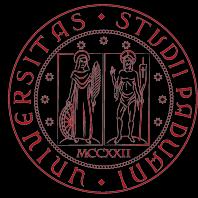
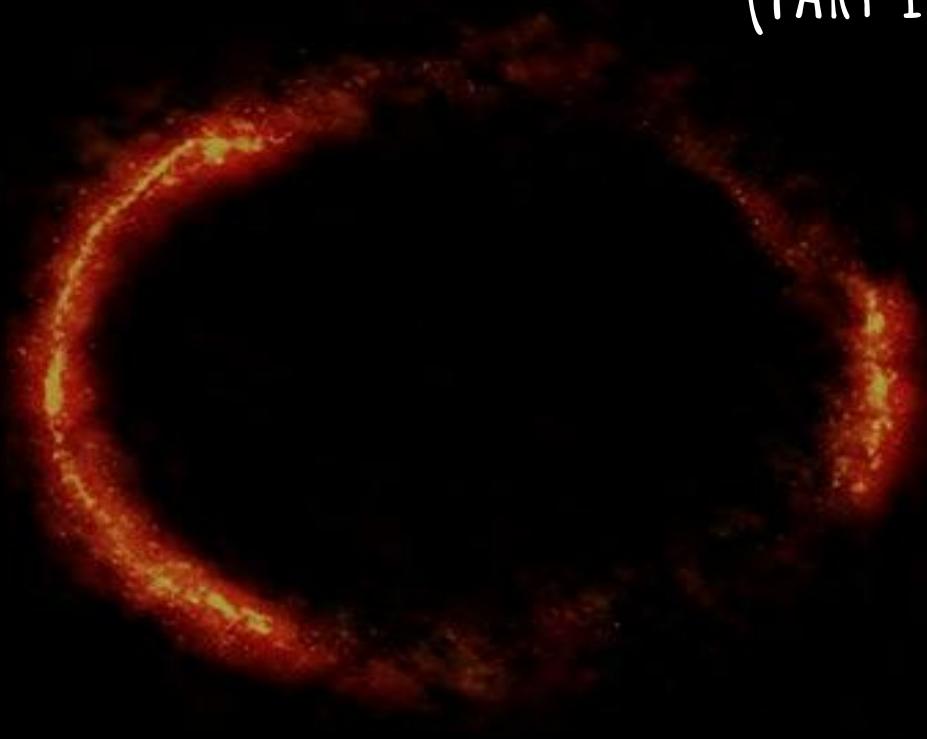


GRAVITATIONAL LENSING (PART 1: LENSING THEORY)



OBSERVATIONAL COSMOLOGY COURSE, MASTER'S DEGREE IN ASTROPHYSICS, A.A 2022-2023

Theory of Gravitational Lensing

$$\hat{\vec{\alpha}}(\xi) = \frac{4GM}{c^2\xi} \vec{e}_\xi = \frac{4GM}{c^2\xi^2} \vec{\xi}$$

$$\hat{\Psi}(\vec{\theta}) = \frac{D_{\text{LS}}}{D_{\text{L}}D_{\text{S}}} \frac{2}{c^2} \int \Phi(D_{\text{L}}\vec{\theta}, z) dz$$

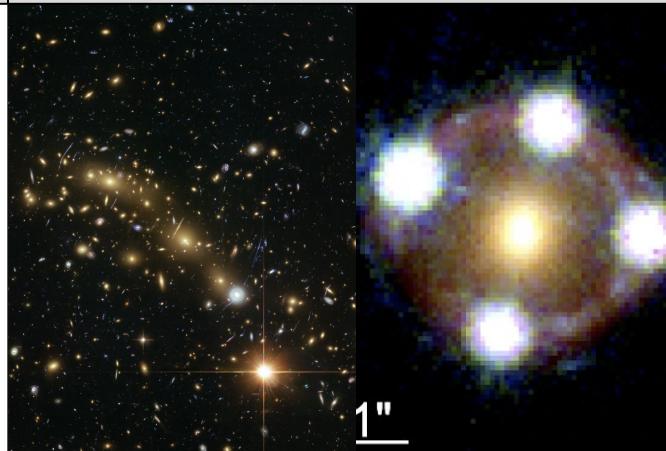
$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i (\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}') \Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2} d^2\xi'$$

$$t(\vec{\theta}) = \frac{(1+z_{\text{L}})}{c} \frac{D_{\text{L}}D_{\text{S}}}{D_{\text{LS}}} \left[\frac{1}{2} (\vec{\theta} - \vec{\beta})^2 - \hat{\Psi}(\vec{\theta}) \right]$$

$$\hat{\alpha}(\theta) = \frac{4GM}{c^2 D_{\text{L}} \theta} \quad \theta_E \approx (10^{-3})'' \left(\frac{M}{M_{\odot}} \right)^{1/2} \left(\frac{D}{10 \text{kpc}} \right)^{-1/2}$$

Lensing, applied



Outline

Notes/References

Section 2

2. DEVIATIONS FROM HOMOGENEITY: THE GRAVITATIONAL LENSING

1. THE GRAVITATIONAL LENSING

We have seen during the third year course how to treat the space-time geometry of an expanding universe on the basis of very general considerations referring to the Cosmological Principle, and essentially making use of the symmetry properties of the space-time inherent the Principle. It has thus been possible to define a metrics that has allowed us to set up in a rigorous way the concept of distances and, based on this, all cosmological observables.

The Cosmological Principle and the consequent Robertson-Walker metrics assume a rigorously homogeneous and isotropic universe. As we have seen in the previous Section, this is valid on the large scales (essentially on scales larger than at least a hundred Mpc), but obviously breaks down on smaller scales where we have seen a high degree of inhomogeneity. In this Section we discuss the phenomenon of **gravitational lensing**, originating from deformations of the space-time metrics due to the inhomogeneities of the gravitational field induced by cosmic structures.

The gravitational lensing offers not only a solid verification of the General Relativity theory (indeed it made its first experimental test), but even a unique instrument for studying the gravitational matter (baryons + dark matter) distribution in the Universe (hence potentially bypassing the very serious problem of cosmological *bias* mentioned in Sect. 1.4) and, finally, a method for constraining the cosmological parameters, in particular the Ω_Λ parameter that so far we have found difficult to quantify in a direct way. However, gravitational lensing has countless applications in astrophysics and cosmology.

For our cosmological applications, gravitational lensing offers a unique, model independent way to probe directly the dark matter distribution, and addressing then the bias problem discussed in Sect. 2.5 below.

Introduction

In agreement with the Einstein theory of General Relativity, an object of a given mass produces a curvature in the space-time with its gravitational field, and imposes a



Lecture Notes in Physics

Massimo Meneghetti

Introduction to Gravitational Lensing

With Python Examples

Springer

<https://link.springer.com/book/10.1007/978-3-030-73582-1>

These slides will be available in a GitHub repository here:
<https://github.com/AndreaEnia>

for anything, contact me at andrea.enia@unibo.it

Outline

Notes/References

Brief History of GL

Outline

Notes/References

Brief History of GL

The idea that gravity can deflect light is way older than you think.

Michell (1784) & Laplace (1796) independently assumed that the mass of a star could be measured from how much the light slows down in its proximity (pre-Einstein...)



John Michell



Pierre Simon de Laplace

Cavendish (late XVII century) and von Soldner (1802) actually did the math:

$$\hat{\alpha}(R) = \frac{2GM}{c^2 R}$$

~ 0.875" for the Sun, which is surprisingly close to the correct value for the deflection angle.



Henry Cavendish

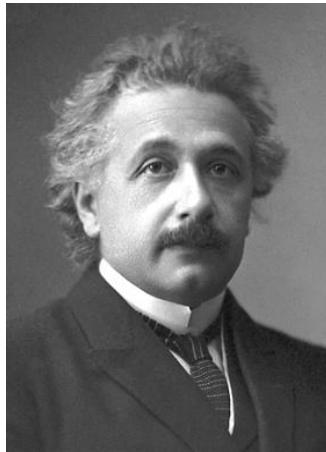


Johann von Soldner

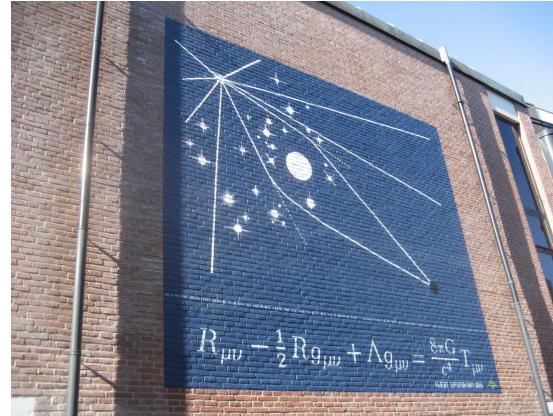
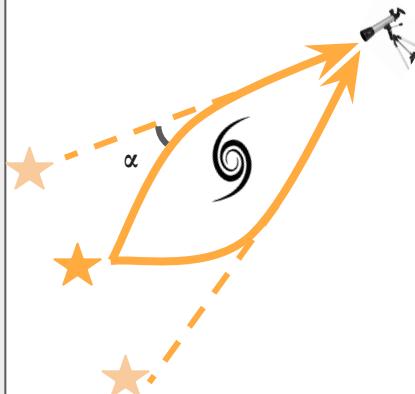
Of course, everything changed after 1915, with the publication of Einstein's Theory of Gravity
(but you may also know it as General Relativity)

1916.

Nº 7.



Albert Einstein



ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 49.

1. *Die Grundlage
der allgemeinen Relativitätstheorie;
von A. Einstein.*

Die im nachfolgenden dargelegte Theorie bildet die denkbar weitgehendste Verallgemeinerung der heute allgemein als

With the new formalism, Einstein predicted that a point source with radius R and mass M would deflect the light of a background object with a deflection angle of:

$$\hat{\alpha}(R) = \frac{4GM}{c^2 R}$$

~ 1.75" for the Sun.

Outline

Notes/References

Brief History of GL

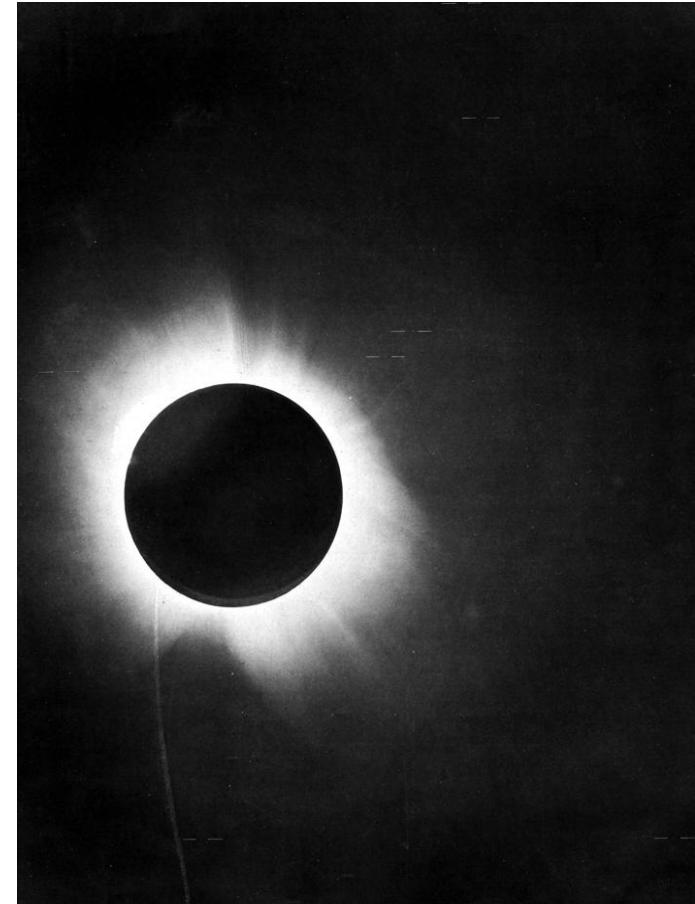
Gravitational lensing played a fundamental role in showing that Einstein's General Relativity is true.

(well, Popper would argue that it just probed Newton's Gravity as false, since you can't *really* probe something as true, but whatever, the concept is clear)



Arthur Eddington

After a series of failed attempts (bad weather, Crimean war, WWI, lousy instrumentation, any other business), Eddington's expedition in Principe Island found a deflection angle of 1.60 ± 0.31 arcsec, in spectacular accord with what Einstein predicted.



ESO/Landessternwarte Heidelberg-Königstuhl
F. W. Dyson, A. S. Eddington & C. Davidson

Outline

Notes/References

Brief History of GL

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All hell broke loose, Einstein became a celebrity, GR one of the two pillars of modern physics...

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Outline

Notes/References

Brief History of GL

Gravitational lensing played a fundamental role in showing that Einstein's General Relativity is true.

(well, Popper would argue that it just probed Newton's Gravity as false, since you can't *really* probe something as true, but whatever, the concept is clear)



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All hell broke loose, Einstein became a celebrity, GR one of the two pillars of modern physics...

... and as for lensing, nothing new happened for 50 years.

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

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Notes/References

Brief History of GL

Observing a gravitational lensing phenomena has been impossible for decades, for two reasons: insufficient instrumentation and (at the time) the need of pure serendipitous observation.

However, theoretical studies on gravitational lensing kept going:

- Chwolson, 1924 and Einstein, 1936 both theorized Einstein's rings



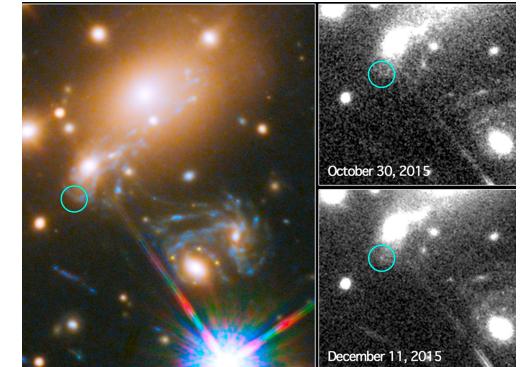
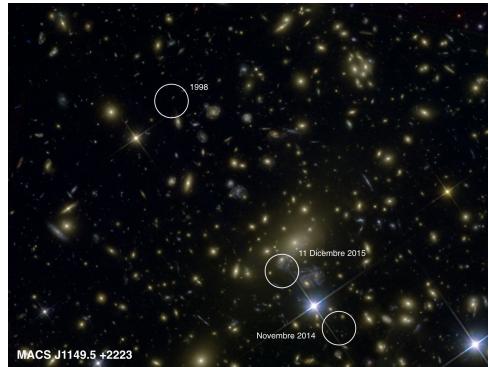
- Zwicky, 1937a, b “*Nebulae** as gravitational lenses”

- Klimov 1963, galaxy-galaxy strong lensing
&
- Liebes 1964, stars and globular clusters as lenses



* at that time *nebula* was still used in lieu of galaxy

- Refsdal, 1964, discussed on multiple images, magnifications, time delays, and firstly theorized lensed SNe



