19 marzo 2025

mercoledì 19 marzo 2025 14:11

$z = (a, b)$

$$\underset{a}{(a,0)} + \underset{i}{(0,1)}\,\underset{b}{(b,0)} = (a,0) + (0-0,\, 0+b) = (a,0) + (0,b) = (a,b) = z$$

$z = a + i\,b$ FORMA ALGEBRICA $\quad z = \operatorname{Re} z + i \operatorname{Im} z$

parte reale $\operatorname{Re} z$

parte immaginaria

i = unità imm.

$b = \operatorname{Im} z$

$$i^2 = (0,1)(0,1) = (0-1,\, 0+0) = -1$$

$$(2+6i)(1-4i) = 2 - 8i + 6i - 24(-1) = 26 - 2i$$

$z = 3+i \quad w = 4 - 2i \quad$ calc. $\operatorname{Re}(zw) = 3 \cdot 4 + (-2) \cdot i^2 = 14$

$$\frac{3-i}{2+3i} = \frac{(3-i)(2-3i)}{(2+3i)(2-3i)} = \frac{3-11i}{4+9} = \frac{3}{13} - \frac{11}{13} i \qquad z\bar{z} = |z|^2$$

$$|a + ib| = a^2 + b^2$$

[SI MOLTIPLICANO NUMER. E DENOM. PER IL CONIUGATO DEL DENOM.]

$$\frac{2+5i}{3i-1} = \frac{(2+5i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{13-11i}{10} = \frac{13}{10} - \frac{11}{10} i$$

$$(2-i)i - (2+i)\frac{4}{5-3i} = 2i + 1 - \frac{(8+4i)(5+3i)}{34} = 2i + 1 - \frac{\cancel{28}^{\,14}}{\cancel{34}_{\,17}} - \frac{\cancel{44}^{\,22}}{\cancel{34}_{\,17}} i =$$

$$= \frac{3}{17} + \frac{12}{17} i$$

Potenze di i

$$\left.\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \end{aligned}\right\}$$

$i^4 = 1$

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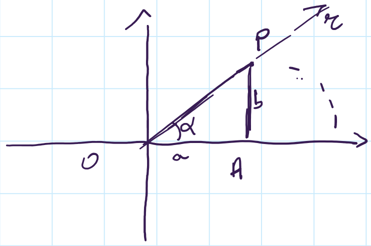
Risolviamo l'eq. binomia $w^n = z$ nel campo complesso. Per farlo occorre introdurre una nuova espressione dei numeri compl.



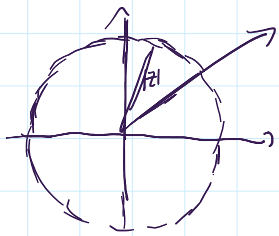
$z = a + ib \neq 0 \quad P(a,b)$

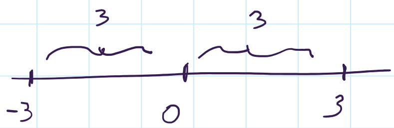
$\overline{OP} = |z|$

$r =$ semiretta OP orientata da O verso P

$\alpha =$ misura in radianti del minimo angolo di cui deve ruotare in verso antiorario il semiasse delle ascisse positive per sovrapporsi in direz. e verso ad r

Def. $\arg z = \alpha + 2k\pi \quad (k \in \mathbb{Z})$ ARGOMENTO DI Z





$OP = |z|$

$OA = a = OP \cos\alpha = |z| \cos(\arg z)$

$BA = b = OP \sin\alpha = |z| \sin(\arg z)$

$z = a + ib = |z|\left(\cos\alpha + i \sin\alpha\right)$ FORMA TRIGONOMETRICA

$z = |z|(\cos\alpha + i \sin\alpha)$

$w = |w|(\cos\beta + i \sin\beta)$

$z = w \iff \begin{cases} |z| = |w| \\ \beta = \alpha + 2k\pi \end{cases}$

$zw = |z|(\cos\alpha + i\sin\alpha)\,|w|(\cos\beta + i \sin\beta) =$

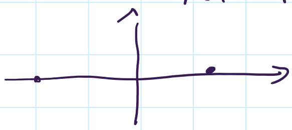
$= |z||w|\Big(\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\Big) =$

