PageRank centrality and algorithms for weighted, directed networks with applications to World Input-Output Tables

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Abstract

PageRank (PR) is a fundamental tool for assessing the relative importance of the nodes in a network. In this paper, we propose a measure, weighted PageRank (WPR), extended from the classical PR for weighted, directed networks with possible non-uniform node-specific information that is dependent or independent of network structure. A tuning parameter leveraging node degree and strength is introduced. An efficient algorithm based on R program has been developed for computing WPR in large-scale networks. We have tested the proposed WPR on widely used simulated network models, and found it outperformed other competing measures in the literature. By applying the proposed WPR to the real network data generated from World Input-Output Tables, we have seen the results that are consistent with the global economic trends, which renders it a preferred measure in the analysis.

Keywords: node centrality, weighted directed networks, weighted PageRank, World

Input-Output Tables

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1. Introduction

Centrality measures are widely accepted tools for assessing the relative importance of the entities in networks. A variety of centrality measures have been developed in the literature, including position/degree centrality [1], closeness centrality [2], betweenness centrality [2], eigenvector centrality [3], Katz centrality [4], and PageRank [5], among others. Centrality measures have been applied to different types of real networks; for instance, ranking the cities with at least one operating airport in the air transport network of China [6] and evaluating the impact of research papers in a citation network [7]. See Das et al. [8] for a concise review and Newman [9, Chapter 7] for a text-style elaboration.

The classical PageRank [PR, 10] was designed to precisely rank web pages in Google search via hyper-textual information (primarily link structure). Today, PR and its extensions are popular tools for the analyses of all kinds of networks, such as co-authorship networks [11], citation networks [12], and biological networks [13]. One limitation of the classical PR is that it does not account for edge weight in definition. Although ignoring edge weight may sometimes help with a quick exploration of the fundamental structure of a network, the discarded edge weight can lead to incorrect inference [14]. Only a limited number of works

considered weight for PR. Xing and Ghorbani [15] asserted that the popularity of a web page should be based on the numbers of its in-links and out-links. Ding [12] suggested replacing the random restart of the new process with a probability distribution based on the weights assigned to the nodes. No PR centrality measures have put edge weight and node weight in a unified framework.

As most real networks are weighted and directed, possibly with node-specific auxiliary information, here we consider a weighted PageRank (WPR) measure. The WPR uses edge direction and weight as well as auxiliary information at the node level to better characterize the centrality of a network. The computation of WPR boils down to finding the principal eigenvector of a big matrix, which can be efficiently done for large networks. We assess the performance of the proposed WPR in comparison with a few extended PR measures in the literature though numerical studies with synthetic networks. In the applications to the World Input-Output networks (WIONs) constructed from World Input-Output Tables [WIOTs, 16], the proposed WPR measure gives more intuitive results than the existing PR measures. The implementation of the proposed and competing measures is publicly available in an open-source R package wdnet [17].

The rest of the manuscript is organized as follows. In Section 2, we propose a WPR measure and demonstrate the computation strategy. We carry out some synthetic data analyses in Section 3, where two classes of widely used network models, namely scale-free networks and stochastic block models, are adopted. We apply the proposed WRP measure to the WIONs in Section 4, followed by some concluding remarks and discussions in Section 5.

2. Weighted PageRank

We begin with some basic network notations. Let G(V, E) denote a weighted and directed network, where V and E are respectively its node and edge sets. The structure of G is characterized by its weighted adjacency matrix $\mathbf{W} := (w_{ij})$, where w_{ij} is the weight of the directed edge from $i \in V$ to $j \in V$. If no edge exists from i to j, then $w_{ij} = 0$. When edge weight is ignored, \mathbf{W} is reduced to the standard adjacency matrix $\mathbf{A} := (a_{ij})$, where a_{ij} takes value 0 or 1. For any $i \in V$, let $d_i^{(\text{out})} := \sum_{j \in V} a_{ij}$ and $d_i^{(\text{in})} := \sum_{j \in V} a_{ji}$ respectively denote the out-degree and in-degree of i, referring to the numbers of edges emanating out from and pointing into i. Analogously, we have $s_i^{(\text{out})} := \sum_{j \in V} w_{ij}$ and $s_i^{(\text{in})} := \sum_{j \in V} w_{ji}$, respectively called the out-strength and in-strength of i when weight is accounted.

2.1. Formulation

Brin and Page [10] defined PR recursively as

$$PR(i) = \gamma \sum_{j \in V} \frac{a_{ji}}{d_j^{(\text{out})}} PR(j) + \frac{1 - \gamma}{n},$$
(1)

where PR(i) is the PR of node i, n = |V| counts the number of nodes in G(V, E), and $\gamma \in [0, 1)$ is a damping factor ensuring the algorithm never gets stuck in a "sinking node". This definition is is based on a random surfer model. Suppose that an Internet surfer keeps clicking on links bringing her to different web pages. With probability $(1 - \gamma)$ she restarts the process by randomly selecting a web page as the new initial state, where γ is the probability

that she continues in the current process. The inclusion of damping factor in the model ensures that the process will not be forced to terminate when the surfer arrives at a web page with no outbound link, called sinking node. Equation (1) suggests that a node would get a high PR score if: (1) it receives a large number of incoming edges; (2) senders of those incoming edges have small out-degrees; or (3) the PR scores of the senders are high.

In practice, lots of real networks are directed, weighted, and affiliated with important node-specific information. For instance, in Section 4, we consider the WIONs whose nodes correspond to different region-sectors, and edge weights are determined by transaction volumes. In addition, the total value added for each region-sector is considered as node-specific information. Here we extend the classical PR to a weighted version by simultaneously considering edge weights and auxiliary information contained in nodes. Let $\phi(i)$ denote the weighted PR of i, and β_i be some node-specific quantifiable information attached to i. We assume that β_i is independent of w_{ij} for all $i, j \in V$.

