6 novembre 2025_AL

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1\.

1. $y' = \frac{y}{x}$
2. $y' = \frac{x}{y}$
3. $y' = \log(xy)$
4. $y' = y(\log x)$
5. $y'' + \frac{2x}{y''} - y' = \cos x$
6. $y'' + \frac{2y'}{x^2} + 6y = \log x$
7. $(y'')^2 = -2y' + 3y$
8. $4y^{IV} - 3y''' + 2y'' - y' = 0$
9. $x^2 y'' + 3xy' - y''' = 0$
10. $x(y'')^2 + 3xy' - y''' = 0$

Quali sono lineari? 1, 4, 6, 8, 9

2\. $y^{IV} - 2y''' + y'' = e^x$

eq. omog. $y^{IV} - 2y''' + y'' = 0$

eq. caratt. $\alpha^4 - 2\alpha^3 + \alpha^2 = 0 \Rightarrow \alpha^2(\alpha^2 - 2\alpha + 1) = 0$

$\alpha^2(\alpha - 1)^2 = 0$

$\alpha = 0 \quad s = 2 \quad \rightarrow e^{0x}, \; x e^{0 \cdot x}$

$\alpha = 1 \quad s = 2 \quad \rightarrow e^{1 \cdot x}, \; x e^{1 \cdot x}$

int. gen. om. $y(x) = k_1 + k_2 x + k_3 e^x + k_4 x e^x$

$f(x) = e^x \qquad m = 0 \qquad h = 1 \qquad s = 2$

cerco $y(x) = k e^x x^2$

$y'(x) = k e^x (x^2 + 2x)$

$y''(x) = k e^x (x^2 + 4x + 2)$

$y'''(x) = k e^x (x^2 + 4x + 2 + 2x + 4) = k e^x (x^2 + 6x + 6)$

$y^{IV}(x) = k e^x (x^2 + 6x + 6 + 2x + 6) = k e^x (x^2 + 8x + 12)$

sost. nell' eq.

$k\cancel{e^x}(\cancel{x^2} + \cancel{8x} + \cancel{12} - \cancel{2x^2} - \cancel{12x} - \cancel{12} + \cancel{x^2} + \cancel{4x} + 2) = \cancel{e^x} \Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$

INT GEN COMPL $y(x) = k_1 + k_2 x + k_3 e^x + k_4 x e^x + \frac{1}{2} x^2 e^x$

3\. $y'' - 6y' + 9y = x e^{3x}$

eq. omog. $y'' - 6y' + 9y = 0$

eq. caratt. $\alpha^2 - 6\alpha + 9 = 0 \qquad \alpha = 3 \quad s = 2$

$(\alpha - 3)^2 = 0$

int gen om. $y(x) = k_1 e^{3x} + k_2 x e^{3x}$

$f(x) = x e^{3x} \qquad m = 1 \qquad h = 3 \qquad s = 2$

cerco $y(x) = e^{3x} x^2 (ax + b) = e^{3x}(ax^3 + bx^2)$

$y'(x) = e^{3x}(3ax^3 + 3bx^2 + 3ax^2 + 2bx)$

$y''(x) = e^{3x}(9ax^3 + 9bx^2 + 9ax^2 + 6bx + 9ax^2 + 6bx + 6ax + 2b)$

sost. nell' eq.

$\cancel{e^{3x}}(\cancel{9ax^3} + \cancel{9bx^2} + \cancel{18ax^2} + \cancel{12bx} + 6ax + 2b - \cancel{18ax^3} - \cancel{18bx^2} - \cancel{18ax^2} -$

$- \cancel{12bx} + \cancel{9ax^3} + \cancel{9bx^2}) = x\cancel{e^{3x}} \qquad 6ax + 2b = x$

$\begin{cases} 6a = 1 \\ 2b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{6} \\ b = 0 \end{cases}$

$$\begin{cases} 6a = 1 \\ 2b = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{6} \\ b = 0 \end{cases}$$

INT GEN $\quad y(x) = k_1 e^{3x} + k_2 x e^{3x} + \frac{1}{6} e^{3x} x^3 = e^{3x}\left(\frac{1}{6}x^3 + k_2 x + k_1\right)$

