Contents

Chapter 1

Introduction to optimal control

1.1 Optimal control problem formulation

Consider the continuous-time system ($t \in \mathbb{R}$)

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned} \tag{1.1}$$

- $x(t) \in \mathbb{R}^n$ state of the system at time t
- $u(t) \in \mathbb{R}^m$ input of the system at time t
- $y(t) \in \mathbb{R}^p$ output of the system at time t

We will mainly work with time invariant systems, $\dot{x}(t) = f(x(t), u(t))$.

We consider nonlinear, discrete-time systems described by

$$x(t+1) = f_t(x(t), u(t)) \quad t \in \mathbb{N}_0$$

but from now on we will use the compact notation

$$x_{t+1} = f_t(x_t, u_t) \quad t \in \mathbb{N}_0$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the state and the input of the system at time t.

Consider a nonlinear, discrete-time system on a finite time horizon

$$x_{t+1} = f_t(x_t, u_t) \quad t = 0, \ldots, T-1$$

We use $\mathbf{x} \in \mathbb{R}^{nT}$ and $\mathbf{u} \in \mathbb{R}^{mT}$ to denote, respectively, the stack of the states x_t for all $t \in \{1, \ldots, T\}$ and the unputs u_t for all $t \in \{0, \ldots, T-1\}$, that is:

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} \qquad \mathbf{u} := \begin{bmatrix} u_0 \\ \vdots \\ u_{T-1} \end{bmatrix}$$

Trajectory of a system

Definition: A pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT}$ is called a trajectory of system (1.1) if $\bar{x}_{t+1} = f_t(\bar{x}_t, \bar{u}_t)$ for all $t \in \{0, \ldots, T-1\}$., That is, if $\bar{\mathbf{x}}, \bar{\mathbf{u}})$ satisfies the system dynamics (the same holds for continuous time systems with proper adjustments). In particular, $\bar{\mathbf{x}}$ is the state trajectory, while $\bar{\mathbf{u}}$ is the input trajectory.

Equilibrium

Definition: A state-input pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of (1.1) if $(x_t, u_t) = (x_e, u_e), t \in \mathbb{N}_0$ is a trajectory of the system.
Equilibria of time-invariant systems satisfy $x_e = f(x_e, u_e)$

Linearization of a system about a trajectory

Given the dynamics (1.1) and a trajectory $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$, the linearization of (1) about $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is given by the linear (possibly) time-varying system

$$\Delta x_{t+1} = A_t \Delta x_t + B_t \Delta u_t \quad t \in \mathbb{N}_0$$

with A_t and B_t the Jacobians of f_t, with respect to state and input respectively, evaluated at $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$

$$A_t = \frac{\partial}{\partial x} f(\bar{x}_t, \bar{u}_t)\bigg|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})} \qquad B_t = \frac{\partial}{\partial u} f(\bar{x}_t, \bar{u}_t)\bigg|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}$$

1.1.1 Optimization

Main ingredients

- Decision variable: $x \in \mathbb{R}^n$
- Cost function: $\ell(x) : \mathbb{R}^n \to \mathbb{R}$ cost associated to decision x
- Constraints (constraint sets): for some given functions $h_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m,$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, the decision vector $x \in \mathbb{R}^n$ needs to satisfy

  $$\begin{aligned} h_i(x) &= 0 \quad i = 1, \ldots, m \\ g_j(x) &= 0 \quad j = 1, \ldots, r \end{aligned}$$

  equivalently we can say that we require $x \in X$ with

  $$X = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0\},$$

  where we compactly denoted $h(x) = \text{col}(h_1(x), \ldots, h_m(x))$ and $g(x) = \text{col}(g_1(x), \ldots, g_r(x))$

Minimization

We can write our optimization problem as

$$\min_{x \in \mathbb{R}^n} \ell(x) \tag{1.2}$$

$$\text{subj. to } h_i(x) = 0 \quad i = 1, \ldots, m \tag{1.3}$$

$$g_j(x) \leq 0 \quad j = 1, \ldots, r \tag{1.4}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$
We can write it more compactly as

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \ell(x) \\ &\text{subj. to } h(x) = 0 g(x) \leq 0 \end{aligned}$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^r$

1.1.2 Discrete-time optimal control

main ingredients

- Dynamics: a discrete-time system in state space form

  $$x_{t+1} = f_t(x_t, u_t) \quad t = 0, 1, \ldots, T-1$$

- the dynamics introduce T equality constraints

  $$\begin{array}{lll} x_1 = f(x_0, u_0) & \text{i.e.} & x_1 - f_t(x_0, u_0) = 0 \\ x_2 = f(x_1, u_1) & \text{i.e.} & x_1 - f_t(x_1, u_1) = 0 \\ \vdots & & \\ x_T = f(x_{T-1}, u_{T-1}) & \text{i.e.} & x_T - f_t(x_{T-1}, u_{T-1}) = 0 \end{array}$$

  This is equivalent to nT scalar constraints

- Cost function: a cost "to be payed" for a chosen trajectory. We consider an additive structure in time

$$\ell(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T)$$

where $\ell_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is called stage-cost, while $\ell_T : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost.

- End-point constraints: function of the state variable prescribed at initial and/or final point

$$r(x_0, x_T) = 0$$

- Path constraints: point-wise (in time) constraints representing possible limits on states and inputs at each time t

$$g_t(x_t, u_t) \leq 0, \quad t \in \{0, \dots, T-1\}$$

A discrete-time optimal control problem can be written as

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_T \\ u_0, \dots, u_{T-1}}} \; & \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) \\ \text{subj. to} \quad & x_{t+1} = f_t(x_t, u_t), \quad t \in \{0, \dots, T-1\} \\ & r(x_0, x_T) = 0 \\ & g_t(x_t, u_t) \leq 0, \quad t \in \{0, \dots, T-1\} \end{aligned}$$

Optimal control for trajectory generation

We can pose a trajectory generation problem as

$$\min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m} \sum_{t=0}^{T-1} \frac{1}{2} \|x_t - x_t^{\text{des}}\|_Q^2 + \frac{1}{2} \|u_t - u_t^{\text{des}}\|_R^2 + \frac{1}{2} \|x_T - x_T^{\text{des}}\|_{P_f}^2$$

Continuous-time Optimal Control problem

A continuous-time optimal control problem, i.e., $t \in \mathbb{R}$ can be written as

$$\begin{aligned} \min_{(x(\cdot), u(\cdot)) \in \mathcal{F}} \; & \int_0^T \ell_\tau(x(\tau), u(\tau)) d\tau + \ell_T(x(T)) \\ \text{subj. to} \quad & \dot{x}(t) = f_t(x(t), u(t)) \quad t \in [0, T] \\ & r(x(0), x(T)) = 0 \\ & g_t(x(t), u(t)) \leq 0 \quad t \in [0, T) \end{aligned}$$

Note that $\mathcal{F}$ is a space of functions (function space). This is an infinite dimensional optimization problem

- Cost functional $\ell : \mathcal{F} \to \mathbb{R}$

$$\ell(x(\cdot), u(\cdot)) = \int_0^T \ell_\tau(x(\tau), u(\tau)) d\tau + \ell_T(x(T))$$

- Space of trajectories (or trajectory manifold)

$$\mathcal{T} = \{(x(\cdot), u(\cdot)) \in \mathcal{F} | \dot{x}(t) = f_t(x(t), u(t)), t \geq 0\}$$

Chapter 2

Nonlinear Optimization

2.1 Unconstrained Optimization

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

with $\ell : \mathbb{R}^n \to \mathbb{R}$ a cost function to be minimized and x a decision vector

We say that x^* is a

- global minimum if $\ell(x^*) \leq \ell(x)$ for all $x \in \mathbb{R}^n$
- strict global minimum if $\ell(x^*) < \ell(x)$ for all $x \neq x^*$
- local minimum if there exists $\epsilon > 0$ such that $\ell(x^*) \leq \ell(x)$ for all $x \in B(x^*, \epsilon) = \{x \in \mathbb{R}^n | \|x - x^*\| < \epsilon\}$
- strict local minimum if there exists $\epsilon > 0$ such that $\ell(x^*) < \ell(x)$ for all $x \in B(x^*, \epsilon)$