Analogous to Equation (1), we define the weighted PR recursively by

$$\phi(i) = \gamma \sum_{j \in V} \left(\theta \frac{w_{ji}}{s_i^{\text{(out)}}} + (1 - \theta) \frac{a_{ji}}{d_i^{\text{(out)}}} \right) \phi(j) + \frac{(1 - \gamma)\beta_i}{\sum_{i \in V} \beta_i}, \tag{2}$$

where $\theta \in [0, 1]$ is a tuning parameter adjusting the relative importance of weights in the definition. The value of θ can be chosen according to practical needs and actual interpretations. For instance, the value of a business project may be heavily reflected in the investment amount it has received rather than the number of investors, and the popularity of a product mainly depends on the sales volume. In many situations, a balance between the two factors is needed. For example, the strength of a researcher is related to the number of publications as well as the prestige of the journals (measured by a unified metric such as impact factor) of the publications simultaneously. The tuning parameter θ controls the balance between weight and degree. For the special case of $\theta = 0$, the proposed WPR is equivalent to the weighted PR introduced by Ding [12]. The vector $\boldsymbol{\beta} := (\beta_1, \beta_2, \dots, \beta_n)^{\top}$ usually takes the non-uniform relative importance of the nodes into account. When no such information is available, we let $\beta_i = 1, i = 1, \dots, n$, so that the second term on the right-hand-side (RHS) of Equation (2) coincides with what has been defined in Equation (1).

2.2. Computation

We propose an efficient algorithm for the computation of the proposed WPR in large networks. The standard method to compute classical PR is the power iteration. It is well known that the power iteration converges slowly, especially for massive and dense networks. This leads to the development of accelerated algorithms, many of which have been surveyed in Berkhin [18]. When $\gamma \neq 1$, the underlying process of the random surfer model for the classical PR is a irreducible Markov chain [18], where every state in the chain can be accessed with positive probability from other states. Here we regard nodes in a network as states in a Markov chain.

A similar argument can be applied to the proposed WPR. Let $\mathbf{M} := (m_{ij})$ be the

transition matrix of the associated Markov chain for WPR, where

$$m_{ij} = \begin{cases} \theta w_{ji} / s_j^{\text{(out)}} + (1 - \theta) a_{ji} / d_j^{\text{(out)}}, & \text{if } d_j^{\text{(out)}} \neq 0; \\ \beta_i / \sum_{i \in V} \beta_i, & \text{if } d_j^{\text{(out)}} = 0. \end{cases}$$

Notice that

$$\sum_{i \in V} \left(\theta \frac{w_{ji}}{s_j^{\text{(out)}}} + (1 - \theta) \frac{a_{ji}}{d_j^{\text{(out)}}} \right) = \theta + (1 - \theta) = 1.$$

That is, matrix M is non-negative and column stochastic. Let $P := (\phi(1), \phi(2), \dots, \phi(n))^{\top}$ be a column vector collecting the WPR for each node. Equation (2) is equivalent to

$$P = \gamma MP + (1 - \gamma)\beta^*, \tag{3}$$

where $\beta^* = \beta/\|\beta\|_1$ is the normalization of β .

Since M is column stochastic, we can rewrite Equation (3) as

$$P = (\gamma M + (1 - \gamma)B)P =: M^*P, \tag{4}$$

where \mathbf{B} is an $(n \times n)$ matrix such that the i-th column is given by $\beta_i^* \cdot \mathbf{1}$ for i = 1, 2, ..., n. It is obvious that \mathbf{B} is also column stochastic, rendering that \mathbf{M}^* is strictly positive and column stochastic provided that $\boldsymbol{\beta} \neq \mathbf{0}$. By the Perron-Frobenius theorem [19], the largest eigenvalue of \mathbf{M}^* is equal to 1, and the solution to Equation (4) is the corresponding eigenvector. In the context of stochastic process, we regard the normalized solution of \mathbf{P} as a stationary distribution of the Markov chain associated with the probability transition matrix \mathbf{M}^* . Based on the above representation, the computation of \mathbf{P} in a massive network is converted to finding the principal eigenvector of a large-scale matrix. One of the most efficient approaches is the ARPACK software [20], with a recent interface for R through package $\mathbf{rARPACK}$ [21].

3. Synthetic Data Examples

We assess the performance of the proposed WPR measure with scale-free networks and stochastic block networks. Comparison is done with respect to classical PR [5] and two existing weighted PR measures respectively introduced by Xing and Ghorbani [15] and Ding [12]. The definition of the former is relegated Appendix Appendix A, where the latter is a special case of the proposed WPR as mentioned. Under each network model, we introduced a parameter $\rho \in [0,1]$ to control the strength of the dependency between the node-specific prior information and the node strength. Specifically, let S_i be the strength of node $i, i \in V$. Then, the prior information β_i of this node was generated such that the correlation between S_i and β_i is $\rho \in [0,1]$. This can be done by setting $\beta_i = \alpha S_i + (1-\alpha)X_i$, where X_i is a postive random variable independent of S_i and

$$\alpha = \left(1 + \sqrt{\frac{(1 - \rho^2)\operatorname{Var}(S_i)}{\rho^2\operatorname{Var}(X_i)}}\right)^{-1}.$$
 (5)

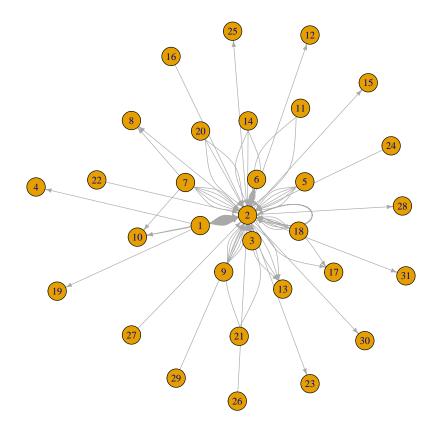


Figure 1: A simulated (weighted and directed) PA network mimicking a Facebook wall post data.

