V, W spazi vettoriali $\quad B_V$ base di V, B_W base di W

$f: V \to W \quad B'_V \quad , B'_W$

$$[M(f)]^{B_V}_{B_W} \neq [M(f)]^{B'_V}_{B'_W}$$

$$B_V = \{v_1 \dots v_n\} \qquad B_W = \{w_1 \dots w_m\}$$

$$B'_V = \{b_1 \dots b_n\} \qquad B'_W = \{c_1 \dots c_m\}$$

$$v \in V \Rightarrow v = \lambda_1 v_1 + \dots + \lambda_n v_n = [v]_{B_V} = (\lambda_1 \dots \lambda_n)$$

$$b_j = b_{1j} \cdot v_1 + b_{2j} \cdot v_2 + \dots + b_{nj} v_n$$

$$B = (b_{ij})_{1 \leq ij \leq n} \in M_{n,n}(\mathbb{R}) \Rightarrow v = \mu_1(b_{11} v_1 + \dots b_{n1} v_n) + \dots + \mu_n(b_{1n} v_1 + \dots + b_{nn} v_n) =$$

$$\begin{pmatrix} [b_1]_{B_V} & \cdots & [b_n]_{B_V} \end{pmatrix} = B$$

$$v \in V \Rightarrow \exists \mu_1 \dots \mu_n \text{ t.c. } v = \mu_1 b_1 + \dots + \mu_1 b_n \Leftrightarrow [v]_{B'_V}$$

$$(\mu_1 b_{11} + \mu_2 b_{12} + \dots + \mu b_n) v_1 + \dots + (\mu_1 b_{n1} + \dots + \mu_n b_{nn}) v_n$$

$$\Rightarrow [v]_{B_V} = \begin{pmatrix} \mu_1 b_{11} + \dots + \mu_n b_{1n} \\ \vdots \\ \mu_1 b_{n1} + \dots + \mu_n b_{nn} \end{pmatrix} = B \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = B[v]_{B'_V}$$

$$[w]_{B_W} = C[w]_{B'_W}$$

$v \in V \Rightarrow f(v) \in W \qquad \Rightarrow C[f(v)]_{B'_W} = [M(f)]^{B_v}_{B_W} \cdot B[v]_{B_V}$

$$\begin{array}{ccccc} [f(v)]_{B_W} & = & [M(f)]^{B_V}_{B_W} & \cdot & [v]_{B_V} \\ \| & & & & \| \\ C[f(v)]_{B'_W} & & & & B \cdot [v]_{B'_V} \end{array} \Rightarrow [f(v)]_{B'_W} = \underbrace{\boxed{C^{-1} \cdot [M(f)]^{B_V}_{B_W} \cdot B}}_{= [M(f)]^{B'_V}_{B'_W}} \cdot [v]_{B'_V}$$

Definizione

$M_1, M_2 \in M_{m,n}(\mathbb{R})$ si dicono equivalenti se

$\exists C \in M_{m,m}(\mathbb{R}) \, \exists B \in M_{n,n}(\mathbb{R})$ invertibili tale che $M_1 = C^{-1} M_2 B$

Proposizione

$f: V \to W$ omomorfismo

M_1 e M_2 rappresentano f rispetto a basi diverse

M_1 e M_2 sono simili

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "{[M(f)]^{B_v}_{B_W}}"] & & \mathbb{R}_m \\ V \arrow[u, "B_V \cong"] \arrow[d, "\cong B'_V"'] \arrow[rr, "f"] & & W \arrow[u, "B_W \cong"'] \arrow[d, "\cong B'_W"] \\ \mathbb{R}^n \arrow[uu, bend left, "B"] \arrow[rr, "{[M(f)]^{B_V}_{B_W}}"'] & & \mathbb{R}_m \arrow[uu, bend right, "C"'] \end{tikzcd}$$

Come funziona con gli endomorfismi?

