

# Industrial Robotics M

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# Chapter 1

## Elements of Kinematics and Dynamics of Industrial Robots

### 1.1 Introduction

A manipulator may be mechanically described as the interconnection of rigid bodies (*links*) through kinematic pairs (*joints*). In this course we deal with open chains of links (serial configuration). The first link is the *base*, while the last link is the *end effector*. We will consider the links to be rigid bodies for the sake of simplicity.

A joint is an element that constrains one or more relative motion directions between two links. In robotics only two types of joints are used: rotoidal joints and prismatic joints.

#### 1.1.1 Degrees of Freedom of a Manipulator

Each joint is characterised by the number of independent motion directions allowed between two consecutive links. This number defines the *degrees of freedom* of the joint. Rotoidal and prismatic joints have 1 degree of freedom (dof) of the joint. Rotoidal and prismatic joints have 1 degree of freedom (dof). In place of spherical joints, 3 rotoidal joints are used, forming a *spherical wrist*. The spherical wrist is typically at the end of the kinematic chain. If a joint has  $k$  dof, then the relative configuration between two rigid bodies may be expressed as a function of  $k$  variables  $q_1, q_2, \dots, q_k$ , called *joint variables*.

#### 1.1.2 Joint Space and Work Space

Joint Space: Let us stack all the joint variables in a vector

$$q = [q_1 \quad q_2 \quad \dots \quad q_n]^T$$

with  $q \in \mathcal{Q} \subset \mathbb{R}^{N_{dof}}$ . The set  $\mathcal{Q}$  is called *joint space*

Work Space: The work space is a subset of the Euclidean space  $\mathbb{E}$  in which the robot executes its tasks. It is the set of all the points that the mechanical structure may assume, and in general is a 3D subset of  $\mathbb{E}$ . Each point of the work space is indicated by a vector  $x$  of proper dimension,  $x \in \mathbb{R}^y$ , where usually  $y = \{3, 6\}$

configuration of a manipulator: It takes into account both the position and orientation of a reference frame fixed to the manipulator extremity (end effector). Then (locally):

- $x \in \mathbb{R}^3$  in a plane
- $x \in \mathbb{R}^6$  in space

Classification of manipulators: if  $\mathbb{R}^n$  is the joint space and  $\mathbb{R}^m$  is the work space:

- $n = m$ : "normal" cases
- $n < m$ : *defective* manipulators
- $n > m$ : *redundant* manipulators

### 1.1.3 Types of Robotic Manipulators

Common kinematic structures for robotic manipulators:

- Cartesian robots: 3 perpendicular prismatic joints. Simplest robot structure. Structure can be made to be very robust and therefore very large. Usually meant to move large payloads. Simple to control (1 motor for each direction of motion). Simple mapping between joint space and work space.
- Antropomorphic robots (majority): 3 rotational joints, 1 with a vertical axis and 2 with horizontal axes. More dexterous, harder to control, more complex coupling between joint space and work space. Errors on each motor stack in the final motion
- SCARA robots: Selective Compliance Assembly Robot Arm. Rigid along z, more compliant along x and y.
- Cylindrical and spherical robots: not used
- Spherical wrists: typically attached at the end of Cartesian, Antropomorphic or SCARA bots.

The first 3 joints define the basic kinematic structure

## 1.2 Kinematic model of a manipulator

x is made of 3 elements determining position and 3 determining orientation. These 3 represent angles, that may be Roll-Pitch-yaw, or Euler angles (Z-Y-Z Euler angles are used in robotics)

### 1.2.1 Forward kinematics

mapping of joint space variables into work space variables

$$x = f(q) \quad q \in \mathbb{R}^n, x \in \mathbb{R}^m$$

It is possible to define different kinematic models for a robotic manipulator, i.e. different  $f(q)$  functions. They are however mathematically equivalent

### Homogenous transformation matrix

the configuration of a rigid body  $b_i$  in a 3D space can be described by a *homogenous transformation matrix* between a reference frame  $FF_i$  and the base frame  $FF_0$  thus constructed:

$${}^0T_i = \begin{pmatrix} {}^0R_i & {}^0o_i \\ \mathbf{0} & 1 \end{pmatrix}$$

where  ${}^0R_i$  is a rotation matrix belonging to  $SO(3)$ , and  ${}^0o_i \in \mathbb{R}^3$  is a position vector. It can be noted that the determinant of the homogenous transformation matrix is also equal to 1.

For computational reasons, a new position vector  $p = [p_x, p_y, p_z, 1]^T$  is defined. This way, the position  ${}^0p$  of a generic point  ${}^1p$  of the rigid body can be obtained as:

$${}^0p = {}^0T_1 {}^1p$$

or, with some abuse of notation:

$$\begin{bmatrix} {}^0p \\ 1 \end{bmatrix} = {}^0T_1 \begin{bmatrix} {}^1p \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0R_1 {}^1p + {}^0o_1 \\ 1 \end{bmatrix}$$

in general

$${}^0T_n = {}^0T_1 {}^1T_2 \dots {}^{n-1}T_n$$

due to homogenous transformation being a linear operation.

### Denavit-Hartenberg convention

- each link is numbered, from 0 to  $n$ , so that it is uniquely identified in the mechanical chain:  $L_0, L_1, \dots, L_n$ . The base link is conventionally identified as  $L_0$ , the distal link as  $L_n$ . A manipulator with  $n + 1$  links has  $n$  joints.
- Joints are numbered from 1 to  $n$ .  $J_i$  connects links  $L_{i-1}$  and  $L_i$
- Frames  $\mathcal{F}_i$  are associated to each joint, according to the Denavit-Hartenberg procedure, and matrices  ${}^{i-1}T_i$  are computed:
  - all the  $z$  axes are directed along the direction of motion
  - axis  $x + 1$  intersects and is perpendicular to axis  $z_i$
- The Denavit-Hartenberg procedure requires 4 parameters for each link  $(d_i, \theta_i, a_i, \alpha_i)$

This procedure provides a 4x4 matrix, not a vector, for  $x$ . The vector  $x$  can be computed as a function of the homogeneous transformation matrix.

### 1.2.2 Inverse kinematics

mapping of work space variables into joint space variables

$$q = g(x) = f^{-1}(x) \quad q \in \mathbb{R}^n, x \in \mathbb{R}^m$$

A standard procedure to obtain the inverse kinematic model does not exist. Moreover, we may have

- no solution (if  $x$  does not belong to the workspace)
- a finite set of solutions
- infinite solutions

A *closed form* solution is preferable to numerical procedures (better computation, easier to select a particular solution in the set of solutions)

A closed form solution to the inverse kinematic problem may be obtained, if it exists, by trying one of these approaches:

- Algebraic approach: elaboration of the kinematic equations in order to obtain a set of 'simple' equations that can be solved in the unknown joint variables  $q$
- Geometric approach: based on considerations dependent on the geometric structure of the manipulator

#### Algebraic approach

The values of the homogeneous transformation matrix are known, and the homogeneous transformation matrix can be written as a function of joint variables. This yields 12 scalar equations (usually more than needed). Hopefully there are a sufficient number of equations that can be solved easily enough. As an additional resource, one can perform algebraic manipulations on the involved matrices to try and find a better set of equations. This procedure is in general not so simple

#### Geometric approach

Also called Pieper approach.

