14 maggio 2025

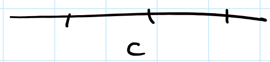
mercoledì 14 maggio 2025 14:00

DIM. DEL TEOR. DI FERMAT

$c \in]a,b[\Rightarrow f'(c) = f'_-(c) = f'_+(c)$

supp. c p. di max rel. $\Rightarrow \exists\, \delta > 0 :$ in $]c-\delta, c+\delta[$ $\quad f(x) \le f(c) \Rightarrow f(x) - f(c) \le 0$

Oss. $\quad f'_-(c) = \lim_{x \to c^-} \dfrac{\overbrace{f(x)-f(c)}^{\le 0}}{\underbrace{x-c}_{<0}} \ge 0$



$\quad ||$

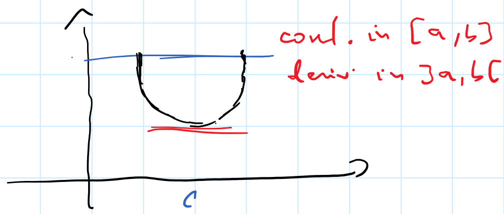
$f'_+(c) = \lim_{x \to c^+} \dfrac{\overbrace{f(x)-f(c)}^{\le 0}}{\underbrace{x-c}_{>0}} \le 0$

$\Big\} \Rightarrow f'(c) = 0$

TEOREMA DI ROLLE

IP. $f : [a,b] \to \mathbb{R}$ cont. in $[a,b]$
deriv. in $]a,b[$
$f(a) = f(b)$

TS. $\exists\, c \in [a,b] : f'(c) = 0$



DIM. $\quad f$ cont. in $[a,b] \Rightarrow$ per il teor. di Weierstrass f è dotata di min e max assoluti

Siano $x_1, x_2 \in [a,b] : \; f(x_1) = m \;, \; f(x_2) = M$

Se $x_1 = a$, $x_2 = b$ (o viceversa) ne segue $m = M \Rightarrow f$ costante $\Rightarrow f'(x) = 0 \;\; \forall x \in [a,b]$

Se almeno uno fra x_1, x_2, ad es. $x_1 \in]a,b[\;\Rightarrow f'(x_1) = 0$ per il teor. di Fermat.

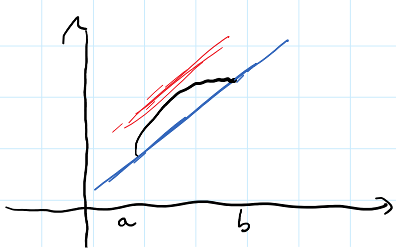
TEOREMA DI LAGRANGE

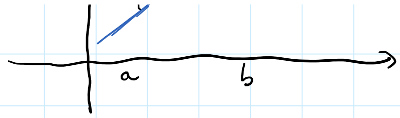
IP. $f : [a,b] \to \mathbb{R}$ cont. in $[a,b]$
deriv. in $]a,b[$

TS. $\exists\, c \in]a,b[: \; f(b) - f(a) = f'(c)(b-a)$

$\Updownarrow$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$





DIM. Cons. in $[a,b]$ la funz. $g(x) = \big(f(b)-f(a)\big)x + (a-b)f(x)$

g è cont. in $[a,b]$ e deriv. in $]a,b[$

$g(a) = f(b)a - \cancel{f(a)a} + \cancel{af(a)} - bf(a) \quad = \quad g(b) = \cancel{f(b)b} - f(a)b + af(b) - \cancel{bf(b)}$

Per il teor. di Rolle $\exists\, c \in\,]a,b[\ : \ g'(c) = 0$

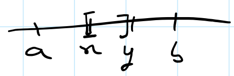
$g'(x) = f(b) - f(a) + (a-b)f'(x) \qquad g'(c) = 0 \Rightarrow f(b) - f(a) = (b-a)f'(c)$ Ts.

Applicazioni del teor. di Lagrange

Criterio di monotonia

$f: (a,b) \to \mathbb{R}$ derivabile
$f'(x) \geq 0 \quad \forall x \in (a,b)$ $\Rightarrow$ f crescente in (a,b)

DIM. Siano $x, y \in (a,b)$ con $x < y$, dim. che $f(x) \leq f(y)$



f deriv. in $[x,y] \Rightarrow$ per il teor di Lagrange $\exists\, c \in\,]x,y[\,:$

$$f(y) - f(x) = \underbrace{f'(c)}_{\geq 0}\underbrace{(y-x)}_{+} \geq 0$$

Criterio di stretta monotonia

Sia $f: (a,b) \to \mathbb{R}$ deriv.
CNS affinché f sia strett. cresc. in (a,b) è che
$f'(x) \geq 0 \quad \forall x$
$\nexists\, (c,d) \subseteq (a,b) : f'(x) = 0 \quad \forall x \in (c,d)$

NO DIM.

