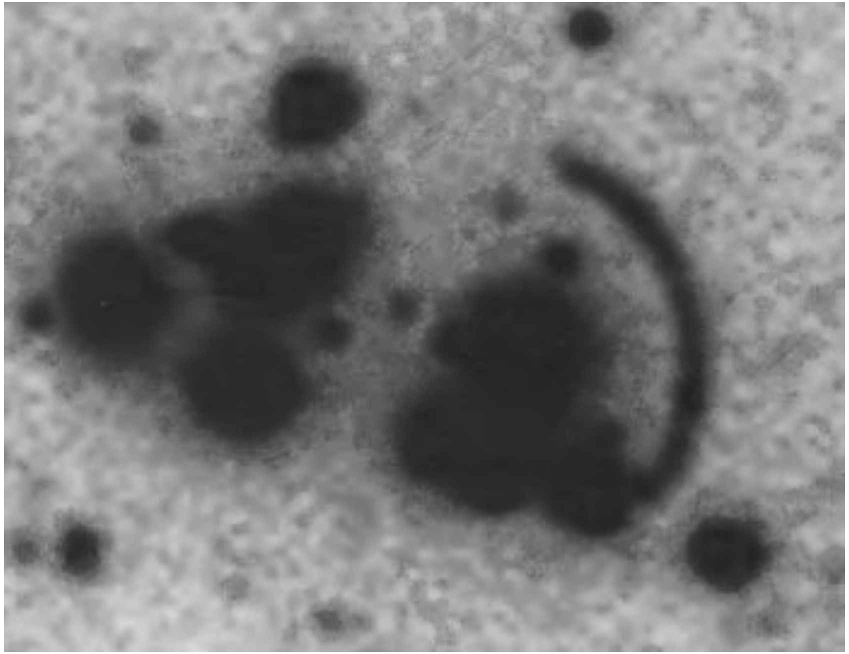
Notes/References

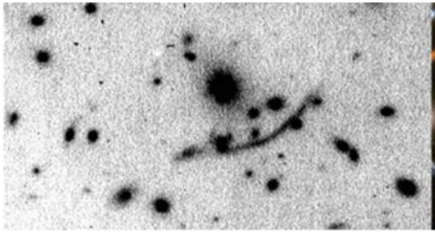
Brief History of GL

First detections

First ever gravitational arcs: Lynds and Petrosian 1986, Soucail et al., 1987, 1988.







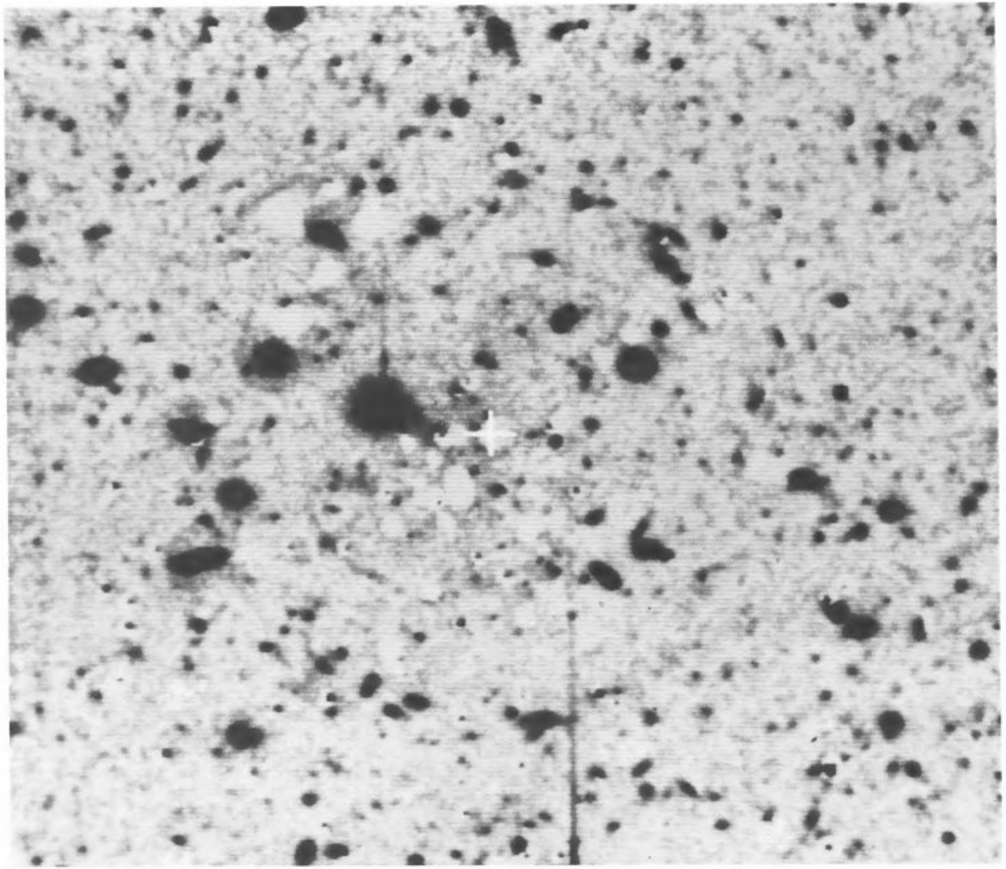
Outline

Notes/References

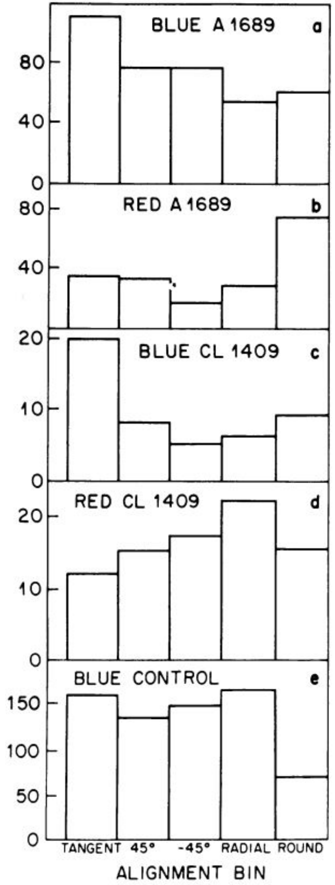
Brief History of GL

First detections

First ever systematic weak lensing detection (Taylor et al., 1990)



Abell 1689



Taylor et al., 1990

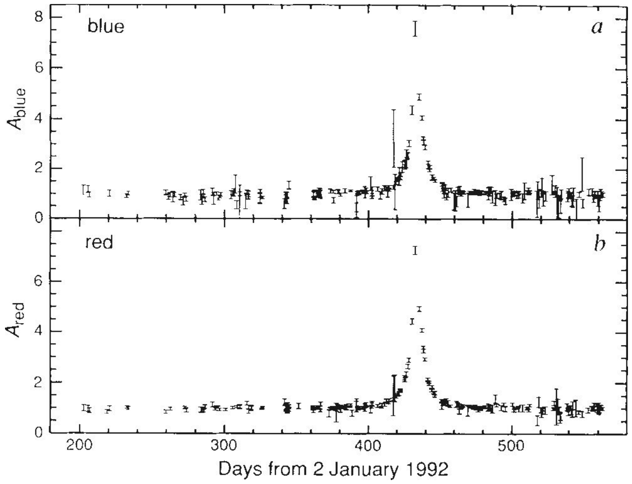
Outline

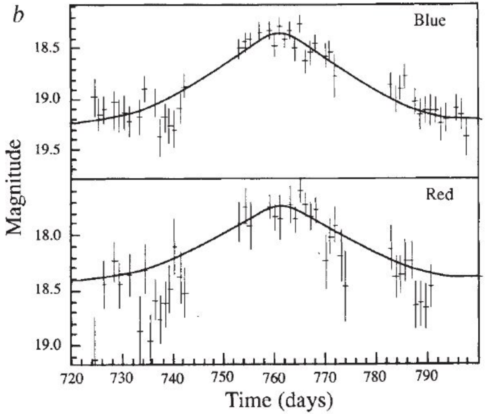
Notes/References

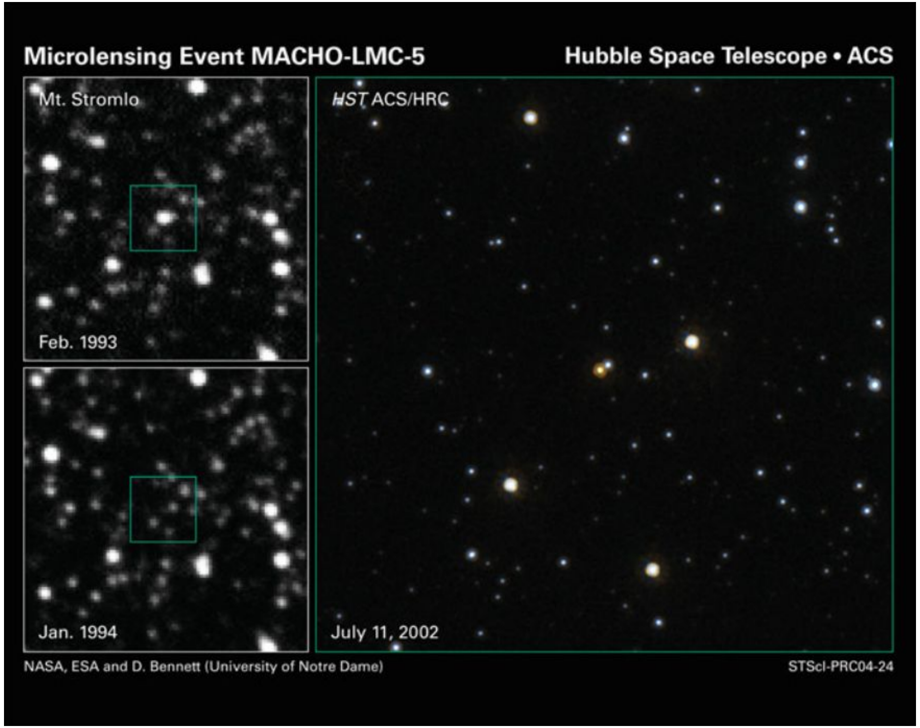
Brief History of GL

First detections

First ever microlensing event: Alcock et al. 1993, Aubourg et al. 1993.







"I don't understand; you're looking for planets you can't see around stars you can't see" (D. Fisher, reported)

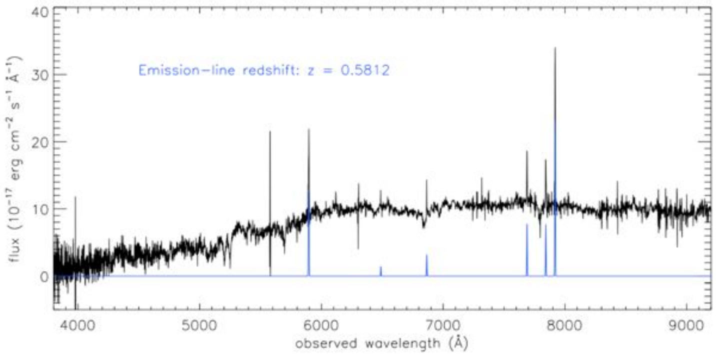
Outline

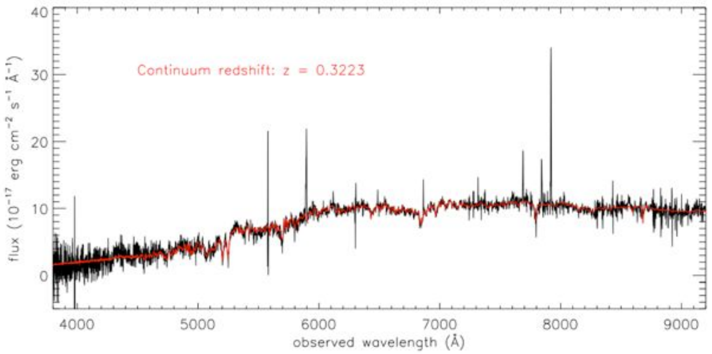
Notes/References

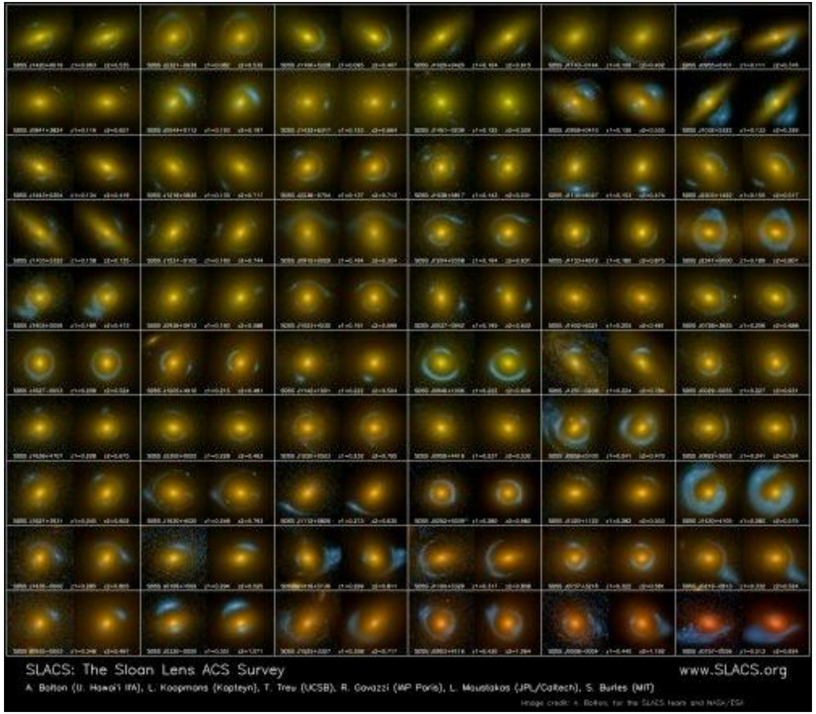
Brief History of GL

First detections

First ever systematic lensing surveys in the optical/near-IR (2000s)