As $\rho \to 0$, we have $\beta_i \to X_i$, which coincides with the case of Ding [12]. When $\rho = 1$, $\beta_i = S_i$. For any $\rho \in (0,1)$, β_i 's generated with Equation (5) were used as prior information for WPR computations.

3.1. Scale-free Network

In the literature, the preferential attachment (PA) rule [22] is one way to generate scale-free networks. We used the algorithm of Yuan et al. [23] to generate weighted directed PA networks via R package wdnet [17]. Specifically, the simulated PA network initiates with a directed edge from node 1 to 2, where the weight is drawn from Bin(100, 0.75). At each subsequent step, an edge is added according to one of the following three scenarios: With probability $\alpha = 0.05$, the edge is added from a new node to an existing one; with probability $\beta = 0.9$, the edge is added between two existing nodes; with probability $\gamma = 0.05$, the edge is added from an existing node to a new one. The source and target nodes of the added edge are selected proportional to their current out- and in-strengths, respectively. Upon the edge being added, its weight is independently drawn from Bin(100, 0.75). The leveraging parameters $\delta_{\rm in}$ and $\delta_{\rm out}$ that respectively control the growth rates of in-strengths and outstrengths are fixed, both taking value 1. We refer the readers to Wang and Zhang [24] for the statistical properties and detailed interpretations of these parameters. The evolution proceeds in this fashion for 300 steps, where the resulting PA network is depicted in Figure 1.

Table 1: A comparison of the proposed WPR measure with those respectively proposed in Ding [12] (equivalent to the case of $\theta = 0$) and Xing and Ghorbani [15] for the simulated PA network; the damping factor for the proposed WPR is fixed $\gamma = 0.85$.

| $\phi \ (\theta = 0)$ | | ϕ (| $\theta = 0.5$) | $\phi \ (\theta = 1)$ | | X-G's measure | |
|-----------------------|---------|----------|------------------|-----------------------|---------|---------------|---------|
| Node | WPR (%) | Node | WPR (%) | Node | WPR (%) | Node | WPR (%) |
| 2 | 26.216 | 2 | 38.032 | 2 | 66.030 | 3 | 1.710 |
| 10 | 4.415 | 10 | 3.404 | 13 | 2.177 | 5 | 1.710 |
| 8 | 4.051 | 8 | 3.225 | 17 | 1.643 | 6 | 1.710 |
| 13 | 3.567 | 13 | 3.185 | 10 | 1.429 | 9 | 1.710 |
| 31 | 3.567 | 17 | 3.031 | 8 | 1.377 | 11 | 1.710 |
| 17 | 3.567 | 28 | 2.931 | 28 | 1.294 | 14 | 1.710 |
| 25 | 3.567 | 23 | 2.928 | 23 | 1.286 | 16 | 1.710 |
| 28 | 3.567 | 30 | 2.928 | 30 | 1.286 | 18 | 1.710 |
| 30 | 3.567 | 25 | 2.927 | 25 | 1.282 | 20 | 1.710 |
| 12 | 3.567 | 15 | 2.925 | 15 | 1.273 | 21 | 1.710 |

In the first experiment, we did not consider any kind of node-specific prior information, that is, $\beta_i = 1$ for all $i \in V$. Table 1 summarizes the top 10 nodes based on Xing and Ghorbanis' PR measure and the proposed WPR measures (with $\gamma = 0.85$ and $\theta = \{0, 0.5, 1\}$) With Xing and Ghorbanis' PR measure, all of the top 10 nodes have the same score, so they are simply ordered by their appearance timing. This does not provide much practical guidance. In particular, node 2, which emerges at the central position in Figure 1, does not appear in the top 10 list. Hence, Xing and Ghorbanis' PR measure will not be considered in the sequel.

Node 2 ranks first in all three lists in Table 1, which is consistent with the observation from Figure 1. The nodes of rank 2 and 3 are the same for $\theta = 0$ and $\theta = 0.5$, but not for $\theta = 1$. Ding's PR measure ($\theta = 0$) could not distinguish the nodes from rank 4 to 10, as they have exactly the same score. This is expected; a measure not accounting for edge weight is not suitable for weighted networks. From the list of $\theta = 0.5$, nodes of lower ranks (in top 10) have become identifiable, albeit with tiny gaps. When edge weights are fully accounted, node 2 is extensively dominant in the network with a much higher WPR score than the rest. Meanwhile, the normalized WPR score of node 2 in the list of $\theta = 1.0$ is higher than the counterparts in the lists of $\theta = 0$ and $\theta = 0.5$ as well. The ranks of nodes 13 and 17 both rise in the list of $\theta = 1.0$, while those of nodes 10 and 8 drop. Further investigation reveals that node 13 and 17 both have links of high weight 295 and 165, respectively, from node 2. Except for edges pointing towards node 2, these two edges are the most weighted in the network. Though node 10 receives three links respectively with weight 72 from node 1, weight 80 from node 2, and weight 72 from node 7, and node 8 receives two links respectively with weight 71 from node 2 and weight 76 from node 7, the incoming edges from node 2 are relatively small and the WPR scores of all of the other nodes are much smaller than that of node 2. As a result, nodes 10 and 8 are ranked lower than nodes 13 and 17.