Outline

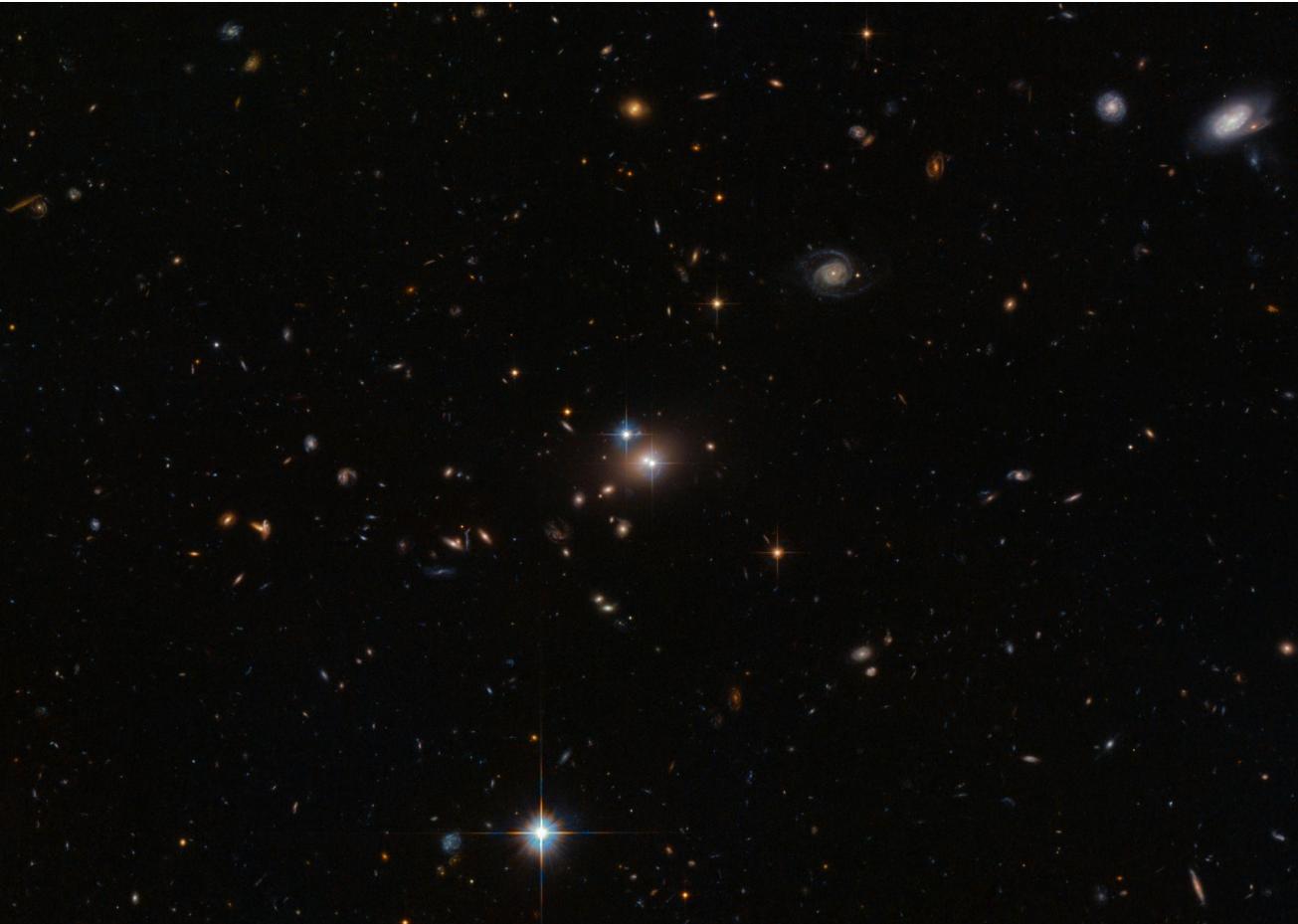
Notes/References

Brief History of GL

First detections

QSO 0957+561: the first ever* detected gravitational lensing event (Walsh et al., 1979)

* well, technically Eddington's measurements were the first one ever



ESA/Hubble & NASA

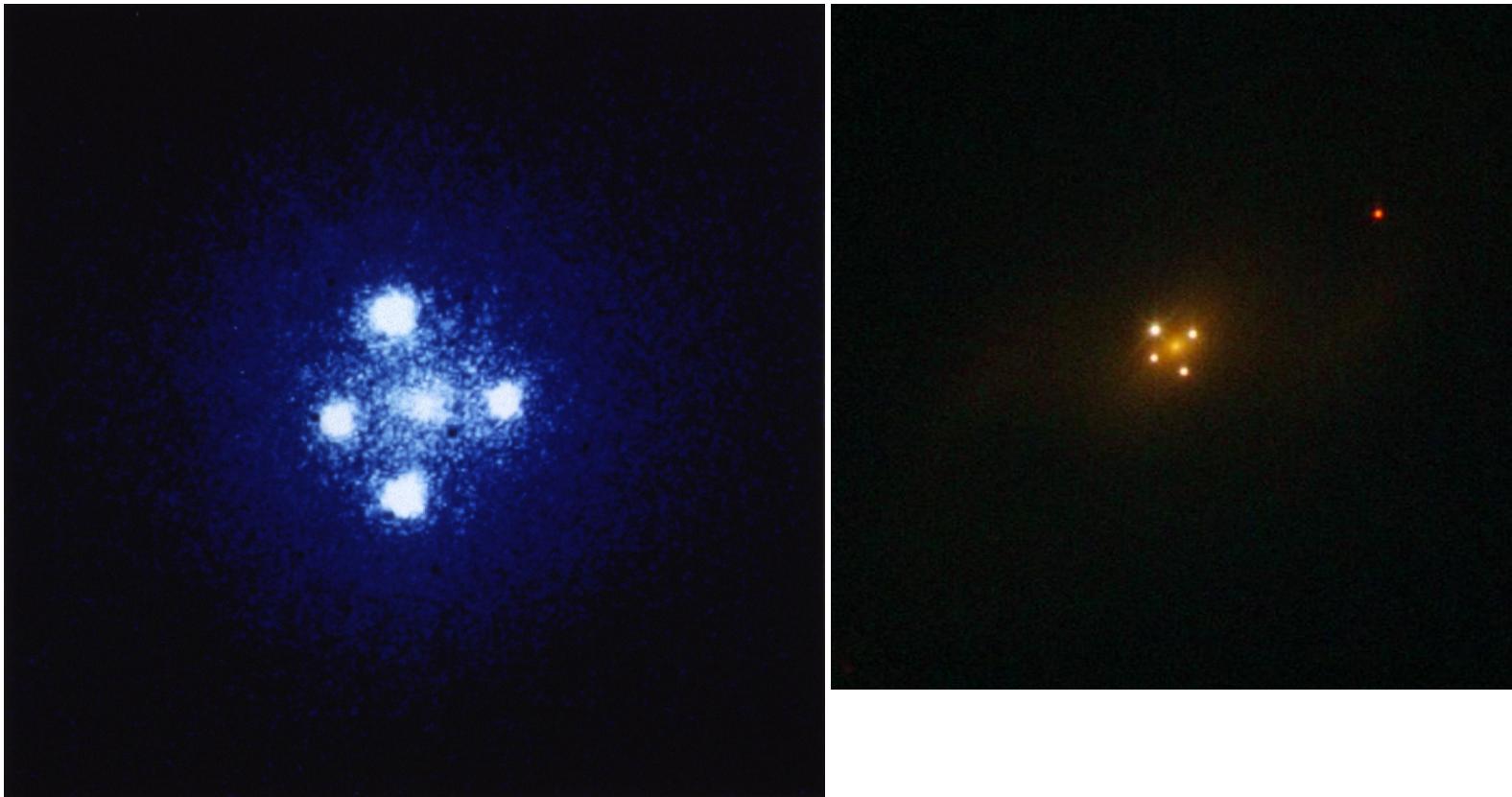
Outline

Notes/References

Brief History of GL

First detections

QSO-Q2237+0305: the first Einstein cross (Huchra et al., 1985)



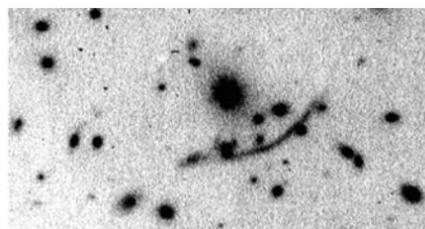
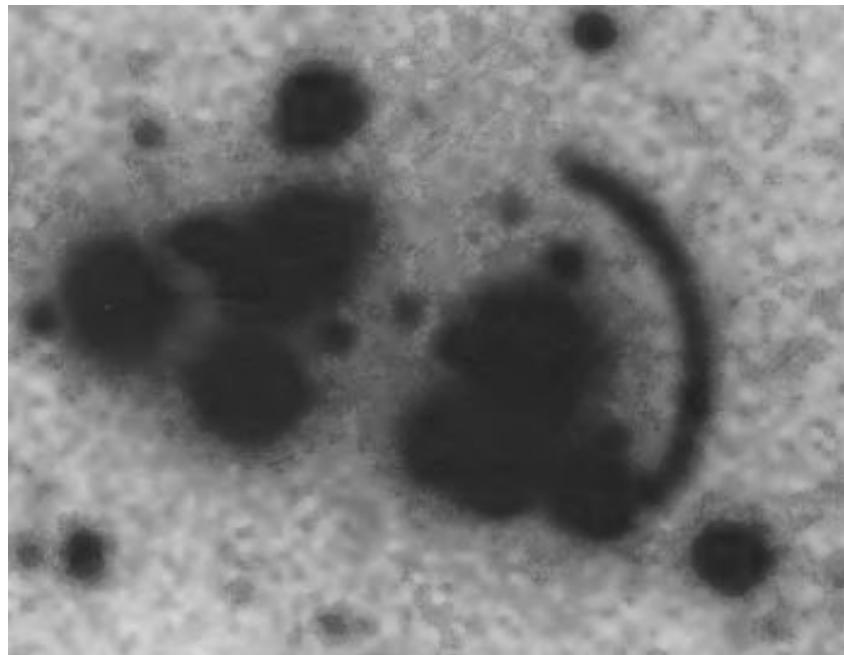
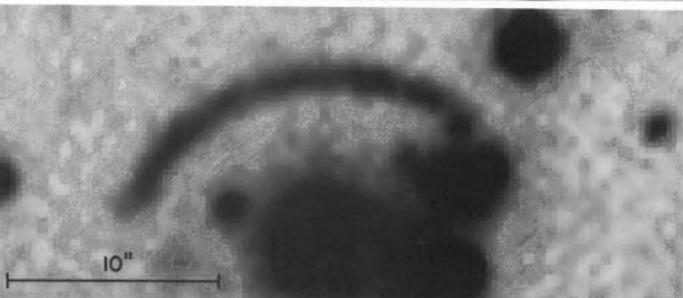
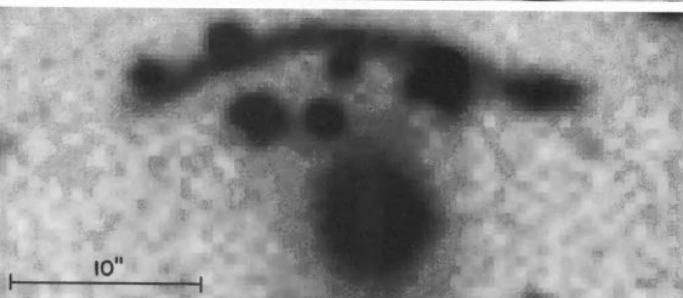
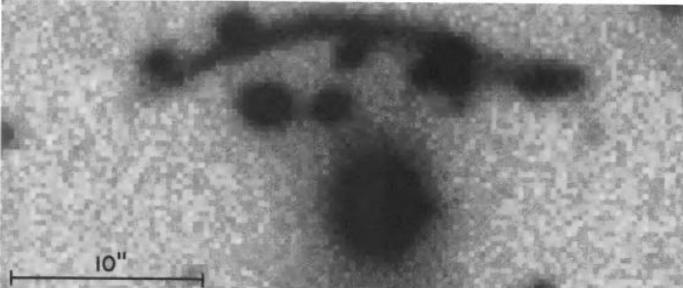
Outline

Notes/References

Brief History of GL

First detections

First ever gravitational arcs: Lynds and Petrosian 1986, Soucail et al., 1987, 1988.



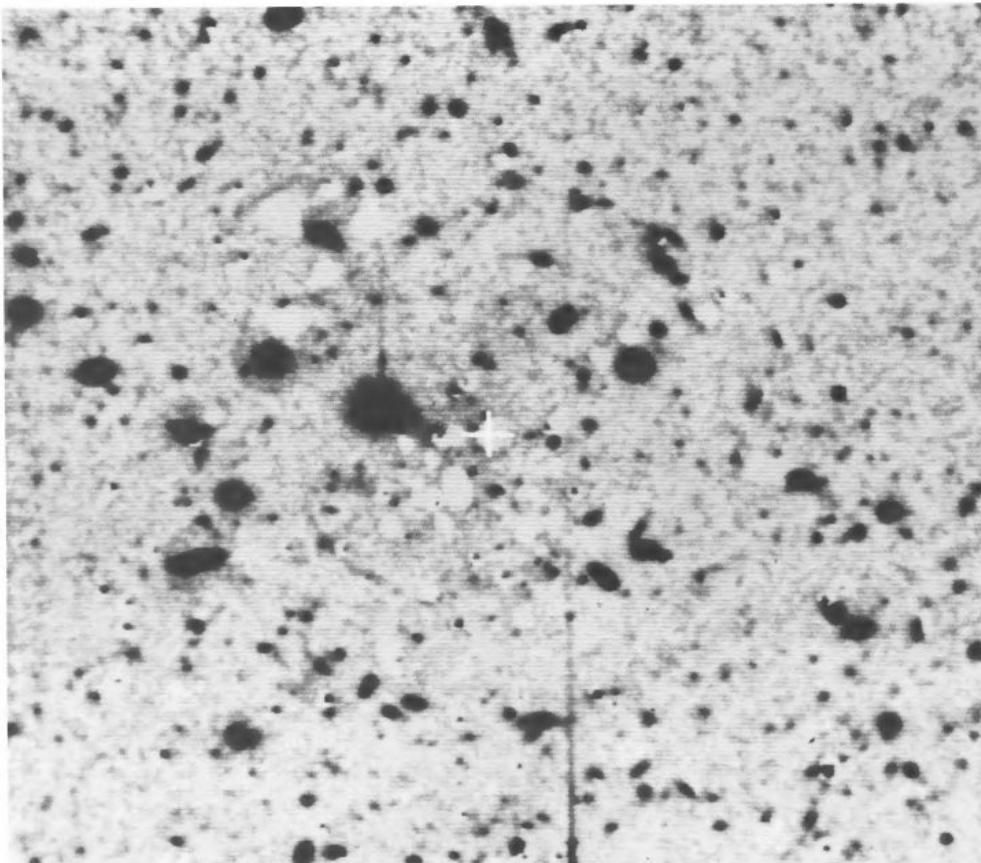
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Notes/References

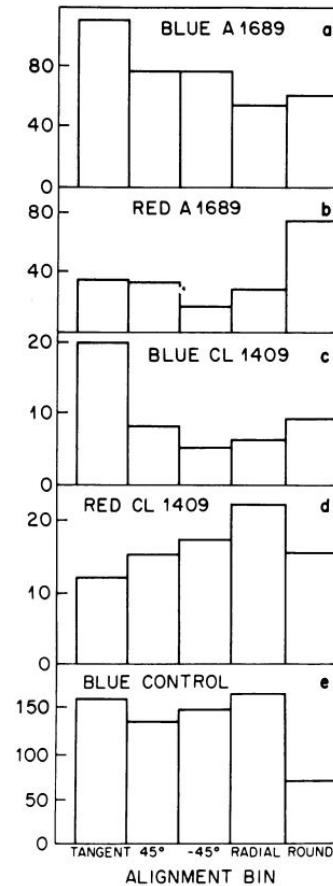
Brief History of GL

First detections

First ever systematic weak lensing detection (Taylor et al., 1990)