For several kinematic structures it is possible to apply the *kinematic decoupling* principle, that allows to decompose the overall 6Dof problem into two 3 Dof sub-problems:

- inverse kinematic problem for position
- inverse kinematic problem for orientation

Sufficient condition to solve the inverse kinematic problem for a 6DoF manipulator:

- 3 consecutive rotational joints whose axes intersect in a point (spherical wrist: notice that the position of the center of rotation of the spherical wrist does not depend on the joint variables of the spherical wrist but rather on the other joints)
- 3 consecutive rotation joints with parallel axes

### 1.2.3 Differential kinematics - the Jacobian

Other relations of interest in robotics are those between:

- end-effector velocity and joint velocities

$$\begin{pmatrix} v \\ \omega \end{pmatrix} \Longleftrightarrow \dot{q}$$

- force applied on the environment by the manipulator and corresponding joint torques:

$$\begin{pmatrix} f \\ n \end{pmatrix} \Longleftrightarrow \tau$$

These relations are based on a matrix known as the *Jacobian of the manipulator*. The Jacobian is also used for:

- studying the singularities of the mechanical structures
- defining numerical algorithms to solve the inverse kinematics
- studying manipulability properties

The six-dimensional velocity vector (twist) in the work-space is defined with three linear and three rotational components

$$\dot{x} = \begin{pmatrix} v \\ \omega \end{pmatrix}$$

Two expressions of the Jacobian matrix can be defined:

- the analytic Jacobian, used when the rotational component in  $\dot{x}$  is defined as  $\dot{\gamma}$ , the time-derivative of a triple of orientation variables
- the geometric Jacobian, used when the rotational component in  $\dot{x}$  is defined as  $\omega$ , the rotational velocity vector.

It is possible to relate the two expressions by means of a proper matrix  $T(\gamma)$

$$J_G(q) = T(\gamma)J_A(q) = \begin{bmatrix} I_3 & 0 \\ 0 & \hat{T}(\gamma) \end{bmatrix} J_A(q)$$

#### Analytic Jacobian

The analytic expression of the Jacobian is obtained by differentiating the six-dimensional vector  $x = f(q) = [p^T, \gamma^T]^T$  containing the position ( $p$ ) and orientation ( $\gamma$ ) of the end-effector in the work-space wrt the base frame FF0; usually  $\gamma$  are the Euler or RPY angles

By differentiating  $f(q)$  we get:

$$dx = \frac{\partial f(q)}{\partial q} dq = J(q) dq \quad (\dot{x} = J(q)\dot{q})$$

where the  $m \times n$  matrix

$$J(q) = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{pmatrix}, \quad J(q) \in \mathbb{R}^{m \times n}$$

Is called the *Jacobian matrix*

We have that the relationship between  $\dot{x}$  and  $\dot{q}$  is linear. The Jacobian can be conceptually split in two blocks, the first concerning the linear part of the velocity, the second concerning the angular part:

$$\dot{x}_A = J_A(q)\dot{q} = \begin{bmatrix} \dot{p} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} J_p \\ J_\gamma \end{bmatrix} \dot{q}$$

### Geometric expression

In practice, the Jacobian is always computed in its geometric expression as it is much simpler. The geometric expression of the Jacobian is obtained considering as rotational components of the velocity vector  $\dot{x}$  the rotational vector  $\omega$  and not the derivative of a triple of angles  $\gamma$

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{\gamma} \end{bmatrix} \iff \dot{x} = \begin{bmatrix} \dot{p} \\ \omega \end{bmatrix}$$

$$\dot{x}_G = J_G(1)\dot{q} = \dot{x} = \begin{bmatrix} \dot{p} \\ \omega \end{bmatrix} = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \dot{q} = \begin{bmatrix} J_{v_1} & J_{v_2} & \dots & J_{v_n} \\ J_{\omega_1} & J_{\omega_2} & \dots & J_{\omega_n} \end{bmatrix}$$

this leads to

$$v = J_{v_1}\dot{q}_1 + J_{v_2}\dot{q}_2 + \dots + J_{v_n}\dot{q}_n$$

$$\omega = J_{\omega_1}\dot{q}_1 + J_{\omega_2}\dot{q}_2 + \dots + J_{\omega_n}\dot{q}_n$$

it is possible to show, applying the superimposition principle by considering only the  $i$ -th joint to be moving, that the generic  $i$ -th column of the geometric Jacobian is defined as

$$\begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix} = \begin{bmatrix} {}^0z_{i-1} \times ({}^0p_n - {}^0p_{i-1}) \\ {}^0z_{i-1} \end{bmatrix} \quad \text{rotational joint}$$

$$\begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix} = \begin{bmatrix} {}^0z_{i-1} \\ 0 \end{bmatrix} \quad \text{rotational joint}$$

Once the forward kinematics of the manipulator is known these expressions are simple to obtain as the vectors  ${}^0p_i$  and  ${}^0z_i$  are obtained from the matrices  ${}^0T_i$ . This does not represent a computational burden with respect to the computation of the forward kinematic model.

### Force domain

By exploiting the *virtual work principle*, a mapping similar to that in the velocity domain can be obtained for the force domain.

Because work is invariant wrt the frame in which it is computed:

$$w^T dx = \tau^T dq$$

where  $w = (f^T n^T)^T$  is a 6-dimensional vector (*wrench*) collecting the linear forces  $f$  and the torques  $n$  applied to the manipulator.  $\tau$  is the  $n$ -dimensional vector of the *joint forces/torques* since

$$dx = J(q)dq$$

we obtain by substituting

$$\tau = J^T(q)w$$

### Singularities

When the Jacobian is rank deficient, there are directions in  $\mathbb{R}^6$  that the robot cannot move in. Furthermore, limited velocities of the end-effector may correspond to infinite velocities in the joint space. There is no well defined solution to the inverse kinematic problem: either no solution or infinite solutions. Singularities typically correspond to positions at the border of the workspace.

### Inverse differential kinematics

The forward mapping between joint and work-space is given by

$$\dot{x} = J(q)\dot{q}$$

Then, the inverse mapping is given as the solution of a linear algebraic equation

- $m = n$ : if the manipulator is not in a singular configuration, it is possible to use the inverse of the jacobian:

$$\dot{q} = J^{-1}(q)\dot{x}$$

- $m \neq n$ : the Moore-Penrose pseudoinverse of the Jacobian is used:

$$\dot{q} = J^\dagger(q)\dot{x}$$

if  $m < n$  (right pseudoinverse):

$$J^\dagger = J^T(JJ^T)^{-1}$$

$$J^\dagger = I_m$$

if  $m > n$  (left pseudoinverse):

$$J^\dagger = (J^T J)^{-1} J^T$$

$$J^\dagger = I_n$$

When the joint space is larger than the work space ( $n > m$ ), the solution of  $\dot{x} = J\dot{q}$ :

- $\text{rank}(J) = \min(m, n) = m \implies \mathcal{R}(J) = \mathbb{R}^m$
- $\forall \dot{x}, \exists \dot{q}$  such that  $\dot{x} = J\dot{q}$  more than one exists
- $\dot{q} = J^\dagger \dot{x} + \dot{q}_N \implies \dot{x} = J(J^\dagger \dot{x} + \dot{q}_N) = \dot{x} \quad \forall \dot{q}_N \in \mathcal{N}(J)$   
 $\implies \dot{q} = J^\dagger \dot{x} + (I - J^\dagger J)y$  is a general expression of the solution  
 $(I - J^\dagger J)$  is a base of  $\mathcal{N}(J)$
- $\dot{q} = J^\dagger \dot{x}$  has *minimum norm*

### Inverse kinematics with the Jacobian

The Jacobian may be used to solve the inverse kinematic problem if a closed form solution is not available. A method could be to compute

$$q_{k+1} = q_k + J^{-1}(q_k)v_k dt$$

where a numerical integration of the position is performed. Unfortunately, this method is subject to numerical drifts and initialization problems, and it is therefore difficult to obtain the desired solution.

