Chapter 1

Introduction to optimal control

1.1 Optimal control problem formulation

Consider the continuous-time system ($t \in \mathbb{R}$)

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1.1}$$
$$y(t) = h(x(t), u(t), t) \tag{1.2}$$

- $x(t) \in \mathbb{R}^n$ state of the system at time t
- $u(t) \in \mathbb{R}^m$ input of the system at time t
- $y(t) \in \mathbb{R}^p$ output of the system at time t

We will mainly work with time invariant systems, $\dot{x}(t) = f(x(t), u(t))$.

We consider nonlinear, discrete-time systems described by

$$x(t+1) = f_t(x(t), u(t)) \quad t \in \mathbb{N}_0$$

but from now on we will use the compact notation

$$x_{t+1} = f_t(x_t, u_t) \quad t \in \mathbb{N}_0$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the state and the input of the system at time t.

Consider a nonlinear, discrete-time system on a finite time horizon

$$x_{t+1} = f_t(x_t, u_t) \quad t = 0, \dots, T-1$$

We use $\mathbf{x} \in \mathbb{R}^{nT}$ and $\mathbf{u} \in \mathbb{R}^{mT}$ to denote, respectively, the stack of the states x_t for all $t \in \{1, \dots, T\}$ and the unputs u_t for all $t \in \{0, \dots, T-1\}$, that is:

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} \qquad \mathbf{u} := \begin{bmatrix} u_0 \\ \vdots \\ u_{T-1} \end{bmatrix}$$

Trajectory of a system

Definition: A pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT}$ is called a trajectory of system (1) if $\bar{x}_{t+1} = f_t(\bar{x}_t, \bar{u}_t)$ for all $t \in \{0, \dots, T-1\}$., That is, if $\bar{\mathbf{x}}, \bar{\mathbf{u}})$ satisfies the system dynamics (the same holds for continuous time systems with proper adjustments). In particular, $\bar{\mathbf{x}}$ is the state trajectory, while $\bar{\mathbf{u}}$ is the input trajectory.

Equilibrium

Definition: A state-input pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of (1) if $(x_t, u_t) = (x_e, u_e), t \in \mathbb{N}_0$ is a trajectory of the system.
Equilibria of time-invariant systems satisfy $x_e = f(x_e, u_e)$

Linearization of a system about a trajectory

Given the dynamics (1)and a trajectory $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$, the linearization of (1) about $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is given by the linear (possibly) time-varying system

$$\Delta x_{t+1} = A_t \Delta x_t + B_t \Delta u_t \quad t \in \mathbb{N}_0$$

with A_t and B_t the Jacobians of f_t, with respect to state and input respectively, evaluated at $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$

$$A_t = \frac{\partial}{\partial x} f(\bar{x}_t, \bar{u}_t)\Big|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})} \qquad B_t = \frac{\partial}{\partial u} f(\bar{x}_t, \bar{u}_t)\Big|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}$$

1.1.1 Optimization

Main ingredients

- Decision variable: $x \in \mathbb{R}^n$
- Cost function: $\ell(x) : \mathbb{R}^n \to \mathbb{R}$ cost associated to decision x
- Constraints (constraint sets): for some given functions $h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, m,$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, the decision vector $x \in \mathbb{R}^n$ needs to satisfy

  $$\begin{aligned} h_i(x) = 0 \quad i = 1, \dots, m \\ g_j(x) = 0 \quad j = 1, \dots, r \end{aligned}$$

  equivalently we can say that we require $x \in X$ with

  $$X = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0\},$$

  where we compactly denoted $h(x) = \mathrm{col}(h_1(x), \dots, h_m(x))$ and $g(x) = \mathrm{col}(g_1(x), \dots, g_r(x))$

Minimization

We can write our optimization problem as

$$\min_{x \in \mathbb{R}^n} \ell(x) \tag{1.3}$$

$$\text{subj. to } h_i(x) = 0 \quad i = 1, \dots, m \tag{1.4}$$

$$g_j(x) \leq 0 \quad j = 1, \dots, r \tag{1.5}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$
We can write it more compactly as

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

$$\text{subj. to } h(x) = 0 g(x) \leq 0$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^r$