A. Bolton (UH IfA) for SLACS and NASA/ESA

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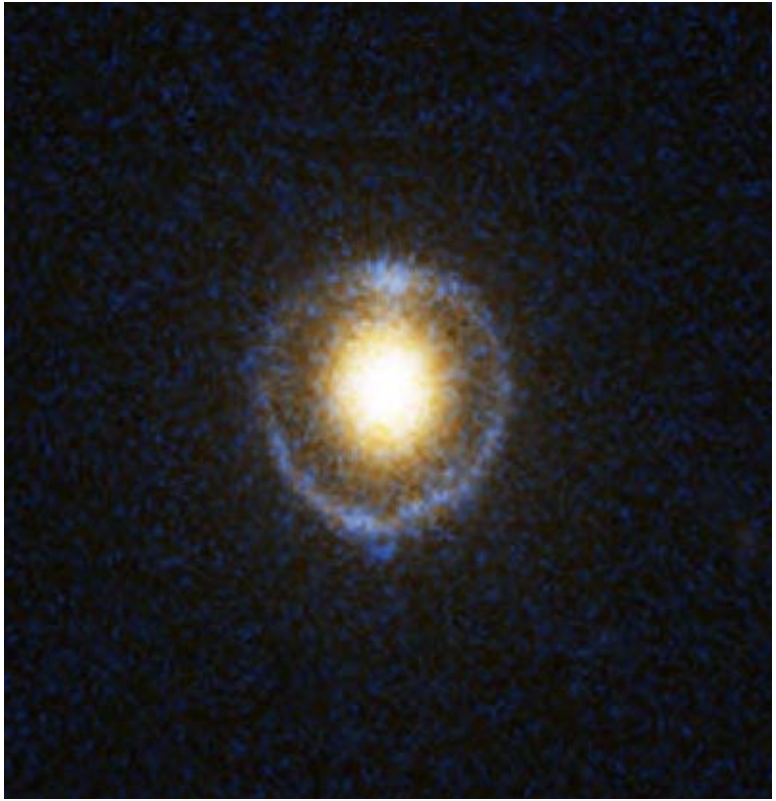

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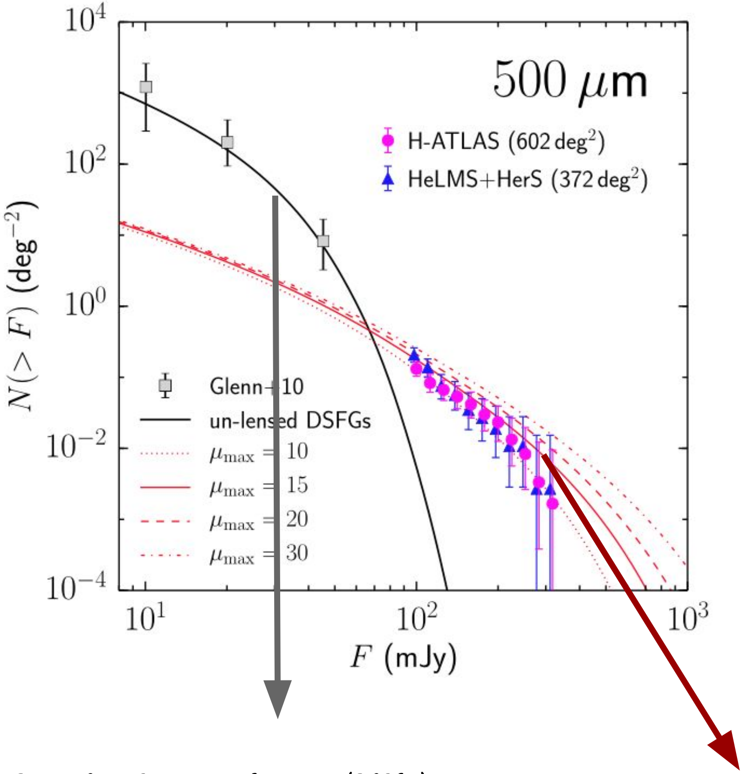
Outline

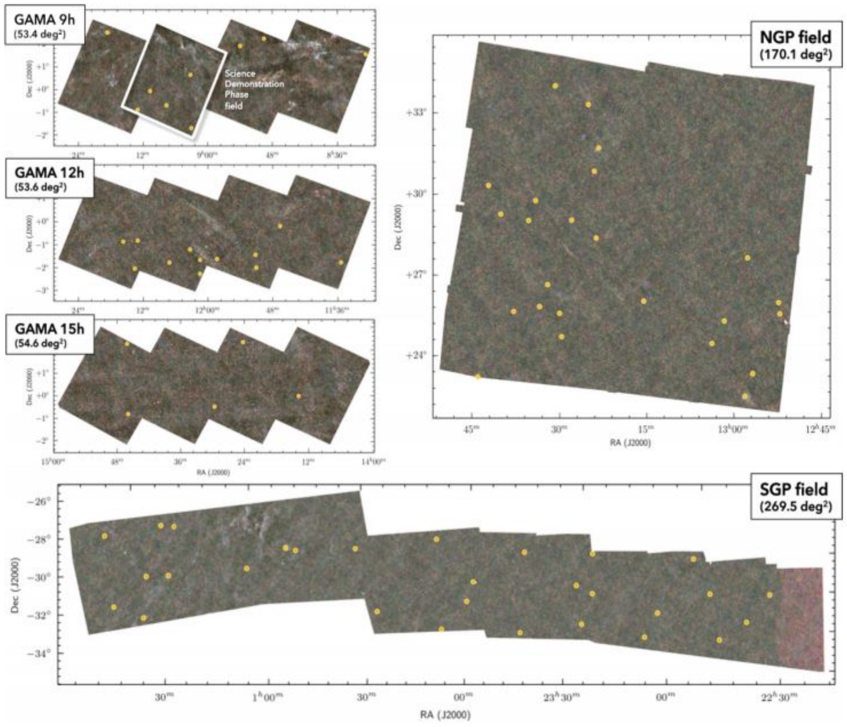
Notes/References

Brief History of GL

First detections

First ever systematic lensing surveys in sub-mm (2010s)





Negrello et al., 2017

DUSTY STAR FORMING GALAXIES (DSFGs) NUMBER COUNTS AT Z = 1.5 - 4

STRONGLY LENSED DSFGs CANDIDATES

Outline

Notes/References

Brief History of GL

First detections

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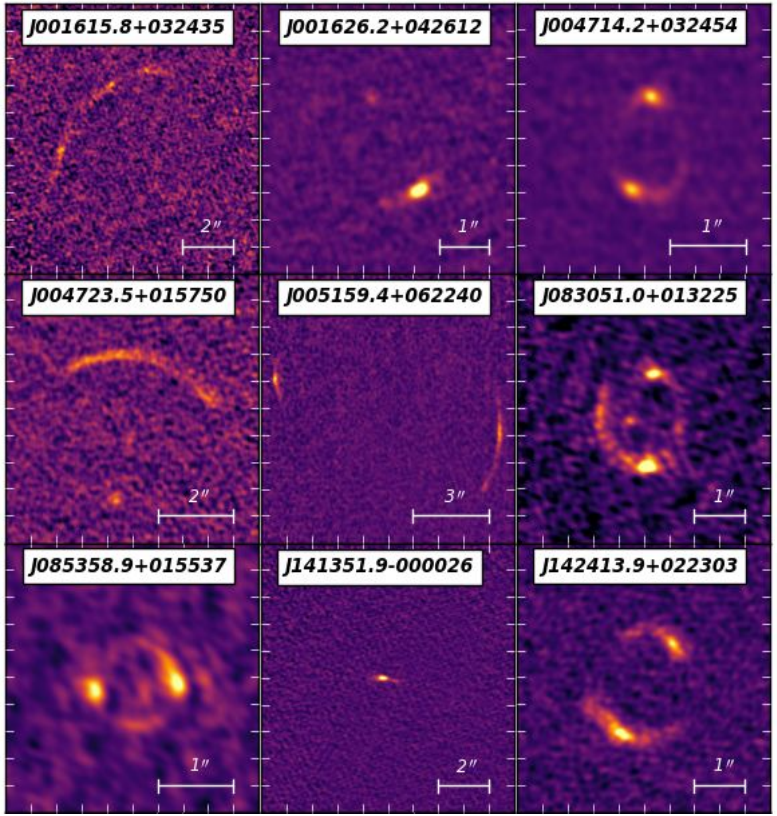

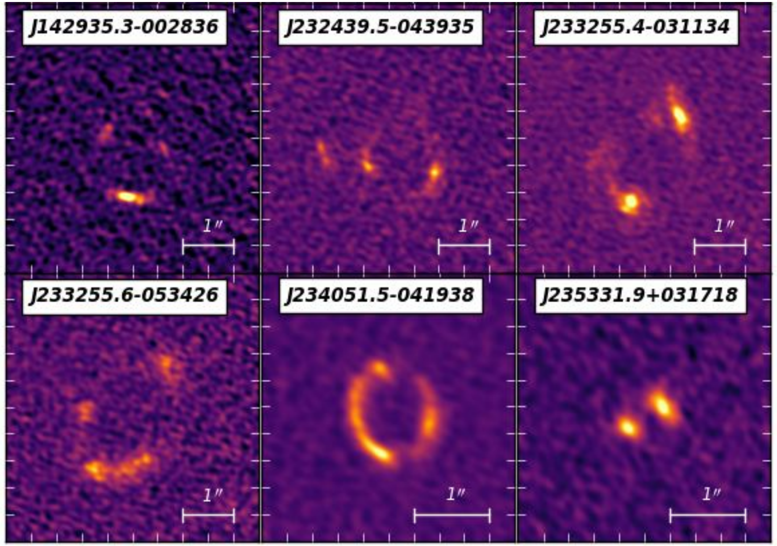

An always increasing number of sub-mm selected strongly lensed galaxies are observed nowadays in sub-mm/mm bands at high resolution, thanks to interferometers such as ALMA.

Amvrosiadis et al., MNRAS, 2017

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

Light deflection according to Newton

Outline

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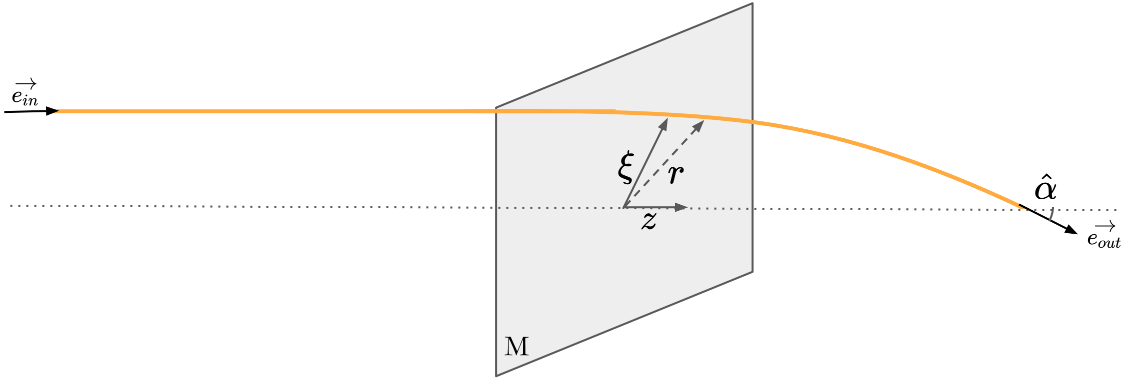
Light deflection according to Newton