4. $$\begin{cases} y'' - 2y' - 2y = \dfrac{2x+1}{e^x} = e^{-x}(2x+1) \\ y(0) = 1 \\ y'(0) = \frac{8}{8} - 7 \end{cases}$$

eq. assoc. $\quad y'' - 2y' - 2y = 0$

eq. caratt. $\quad \alpha^2 - 2\alpha - 2 = 0 \qquad \alpha = 1 \pm \sqrt{3}$

int gen omog $\quad y(x) = k_1 e^{(1-\sqrt{3})x} + k_2 e^{(1+\sqrt{3})x}$

$f(x) = e^{-x}(2x+1) \qquad h = -1 \quad s = 0 \quad m = 1$

cerco $\quad y(x) = e^{-x}(ax+b)$

$\quad y'(x) = e^{-x}(-ax - b + a)$

$\quad y''(x) = e^{-x}(ax + b - 2a)$

sost. nell'eq. $\quad e^{-x}(ax + b - 2a + \cancel{2ax} + \cancel{2b} - 2a - \cancel{2ax} - \cancel{2b}) = e^{-x}(2x+1)$

$ax - 4a + b = 2x + 1 \qquad \begin{cases} a = 2 \\ -4a + b = 1 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = 9 \end{cases}$

INT GEN $\quad y(x) = k_1 e^{(1-\sqrt{3})x} + k_2 e^{(1+\sqrt{3})x} + e^{-x}(2x+9)$

$y'(x) = k_1(1-\sqrt{3})e^{(1-\sqrt{3})x} + k_2(1+\sqrt{3})e^{(1+\sqrt{3})x} + e^{-x}(-2x - 9 + 2)$

$$\begin{cases} y(0) = 1 \\ y'(0) = -7 \end{cases} \qquad \begin{cases} k_1 + k_2 + 9 = 1 \\ k_1(1-\sqrt{3}) + k_2(1+\sqrt{3}) - 7 = -7 \end{cases} \Rightarrow$$

$$\begin{cases} k_1 + k_2 = -8 \\ (1-\sqrt{3})k_1 + (1+\sqrt{3})k_2 = 0 \end{cases} \qquad 1+\sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3}$$

$$k_1 = \frac{\begin{vmatrix} -8 & 1 \\ 0 & 1+\sqrt{3} \end{vmatrix}}{2\sqrt{3}} = \frac{-4 - 4\sqrt{3}}{\sqrt{3}} = -4\left(\frac{1}{\sqrt{3}} + 1\right)$$

$$k_2 = \frac{\begin{vmatrix} 1 & -8 \\ 1-\sqrt{3} & 0 \end{vmatrix}}{2\sqrt{3}} = \frac{8 - 8\sqrt{3}}{2\sqrt{3}} = 4\left(\frac{1}{\sqrt{3}} - 1\right)$$

SOL. $\quad y(x) = -4\left(\frac{1}{\sqrt{3}} + 1\right) e^{(1-\sqrt{3})x} + 4\left(\frac{1}{\sqrt{3}} - 1\right) e^{(1+\sqrt{3})x} + e^{-x}(2x+9)$

5. $\quad y'' - 9y = (x+1)e^x + \dfrac{x}{e^{3x}} \qquad (1)$

$y'' - 9y = (x+1)e^x \qquad (2) \qquad$ dovremo sommare un int part di (2) e uno di (3)

$y'' - 9y = e^{-3x} x \qquad (3)$

EQ OMOG $\quad y'' - 9y = 0$

" CARATT $\quad \alpha^2 - 9 = 0 \qquad \alpha = 3, \ \alpha = -3$

INT GEN OMOG $\quad y(x) = k_1 e^{3x} + k_2 e^{-3x}$

(2) $\quad f(x) = (x+1)e^x \qquad m = 1 \quad h = 1 \quad s = 0$

cerco $\quad y(x) = e^x(ax+b)$

$\quad y'(x) = e^x(ax + b + a)$

$\quad y''(x) = e^x(ax + b + 2a)$

sost. nell'eq. $\quad \cancel{e^x}(ax + b + 2a - 9ax - 9b) = \cancel{e^x}(x+1)$

$-8ax + 2a - 8b = x + 1 \qquad a = -\frac{1}{8} \qquad b = -\frac{5}{32}$

$y(x) = e^x\left(-\frac{1}{8}x - \frac{5}{32}\right)$

(3) $\quad f(x) = e^{-3x} x \qquad m = 1 \quad h = -3 \quad s = 1$

cerco $y(x) = e^{-3x}(ax^2+bx)$

$y'(x) = e^{-3x}(-3ax^2-3bx+2ax+b)$

$y''(x) = e^{-3x}(9ax^2+9bx-6ax-3b-6ax-3b+2a)$

sost nell'eq

$\cancel{e^{-3x}}(\cancel{9ax^2}+\cancel{9bx}-12ax-6b+2a-\cancel{9ax^2}-\cancel{9bx}) = \cancel{e^{-3x}}\,x$