1.1.2 Discrete-time optimal control

main ingredients

- Dynamics: a discrete-time system in state space form

  $$x_{t+1} = f_t(x_t, u_t) \quad t = 0, 1, \dots, T-1$$

- the dynamics introduce T equality constraints

  $$\begin{array}{lll} x_1 = f(x_0, u_0) & \text{i.e.} & x_1 - f_t(x_0, u_0) = 0 \\ x_2 = f(x_1, u_1) & \text{i.e.} & x_1 - f_t(x_1, u_1) = 0 \\ \vdots & & \\ x_T = f(x_{T-1}, u_{T-1}) & \text{i.e.} & x_T - f_t(x_{T-1}, u_{T-1}) = 0 \end{array}$$

  This is equivalent to nT scalar constraints

- Cost function: a cost "to be payed" for a chosen trajectory. We consider an additive structure in time

$$\ell(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T)$$

  where $\ell_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is called stage-cost, while $\ell_T : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost.

- End-point constraints: function of the state variable prescribed at initial and/or final point

$$r(x_0, x_T) = 0$$

- Path constraints: point-wise (in time) constraints representing possible limits on states and inputs at each time t

$$g_t(x_t, u_t) \leq 0, \quad t \in \{0, \ldots, T-1\}$$

A discrete-time optimal control problem can be written as

$$\begin{aligned}
\min_{\substack{x_0, x_1, \ldots, x_T \\ u_0, \ldots, u_{T-1}}} & \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) \\
\text{subj. to} \quad & x_{t+1} = f_t(x_t, u_t), \quad t \in \{0, \ldots, T-1\} \\
& r(x_0, x_T) = 0 \\
& g_t(x_t, u_t) \leq 0, \quad t \in \{0, \ldots, T-1\}
\end{aligned}$$

Optimal control for trajectory generation

We can pose a trajectory generation problem as

$$\min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m} \sum_{t=0}^{T-1} \frac{1}{2} \|x_t - x_t^{\text{des}}\|_Q^2 + \frac{1}{2} \|u_t - u_t^{\text{des}}\|_R^2 + \frac{1}{2} \|x_T - x_T^{\text{des}}\|_{P_f}^2$$

Continuous-time Optimal Control problem

A continuous-time optimal control problem, i.e., $t \in \mathbb{R}$ can be written as

$$\begin{aligned}
\min_{(x(\cdot), u(\cdot)) \in \mathcal{F}} & \int_0^T \ell_\tau(x(\tau), u(\tau)) d\tau + \ell_T(x(T)) \\
\text{subj. to} \quad & \dot{x}(t) = f_t(x(t), u(t)) \quad t \in [0, T] \\
& r(x(0), x(T)) = 0 \\
& g_t(x(t), u(t)) \leq 0 \quad t \in [0, T)
\end{aligned}$$

Note that $\mathcal{F}$ is a space of functions (function space). This is an infinite dimensional optimization problem

- Cost functional $\ell : \mathcal{F} \to \mathbb{R}$

$$\ell(x(\cdot), u(\cdot)) = \int_0^T \ell_\tau(x(\tau), u(\tau)) d\tau + \ell_T(x(T))$$

- Space of trajectories (or trajectory manifold)

$$\mathcal{T} = \{(x(\cdot), u(\cdot)) \in \mathcal{F} | \dot{x}(t) = f_t(x(t), u(t)), t \geq 0\}$$

Chapter 2

Nonlinear Optimization

2.1 Unconstrained Optimization

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

with $\ell : \mathbb{R}^n \to \mathbb{R}$ a cost function to be minimized and x a decision vector

We say that x^* is a

- global minimum if $\ell(x^*) \leq \ell(x)$ for all $x \in \mathbb{R}^n$
- strict global minimum if $\ell(x^*) < \ell(x)$ for all $x \neq x^*$
- local minimum if there exists $\epsilon > 0$ such that $\ell(x^*) \leq \ell(x)$ for all $x \in B(x^*, \epsilon) = \{x \in \mathbb{R}^n | \|x - x^*\| < \epsilon\}$
- strict local minimum if there exists $\epsilon > 0$ such that $\ell(x^*) < \ell(x)$ for all $x \in B(x^*, \epsilon)$

Notation

We denote $\ell(x^*)$ the optimal (minimum) value of a generic optimization problem, i.e.