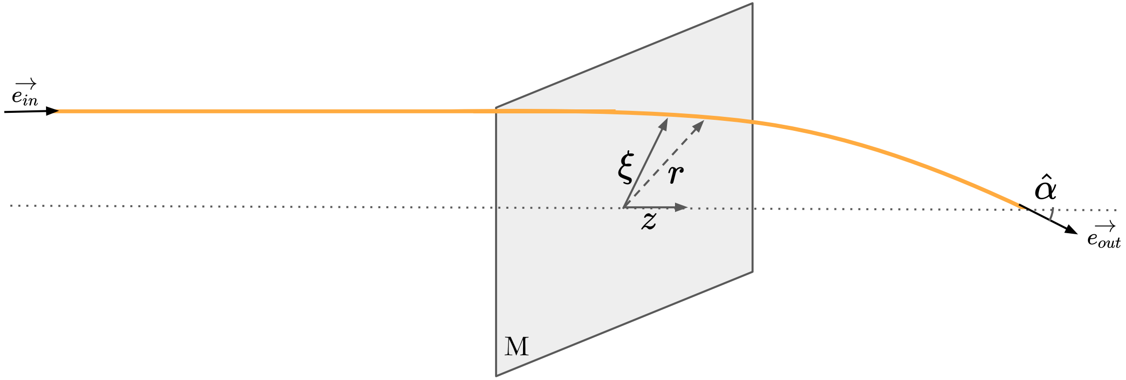


$$\hat{\alpha} = \vec{e_{in}} - \vec{e_{out}}$$

$$z = ct \longrightarrow dz = cdt$$

$$r^2 = \xi^2 + z^2$$

Light deflection according to Newton



$$\hat{\alpha} = \vec{e_{in}} - \vec{e_{out}}$$

$$z = ct \longrightarrow dz = cdt$$

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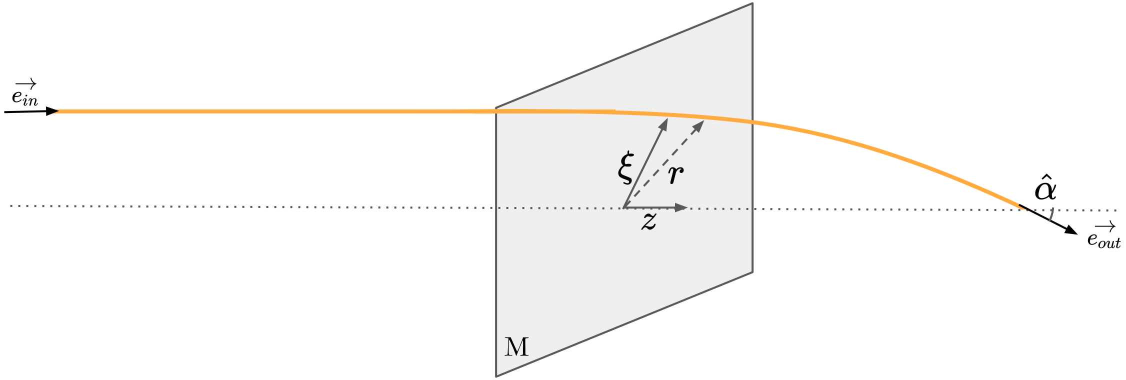
We know that:

$$\vec{F} = -\cancel{m_\gamma}\nabla\phi = \cancel{m_\gamma}\vec{a} \Longrightarrow \vec{a} = -\nabla\phi \qquad \text{and} \qquad \phi = \frac{GM}{r}$$

Light deflection according to Newton



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As such, the photon will experience a change in velocity:

$$\Delta\vec{v} = \int_{t_{in}}^{t_{out}} \vec{a}dt \simeq -\frac{1}{c}\int_{z_{in}}^{z_{out}} \nabla\phi dz$$

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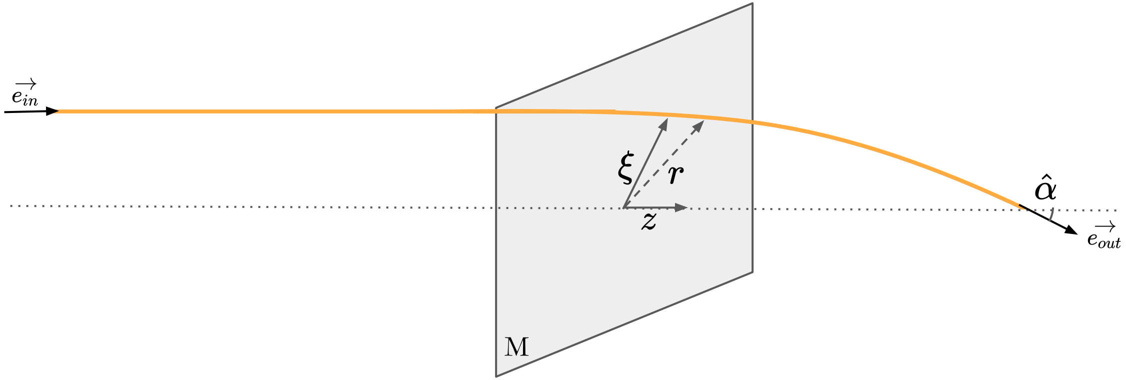
Light deflection according to Newton



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we can safely assume $|z_{in}| = |z_{out}| = \infty$

as such $\phi(\infty) \rightarrow 0 \Longrightarrow \Delta v_{//} = 0$

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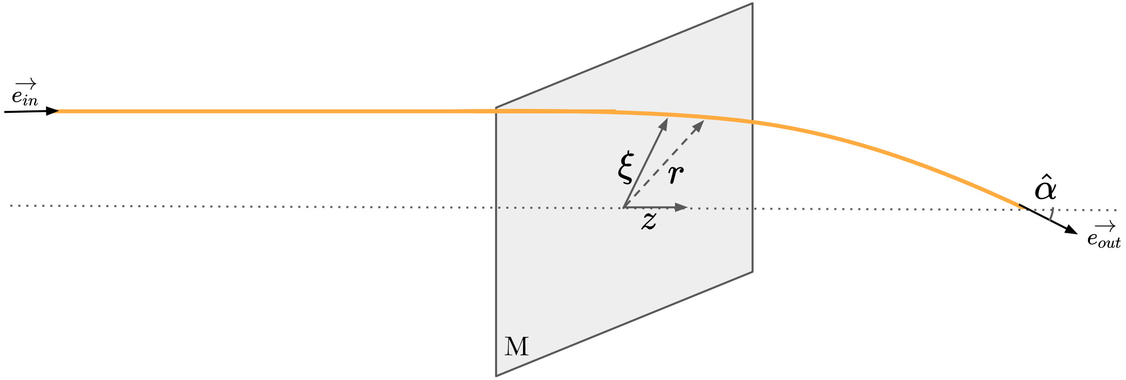
Light deflection according to Newton



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$$\Delta\vec{v} = \int_{t_{in}}^{t_{out}} \vec{a}dt \simeq -\frac{1}{c}\int_{z_{in}}^{z_{out}} \nabla\phi dz \longrightarrow \Delta v_{//} = -\frac{1}{c}\int_{z_{in}}^{z_{out}} \frac{d\phi}{dz}dz = \frac{1}{c}(\phi_{in} - \phi_{out})$$

we can safely assume $|z_{in}| = |z_{out}| = \infty$

as such $\phi(\infty) \to 0 \Longrightarrow \Delta v_{//} = 0$

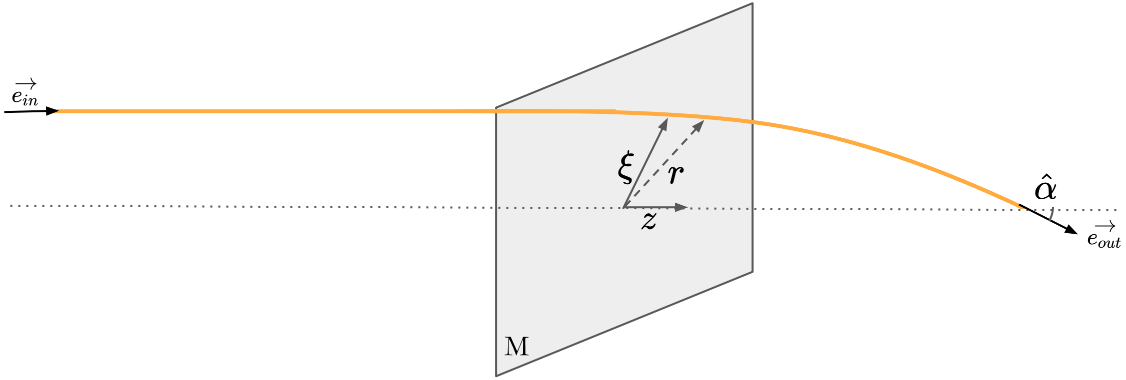
$$\Delta v_\perp = -\frac{1}{c}\int_{z_{in}}^{z_{out}} \frac{d\phi}{d\xi}dz = -\frac{GM\xi}{c}\int_{z_{in}}^{z_{out}} (\xi^2 + r^2)^{-3/2}dz$$

$\tan\theta = z/\xi$

$$= -\frac{GM}{c\xi}\int_{-\pi/2}^{\pi/2} \cos\theta d\theta = -\frac{2GM}{c\xi}$$

and the photon's velocity will change accordingly in the perpendicular direction

Light deflection according to Newton



$$\hat{\alpha} = \vec{e_{in}} - \vec{e_{out}}$$

$$z = ct \longrightarrow dz = cdt$$

$$r^2 = \xi^2 + z^2$$

Therefore:

$$\vec{v_{out}} = |v_{out}|\vec{e_{out}} = c\vec{e_{in}} - \frac{2GM}{c\xi}\vec{e_{\xi}}$$

but $\quad |v_{out}| = \sqrt{c^2 + \frac{4G^2M^2}{c^2\xi^2}} \sim c$

and as such: $\hat{\alpha} = \vec{e_{in}} - \vec{e_{out}} = \frac{2GM}{c^2\xi}\vec{e_{\xi}}$

$\Longrightarrow$

If the impact parameter is $R_\odot$

$$\hat{\alpha} = \frac{2GM}{c^2 R_\odot} \simeq 0.875''$$

Light deflection according to General Relativity

Gravity is a manifestation of spacetime geometry.

Matter/energy just want to move along geodesics, and they actually do, is just the underlying geometry that is curved, thus deflecting their path from the pure straight-Minkowskian-line.

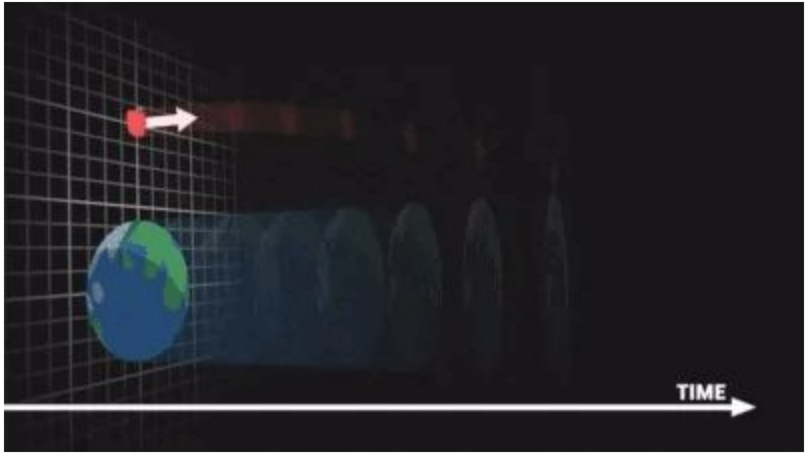
This applies to people on Earth (or whatever exoplanet you could think of), planets around stars, stars in galaxies, galaxies in clusters and DM wells, the Universe as a whole, and as such, also to light.

"Spacetime tells matter how to move;
matter tells spacetime how to curve"
J. A. Wheeler

$$\nabla_\nu A^\mu = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

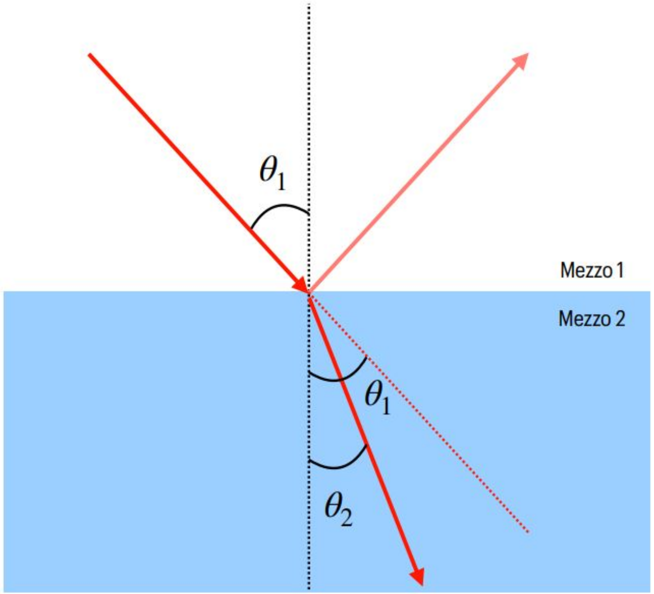
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

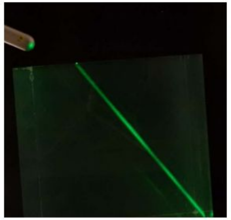


Light deflection according to General Relativity

We can study the light deflection according to GR (and other useful intuitions along the path) treating the problem as in geometrical optics: a refraction problem.