In the second experiment, we incorporated an independent node-specific prior ranking weight ($\rho = 0$) into the WPR computation. The priors were generated independently from

Table 2: Nodes of top 10 WPR scores with/without (independent) prior information in the simulated PA network for $\theta \in \{0, 0.5, 1\}$ and $\gamma = 0.85$.

| $\theta = 0$ | | θ = | = 0.5 | $\theta = 1$ | | |
|--------------|---------------------|------------|---------------------|--------------|------------|--|
| no prior | no prior with prior | | no prior with prior | | with prior | |
| 2 | 2 | 2 | 2 | 2 | 2 | |
| 10 | 19 | 10 | 19 | 13 | 19 | |
| 8 | 10 | 8 | 10 | 17 | 7 | |
| 13 | 8 | 13 | 7 | 10 | 10 | |
| 31 | 7 | 17 | 17 | 8 | 4 | |
| 17 | 17 | 28 | 4 | 28 | 17 | |
| 25 | 4 | 23 | 8 | 23 | 29 | |
| 28 | 23 | 30 | 23 | 30 | 23 | |
| 30 | 25 | 25 | 25 | 25 | 25 | |
| 12 | 29 | 15 | 29 | 15 | 8 | |

the network with an exponential distribution with mean 5; see Figure 2. Table 2 summarizes the top 10 nodes based on proposed WPRs obtained under $\theta \in \{0, 0.5, 1\}$ and $\gamma = 0.85$ in comparison to those without considering the prior information. Drastic changes are observed. Take $\theta = 1$ as an example. Four nodes in the top 10 list are new. Two of them, nodes 19 and 7 rank 2 and 3, respectively, compared to 13 and 31 without the prior information. Indeed, these two nodes indeed have the top two prior scores as shown in Figure 2. Although node 2 remains at the top, inclusion of the prior information has brought several low rank nodes to much higher ranks. This suggest that a non-negligible impact of the prior information on WPR and the ultimate ranking.

Lastly, we carried out a sensitivity analysis for $\theta \in \{0, 0.5, 1\}$ and $\rho \in \{0.25, 0.5, 0.75\}$, where ρ is the correlation between the prior information and the in-strength. Table 3 summarizes the results of the top 10 nodes. Especially for $\theta = 1$, the ranks are almost identical for different choices of ρ , suggesting that the proposed WPR is robust when edge weight is fully accounted in the computation. For $\theta = 0.5$, we do not observe significant difference in node labels across the three lists. The new participant for the list of $\rho = 0.75$, node 13, is ranked 12 and 11 respectively in the lists of $\rho = 0.25$ and $\rho = 0.5$, and node 29 has dropped to rank 11 in the list of $\rho = 0.75$. There have been some changes in the rank orders as expected, as the quantities of resulting priors change with the value of ρ . For instance, the prior of node 7 is much higher than that of node 17 for small ρ , but the deviation gets smaller as ρ gets larger. As node 17 receives a moderately weighted link from node 2 (the one with the largest WPR score), it takes the fourth place in the lists of $\rho = 0.5$ and $\rho = 0.75$. Furthermore, node 8 has surpassed node 7 in the list of $\rho = 0.75$, too, as it also gets a link from node 2, albeit its small prior. When weight is not considered, we again only see difference in the order of ranks in the presented lists. For $\rho = 0.75$, node 10 has replaced node 19 taking the second place. Node 19 is always ranked high since it has the largest prior of all, but there is only one link pointing to it. The difference between the priors of node 10 and node 19 is not big for $\rho = 0.75$, but node 10 receives more links from the others in the network, including

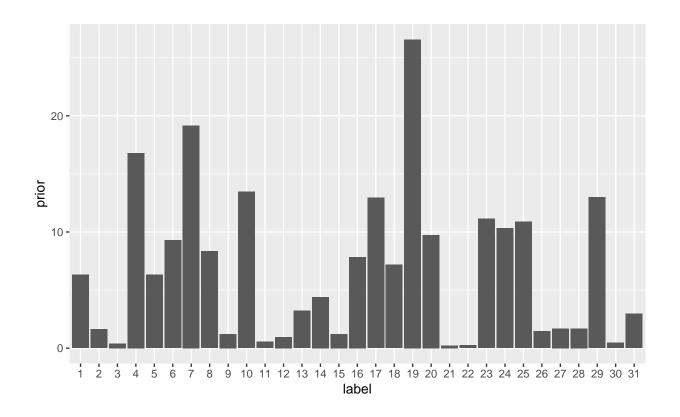


Figure 2: Generated scores (as prior information) for the nodes in the simulated PA network.

Table 3: Nodes of top 10 WPR scores with a variety of correlated prior information, $\rho \in \{0.25, 0.5, 0.75\}$, in the simulated PA network for $\theta \in \{0, 0.5, 1\}$ and $\gamma = 0.85$.

| $\theta = 0$ | | | $\theta = 0.5$ | | | $\theta = 1$ | | |
|---------------|--------------|---------------|----------------|--------------|---------------|---------------|--------------|---------------|
| $\rho = 0.25$ | $\rho = 0.5$ | $\rho = 0.75$ | $\rho = 0.25$ | $\rho = 0.5$ | $\rho = 0.75$ | $\rho = 0.25$ | $\rho = 0.5$ | $\rho = 0.75$ |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 19 | 19 | 10 | 19 | 19 | 19 | 19 | 19 | 19 |
| 10 | 10 | 19 | 10 | 10 | 10 | 7 | 7 | 7 |
| 8 | 8 | 8 | 7 | 17 | 17 | 10 | 10 | 10 |
| 7 | 17 | 17 | 17 | 7 | 8 | 17 | 17 | 17 |
| 17 | 7 | 23 | 8 | 8 | 7 | 4 | 4 | 4 |
| 4 | 4 | 25 | 4 | 23 | 23 | 23 | 23 | 13 |
| 23 | 23 | 7 | 23 | 4 | 25 | 29 | 25 | 23 |
| 25 | 25 | 4 | 25 | 25 | 4 | 25 | 29 | 25 |
| 29 | 29 | 29 | 29 | 29 | 13 | 8 | 8 | 8 |

one from node 2, rendering it to take a higher rank ultimately.