$$\ell(x^*) = \min_{x \in \mathbb{R}^n} \ell(x)$$

where x^* is the minimum point (optimal value for the optimization variable) i.e.

$$x^* = \arg\min_{x \in \mathbb{R}^n} \ell(x)$$

Gradient and Hessian

Gradient of a function: for a function $r : \mathbb{R}^n \to \mathbb{R}$ the gradient is denoted as

$$\nabla r(x) = \begin{bmatrix} \dfrac{\partial r(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial r(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Hessian matrix of a function: for a fcuntion $r : \mathbb{R}^n \to \mathbb{R}$ the Hessian matrix is denoted as

$$\nabla^2(r(x)) = \begin{bmatrix} \dfrac{\partial^2 r(x)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 r(x)}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 r(x)}{\partial x_n x_1} & \cdots & \dfrac{\partial^2 r(x)}{\partial x_1^2} \end{bmatrix}$$

Gradient of a vector-valued function: for a vector field $r : \mathbb{R}^n \to \mathbb{R}^m$, the gradient is denoted as

$$\nabla r(x) = \begin{bmatrix} \nabla r_1(x) & \cdots & \nabla r_m(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1(x)}{\partial x_1} & \cdots & \frac{\partial r_m(x)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1(x)}{\partial x_n} & \cdots & \frac{\partial r_m(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

which is the transpose of the Jacobian matrix of r

2.1.1 Conditions of optimality

First order necessary condition (FNC) of optimality (unconstrained)

Let x^* be an unconstrained local minimum of $\ell : \mathbb{R}^n \to \mathbb{R}$ and assume that ℓ is continuously differentiable ($\mathcal{C}^1$) in $B(x^*, \epsilon)$ for some $\epsilon > 0$. Then $\nabla \ell(x^*) = 0$

Second order necessary condition (FNC) of optimality (unconstrained)

If additionally ℓ is twice continuously differentiable ($\mathcal{C}^2$) in $B(x^*, \epsilon)$, then $\nabla^2 \ell(x^*) \geq 0$ (The Hessian of ℓ is positive semidifinite)

Second order sufficient conditions of optimality (unconstrained)

Let $\ell : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^2$ in $b(x^*, \epsilon)$ for some $\epsilon > 0$. Suppose that $x^* \in \mathbb{R}^n$ satisfies

$$\nabla \ell(x^*) = 0 and \nabla^2 \ell(x^*) > 0$$

Then x^* is a strict (unconstrained) local minimum of ℓ

Convex set

A set $X \subset \mathbb{R}^n$ is convex if for any two points x_A and x_B in X and for all $\lambda \in [0, 1]$, then

$$\lambda x_a + (1 - \lambda) x_B \in X$$

Convex functions

Let $X \subset \mathbb{R}^n$ be a convex set. A function $\ell : X \to \mathbb{R}$ is convex if for any two points x_A and x_B in X and for all $\lambda \in [0, 1]$, then

$$\ell(\lambda x_A + (1 - \lambda) x_B) \leq \lambda \ell(x_A) + (1 - \lambda) \ell(x_B)$$

2.1.2 Minimization of convex functions

Proposition

Let $X \subset \mathbb{R}^n$ be a convex set and $\ell : X \to \mathbb{R}$ a convex function. Then a local minimum of ℓ is also a global minimum
Proof: not done in class but present in slides for funsies

Necessary and sufficient condition of optimality (unconstrained)

For the unconstrained minimization of a convex function it can be shown that the first order necessary condition of optimality is also sufficient (for a global minimum).

Proposition

Let $\ell^n_{\mathbb{R}} \to \mathbb{R}$ be a convex function. Then x^* is a global minimum if and only if $\nabla \ell(x^*) = 0$

Proof: not done in class but present in slides for funsies

2.1.3 Quadratic programming

Let us consider a special class of optimization problems, namely quadratic optimization problems or quadratic programs:

$$\min_{x \in \mathbb{R}^n} x^T Q x + b^t x$$

with $Q = Q^T \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$

optimality conditions

First-order necessary condition for optimality: if x^* is a minimum then

$$\nabla \ell(x^*) = 0 \implies 2Qx^* + b = 0$$

Second-order necessary condition for optimality: if x^* is a minimum then

$$\nabla^2 \ell(x^*) \geq 0 \implies 2Q > 0$$

A necessary condition for the existence of minima for a quadratic program is that $Q \geq 0$. Thus, quadratic programs admitting at least a minimum are convex optimization problems.