In geometrical optics, when light travels between two mediums with different refractive indexes n_1 and n_2, it changes speed and direction.





The same intuition can be applied when lights travels from an unperturbed Minkowski spacetime to a perturbed (e.g. Schwarzschild) spacetime.

Light deflection according to General Relativity

In the perturbed Schwarzschild metric light travels with speed c' = c/n

N.B. this does not mean that light *actually* changes its speed, which locally is always c; this means that an external observer sees the action of an effective refraction index that gives *the illusion* of a speed variation, which is always a slowdown

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (d\vec{x})^2 = c^2 dt^2 - (d\vec{x})^2$$

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1+\frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1-\frac{2\Phi}{c^2}\right)(d\vec{x})^2$$

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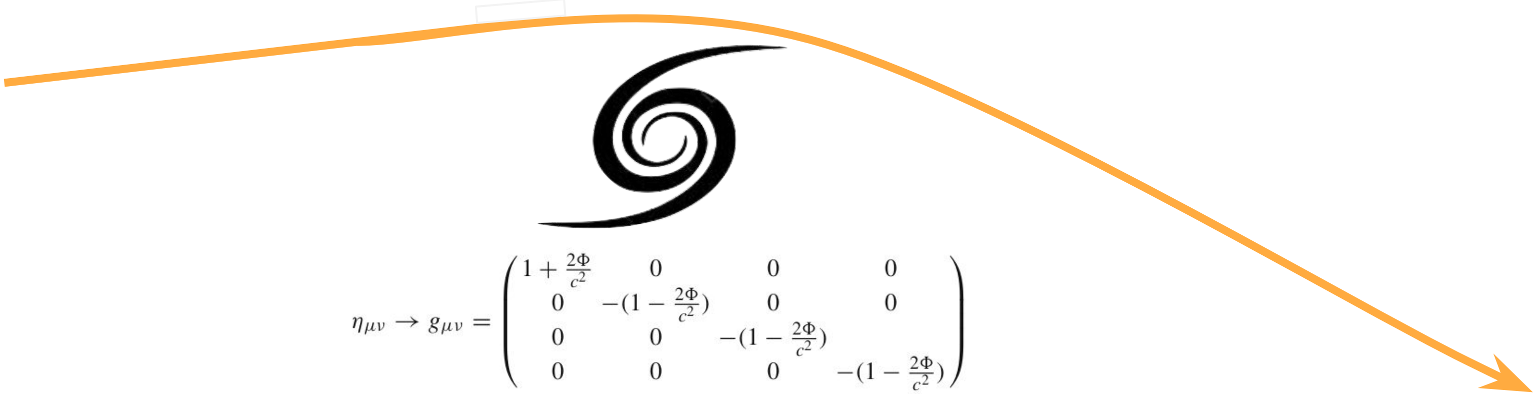

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$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2$$

Light propagates with ds = 0, as such (in the weak field assumption* $\phi/c^2 << 1$):

* which astrophysically holds almost everywhere (in galaxy clusters $\phi/c^2 \sim 10^{-4}$), with the notable exception of black holes

$$\left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 = \left(1-\frac{2\Phi}{c^2}\right)(\mathrm{d}\vec{x})^2 \longrightarrow c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \approx c\left(1+\frac{2\Phi}{c^2}\right) \longrightarrow \boxed{n = \frac{c}{c'} = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}}$$

Outline

Light deflection according to General Relativity

In the perturbed Schwarzschild metric light travels with speed c’ = c/n

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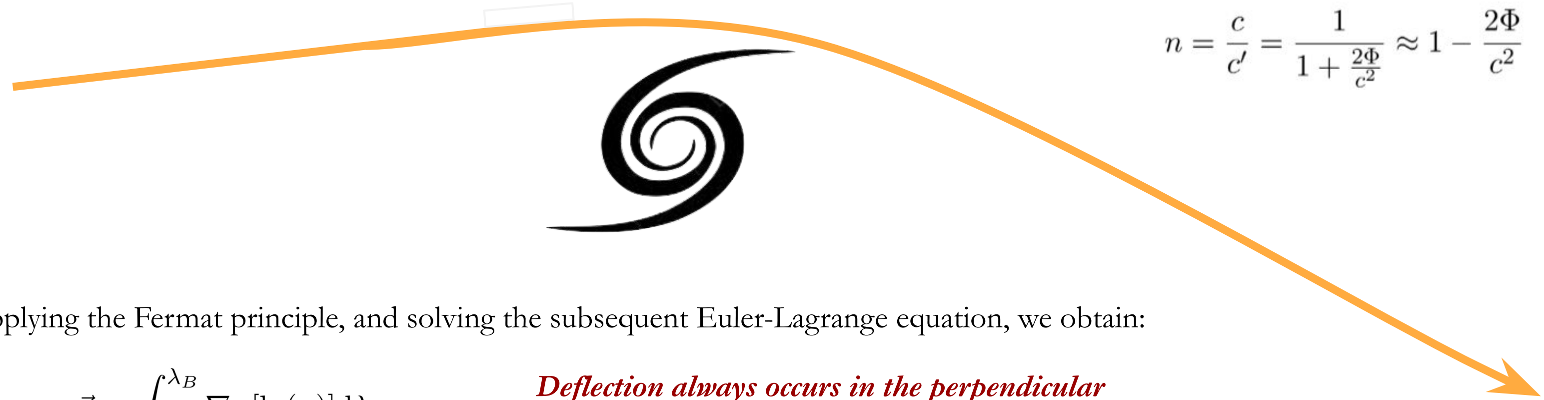

$$n = \frac{c}{c'} = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

Applying the Fermat principle, and solving the subsequent Euler-Lagrange equation, we obtain:

$$\vec{\alpha} = \int_{\lambda_A}^{\lambda_B} \nabla_\perp[\ln(n)]\mathrm{d}\lambda$$

Deflection always occurs in the perpendicular direction to the light trajectory.

and since $\ln(n) \sim -2\phi/c^2$, we find:

$$\vec{\alpha} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}\Phi\mathrm{d}\lambda$$

Deflection always occurs toward the deflector's center.

This is the most general expression for the deflection angle (under the aforementioned assumption of weak field!). However, it is not useful yet, since it requires an integration over the actual light path.

Light deflection according to General Relativity

In the perturbed Schwarzschild metric light travels with speed c' = c/n

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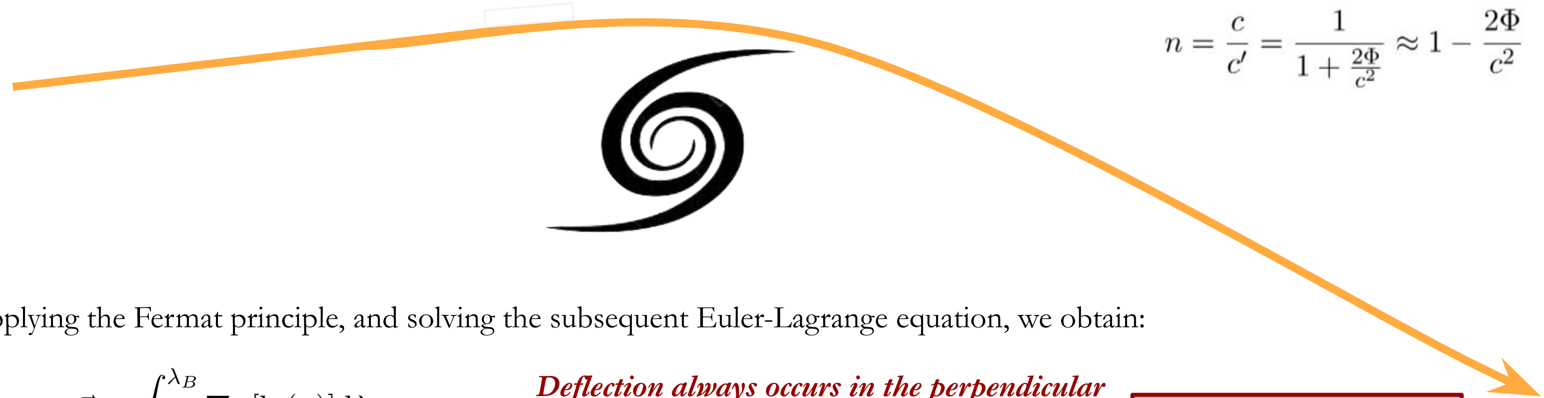

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Deflection always occurs toward the deflector's center.

Notice that the deflection angle does NOT depend on the photon energy: lensing is achromatic!

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Light deflection according to General Relativity

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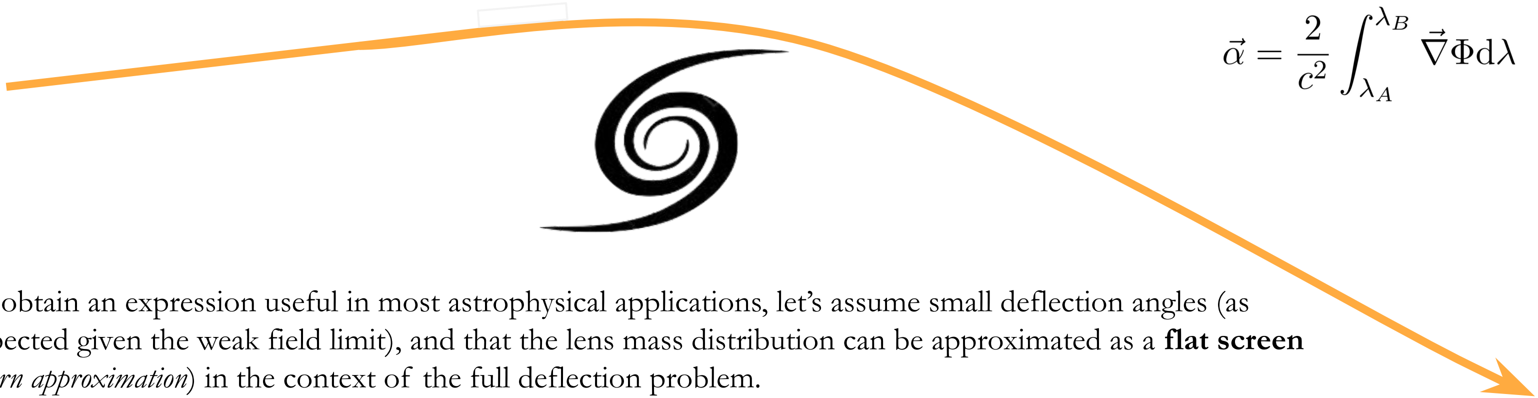


$$\vec{\alpha} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}\Phi \mathrm{d}\lambda$$

To obtain an expression useful in most astrophysical applications, let's assume small deflection angles (as expected given the weak field limit), and that the lens mass distribution can be approximated as a **flat screen** (*Born approximation*) in the context of the full deflection problem.

As such, we obtain:

$$\hat{\vec{\alpha}}(\xi) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_{\perp}\phi(\xi, z)\mathrm{d}z$$

For point mass: $\Phi = -GM\sqrt{\xi^2 + z^2}$

And therefore:

$$\hat{\vec{\alpha}}(\xi) = \frac{4GM}{c^2\xi}\vec{e}_{\xi} = \frac{4GM}{c^2\xi^2}\vec{\xi}$$

which is just two times the one obtained under Newtonian gravity $\longrightarrow$

If the impact parameter is $R_{\odot}$

$$\hat{\alpha} = \frac{4GM}{c^2 R_{\odot}} \simeq 1.75''$$

Outline

Notes/References

Brief History of GL

First detections

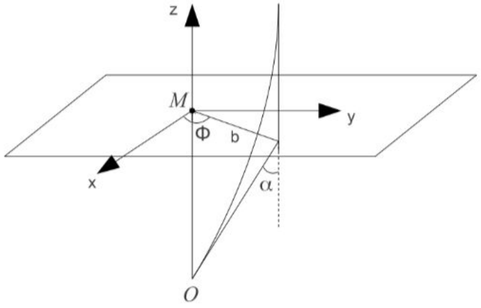
Light deflection:

- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

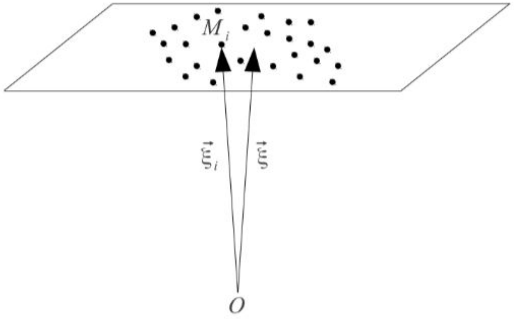
Point mass



$$\hat{\vec{\alpha}}(\xi) = \frac{4GM}{c^2\xi}\vec{e}_\xi = \frac{4GM}{c^2\xi^2}\vec{\xi}$$

Ensemble of point masses

Total deflection is the sum of every single mass contribution



$$\vec{\alpha}(\vec{\xi}) = \sum_i^N \vec{\alpha}(\vec{\xi}-\vec{\xi}_i) = \frac{4G}{c^2}\sum_i^N M_i \frac{\vec{\xi}-\vec{\xi}_i}{|\vec{\xi}-\vec{\xi}_i|^2}$$

Extended (continuous) mass distribution

Thin screen approximation: the (formally 3D) lens is approximated by a planar distribution of matter (lens plane). Once defined the lens surface density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\,\mathrm{d}z$$

the deflection angle becomes:

$$\vec{\alpha}(\vec{\xi}) = \frac{4M}{c^2}\int \frac{\vec{\xi}-\vec{\xi}'}{|\vec{\xi}-\vec{\xi}'|^2}\Sigma(\vec{\xi}')\mathrm{d}^2\vec{\xi}'$$

ξ' is the distance at which the light passes with respect to the lens' barycenter

For a circularly symmetric mass distribution:

$$\vec{\alpha}(\xi) = \frac{4GM(<\xi)}{c^2\xi}$$

with $M(<\xi) = 2\pi\int d\xi'\xi'\Sigma(\xi')$

Only the mass contained within the impact parameter actually enters the deflection angle definition.