Notation

We denote $\ell(x^*)$ the optimal (minimum) value of a generic optimization problem, i.e.

$$\ell(x^*) = \min_{x \in \mathbb{R}^n} \ell(x)$$

where x^* is the minimum point (optimal value for the optimization variable) i.e.

$$x^* = \arg\min_{x \in \mathbb{R}^n} \ell(x)$$

Gradient and Hessian

Gradient of a function: for a function $r : \mathbb{R}^n \to \mathbb{R}$ the gradient is denoted as

$$\nabla r(x) = \begin{bmatrix} \dfrac{\partial r(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial r(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Hessian matrix of a function: for a fcuntion $r : \mathbb{R}^n \to \mathbb{R}$ the Hessian matrix is denoted as

$$\nabla^2(r(x)) = \begin{bmatrix} \dfrac{\partial^2 r(x)}{\partial x_1^2} & \cdots & \dfrac{\partial^2 r(x)}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 r(x)}{\partial x_n x_1} & \cdots & \dfrac{\partial^2 r(x)}{\partial x_1^2} \end{bmatrix}$$

Gradient of a vector-valued function: for a vector field $r : \mathbb{R}^n \to \mathbb{R}^m$, the gradient is denoted as

$$\nabla r(x) = \begin{bmatrix} \nabla r_1(x) & \cdots & \nabla r_m(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1(x)}{\partial x_1} & \cdots & \frac{\partial r_m(x)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1(x)}{\partial x_n} & \cdots & \frac{\partial r_m(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

which is the transpose of the Jacobian matrix of r

2.1.1 Conditions of optimality

First order necessary condition (FNC) of optimality (unconstrained)

Let x^* be an unconstrained local minimum of $\ell : \mathbb{R}^n \to \mathbb{R}$ and assume that ℓ is continuously differentiable ($\mathcal{C}^1$) in $B(x^*, \epsilon)$ for some $\epsilon > 0$. Then $\nabla \ell(x^*) = 0$

Second order necessary condition (FNC) of optimality (unconstrained)

If additionally ℓ is twice continuously differentiable ($\mathcal{C}^2$) in $B(x^*, \epsilon)$, then $\nabla^2 \ell(x^*) \geq 0$ (The Hessian of ℓ is positive semidifinite)

Second order sufficient conditions of optimality (unconstrained)

Let $\ell : \mathbb{R}^n \to \mathbb{R} \in \mathcal{C}^2$ in $b(x^*, \epsilon)$ for some $\epsilon > 0$. Suppose that $x^* \in \mathbb{R}^n$ satisfies

$$\nabla \ell(x^*) = 0 and \nabla^2 \ell(x^*) > 0$$

Then x^* is a strict (unconstrained) local minimum of ℓ

Convex set

A set $X \subset \mathbb{R}^n$ is convex if for any two points x_A and x_B in X and for all $\lambda \in [0, 1]$, then

$$\lambda x_a + (1 - \lambda) x_B \in X$$

Convex functions

Let $X \subset \mathbb{R}^n$ be a convex set. A function $\ell : X \to \mathbb{R}$ is convex if for any two points x_A and x_B in X and for all $\lambda \in [0, 1]$, then

$$\ell(\lambda x_A + (1 - \lambda) x_B) \leq \lambda \ell(x_A) + (1 - \lambda) \ell(x_B)$$

2.1.2 Minimization of convex functions

Proposition

Let $X \subset \mathbb{R}^n$ be a convex set and $\ell : X \to \mathbb{R}$ a convex function. Then a local minimum of ℓ is also a global minimum
Proof: not done in class but present in slides for funsies

Necessary and sufficient condition of optimality (unconstrained)

For the unconstrained minimization of a convex function it can be shown that the first order necessary condition of optimality is also sufficient (for a global minimum).

Proposition

Let $\ell^n_{\mathbb{R}} \to \mathbb{R}$ be a convex function. Then x^* is a global minimum if and only if $\nabla \ell(x^*) = 0$

Proof: not done in class but present in slides for funsies

2.1.3 Quadratic programming

Let us consider a special class of optimization problems, namely quadratic optimization problems or quadratic programs:

$$\min_{x \in \mathbb{R}^n} x^T Q x + b^t x$$

with $Q = Q^T \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$

optimality conditions

First-order necessary condition for optimality: if x^* is a minimum then

$$\nabla \ell(x^*) = 0 \implies 2Qx^* + b = 0$$

Second-order necessary condition for optimality: if x^* is a minimum then

$$\nabla^2 \ell(x^*) \geq 0 \implies 2Q > 0$$

A necessary condition for the existence of minima for a quadratic program is that $Q \geq 0$. Thus, quadratic programs admitting at least a minimum are convex optimization problems.

properties

Since quadratic programs are convex programs ($Q \geq 0$ is necessary to have a local minimum), then the following holds:

- For a quadratic program necessary conditions of optimality are also sufficient and minima are global

If $Q > 0$, then there exists a unique global minimum given by

$$x^* = -\frac{1}{2} Q^{-1} b$$

2.2 Unconstrained Optimization Algorithms

2.2.1 Iterative descent methods

We consider optimization algorithms relying on the iterative descent idea. We denote $x^k \in \mathbb{R}^n$ an estimate of a local minimum at iteration $k \in \mathbb{N}$. The algorithm starts at a given initial guess x^0 and iteratively generates vectors $x^1, x^2, \dots$ such that ℓ is decreased at each iteration, i.e.

$$\ell(x^{k+1}) < \ell(x^k) \qquad k = 1, 2, \dots$$

two-step procedure

We consider a general two-step procedure that reads as follows

$$x^{k+1} = x^k + \gamma^k d^k, \qquad k = 1, 2, \dots$$

in which

1. each $\gamma^k > 0$ is a ”step-size”
2. $d^k \in \mathbb{R}^n$ is a ”direction”

The goal is to

1. choose a direction d^k along which the cost decreases for γ^k sufficiently small;
2. select a step-size γ^k guaranteeing a sufficient decrease.

In oher references these are called line-search methods.

2.2.2 Gradient methods

Let x^k be such that $\nabla\ell(x^k) \neq 0$. We start by considering the update rule

$$x^{k+1} = x^k - \gamma^k \nabla\ell(x^k)$$

i.e., we choose $d^k = \nabla\ell(x^k)$

From the first order Taylor expansion of ℓ at x we have

$$\begin{aligned} \ell(x^{k+1}) &= \ell(x^k) + \nabla\ell(x^k)^T(x^{k+1} - x^k) + o(\|x^{k+1} - x^k\|) \\ &= \ell(x^k) - \gamma^k\|\nabla\ell(x^k)\|^2 + o(\gamma^k) \end{aligned}$$

Thus, for $\gamma^k > 0$ sufficiently small it can be shown that $\ell(x^k + 1) < \ell(x^k)$

The update rule

$$x^{k+1} = x^k - \gamma^k \nabla\ell(x^k)$$

can be generalized to so called *gradient methods*

$$x^{k+1} = x^k + \gamma^k d^k$$

with d^k such that

$$\nabla\ell(x^k)^T d^k < 0$$

Also, d^k must be gradient related, i.e. d^k must not asymptotically become perpendicular to $\nabla\ell$

selecting the descent direction

Several gradient methods can be written as

$$x^{k+1}0x^k - \gamma^k D^k \nabla\ell(x^k) \quad k = 1, 2, \ldots$$

where $D^k \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. It can be immediately seen that

$$-\nabla\ell(x^k)^T D^k \nabla\ell(x^k) < 0$$

i.e. $d^k = -D^k\nabla\ell(x^k)$ is a descent direction. The choice of D^k must be made such that there exist d_1, d_2 positive real, such that $d_1 I \leq D^k \leq d_2 I$

Some choices for D^k:

- Steepest descent $D^k = I_n$
- Newton's method $D^k = (\nabla^2\ell(x^k))^{-1}$
  It can be used when $\nabla^2\ell(x^k) > 0$. It typically converges very fast asymptotically. For $\gamma^k = 1$ pure Newton's method
- Discretized Newton's method $D^k = (H(x^k))^{-1}$, where $H(x^k)$ is a positive definite symmetric approximation of $\nabla^2\ell(x^k)$ obtained by using finite difference approximations of the second derivatives
- Some regularized version of the Hessian

2.2.3 gradient method

The update rule obtained for $D^k = I$ is called steepest descent. The name steepest descent is due to the following property: the normalized negative gradient direction

$$d^k = -\frac{\nabla\ell(x^k)}{\|\nabla\ell(x^k)\|}$$

minimizes the slope $\nabla\ell(x^k)^T d^k$ among all normalized directions, i.e. it gives the steepest descent.