Abell 1689



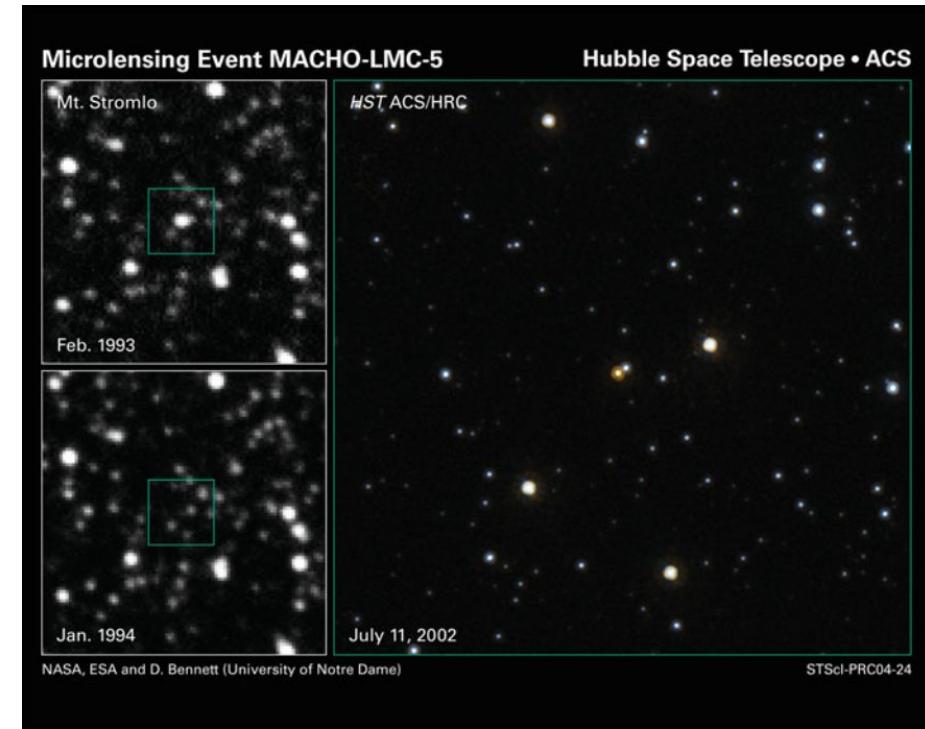
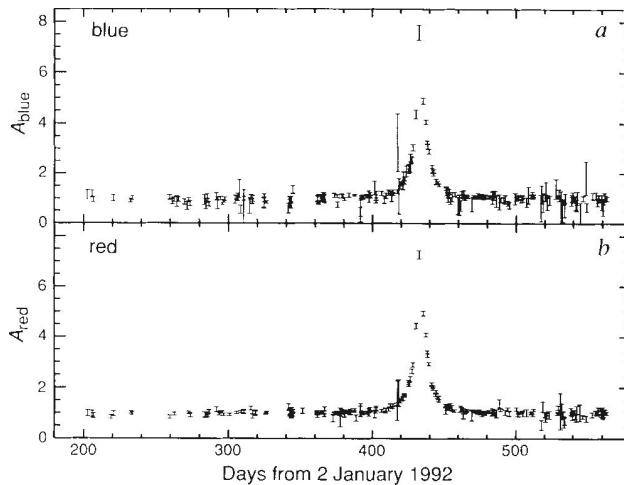
Outline

Notes/References

Brief History of GL

First detections

First ever microlensing event: Alcock et al. 1993, Aubourg et al. 1993.



"I don't understand; you're looking for planets you can't see around stars you can't see" (D. Fisher, reported)

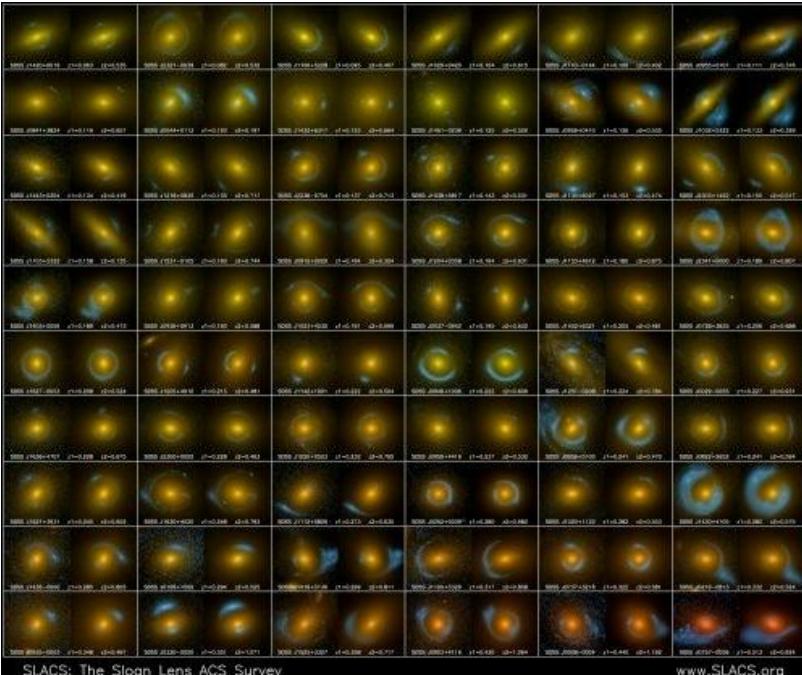
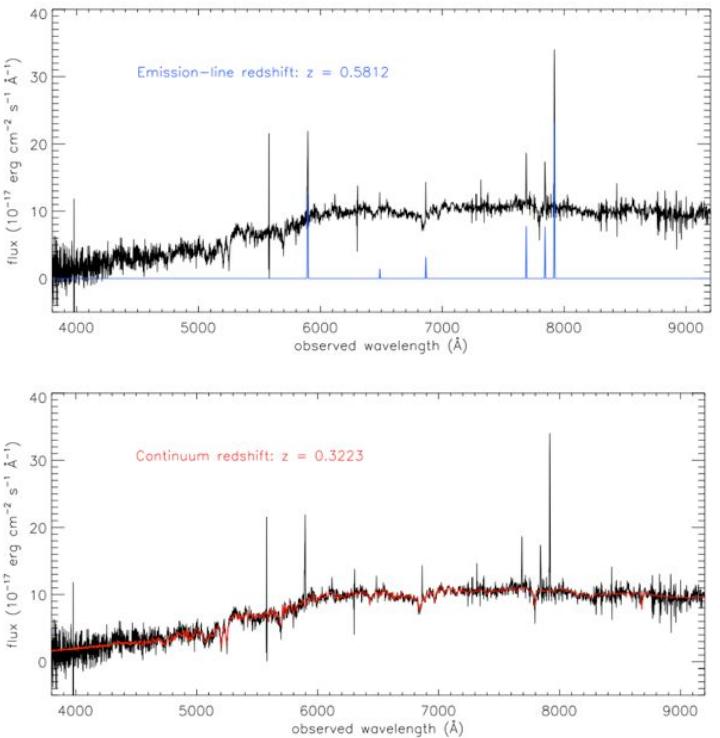
Outline

First ever systematic lensing surveys in the optical/near-IR (2000s)

Notes/References

Brief History of GL

First detections



SLACS: The Sloan Lens ACS Survey

A. Bolton (U. Hawaii IIA), L. Koopmans (Kapteyn), T. Treu (UCSB), R. Gavazzi (IPN Paris), L. Moustakas (JPL/Caltech), S. Baltes (MIT)

Image credit: SLACS survey and NASA/ESA

A. Bolton (UH IIA) for SLACS and NASA/ESA

Outline

Notes/References

Brief History of GL

First detections

First ever systematic lensing surveys in the optical/near-IR (2000s)



A. Bolton (UH IfA) for SLACS and NASA/ESA

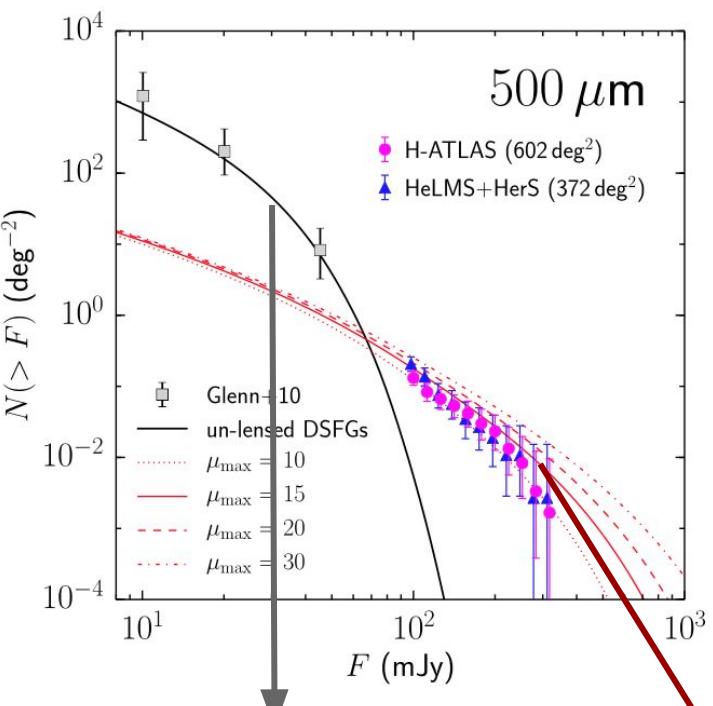
Outline

Notes/References

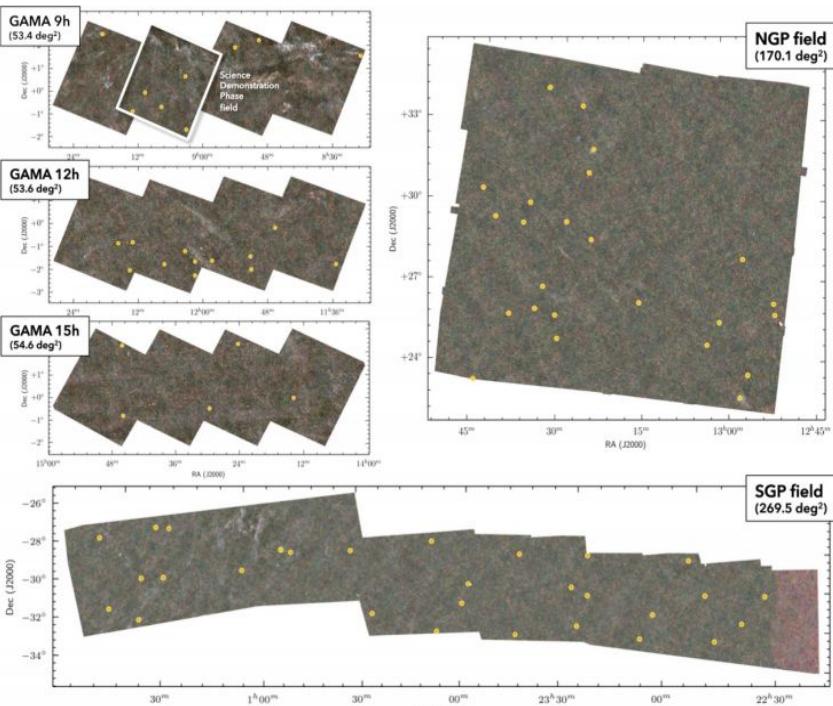
Brief History of GL

First detections

First ever systematic lensing surveys in sub-mm (2010s)



DUSTY STAR FORMING GALAXIES (DSFGs) NUMBER
COUNTS AT $Z = 1.5 - 4$



STRONGLY LENSED DSFGs CANDIDATES

Negrello et al., 2017

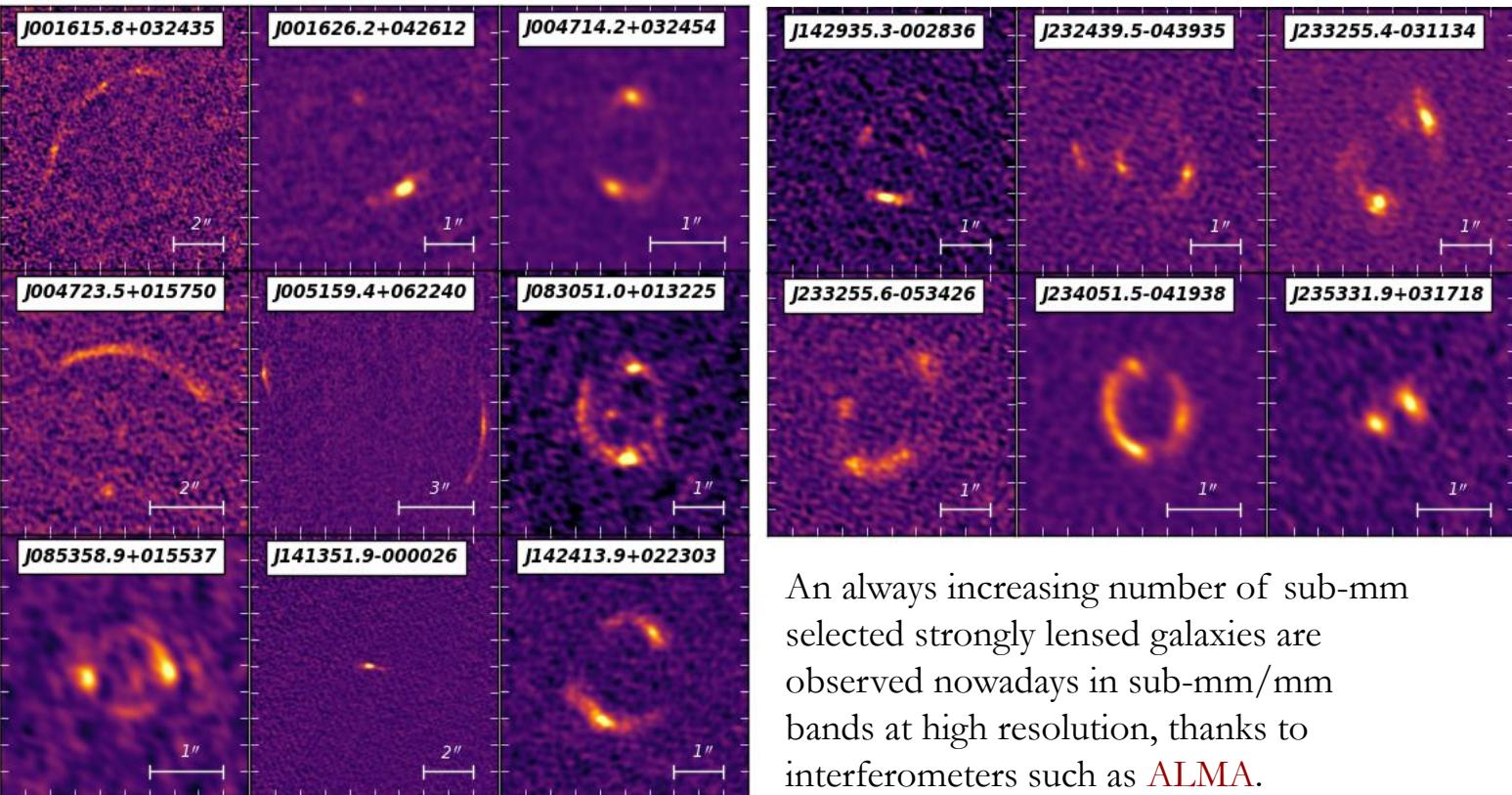
Outline

Notes/References

Brief History of GL

First detections

First ever systematic lensing surveys in sub-mm (2010s)



An always increasing number of sub-mm selected strongly lensed galaxies are observed nowadays in sub-mm/mm bands at high resolution, thanks to interferometers such as [ALMA](#).