Esercizi sulle derivate

$f(x) = \log\left(x^2 - \sqrt{x+1}\right) \qquad f'(x) = \dfrac{1}{x^2 - \sqrt{x+1}}\left(2x - \dfrac{1}{2}\dfrac{2}{\sqrt{x+1}}\right)$

$f(x) = \dfrac{x^2 \sin x}{(x+1)e^x} \qquad f'(x) = \dfrac{(2x\sin x + x^2\cos x)(x+1)e^x - \left(e^x + e^x(x+1)\right)x^2 \sin x}{\left[(x+1)e^x\right]^2}$

$$f(x) = e^{x^2 \cos x} \qquad f'(x) = e^{x^2 \cos x}\left(2x \cos x - \sin x \cdot x^2\right)$$

$$f(x) = \log(\log(\log x)) \qquad f'(x) = \frac{\frac{1}{\log x}\frac{1}{x}}{\log(\log x)} = \frac{1}{x(\log x)(\log(\log x))}$$

$$f(x) = \operatorname{arctg} \frac{x^2+1}{e^{2x}} \qquad f'(x) = \frac{1}{1+\left(\frac{x^2+1}{e^{2x}}\right)^2} \; \frac{2x e^{2x} - (x^2+1)\, 2 e^{2x}}{e^{4x}}$$

$$f(x) = \frac{x^2-1}{2x^2+3} \qquad f'(x) = \frac{2x(2x^2+3) - (x^2-1)4x}{(2x^2+3)^2}$$

$$f(x) = \frac{\sin x^2}{e^x} \qquad f'(x) = \frac{(\cos x^2) e^x 2x - e^x \sin x^2}{e^{2x}}$$

$y^3 \qquad \sin x$

$3y^2 \qquad \cos x$

$$f(x) = \sin^3 x \qquad f'(x) = 3 \sin^2 x \cos x$$

$$f(x) = \log^4 \frac{1}{y} \qquad f'(x) = 4\left(\log^3 \frac{1}{y}\right) y \left(\frac{-1}{y^2}\right)$$

$$f(x) = \log \left|\frac{2x}{x-1}\right| \qquad f'(x) = \frac{x-1}{2x}\,\frac{2(x-1) - 1 \cdot 2x}{(x-1)^2}$$

$$D\left(\log |f(x)|\right) = \frac{f'(x)}{f(x)}$$

$$f(x) = \arcsin(2-x^2) \qquad f'(x) = \frac{-2x}{\sqrt{1-(2-x^2)^2}}$$

EST $\qquad$ INT

$\arcsin y \qquad \operatorname{arctg} x$

$$\frac{1}{\sqrt{1-y^2}} \qquad \frac{1}{1+x^2}$$

$$f(x) = \arcsin(\operatorname{arctg} x) \qquad f'(x) = \frac{1}{\sqrt{1-\operatorname{arctg}^2 x}} \; \frac{1}{1+x^2}$$

$$f(x) = \sin^2(\cos^2 x^2) \qquad f'(x) = \left(2 \sin(\cos^2 x^2)\right)\left(\cos(\cos^2 x^2)\right)\left(2\cos x^2\right)\left(-\sin x^2\right)(2x)$$

$$f(x) = \log\left(\operatorname{tg} \frac{1}{x^2}\right) \qquad f'(x) = \frac{\left(1+\operatorname{tg}^2 \frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right)}{\operatorname{tg} \frac{1}{x^2}}$$

$\operatorname{tg} x \to 1 + \operatorname{tg}^2 x$

$$f(x) = \cos(\log x^3) \qquad f'(x) = \left(-\sin(\log x^3)\right)\frac{1}{x^3}\cdot 3x^2$$

$$f(x) = \operatorname{tg}\left(\operatorname{arctg}\frac{2}{x}\right) \qquad f'(x) = \left(1+\operatorname{tg}^2\left(\operatorname{arctg}\frac{2}{x}\right)\right)\frac{1}{1+\frac{4}{x^2}}\left(-\frac{2}{x^2}\right)$$

x^{-1}

$-1\cdot x^{-2}$

$-\frac{1}{x^2}$

$$f(x) = e^{\sin x \cos x} \qquad f'(x) = e^{\sin x \cos x}\left(\cos^2 x - \sin^2 x\right)$$

$f'g + fg'$

$$f(x) = \log(\cos x^2) \qquad f'(x) = \frac{1}{\cos x^2}\left(-\sin x^2\right)2x$$

$$f(x) = e^{e^x} \qquad f'(x) = e^{e^x} e^x$$