2.2.4 Newton's method for root finding

Consider the nonlinear root finding problem

$$r(x) = 0$$

Idea: iteratively refine the solution such that the improved guess x^{k+1} represents a root of the linear approximation of r about the current tentative solution x^k. Consider the linear approximation of r about x^k, we have

$$r^k(x^k + \Delta x^k) = r(x^k) + \nabla r(x^k)^T \Delta x^k$$

then, finding the zeros of the approximation, we have

$$\Delta x^k = -(\nabla r(x^k)^T)^{-1} r(x^k)$$

Thus, the solution is improved as

$$x^{k+1} = x^k - (\nabla r(x^k)^T)^{-1} r(x^k)$$

2.2.5 Newton's method for unconstrained optimization

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

stationary points $\bar{x}$ satisfy the first order optimality condition

$$\nabla \ell(\bar{x}) = 0$$

We can look at it as a root finding problem, with $r(x) = \nabla \ell(x)$, and solve it via Newton's method. Therefore, we can compute Δx^k as the solution of the linearization of $r(x) = \nabla \ell(x)$ at x^k, i.e.

$$\nabla \ell(x^k) + \nabla^2 \ell(x^k) \Delta x^k = 0$$

and run the update

$$x^{k+1} = x^k - (\nabla^2 \ell(x^k))^{-1} \nabla \ell(x^k)$$

We can introduce a variable step-size

$$x^{k+1} = x^k - \gamma^k (\nabla^2 \ell(x^k))^{-1} \nabla \ell(x^k)$$

This is called generalized Newton's method

Newton's method via Quadratic Optimization

Observe that

$$\nabla \ell(x^k) + \nabla^2 \ell(x^k) \Delta x^k = 0$$

is the first-order necessary and sufficient condition of optimality for the quadratic program

$$\Delta x^k = \arg\min_{\Delta x} \nabla \ell(x^k)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 \ell(x^k) \Delta x \tag{2.1}$$

Thus, the k-th iteration of Newton's method can be seen as

$$x^{k+1} = x^k + \Delta x^k$$

with Δx^k solution of the quadratic problem (2.1). Generalized version:

$$x^{k+1} = x^k + \gamma^k \Delta x^k$$

2.2.6 Gradient methods via quadratic optimization

Similarly to Newton's method, a descent direction $\Delta x^k = D^k \nabla \ell(x^k)$ can be seen as the direction that minimizes at each iteration a different quadratic approximation of ℓ about x^k. In fact, consider the quadratic approximation $\ell^k(x)$ about x^k given by

$$\ell^k(x) = \ell(x^k) + \nabla \ell(x^k)^T (x - x^k) + \frac{1}{2}(x - x^k)^T (D^k)^{-1} (x - x^k)$$

By setting the derivative to zero, we have

$$\nabla \ell(x^k) + (D^k)^{-1}(x - x^k) = 0$$

we can calculate the minimum of $\ell^k(x)$ and set it as the next iterate x^{k+1}

$$\Delta x^k = -D^k \nabla \ell(x^k)$$

2.2.7 step-size selection rules

- Constant step-size: $\gamma^k = \gamma > 0$
- Diminishing step-size: $\gamma^k \to 0$ as $k \to \infty$. It must hold that

$$\sum_{k=0}^{\infty} \gamma^k = +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} (\gamma^k)^2 < +\infty$$

  The above conditions avoid pathological choices of γ^k
- minimization rule
- Armijo rule

Armijo rule

Step-size is selected following the procedure:

1. set $\bar{\gamma}^0 > 0, \quad \beta \in (0, 1), \quad c \in (0, 1)$

given d^k descent direction we can consider

$$g^k(\gamma) = \ell(x^k + \gamma d^k), \quad g : \mathbb{R} \to \mathbb{R}$$

The value of $g^k(\gamma)$ for $\gamma = 0$ is $\ell(x^k)$. The minimization rule chooses as the value for γ the value that minimizes $g^k(\gamma)$. The partial minimization rule would search for a minimum in a restricted set of values for γ. Let us differentiate g wrt γ:

$$g'(\gamma) = \frac{d}{d\gamma} g(\gamma) = \frac{d}{d\gamma} \ell(x^k + \gamma d^k)$$
$$g'(0) = \frac{d}{d\gamma} \ell(x^k + \gamma d^k)\ \gamma = 0|_= \nabla \ell(x^k)^T d^k$$

We compute a linear approximation of $g(\gamma)$:

$$g(\gamma) = g(0) + g'(0)\gamma + o(\gamma)$$
$$\ell(x^k + \gamma d^k) = \ell(x^k) + \nabla \ell(x^k) d^k \gamma + o(\gamma)$$

This is the tangent to the $g(\gamma)$ curve at $\gamma = 0$. We also consider the line

$$\ell(x^k) + c\gamma \nabla \ell(x^k)^T d^k$$

which is a line with a slightly less negative slope given that $c \in (0, 1)$ The Armijo rule is applied as follows:

1. Set $\bar{\gamma}^0 > 0, \quad \beta \in (0,1), \quad c \in (0,1)$
2. While $\ell(x^k + \bar{\gamma}^1 d^k) \geq \ell(x^k) + c\bar{\gamma}^i \nabla \ell(x^k)^T d^k$:
$$\bar{\gamma}^{i+1} = \beta \bar{\gamma}^i$$
3. Set $\gamma^k = \bar{\gamma}^i$

Typical values are $\beta = 0.7$ and $c = 0.5$

Proposition: convergence with Armijo step-size

Let $\{x^k\}$ be a sequence generated by a gradient method $x^{k+1} = x^k - \gamma^k D^k \nabla \ell(x^k)$ with $d_1 I \leq D^k \leq d_2 I, \quad d_1, d_2 > 0$. Assume that γ^k is chosen by the Armijo rule and $\ell(x) \in \mathcal{C}^1$. Then, every limit point $\bar{x}$ of the sequence $\{x^k\}$ is a stationary point, i.e. $\nabla \ell(\bar{x}) = 0$

Recall that a vector $x \in \mathbb{R}^n$ is a limit point of a sequence $\{x^k\}$ in $\mathbb{R}^n$ if there exists a subsequnce of $\{x^k\}$ that converges to x.

convergene with constant or diminishing step-size

Let $\{x^k\}$ be a sequence generated by a gradient method $x^{k+1} = x^k - \gamma^k D^k \nabla \ell(x^k)$ with $d_1 I \leq D^k \leq d_2 I, \quad d_1, d_2 > 0$. Assume that for some $L > 0$

$$\|\nabla \ell(x) - \nabla \ell(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R}^n$$

i.e. The gradient is a Lipschitz continuous function. Assume either

1. $\gamma^k = \gamma > 0$ sufficiently small, or
2. $\gamma^k \to 0$ and $\displaystyle\sum_{t=0}^{\infty} \gamma^k = \infty$

Then, every limit point $\bar{x}$ of the sequence $\{x^k\}$ is a stationary point, i.e. $\nabla \ell(\bar{x}) = 0$

Remarks on gradient methods

- The propositions do not guarantee that the sequence converges and not even existence of limit points. Then either $\ell(x^k) \to -\infty$ or $\ell(x^k)$ converges to a finite value and $\nabla \ell(x^k) \to 0$. In the second case, one can show that any subsequnce $\{x^{k_p}\}$ converges to some stationary point $\bar{>}$ satisfying $\nabla \ell(\bar{x}) = 0$
- Existence of minima can be guaranteed by excluding $\ell(x^k) \to -\infty$ via suitable assumptions. Assume, e.g., ℓ coercive (radially unboundend)
- For general (nonconvex) problems, assuming coecivity, only convergence (of subsequences) to stationary points can be proven.
- for convex programs, assuming coercivity, convergence to global minima is guardanteed since necessary conditions of optimality are also sufficient.