$-12ax+2a-6b = x \qquad -12a = 1 \Rightarrow a = -\frac{1}{12}$

$\qquad\qquad\qquad\qquad 2a-6b = 0 \qquad b = -\frac{1}{36}$

INT GEN $\quad y(x) = k_1 e^{3x} + k_2 e^{-3x} + e^{x}\left(-\frac{1}{8}x - \frac{5}{32}\right) + e^{-3x}\left(-\frac{1}{12}x^2 - \frac{1}{36}x\right)$

6. $\quad y'' + y = x\cos x \qquad (1)$

eq omog $\quad y''+y=0$

" caratt $\quad \alpha^2+1=0 \qquad \alpha = \pm i \qquad$ int gen omog $y(x) = k_1\cos x + k_2 \sin x$

risolv. $\quad y''+y = x e^{ix} \qquad\qquad (e^{ix} = \cos x + i\sin x)$

cerchiamo un suo int. partic. $\quad u+iv$ e prenderemo u

$f(x) = x e^{ix} \qquad m=1 \qquad h=i \qquad s=1$

cerco $\quad y(x) = e^{ix}(ax^2+bx)$

$y'(x) = e^{ix}(iax^2+ibx+2ax+b)$

$y''(x) = e^{ix}(-ax^2-bx+4iax+2ib+2a)$

sost nell'eq

$\cancel{e^{ix}}(-\cancel{ax^2}-\cancel{bx}+4iax+2ib+2a+\cancel{ax^2}+\cancel{bx}) = x\,\cancel{e^{ix}}$

$4iax+2a+2ib = x \qquad \begin{cases} 4ia = 1 \\ 2a+2ib = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{4}i \\ \left(-\frac{1}{2}+2b\right)i = 0 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{4}i \\ b = \frac{1}{4} \end{cases}$

$y(x) = e^{ix}(ax^2+bx) = (\cos x + i\sin x)\left(\frac{1}{4}x - \frac{1}{4}ix^2\right)$

$u(x) = \frac{1}{4}x\cos x + \frac{1}{4}x^2\sin x \quad$ è l'int. partic. della (1)

7. $\begin{cases} y''+4y = \sin 2x + e^{2x} \qquad (1) \\ y\left(\frac{\pi}{4}\right) = \frac{1}{4}e^{\frac{\pi}{2}} \\ y'\left(\frac{\pi}{4}\right) = \frac{1}{4}e^{\frac{\pi}{2}} \end{cases}$

dobbiamo risolv.

$y''+4y = \sin 2x \qquad (2)$

$y''+4y = e^{2x} \qquad (3)$

Per risolvere la (2) risolviamo la $\quad y''+4y = e^{2ix} \quad (4)$

se y è un suo int. part. $\quad y = u+iv$, v sarà int partic. di (2)

eq. omog. $\quad y''+4y=0 \qquad$ eq. caratt. $\quad \alpha^2+4=0 \qquad \alpha = \pm 2i$

int. gen. omog $\quad y(x) = k_1\cos 2x + k_2\sin 2x$

(4) $\quad f(x) = e^{2ix} \qquad m=0 \qquad h=2i \qquad s=1$

cerco $\quad y(x) = k\,x\,e^{2ix}$

$y'(x) = k\,e^{2ix}(1+2ix)$

$y''(x) = k\,e^{2ix}(4i-4x)$

sost nell'eq $\qquad k\,e^{2ix}(4i-\cancel{4x}+\cancel{4x}) = e^{2ix}$

$4ik = 1 \Rightarrow k = -\frac{1}{4}i$

$y(x) = -\frac{1}{4}ix\,e^{2ix} = -\frac{1}{4}ix(\cos 2x + i\sin 2x)$

$v(x) = -\frac{1}{4}x\cos 2x$

(3) $\quad f(x) = e^{2x} \qquad m=0 \qquad h=2 \qquad s=0$

cerco $\quad y(x) = k\,e^{2x}$

$y'(x) = 2k\,e^{2x}$

$y''(x) = 4k\,e^{2x}$

sost nell'eq

$4k e^{2x} + 4k e^{2x} = e^{2x} \Rightarrow k = \frac{1}{8}$ $y(x) = \frac{1}{8} e^{2x}$