$= |z||w|\left(\cos(\alpha+\beta) + i \sin(\alpha+\beta)\right) \Rightarrow \begin{array}{l} |zw| = |z||w| \\ \arg(zw) = \arg z + \arg w \end{array}$

in modo simile si trova che $\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad \arg\left(\frac{z}{w}\right) = \arg z - \arg w$

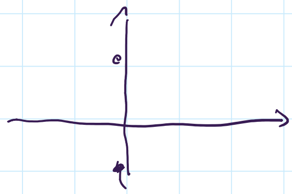
se $z = a \in \mathbb{R}$, $|z| = |a|$ $\quad \arg z = $ 0 se $a > 0$

se $z = a \in \mathbb{R}$ $\qquad |z| = |a| \qquad \arg z = \begin{cases} 0 & \text{se } a > 0 \\ \pi & \text{se } a < 0 \end{cases}$



se $z = ib \quad (b \in \mathbb{R})$

$|z| = |b|$



$\arg z = \begin{cases} \frac{\pi}{2} & \text{se } b > 0 \\ -\frac{\pi}{2} \ \left(\frac{3}{2}\pi\right) & \text{se } b < 0 \end{cases}$

Formula di MOIVRE (potenze intere di un num. compl.)

$z \in \mathbb{C} \quad z \neq 0 \qquad z = |z|(\cos\alpha + i \sin\alpha)$

$n \in \mathbb{Z} \qquad z^n = |z|^n (\cos(n\alpha) + i \sin(n\alpha))$

es. $i^2 = -1 \qquad i = |i|(\cos\alpha + i\sin\alpha) = 1\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$

$i^2 = 1^2\left(\cos 2\frac{\pi}{2} + i \sin 2\frac{\pi}{2}\right) = 1(\cos\pi + i\sin\pi) = -1$

Radici $\quad z \in \mathbb{C} \quad n \in \mathbb{N},\ n \geq 2 \quad$ si cerca $w \in \mathbb{C} : w^n = z$

se $z = 0 \Rightarrow w = 0$ è l'unica sol.

se $z \neq 0 \qquad z = |z|(\cos\alpha + i\sin\alpha)$

$w = 0$ non è sol., sia $w \neq 0$ una sol. $\qquad w = |w|(\cos\beta + i\sin\beta)$

$w^n = z \iff |w|^n(\cos(n\beta) + i\sin(n\beta)) = |z|(\cos\alpha + i\sin\alpha) \Rightarrow$

$\Rightarrow \begin{cases} |w|^n = |z| \\ n\beta = \alpha + 2k\pi \end{cases} \Rightarrow \begin{array}{l} |w| = \sqrt[n]{|z|} \\ \beta = \dfrac{\alpha + 2k\pi}{n} \end{array} \Rightarrow w$, se è sol., è del tipo

$$w_k = \sqrt[n]{|z|}\left(\cos\frac{\alpha+2k\pi}{n} + i\sin\frac{\alpha+2k\pi}{n}\right) \quad \text{per qualche } k \in \mathbb{Z}$$

infatti $(w_k)^n = \left(\sqrt[n]{|z|}\right)^n\left(\cos \cancel{n}\frac{\alpha+2k\pi}{\cancel{n}} + i\sin \cancel{n}\frac{\alpha+2k\pi}{\cancel{n}}\right) = |z|(\cos\alpha + i\sin\alpha) = z$

i numeri w_k sono distinti solo per n valori di k, $k \in I = \{0; 1; \dots; n-1\}$

$w^n = z$ ha le sol. $w_0, w_1, \dots, w_{n-1}$

$\forall k \in I \quad |w_k| = \sqrt[n]{|z|} \qquad \arg w_k = \frac{\arg z + 2k\pi}{n}$

se $z \in \mathbb{R}$ le eventuali sol. reali sono fra queste.

es. $z = 16 \quad n = 4 \qquad w_k = \sqrt[4]{16}\left(\cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}\right) \qquad k = 0, 1, 2, 3$

$w_0 = 2(\cos 0 + i\sin 0) = 2 \qquad w_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$

$w_2 = 2(\cos\pi + i\sin\pi) = -2 \qquad w_3 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$

$z = +16 \quad n = 2 \qquad w_k = \sqrt{16}\left(\cos\frac{0+2k\pi}{2} + i\sin\frac{0+2k\pi}{2}\right) \qquad k = 0, 1$

$w_0 = 4(\cos 0 + i\sin 0) = 4 \qquad w_1 = 4(\cos\pi + i\sin\pi) = -4$

$z = -16 \quad n = 2 \qquad w_k = \sqrt{16}\left(\cos\frac{\pi+2k\pi}{2} + i\sin\frac{\pi+2k\pi}{2}\right)$