2.3 Constrained optimization over convex sets

consider the optimization problem
$$\min_{x \in X} \ell(x)$$
where $X \subset \mathbb{R}^n$ is nonempty, convex, and closed, and ℓ is continuously differentiable on X.

Optimality conditions

If a point $x^* \in X$ is a local minimum of $\ell(x)$ over X, then

$$\nabla \ell(x^*)^T (\bar{x} - x^*) \geq 0 \qquad \forall \bar{x} \in X$$

Projection over a convex set

Given a point $x \in \mathbb{R}^n$ and a closed convex set X, it can be shown that

$$P_X(x) := \arg\min_{z \in X} \|z - x\|^2$$

exists and is unique. The point $P_X(x)$ is called the projection of x on X.

2.3.1 Projected gradient method

Gradient methods can be generalized to optimization over convex sets

$$x^{k+1} = P_X(x^k - \gamma^k \nabla \ell(x^k))$$

The algorithm is based on the idea of generating at each t feasible points (i.e. belonging to X) that give a descent in the cost. The analysis follows similar arguments to the one of unconstrained gradient methods.

2.3.2 Feasible direction method

Find $\tilde{x} \in \mathbb{R}^n$ such that

$$\tilde{x} = \arg\min_{x \in X} \ell(x^k) + \nabla \ell(x^k)^T (x - x^k) + \frac{1}{2}(x - x^k)^T (x - x^k)$$

Update the solution

$$x^{k+1} = x^k + \gamma^k (\tilde{x} - x^k)$$

where $(\tilde{x} - x^k)$ is a feasible direction as it is contained in the set by construction. For γ^k sufficiently small, $x^{k+1} \in X$

Barrier function strategy for inequality constraints

Consider the inequality constrained optimization problem

$$\min_{x \in \mathbb{R}^d} \ell(x) \text{subj. to } g_j(x) \leq 0 \quad j \in \{1, \ldots, r\}$$

inequality constraints can be relaxed and embedded in the cost function by means of a barrier function $-\varepsilon \log(x)$. The resulting unconstraind problem reads as

$$\min_{x \in \mathbb{R}^d} \ell(x) + \varepsilon \sum_{j=1}^{r} - \log(-g_j(x))$$

Implementation: every few iterations shring the barrier parameters ε

Methods such as this go by the name of *interior point methods*

2.4 Constrained optimization (equality and inequality constraints): optimality conditions

$$\begin{aligned} &\min_{x \in X} \ell(x) \\ \text{subj. to} \quad & g_j(x) \leq 0 \quad j \in \{1, \ldots, r\} \\ & h_i(x) = 0 \quad i \in \{1, \ldots, m\} \end{aligned}$$

Definition 2.4.1 (Set of active inequality constraints)**.** For a point x, the set of active inequality constraints at x is $A(x) = \{j \in \{1, \ldots, r\} | g_j(x) = 0\}$

Definition 2.4.2 (Regular point)**.** A point x is regular if the vectors $\nabla h_i(x), i \in \{1, \ldots, m\}$ and $\nabla g_j(x), j \in A(x)$, are linearly independent

Lagrangian function

In order to state the first-order necessary conditions of optimality for (equality and inequality) constrained problems it is useful to introduce the Lagrangian function

$$L(x,\mu,\lambda) = \ell(x) + \sum_{j=1}^{r} \mu_i g_j(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

Theorem 2.4.1 (Karush-Kuhn-Tucker necessary conditions)**.** Let x^* be a regular local minimum of

$$\begin{array}{rll} & \min\limits_{x\in\mathbb{R}^d} \ell(x) & \\ \text{subj. to} & g_j(x) \le 0 & j \in \{1,\dots,r\} \\ & h_i(x) = 0 & i \in \{1,\dots,m\} \end{array}$$

where ℓ, g_j and h_i are $\mathcal{C}^1$.
Then $\exists!$ μ_j^* and λ_i^*, called Lagrange multipliers, s.t.[1]

$$\begin{aligned} \nabla_1 L(x^*,\mu^*,\lambda^*) &= 0 \\ \mu_j^* &\ge 0 \\ \mu_j^* g_j(x^*) &= 0 \qquad j \in \{1,\dots,r\} \end{aligned}$$

Moreover, if ℓ, g_j and h_i are $\mathcal{C}^2$ it holds

$$y^T \nabla_{11}^2 \mathcal{L}(x^*,\mu^*,\lambda^*) y \ge 0$$

for all $y \in \mathbb{R}^n$ such that

$$\nabla h_i(x)^T y = 0, \quad i \in \{1,\dots,m\}, \qquad \nabla g_j(x)^T y = 0, \quad j \in A(x) \quad (\text{i.e.} j \in \{1,\dots,r\} \text{ s.t. } g_j(x) = 0\}$$

Remark. The condition $\mu^* g_j^*(x^*) = 0, j \in \{1,\dots,r\}$, is called complementary slackness

Notation. Points satisfying the KKT necessary conditions of optimality are referred to as KKT points. They are the counterpart of stationary points in constrained optimization.

Notation. note that ∇_{11} denotes the hessian of a function wrt the first variable

2.4.1 Quadratic programming (constrained)

Let us consider quadratic optimization problems with linear equality constraints

$$\begin{array}{cl} \min_{x\in\mathbb{R}^n} & x^T Q x + q^T x \\ \text{subj. to} & Ax = b \end{array}$$

with $q = Q^T \in \mathbb{R}^{n\times n}, q \in \mathbb{R}^n, a \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ The Lagrangian function is

$$L(x,\lambda) = x^T Q x + q^T x + \sum_{i=1}^{m} \lambda_i (A_i x + b_i) = x^T Q x + q^T x + \lambda^T (Ax - b)$$

And the gradient computes as

$$\nabla_1 L(x^*,\lambda^*) = 2Qx^* + q + \sum_{i=1}^{m} \lambda_i^* A_i^T = 2Qx^* + q + A^T \lambda^*$$

The equality constraints must also be enforced:

$$Ax^* - b = 0$$

We can note that

$$\nabla_2 L(x^*,\lambda^*) = Ax - b$$

[1] ∇_1 denotes the gradient wrt the first variable of the function

Therefore, first order conditions of optimality may be written as

$$\begin{bmatrix} \nabla_1 L(x^*, \lambda^*) \\ \nabla_2 L(x^*, \lambda^*) \end{bmatrix} = 0$$