INT GEN $y(x) = k_1 \cos 2x + k_2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} e^{2x}$

$y'(x) = -2k_1 \sin 2x + 2k_2 \cos 2x - \frac{1}{4}\cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} e^{2x}$

$$\begin{cases} y\left(\frac{\pi}{4}\right) = \frac{1}{4} e^{\frac{\pi}{2}} & k_2 + \frac{1}{8} e^{\frac{\pi}{2}} = \frac{1}{4} e^{\frac{\pi}{2}} \Rightarrow k_2 = \frac{1}{8} e^{\frac{\pi}{2}} \\ y'\left(\frac{\pi}{4}\right) = \frac{1}{4} e^{\frac{\pi}{2}} & -2k_1 + \frac{\pi}{2} + \frac{1}{4} e^{\frac{\pi}{2}} = \frac{1}{4} e^{\frac{\pi}{2}} \Rightarrow k_1 = \frac{\pi}{4} \end{cases}$$

SOL DEL PC

$y(x) = -\frac{\pi}{2} \sin 2x + \frac{1}{4} e^{\frac{\pi}{2}} \cos 2x - \frac{1}{4} \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} e^{2x}$

8. $$\begin{cases} y''' + 4y'' - 5y' = (x-3)e^x \\ y(0) = 1 \\ y'(0) = -1 \\ y''(0) = 3 \end{cases}$$

EQ. OMOG. $y''' + 4y'' - 5y' = 0$

" CARATT. $\alpha^3 + 4\alpha^2 - 5\alpha = 0$

$\alpha(\alpha^2 + 4\alpha - 5) = 0$ $\quad \alpha = -2 \pm 3 \begin{matrix} -5 \\ 1 \end{matrix}$ $\quad \alpha = 0$

INT GEN OMOG $y(x) = k_1 + k_2 e^{-5x} + k_3 e^x$

$f(x) = e^x(x-3)$ $\quad h = 1 \quad s = 1 \quad m = 1$

cerco $y(x) = e^x(ax^2 + bx)$

$y'(x) = e^x(ax^2 + bx + 2ax + b)$

$y''(x) = e^x(ax^2 + bx + 4ax + 2b + 2a)$

$y'''(x) = e^x(ax^2 + bx + 6ax + 3b + 6a)$

sost nell'eq.

$e^x(\cancel{ax^2} + \cancel{bx} + 6ax + 3b + 6a + \cancel{4ax^2} + \cancel{4bx} + 16ax + 8b + 8a - \cancel{5ax^2} -$

$- \cancel{5bx} - 10ax - 5b) = e^x(x-3)$ $\quad 12ax + 14a + 6b = x - 3$

$a = \frac{1}{12}$

$\frac{7}{6} + 6b = -3 \Rightarrow b = -\frac{25}{6}$

$y(x) = e^x\left(-3x^2 - \frac{25}{6}x\right)$

INT GEN $y(x) = e^x\left(k_3 - 3x^2 - \frac{25}{6}x\right) + k_1 + k_2 e^{-5x}$

$y'(x) = e^x\left(k_3 - 3x^2 - \frac{25}{6}x - 6x - \frac{25}{6}\right) - 5k_2 e^{-5x}$

$y''(x) = e^x\left(k_3 - 3x^2 - \frac{61}{6}x - \frac{25}{6} - 6x - \frac{61}{6}\right) + 25k_2 e^{-5x}$

$$\begin{cases} y(0) = 1 \\ y'(0) = -1 \\ y''(0) = 3 \end{cases} \quad \begin{cases} k_3 + k_1 + k_2 = 1 \\ k_3 - \frac{25}{6} - 5k_2 = -1 \\ k_3 - 15 + 25k_2 = 3 \end{cases} \quad \begin{cases} k_1 + k_2 + k_3 = 1 \\ -5k_2 + k_3 = -\frac{23}{6} \\ 25k_2 + k_3 = 18 \end{cases}$$

