$V, W$ spazi vettoriali $\quad B_V$ base di $V$, $B_W$ base di $W$

$f: V \longrightarrow W \quad B_V'$ // , $B_W'$ //

SONO ALTRE BASI DELLO STESSO VETTORE

$$[M(f)]_{B_W}^{B_V} \neq [M(f)]_{B_W'}^{B_V'}$$

$$B_V = \{v_1 \dots v_n\} \qquad B_W = \{w_1 \dots w_m\}$$

$$B_V' = \{b_1 \dots b_n\} \qquad B_W' = \{c_1 \dots c_m\}$$

$$v \in V \Rightarrow v = \lambda_1 \cdot v_1 + \dots \lambda_n \cdot v_n \qquad [v]_{B_V} = (\lambda_1 \dots \lambda_n)$$

$$b_j = b_{1j} \cdot v_1 + b_{2j} \cdot v_2 + \dots b_{nj} \cdot v_n$$

$$B = (b_{ij})_{1 \leq i j \leq n} \in M_{n,n}(\mathbb{R}) \Rightarrow v = \mu_1(b_{11} v_1 + \dots b_{n1} v_n) + \dots + \mu_n(b_{1n} v_1 + \dots b_{nn} v_n) =$$

$$\left([b_1]_{B_V} \dots [b_n]_{B_V}\right) = B$$

$$v \in V \Rightarrow \exists \mu_1 \dots \mu_n \text{ tale che } v = \mu_1 b_1 + \mu_n b_n \Leftrightarrow [v]_{B_V'}$$

$$\left(\mu_1 b_{11} + \mu_2 b_{12} + \dots \mu b_n\right) v_1 + \dots + \left(\mu_1 b_{n1} + \dots + \mu_n b_{nn}\right) v_n$$

$$\Rightarrow [v]_{B_V} = \begin{pmatrix} \mu_1 b_{11} + \dots + \mu_n b_{1n} \\ \vdots \qquad\qquad \vdots \\ \mu_1 b_{n1} + \dots + \mu_n b_{nn} \end{pmatrix} = B \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = B[v]_{B_V'}$$

$$[w]_{B_W} = C[w]_{B_W'}$$

$$v \in V \Rightarrow f(v) \in W \Rightarrow C[f(v)]_{B_W'} = [M(f)]^{B_V}_{B_W} \cdot B[v]_{B_V}$$

$$\begin{array}{ccccc} [f(v)]_{B_W} & = & [M(f)]^{B_V}_{B_W} & \cdot & [v]_{B_V} \\ \| & & & & \| \\ C[f(v)]_{B_W'} & & & & B\cdot[v]_{B_V'} \end{array} \Rightarrow [f(v)]_{B_W'} = \boxed{C^{-1}[M(f)]^{B_V}_{B_W}\cdot B}\cdot[v]_{B_V'}$$

$$\boxed{C^{-1}[M(f)]^{B_V}_{B_W}\cdot B} = [M(f)]^{B_V'}_{B_W'}$$

- definizione

$M_1, M_2 \in M_{m,n}(\mathbb{R})$ si dicono equivalenti se:

$\exists C \in M_{m,m}(\mathbb{R})\ \exists B \in M_{n,n}(\mathbb{R})$ invertibili tale che $M_1 = C^{-1} \cdot M_2 \cdot B$

- Proposizione

$f: V \longrightarrow W$ omomorfismo

$M_1$ e $M_2$ rappresentano $f$ rispetto a basi diverse

$M_1$ e $M_2$ sono simili

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "{[M(f)]^{B_V}_{B_W}}"] & & \mathbb{R}^m \\
V \arrow[u, "B_V", "\cong"'] \arrow[d, "\cong"'] \arrow[rr, "f"] & & W \arrow[u, "B_W"', "\cong"] \arrow[d, "\cong"] \\
\mathbb{R}^n \arrow[uu, bend left=60, "B"] \arrow[rr, "{[M(f)]^{B_V}_{B_W}}"'] & B_V' \quad\quad B_W' & \mathbb{R}^m \arrow[uu, bend right=60, "e"']
\end{tikzcd}$$

Come funziona con gli endomorfismi?