$w_0 = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4i \qquad w_1 = -4i \quad \dots$

in generale se $a \in \mathbb{R}$, $a < 0 \qquad \sqrt{a} = \pm i\sqrt{-a}$

cons. l'eq. di II grado con $\Delta < 0 \qquad \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm i\sqrt{-\Delta}}{2a}$

$x^2 + x + 4 = 0 \qquad \Delta = -15$

$x = \frac{-1 \pm \sqrt{-15}}{2} = -\frac{1}{2} + \frac{\sqrt{15}}{2}i, \ -\frac{1}{2} - \frac{\sqrt{15}}{2}i$

$\sqrt[3]{8} \qquad |z| = 2 \quad \arg z = 0 \qquad w_k = \sqrt[3]{8}\left(\cos\frac{0+2k\pi}{3} + i\sin\frac{0+2k\pi}{3}\right) \quad k = 0, 1, 2$

$w_0 = 2 \qquad w_1 = 2\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) \qquad w_2 = 2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right)$

$\sqrt[4]{i} \qquad |z| = 1 \quad \arg z = \frac{\pi}{2} \qquad w_0 = \left(\cos\frac{\frac{\pi}{2}}{\ } + i\sin\frac{\frac{\pi}{2}}{\ }\right) \qquad w_1 = \cos\frac{3}{8}\pi + i\sin\frac{3}{8}\pi$

$\sqrt[4]{i}$ $\quad |z| = 1 \quad \arg z = \frac{\pi}{2}$ $\quad w_0 = \left(\cos\frac{\frac{\pi}{2}}{4} + i \sin\frac{\frac{\pi}{2}}{4}\right)$ $\quad w_1 = \cos\frac{3}{8}\pi + i \sin\frac{3}{8}\pi$

$w_2 = \ldots \qquad w_3 = \ldots$

CASO $n = 2$ $\quad w_k = \sqrt{|z|}\left(\cos\left(\frac{\alpha + 2k\pi}{2}\right) + i \sin\frac{\alpha + 2k\pi}{2}\right) \quad k = 0, 1$

$w_0 = \sqrt{|z|}\left(\cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2}\right) \qquad w_1 = \sqrt{|z|}\left(\cos\left(\frac{\alpha}{2} + \pi\right) + i \sin\left(\frac{\alpha}{2} + \pi\right)\right) = -w_0$

OSSERV. se $z = a + ib$ $\quad z = |z|(\cos\alpha + i \sin\alpha)$

$a = |z|\cos\alpha \qquad b = |z| \sin\alpha$

forma alg $\to$ forma trig $\quad |z| = \sqrt{a^2 + b^2} \quad \cos\alpha = \frac{a}{|z|} \quad \sin\alpha = \frac{b}{|z|}$

$\sqrt[3]{1+i}$ $\quad z = 1 + i \quad |z| = \sqrt{2}$ $\quad \begin{array}{l} 1 = |z|\cos\alpha \\ 1 = |z|\sin\alpha \end{array} \Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}, \ \sin\alpha = \frac{1}{\sqrt{2}}$

$\alpha = \frac{\pi}{4}$

$w_0 = \sqrt[6]{2}\left(\cos\frac{\frac{\pi}{4}}{3} + i \sin\frac{\pi}{12}\right) \qquad w_1 = \sqrt[6]{2}\left(\cos\frac{\frac{\pi}{4} + \pi}{3} + i \sin \underline{\qquad}\right)$

$\frac{\frac{\pi}{4} + \pi}{3} = \frac{5}{12}\pi$

$w_2 = \sqrt[6]{2}\left(\cos\frac{\frac{\pi}{4} + 2\pi}{3} + i \sin \underline{\qquad}\right)$

$\frac{\frac{\pi}{4} + 2\pi}{3} = \frac{9}{12}\pi$

eq. di II grado: la formula risolutiva vale anche nel campo complesso

$i z^2 + 2z - 2i = 0 \qquad z = \frac{-1 \pm \sqrt{1 - 2}}{i} = \frac{-1 \pm \sqrt{-1}}{i} = \frac{-1 \pm i}{i} =$

$= \frac{-1}{i} \pm \frac{i}{i} = i \pm 1 \qquad$ le sol. sono $1 + i, \ -1 + i$