Outline

Light deflection according to Newton

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

Outline

Notes/References

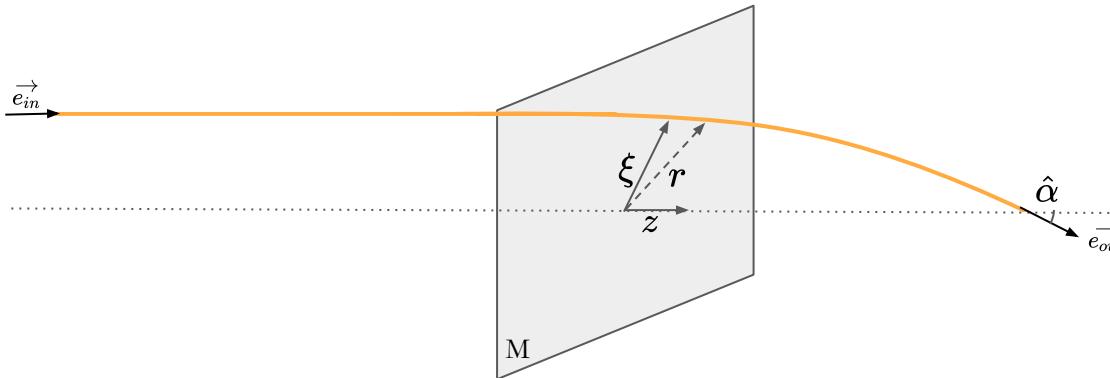
Brief History of GL

First detections

Light deflection:

- Newtonian gravity

Light deflection according to Newton



$$\hat{\alpha} = \vec{e}_{in} - \vec{e}_{out}$$

$$z = ct \longrightarrow dz = cdt$$

$$r^2 = \xi^2 + z^2$$

Outline

Notes/References

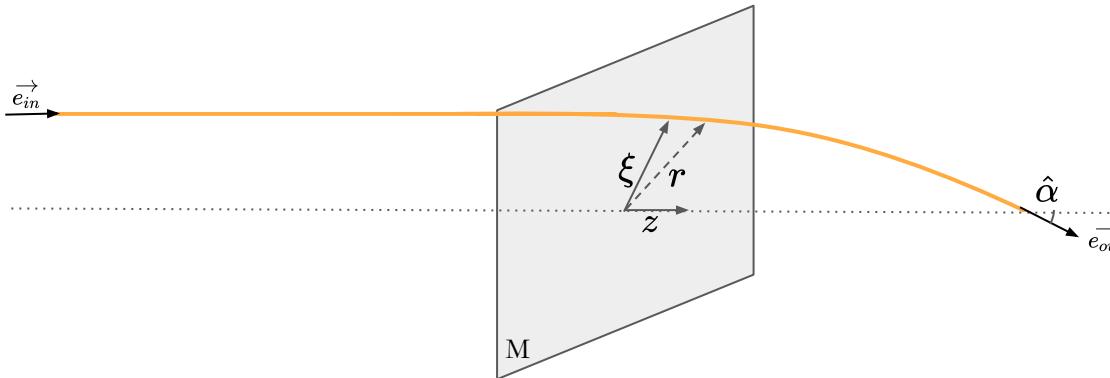
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Light deflection according to Newton



$$\begin{aligned}\hat{\alpha} &= \vec{e}_{in} - \vec{e}_{out} \\ z = ct &\longrightarrow dz = cdt \\ r^2 &= \xi^2 + z^2\end{aligned}$$

We know that:

$$\vec{F} = -m\nabla\phi = m\vec{a} \implies \vec{a} = -\nabla\phi \quad \text{and} \quad \phi = \frac{GM}{r}$$

Outline

Notes/References

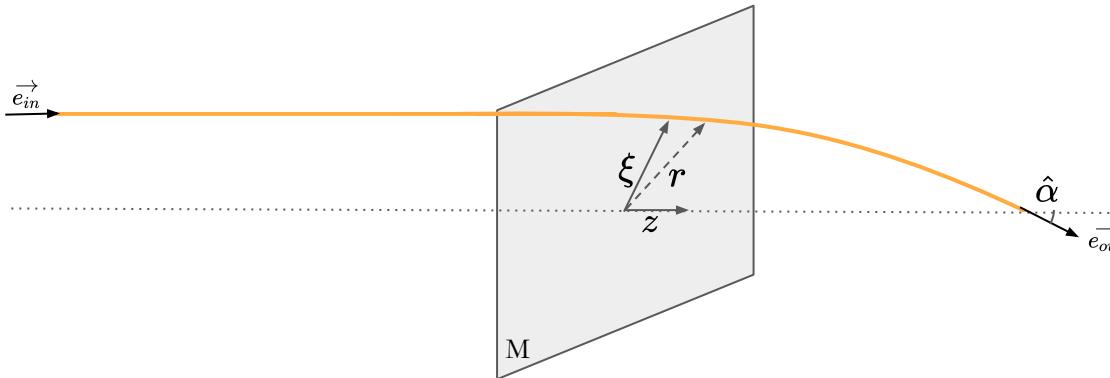
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As such, the photon will experience a change in velocity:

$$\Delta\vec{v} = \int_{t_{in}}^{t_{out}} \vec{a} dt \simeq -\frac{1}{c} \int_{z_{in}}^{z_{out}} \nabla\phi dz$$

Outline

Light deflection according to Newton

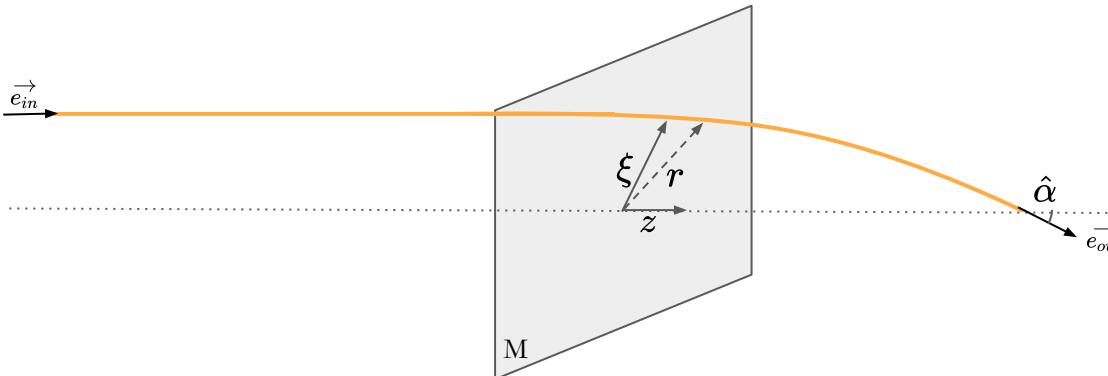
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Brief History of GL

First detections

Light deflection:

- Newtonian gravity



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As such, the photon will experience a change in velocity:

$$\Delta\vec{v} = \int_{t_{in}}^{t_{out}} \vec{a} dt \simeq -\frac{1}{c} \int_{z_{in}}^{z_{out}} \nabla\phi dz \longrightarrow \Delta v_{||} = -\frac{1}{c} \int_{z_{in}}^{z_{out}} \frac{d\phi}{dz} dz = \frac{1}{c}(\phi_{in} - \phi_{out})$$

we can safely assume $|z_{in}| = |z_{out}| = \infty$
as such $\phi(\infty) \rightarrow 0 \implies \Delta v_{||} = 0$

Outline

Light deflection according to Newton

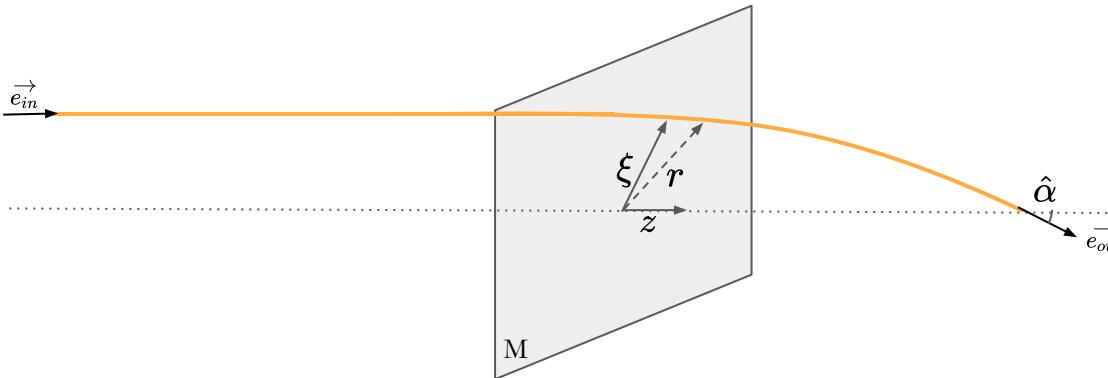
Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity



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↓
we can safely assume $|z_{in}| = |z_{out}| = \infty$
as such $\phi(\infty) \rightarrow 0 \implies \Delta v_{||} = 0$

$$\begin{aligned}\Delta v_\perp &= -\frac{1}{c} \int_{z_{in}}^{z_{out}} \frac{d\phi}{d\xi} dz = -\frac{GM\xi}{c} \int_{z_{in}}^{z_{out}} (\xi^2 + r^2)^{-3/2} dz \\ \tan\theta &= z/\xi \\ &= -\frac{GM}{c\xi} \int_{-\pi/2}^{\pi/2} \cos\theta d\theta = -\frac{2GM}{c\xi}\end{aligned}$$

and the photon's velocity will change
accordingly in the perpendicular direction

Outline

Light deflection according to Newton

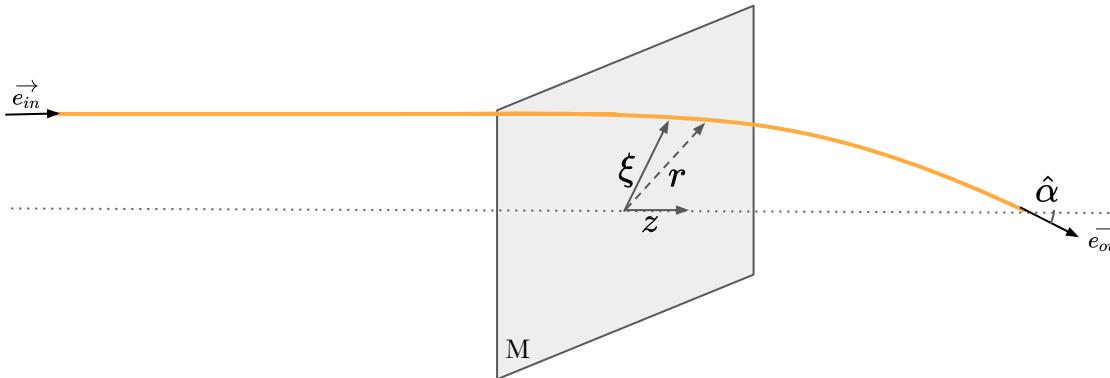
Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity



$$\begin{aligned}\hat{\alpha} &= \vec{e}_{in} - \vec{e}_{out} \\ z = ct &\rightarrow dz = cdt \\ r^2 &= \xi^2 + z^2\end{aligned}$$

Therefore:

$$\vec{v}_{out} = |v_{out}| \vec{e}_{out} = c \vec{e}_{in} - \frac{2GM}{c\xi} \vec{e}_\xi$$

but $|v_{out}| = \sqrt{c^2 + \frac{4G^2 M^2}{c^2 \xi^2}} \sim c$

If the impact parameter is R_\odot

$$\hat{\alpha} = \frac{2GM}{c^2 R_\odot} \simeq 0.875''$$

and as such: $\hat{\alpha} = \vec{e}_{in} - \vec{e}_{out} = \frac{2GM}{c^2 \xi} \vec{e}_\xi$

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

Gravity is a manifestation of spacetime geometry.

Matter/energy just want to move along geodesics, and they actually do, is just the underlying geometry that is curved, thus deflecting their path from the pure straight-Minkowskian-line.

This applies to people on Earth (or whatever exoplanet you could think of), planets around stars, stars in galaxies, galaxies in clusters and DM wells, the Universe as a whole, and as such, also to light.

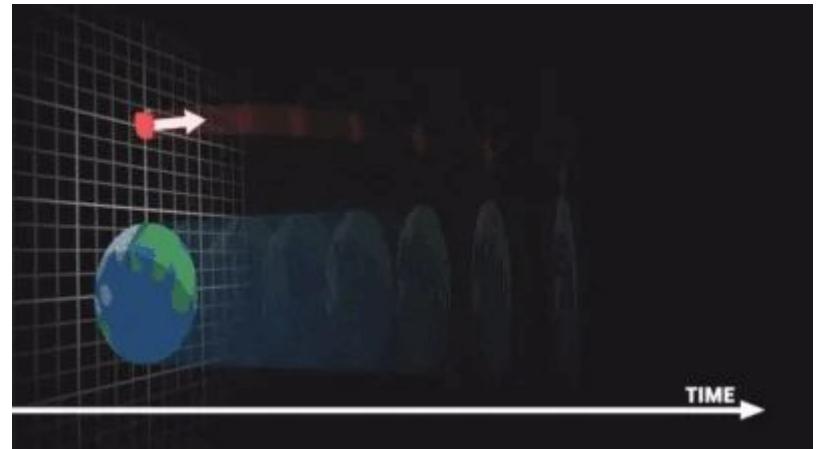
*"Spacetime tells matter how to move;
matter tells spacetime how to curve"*

J. A. Wheeler

$$\nabla_\nu A^\mu = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma_{\alpha\nu}^\mu A^\alpha$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\nu\sigma}^\mu \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$



Outline

Notes/References

Brief History of GL

First detections

Light deflection:

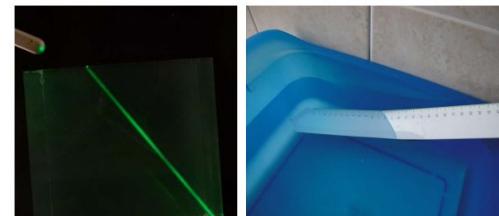
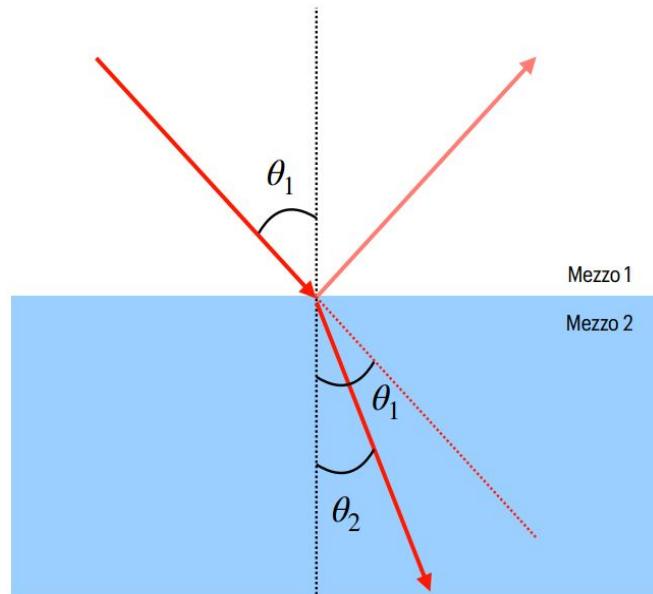
- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

We can study the light deflection according to GR (and other useful intuitions along the path) treating the problem as in geometrical optics: a refraction problem.

In geometrical optics, when light travels between two media with different refractive indexes n_1 and n_2 , it changes speed and direction.



The same intuition can be applied when lights travels from an unperturbed Minkowski spacetime to a perturbed (e.g. Schwarzschild) spacetime.