This is always the case when only equality constraints are present. Second order necessary conditions for optimality impose that, if x^* is a minimum then

$$y^T \nabla^2 \mathcal{L}(x^*, \lambda^*) y = y^T Q y \geq 0$$

for all $y \in \mathbb{R}^n$ such that

$$\nabla h_i(x)^T y = 0 \qquad i \in \{1, \ldots, p\} \qquad \Longrightarrow \qquad A^T y = 0$$

namely, for all $y \in \mathbb{R}^n$ in the null-space of A^t

2.5 Constrained optimization (equality and inequality constraints): optimization algorithms

2.5.1 Newton's method for equality constrained problems

KKT points can be found by solving a root finding problem in variables x, λ wrt $r(x, \lambda) = \nabla L(x, \lambda)$. Newton's method for this root finding problem reads as

$$\begin{bmatrix} x^{k+1} \\ \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ \lambda^k \end{bmatrix} + \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix}$$

with

$$\begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = -(\nabla^2 L(x^k, \lambda^k))^{-1} \nabla L(x^k, \lambda^k)$$

where

$$\nabla^2 L(x^k, \lambda^k) = \begin{bmatrix} \nabla_{11} L(x^*, \lambda^*) & \nabla_{12} L(x^*, \lambda^*) \\ \nabla_{21} L(x^*, \lambda^*) & \nabla_{22} L(x^*, \lambda^*) \end{bmatrix} = \begin{bmatrix} H^k & \nabla h(x^k) \\ \nabla h(x^k)^T & 0 \end{bmatrix}$$

$$\nabla L(x^k, \lambda^k) = \begin{bmatrix} \nabla \ell(x^k) + \nabla h(x^k) \lambda^k \\ h(x^k) \end{bmatrix}$$

$$H^k = \nabla_{11}^2 L(x^k, \lambda^k) \qquad \nabla_{11} L(x, \lambda) = \nabla^2 \ell(x) + \sum_{i=1}^{m} \lambda_i \nabla^2 h_i(x)$$

We can write

$$\nabla^2 L(x^k, \lambda^k) \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = -\nabla L(x^k, \lambda^k)$$

namely

$$\begin{bmatrix} H^k & \nabla h(x^k) \\ \nabla h(x^k)^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \end{bmatrix} = - \begin{bmatrix} \nabla \ell(x^k) + \nabla h(x^k) \lambda^k \\ h(x^k) \end{bmatrix}$$

thus, $\Delta x^k, \Delta \lambda^k$ can be obtained as solution of a linear system of equations in the variables $\Delta x, \Delta \lambda$. The linear system of equations can be rewritten as

$$H^k \Delta x^k + \nabla H(x^k) \Delta \lambda^k = -\nabla \ell(x^k) - \nabla h(x^k) \lambda^k$$
$$\nabla h(x^k)^T \Delta x^k = -h(x^k)$$

and equivalently as

$$\nabla \ell(x^k) + H^k \Delta x^k + \nabla H(x^k) \Delta \lambda^{k+1}$$
$$h(x^k) + \nabla h(x^k)^T \Delta x^k = 0$$

We can observe that the above equations are the necessary and sufficient optimality conditions for the Quadratic Program (QP)

$$\min_{\Delta x} \nabla \ell(x^k)^T \Delta x + \frac{1}{2} \Delta x^T H^k \Delta x \ \text{ subj. to } \quad h(x^k) + \nabla h(x^k)^T \Delta x = 0$$

Therefore, in the Newton's update, we can obtain $(\Delta x^k, \lambda^{k+1})$ by solving this QP.

2.5.2 Sequential Quadratic Programming (SQP)

Start from a tentative solution x^0. For $k = 0, 1, \dots$ (up to convergence)

1. Compute $\nabla \ell(x^k), H^k, \nabla h(x^k)$
2. Obtain $(\Delta x^k, \Delta \lambda^+_{QP})$ from

$$\begin{array}{rl} \Delta x^k = \arg\min_{\Delta x} & \nabla \ell(x^k)^T \Delta x + \frac{1}{2} \Delta x^T H^k \Delta x \\ textsubj.to & h(x^k) + \nabla h(x^k)^T \Delta x = 0 \end{array} \tag{2.2}$$

   with Δ^*_{QP} the Lagrange multiplier associated to the optimal solution of (2.2)

3. Choose γ^k using Armijo's rule on *merit function* $M_1(x^k + \gamma \Delta x^k)$
4. Update

$$x^{k+1} = x^k + \gamma^k \Delta x^k$$
$$\lambda^{k+1} = \Delta \lambda^*_{QP}$$

2.5.3 Barrier function strategy for inequality constraints

Consider the inequality optimization problem

$$\begin{array}{rll} \min_{x \in \mathbb{R}^d} & \ell(x) & \\ textsubj.to & g_j(x) \leq 0 & j \in \{1, \dots, r\} \\ & h(x) = 0 & \end{array}$$

Inequality constraints can be embedded in the const function by means of a barrier function $-\varepsilon \log(x)$. The resulting unconstrained problem reads as

$$\min_{x \in \mathbb{R}^d} \ell(x) + \varepsilon \sum_{j=1}^{r} - \log(-g_j(x))$$
$$h(x) = 0$$

Implementation: every few iterations shrink the barrier parameters ε

Chapter 3

Optimality conditions for optimal control

3.1 boh

3.1.1 Dynamics as equality constraints

Let us rerwrite the nonlinear dynamics of a dt system as an implicit equality constraint $h : \mathbb{R}^{nT} \times \mathbb{R}^{mT} \to \mathbb{R}^{nT}$

$$h(\bar{\mathbf{x}}, \bar{\mathbf{u}}) := \begin{bmatrix} f_0(x_0, u_0) - x_1 \\ \vdots \\ f_{T-1}(x_{T-1}, u_{T-1}) - x_T \end{bmatrix}$$

so that a curve $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is a trajectory of the system if it satisfies the (possibly nonlinear) equality constraint

$$h(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = 0$$

3.1.2 system trajectories and trajectory manifold

We can now define the trajectory manifold $\mathcal{T} \subset \mathbb{R}^{nT} \times \mathbb{R}^{mT}$ of (ref)

$$\mathcal{T} := \{((x),(u)) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT} | h((x),(u)) = 0\} = \{((x),(u)) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT} | x_{t+1} = f_y(x_t, u_t), t = 0, \ldots, T-1\}$$

Let $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathcal{T}$ be a trajectory of the system, i.e. a point on the trajectory manifold $\mathcal{T}$. The tangent space to $\mathcal{T}$ at a given trajectory (point) $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$, denoted as $T_(\bar{\mathbf{x}}, \bar{\mathbf{u}})\mathcal{T}$, is the set of trajectories satisfying the linearization of $x_{t+1} = f_t(x_t, u_t)$ about the trajectory $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ That is, $T_(\bar{\mathbf{x}}, \bar{\mathbf{u}})\mathcal{T} = \{(\boldsymbol{\Delta}\mathbf{x}, \boldsymbol{\Delta}\mathbf{u}) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT} | \nabla_1 h(\mathbf{x}, \mathbf{u})^T \boldsymbol{\Delta}\mathbf{x} + \nabla_2 h(\mathbf{x}, \mathbf{u})^T \boldsymbol{\Delta}\mathbf{u} = 0\}$ is the set of trajectories $(\boldsymbol{\Delta}\mathbf{x}, \boldsymbol{\Delta}\mathbf{u})$ of

$$\Delta x_{t+1} = A_t \Delta x_t + B_t \Delta u_t$$

with

$$A_t = \nabla_1 f_t(\bar{x}_t, \bar{u}_t)^T B_t = \nabla_2 f_t(\bar{x}_t, \bar{u}_t)^T$$

3.2 Unconstrained optimal control problem (d-t)

We look for a solution of the discrete-time optimal control problemm

$$\begin{array}{rl} \min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m} & \displaystyle\sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) \\ \text{subj. to} & x_{t+1} = f_t(x_t, u_t), \quad t \in \{0, \ldots, T-1\} \end{array}$$

with given initial condition $x_0 = x_{\text{init}} \in \mathbb{R}^n$.