$f: V \to V$ $\quad \underbrace{\mathcal{B}, \mathcal{B}'}_{\text{base}}$ basi di V $\quad \underbrace{B}_{\text{matrice}}$ = matrice di cambiamento di base

sono 2 cose diverse

$$[M(f)]^{\mathcal{B}}_{\mathcal{B}}, \quad [M(f)]^{\mathcal{B}'}_{\mathcal{B}}, \quad \text{ecc...}$$

$$\begin{tikzcd}
R^n \arrow[rr, "{[M(f)]^B_B}"] & & R_n \\
V \arrow[u, "\mathcal{B}\,\cong"] \arrow[d, "\mathcal{B}'\,\cong"'] \arrow[rr, "f"] & & V \arrow[u, "\mathcal{B}\,\cong"'] \arrow[d, "\mathcal{B}'\,\cong"] \\
R^n \arrow[uu, bend left, "B"] \arrow[rr, "{[M(f)]^B_B}"'] & & R_n \arrow[uu, bend right, "B"']
\end{tikzcd}$$

quindi $$[M(f)]^{\mathcal{B}'}_{\mathcal{B}'} = B^{-1} [M(f)]^{\mathcal{B}}_{\mathcal{B}} \cdot B$$

$$[M(f)]^{\mathcal{B}'}_{\mathcal{B}} = [M(f)]^{\mathcal{B}}_{\mathcal{B}} \cdot B$$

$$[M(f)]^{\mathcal{B}}_{\mathcal{B}'} = B^{-1} [M(f)]^{\mathcal{B}}_{\mathcal{B}}$$

Due matrici $M_1, M_2 \in M_{n,n}(\mathbb{R})$ sono simili se $\exists B \in M_{n,n}(\mathbb{R})$ invertibile tale che $M_1 = B^{-1} M_2 B$

Esercizi

$$f: \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x+2y+z \\ y+z \end{pmatrix}$$

1) f è lineare? Si perché è data da polinomi omogenei di primo grado

2) Scrivere $M(f)$ rispetto a $\mathbb{R}^3$ e $\mathbb{R}^2$

$$\mathcal{E}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{E}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M(f) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

3) $\operatorname{Ker} f \neq 0$?

$$\operatorname{Ker}(f) = \operatorname{Ker}(M(f))$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{cases} x - z = 0 \\ y + z = 0 \end{cases} \to \begin{cases} x = z \\ y = -z \end{cases} \Rightarrow \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$$

$$\operatorname{Ker} \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle \to$$ è diverso da 0

$$\dim(\operatorname{Ker}(f)) = 1 = \operatorname{null}(f) \Rightarrow 2 = rk(A) = rk(f) = \dim(\operatorname{Im}(f))$$

2)

$$w = f\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}$$

$$f^{-1}(w) = \left\{ v \in V \;\middle|\; f(v) = \begin{pmatrix}7\\9\end{pmatrix} \right\} = \begin{pmatrix}2\\1\\3\end{pmatrix} + \operatorname{Ker} f = \begin{pmatrix}2\\1\\3\end{pmatrix} + \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix} \right\rangle$$

$$v = \begin{pmatrix}x\\y\\z\end{pmatrix} \qquad f(v) = \begin{pmatrix}x+2y+z\\y+z\end{pmatrix} = \begin{pmatrix}7\\9\end{pmatrix}$$

$$\begin{cases} x + 2y + z = 7 \\ y + z = 9 \end{cases}$$

Esercizio 8 $T: \mathbb{R}^4 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 2x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ x_2 + 2x_3 \end{pmatrix}$$

Scrivere $\left[M(f) \right]^{\mathcal{E}_4}_{\mathcal{E}_3}$

$$x_1 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[M(f) \right]^{\mathcal{E}_4}_{\mathcal{E}_3} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$\operatorname{Ker} f = \operatorname{Ker}(M(f))$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xrightarrow{G-J} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{pmatrix} \to x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\operatorname{Ker} f = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $\quad \dim(\operatorname{Ker}.(f)) = 1 = \operatorname{null}(f)$

$\operatorname{Im}(f) \left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \right\rangle$

↳ mettiamo a matrice e troviamo $rk = 3$ quindi è lin ind ciò vuol dire che è una base di f trovare $[M(t)]^{\mathcal{B}}_{\mathcal{C}}$

$t: \mathbb{R}^4 \to \mathbb{R}^3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} 2x_1 + x_3 \\ 2x_1 + x_2 + x_3 \\ x_2 + 2x_3 \end{pmatrix}$

relazione alla quale facciamo riferimento

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Poi risolvere usando queste formule

$C^{-1} M(t)^{\mathcal{E}}_{\mathcal{E}} B$

$t(b_1) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad t(b_2) = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \quad t(b_3) = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \quad t(b_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$[t(b_1)]_C = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad [t(b_2)]_C = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \quad [t(b_3)]_C = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix},$

$[t(b_4)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$

$[M(t)]^{\mathcal{B}}_{C} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}$