Outline

Light deflection according to General Relativity

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

- General Relativity

In the perturbed Schwarzschild metric light travels with speed $c' = c/n$

N.B. this does not mean that light *actually* changes its speed, which locally is always c ; this means that an external observer sees the action of an effective refraction index that gives *the illusion* of a speed variation, which is always a slowdown

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (d\vec{x})^2 = c^2 dt^2 - (d\vec{x})^2$$

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1 - \frac{2\Phi}{c^2}) & 0 \\ 0 & 0 & 0 & -(1 - \frac{2\Phi}{c^2}) \end{pmatrix}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) (d\vec{x})^2$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (d\vec{x})^2 = c^2 dt^2 - (d\vec{x})^2$$

Outline

Light deflection according to General Relativity

Notes/References

Brief History of GL

First detections

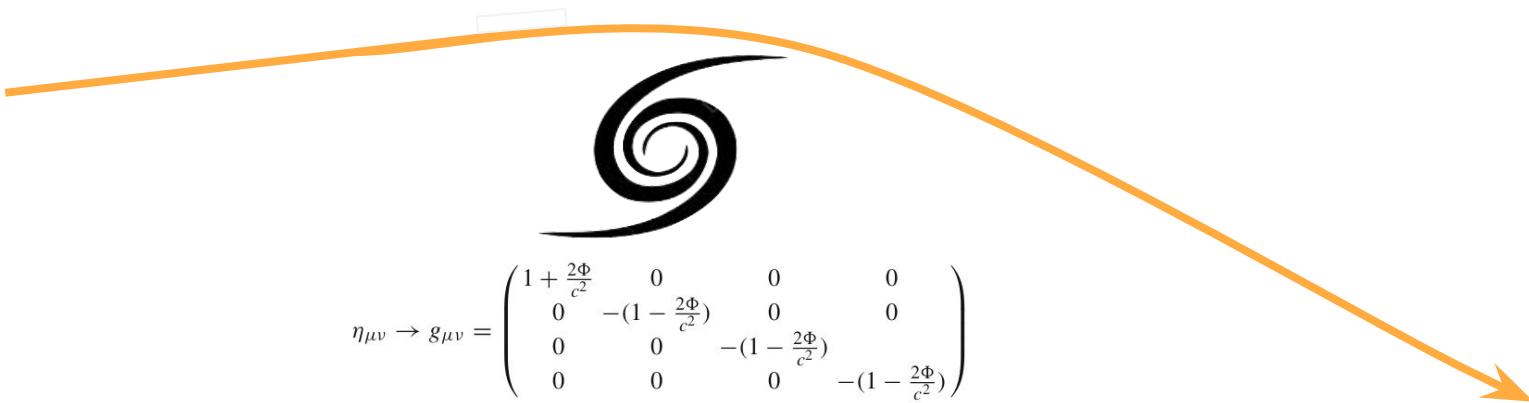
Light deflection:

- Newtonian gravity

- General Relativity

In the perturbed Schwarzschild metric light travels with speed $c' = c/n$

N.B. this does not mean that light *actually* changes its speed, which locally is always c ; this means that an external observer sees the action of an effective refraction index that gives *the illusion* of a speed variation, which is always a slowdown



$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1 - \frac{2\Phi}{c^2}) & 0 \\ 0 & 0 & 0 & -(1 - \frac{2\Phi}{c^2}) \end{pmatrix}$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) (d\vec{x})^2$$

Light propagates with $ds = 0$, as such (in the weak field assumption* $\Phi/c^2 \ll 1$):

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 = \left(1 - \frac{2\Phi}{c^2}\right) (d\vec{x})^2 \longrightarrow c' = \frac{|d\vec{x}|}{dt} = c \sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c \left(1 + \frac{2\Phi}{c^2}\right) \longrightarrow n = \frac{c}{c'} = \frac{1}{1 + \frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

* which astrophysically holds almost everywhere (in galaxy clusters $\Phi/c^2 \sim 10^{-4}$), with the notable exception of black holes

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

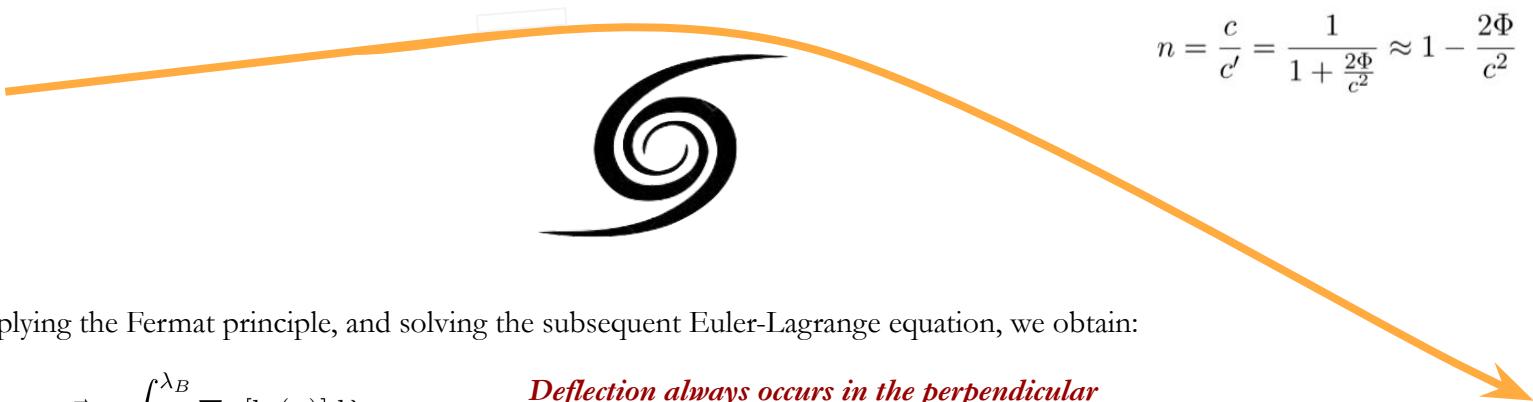
- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

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$$n = \frac{c}{c'} = \frac{1}{1 + \frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

Applying the Fermat principle, and solving the subsequent Euler-Lagrange equation, we obtain:

$$\vec{\alpha} = \int_{\lambda_A}^{\lambda_B} \nabla_{\perp} [\ln(n)] d\lambda$$

Deflection always occurs in the perpendicular direction to the light trajectory.

and since $\ln(n) \sim -2\phi/c^2$, we find:

$$\vec{\alpha} = \frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla} \Phi d\lambda$$

Deflection always occurs toward the deflector's center.

This is the most general expression for the deflection angle (under the aforementioned assumption of weak field!).
However, it is not useful yet, since it requires an integration over the actual light path.

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

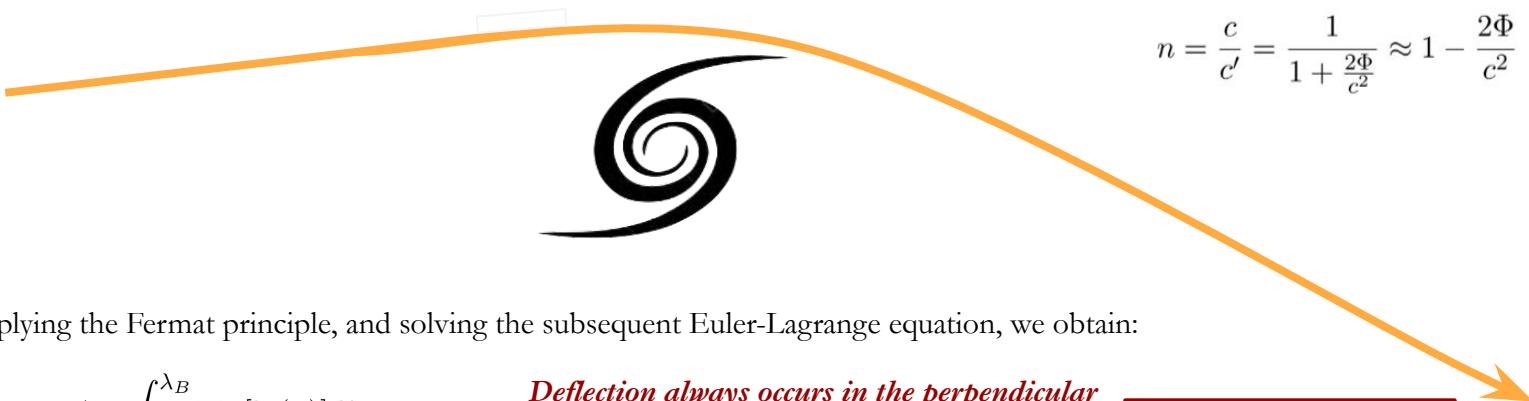
- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

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$$\vec{\alpha} = \frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla} \Phi d\lambda$$

Deflection always occurs toward the deflector's center.

Notice that the deflection angle does NOT depend on the photon energy: lensing is achromatic!

This is the most general expression for the deflection angle (under the aforementioned assumption of weak field!).
However, it is not useful yet, since it requires an integration over the actual light path.

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

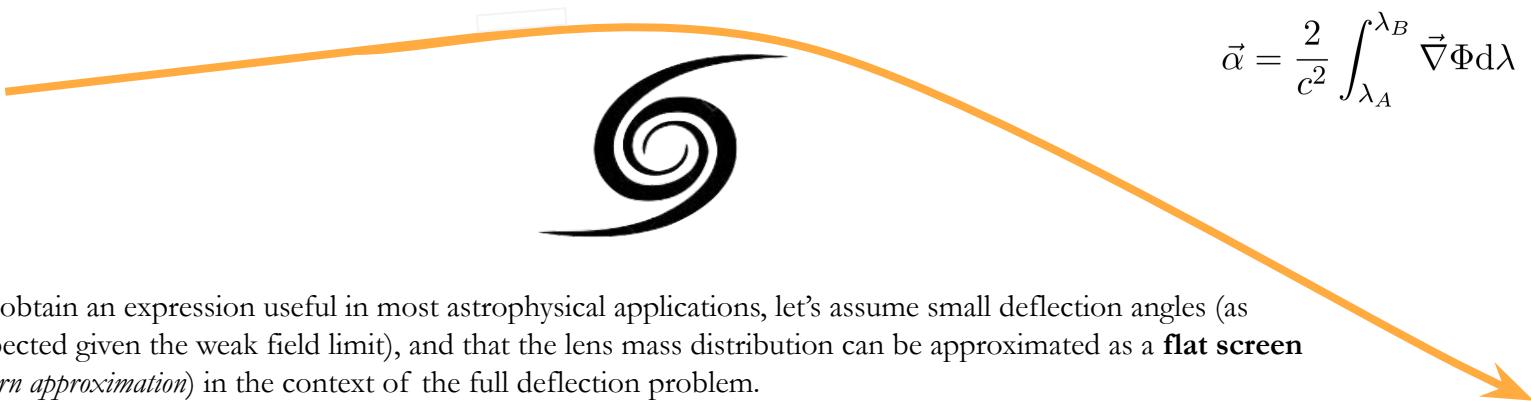
- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

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N.B. this does not mean that light *actually* changes its speed, which locally is always c ; this means that an external observer sees the action of an effective refraction index that gives *the illusion* of a speed variation, which is always a slowdown



To obtain an expression useful in most astrophysical applications, let's assume small deflection angles (as expected given the weak field limit), and that the lens mass distribution can be approximated as a **flat screen** (*Born approximation*) in the context of the full deflection problem.

As such, we obtain:

$$\hat{\alpha}(\xi) = \frac{2}{c^2} \int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi(\xi, z) dz \quad \text{For point mass: } \Phi = -GM\sqrt{\xi^2 + z^2}$$

And therefore:

$$\hat{\alpha}(\xi) = \frac{4GM}{c^2\xi} \vec{e}_\xi = \frac{4GM}{c^2\xi^2} \vec{\xi}$$

which is just two times
the one obtained under
Newtonian gravity

If the impact parameter is R_\odot

$$\hat{\alpha} = \frac{4GM}{c^2 R_\odot} \simeq 1.75''$$

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Brief History of GL

First detections

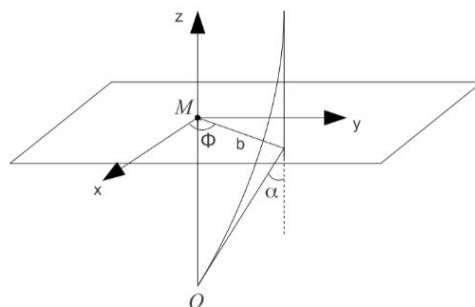
Light deflection:

- Newtonian gravity

- General Relativity

Light deflection according to General Relativity

Point mass



$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4GM}{c^2\vec{\xi}} \vec{e}_{\vec{\xi}} = \frac{4GM}{c^2\vec{\xi}^2} \vec{\xi}$$

Extended (continuous) mass distribution

Thin screen approximation: the (formally 3D) lens is approximated by a planar distribution of matter (lens plane). Once defined the lens surface density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz$$

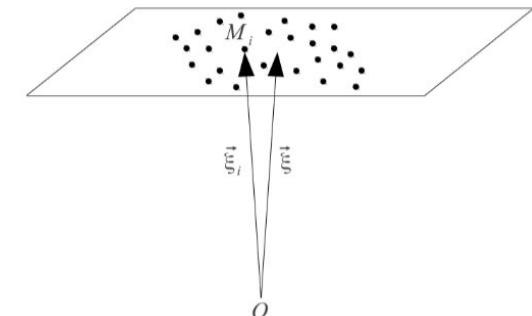
the deflection angle becomes:

$$\vec{\alpha}(\vec{\xi}) = \frac{4M}{c^2} \int \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \Sigma(\vec{\xi}') d^2\vec{\xi}'$$

$\vec{\xi}'$ is the distance at which the light passes with respect to the lens' barycenter

Ensemble of point masses

Total deflection is the sum of every single mass contribution



$$\vec{\alpha}(\vec{\xi}) = \sum_i^N \vec{\alpha}(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i^N M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

For a circularly symmetric mass distribution:

$$\vec{\alpha}(\vec{\xi}) = \frac{4GM(<\xi)}{c^2\vec{\xi}}$$

with $M(<\xi) = 2\pi \int d\xi' \xi' \Sigma(\xi')$

Only the mass contained within the impact parameter actually enters the deflection angle definition.

Outline

Notes/References

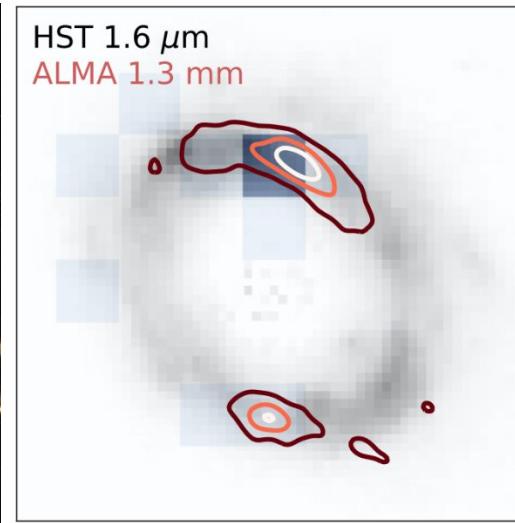
Brief History of GL

First detections

Light deflection:

- Newtonian gravity
- General Relativity

Light deflection according to General Relativity



Extended (continuous) mass distribution

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Brief History of GL

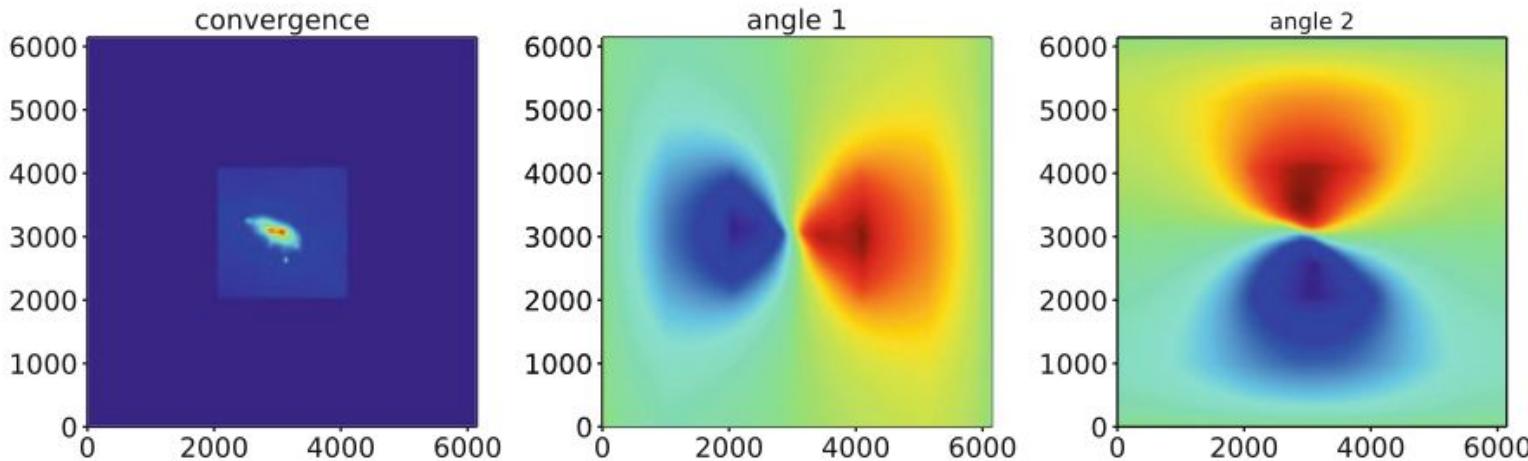
First detections

Light deflection:

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- General Relativity

Light deflection according to General Relativity



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Notes/References

Brief History of GL

First detections

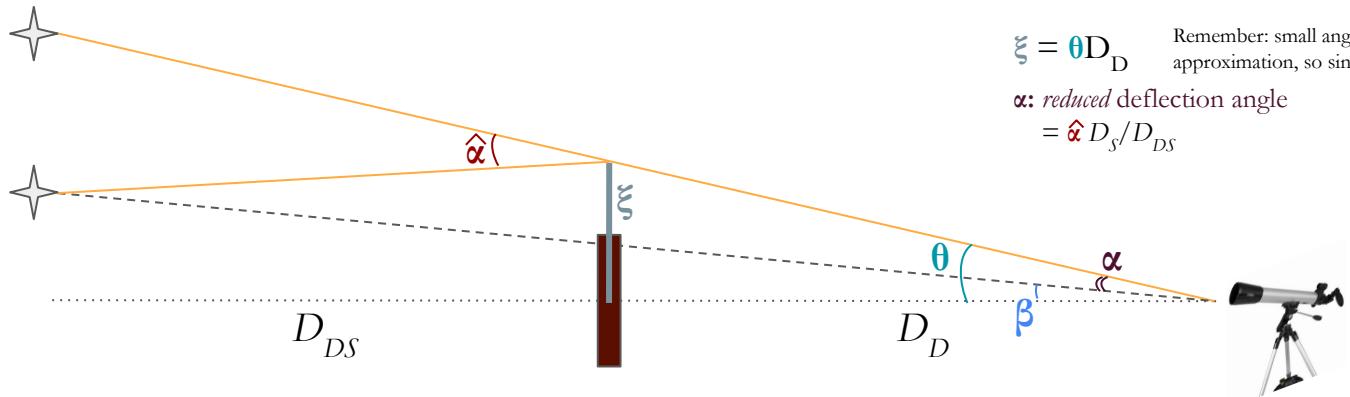
Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

The lens equation



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Notes/References

Brief History of GL

First detections

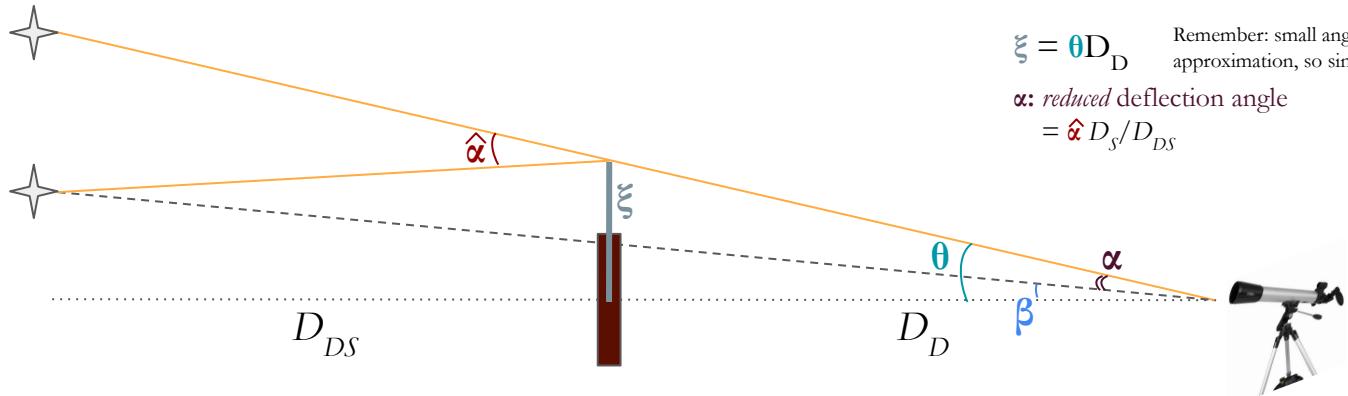
Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

The lens equation



$$\xi = \theta D_D$$

Remember: small angle approximation, so $\sin\theta = \theta$

α : reduced deflection angle
 $= \hat{\alpha} D_s / D_{DS}$

It is immediately clear that $\alpha + \beta = \theta$ or, in terms of the segments on the source plane: $\hat{\alpha} D_{DS} + \beta D_s = \theta D_s$

Outline

Notes/References

Brief History of GL

First detections

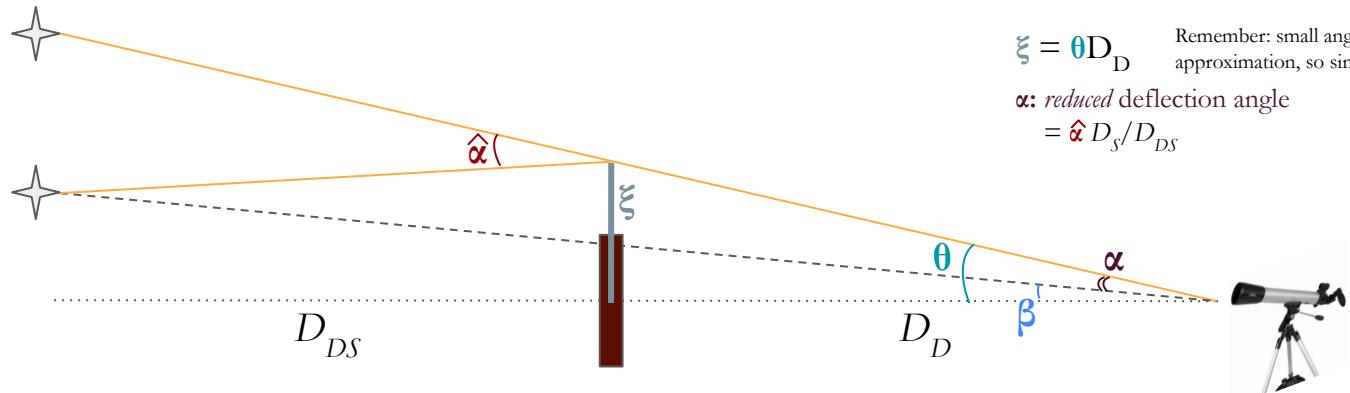
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The lens equation



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Circularly symmetric lens

$$\begin{aligned}\beta &= \theta - \alpha = \theta - \frac{D_{DS}}{D_S} \hat{\alpha} && \text{We know that } \vec{\alpha}(\xi) = \frac{4GM(<\xi)}{c^2\xi} \\ &= \theta - \frac{D_{DS}}{D_S} \frac{4GM(\xi)}{\xi c^2} = \theta - \frac{D_{DS}}{D_D D_S} \frac{4GM(\theta)}{\theta c^2}\end{aligned}$$

Outline

Notes/References

Brief History of GL

First detections

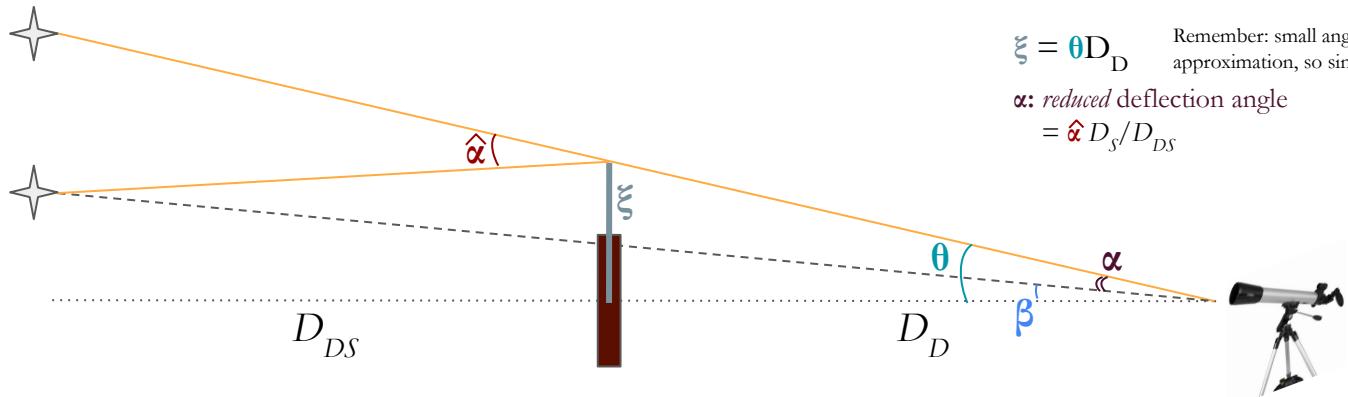
Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

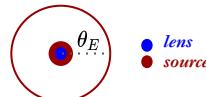
The lens equation



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If we call:

$$\theta_E = \left(\frac{D_{DS}}{D_D D_S} \frac{4GM(\theta)}{c^2} \right)^{1/2} \quad \text{Einstein radius}$$

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Brief History of GL

First detections

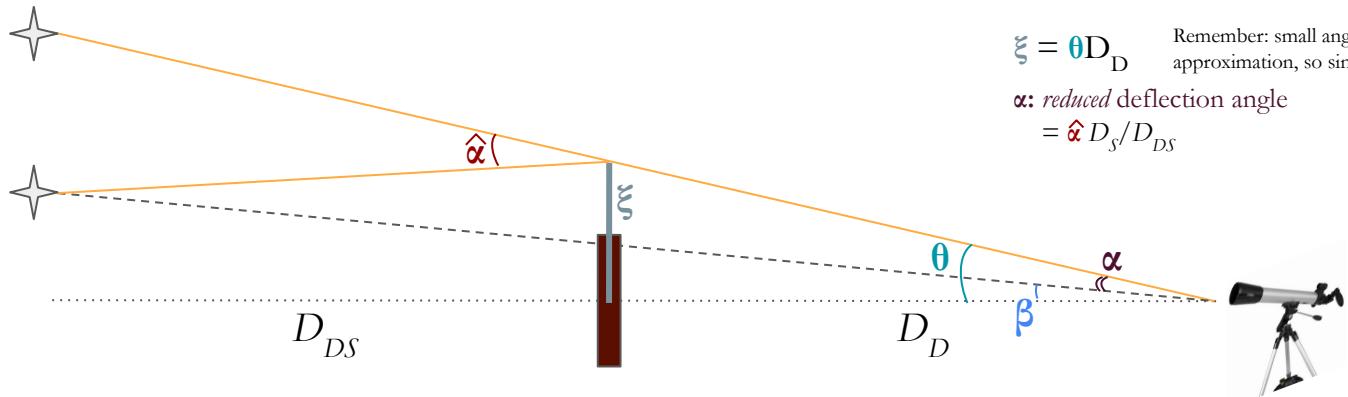
Light deflection:

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Lens equation

The lens equation

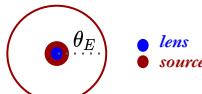


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If: $\beta = 0 \implies \theta = \theta_E$

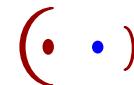


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else:

$$\theta_{\pm} = \frac{1}{2} (\beta \pm \sqrt{\beta^2 - 4\theta_E^2})$$



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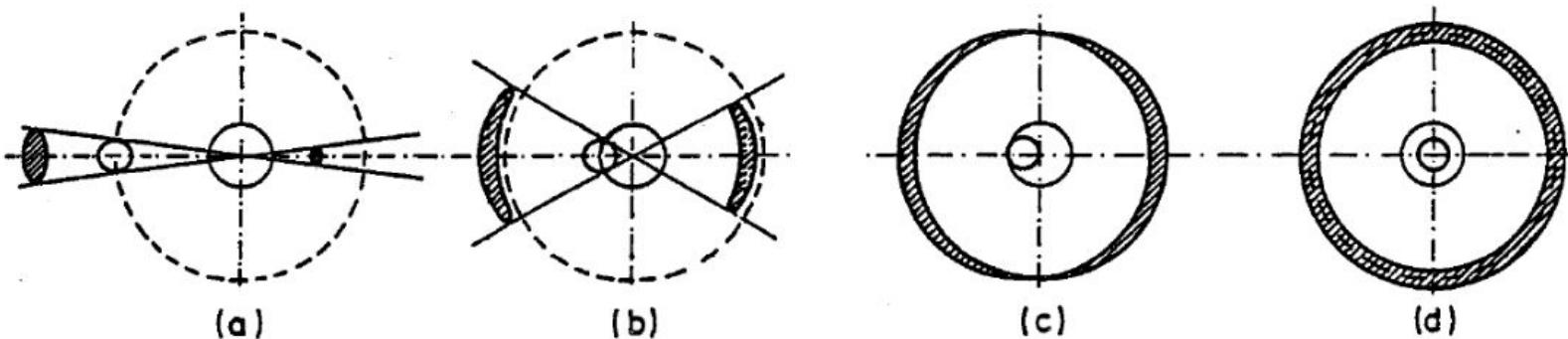
Light deflection:

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Lens equation

The lens equation

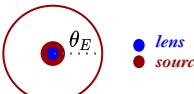


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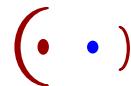


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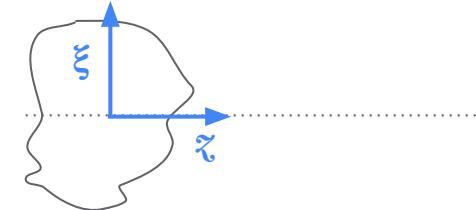
Lensing potential

General lensing solution

What happens in the general case of a lens without any particular symmetry (though still in the thin lens hypothesis)?

The fundamental quantity here is the **lensing potential**:

$$\psi(\theta) = \frac{D_{LS}}{D_L D_S} \frac{2}{c^2} \int_{-\infty}^{+\infty} \phi(\xi, z) dz$$



The lensing potential satisfies two fundamental properties in gravitational lensing:

$$\nabla_\theta \psi(\theta) = \alpha$$

The angular gradient of the lens potential is the reduced deflection angle

$$\nabla_\theta^2 \psi(\theta) = 2 \frac{\Sigma(\theta)}{\Sigma^*} \equiv 2K(\theta)$$

The laplacian of the lensing potential is the convergence K , that is directly related to the lens surface density (\rightarrow its mass distribution)

with:

$$\Sigma^* = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$$

the critical surface density, such that $\alpha(\theta) = \theta$ for any θ

Outline

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

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Lens equation

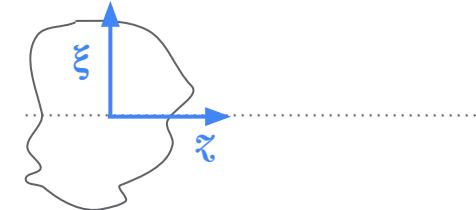
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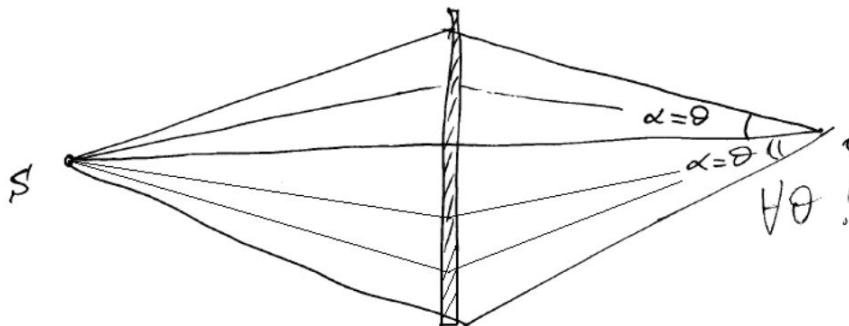
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The laplacian of the lensing potential is the convergence K, that is directly related to the lens surface density (\rightarrow its mass distribution)



Lenses for which $\Sigma > \Sigma^*$ are said to be *strong lenses*; the opposite case is the one for *weak lenses*

with:

$$\Sigma^* = \frac{c^2}{4\pi G} \frac{D_S}{D_L D_{LS}}$$

the critical surface density, such that $\alpha(\theta) = \theta$ for any θ



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Light deflection:

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Lens equation

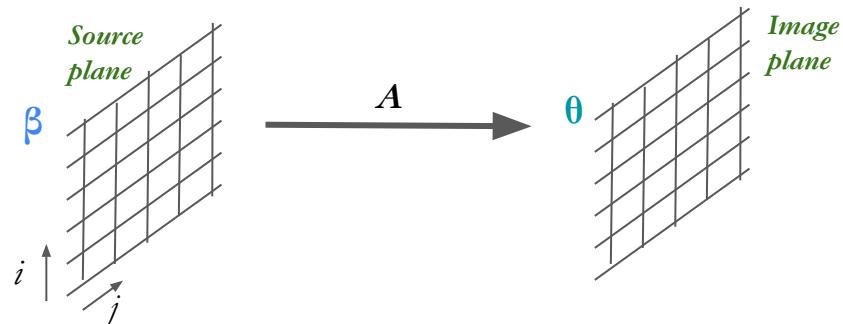
Lensing potential

Linear solution

Lensing at the first order: the linear solution

At the first order (“linear”) a lensing phenomena is approximated as a mapping, a transformation between two planes: the **source plane**, where the source lies, and the **image plane**, that is what we observe.