From now on, we will assume that functions $\ell_t(\cdot,\cdot), \ell_T(\cdot), f_t(\cdot,\cdot)$ are twice continuosly differentiable, i.e. the are $\mathcal{C}^2$

3.3 KKT conditions for unconstrained optimal control

the Lagrangian function has the form

$$
\begin{aligned}
\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda) = \ell(\mathbf{x}, \mathbf{u}) + \lambda^T h(\mathbf{x}, \mathbf{u}) = \\
\sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) + \sum_{t=0}^{T-1} \lambda_{t+1}^T (f_t(x_t, u_t) - x_{t+1}) \\
= \sum_{t=0}^{T-1} (\ell_t(x_t, u_t) + \lambda_{t+1}^T (f_t(x_t, u_t) - x_{t+1}) + \ell_T(x_T) \\
= \sum_{t=0}^{T} \mathcal{L}_t(x_t, u_t, \lambda)
\end{aligned}
$$

where $\lambda \in \mathbb{R}^{nT}$ and

$$
\begin{aligned}
\mathcal{L}_0(x_0, u_0, \lambda) &= \ell_0(x_0, u_0) + \lambda_1^T f_0(x_0, u_0) \\
\mathcal{L}_t(x_t, u_t, \lambda) &= \ell_t(x_t, u_t) + \lambda_1^T f_t(x_t, u_t) - \lambda_t x_t \\
\mathcal{L}_T(x_T, \lambda) &= \ell_T(x_T) - \lambda_T^T x_T
\end{aligned}
$$

Let $(\mathbf{x}^*, \mathbf{u}^*)$ be a regular point for the dynamics constraints and an optimale (state-input) trajectory. Then there exists λ^* such that $\nabla \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) = 0$

Let us explicitly write condition $\nabla_{(1,2)} \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) = 0$

$$
\nabla_{(1,2)} \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) = \begin{bmatrix} \nabla_1 \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) \\ \nabla_2 \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) \end{bmatrix} = 0
$$

Let us note that

$$
\nabla_1 \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*) = \left. \begin{bmatrix} \dfrac{\partial \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda)}{\partial (x_1)_1} \\ \vdots \\ \dfrac{\partial \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda)}{\partial (x_1)_n} \\ \vdots \\ \dfrac{\partial \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda)}{\partial (x_T)_1} \\ \vdots \\ \dfrac{\partial \mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda)}{\partial (x_T)_n} \end{bmatrix} \right|_{\mathbf{x}=\mathbf{x}^*}
$$

Since $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda) = \sum_{t=0}^{T} \mathcal{L}(x_t, u_t, \lambda)$, we can explpot this sparsity and write

$$
\nabla_2 \mathcal{L}_0(x_0, u_0, \lambda) = 0 \qquad \nabla_2 \ell_0(x_0, u_0) \nabla_2 f_0(x_0, u_0) \lambda_0
$$

$$
\begin{bmatrix} \nabla_1 \mathcal{L}_t(x_t, u_t, \lambda) \\ \nabla_2 \mathcal{L}_t(x_t, u_t, \lambda) \end{bmatrix} = 0 \quad \begin{bmatrix} \nabla_1 \ell_t(x_t, u_t) + \nabla_1 f_t(x_t, u_t) \lambda_{t+1} - \lambda_t \\ \nabla_2 \ell_t(x_t, u_t) + \nabla_2 f_t(x_t, u_t) \lambda_{t+1} \end{bmatrix} = 0 \quad t = 1, \dots, T-1
$$

$$
\nabla_1 \mathcal{L}_t(x_t, \lambda) = 0 \qquad \nabla \ell_T(x_T) - \lambda_T = 0
$$

Let us introduce some notation:

$$
\begin{aligned}
\nabla_1 \ell_t(x_t^*, u_t^*) &= a_t \in \mathbb{R}^n \\
\nabla_1 f_t(x_t^*, u_t^*) &= A_t^T \\
\nabla_2 \ell_t(x_t^*, u_t^*) &= b_t \in \mathbb{R}^n \\
\nabla_2 f_t(x_t^*, u_t^*) &= B_t^T
\end{aligned}
$$

So we can rewrite the KKT conditions for unconstrained optimal control as:

$$\begin{aligned}
&\lambda_t^* = A_t^T \lambda_{t+1}^* + a_t \qquad t = T-1, \ldots, 1 \\
&\lambda_T^* = \nabla \ell(x_T^*) \\
&B_t^T \lambda_{t+1}^* + b_t = 0 \qquad t = 0, \ldots, T-1
\end{aligned}$$

3.3.1 Indirect methods for optimal control

Solving the optimality conditions:

- Guess some $u_t^0, \quad t = 0, \ldots, T-1 \quad k = 0$
- run "forward"
$$x_{t+1}^0 = f - t(x_t^0, u_t^0) \quad x_0$$
- run "backward"
$$a$$
- given $\lambda_t^0 \quad t = 1, \ldots, T$ solve:
$$\nabla_2 \ell(x_t^0, u_t) + \nabla_2 f(x_t^0, u_t)\lambda_{t+1}^0 = 0 \quad t = 0, \ldots, T-1$$
to get $u_t^1 \quad t = 0, \ldots, T-1$

3.4 KKT conditions for constrained optimal control

We look for a solution of the discrete-time optimal control problem

$$\begin{aligned}
\min_{x_0 \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^{nT}, \mathbf{U} \in \mathbb{R}^{mT}} \quad & \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) \\
\text{subj. to} \quad & x_{t+1} = f_t(x_t, u_t), \quad t = 0, \ldots, T-1 \\
& r(x_0, x_T) = 0 \\
& g_t(x_t, u_t) \leq 0, \quad t = 0, \ldots, T-1
\end{aligned}$$

where

- $\ell_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the stage cost,
- $\ell_T : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost,
- $r\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{p_0}$ identifies a boundary constraint on initial and final states,
- $g_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ for each t identifies point-wise constraints on state and input at some time t

The Lagrangian function has the form

$$\begin{aligned}
\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda, \mu) &= \ell(\mathbf{x}, \mathbf{u}) + \lambda_d^T h(\mathbf{x}, \mathbf{u}) + \lambda_b^T r(x_0, x_T) + \mu^T g(\mathbf{x}, \mathbf{u}) \\
&= \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) + \sum_{t=0}^{T-1} \lambda_{d,t+1}(f_t(x_t, u_t) - x_{t+1}) + \lambda_b^T r(x_0, x_T) + \sum_{t=0}^{T-1} \mu_t^T g_t(x_t, u_t) \\
&= \sum_{t=0}^{T} \mathcal{L}_t(x_t, u_t, \lambda, \mu)
\end{aligned}$$

Chapter 4

Linear Quadratic (LQ) optimal control

Consider a linear quadratic optimal control problem as:

$$\begin{aligned} \min_{\substack{x_1,\ldots,x_T \\ u_0,\ldots,u_{T-1}}} \quad & \sum_{t=0}^{T-1} \frac{1}{2}[x_t^T q_t x_t + u_t^T R_t u_t] + \frac{1}{2} x_T^t q_T x_T \\ \text{subj. to} \quad & x_{t+1} = A_t x_t + B_t u_t \quad t = 0, \ldots, T-1 \\ & x_0 = x_{\text{init}} \end{aligned}$$

We assume $Q_t = Q_t^T \geq 0 \forall t = 0, \ldots, T-1$, $Q_t = Q_t^T \geq 0$, and $R_t = R_t^T > 0 \forall t = 0, \ldots, T-1$

4.1 First order optimality condition

$$\nabla_1 f_t(x_t, u_t) = A_t^T$$
$$\nabla_1 \ell(x_t, u_t) = \nabla_1 (\frac{1}{2} x_t^T Q_t x_t + \frac{1}{2} u_t^T R_t u_t) = Q_t x_t$$
$$\nabla_2 f_t(x_t, u_t) = B_t^T$$
$$\nabla_2 \ell_t(x_t, u_t) = R_t u_t$$

therefore

$$\lambda_t^* = A_t^T \lambda_t + 1^* + Q_t x_t^* \quad t = T_1, \ldots, 0$$
$$\lambda_T^* = Q_T x_T^*$$
$$B_t^T \lambda_{t+1}^* + R_t u_t^* = 0 \quad t = 0, \ldots, T-1$$

Remark: second order optimality conditions

$$y^T \nabla^2_{(1,2)(1,2)} \mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, \lambda^*,) y \geq 0$$

For vectors y satisfying the "linear approximation of the constraint". The hessian turns out as

$$\begin{bmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_n \end{bmatrix}$$

