• Proposizione:

V spazio vett.

$\mathcal{B} = \{b_1 \dots b_n\}$ base di V, $v \in V$, $n = \dim V$

$\to \exists! \ \lambda_1 \dots \lambda_n$ tale che $v = \lambda_1 \cdot v_1 + \dots \lambda_n \cdot v_n$

$(\lambda_1 \dots \lambda_n)$ coordinate di v

$$\varphi : V \longrightarrow \mathbb{R}^n \qquad \varphi(v) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$b_i \longrightarrow e_i$$

$$v = \lambda_1 \cdot b_1 \dots + \lambda_n \cdot b_n \Rightarrow \varphi(v) = \varphi(\lambda_1 \cdot b_1 \dots + \lambda_n \cdot b_n) =$$

$$\lambda_1 \varphi(b_1) + \dots \lambda_n \varphi(b_n) = \lambda_1 e_1 + \dots \lambda_n e_n = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad [v]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Come faccio a scrivere questo vettore rispetto alla base $\mathcal{B}$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$V = \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right]_{\mathcal{E}} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow [V]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

la scelta di una base determina la scrittura del vettore

V, W spazi vettoriali $\dim V = n$ $\dim W = m$

$B_V = \{ V_1 \ldots\ldots V_n \}$ base di V, $B_W = \{ W_1 \ldots W_m \}$ base di W

$f : V \longrightarrow W$ omomorfismo

$\{ f(v_1) \ldots\ldots f(v_n) \} \subseteq W$

$\forall \; f(v_j) = a_{1j} \cdot W_1 + \ldots a_{mj} \cdot W_m$

$v \in V \qquad v = \lambda_1 \cdot v_1 \dots + \lambda_n \cdot v_n \Rightarrow \varphi(v) = \varphi(\lambda_1 \cdot v_1 \dots + \lambda_n \cdot v_n) =$

$= \lambda_1 \varphi(v_1) + \dots \lambda_n \varphi(v_n) = \lambda_1 (a_{1j} \cdot w_1 + \dots a_{mj} w_m) =$

$= \lambda_n (a_{1n} \cdot w_1 + \dots a_{mn} \cdot w_m)$

$$[v]_{B_V} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \qquad [f(v)]_{B_W} = \begin{pmatrix} \lambda_1 a_n + \dots \lambda_n a_m \\ \vdots \\ \lambda a_{m+1} + \dots \lambda_n a_{mn} \end{pmatrix}$$

$$\downarrow$$

$$(a_{ij}) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$[f(v)]_{B_W} = (a_{ij}) \cdot [v]_{B_V}$$

matrice che rappresenta l'omomorfismo

$$\begin{pmatrix} a_{11} & a_{1j} & a_{1n} \\ a_{21} & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{mj} & a_{mn} \end{pmatrix} \in M_{m,n}(\mathbb{R})$$

$$\longrightarrow [M(f)]^{B_V}_{B_W}$$

ESEMPIO

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ 2x+2y \\ x+y \end{pmatrix} \qquad f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left[M(f)\right]_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$$

$$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y-3z \\ 2x \end{pmatrix} \qquad g\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad g\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$g\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$\left[M(f)\right]_{\mathcal{E}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 0 & 0 \end{pmatrix}$$

TEOREMA

V, W spazi vett.

$\dim V = n \quad \dim W = m$

$$\mathrm{Hom}(V, W) \simeq M_{m,n}(\mathbb{R})$$

$\Downarrow$

POSSO FARE ANCHE IL CONTRARIO

$$\begin{pmatrix} 4 & 8 & 2 \\ 0 & 1 & 5 \end{pmatrix}$$

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow x\begin{pmatrix} 4 \\ 0 \end{pmatrix} + y\begin{pmatrix} 8 \\ 1 \end{pmatrix} + z\begin{pmatrix} 2 \\ 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 4x + 8y + 2x \\ y + 5z \end{pmatrix}$$

$$\mathrm{Hom}(V, W) \cong M_{m,n}(\mathbb{R})$$

$$f \longrightarrow [M(f)]^{B_V}_{B_W}$$

isomorfismo dipende dalle basi scelte

$End(V) \simeq M_{n,n}(\mathbb{R})$

$id : V \longrightarrow V$

$$\left[M(id)\right]_B^B = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

• Proposizione

V, W, U spazi vett dim n, m, p

con basi finite.

$f : V \longrightarrow W$, $g : W \longrightarrow U$

siano $M(f) \in M_{m,n}(\mathbb{R})$, $M(g) \in M_{p,m}(\mathbb{R})$

e $g \circ f : V \longrightarrow U$

$$M(g \circ f) \in M_{p,n}(\mathbb{R}) = M(g) \cdot M(f)$$

- Corollario $f \in End(V)$

f invertibile $\iff M(f^{-1}) = M(f)^{-1}$

POSSIAMO USARE GAUSS JORDAN PER CAPIRE SE È INVERTIBILE

- Proposizione

$f : V \longrightarrow W$ omomorfismo

fissiamo basi di V e W e consideriamo $M(f)$

$Ker\, f \simeq Ker\, M(f)$

$Im\, f \simeq col(M(f))$

$\|$

$\langle f(b_1) \dots f(b_n) \rangle \subseteq W \simeq \mathbb{R}^n$

$B = \{b_1 \dots b_n\}$ base di V

$\dim W = n$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ base standard

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ y+z \end{pmatrix}$$

$$M(f) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\operatorname{Ker} f \cong \operatorname{Ker} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ FACCIO GAUSS-JORDAN} \dots \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{cases} x = z \\ y = -z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\operatorname{Ker} f \cong \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$ → è il ker$(M(f))$

$$\operatorname{Im}(f) \cong \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \cong \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \cong \mathbb{R}^2$$

è suriettiva

- Corollario

$f: V \longrightarrow W$ omomorfismo, fissiamo basi di V e W

$M(f)$ matrice associata

$null(f) = null(M(f))$, $Rk(M(f)) = Rk(f)$

Cambiamento di base

V, W spazi vettoriali $\dim V = n$, $\dim W = m$

$f : V \longrightarrow W$

$B_V = \{v_1 \dots v_n\}$ base di V, $B_W = \{w_1 \dots w_m\}$ base di W

$$[M(f)]_{B_W}^{B_V} \in M_{m,n}(\mathbb{R})$$

$B_V' = \{b_1 \dots b_n\}$ base di V, $B_W' = \{c_1 \dots c_m\}$ base di W

$$[M(f)]_{B_W'}^{B_V'}$$

che relazione c'è tra loro?

$v \in V$

$$[f(v)]_{B_W} = [M(f)]^{B_V}_{B_W} \cdot [v]_{B_V}$$

$$[f(v)]_{B_W'} = [M(f)]^{B_V'}_{B_W'} \cdot [v]_{B_V'}$$

poiché $\{v_1 \dots v_n\}$ base di V

$$b_j = b_{1j} \cdot v_1 + \dots \; b_{nj} \cdot v_n$$

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{m,n} \end{pmatrix} = \begin{pmatrix} [b_1]_{B_V} & [b_2]_{B_V} & [b_n]_{B_V} \end{pmatrix}$$

$$[b_j]_{B_V} = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

Un cambiamento di base corrisponde ad un endomorfismo