No significant change in WPR scores is observed for different selections of θ and ρ . This is due to the characteristic of PA rule that nodes of high in-degree (in-strength) are likely to attract more incoming connections. When there is a subset of nodes that have

received the majority of incoming edges at an early stage, the newly generated edges will be connected towards these nodes with high probabilities. While the in-strengths of these nodes keep growing, their in-degrees increase as well, which results in high scores of classical PR measure. Accordingly, the effect of edge weight on the final ranking results, especially the top 5, has become limited for PA networks. Rankings sensitive to θ are illustrated in the next example.

3.2. Stochastic Block Model

Stochastic block models (SBMs) are a class of network models for characterizing community structure [25, 26, 27]. In essence, an SBM is comprised of a certain number of within-block Erdös-Renyi [ER, 28] models, where the cross-block structure is specified by Bernoulli models. We generated a weighted and directed SBM network consisting of two communities, C_1 and C_2 , each containing 50 members. The link densities within C_1 and C_2 were respectively 0.2 and 0.3, whereas the link density between C_1 and C_2 was 0.02. The weights for edges in C_1 and C_2 were were independently drawn from Bin(500, 0.75) and Bin(20, 0.5), respectively. The weights for edges C_1 and C_2 (either from C_1 to C_2 or from C_2 to C_1) were independently drawn from Bin(5, 0.5). Community C_1 is sparser than community C_2 , but their within-community links are both much denser than the between-community links. In addition, edge weights in C_1 are much larger than those in C_2 , and edge weights between the communities are the smallest. The generated SBM network is presented in Figure 3, where the node sizes are proportional to the logarithm of their strengths. The nodes in C_1 and C_2 are colored with blue and red, respectively.

Table 4 summarizes the top 10 nodes based on the WPR scores with $\theta \in \{0, 0.5, 1\}$ and $\gamma = 0.85$. When edge weight is not accounted for (i.e., $\theta = 0$), nine of the top 10 nodes come from C_2 since the nodes therein are more densely connected. A close inspection shows that the top 3 nodes 59, 63 and 93 have the largest in-degrees. For $\theta = 1$, all top 10 nodes belong to C_1 . Node 39 has in-strength 2,230, which is less than that of node 50 (2,415), but still ranks higher than node 50. This is desired as node 50 has a large amount of inputs from insignificant nodes, like nodes 22 (rank 87), 23 (rank 100), 28 (rank 85) and 29 (rank 90), whereas node 39 has inputs mostly from high rank nodes including itself. The top 3 nodes in the list of $\theta = 0$ rank only, respectively, 56, 97 and 54, as they have low in-strengths, and the WPR scores of the nodes linking towards them are relatively low. For the hybrid case of $\theta = 0.5$, we observe a mixture of the nodes from the top 10 lists of $\theta = 0$ and $\theta = 1$ with five each. The top 1 is node 59 (top 1 from $\theta = 0$), whose WPR score is slightly higher than that of the second largest, node 39 (top 1 form $\theta = 1$). The node with third highest WPR score is node 8, which also comes from the $\theta = 1$ list. We see that the difference between the WPR scores between nodes 39 and 8 is smaller than that in the list of $\theta = 1$ since the edge weight is not yet fully accounted.

The SBM example provides strong evidence for the necessity of accounting for edge weight in the WPR computation. Unlike the example of PA network, the top 10 nodes here are almost completely different between $\theta = 0$ and $\theta = 1$. Noticeable changes have been observed in the WPR scores as well. Such drastic changes in node ranks are primarily due to the structure of the network. There is no preferential attachment feature in the generation of SBMs, so the impact of edge weight on the WPR scores remains compelling.

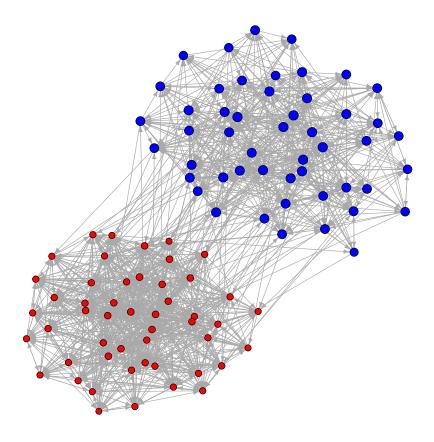


Figure 3: A simulated (weighted and directed) SBM network consisting of two communities; the blue nodes are from C_1 and the red ones are from C_2 ; self-loops exist but are not presented.