$-\frac{23}{6} + 5k_2 = 18 + 25k_2 \Rightarrow 30k_2 = -\frac{23}{6} - 18 = -\frac{131}{6} \Rightarrow k_2 = -\frac{131}{180}$

$k_3 = 18 - 25k_2 = \dots$

9. $$\begin{cases} y'' + y = x^2 + 3 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$ $$\begin{cases} y'' + y' = x^2 + 3 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$ (esercizio proposto)

omog $y'' + y = 0$

caratt $\alpha^2 + 1 = 0$ $\quad \alpha = \pm i$ $\quad$ int gen omog $y(x) = k_1 \cos x + k_2 \sin x$

$f(x) = x^2 + 3$ $\quad m = 2 \quad h = 0 \quad s = 0$

cerco $y(x) = (ax^2 + bx + c)$

$y'(x) = 2ax + b$

$y''(x) = 2a$ $\quad h = 1$

sost nell'eq $\quad 2a + ax^2 + bx + c = x^2 + 3 \Rightarrow \begin{cases} a = 1 \\ b = 0 \\ 2a + c = 3 \Rightarrow c = 1 \end{cases}$

INT GEN $\quad y(x) = k_1 \cos x + k_2 \sin x + x^2 + 1$

$y'(x) = -k_1 \sin x + k_2 \cos x + 2x$

$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \qquad \begin{matrix} k_1 + 1 = 1 \Rightarrow k_1 = 0 \\ k_2 = 0 \end{matrix}$

SOL
$y(x) = x^2 + 1$

10. $\quad y'' - y' - 2y = (3x+1)\cos x + (x-1)e^{2x} \qquad (1)$

$y'' - y' - 2y = (3x+1)\cos x \qquad (2)$

$y'' - y' - 2y = (x-1)e^{2x} \qquad (3)$

$y'' - y' - 2y = (3x+1)e^{ix} \qquad (4)$

se $y = u + iv$ è sol di (4) allora u è sol di (2)

eq omog $\quad y'' - y' - 2y = 0$

caratt $\quad \alpha^2 - \alpha - 2 = 0 \qquad \alpha = \frac{1 \pm 3}{2} \begin{matrix} \nearrow 2 \\ \searrow -1 \end{matrix}$

int gen omog $\quad y(x) = k_1 e^{2x} + k_2 e^{-x}$

(4) $\quad f(x) = (3x+1)e^{ix} \qquad m = 1 \qquad h = i \qquad s = 0$

(3) $\quad f(x) = (x-1)e^{2x} \qquad m = 1 \qquad h = 2 \qquad s = 1$

(4) cerco $\quad y(x) = e^{ix}(ax + b)$

$y'(x) = e^{ix}(iax + ib + a)$

$y''(x) = e^{ix}(-ax - b + 2ia)$

sost nell'eq $\quad \cancel{e^{ix}}(-ax - b + 2ia - iax - ib - a - 2ax - 2b) = (3x+1)\cancel{e^{ix}}$

$a(-3-i)x + 2ia - a - 3b - ib = 3x + 1$

$a = \frac{3}{-3-i} = \frac{3(-3+i)}{9+1} = -\frac{9}{10} + \frac{3}{10}i = \frac{3}{10}(-3+i)$

$2i\left(-\frac{9}{10} + \frac{3}{10}i\right) + \frac{9}{10} - \frac{3}{10}i + b(-3-i) = 1$

$-\frac{9}{5}i - \frac{3}{5} + \frac{9}{10} - \frac{3}{10}i + b(-3-i) = 1$

$\frac{3}{10} - \frac{21}{10}i - 1 = b(3+i)$

$b(3+i) = -\frac{7}{10}(1+3i) \qquad b = -\frac{7}{10}\frac{1+3i}{3+i} =$

$= -\frac{7}{10}\frac{(1+3i)(3-i)}{9+1} = -\frac{7}{100}(6+8i)$

$y(x) = \left(-\frac{3}{10}(3+i)x - \frac{7}{50}(3+4i)\right)(\cos x + i\sin x) =$

$u(x) = \left(-\frac{9}{10}x\cos x - \frac{21}{50}\cos x + \frac{3}{10}x\sin x + \frac{14}{25}\sin x\right)$ sol di (2)

(3) cerco $\quad y(x) = e^{2x}(ax^2 + bx)$ da completare