Mapping between two planes is always obtained from the Jacobian of the transformation: $A \equiv \partial \vec{\beta} / \partial \vec{\theta}$



$$A = \partial \vec{\beta} / \partial \vec{\theta} = \delta_{ij} - \frac{\partial \vec{\alpha}}{\partial \vec{\theta}} \quad \text{but} \quad \vec{\alpha} = \frac{\partial \psi}{\partial \vec{\theta}} \implies A = \delta_{ij} - \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} = \delta_{ij} - \Psi_{ij}$$

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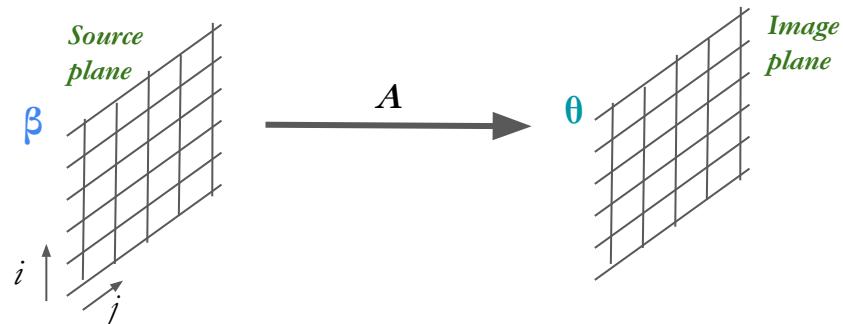
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Now we split this matrix into its *isotropic* and *anisotropic* components:

$$\frac{1}{2} \text{Tr} A \cdot \mathbb{I} = \left[1 - \frac{1}{2} (\Psi_{11} + \Psi_{22}) \right] \delta_{ij} \equiv (1 - K) \delta_{ij}$$

$$A - \frac{1}{2} \text{Tr} A \cdot \mathbb{I} = \delta_{ij} - \Psi_{ij} - \frac{1}{2} (2 - \Psi_{11} - \Psi_{22}) \delta_{ij} = \begin{pmatrix} -\frac{1}{2}(\Psi_{11} - \Psi_{22}) & \Psi_{12} \\ -\Psi_{21} & \frac{1}{2}(\Psi_{11} - \Psi_{22}) \end{pmatrix}$$

Components of a vector called **shear**

$$\begin{pmatrix} \gamma_1 & \\ & \gamma_2 \end{pmatrix}$$

Therefore the mapping we’re looking for is:

$$A = \begin{pmatrix} 1 - K - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - K + \gamma_1 \end{pmatrix}$$

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Brief History of GL

First detections

Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

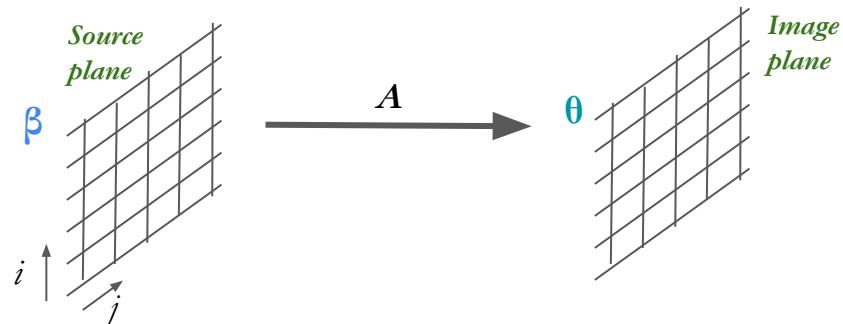
Lensing potential

Linear solution

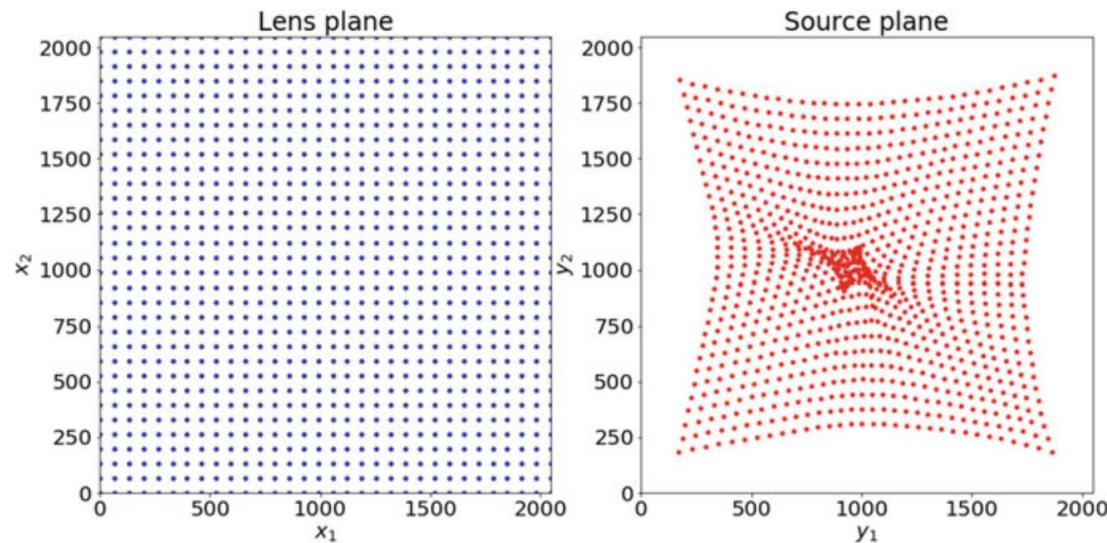
Lensing at the first order: the linear solution

At the first order (“linear”) a lensing phenomena is approximated as a mapping, a transformation between two planes: the **source plane**, where the source lies, and the **image plane**, that is what we observe.

Mapping between two planes is always obtained from the Jacobian of the transformation: $A \equiv \partial \vec{\beta} / \partial \vec{\theta}$



The example on the right shows how points on the lens/image plane are mapped onto the source plane, once a lens model is assumed



Outline

Notes/References

Brief History of GL

First detections

Light deflection:

- Newtonian gravity

- General Relativity

Lens equation

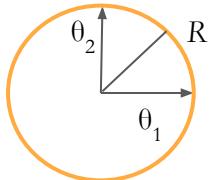
Lensing potential

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Convergence and shear

Lensing at the first order: the linear solution

What is the meaning of the convergence \mathbf{K} and shear γ ? Let's consider a simple circular source.



Circle of radius R :

$$\theta_1^2 + \theta_2^2 = R^2$$

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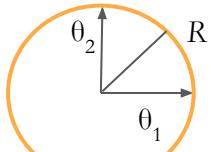
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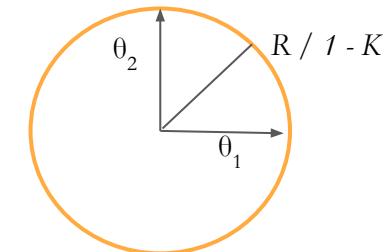
Circle of radius R:

$$\theta_1^2 + \theta_2^2 = R^2$$

If $\gamma = 0$ then $A = \begin{pmatrix} 1 - K & 0 \\ 0 & 1 - K \end{pmatrix}$ so the transformation is:

$$\vec{\beta} = (1 - K)\delta_{ij}\vec{\theta} \implies (1 - k)^2(\theta_1^2 + \theta_2^2) = R^2$$

the result is still a circle, with radius $R/(1-K)$



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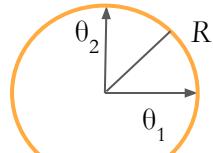
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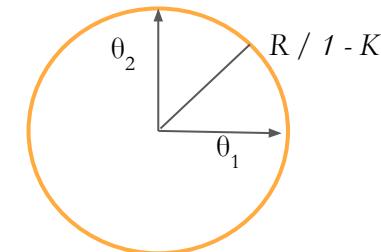


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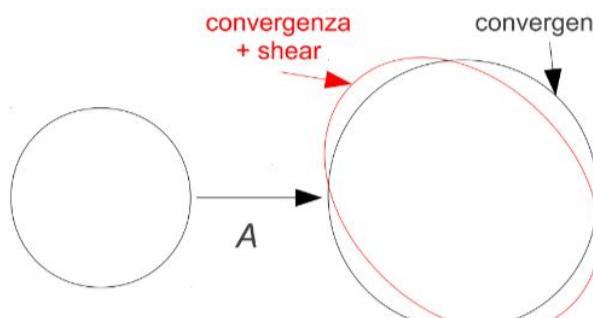
$$\vec{\beta} = (1 - K)\delta_{ij}\vec{\theta} \implies (1 - k)^2(\theta_1^2 + \theta_2^2) = R^2$$



the result is still a circle, with radius $R/(1-K)$

Now, let's consider the shear:

$$\vec{\beta} = (1 - K)\delta_{ij}\vec{\theta} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix}\vec{\theta} \quad \text{and the equation becomes the one of an ellipse:}$$



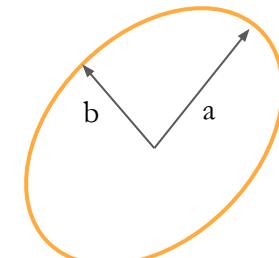
LP

$$(1 - k - \gamma)^2\theta_1^2 + (1 - k + \gamma)^2\theta_2^2 = R^2$$

with:

$$a = \frac{R}{1-k-\gamma}$$

$$b = \frac{R}{1-k+\gamma}$$



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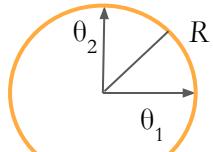
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What is the meaning of the convergence K and shear γ ? Let's consider a simple circular source.

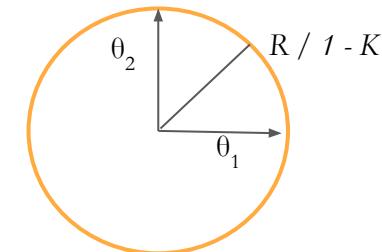


Circle of radius R :

$$\theta_1^2 + \theta_2^2 = R^2$$

If $\gamma = 0$ then $A = \begin{pmatrix} 1 - K & 0 \\ 0 & 1 - K \end{pmatrix}$ so the transformation is:

$$\vec{\beta} = (1 - K)\delta_{ij}\vec{\theta} \implies (1 - k)^2(\theta_1^2 + \theta_2^2) = R^2$$

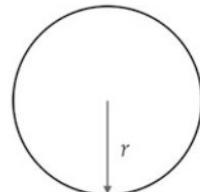


the result is still a circle, with radius $R/(1-K)$

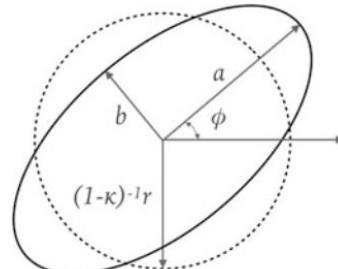
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$$(1 - k - \gamma)^2\theta_1^2 + (1 - k + \gamma)^2\theta_2^2 = R^2$$



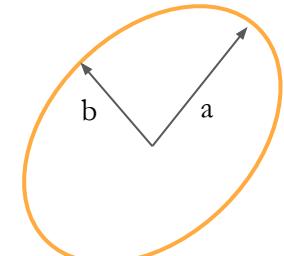
A^{-1}



with:

$$a = \frac{R}{1-k-\gamma}$$

$$b = \frac{R}{1-k+\gamma}$$



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Lensing at the first order: the linear solution

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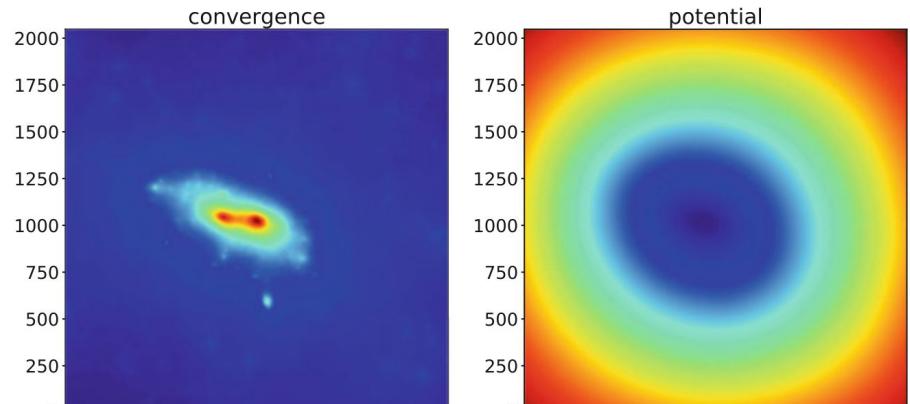
Lensing potential

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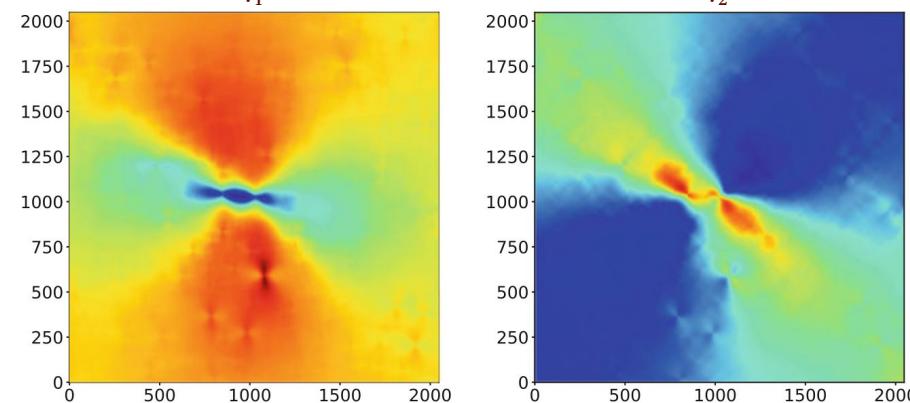
Convergence and shear

What is the meaning of the convergence \mathbf{K} and shear γ ?

Convergence closely follows the lens mass distribution



Shear is just a mess



Potential immediately gives an intuition on deflection angle

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Magnification

Lensing at the first order: magnification, caustics, critics

The surface brightness S of a source is defined as:

$$S \equiv \frac{F}{\Omega} = \frac{L}{4\pi A}$$

where A is the area occupied by the source in the source plane.

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Lensing at the first order: magnification, caustics, critics

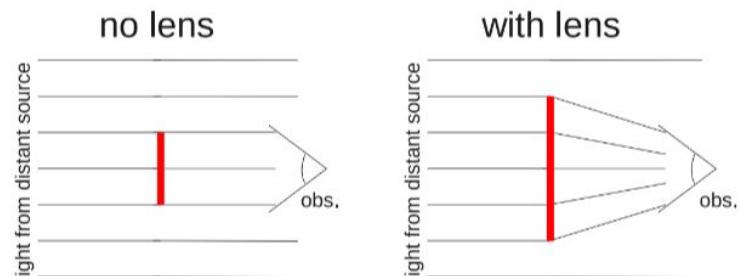
The surface brightness S of a source is defined as:

$$S \equiv \frac{F}{\Omega} = \frac{L}{4\pi A}$$

where A is the area occupied by the source in the source plane.