An alternative method is based on a feedback loop, by defining an error in the work-space. Given

$$e = x_d - x$$

by differentiating, we obtain

$$\dot{e} = \dot{x}_d - J(q)\dot{q}$$

it is necessary to define  $\dot{q}$  in such a way that the error  $e$  converges to 0

### Algorithm 1

if

$$\dot{q} = J^{-1}(q)(\dot{x}_d + Ke)$$

with  $K = K^T > 0$ , then

$$\dot{e} = \dot{x}_d - J J^{-1}(\dot{x}_d + Ke)$$

$$\dot{e} = -Ke$$

Typically  $K$  is chosen as a diagonal matrix, leading to

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} K_1 & \dots & & \\ \vdots & K_2 & & \\ & & \ddots & \\ & & & K_n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Which is completely decoupled and converging to 0.



**Algorithm 2**

We use a Lyapunov technique:

Given  $\dot{x}_d = 0$ , if

$$\dot{q} = J^T(q)Ke$$

with  $K = K^T > 0$ , then

$$\begin{aligned} V(e) = \frac{1}{2}e^T Ke &\implies \dot{V}(e) = e^T K\dot{e} = e^T K(\dot{x}_d - J\dot{q}) \\ &= -e^T KJ(q)\dot{q} \end{aligned}$$

we choose  $\dot{q} = J^T Ke$ , leading to:

$$\dot{V}(e) = -e^T KJJ^T Ke \leq 0$$

**1.2.4 Manipulability Measures**

The Jacobian may be used to evaluate the achievable performances of a manipulator, in terms of velocities and applicable forces, through the *manipulability ellipsoids*

**Velocity manipulability ellipsoids**

Given a sphere with unit radius in the joint velocity space,  $\dot{q}^T \dot{q} = 1$ , that may be considered as a "cost" (energy input to the manipulator), we want to obtain its equivalent expression in the work-space. From  $\dot{x} = J(q)\dot{q}$  we obtain:

$$\begin{aligned} \dot{q} &= J^\dagger \dot{x} \\ \implies \dot{x} J^{\dagger T} J^\dagger \dot{x} &= 1 \\ \implies \dot{x}^T (JJ^T)^\dagger \dot{x} &= 1 \end{aligned}$$

which is an ellipsoid in the work-space  $\mathbb{R}^m$

- the principal axes are directed along the eigenvectors of  $JJ^T$
- the length of the principal axes are given by the singular values of J,  $\sigma_i = \sqrt{\lambda_i(JJ^T)}$

**Force manipulability ellipsoids**

Let us now consider the equation  $\tau^T \tau = 1$  from which we obtain:

$$w^T JJ^T w = 1$$

This equation describes an ellipsoid in the  $\mathbb{R}^m$  force space

- the principal axes are directed along the eigenvectors of  $JJ^T$
- the length of the principal axes are given by the inverse of the singular values of J,  $\sigma_i = \sqrt{\lambda_i(JJ^T)}$

This result confirms the duality of the velocity/force spaces: along the directions in which high velocity performances are possible, low force performances are achievable and viceversa



## Chapter 2

# Dynamic Model of Robotic Manipulators

Dynamics of a robot manipulator: analysis of the relationship between the applied forces/torques and the resulting motion of a robot. For the dynamics, it is possible to define two "models":

- Direct model: given the forces/torques applied at joints, the joint position and velocity, compute the acceleration

$$\ddot{q} = f(q, \dot{q}, \tau)$$

and then

$$\dot{q} = \int \ddot{q} dt \quad q = \int \dot{q} dt$$

- Inverse model: given the desired joint acceleration, velocity and position, compute the corresponding force/torque to be applied

$$\tau = f^{-1}(\ddot{q}, \dot{q}, q) = g(\ddot{q}, \dot{q}, q)$$

In deriving the dynamic model of a robot, we refer to a chain of rigid bodies mutually, and ideally, connected among them. The dynamic model of a robot is used for:

- simulation: testing motion behaviour without resorting to physical experiments
- analysis and synthesis of control laws
- analysis of the structural properties of a manipulator in the design phase

### 2.1 Euler-Lagrange model

The dynamic model obtained through this method is simpler and more intuitive, and also better suited to understanding the effects of changes in the mechanical parameters. The links are considered altogether, and the model is obtained analytically. Drawbacks: the model is obtained starting from the kinetic and potential energies (non intuitive) and the model is not computationally efficient.

Lagrange coordinates: independent variables used to describe the position of rigid bodies in space. For the same physical system, several choices of Lagrangian coordinates are usually possible. In robotics, we use *joint variables*  $q_1, q_2, \dots, q_n$ . Input torques should be defined according to the chosen Lagrangian coordinates  $\implies$  joint torques  $\tau_1, \tau_2, \dots, \tau_n$

From physics, we know it is possible to define:

- the Kinetic Energy function  $K(q, \dot{q})$
- the Potential Energy function  $P(q)$

And therefore the Lagrangian function

$$\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

The Euler-Lagrange equations for a system described by  $n$  Lagrange coordinates  $q_i$  are

$$\psi_i = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} \quad i = 1, \dots, n$$

Being  $\psi_i$  the non-conservative (external or dissipative) generalized forces performing work on  $q_i$ . In robotics, we consider:

- $\tau_i$  joint actuator torque
- $[J^T F_c]_i$  term due to external (contact) forces
- $d_{ii} \dot{q}_i$  joint friction torque

Therefore:

$$\psi_i = \tau_i + [J^T F_c]_i - d_{ii} \dot{q}_i$$

It can be noted that

$$K = \sum_{i=1}^n K_i \quad P = \sum_{i=1}^n n P_i$$

### 2.1.1 obtained model

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = \tau$$

Non-linear wrt the state. It can be made linear wrt a set of parameters  $\alpha$

### 2.1.2 Identification of the dynamic parameters

From:

$$Y(q, \dot{q}, \ddot{q})\alpha = \tau$$

by measuring along a proper trajectory the quantities  $q, \dot{q}, \ddot{q}, \tau$  ( $N > p$  samples) one obtains:

$$\begin{aligned} \tau_0 &= Y_0 \alpha \\ Y_0^T \tau_0 &= Y_0^T Y_0 \alpha \\ \alpha &= (Y_0^T Y_0)^{-1} Y_0^T \tau_0 = Y_0^\dagger \tau_0 \end{aligned}$$

### 2.1.3 Dynamic model in simulink

When only one of the two of the joints has friction energy still leaks out of the system because the system is coupled so movement of each link affects the other link causing the energy to be lost.

### 2.1.4 Linearization of the Euler-Lagrange model

By neglecting friction effects ( $D = 0$ ), and considering the vector  $C(q, \dot{q})\dot{q} = c(q, \dot{q})$ , the non linear dynamic model of a robot manipulator is:

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$$

It is possible to obtain a linear dynamic model which is valid only locally around a given operative condition (not used, very bad, useless, stupid, why even bother):

- linearization around a (constant) equilibrium state
- linearization along a (nominal, time-varying) equilibrium trajectory

skipped because excessively useless and pointless

### 2.1.5 Symbolic computation of the dynamic model

The Euler-Lagrange formulation of the dynamic model of a robot arm can be expressed in a form suitable to be implemented in a symbolic package (Mathematica or Matlab Symbolic toolbox) in order to obtain, given the kinematic and mechanical parameters of each link/joint, the model expressed in a symbolic form. Parameters needed: DH parameters, masses, inertia matrices, position of centers of mass.

