We will treat Model-based control.

- Ideas
- examples
- mrac (model reference adaptive control)
- stability of adaptive control systems

Let us call $\mathcal{P}$ the plant model. We define a number of signals as input to the model (control input u, disturbance $d \in \mathbb{R}^q$, reference signals $r \in \mathbb{R}^s$) and as output (measured output $y \in \mathbb{R}^p$, error to the reference $e \in \mathbb{R}^r$). The goal is to determine a feedback control $\mathcal{C}$ in a way st certain criteria are met for the system. We start by restricting the class of signals d and r

- $r, d \in \mathcal{L}_\infty$

Desired performance:

- $e(t), t \geq 0$ bounded ("small" if d, r are "small")
- $d(t) = 0 \implies \lim_{t\to\infty} \|e(t)\| = 0$

For adaptive control we assume to have a parameterized family of plant models $\mathcal{P}(\theta)$ where θ is a set of parameters that range within bounded sets, in order to deal with the issue of parametric uncertainties. One option to deal with having infinitely many plants given by the variation of the parameters is to have infinitely many parameterized controllers that are tuned to solve the control problem.

We state 2 problems:

1. Given $r(\cdot) \in \mathcal{L}_\infty$ design a controller $\mathcal{C}(k*)$ with k^* fixed values such that when $d(\cdot) = 0$ the closed loop system is such that
   (a) all "internal" trajectories are bounded
   (b) $\lim_{t\to\infty} \|e(t)\| = 0$
2. when $d(\cdot) \neq 0, d(\cdot) \in \mathcal{L}_\infty$
   (a) All internal trajectories are bounded
   (b) $e(t)$ is ultimately bounded, meaning

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall d(\cdot) \in \mathcal{L} : \|d(\cdot)\| \leq \delta then lim sup_{t\to\infty} \|e(t)\| \leq \varepsilon$$

0.1 Model Reference Adaptive control

we start with a system called the reference model $\mathcal{M}_r$ which has as input the reference signal r and as output $y_r(t)$, while the actual ssytem $\mathcal{P}(\theta)$ takes as input the control input u and as output the measured output $y(t)$. We define the tracking error as

$$e = y - y_r$$

The goal for the control system $\mathcal{C}(k)$ is

1. All internal signlas bounded if $r(t) \in \mathcal{L}_{infty}, y_r(t) \in \mathcal{L}_{infty}$
2. $\lim_{t\to\infty} \|e(t)\| = 0$

$\mathcal{M}_r$ is typically a stable filter, and is called the "handling quality" in aerospace. The reference model defines the desired behaviour of the system. $\mathcal{C}(k)$ must have a structure such that the control problem is solvable. We introduce the

Certainty equivalence princlple

You need to find matching conditions between the gains of the controller and the parameter vector of the plant, meaning a function between k and θ :

$$k^* = k^*(\theta) k^* : \mathbb{R}^p \to \mathbb{R}^\mu$$

if $k = k^*(\theta)$ then $\mathcal{C}(k)$ solves the mrac problem for $\mathcal{P}(\theta)$ A tuning mechanisim τ (*update law*) is used to fix k in real time $k \to \hat{k}(t)$:

$$\dot{\hat{k}}(t) = \tau$$

1. start with a matching condition $k^* = k^*(\theta)$ (the problem will be solved if i know θ)
2. replace k with a time varying $\hat{k}(t)$
3. find τ in $\dot{\hat{k}}(t) = \tau(\dots)$ such that $\hat{k}(t) \to k^*$ as $t \to \infty$
4. use $\mathcal{C}(\hat{k})$ to control the system

Attention must be paid to the fact that the controller becomes a dynamical system and its transient may destroy the stabilty of the system.

Two ways of tackling the problem are available

- Direct adaptive control: we find the matching condition and design $\tau : \hat{k}(t) \to k^*$ (directly updating the controller gains)
- Indirect adaptive control: estimate the parameters of the plant $\hat{\theta}$ online and obtain the estimate of the gain from the matching condition $\hat{k}(t) = k^*(\hat{\theta})$

