

Chapter 1

Introduction to optimal control

1.1 Optimal control problem formulation

Consider the continuous-time system ($t \in \mathbb{R}$)

$$\dot{x}(t) = f(x(t), u(t), t) \quad (1.1)$$

$$y(t) = h(x(t), u(t), t) \quad (1.2)$$

- $x(t) \in \mathbb{R}^n$ state of the system at time t
- $u(t) \in \mathbb{R}^m$ input of the system at time t
- $y(t) \in \mathbb{R}^p$ output of the system at time t

We will mainly work with time invariant systems, $\dot{x}(t) = f(x(t), u(t))$.

We consider nonlinear, discrete-time systems described by

$$x(t+1) = f_t(x(t), u(t)) \quad t \in \mathbb{N}_0$$

but from now on we will use the compact notation

$$x_{t+1} = f_t(x_t, u_t) \quad t \in \mathbb{N}_0$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the state and the input of the system at time t .

Consider a nonlinear, discrete-time system on a finite time horizon

$$x_{t+1} = f_t(x_t, u_t) \quad t = 0, \dots, T-1$$

We use $\mathbf{x} \in \mathbb{R}^{nT}$ and $\mathbf{u} \in \mathbb{R}^{mT}$ to denote, respectively, the stack of the states x_t for all $t \in \{1, \dots, T\}$ and the inputs u_t for all $t \in \{0, \dots, T-1\}$, that is:

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} \quad \mathbf{u} := \begin{bmatrix} u_0 \\ \vdots \\ u_{T-1} \end{bmatrix}$$

Trajectory of a system

Definition: A pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \in \mathbb{R}^{nT} \times \mathbb{R}^{mT}$ is called a trajectory of system (1) if $\bar{x}_{t+1} = f_t(\bar{x}_t, \bar{u}_t)$ for all $t \in \{0, \dots, T-1\}$. That is, if $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ satisfies the system dynamics (the same holds for continuous time systems with proper adjustments). In particular, $\bar{\mathbf{x}}$ is the state trajectory, while $\bar{\mathbf{u}}$ is the input trajectory.

Equilibrium

Definition: A state-input pair $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of (1) if $(x_t, u_t) = (x_e, u_e), t \in \mathbb{N}_0$ is a trajectory of the system.

Equilibria of time-invariant systems satisfy $x_e = f(x_e, u_e)$

Linearization of a system about a trajectory

Given the dynamics (1) and a trajectory (\bar{x}, \bar{u}) , the linearization of (1) about (\bar{x}, \bar{u}) is given by the linear (possibly) time-varying system

$$\Delta x_{t+1} = A_t \Delta x_t + B_t \Delta u_t \quad t \in \mathbb{N}_0$$

with A_t and B_t the Jacobians of f_t , with respect to state and input respectively, evaluated at (\bar{x}, \bar{u})

$$A_t = \left. \frac{\partial}{\partial x} f(\bar{x}_t, \bar{u}_t) \right|_{(\bar{x}, \bar{u})} \quad B_t = \left. \frac{\partial}{\partial u} f(\bar{x}_t, \bar{u}_t) \right|_{(\bar{x}, \bar{u})}$$

1.1.1 Optimization

Main ingredients

- Decision variable: $x \in \mathbb{R}^n$
- Cost function: $\ell(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ cost associated to decision x
- Constraints (constraint sets): for some given functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, m$, and $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$, the decision vector $x \in \mathbb{R}^n$ needs to satisfy

$$h_i(x) = 0 \quad i = 1, \dots, m \quad g_j(x) = 0 \quad j = 1, \dots, r$$

equivalently we can say that we require $x \in X$ with

$$X = \{x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0\},$$

where we compactly denoted $h(x) = \text{col}(h_1(x), \dots, h_m(x))$ and $g(x) = \text{col}(g_1(x), \dots, g_r(x))$

Minimization

We can write our optimization problem as

$$\min_{x \in \mathbb{R}^n} \ell(x) \tag{1.3}$$

$$\text{subj. to } h_i(x) = 0 \quad i = 1, \dots, m \tag{1.4}$$

$$g_j(x) \leq 0 \quad j = 1, \dots, r \tag{1.5}$$

where $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$

We can write it more compactly as

$$\min_{x \in \mathbb{R}^n} \ell(x)$$

$$\text{subj. to } h(x) = 0, g(x) \leq 0$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^r$

1.1.2 Discrete-time optimal control

main ingredients

- Dynamics: a discrete-time system in state space form

$$x_{t+1} = f_t(x_t, u_t) \quad t = 0, 1, \dots, T-1$$

- the dynamics introduce T equality constraints

$$\begin{array}{ll} x_1 = f(x_0, u_0) & \text{i.e.} \quad x_1 - f(x_0, u_0) = 0 \\ x_2 = f(x_1, u_1) & \text{i.e.} \quad x_2 - f(x_1, u_1) = 0 \\ \vdots & \\ x_T = f(x_{T-1}, u_{T-1}) & \text{i.e.} \quad x_T - f(x_{T-1}, u_{T-1}) = 0 \end{array}$$

- Cost function: a cost "to be payed" for a chosen trajectory. We consider an additive structure in time

$$\ell(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T)$$

where $\ell_t : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is called stage-cost, while $\ell_T : \mathbb{R}^n \rightarrow \mathbb{R}$ is the terminal cost.

- End-point constraints: function of the state variable prescribed at initial and/or final point

$$r(x_0, x_T) = 0$$

- Path constraints: point-wise (in time) constraints representing possible limits on states and inputs at each time t

$$g_t(x_t, u_t) \leq 0, \quad t \in \{0, \dots, T-1\}$$

A discrete-time optimal control problem can be written as

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_T \\ u_0, \dots, u_{T-1}}} \sum_{t=0}^{T-1} \ell_t(x_t, u_t) + \ell_T(x_T) \\ \text{subj. to } & x_{t+1} = f_t(x_t, u_t), \quad t \in \{0, \dots, T-1\} \\ & r(x_0, x_T) = 0 \\ & g_t(x_t, u_t) \leq 0, \quad t \in \{0, \dots, T-1\} \end{aligned}$$

Optimal control for trajectory generation

We can pose a trajectory generation problem as

$$\min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m} \sum_{t=0}^{T-1} \frac{1}{2} \|x_t - x_t^{\text{des}}\|_Q^2 + \frac{1}{2} \|u_t - u_t^{\text{des}}\|_R^2 + \frac{1}{2} \|x_T - x_T^{\text{des}}\|_{P_f}^2$$

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Nonlinear Optimization

Chapter 3

Optimality conditions for optimal control

Chapter 4

Linear Quadratic (LQ) optimal control

Chapter 5

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Chapter 6

Numerical methods for nonlinear optimal control

Chapter 7

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