Outline

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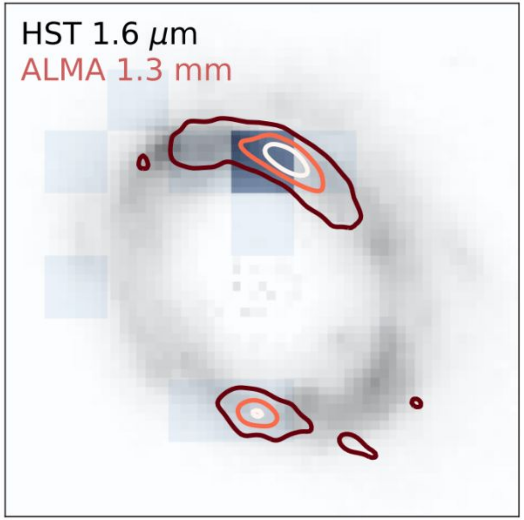
Light deflection according to General Relativity



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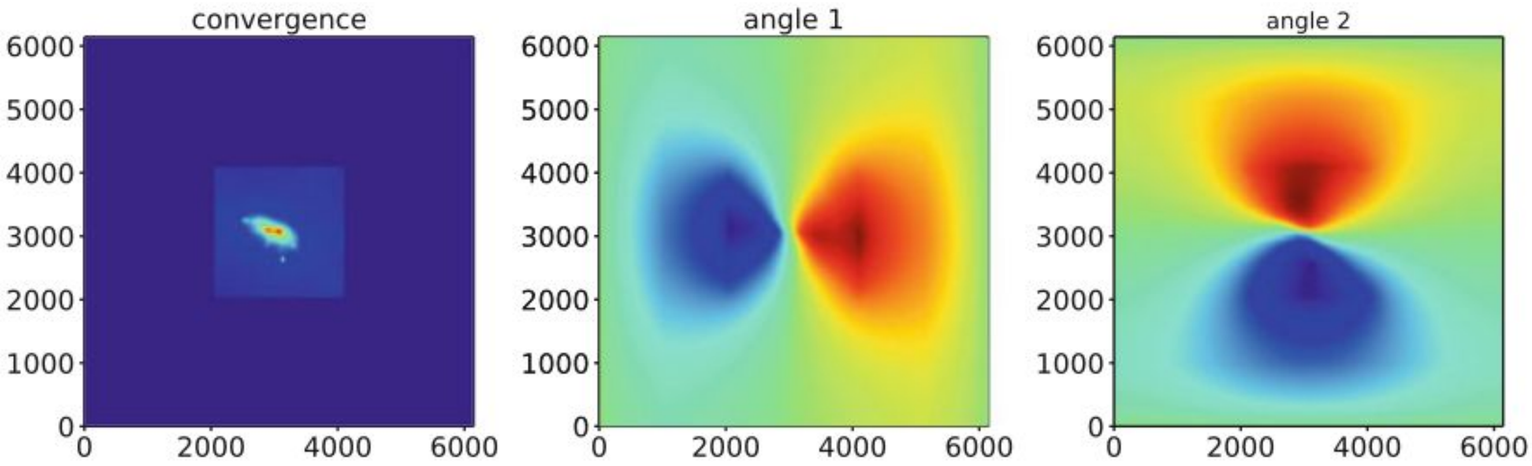
Light deflection according to General Relativity



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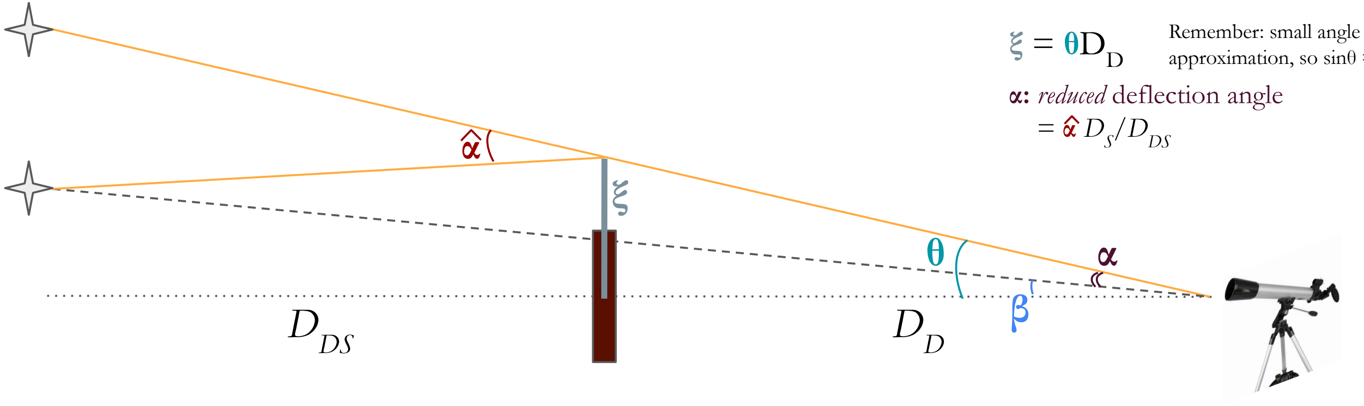
Outline

The lens equation



$\xi = \theta D_D$ Remember: small angle approximation, so $\sin\theta = \theta$

α: *reduced* deflection angle $= \hat{\alpha}\, D_S/D_{DS}$

Outline

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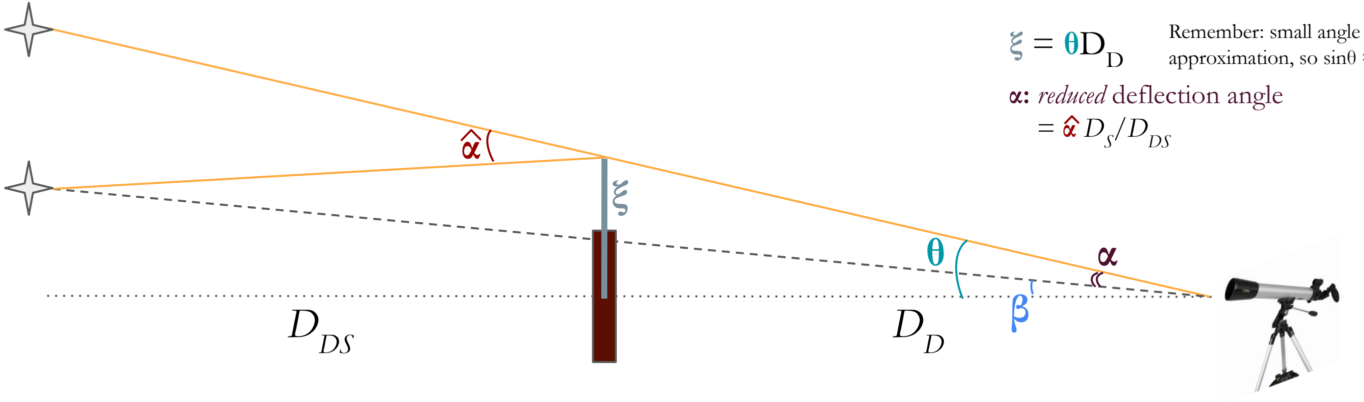
Lens equation

The lens equation



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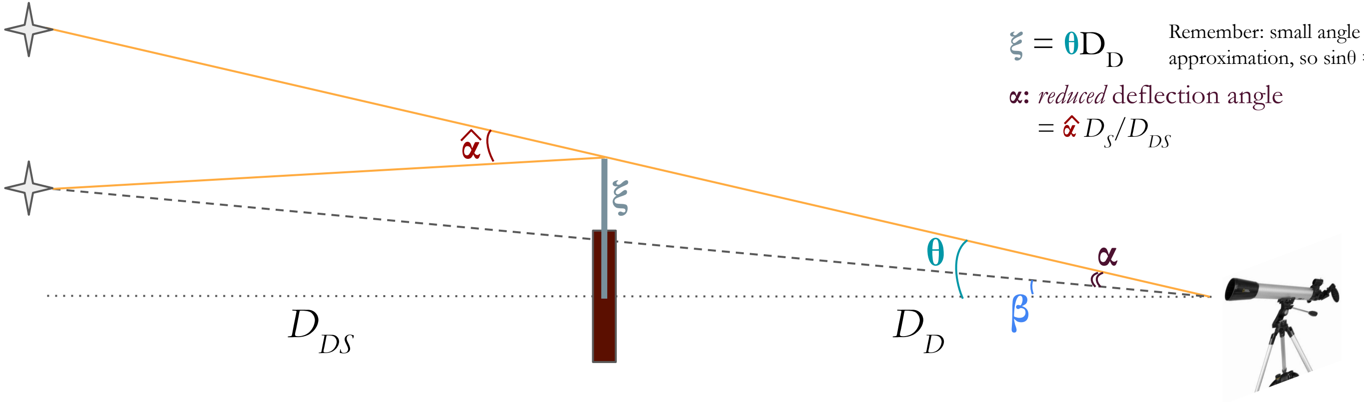
It is immediately clear that $\alpha + \beta = \theta$ or, in terms of the segments on the source plane: $\hat{\alpha}\, D_{DS} + \beta\, D_S = \theta\, D_S$

Outline

The lens equation



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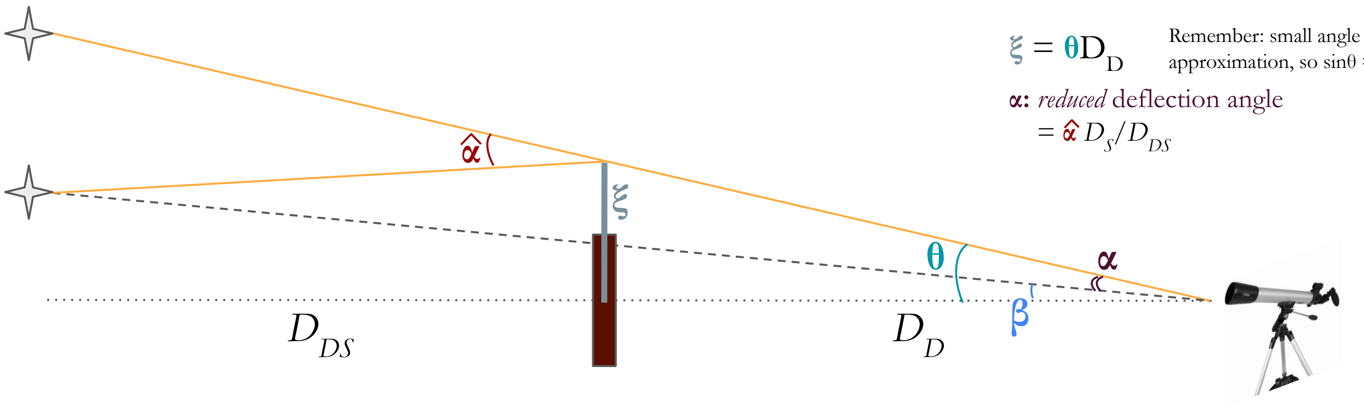
Circularly symmetric lens

$$\beta = \theta - \alpha = \theta - \frac{D_{DS}}{D_S}\hat{\alpha}$$

We know that $\vec{\alpha}(\xi) = \frac{4GM(<\xi)}{c^2\xi}$

$$= \theta - \frac{D_{DS}}{D_S}\frac{4GM(\xi)}{\xi c^2} = \theta - \frac{D_{DS}}{D_D D_S}\frac{4GM(\theta)}{\theta c^2}$$