## 2.2 Newton-Euler model

Based on a computationally efficient recursive technique that exploits the serial structure of an industrial manipulator. On the other hand, the model is not expressed in closed form. The model is based on the balance of forces and torques acting on each link.

The dynamic equations are written separately for each link and a *recursive formulation* is obtained: forward recursion (from Link 0 to Link  $n$ ) for the computation of the "propagation" of accelerations and velocities, backward recursion (from Link  $n$  to Link 0) for the computation of the "propagation" of forces and torques. The basic equations are:

- Newton's second law of motion: (conservation of momentum  $mv$ )

$$f = \frac{d(mv)}{dt}$$

where  $m$  is mass,  $v$  is the linear velocity,  $f$  the sum of all the applied external forces and  $mv$  the momentum. Since in robotics the masses of the links are constant (and considered as concentrated in the center of mass) we have the well known Newton equation  $f = ma = m\ddot{p}_C$

- Euler's law of motion (conservation of the angular momentum  $L = I\omega$ )

$$\ni = \frac{d({}^0I^0\omega)}{dt}$$

where  ${}^0I$  is the moment of inertia (rotational inertia) expressed in an inertial frame FF0 placed in the center of mass,  ${}^0\omega$  the rotational velocity,  $\tau$  the sum of the torques applied to the rigid body.

Let us consider the generic  $i$ -th link of the manipulator. Define, in an inertial frame, the following parameters:

NOTE: write them at some point please (slide 102)	$m_i$	mass of the link	$\ddot{p}_{C_i}$	acceleration of $C_i$
	$I_i$	rotational inertia	$\ddot{\omega}_i$	acceleration of $C_i$
	$R_{i-1,C_i}$	vector from FF $i-1$ to the CoM $C_i$	$g_i$	gravity acceleration
	$r_{i,C_i}$	vector from FF $i$ to the CoM $C_i$	$f_i$	force applied on $C_i$
	$r_{i-1,i}$	vector from FF $i-1$ to FF $i$	$-f_{i+1}$	force applied on $C_i$
	$\dot{p}_{C_i}$	linear velocity of $C_i$	$\nu_i$	torque applied on $C_i$
	$\dot{p}_i$	linear velocity of FF $i$	$-\nu_{i+1}$	torque applied on $C_i$
	$\omega_i$	rotational velocity of the link		

Therefore, for each link we have:

Newton law:

$$f_i - f_{i+1} + m_i g = m_i \ddot{p}_{C_i}$$

Euler law:

$$\nu_i + f_i \times r_{i-1,C_i} - \nu_{i+1} - f_{i+1} \times r_{i,C_i} = \frac{d}{dt}(I_i \omega_i)$$

The Euler law results:

$$\begin{aligned} \nu_i + f_i \times r_{i-1,C_i} - \nu_{i+1} - f_{i+1} \times r_{i,C_i} &= \frac{d}{dt}(I_i \omega_i) = I_i \dot{\omega}_i + \frac{d}{dt}(R_i^i I_i R_i^T) \omega_i \\ &= I_i \dot{\omega}_i + (\dot{R}_i^i I_i R_i^T + R_i^i I_i \dot{R}_i^T) \omega_i \\ &= I_i \dot{\omega}_i + S(\omega_i)(R_i^i I_i R_i^T(\omega_i) + R_i^i I_i R_i^T S(\omega_i) \omega_i) \\ &= I_i \dot{\omega}_i + \omega_i \times (I_i \omega_i) \quad [S(\omega_i) \omega_i = \omega_i \times \omega_i = 0] \end{aligned}$$

slide 104-105-106-107 couldn't keep up

After forward and backward recursion, we obtain forces/torques applied on the links. We require torques to be applied by the motors. The generalized force  $\tau_i$  applied on the  $i$ -th joint by the link  $L_{i-1}$  can be computed as the projection of  $f_i$  or  $\nu_i$  along the  $z_{i-1}$  axis. We have:

$$\begin{aligned} \tau_i &= \nu_i^T z_{i-1} \\ \tau_i &= f_i^T z_{i-1} \end{aligned}$$

Friction may be considered at this point, for example

$$\begin{aligned} \tau_i &= \nu_i^T z_{i-1} + \mu_i \dot{q}_i \\ \tau_i &= f_i^T z_{i-1} + \mu_i \dot{q}_i \end{aligned}$$

Overall algorithm

1. given joint positions, velocities and accelerations link velocities and accelerations are computed through forward recursion
2. forces and torques are computed through backward recursion
3. joint torques are computed through force/torque projection

This algorithm allows for a variety of useful outputs: slide 113

## 2.3 Dynamic model in the work-space

It may be of interest to obtain the dynamic model in the work-space and not in the joint-space. For the sake of simplicity, let us consider a 6 dof manipulator, not in a singular configuration. Since  $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = \tau + J(q)^T F_a$ , then

$$\ddot{q} = M^{-1}(q) [\tau + J^T(q)F_a - C(q, \dot{q})\dot{q} - D\dot{q} - g(q)]$$

Moreover, by deriving  $\dot{x} = J(q)\dot{q}$  with respect to time, we have:

$$\ddot{x} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$$

Therefore

$$\hat{M}(x)\ddot{x} + \hat{C}(c, \dot{x})\dot{x} + \hat{D}\dot{x} + \hat{g}(x) = F + F_a \quad \tau = J^T F$$

with

$$\begin{aligned} \hat{M} &= (JM^{-1}J^T)^{-1} \\ \hat{C} &= \hat{M} \end{aligned}$$

## Chapter 3

# Control of robot manipulators

### 3.1 Nominal case

All the parameters are known exactly and they are assumed to be constant in time.

Control problem: definition of the input signals for the joints (e.g. torques or actuator input voltages) in order to achieve a predefined behaviour for the manipulator. The achievable performances can vary due to:

- the many control techniques available to solve such a problem
- the hardware used to implement the control algorithms
- the mechanical configuration of the robot (anthropomorphic, cartesia, ...)

The robot performances are mainly influenced by the mechanical design and by the actuation system. For example, gearboxes "decouple" the load from the motor (disturbances on the load are diminished towards the motor through the gearbox), however they usually introduce non linear effects such as dead-zones, friction, elasticity,...; Direct Drive motors (can provide high torque with low speed without reduction gears) ensure better performances and do not introduce non-linearities in the transmission chain; however a more relevant dynamic coupling between joints is present.