Table 4: The nodes with top 10 proposed WPR scores in the simulated SBM network for $\theta \in \{0, 0.5, 1\}$; the damping factor is fixed $\gamma = 0.85$.

| $\phi \ (\theta = 0)$ | | φ (| $\theta = 0.5)$ | $\phi \ (\theta = 1)$ | | |
|-----------------------|---------|------|-----------------|-----------------------|---------|--|
| Node | WPR (%) | Node | WPR (%) | Node | WPR (%) | |
| 59 | 1.775 | 59 | 1.536 | 39 | 1.750 | |
| 63 | 1.500 | 39 | 1.492 | 8 | 1.605 | |
| 93 | 1.494 | 8 | 1.466 | 50 | 1.583 | |
| 98 | 1.444 | 63 | 1.424 | 32 | 1.576 | |
| 78 | 1.429 | 32 | 1.367 | 36 | 1.525 | |
| 57 | 1.417 | 93 | 1.364 | 5 | 1.421 | |
| 86 | 1.403 | 36 | 1.352 | 25 | 1.420 | |
| 71 | 1.390 | 98 | 1.343 | 38 | 1.384 | |
| 8 | 1.390 | 78 | 1.328 | 46 | 1.353 | |
| 72 | 1.372 | 50 | 1.328 | 30 | 1.327 | |

Table 5: The region-sectors with top 10 WPR scores (no prior information) in the WIONs from 2000, 2007 and 2014, with $\gamma = 0.85$ and $\theta \in \{0, 0.5, 1\}$.

| | 2000 | | | 2007 | | | 2014 | | |
|------|--------------|----------------|--------------|--------------|----------------|--------------|--------------|----------------|--------------|
| Rank | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ | $\theta = 0$ | $\theta = 0.5$ | $\theta = 1$ |
| 1 | IND39 | ROW27 | USA51 | IND39 | ROW27 | ROW27 | IND39 | ROW27 | ROW27 |
| 2 | IND31 | DEU20 | ROW27 | IND31 | DEU20 | GBR53 | IND31 | ROW4 | USA51 |
| 3 | IND40 | USA51 | GBR53 | MEX5 | ROW4 | USA51 | IND40 | DEU20 | GBR53 |
| 4 | IND27 | ROW24 | USA53 | MEX50 | USA51 | USA53 | MEX50 | USA51 | CHN27 |
| 5 | IND32 | ROW6 | DEU20 | MEX27 | ROW24 | ROW24 | MEX5 | ROW24 | ROW4 |
| 6 | IND1 | ROW29 | USA20 | MEX28 | ESP27 | DEU20 | MEX27 | ROW5 | ROW24 |
| 7 | IND42 | USA27 | ROW6 | MEX29 | FRA27 | ESP27 | MEX30 | ROW51 | USA53 |
| 8 | MEX5 | DEU27 | USA27 | MEX30 | ROW6 | ROW4 | IND27 | GBR53 | ROW5 |
| 9 | MEX50 | USA20 | USA44 | IND40 | GBR53 | ROW6 | MEX29 | CHN27 | DEU20 |
| 10 | MEX27 | ROW31 | ROW24 | MEX45 | ROW17 | USA44 | MEX45 | ROW29 | ROW6 |

4. World Input-Output Networks

In economics, a World Input-Output Table (WIOT) is a multi-regional input-output table, which records the intermediate transaction volumes among the sectors from different countries/regions. It has great research value in analyzing the inter-dependency across multi-regional sectors in the global economy. We applied the proposed WPR to the WIONs constructed from the annual WIOTs [WIOTs, 16] from 2000 to 2014 using the 2016 release of the World Input-Output Database. The 2016 release covers 56 sectors from 44 countries/regions, including a region called "the rest of the world" (ROW). The dictionary for the sector codes are given in Table B.7 in Appendix Appendix B. The 2,464 region-sectors are the nodes of the WIONs. A transaction from one region-sector to another forms a weighted, directed edge, where the edge weight is represented by the transaction volume (in the unit of 1 million USD). The link densities of the WIONs are high with average 83%. A sub-network consisting of seven major economies for 2014 is depicted in Figure 4. Only edges of weight ≥ 500 are presented, while the self-loops and isolated nodes have been removed. The node size is proportional to the natural logarithm of its total strength. A few studies have investigated centrality measures (not limited to PR and its variants) of the WIONs [29, 30, 31].

Table 5 presents the top 10 region-sectors ranked by the proposed WPR with $\theta \in \{0, 0.5, 1\}$ and $\gamma = 0.85$ for the WIONs from 2000, 2007 and 2014. When edge weights are not taken into account (i.e. $\theta = 0$), all top 10 nodes are sectors from India (IND) and Mexico (MEX) for all three years. These results are counter-intuitive as neither of the two countries was regarded as the most influential in the world economy during the study period. When weight is partially accounted ($\theta = 0.5$), results are more reasonable, but completely different from those with $\theta = 0$. Construction (27) from ROW took the first place in all three years. Manufacture of motor vehicles, trailers and semi-trailers (20) from Germany (DEU) took the second place in 2000 and 2007, but the third place in 2014, and the second

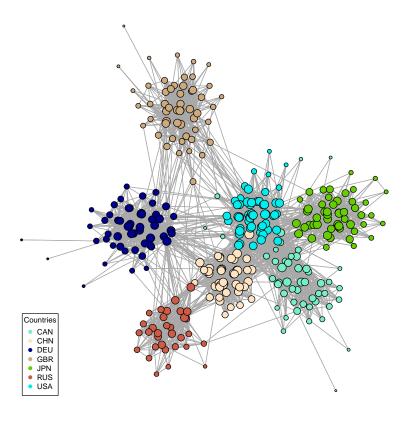


Figure 4: A example of sub-network of the WION consisting of seven major economies in 2014; self-loops and isolated nodes have been removed; node sizes are proportional to the natural logarithm of their strengths; the edges of weight greater than or equal to 500 (with unit 1 million USD) are presented.

place in 2014 was taken by mining and quarrying (4) from ROW. Several other traditional leading sectors from influential economies are also included in the top 10 list, such as public administration and defense and compulsory social security (51) from the United States of America (USA) and human health and social work activities (53) from the United Kingdom (GBR). Quite a few sectors from ROW besides construction (27) are in the top 10 lists, which is understandable since ROW aggregates over those outside of the 43 countries/regions.