properties

Since quadratic programs are convex programs ($Q \geq 0$ is necessary to have a local minimum), then the following holds:

- For a quadratic program necessary conditions of optimality are also sufficient and minima are global

If $Q > 0$, then there exists a unique global minimum given by

$$x^* = -\frac{1}{2} Q^{-1} b$$

2.2 Unconstrained Optimization Algorithms

2.2.1 Iterative descent methods

We consider optimization algorithms relying on the iterative descent idea. We denote $x^k \in \mathbb{R}^n$ an estimate of a local minimum at iteration $k \in \mathbb{N}$. The algorithm starts at a given initial guess x^0 and iteratively generates vectors $x^1, x^2, \ldots$ such that ℓ is decreased at each iteration, i.e.

$$\ell(x^{k+1}) < \ell(x^k) \qquad k = 1, 2, \ldots$$

two-step procedure

We consider a general two-step procedure that reads as follows

$$x^{k+1} = x^k + \gamma^k d^k, \qquad k = 1, 2, \ldots$$

in which

1. each $\gamma^k > 0$ is a "step-size"
2. $d^k \in \mathbb{R}^n$ is a "direction"

The goal is to

1. choose a direction d^k along which the cost decreases for γ^k sufficiently small;
2. select a step-size γ^k guaranteeing a sufficient decrease.

In oher references these are called line-search methods.

2.2.2 Gradient methods

Let x^k be such that $\nabla \ell(x^k) \neq 0$. We start by considering the update rule

$$x^{k+1} = x^k - \gamma^k \nabla \ell(x^k)$$

i.e., we choose $d^k = \nabla \ell(x^k)$

From the first order Taylor expansion of ℓ at x we have

$$\begin{aligned} \ell(x^{k+1}) &= \ell(x^k) + \nabla \ell(x^k)^T (x^{k+1} - x^k) + o(\|x^{k+1} - x^k\|) \\ &= \ell(x^k) - \gamma^k \|\nabla \ell(x^k)\|^2 + o(\gamma^k) \end{aligned}$$

Thus, for $\gamma^k > 0$ sufficiently small it can be shown that $\ell(x^k + 1) < \ell(x^k)$

The update rule

$$x^{k+1} = x^k - \gamma^k \nabla \ell(x^k)$$

can be generalized to so called *gradient methods*

$$x^{k+1} = x^k + \gamma^k d^k$$

with d^k such that

$$\nabla \ell(x^k)^T d^k < 0$$

Also, d^k must be gradient related, i.e. d^k must not asymptotically become perpendicular to $\nabla \ell$

selecting the descent direction

Several gradient methods can be written as

$$x^{k+1} 0 x^k - \gamma^k D^k \nabla \ell(x^k) \quad k = 1, 2, \ldots$$

where $D^k \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. It can be immediately seen that

$$-\nabla \ell(x^k)^T D^k \nabla \ell(x^k) < 0$$

i.e. $d^k = -D^k \nabla \ell(x^k)$ is a descent direction. The choice of D^k must be made such that there exist d_1, d_2 positive real, such that $d_1 I \leq D^k \leq d_2 I$

Some choices for D^k:

- Steepest descent $D^k = I_n$
- Newton's method $D^k = (\nabla^2 \ell(x^k))^{-1}$
  It can be used when $\nabla^2 \ell(x^k) > 0$. It typically converges very fast asymptotically. For $\gamma^k = 1$ pure Newton's method
- Discretized Newton's method $D^k = (H(x^k))^{-1}$, where $H(x^k)$ is a positive definite symmetric approximation of $\nabla^2 \ell(x^k)$ obtained by using finite difference approximations of the second derivatives
- Some regularized version of the Hessian

2.2.3 gradient method

The update rule obtained for $D^k = I$ is called steepest descent. The name steepest descent is due to the following property: the normalized negative gradient direction

$$d^k = -\frac{\nabla \ell(x^k)}{\|\nabla \ell(x^k)\|}$$

minimizes the slope $\nabla \ell(x^k)^T d^k$ among all normalized directions, i.e. it gives the steepest descent.