The lens equation



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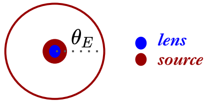
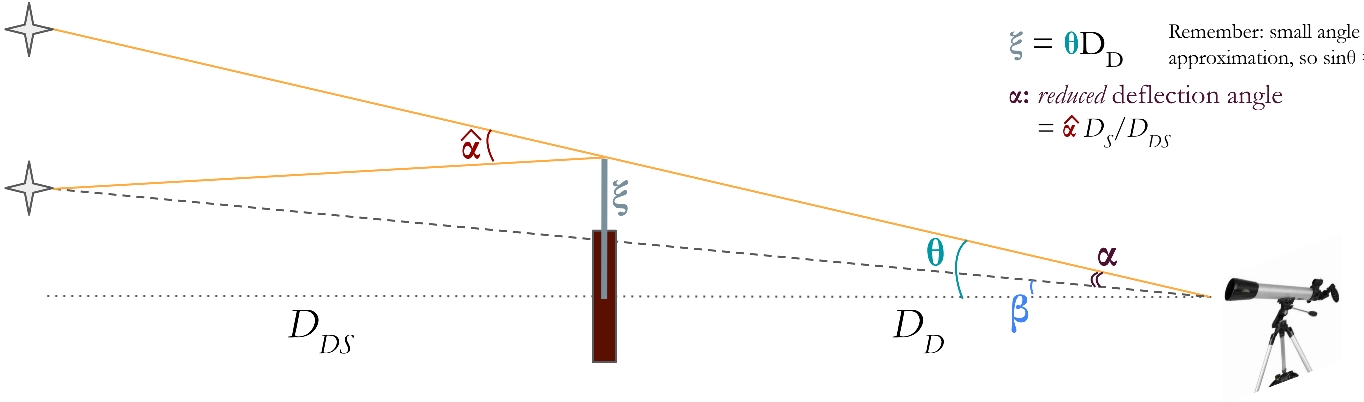


If we call:

$$\theta_E = \left(\frac{D_{DS}}{D_D D_S}\frac{4GM(\theta)}{c^2}\right)^{1/2}$$

Einstein radius

The lens equation



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If we call:

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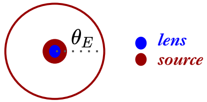
Einstein radius

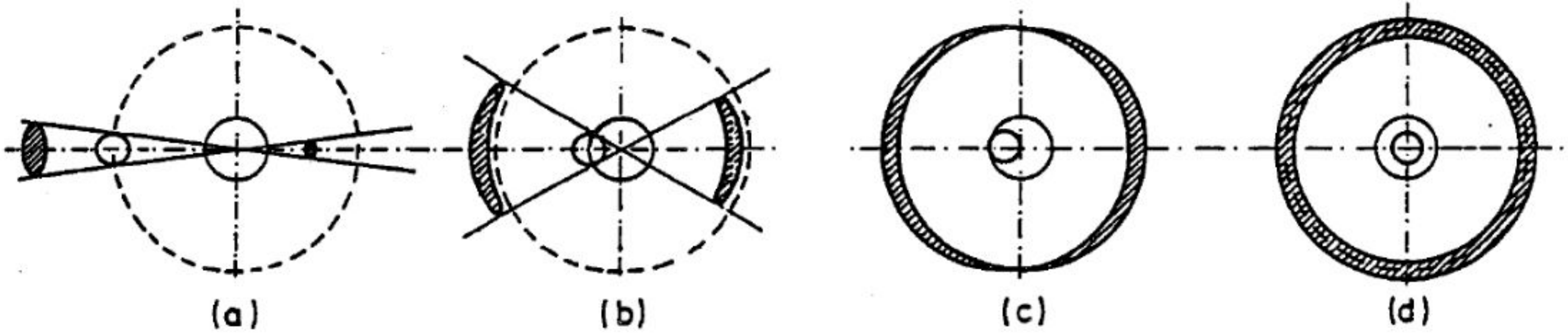
If: $\beta = 0 \Longrightarrow \theta = \theta_E$



else:

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 - 4\theta_E^2}\right)$$

The lens equation



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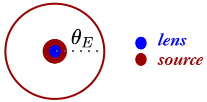

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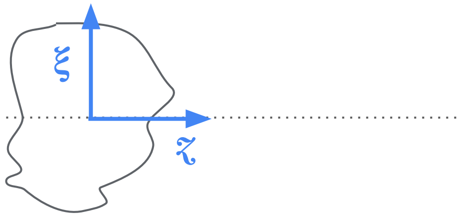
Outline

General lensing solution

What happens in the general case of a lens without any particular symmetry (though still in the thin lens hypothesis)?

The fundamental quantity here is the lensing potential:

$$\psi(\theta) = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int_{-\infty}^{+\infty} \phi(\xi, z) dz$$



The lensing potential satisfies two fundamental properties in gravitational lensing:

$$\nabla_\theta \psi(\theta) = \alpha$$

The angular gradient of the lens potential is the reduced deflection angle

$$\nabla_\theta^2 \psi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma^\star} \equiv 2K(\theta)$$

The laplacian of the lensing potential is the convergence K, that is directly related to the lens surface density (→ its mass distribution)

with:

$$\Sigma^\star = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$$

the critical surface density, such that $\alpha(\theta) = \theta$ for any θ

Outline

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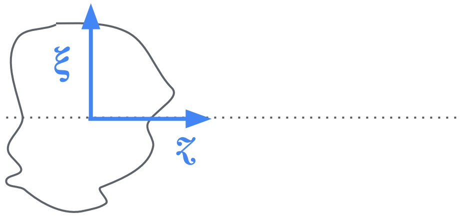

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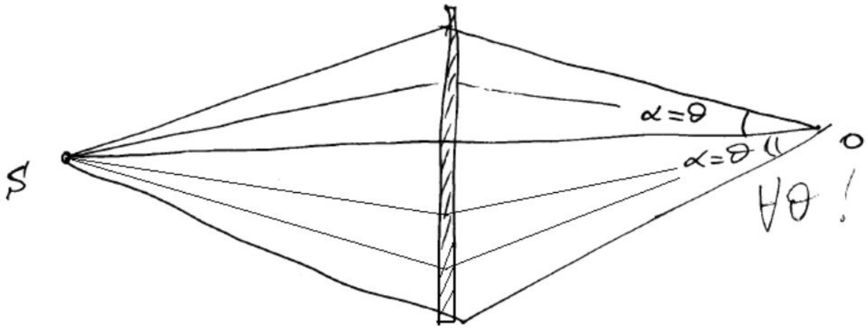
The laplacian of the lensing potential is the convergence K, that is directly related to the lens surface density (→ its mass distribution)

with:

$$\Sigma^\star = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$$

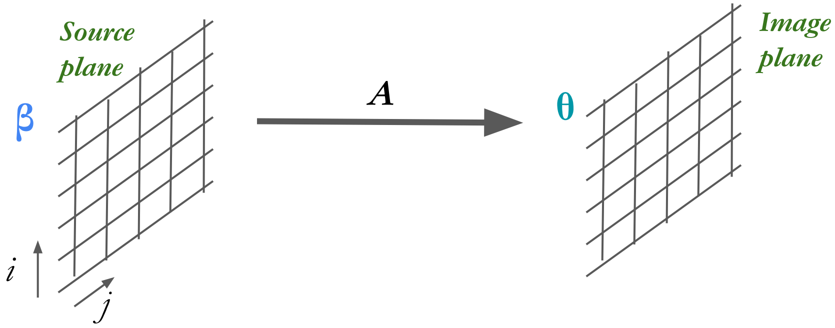
the critical surface density, such that $\alpha(\theta) = \theta$ for any θ



Lenses for which $\Sigma > \Sigma^*$ are said to be *strong lenses*; the opposite case is the one for *weak lenses*

Lensing at the first order: the linear solution

At the first order ("linear") a lensing phenomena is approximated as a mapping, a transformation between two planes: the ***source plane***, where the source lies, and the ***image plane***, that is what we observe.
Mapping between two planes is always obtained from the Jacobian of the transformation: $A \equiv \partial\vec{\beta}/\partial\vec{\theta}$



$$A = \partial\vec{\beta}/\partial\vec{\theta} = \delta_{ij} - \frac{\partial\vec{\alpha}}{\partial\hat{\theta}} \qquad \text{but} \quad \vec{\alpha} = \frac{\partial\psi}{\partial\vec{\theta}} \quad \Longrightarrow A = \delta_{ij} - \frac{\partial^2\psi}{\partial\theta_i\partial\theta_j} = \delta_{ij} - \Psi_{ij}$$

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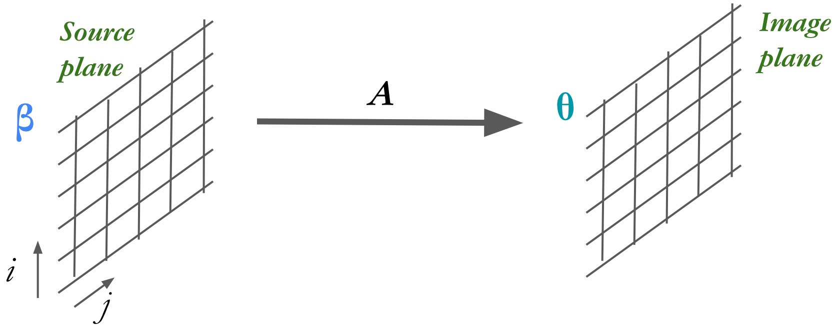

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Now we split this matrix into its *isotropic* and *anisotropic* components:

$$\tfrac{1}{2}\mathrm{Tr}A \cdot \mathbb{I} = \left[1 - \tfrac{1}{2}(\Psi_{11} + \Psi_{22})\right]\delta_{ij} \equiv (1 - K)\delta_{ij}$$

Components of a vector called ***shear***

$$A - \frac{1}{2}\mathrm{Tr}A \cdot \mathbb{I} = \delta_{ij} - \Psi_{ij} - \frac{1}{2}(2 - \Psi_{11} - \Psi_{22})\delta_{ij} = \begin{pmatrix} -\overset{\gamma_1}{\boxed{\frac{1}{2}(\Psi_{11} - \Psi_{22})}} & -\overset{\gamma_2}{\boxed{\Psi_{12}}} \\ -\Psi_{21} & \frac{1}{2}(\Psi_{11} - \Psi_{22}) \end{pmatrix}$$

Therefore the mapping we’re looking for is:

$$A = \begin{pmatrix} 1 - K - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - K + \gamma_1 \end{pmatrix}$$

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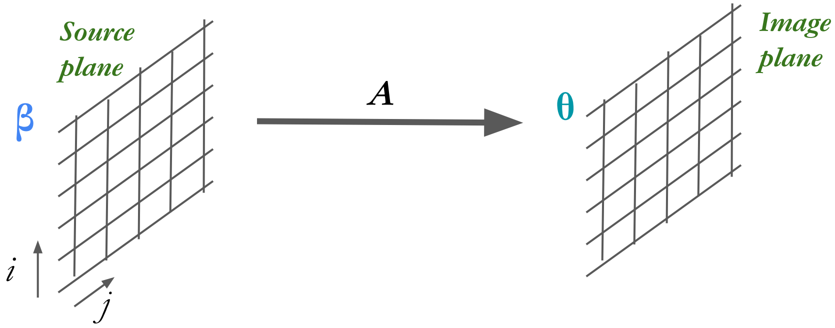
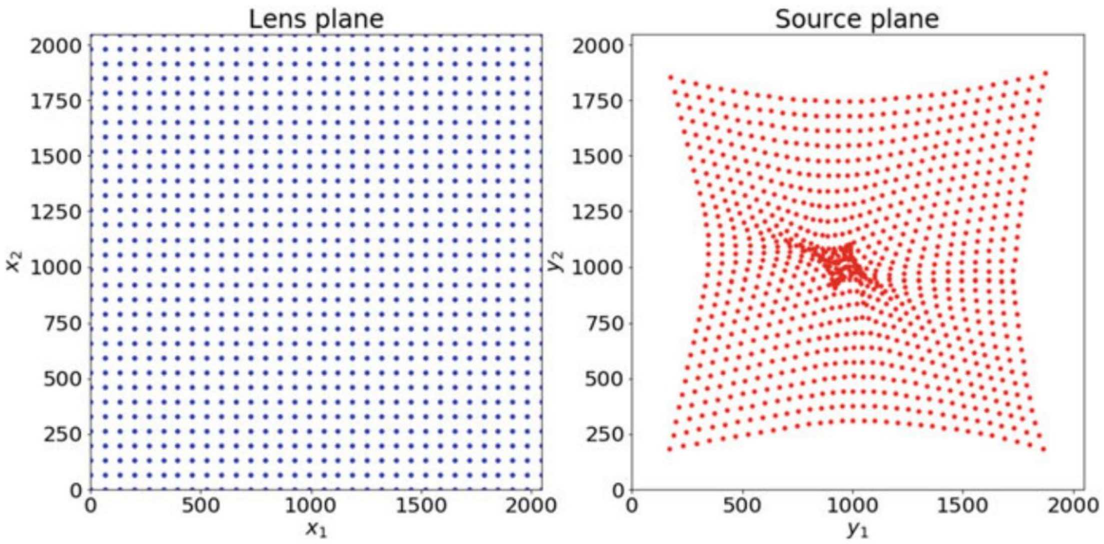
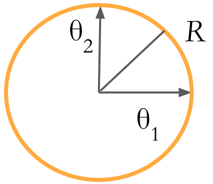


The example on the right shows how points on the lens/image plane are mapped onto the source plane, once a lens model is assumed



Lensing at the first order: the linear solution

What is the meaning of the convergence **K** and shear **γ**? Let's consider a simple circular source.