Control problems:

- Control of the robot's motion (position control schemes)
  - joint-space control
  - workspace control
- Control of the interaction with the workspace (force control schemes)

Control schemes:

- Decentralized (or independent) control schemes (SISO)
- Centralized control schemes (MIMO)

Dynamic model of a manipulator:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(\dot{q} + g(q)) = \tau + J^T(q)F_a$$

Control problem: define the generalized forces  $\tau$  to be applied to the joints in order to obtain a desired trajectory  $q_d(t)$

In non-linear control schemes it is best to choose a sampling frequency as large as possible

### 3.2 Decentralized position control

Actuators: velocities  $\omega_m(t)$ , positions  $q_m$ , torques  $\tau_m(t)$

Gearboxes: reduction ratio  $K_r$

Joints: velocities  $\omega(t)$ , positions  $q$ , generalized forces  $\tau(t)$

$$K_r\omega = \omega_m \quad K_rq = q_m \quad \tau_m = K_r^{-1}\tau$$

$K_r$  is a diagonal matrix with elements  $\gg 1$  we can observe that:

$$\begin{aligned} q &= K_r^{-1} q_m \\ \dot{q} &= K_r^{-1} \dot{q}_m \\ \ddot{q} &= K_r^{-1} \ddot{q}_m \\ \tau &= K_r \tau_m \end{aligned}$$

we get

$$M(q)K_r^{-1}\ddot{q}_m + CK_r^{-1}\dot{q}_m + DK_r^{-1}\dot{q}_m + g(q_m) = K_r\tau_m$$

and therefore

$$K_r^{-1}M(q)K_r^{-1}\ddot{q}_m + K_r^{-1}CK_r^{-1}\dot{q}_m + K_r^{-1}DK_r^{-1}\dot{q}_m + K_r^{-1}g(q_m) = K_r\tau_m$$

The diagonal of the matrix  $M(q)$  is composed by two kinds of elements

- inertia terms that do not depend on the robot's configuration
- terms that depend on the robot's configuration

Therefore:

$$M(q) = \bar{M} + \Delta M(q)$$

where  $\bar{M}$  is a diagonal matrix with constant elements (i.e. the mean values of the joints inertia). From the robot dynamic model it follows that

$$\tau_m = (K_r^{-1}\bar{M}K_r^{-1})\ddot{q}_m + D_m\dot{q}_m + d$$

having renamed

$$\begin{aligned} D_m &= K_r^{-1}DK_r^{-1} \\ d &= (K_r^{-1}\Delta M(q)K_r^{-1})\ddot{q}_m + (K_r^{-1}C(q, \dot{q})K_r^{-1})\dot{q}_m + K_r^{-1}g(q_m) \end{aligned}$$

$D_m$  is a matrix collecting the motors friction coefficients,  $d$  is a term that can be considered as a disturbance that collects nonlinearities and coupling effects. The disturbance terms are divided by  $K_r^2$  and therefore greatly reduced (the gravity is only divided by  $K_r$ ). Based on this we can design a linear controller, with the hope that  $d$  may be small enough.  $d$  grows with the speed and acceleration of the robot.

How to reduce the effects of the disturbance  $d$ :

- high gain
- integral action to annihilate the steady state error (e.g. the effect of the gravitational term)

Possible solution: PI controller

$$C(s) = K_c \frac{1 + sT_c}{s}$$

### 3.2.1 possible control scheme

A cascade configuration can be employed. If inertia is constant, current control implies acceleration control as torque is directly proportional to acceleration.

In robotics (and in general in motion control systems), feedforward control actions are usually adopted with very positive results:

- velocity ff action
- acceleration (torque) ff action

With ff actions, it is possible to compensate (partially) also some parts of the "disturbance" term  $d(t)$ . Since  $d(t)$  is known from the dynamic model, given the desired position, velocity and acceleration signals it is possible to compute a ff control action to compensate for  $d(t)$ :

$$d_d = (K_r^{-1}\Delta M(q_d)K_r^{-1})\ddot{q}_{md} + (K_r^{-1}C(q_d, \dot{q}_d)K_r^{-1})\dot{q}_{md} + K_r^{-1}g(q_{md})$$

It is to be noted that in any case  $d \neq d_d$

## 3.3 Centralized control

We introduce two centralized control schemes:



### 3.3.1 PD + gravity compensation

Given a desired reference configuration  $q_d$ , the goal is to define a controller ensuring the global asymptotic stability of the nonlinear dynamical system described by:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(\dot{q} + g(q)) = \tau + J^T(q)F_a$$

For this purpose, let us define the error as

$$\tilde{q} = q_d - q$$

and consider a dynamic system with state  $x$  given by

$$x = \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix}$$

The *direct Lyapunov method* is exploited for the control law definition. Let us consider the following candidate Lyapunov function:

$$V(x) = V(\tilde{q}, \dot{\tilde{q}}) = \frac{1}{2}\dot{\tilde{q}}^T M(q)\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^T K_P \tilde{q} > 0 \quad \forall \tilde{q}, \dot{\tilde{q}} \neq 0$$

Where  $K_P$  is a square  $(n \times n)$  positive definite matrix. Usually, it is chosen as a diagonal matrix. Function  $V(\tilde{q}, \dot{\tilde{q}})$  is composed by two terms, the first expressing the kinetic energy of the system, the second can be interpreted as elastic energy stored by springs with stiffness  $K_P$ . These springs are a physical interpretation of the proportional control.

We compute the time derivative of the candidate Lyapunov function:

$$\begin{aligned} \dot{V}(x) &= \dot{\tilde{q}}^T M\ddot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T \dot{M}\dot{\tilde{q}} - \dot{\tilde{q}}^T K_P \tilde{q} \\ &= \dot{\tilde{q}}^T (u - C\dot{\tilde{q}} - D\dot{\tilde{q}} - g(q)) + \frac{1}{2}\dot{\tilde{q}}^T \dot{M}\dot{\tilde{q}} - \dot{\tilde{q}}^T K_P \tilde{q} \\ &= \frac{1}{2}\dot{\tilde{q}}^T (\dot{M} - 2C)\dot{\tilde{q}} - \dot{\tilde{q}}^T D\dot{\tilde{q}} + \dot{\tilde{q}}^T (u - g(q) - K_P \tilde{q}) \end{aligned}$$

note that  $\frac{1}{2}\dot{\tilde{q}}^T (\dot{M} - 2C)\dot{\tilde{q}} = 0$  due to the choice of  $C$  (Christoffel symbols). Also, by choosing

$$u = g(q) + K_P \tilde{q} + K_D \dot{\tilde{q}}$$

where  $K_D$  is a positive definite matrix, we obtain a proportional and derivative control with gravity compensation, and we have:

$$\dot{V}(x) = -\dot{\tilde{q}}^T D\dot{\tilde{q}} - \dot{\tilde{q}}^T K_D \dot{\tilde{q}}$$

note that the term introduced by the derivative control acts like a friction, bringing the energy of the system towards 0 faster.

The PD control scheme can be interpreted as a spring and a damper.

$$\dot{V}(x) = -\dot{\tilde{q}}^T (D + K_D)\dot{\tilde{q}}$$

So far, we have guaranteed that the system stops, but we do not know where.

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(\dot{q} + g(q)) = K_P \tilde{q} - K_D \dot{\tilde{q}} + g(q)$$

we observe that all the terms other than the one with  $\tilde{q}$ , in steady state cancel out:

$$K_P \tilde{q} = 0$$

That is, as  $K_P$  is positive definite

$$q = q_d$$

A perfect compensation of the gravity term is necessary, otherwise it is not possible to guarantee the stability of the system.

The "PD+g(q)" approach can be extended to the position control in work-space.