Because $R_t > 0$ it is invertible. Therefore, we can write

$$u_t^* = -R_t^{-1} B_t^T \lambda_{t+1}^*$$

Introducing a matrix $P_t = P_t^T \geq 0$, it can be proven that

$$\lambda_t^* = P_t x_t^*$$

Assuming that it holds for some $t \leq T-1$, then we have

$$u_t^* = -R_t^{-1} B_t^T P_{t+1} x_{t+1}^*$$

Now, considering the constraint represented by the dynamics

$$u_t^* = -R_t^{-1} N_t^T P_{t+1}(A_t x_t^* + B_t u_t^*)$$

Solving by u_t^* yields

$$u_t^* = -(R_t + B_t^T p_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t x_t^* \quad t = 0, \dots, T-1$$

we now get

$$\begin{aligned} u_t^* &= -R_t^{-1} B_t^T p_{t+1} x_{t+1}^* \\ &= -R_t^{-1} B_t^T p_{t+1}(A_t x_t^* + B_t u_t^*) \end{aligned}$$

we multiply both sides by R_t:

$$\begin{aligned} R_t u_t^* &= -B_t^T P_{t+1}(A_t x_t^* + B_t u_t^*) \\ R_t u_t^* &= -B_t^T P_{t+1} A_t x_t^* - B_t^T P_{t+1} B_t u_t^*) \\ (R_t + B_t^T P_{t+1} B_t) u_t^* &= -B_t^T P_{t+1} A_t x_t^* \end{aligned}$$

The matrix on the left is clearly positive definite, therefore:

$$u_t^* = -(R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t x_t^*$$

which we can write as

$$u_t^* = K_t^* x_t^*$$

that is, the optimal control is a state feedback with gain $-(R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1}$

$$x_{t+1} = A_t x_t^* - B_t (R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t x_t^*$$

which we can rewrite as

$$x_{t+1}^* = (A_t - B_t (R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t) x_t^*$$

which is a closed loop system. We multiply both sides by P_{t+1} and obtain

$$P_{t+1} x_{t+1}^* = P_{t+1}(\cdots) x_t^*$$

On the left side of the equation we have obtained λ_{t+1}^*

$$\lambda_{t+1}^* = P_{t+1}(\cdots) x_t^*$$

Remembering that $\lambda_t^* = A_t^T + \lambda_{t+1}^* + Q_t x_t^*$ we multiply both sides by A_t^T and then add $Q_t x_t^*$ and obtain

$$A_t^T \lambda_{t+1}^* + Q_t x_t^* = A_t^T P_{t+1}(\cdots) x_t^* + Q_t x_t^*$$

and because

$$\lambda_t^* = P_t x_t^*$$

then

$$P_t x_t^* = [A_t^T P_{t+1}(\cdots) + Q_t] x_t^*$$

so

$$P_t x_t^* = \left[A_t^T P_{t+1} A_t - a_t^T P_{t+1} B_t (R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t + Q_t\right] x_t^*$$

from which

$$P_T = \left[A_t^T P_{t+1} A_t - a_t^T P_{t+1} B_t (R_t + B_t^T P_{t+1} B_t)^{-1} B_t^T P_{t+1} A_t + Q_t\right] \tag{4.1}$$

because $\lambda_T^* = Q_T x_T^*$ we have that

$$P_T = Q_T$$

Therefore, by propagating equation (4.1) back in time, P_t can be calculated. equation (4.1) is called difference Riccati equation

- gains K_t^* can be precomputed offline and the unsed for different x_0
- It can be shown that if $T \to \infty$ the gains K_t^* converge and asymptotically stabilize the system

4.2 Infinite horizion LQ optimal control

Consider the infinite-horizon optimal control problem

$$\begin{aligned} \min_{\substack{x_1,\dots,x_T \\ u_0,\dots,u_{T-1}}} \quad & \sum_{t=0}^{\infty} \frac{1}{2}[x_t^T Q x_t + u_t^T R u_t] \\ \text{subj. to} \quad & x_{t+1} = Ax_t + Bu_t \quad t = 0,\dots,T-1 \\ & x_0 = x_{\text{init}} \end{aligned}$$

where

- $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$
- $A \in \mathbb{R}^{n\times n}$
- $B \in \mathbb{R}^{n\times m}$
- $Q \in \mathbb{R}^{n\times n}$ and $Q = Q^T \geq 0$
- $R \in \mathbb{R}^{m\times m}$ and $R = R^T > 0$

We assume the pair (A, B) is controllable and the pair (A, C) with $Q = C^T C$ is observable Let us write

$$y_t = Cx_t$$

which leads to

$$\frac{1}{2}x_t^T Q x_t = \frac{1}{2}x_t^T C^T C x_t = \frac{1}{2}y_t^T y_t$$

The controllability assumption guardantees that an optimal controller exists: if (A, B) controllable, then $\exists \bar{u}_0, \dots, \bar{u}_{T-1}$ for T sufficiently large $(T = n)$ such that $\forall x_0 \in \mathbb{R}^n \implies x_T = 0$. Consider the input

$$\bar{u}_0, \dots, \bar{u}_{T-1}, 0, \dots, 0, \dots$$

Let us compute the cost associated to this input

$$\sum_{t=0}^{\infty} \frac{1}{2}[x_t^T Q x_t + u_t^T R u_t] = \sum_{t=0}^{T-1} \frac{1}{2}\bar{x}_t^T Q \bar{x}_t + \frac{1}{2}\bar{u}_t^T R \bar{u}_t$$

We can note that the cost is a finite quantity. Because the cost is finite, There must exist a solution which minimizes the cost.

Proposition 4.2.1. Let the pair (A, B) be controllable and the pair (A, C) with $Q = C^T C$ be observable. Then the following holds:

- there exists a unique positive definite P_∞ equilibrium solution of the Difference Riccati Equation. That is, P_∞ is a solution of

  $$P_\infty = Q + A^T P_\infty A - A^T P\infty B(R + B^T P_\infty B)^{-1} B^T P_\infty A$$

  which is called Algebraic Riccati Equation

- the optimal control is a feedback of the state given by:

  $$\begin{aligned} & K^* = -(R + B^T P_\infty B)^{-1}(B^T P_\infty A) \\ & u_t^* = K^* x_t^* \\ & x_{t+1}^* = Ax_t^* + Bu_t^* \quad t = 1, 2, \dots \quad x_0^* = X_{\text{init}} \end{aligned}$$

Remark. The observability of (A, C) guardantees that if the stage cost goes to zero, then the state trajectory goes to zero.

Chapter 5

Optimality Conditions for Unconstrained Optimal Control via Shooting

Let us consider the system dynamics

$$x_{t+1} = f_t(x_t, u_t) \quad t = 0, \dots, T-1 \quad x_0 \text{ given}$$

and let us suppose we have an input sequence $u_0, \dots, u_{T-1}$. We have:

$$\begin{aligned}
&x_1 = f_0(x_0, u_0) = \tilde{\Phi}_1(\mathbf{u}) \\
&x_2 = f_1(x_1, u_1) = f_1(f_0(x_0, u_0), u_1) = \tilde{\Phi}_2(\mathbf{u}) \\
&\vdots \\
&x_t = \tilde{\Phi}_t(\mathbf{u}) \qquad t = 0, \dots, T-1 && \vdots \\
&x_T = \tilde{\Phi}_T(\mathbf{u}) \qquad t = 0, \dots, T-1
\end{aligned}$$

Idea: express the state x_t at each $t = 1, \dots, T$ as a function of the input sequence $\mathbf{u}$ unly. For all t we can introduce a map $\Phi_t : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$x_t := \Phi_t(\mathbf{u})$$

compact notation

$$\Phi(\mathbf{u}) = \text{col}(\Phi_1(\mathbf{u}), \dots, \Phi_T(\mathbf{u}))$$

so that

$$\mathbf{x} = \Phi(\mathbf{u})$$

Note: Given any arbitrary $\bar{u}_0, \dots, \bar{u}_{T-1}$, we have that $\Phi_{t+1}(\bar{\mathbf{u}}) = f_t(\Phi_t(\bar{\mathbf{u}}), u_t)$ by construction. This is equivalent to the equality constraint for the optimal control problem.