Circle of radius R:

$$\theta_1^2 + \theta_2^2 = R^2$$

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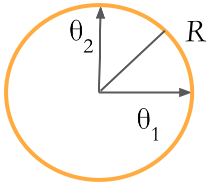


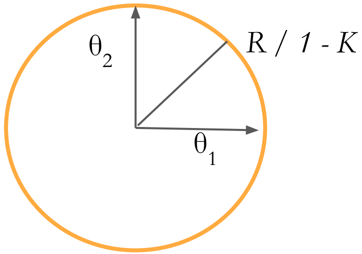
Circle of radius R:

$$\theta_1^2 + \theta_2^2 = R^2$$

If $\gamma = 0$ then $A = \begin{pmatrix} 1-K & 0 \\ 0 & 1-K \end{pmatrix}$ so the transformation is:

$$\vec{\beta} = (1-K)\delta_{ij}\vec{\theta} \Longrightarrow (1-k)^2(\theta_1^2 + \theta_2^2) = R^2$$

the result is still a circle, with radius *R/(1-K)*



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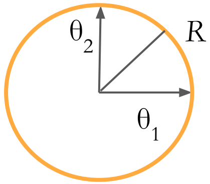

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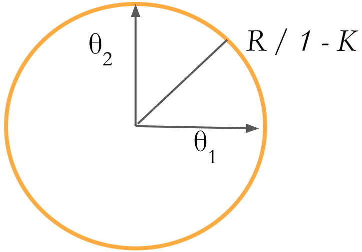
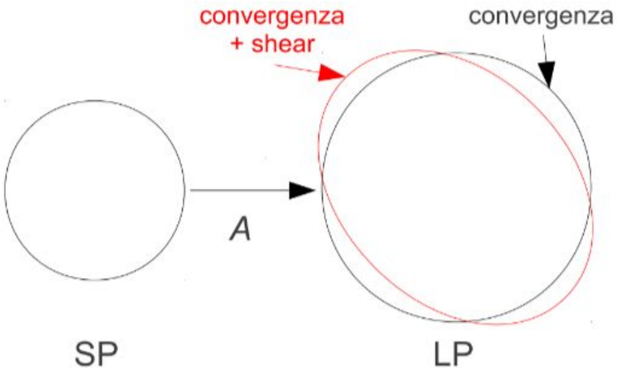


Now, let's consider the shear:

$$\vec{\beta} = (1-K)\delta_{ij}\vec{\theta} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix}\vec{\theta}$$

and the equation becomes the one of an ellipse:

$$(1-k-\gamma)^2\theta_1^2 + (1-k+\gamma)^2\theta_2^2 = R^2$$



with:

$$a = \frac{R}{1-k-\gamma}$$

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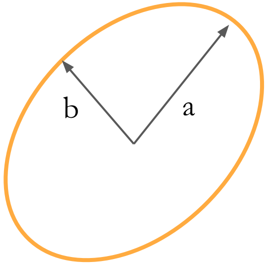

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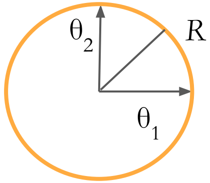

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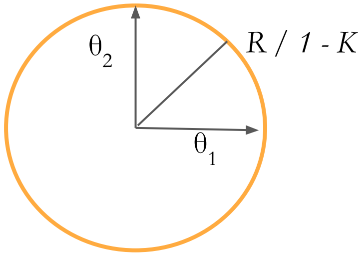

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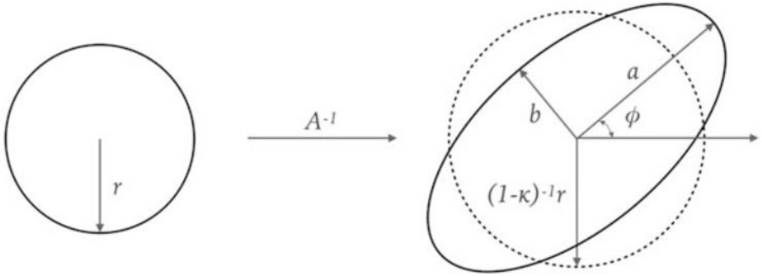

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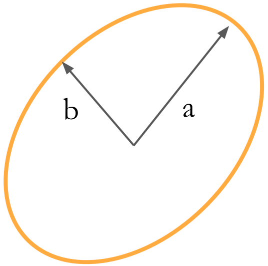


Outline

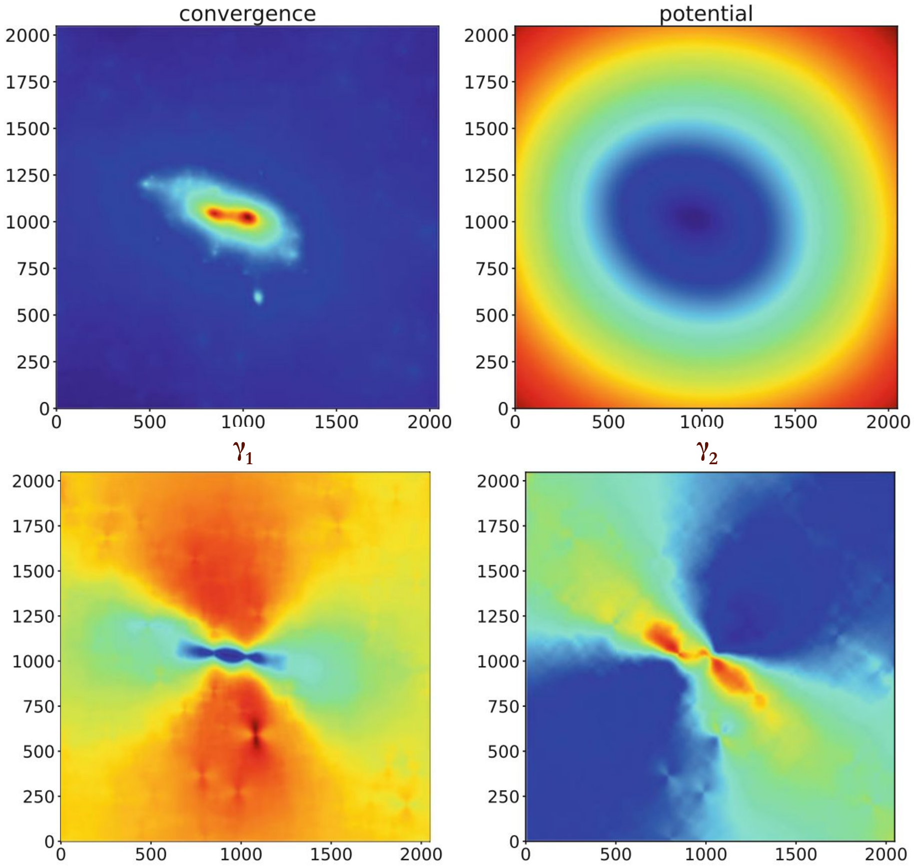
Lensing at the first order: the linear solution

What is the meaning of the convergence **K** and shear **γ**?

Convergence closely follows the lens mass distribution

Potential immediately gives an intuition on deflection angle

Shear is just a mess



Outline

Lensing at the first order: magnification, caustics, critics

The surface brightness S of a source is defined as:

$$S \equiv \frac{F}{\Omega} = \frac{L}{4\pi A}$$

where A is the area occupied by the source in the source plane.

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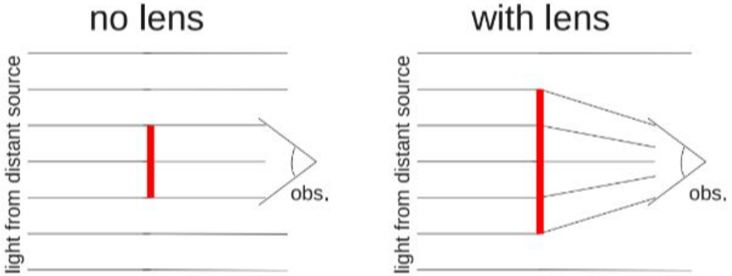
In a lensing phenomena, S remains a constant. Therefore, since the source area A changes, it is necessary that the flux increases (or decreases) proportionally. Considering the full set of multiple images, the result is always an increase in A, and therefore a flux magnification (but the single images could be also de-magnified, i.e. in clusters).

Physically this is motivated by the fact that the observer sees photons that otherwise it would not have seen without the action of the lens.



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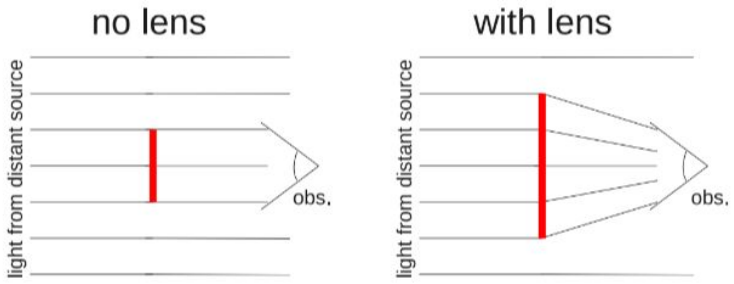


The magnification is obtained as:

$$\mu = \det M = \frac{1}{\det A} = \frac{1}{(1-k)^2-\gamma^2} = \text{image area / source area} \equiv \frac{d\theta^2}{d\beta^2} = \frac{\theta d\theta}{\beta d\beta}$$

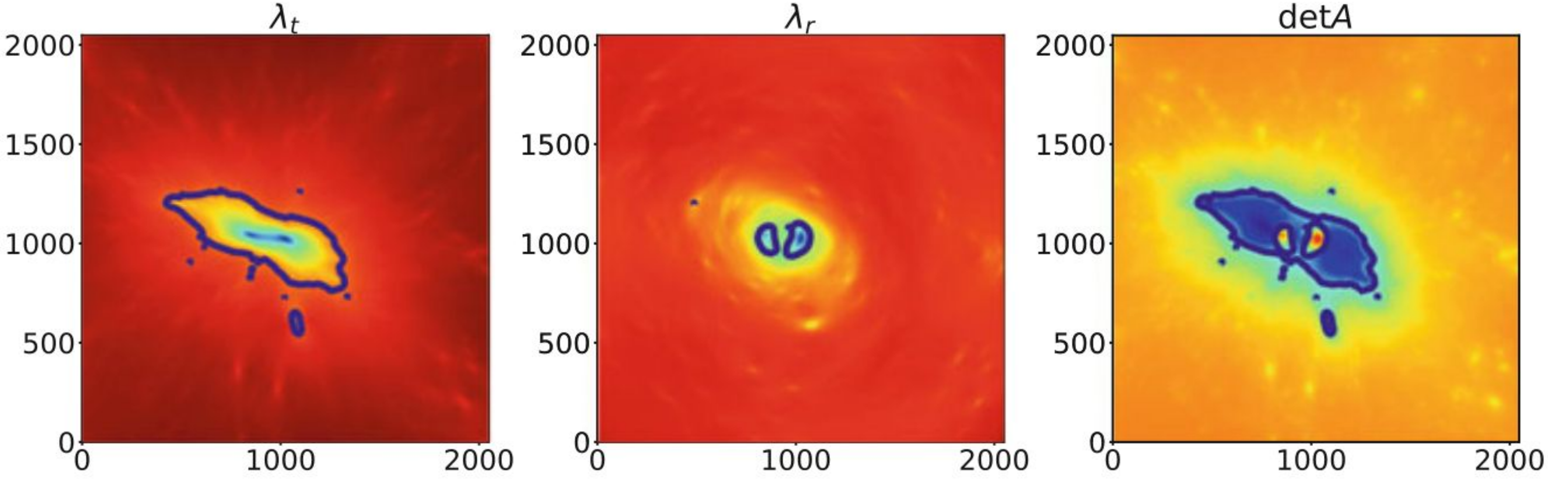
whose eigenvalues are:

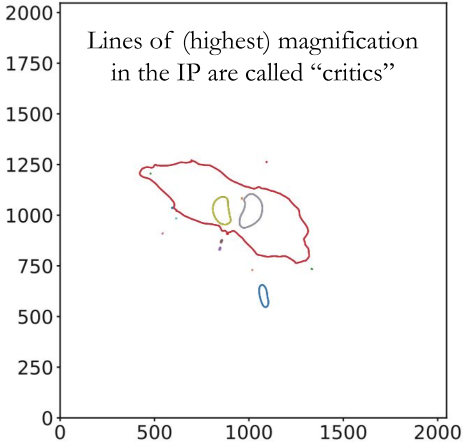
$$\mu_t = \frac{1}{1-k-\gamma}$$

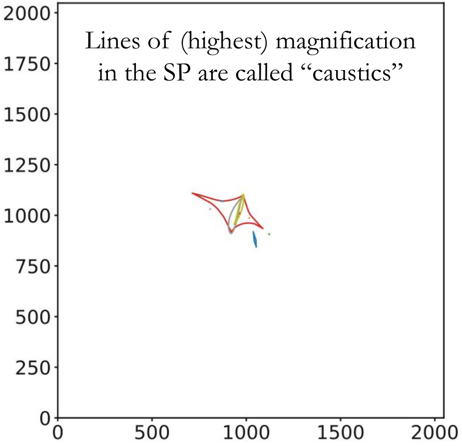
$$\mu_r = \frac{1}{1-k+\gamma}$$

There are points for which $\mu \rightarrow \infty$ that in the source/image planes define lines called ***caustics*** and ***critics***.