### 3.3.2 Inverse dynamics control

The manipulator is considered as a nonlinear MIMO system described by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

or in short:

$$M(q)\ddot{q} + n(q, \dot{q}) = u$$

Two assumptions are necessary:

- the model is linear in the control input:  $\dot{x} = f(x) + \alpha u(t)$
- The matrix  $M(q)$  is invertible for any configuration for the manipulator:  $\exists f^{-1}(x)$

Let us choose the control input  $u$  as:

$$u = M(q)y + n(q, \dot{q})$$

it follows that

$$M\ddot{q} + n = My + n$$

and thus

$$\ddot{q} = y$$

where  $y$  is the new input of the system. The system results to be completely decoupled and linear. However, the resulting system has two poles in the origin and is therefore unstable. In order to stabilize the system  $y$  can be chosen as:

$$y = -K_D\dot{q} - K_Pq + r$$

This means adding proportional and derivative control. From  $\ddot{q} = y$  it follows:

$$\ddot{q} + K_D\dot{q} + K_Pq = r$$

Which is asymptotically stable as long as  $K_P$  and  $K_D$  are positive-definite. If matrices  $K_P, K_D$  are diagonal matrices defined by

$$K_P = \text{diag}\{\omega_n^2\} \quad K_D = \text{diag}\{2\delta\omega_n\}$$

the dynamics of the  $i$ -th component is characterized by the specified natural frequency and damping coefficient. A predefined trajectory  $q_d(t)$  can be tracked by defining

$$r = \ddot{q}_d + K_D\dot{q}_d + K_Pq_d$$

Then the dynamics of the tracking error is

$$\ddot{\tilde{q}} + K_D\dot{\tilde{q}} + K_P\tilde{q} = 0$$

which converges to zero.

## Chapter 4

# Advanced Position Control Schemes

### 4.1 Robust Control

Aims at compensating directly possible errors deriving from model errors and/or external disturbances. Some of the most known robust control techniques:

- H-infinity loop shaping
- Loop transfer recovery (LQG,LTR)
- Sliding mode control (SMC), a variation of variable structure control (VSS)

#### 4.1.1 Variable Structure Control

##### Example

Consider a second order dynamic system described by

$$\ddot{x} = -\psi x \quad \text{with} \quad \psi = \{a_1^2, a_2^2\} \quad (a_1^2 > 1 > a_2^2)$$

Assuming that  $\psi$  is a control input, by using the commutation law:

$$\psi = \begin{cases} a_1^2 & \text{if } x\dot{x} > 0 \\ a_2^2 & \text{if } x\dot{x} < 0 \end{cases}$$

An asymptotically stable system is obtained

##### Example

Consider the system

$$\ddot{x} - \xi \dot{x} + \psi x = 0 \quad \xi > 0$$

where we choose

$$\psi = \begin{cases} -a & (A) \\ a & (B) \end{cases}$$

The system is unstable (different signed elements of the equation).

In case (A), there are two real eigenvalues, given by:

$$\lambda_{1,2} = \frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} + a}$$

one is stable (negative) and one unstable (positive)

In case (B) there are complex conjugate eigenvalues with positive real part given that  $a > \frac{\xi^2}{4}$ . By choosing

$$\psi = \begin{cases} -a & \text{if } xs < 0 \\ a & \text{if } xs > 0 \end{cases} \quad (A) \quad (B) \quad s = -cx + \dot{x}$$

where

$$c = \frac{\xi}{2} - \sqrt{\frac{\xi^2}{4} + a} < 0$$

is the inclination of the line representing the stable mode in the (A) case. The resulting system is asymptotically stable.

This type of control scheme is called Variable structure control or Sliding Mode Control (as the system goes towards a stable mode and "slides" towards the origin) and can be generalized:

We define a *Sliding Surface*  $S(x) = 0$  based on which we decide which one of two control laws is used:

$$u_c = \begin{cases} K_1(x) & \text{if } S(x) > 0 \\ K_2(x) & \text{if } S(x) < 0 \end{cases}$$

Two steps are involved in the design of variable structure control:

- choice of the sliding surface  $S(x)$  so that the control system has the desired dynamic behaviour
- choice of the control law  $u_i$  ( $i = 1, \dots, m$ ) in order to force the state on the sliding surface  $S_i(x)$  even in case of parameter variations or external disturbances.

### Example

Let us consider the second order system

$$\ddot{y} + a\dot{y} + by = \psi + u \quad G(s) = \frac{1}{s^2 + as + b}$$

where  $u$  is the control and  $\psi$  is an external disturbance, supposed to be bounded with bounded first order derivative

$$|\psi| < \Delta_0 \quad |\dot{\psi}| < \Delta_1$$

Choice of the sliding surface:

$$S(x) = \dot{y} + cy = 0 \quad \text{with } x = (y, \dot{y})$$

When the state  $x = (y, \dot{y}) \in S(x) = 0$ :

- the dynamics of the controlled system is exponentially asymptotically stable

$$\dot{y} = -cy \implies y(t) = y(0)e^{-ct}, \quad c > 0$$

- the output of the system  $y(t)$  goes to zero with a velocity that depends only the parameter "c", that is on the chosen sliding surface
- the behaviour of the controlled system does not depend neither on the external disturbance nor on changes in the parameters  $a$  and  $b$ . (ROBUST)

Choice of the control: It is possible to prove that the discontinuous control law

$$u(t) = -K \operatorname{sgn} S(x)$$

is able to force the state towards the sliding surface  $S(x) = 0$  as long as  $K > \Delta_0$ , with total disturbance rejection. This control can only take the state of the system to 0. Substituting  $y$  and  $\dot{y}$  with the tracking error:  $x = (e, \dot{e})$ , we can track a given trajectory with the chosen dynamics, with  $e = y_d - y$ . If  $y_d \neq 0$  the set point is to be considered another disturbance to the system as it is an exogenous input to the system, requiring  $y_d \times G(0)$  to be compensated. The lower bound on  $K$  therefore becomes:

$$K > \Delta_0 + y_d G(0)$$

### Continuous time

When the state  $x$  is on the sliding surface  $S(x) = 0$ , we have the *ideal sliding mode*:

- the control action  $u(t)$  commutes with infinite frequency
- the oscillation produced on the output variable  $y(t)$  has a null amplitude

### Discrete time

If the VS control law is implemented as a discrete-time system we have:

- a finite commutation frequency
- a residual oscillation on the output variable  $y(t)$  (*chattering*)

The amplitude of the residual oscillation is proportional to the sampling period  $T$  and the value of  $K$  of the control action

$$|y(t)| \leq KT$$

Therefore, it is not possible to force the state "exactly" on the surface  $S(x) = 0$ , but it is rather possible to keep it within a sufficiently small neighbourhood of it (*discrete sliding mode*)

#### 4.1.2 The chattering problem

From a practical point of view, it is not possible to commute the control between the two values  $u^+, u^-$  at an infinite frequency. Therefore, "oscillations" are generated in the controlled system (or in the actuator), typically in a frequency range within the system's bandwidth<sup>1</sup>. Then, non modelled dynamics may be excited and undesired behaviours may be obtained. Some methods have been proposed to avoid this problem, among them:

- Boundary layers
- DIC (Discrete Integral Control)

#### Boundary Layers

A region with proper amplitude is defined around the sliding surface; in this region, the control action is not discontinuous, but rather it changes in a continuous way, linearly proportional to the error.