$g: V \longrightarrow V$ $\quad \underbrace{B \,,\, B'}_{\text{BASE}}$ basi di $V$ $\quad \underbrace{B}_{\text{MATRICE}}$ = matrice di cambiamento di base

sono 2 cose diverse

$$[M(g)]^{B}_{B} \quad , \quad [M(g)]^{B'}_{B} \quad , \quad \text{ecc....}$$

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "{[M(g)]^B_B}"] & & \mathbb{R}_M \\
V \arrow[u, "B"] \arrow[d, "B'"'] \arrow[rr, "f"] & & W \arrow[u, "B"'] \arrow[d, "B'"] \\
\mathbb{R}^n \arrow[uu, bend left, "B"] \arrow[rr, "{[M(g)]^B_B}"'] & & \mathbb{R}_m \arrow[uu, bend right, "B"']
\end{tikzcd}
$$

quindi $[M(g)]^{B'}_{B'} = B^{-1}[M(g)]^{B}_{B} \cdot B$

$$[M(g)]^{B'}_{B} = [M(g)]^{B}_{B} \cdot B$$

$$[M(g)]^{B}_{B'} = B^{-1}[M(g)]^{B}_{B}$$

Due matrici $M_1, M_2 \in M_{n,n}(\mathbb{R})$ sono simili se $\exists B \in M_{n,n}(\mathbb{R})$ invertibile tale che $M_1 = B^{-1} \cdot M_2 \cdot B$

Esercizi:

1)

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+2y+z \\ y+z \end{pmatrix}$$

1) $f$ è lineare? Si perché è data da polinomi omogenei di primo grado

2) Scrivere $M(f)$ rispetto a $\mathbb{R}^3$ e $\mathbb{R}^2$

$$\mathcal{E}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{E}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M(f) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

3) $\operatorname{Ker} f \neq 0$?

$$\operatorname{Ker}(f) = \operatorname{Ker}(M(f))$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{GAUSS}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\operatorname{Ker} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{cases} x - z = 0 \\ y + z = 0 \end{cases} \longrightarrow \begin{cases} x = z \\ y = z \end{cases} \Rightarrow \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$$

$\ker < \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} > \rightarrow$ è diverso da 0

$\dim(\ker(f)) = 1 = null(f) \Rightarrow 2 = Rk(A) = Rk(f) = \dim(Im(f))$

2)

$w = f\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$f^{-1}(w) = \{ v \in V \mid f(v) = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \ker f = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + < \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} >$

$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad f(v) = \begin{pmatrix} x + 2y + z \\ y + z \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$\begin{cases} x + 2y + z = 7 \\ y + z = 4 \end{cases}$

Esercizio 8 $\quad T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} 2x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ x_2 + 2x_3 \end{pmatrix}$$

Scrivere $\left[M(f)\right]^{\mathcal{E}_4}_{\mathcal{E}_3}$

$$x_1\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{Ker}\, f = \mathrm{Ker}(M(f))$$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{GAUSS...}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{pmatrix} \longrightarrow x_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

PERCHE'

$$\mathrm{Ker}\, f = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \dim(\mathrm{Ker}(f)) = 1 = \mathrm{null}(f)$$

$$\mathrm{Im}(f) \left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \right\rangle$$

↳ mettiamo a matrice e troviamo Rk = 3 quindi è lin. ind. ciò vuol dire che è una base di f trova $\left[M(t)\right]_{C}^{B}$

$$t : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ x_2 + 2x_3 \end{pmatrix}$$

relazione alla quale faccio riferimento

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} b$$

$$C = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} c$$

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Puoi risolvere usando queste formule

$$C^{-1} \cdot M(t)_{\mathcal{E}}^{\mathcal{E}} \cdot B$$

$$t(b_1) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad t(b_2) = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \quad t(b_3) = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \quad t(b_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[t(b_1)\right]_C = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \left[t(b_2)\right]_C = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \left[t(b_4)\right]_C = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

$$\left[t(b_1)\right]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[M(t)\right]_C^B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$