When edge weight is fully accounted ($\theta = 1$), no significantly different results have been observed from those with $\theta = 0.5$. Over all three years, the top 3 sectors were public administration and defense and compulsory social security (51) from USA, human health and social work activities (53) from GBR, and construction (27) from ROW, except in different orders. Fewer sectors are from ROW in the top 10 lists of $\theta = 1$ compared to those of $\theta = 0.5$. As ROW contains many countries/regions, its sectors have input and output connections with all of the other region-sectors in the networks. Those edge weights are not necessarily large even though they are aggregated counts. Therefore, more leading sectors from the strong economies appear in the top 10 lists. In 2000, four sectors were from the world's largest economy USA, but the number was reduced to three in 2007. Human health and social work activities (53) and real estate activities (44) from USA were indeed world-

leading region-sectors. After the subprime mortgage crisis in 2008, real estate activities (44) did not appear in the top 10 list in 2014 like in 2000 or 2007. Construction (27) from China (CHN) joined the top 10 list in 2014. As China became the world's second largest economic power in 2010, construction as a sector with the largest pulling effect has been expected to have a high rank [32]. The inclusion of a non European Union (EU) country or USA in the top 10 list suggests a changed landscape and increased diversity of the global economy.

We next used the total value-added (TVA) in the WIOTs as prior information in the WPR [32]. The results with edge weights fully accounted (i.e., $\theta = 1$) are summarized in Table 6. Significant changes have been observed over time. In 2000, the top 9 region-sectors were from USA. The top 5 are public administration and defense and compulsory social security (51), human health and social work activities (53), real estate activities (44), construction (27), and manufacture of motor vehicles, trailers and semi-trailers (20). The only non-USA region-sector was construction (27) from Japan, which has been a large component of the Japanese economy in terms of output and employment, and a robust force for the economic recovery and expansion in Japan in the post-war years till today [33, 34]. In 2007, USA was not as dominant as in 2000 but still with six in the top 10. The top 3 remain unchanged. Construction (27) from China ranked the 9th. This result seems to be more reasonable, as the Chinese government provided unlimited support to the construction industry in 2007 in preparation for the 2008 Olympic Game. As the most influential sector in China, construction (27) has driven the development of a large number of domestic sectors as well as international cooperation over that period [35, 36]. In 2014, the top 10 nodes had 4 from China, 3 from USA, and 3 from ROW. Construction (27) of China ranked the first. The other three Chinese sectors were manufacture of motor vehicles, trailers and semi-trailers (20), manufacture of computer, electronic and optical products (17) and manufacture of basic metals (15), which, respectively, ranked 6, 7 and 10. The top 3 USA sectors in 2007 now ranked 2, 4, and 5. No EU sectors showed up in the top 10. This result is consistent with the fact that USA and China are the two largest economies in the world. Compared to results without TVA prior, fewer region-sectors from ROW ranked in the top 10 as their TVA amounts were small in general.

5. Discussions

We propose a weighted PageRank measure that simultaneously accounts for edge weights and prior information on the relative importance of nodes in weighted directed networks. The relative importance of node strengths and edge weights is controlled by a tuning parameter for flexibility. Efficient algorithms are implemented and made publicly available in R package wdnet [17]. Through two simulated network examples and one application to the WIONs, we have observed significant differences between the results from the proposed WPR and other classical measures, where the proposed WPR is preferred. Especially for the WIONs, the proposed WPR has led to conclusions that are more consistent with intuition, providing new insights into the global input-output system. Both synthetic and real data studies suggest the need for considering edge weight and prior information in the node centrality measure for weighted directed networks.

There are several limitations in the present research that merit further studies. So far the proposed measure has been adapted to static networks only. It is of substantial inter-

Table 6: The region-sectors with top 10 WPR scores (with TVA prior information) in the WIONs from 2000, 2007 and 2014, with $\gamma = 0.85$ and $\theta = 1$.

| | 2 | 2000 | 2 | 007 | 2014 | | |
|------|----------|-----------|----------|-----------|----------|-----------|--|
| Rank | no prior | TVA prior | no prior | TVA prior | no prior | TVA prior | |
| 1 | USA51 | USA51 | ROW27 | USA51 | ROW27 | CHN27 | |
| 2 | ROW27 | USA53 | GBR53 | USA53 | USA51 | USA51 | |
| 3 | GBR53 | USA44 | USA51 | USA44 | GBR53 | ROW27 | |
| 4 | USA53 | USA27 | USA53 | ROW27 | CHN27 | USA53 | |
| 5 | DEU20 | USA20 | ROW24 | USA27 | ROW4 | USA44 | |
| 6 | USA20 | USA5 | DEU20 | GBR53 | ROW24 | CHN20 | |
| 7 | ROW6 | USA30 | ESP27 | ROW24 | USA53 | CHN17 | |
| 8 | USA27 | USA36 | ROW4 | USA5 | ROW5 | ROW4 | |
| 9 | USA44 | USA29 | ROW6 | CHN27 | DEU20 | ROW24 | |
| 10 | ROW24 | JPN27 | USA44 | USA29 | ROW6 | CHN15 | |

est to investigate the proposed measure in random network models. Such extension would provide theoretical foundations for statistical inference such as confidence interval and hypothesis testing. Recently, Avrachenkov et al. [37] and Banerjee and Olvera-Cravioto [38] have looked into the asymptotic properties of the classical PR in undirected, unweighted SBMs and directed, unweighted PA networks, respectively. In addition to only being a centrality measure, the classical PR has been used to identify community structure in unweighted networks [39]. Applying the proposed WPR in community detection to weighted networks may lead to fruitful results.