#### DIC (Discrete Integral Control)

Let us assume that the disturbance is bounded with bounded first derivative:

$$|\psi| < \Delta_0 \quad |\dot{\psi}| < \Delta_1$$

In addition to the standard VS control term ( $-ksgnS(x)$ ), two additional contributions are used: a term proportional to the error  $-\lambda S(x)$  and one proportional to the integral of its sign

$$\begin{cases} u(t) = -\lambda S(x) - ksgnS(x) - \tilde{\psi}(t) \\ \dot{\tilde{\psi}}(t) = hsgnS(x) \end{cases}$$

It is possible to prove the following:

If the parameters  $k, h$  and  $\lambda$  of the control law are chosen such that:

$$k > 0, \quad h > \Delta_1, \quad h\lambda k > \Delta_1^2(1 - \ln 2)$$

then the controlled system is globally asymptotically stable, and the state reaches the sliding surface  $S(x) = 0$  in a finite time. Note that there is no lower bound on  $k$ , and it can therefore be chosen to be quite small to reduce the chattering effect. The lower bound on  $h$ , means that the implemented controller must be faster than the disturbance (have a higher derivative).

The resulting controller is composed of a proportional, integral and a switching component. If the proportional part is not present, i.e.  $\lambda = 0$ , the control action is still able to stabilize the system, but only for "small" initial conditions.

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<sup>1</sup>if the control input is discontinuous, all the frequencies are excited and the natural frequencies of the system are necessarily excited

### 4.1.3 Robust control of industrial manipulators

In general, it is difficult to compensate exactly the dynamics of a robot manipulator. In practice, what we get is only a partial compensation. Let us consider the inverse dynamics control:

$$u = \hat{M}(q)y + \hat{n}(q, \dot{q})$$

where  $\hat{M}$  and  $\hat{n}$  represent the known part of the dynamic model. In general, uncertainties may be expressed as

$$\tilde{M} = \hat{M} - M \quad \tilde{n} = \hat{n} - n$$

By using the above equation, we have

$$M\ddot{q} + n = \hat{M} + \hat{n}$$

Since matrix  $M$  is invertible, we have, (adding and subtracting  $y$ )

$$\begin{aligned} \ddot{q} &= y + (M^{-1}\hat{M} - I)y + M^{-1}\tilde{n} \\ &= y - \eta \end{aligned}$$

with

$$\eta = (M^{-1}\hat{M} - I)y + M^{-1}\tilde{n}$$

note that, if  $\hat{M} = M$  and  $\hat{n} = n$  then  $\eta = 0$  and  $\ddot{q} = y$  By using the same control as in the ideal case:

$$y = \ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)$$

The error dynamics  $\tilde{q}$  is given by

$$\ddot{\tilde{q}} + K_D\dot{\tilde{q}} + K_P\tilde{q} = \eta$$

and therefore it is not possible to guarantee that  $\tilde{q}$  goes to zero ( $\eta$  depends on the uncertainties of the system) From

$$\ddot{\tilde{q}} = y - \eta$$

we have

$$\ddot{\tilde{q}} = \ddot{q}_d - y + \eta$$

Therefore, by defining  $\xi = [\tilde{q}^T \quad \dot{\tilde{q}}^T]^T$  as the state vector, we obtain the first order differential equation

$$\dot{\xi} = H\xi + D(\ddot{q}_d - y + \eta)$$

with

$$H = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \quad D = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

The problem with tracking a given trajectory  $(q_d, \dot{q}_d, \ddot{q}_d)$  may be solved by designing a control law  $y$  that stabilizes the error dynamics  $\xi$ , that is nonlinear and time variant (because of  $\eta$ ). Because of the properties of any assigned trajectory (boundedness of velocity/acceleration) and considering the properties of the dynamic model for a robot manipulator (boundedness) it is possible to estimate the range of variability of the uncertainties  $\eta$  affecting the error dynamics.

In order to compensate for the uncertainty term  $\eta$ , we define a control law as:

$$y = \ddot{q}_d + K_D\dot{\tilde{q}} + K_P\tilde{q} + w = \ddot{q}_d + w + [K_P \quad K_D] \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix}$$

the error dynamics in this case results

$$\dot{\xi} = H\xi + D(\eta - w)$$

where the eigenvalues of the matrix

$$\tilde{H} = (H - DK) = \begin{bmatrix} 0 & I \\ -K_P & -K_D \end{bmatrix}$$

have negative real part (as  $K_P, K_D$  are positive definite) By properly choosing the two matrices  $K_P, K_D$  for example as

$$K_P = \text{diag}\{\omega_{n1}^2, \dots, \omega_{nn}^2\} \quad K_D = \text{diag}\{2\delta_1\omega_{n1}, \dots, 2\delta_n\omega_{nn}\}$$

it is possible to obtain a desired (and decoupled) behaviour for the linear part of the error dynamics. Notice that if  $\eta = 0$ , then by choosing  $w = 0$  we obtain the previous control scheme. Viceversa, if  $\eta \neq 0$ , a proper value for  $w$  must be defined. The Lyapunov method is adopted, choosing as a candidate Lyapunov function

$$V(\xi) = \xi^T Q \xi > 0 \quad \forall \xi \neq 0$$

where  $Q$  is a symmetric, positive definite matrix. By derivation, since  $\dot{\xi} = \tilde{H}\xi + D(\eta - w)$ , we have:

$$\begin{aligned} \dot{V} &= \dot{\xi}^T Q \xi + \xi^T Q \dot{\xi} \\ &= (\tilde{H}\xi + D(\eta - w))^T Q \xi + \xi^T Q (\tilde{H}\xi + D(\eta - w)) \\ &= \xi^T (\tilde{H}^T Q + Q \tilde{H}) \xi + 2\xi^T Q D(\eta - w) \\ &= -\xi^T P \xi + 2\xi^T Q D(\eta - w) \\ &= -\xi^T P \xi + 2z^T (\eta - w) \end{aligned}$$

where

$$z = d^T Q \xi$$

and, since  $\tilde{H} < 0$ , the equation

$$\tilde{H}^T Q + Q \tilde{H} = -P, \quad \forall P > 0$$

has a unique solution  $Q$ , symmetric and positive definite. Notice that the first term is negative definite. We need to prove negativity of the second term. The vector  $z$  is known, therefore we can choose the control input:

$$w = \frac{\rho}{\|z\|} z \quad \rho > \|\eta\|$$

we obtain

$$z^T (\eta - w) = z^T \eta - \frac{\rho}{\|z\|} z^T z \leq \|z\| \|\eta\| - \rho \|z\| \leq 0$$

and thus

$$\dot{V} < 0$$

Overall control scheme: This is akin to sliding mode control as the direction of  $z$  cannot be assumed to be continuous. The sliding subspace  $S(\xi) = D^T Q \xi = 0$  depends on the choice of the matrix  $Q$ . The chattering problem may be solved e.g. by the 'boundary layers' method:

$$w = \begin{cases} \frac{\rho}{\|z\|} z & \text{when } \|z\| \geq \epsilon \\ \frac{\rho}{\epsilon} z & \text{when } \|z\| < \epsilon \end{cases}$$

Discrete Integral Control may exhibit some unexpected behaviour and implementative challenges due to the complexity of the system.

## 4.2 Adaptive Control

Aims at modifying the control parameters in order to adapt to different working conditions, or to different values of the model parameters. There are two main families of adaptive controllers:

- Self Tuning Regulator (indirect)
- Implicit regulator (direct)

We assume the dynamics of the process to be linear wrt the parameters to be estimated, and we require the change of parameters to be slower than the dynamics of the system.