2.2.4 Newton's method for root finding

Consider the nonlinear root finding problem

$$r(x) = 0$$

Idea: iteratively refine the solution such that the improved guess x^{k+1} represents a root of the linear approximation of r about the current tentative solution x^k. Consider the linear approximation of r about x^k, we have

$$r^k(x^k + \Delta x^k) = r(x^k) + \nabla r(x^k)^T \Delta x^k$$

then, finding the zeros of the approximation, we have

$$\Delta x^k = -(\nabla r(x^k)^T)^{-1} r(x^k)$$

Thus, the solution is improved as

$$x^{k+1} = x^k - (\nabla r(x^k)^T)^{-1} r(x^k)$$

2.2.5 Newton's method for unconstrained optimization

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

stationary points $\bar{x}$ satisfy the first order optimality condition

$$\nabla \ell(\bar{x}) = 0$$

We can look at it as a root finding problem, with $r(x) = \nabla \ell(x)$, and solve it via Newton's method. Therefore, we can compute Δx^k as the solution of the linearization of $r(x) = \nabla \ell(x)$ at x^k, i.e.

$$\nabla \ell(x^k) + \nabla^2 \ell(x^k) \Delta x^k = 0$$

and run the update

$$x^{k+1} = x^k - (\nabla^2 \ell(x^k))^{-1} \nabla \ell(x^k)$$

We can introduce a variable step-size

$$x^{k+1} = x^k - \gamma^k (\nabla^2 \ell(x^k))^{-1} \nabla \ell(x^k)$$

This is called generalized Newton's method

Newton's method via Quadratic Optimization

Observe that

$$\nabla \ell(x^k) + \nabla^2 \ell(x^k) \Delta x^k = 0$$

is the first-order necessary and sufficient condition of optimality for the quadratic program

$$\Delta x^k = \arg\min_{\Delta x} \nabla \ell(x^k)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 \ell(x^k) \Delta x \tag{2.1}$$

Thus, the k-th iteration of Newton's method can be seen as

$$x^{k+1} = x^k + \Delta x^k$$

with Δx^k solution of the quadratic problem 2.1. Generalized version:

$$x^{k+1} = x^k + \gamma^k \Delta x^k$$

2.2.6 Gradient methods via quadratic optimization

Similarly to Newton's method, a descent direction $\Delta x^k = D^k \nabla \ell(x^k)$ can be seen as the direction that minimizes at each iteration a different quadratic approximation of ℓ about x^k. In fact, consider the quadratic approximation $\ell^k(x)$ about x^k given by

$$\ell^k(x) = \ell(x^k) + \nabla \ell(x^k)^T (x - x^k) + \frac{1}{2}(x - x^k)^T (D^k)^{-1}(x - x^k)$$

By setting the derivative to zero, we have

$$\nabla \ell(x^k) + (D^k)^{-1}(x - x^k) = 0$$

we can calculate the minimum of $\ell^k(x)$ and set it as the next iterate x^{k+1}

$$\Delta x^k = -D^k \nabla \ell(x^k)$$

2.2.7 step-size selection rules

- Constant step-size: $\gamma^k = \gamma > 0$
- Diminishing step-size: $\gamma^k \to 0$ as $k \to \infty$. It must hold that

  $$\sum_{k=0}^{\infty} \gamma^k = +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} (\gamma^k)^2 < +\infty$$

  The above conditions avoid pathological choices of γ^k
- minimization rule
- Armijo rule

Armijo rule

Step-size is selected following the procedure:

1. set $\bar{\gamma}^0 > 0, \quad \beta \in (0,1), \quad c \in (0,1)$

given d^k descent direction we can consider

$$g(\gamma) = \ell(x^k + \gamma d^k), \quad g : \mathbb{R} \to \mathbb{R}$$

The value of $g(\gamma)$ for $\gamma = 0$ is $\ell(x^k)$. The minimization rule chooses as the value for γ the value that minimizes $g(\gamma)$. The partial minimization rule would search for a minimum in a restricted set of values for γ

Chapter 3

Optimality conditions for optimal control

Chapter 4

Linear Quadratic (LQ) optimal control

Chapter 5

Dynamic Programming

Chapter 6

Numerical methods for nonlinear optimal control

Chapter 7

Optimization-based predictive control