In a lensing phenomena, S remains a constant. Therefore, since the source area A changes, it is necessary that the flux increases (or decreases) proportionally. Considering the full set of multiple images, the result is always an increase in A , and therefore a flux magnification (but the single images could be also de-magnified, i.e. in clusters).

Physically this is motivated by the fact that the observer sees photons that otherwise it would not have seen without the action of the lens.



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Lensing at the first order: magnification, caustics, critics

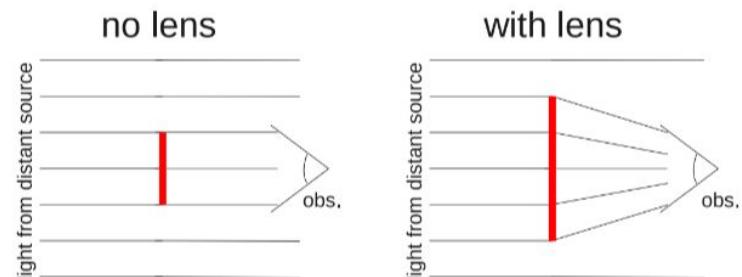
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Physically this is motivated by the fact that the observer sees photons that otherwise it would not have seen without the action of the lens.



The magnification is obtained as:

$$\begin{aligned}\mu &= \det M = \frac{1}{\det A} = \frac{1}{(1-k)^2 - \gamma^2} = \text{image area / source area} \\ &\equiv \frac{d\theta^2}{d\beta^2} = \frac{\theta d\theta}{\beta d\beta}\end{aligned}$$

whose eigenvalues are:

$$\mu_t = \frac{1}{1-k-\gamma}$$

$$\mu_r = \frac{1}{1-k+\gamma}$$

There are points for which $\mu \rightarrow \infty$ that in the source/image planes define lines called **caustics** and **critics**.

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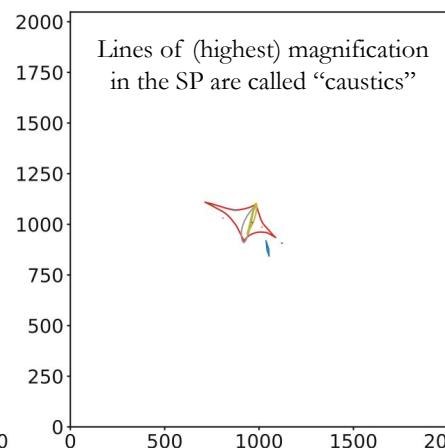
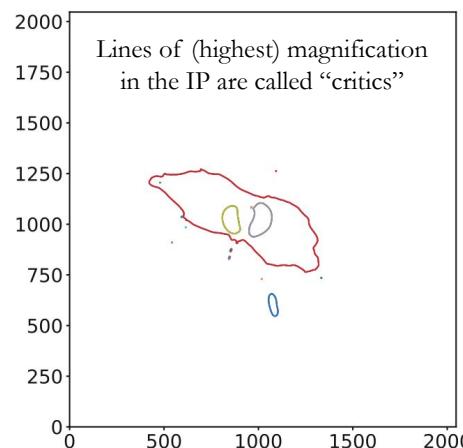
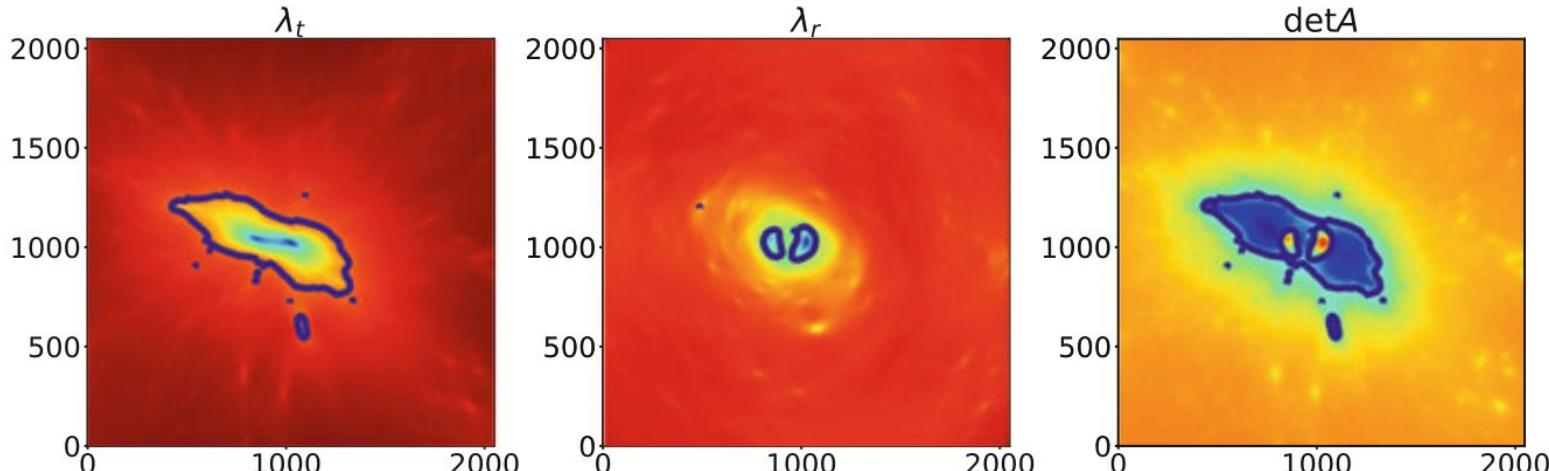
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Caustics and critics

Lensing at the first order: magnification, caustics, critics



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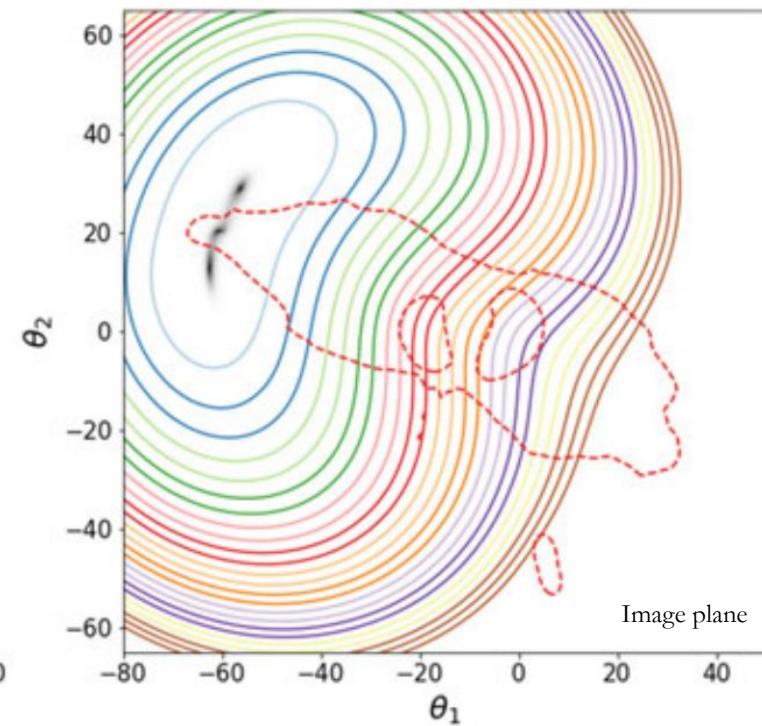
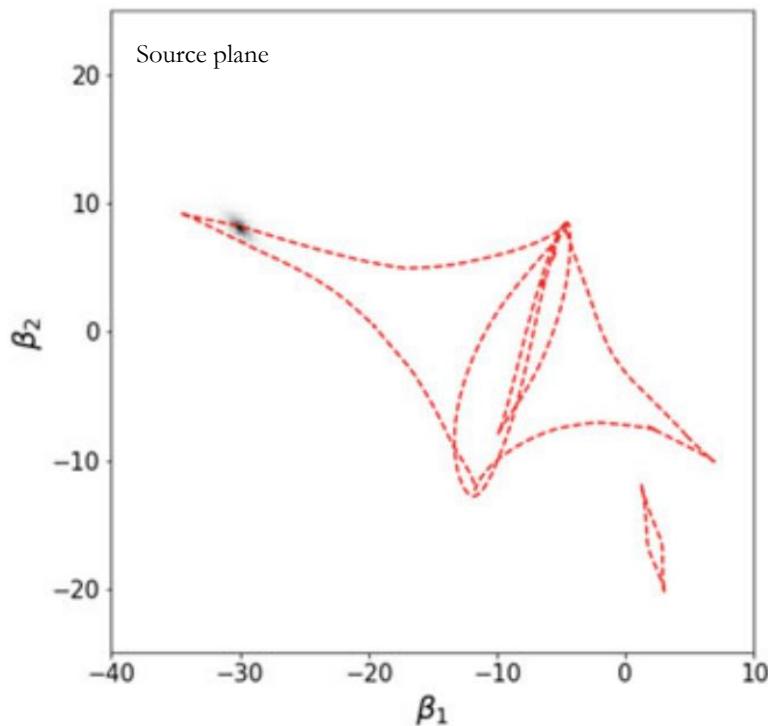
Magnification

Caustics and critics

Lensing at the first order: magnification, caustics, critics

The plot shows the simulation of a (realistic) cluster lens.

The greyscale on the left panel is a Sersic profile, mimicking a background source.



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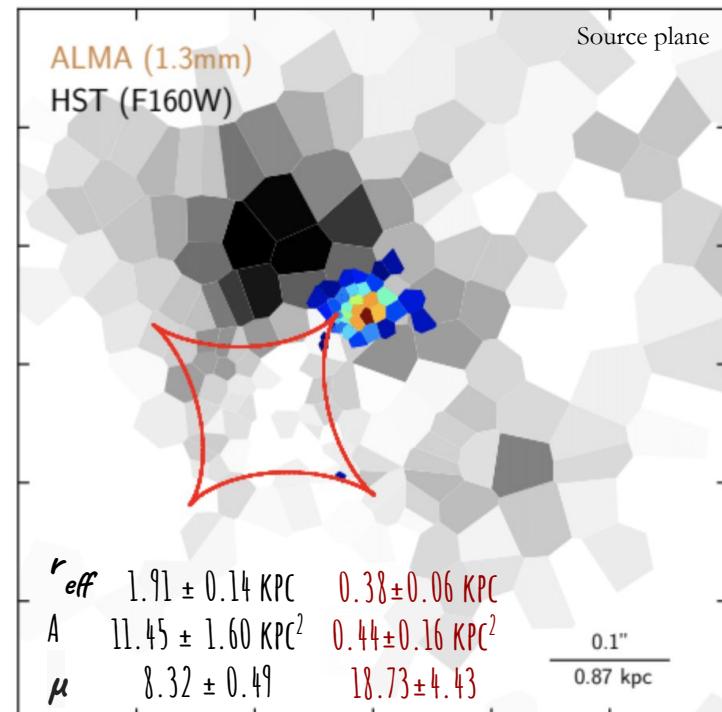
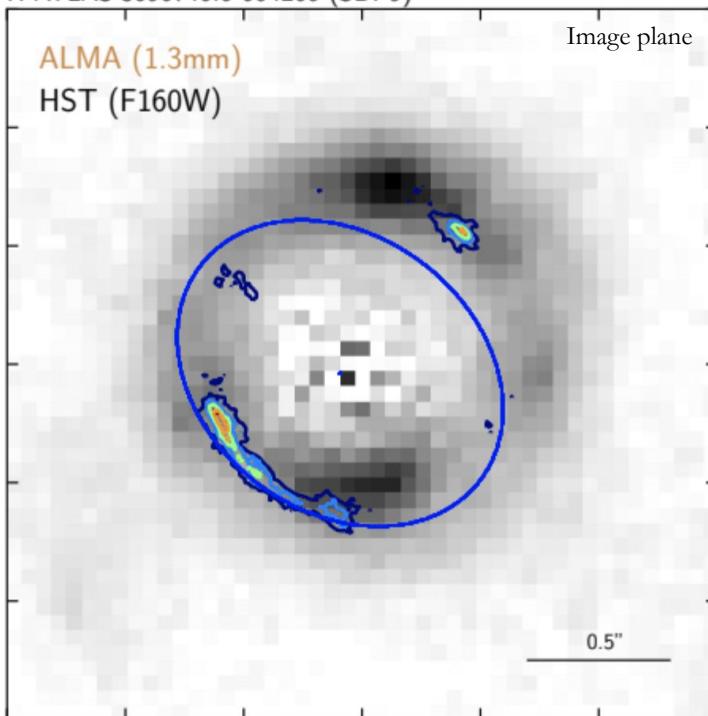
Caustics and critics

Real lenses

Real strong lenses

Now a real lensing application. This is SDP.9, a $z_s = 1.577$ source lensed by a $z_l = 0.6129$ elliptical galaxy (here subtracted to enhance and model the signal from the background source), observed at 1.6 μm with Hubble (probing the unobscured emission from stars) and at 1.3 mm with ALMA (probing the emission from the dust).

H-ATLAS J090740.0-004200 (SDP9)



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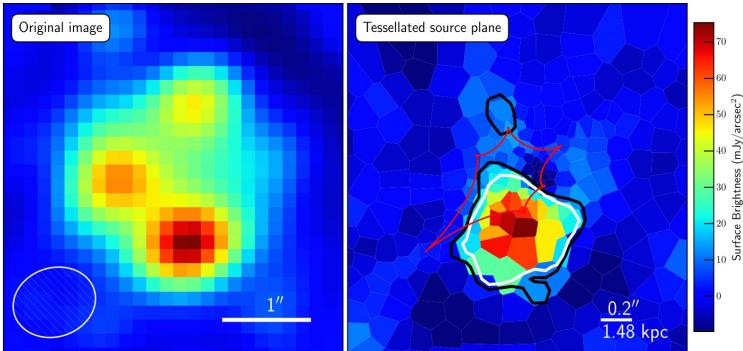
Caustics and critics

Real lenses

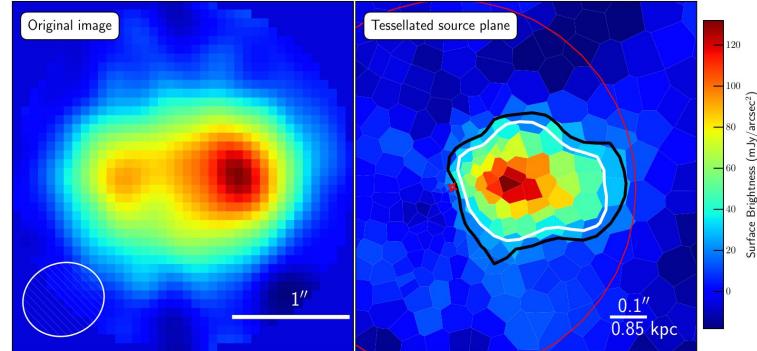
Real strong lenses

Another set of real lens: four Submillimeter Array (SMA) 880 μm observations of sources in H-ATLAS.
Notice that in sub-mm there is no lens emission (why?).

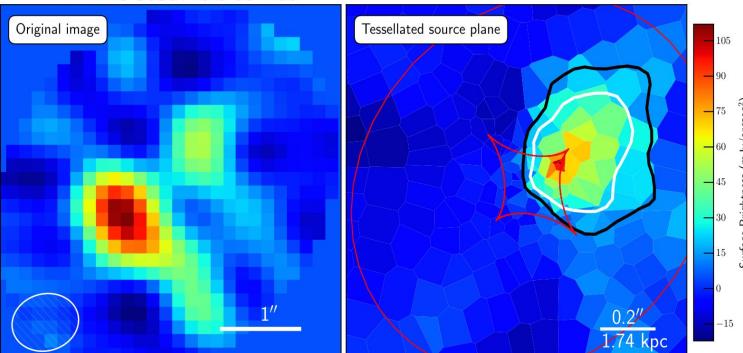
H-ATLAS J083051.0+013225



H-ATLAS J085358.9+015537



H-ATLAS J090740.0-004200



H-ATLAS J091043.0-000322