### 4.2.1 Robot parameters

For design purposes:

- kinematic parameters are assumed to be known (DH parameters, 3 per link)
- uncertain (constant) parameters that can be identified off line: masses(1), positions of CoMs(3), inertia matrices(6) (10 per link)

- parameters that are (slowly) varying during operation: viscous, dry and stiction friction at each joint (3 per joint)
- unknown and abruptly changing parameters: mass, CoM, inertia matrix of the payload wrt the tool center point (10)

so for an  $n$ -dof robot:  $(3 + 10 + 3) \times n + 10 = 16 \times n + 10$  parameters

The main goals of an adaptive control scheme may be defined as:

- given a twice differentiable desired joint trajectory  $q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$
- execute this trajectory under large dynamic uncertainties  $\tilde{\alpha} = \alpha - \hat{\alpha}$  with a tracking error that vanishes asymptotically:

$$\tilde{q} = q_d - q \rightarrow 0, \quad \dot{\tilde{q}} = \dot{q}_d - \dot{q} \rightarrow 0$$

guaranteeing global stability

- parameter identification (i.e.  $\tilde{\alpha} \rightarrow 0$ ) is not of particular concern if the desired trajectory is persistently exciting, one obtains parameter identification as a by-product ( $\tilde{\alpha} \rightarrow 0$ )
- indirect adaptive control schemes are more complex, but in principle allow also systematic convergence of dynamic coefficients to their true values

It is always possible to write the Euler-Lagrange formulation of the dynamics of an  $n$  dof robot as

$$Y(q, \dot{q}, \ddot{q})\alpha = u$$

where:

- vector  $\alpha$  contains only unknown or uncertain coefficients
- each component of  $\alpha$  is in general a combination of the robot physical parameters
- the model regression matrix  $Y$  depends: linearly on  $\ddot{q}$ , quadratically on  $\dot{q}$ , nonlinearly on  $q$

In general, only a partial knowledge of the dynamic parameters is available, and therefore we may describe the dynamics of a robot as:

$$\hat{M}(q)\ddot{q} + \hat{C}(q, \dot{q})\dot{q} + \hat{D}\dot{q} + \hat{g}(q) = Y(q, \dot{q}, \ddot{q})\alpha = u$$

where  $\hat{M}, \hat{C}, \hat{D}, \hat{g}$  estimate  $\alpha$ . Some controllers are better suited to adaptive control with robot manipulators.

#### 4.2.2 Inverse dynamics feedforward + PD

Given a desired trajectory it is possible to implement an inverse dynamics ff + PD control action defined as:

$$u = M(q_d)\ddot{q}_d + C(q_d, \dot{q}_d)\dot{q}_d + D\dot{q}_d + g(q_d) + K_P\tilde{q} + K_d\dot{\tilde{q}}$$

being  $\tilde{q}$  and  $\dot{\tilde{q}}$  the position and velocity errors. Since the dynamic parameters are in general not exactly known, only an approximation of the ff term can be computed. It can be shown that the control law leads to an error dynamics described by

$$\ddot{\tilde{q}} + K_d\dot{\tilde{q}} + K_P\tilde{q} = \eta$$

being  $\eta$  a nonlinear function of the parameter uncertainties and of the nl dynamics of the robot, so the error does not vanish asymptotically

#### 4.2.3 NL control based on fb linearization

The error dynamics behaves analogously to the previous case, and does not vanish asymptotically.



#### 4.2.4 Nonlinear tracking control law without cancelations

Let us consider a candidate control law:

$$u = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + D\dot{q}_d + g(q) + K_p\tilde{q} + K_d\dot{\tilde{q}}$$

that is a nonlinear trajectory tracking control law having global asymptotic stabilization properties. In principle, the control law could be made adaptive by defining

$$\dot{\hat{\alpha}} = \gamma(\hat{\alpha}q, \dot{q}, q_d, \dot{q}_d)$$

where  $\gamma(\cdot)$  is a proper update law for the robot's parameters

It can be shown that with this control law the velocities could track the desired ones (eventually with zero error), but a permanent residual position error may occur if parameters are not exactly known. A possible solution is to modify the velocity reference as

$$\dot{q}_d \rightarrow \dot{q}_r = \dot{q}_d + \Lambda(q_d - q), \quad \Lambda > 0$$

where the value  $\Lambda = K_d^{-1}K_p$  is chosen

In robotics, in order to design an adaptive control law we exploit the property of the dynamic model to be linear wrt the parameters

$$Y(q, \dot{q}, \ddot{q})\alpha = u$$

Let us consider the control law:

$$u = M(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + D\dot{q}_r + g(q) + K_D\sigma$$

where

$$\begin{aligned} \dot{q}_r &= \dot{q}_d + \Lambda\tilde{q} \\ \ddot{q}_r &= \ddot{q}_d + \Lambda\dot{\tilde{q}} \end{aligned}$$

and  $K_D$  a positive definite matrix. The term  $K_D\sigma$  is equivalent to a PD control action on the tracking error assuming

$$\sigma = \dot{q}_r - \dot{q} = \dot{\tilde{q}} + \Lambda\tilde{q}$$

The overall dynamics is described by

$$M(q)\dot{\sigma} + C(q, \dot{q})\sigma + D\sigma + K_D\sigma = 0$$

To prove stability of the system we use the Lyapunov method. We choose as a candidate Lyapunov function

$$V(\sigma, \tilde{q}) = \frac{1}{2}\sigma^T M(q)\sigma + \frac{1}{2}\tilde{q}^T A\tilde{q} > 0 \quad \forall \sigma, \tilde{q} \neq 0$$

where  $A$  is positive definite.

On the basis of the previous result, let us consider the control law

$$u = \hat{M}(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{D}\dot{q}_r + \hat{g}(q) + K_D\sigma = Y(q, \dot{q}, \ddot{q}_r, \dot{q}_r)\hat{\alpha} + K_D\sigma$$

Notice that  $Y$  does not depend on  $\ddot{q}$ . With this control, the overall dynamics are described by: In order to prove the stability of the system under this new control law, we once again use the Lyapunov method:

#### To sum up

Adaptive control:

- does not compensate directly external disturbance
- depends on the considered model
- tries to compensate the effects of disturbances by modifying the parameters
- "smooth" control action

Robust Control:

- direct compensation of disturbances
- chattering phenomenon

### 4.3 Learning/Repetitive Control

If a robot has to repeat a given task cyclically, following every time the same trajectory, possible uncertainties in the system may be compensated by adopting learning or repetitive control schemes, by means of which the control system, by repeating the task, learns the optimal control input to be applied in order to have zero tracking error. There are learning or repetitive control schemes to solve

- trajectory tracking problems
- control to steady state configurations.

Two control actions:

- feedback, based on the error  $e = y_d - y$
- feedforward, updated at each iteration

Assumptions:

- dynamic model of the robot

$$; (q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$

- boundedness of the gradient of the gravity term

$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha$$

- joint-based PD controller (without g compensation)

$$u = K_p(q_d - q) - K_d\dot{q} \quad K_p > 0, K_d > 0$$

- it can be shown that the system is stable and that in steady-state we have:

$$q = \bar{q}, \quad \dot{q} = 0, \quad g(\bar{q}) = K_p(q_d - \bar{q}), \quad \tilde{q} = q_d - \bar{q}$$

Control scheme:

- at the  $i$ -th iteration, the control action is

$$u(t) = \frac{1}{\beta} K_p(q_d - q(t)) - K_d\dot{q}(t) + u_{i-1} \quad \beta > 0, \quad q(0) = \bar{q}_{i-1}$$

**Theorem (sufficient condition for convergence)**

If the following conditions:

1.  $\lambda_{\min}(K_p) \geq \alpha$
2.  $0 < \beta \leq 0.5$

hold, then the sequence converges to  $q_d$  from any initial condition