5.1 Reduced optimal control problem

We can rewrite the optimal control problem as

$$\min_{\mathbf{u} \in \mathbb{R}^{mT}} \sum_{t=0}^{T-1} \ell_t(\Phi_t(\mathbf{u}), u_t) + \ell_T(\Phi_T(\mathbf{u}))$$

as noted before, the equality constraint is satisfied by construction, making this an unconstrained optimization problem. We can rewrite it compactly as

$$\min_{\mathbf{u} \in \mathbb{R}^{mT}} \ell(\Phi(\mathbf{u}), \mathbf{u})$$

and by defining $J(\mathbf{u}) := \ell(\Phi(\mathbf{u}), \mathbf{u})$

$$\min_{\mathbf{u}\in\mathbb{R}^{mT}} J(\mathbf{u}) :$$

This goes by the name of *reduced* or *condensed optimal control problem.* The procedure of writing $\mathbf{x}$ as a function of $\mathbf{u}$ and then plugging it into the optimal control problem is called shooting.

Remark. if we consider path input constraints

$$g_0(u_0) \leq 0 \vdots g_{T-1}(u_{T-1}) \leq 0$$

the problem becomes

$$\min_{\mathbf{u}\in\mathbb{R}^{mT}} J(\mathbf{u}) textsubjto$$

Remark. if we have constraints of the type

$$g_0(x_0, u_0) \leq 0 \vdots g_{T-1}(x_{T-1}, u_{T-1}) \leq 0$$

They can be rewritten as functions of x_0 and $\mathbf{u}$ only, however $\Phi(\cdot)$ must be explicitly known

5.2 Algorithms for optimal control problem solution

We can apply the gradient method, i.e.

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \gamma \nabla J(\mathbf{u}^k)$$

We can formally write the expression of $\nabla J(\mathbf{u}) = \nabla \ell(\Phi(\mathbf{u}), \mathbf{u})$ by using the chain rule of differentiation.

$$\nabla \Phi(\mathbf{u}) = \nabla \begin{bmatrix} \Phi_{1,1}(\mathbf{u}) \\ \Phi_{1,2}(\mathbf{u}) \\ \vdots \\ \Phi_{t,1}(\mathbf{u}) \\ \Phi_{t,2}(\mathbf{u}) \\ \vdots \end{bmatrix}$$

$$\nabla \Phi(\mathbf{u}) = \begin{bmatrix} \frac{\partial \Phi_{1,1}}{\partial u_0} & \frac{\partial \Phi_{1,2}}{\partial u_0} & \cdots & \frac{\partial \Phi_{T,n}}{\partial u_0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_{1,1}}{\partial u_{T-1}} & \frac{\partial \Phi_{1,2}}{\partial u_{T-1}} & \cdots & \frac{\partial \Phi_{T,n}}{\partial u_{T-1}} \end{bmatrix}$$

where $\Phi_{t,j} : \mathbb{R}^{mT} \to \mathbb{R}$, therefore the above matrix is a matrix of scalars. Let us introduce an auxiliary function $\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda) : \mathbb{R}^{nT} \times \mathbb{R}^{mT} \times \mathbb{R}^{nT} \to \mathbb{R}$ given by

$$\mathcal{L}(\mathbf{x}, \mathbf{u}, \lambda) = \ell(\mathbf{x}, \mathbf{u}) + h(\mathbf{x}, \mathbf{u})^T \lambda$$

where $\lambda \in \mathbb{R}^{nT}$ is a "costate vector" and

$$h(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} f_0(x_0, u_0) - x1 \\ \vdots \\ f_{T-1}(x_{T-1} u_{T-1}) - x_T \end{bmatrix}$$

To compute $\nabla J(\mathbf{u})$ let us evaluate $\updownarrow(\cdot)$ for $\mathbf{x} = \Phi(\mathbf{u})$. Since $h(\Phi(\mathbf{u},)\mathbf{u}) = 0$ it holds that

$$\mathcal{L}(\Phi(\mathbf{u}), \mathbf{u}, \lambda) = J(\mathbf{u}) \quad \forall \lambda \in \mathbb{R}^{nT}$$

Therefore

$$\nabla \mathcal{L}(\Phi(\mathbf{u}), \mathbf{u}, \lambda) = \nabla J(\mathbf{u}) \quad \forall \lambda$$

hence we can write

$$\nabla J(\mathbf{u}) = \nabla\Phi(\mathbf{u})(\nabla_1\ell(\Phi(\mathbf{u}),\mathbf{u}) + \nabla_1 h(\Phi(\mathbf{u}),\mathbf{u})\lambda) + \nabla_2\ell(\Phi(\mathbf{u}),\mathbf{u}) + \nabla_2 h(\Phi(\mathbf{u}),\mathbf{u})\lambda$$

which hodls for every λ. Therefore, for a given $\mathbf{u}$, we can cleverly select $\lambda = \lambda(\mathbf{u})$ such that:

$$\nabla_1\ell(\Phi(\mathbf{u}),\mathbf{u}) + \nabla_1 h(\Phi(\mathbf{u}),\mathbf{u})\lambda(\mathbf{u})$$

which leads to

$$\nabla J(\mathbf{u}) = \nabla_2\ell(\Phi(\mathbf{u}),\mathbf{u}) + \nabla_2 h(\Phi(\mathbf{u}),\mathbf{u})\lambda(\mathbf{u})$$

5.2.1 First order necessary condition for optimality

Let $\mathbf{u}^*$ be a local minimum with $\mathbf{x}^* = \Phi(\mathbf{u}^*)$ Then

$$\nabla J(\mathbf{u}^*) = 0$$

that is, if there exists a λ^* such that

$$\nabla_1\mathcal{L}(\Phi(\mathbf{u}^*),\mathbf{u}^*,\lambda^*) = 0$$

it holds

$$\nabla_2\mathcal{L}(\Phi(\mathbf{u}^*),\mathbf{u}^*,\lambda^*) = 0$$

5.2.2 explicit computation of

$$\mathcal{L}(x,u,\lambda) = \sum_{t=0}^{T-1}[\ell_t(x_t,u_t) + \lambda_{t+1}^T f_t(x_t,u_t) - \lambda_{t+1}x_{t+1}] + \ell_T(x_T) = \sum_{t=0}^{T-1}(x_t,u_t)$$

$$\nabla_1\ell_1(x_1,u_1) + \nabla_1 f_1(x_1,u_1)\lambda_2 - \lambda_1 = 0$$
$$\mathcal{L}(x,u,\lambda))\ell_0(x_0,u_0) + \lambda_1^T f_0(x_0,u_0) - \lambda_1 x_1 + \ell_1(x_1,u_1) + \lambda_2^T f_1(x_1,u_1) - \lambda_2 x_2 + \dots$$

Notice we can write

$$A_t^T = \nabla_1 f(x_t,u_t)$$
$$B_t^T = \nabla_2 f(x_t,u_t)$$

so that we obtain

$$\lambda_t = A_t^T\lambda_t + 1 + a_t$$

so given $u_0,\dots,u_{T-1}$ and $x_1,\dots,x_T$ such that $x_{t+1} = f(x_t,u_t)$ we can compute $\lambda_T,\dots,\lambda_1$ running backwards. We can also state that

$$(\nabla J(u))_t = \nabla_2\ell_t(x_t,u_t) + \nabla_2 f_t(x_t,u_t)\lambda_{t+1}$$

which we can rewrite as

$$(\nabla J(u))_t = B_t^T\lambda_{t+1} + b_t$$

Chapter 6

Dynamic Programming

Chapter 7

Numerical methods for nonlinear optimal control

Chapter 8

Optimization-based predictive control