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Appendix A. Weighted PR by Xing and Ghorbani [15]

Xing and Ghorbani [15] proposed a weighted PR to rank the web pages based on their popularity, where the popularity of a web page is reflected in two aspects: a large number of web pages have links to it and a large number of web pages it is linked to. Xing and Ghorbanis' weighted PR measure, after normalization, is

$$\phi^{XG}(i) = \gamma \sum_{j \in V} a_{ji} \left(\frac{d_i^{(in)}}{\sum_{k \in V} a_{jk} d_k^{(in)}} \right) \left(\frac{d_i^{(out)}}{\sum_{k \in V} a_{jk} d_k^{(out)}} \right) \phi^{XG}(j) + \frac{1 - \gamma}{n}. \tag{A.1}$$

One of the distinctive features of this weighted PR measure is that the edges of an unweighted network are actually "weighted". The "weight" of an edge is calculated by accounting for not only the in-degree and out-degree of the target node of the edge, but also the in-degrees and out-degrees of all the nodes that are linked by the source node of the edge. To clarify, let us take the edge from u_1 to v_2 in Figure A.5 as an example. Its in-degree and out-degree generated weight are respectively given by

$$\frac{d_{v_2}^{(\text{in})}}{d_{v_1}^{(\text{in})} + d_{v_2}^{(\text{in})}} = \frac{2}{1+2} = \frac{2}{3} \quad \text{and} \quad \frac{d_{v_2}^{(\text{out})}}{d_{v_1}^{(\text{out})} + d_{v_2}^{(\text{out})}} = \frac{3}{2+3} = \frac{3}{5}.$$

Xing and Ghorbanis' weighted PR does not actually use the information quantified by edge weight; instead, the "weight" is converted from node in- and out-degrees. Thus, the so-called "weight" is always integer-valued, causing the loss of generality. In addition, this PR measure does not seem to be applicable to some social networks. A celebrity may follow only a few users in a social media platform, so is likely to get a low score for Xing and Ghorbanis' PR measure.

Appendix B. Code Dictionary of the WIOTs

Table B.7 summarizes the code and definition of the 56 sectors in the 2016 release of the WIOD.

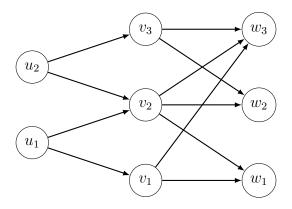


Figure A.5: A toy example for illustrating the edge weighting process in Xing and Ghorbani [15]; all the edges have a unit weight.

Table B.7: Description of the codes in the WIOTs

| Code | Sector |
|------|---|
| 1 | Crop and animal production, hunting and related service activities |
| 2 | Forestry and logging |
| 3 | Fishing and aquaculture |
| 4 | Mining and quarrying |
| 5 | Manufacture of food products, beverages and tobacco products |
| 6 | Manufacture of textiles, wearing apparel and leather products |
| 7 | Manufacture of wood and of products of wood and cork, except furniture; manufacture |
| | of articles of straw and plaiting materials |
| 8 | Manufacture of paper and paper products |
| 9 | Printing and reproduction of recorded media |
| 10 | Manufacture of coke and refined petroleum products |
| 11 | Manufacture of chemicals and chemical products |
| 12 | Manufacture of basic pharmaceutical products and pharmaceutical preparations |
| 13 | Manufacture of rubber and plastic products |
| 14 | Manufacture of other non-metallic mineral products |
| 15 | Manufacture of basic metals |
| 16 | Manufacture of fabricated metal products, except machinery and equipment |
| 17 | Manufacture of computer, electronic and optical products |
| 18 | Manufacture of electrical equipment |
| 19 | Manufacture of machinery and equipment n.e.c. |
| 20 | Manufacture of motor vehicles, trailers and semi-trailers |
| 21 | Manufacture of other transport equipment |
| 22 | Manufacture of furniture; other manufacturing |
| 23 | Repair and installation of machinery and equipment |
| 24 | Electricity, gas, steam and air conditioning supply |
| 25 | Water collection, treatment and supply |
| 26 | Sewerage; waste collection, treatment and disposal activities; materials recovery; remediation activities and other waste management services |
| 27 | Construction |

- 28 Wholesale and retail trade and repair of motor vehicles and motorcycles
- 29 Wholesale trade, except of motor vehicles and motorcycles
- 30 Retail trade, except of motor vehicles and motorcycles
- 31 Land transport and transport via pipelines
- 32 Water transport
- 33 Air transport
- 34 Warehousing and support activities for transportation
- 35 Postal and courier activities
- 36 Accommodation and food service activities
- 37 Publishing activities
- Motion picture, video and television program production, sound recording and music publishing activities; programming and broadcasting activities
- 39 Telecommunications
- 40 Computer programming, consultancy and related activities; information service activities
- 41 Financial service activities, except insurance and pension funding
- 42 Insurance, reinsurance and pension funding, except compulsory social security
- 43 Activities auxiliary to financial services and insurance activities
- 44 Real estate activities
- Legal and accounting activities; activities of head offices; management consultancy activities
- 46 Architectural and engineering activities; technical testing and analysis
- 47 Scientific research and development
- 48 Advertising and market research
- 49 Other professional, scientific and technical activities; veterinary activities
- 50 Administrative and support service activities
- 51 Public administration and defense; compulsory social security
- 52 Education
- 53 Human health and social work activities
- 54 Other service activities
- Activities of households as employers; undifferentiated goods- and services-producing activities of households for own use
- 56 Activities of extraterritorial organizations and bodies