Lensing at the first order: magnification, caustics, critics



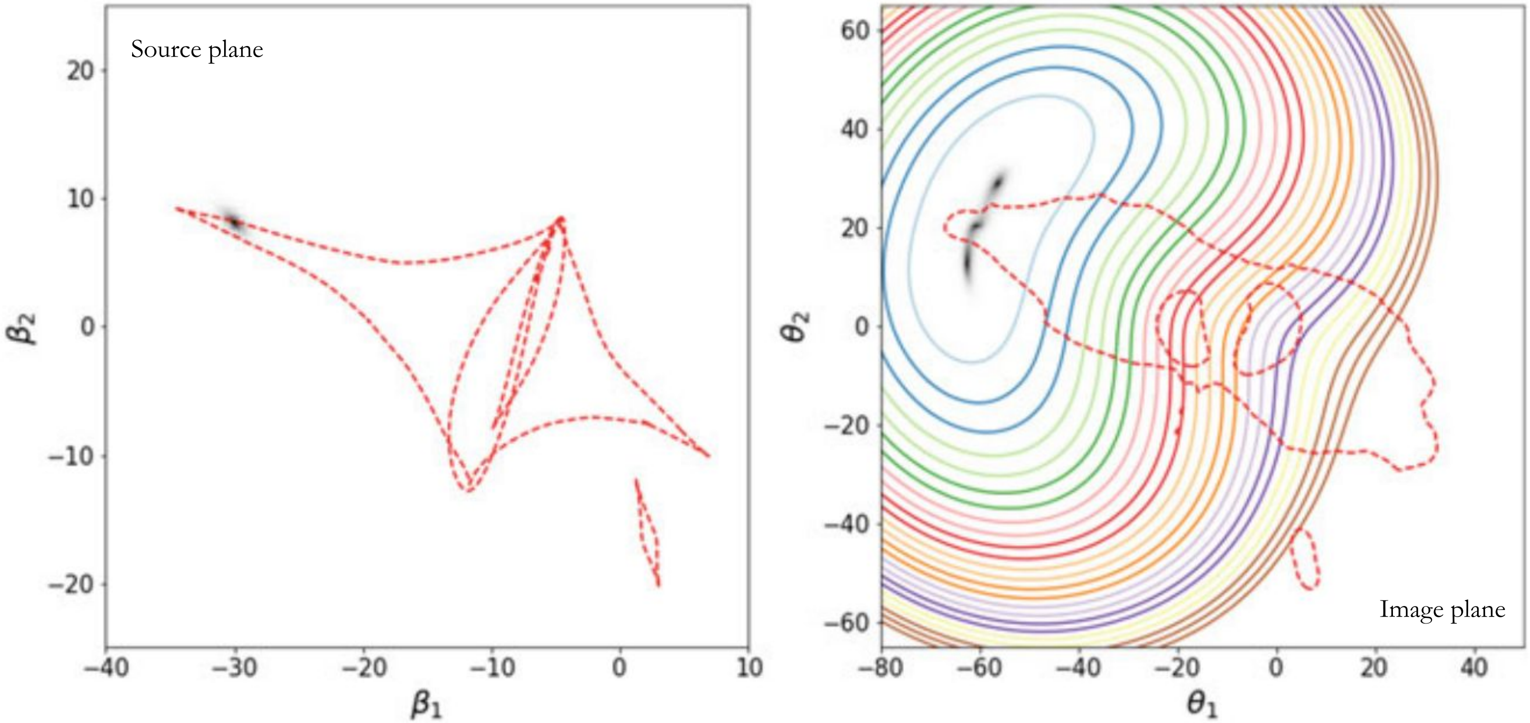




Outline

Lensing at the first order: magnification, caustics, critics

The plot shows the simulation of a (realistic) cluster lens.
The greyscale on the left panel is a Sersic profile, mimicking a background source.



Outline

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

Lensing potential

Linear solution

Convergence and shear

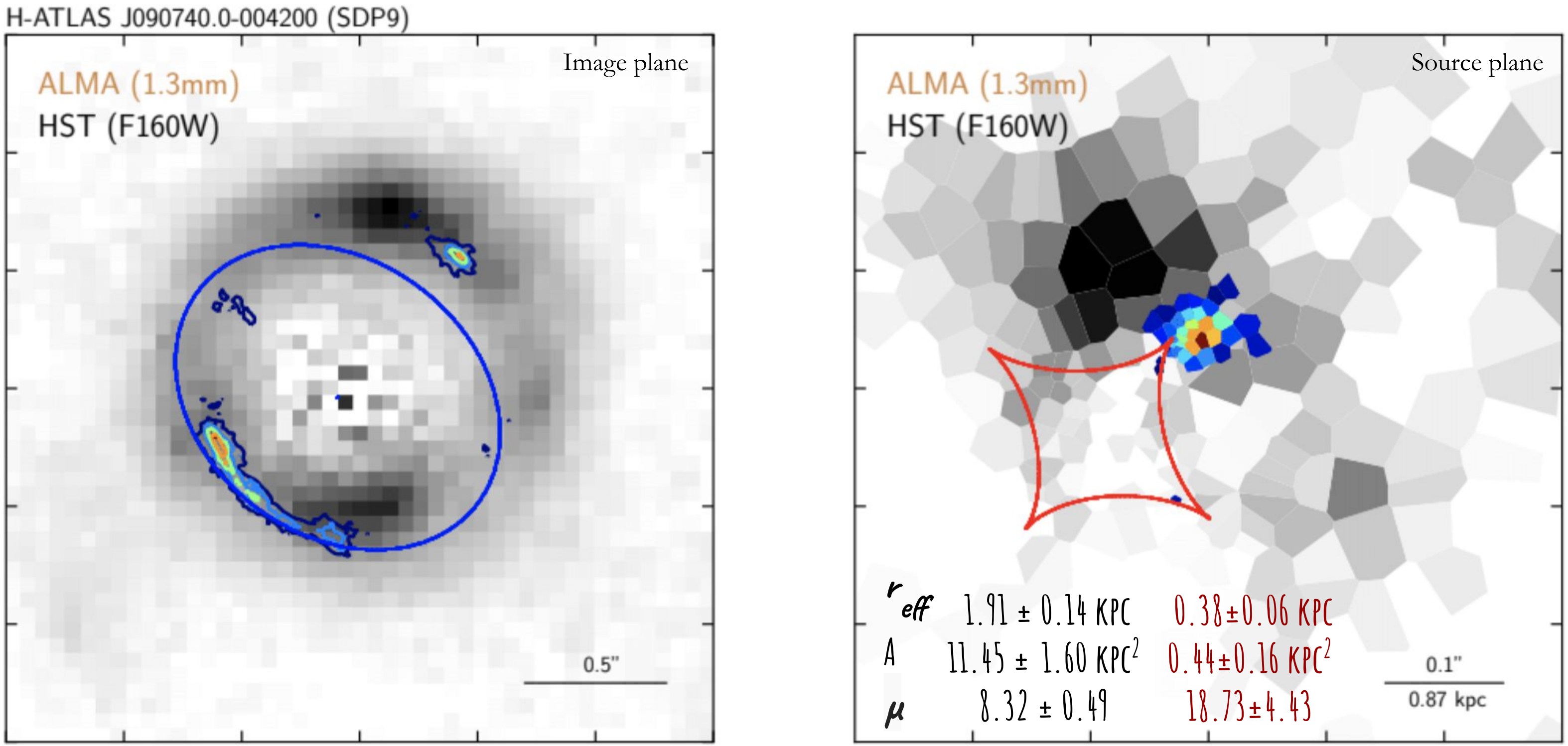
Magnification

Caustics and critics

Real lenses

Real strong lenses

Now a real lensing application. This is SDP.9, a $z_s = 1.577$ source lensed by a $z_l = 0.6129$ elliptical galaxy (here subtracted to enhance and model the signal from the background source), observed at 1.6 μm with Hubble (probing the unobscured emission from stars) and at 1.3 mm with ALMA (probing the emission from the dust).



Massardi et al., 2018

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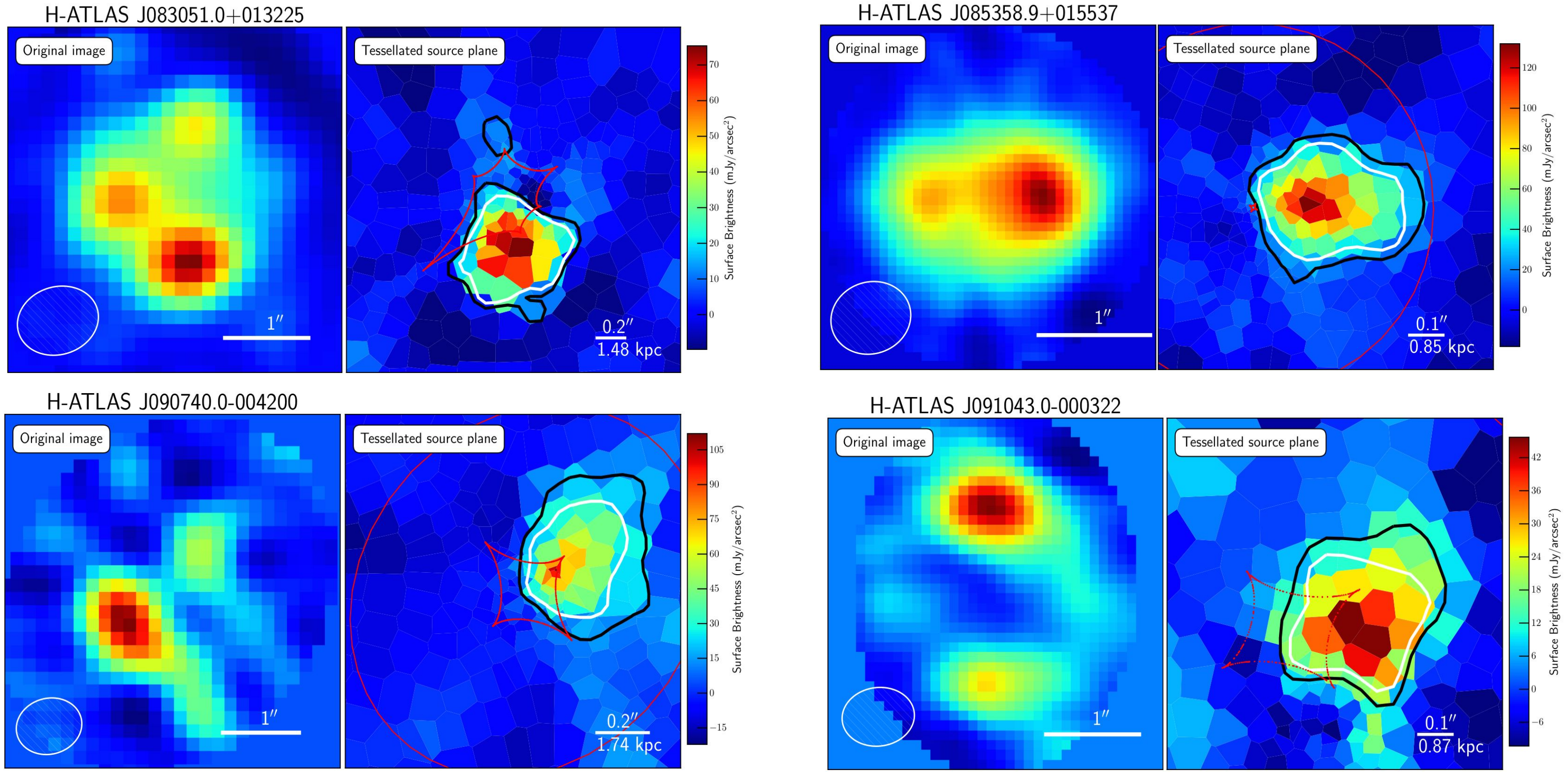
Another set of real lens: four Submillimeter Array (SMA) 880 μm observations of sources in H-ATLAS. Notice that in sub-mm there is no lens emission (why?).



Enia et al., 2018