0.1.1 Example: SISO scalar plant model

$$G(s) = \frac{b}{s+a} \qquad \theta = \begin{pmatrix} a \\ b \end{pmatrix}$$

Assumption: sign of b is known ($b > 0$) and $\exists b_0 > 0 : b \geq b_0$

$$G_r(s) = \frac{\beta_m}{s + \alpha_m} \quad \beta_m > 0, \alpha_m > 0$$

Matching conditions

$$\dot{y}_r = -\alpha_m y_r + \beta_m r$$
$$\dot{y} = -ay + bue = y - y_r$$

Change of coordinates:

$$\dot{e} = -ay + bu + \alpha_m y_r - \beta_m r$$

substituting $y_r = y - e$

$$\dot{e} = -ay + bu\alpha_m y - \alpha_m e - \beta_m r$$

cleaning up the expression

$$\dot{e} = -\alpha_m e + (\alpha_m - a)y + bu - \beta_m r$$

If only the first term on the right was present we wuold have an exponentially descending system. The control can therefore be chosen as

$$u = k_1 y + k_2 r$$

the first term being a fb and the second being ff we obtain

$$\dot{e} = -\alpha_m e + (\alpha_m - a + bk_1)y + (bk_2 - \beta_m)r$$

The matching conditions are therefore

$$k_1^* = \frac{a - \alpha_m}{b} \qquad k_2^* = \frac{\beta_m}{b}$$

Because a and b are not known, we must use estimates for the control:

$$u = \hat{k}_1(t)y + \hat{k}_2(t)r$$

leading to

$$\dot{e} = -\alpha_m e + (\alpha_m - a + b\hat{k}_1)y + (b\hat{k}_2 - \beta_m)r = = -\alpha_m e + (\alpha_m - a + b\hat{k}_1 + bk_1^* - bk_1^*)y + (b\hat{k}_2 - \beta_m + bk_2^* - bk_2^*)r = = -\alpha_m e + b(\hat{k}_1 - k_1^*)y$$

desing of τ

We must find τ_1, τ_2 s.t. for $\dot{\hat{k}}_1 = \tau_1, \dot{\hat{k}}_2 = \tau_2$, $\tilde{k}_1, \tilde{k}_1$ converge. Directly assigning the dynamics is impossible as knowledge of k_1^*, k_2^* is necessary. We rewrite the error derivative as

$$\dot{e} = -\alpha_m e + b\Phi^T(y, r)\tilde{k} \qquad \Phi^T(y, r) = (y \quad r) \qquad \tilde{k} = (\tilde{k}_1 \quad \tilde{k}_2)^T$$

The second term on the right is a regressor. The system is non linear, as any adaptive system.

Stability analysis

We use a Lyapunov approach as the system is non-linear.

$$\dot{e} = -\alpha_m e + b\Phi^T(e + y_r, r)\tilde{k}$$
$$\dot{\tilde{k}} = \tau$$

Goal: find τ so that the equilibrium $(0, 0)$ of the system is asymptotically stable. If we can make the system so, we obtain the goals of the control + $\tilde{k}(t) \to 0$.
Lyapunov candidate function

$$V(e, \tilde{k}) = \frac{1}{2}e^2 + \frac{b}{2\gamma}\tilde{k}^T\tilde{k} \qquad with \gamma > 0$$

The derivative is

$$\dot{V}(e, \tilde{k}) = -\alpha_m e^2 + eb\Phi^T(e + y_r, r)\tilde{k} + \frac{b}{\gamma}\tilde{k}^T r$$

The first term is negative, the second is bad, the third is so-so

$$\dot{V}(e, \tilde{k}) = -\alpha_m e^2 + \tilde{k}^T b\frac{1}{\gamma}[\gamma\Phi(e + y_r, r)e + \tau]$$

γ and the regressor are knwon, so we define

$$\tau = \gamma\Phi(e + y_r, r)e$$

leading to

$$\dot{V}(e, \tilde{k}) = -\alpha_m e^2$$

which is not negative definite as it is identically 0 for $e = 0$ and any value of $\tilde{k}$, it is therefore negative semi-definite. However the system $(e, \tilde{k}) = (0, \vec{0})$ is stable in the Lyapunov sense. This means that, take arbitrarirly $e_0, \tilde{k}_0$ initial conditions, the derivative of the lyapunov function of the corresponding solution $e(t), \tilde{k}(t)$ is negative semi-definite, meaning the energy of the system is non increasing, meaning $V(e(t), \tilde{k}(t)) \leq V(e_0, \tilde{k}_0) = M_0$, giving us global boundedness of the system.

What happens to $e(t) and \tilde{k}(t)$ as $t \to \infty$? Wec cannot apply the La Salle criterion as the system is time variant and the La Salle criterion only applies to time invariant or periodic systems. We use the La Salle-Yoshizawa criterion

La Salle-Yoshizawa criterion

An extension of La Salle