FUNZIONI RAZIONALI FRATTE

$f(x) = \frac{A(x)}{B(x)}$ $\quad$ A polinomio di grado n $(n \in \mathbb{N}_0)$

B " m $(m \in \mathbb{N})$

se $n \geq m$ f funz. raz. fratta NON PROPRIA

se $n < m$ " " PROPRIA

NON PROPRIA $\to$ POLINOMIO + PROPRIA

infatti $A(x) = B(x)\,Q(x) + R(x)$ $\quad$ grado R < grado B

$$\Rightarrow \quad f(x) = \frac{A(x)}{B(x)} = \underset{\substack{\uparrow \\ \text{polin.}}}{Q(x)} + \underset{\substack{\uparrow \\ \text{propria}}}{\frac{R(x)}{B(x)}}$$

es. $\int \frac{x^2+3}{x+1}\,dx = \int \frac{(x^2+2x+1)-2x+2}{x+1}\,dx = \int \frac{(x+1)^2}{x+1}\,dx - 2\int \frac{x-1}{x+1}\,dx =$

$= \int (x+1)\,dx - 2\int \frac{x+1-2}{x+1}\,dx = \int (x+1)\,dx - 2\int \frac{x+1}{x+1}\,dx + 4\int \frac{dx}{x+1} =$

$= \frac{x^2}{2} + x - 2x + 4\log|x+1| + k$

Occupiamoci intanto di alcuni tipi di funz proprie prima di passare al caso generale

$I_n = \int \frac{dx}{(x-c)^n}$ $\quad$ $I_1 = \int \frac{dx}{x-c} = \log|x-c| + k$

$n > 1$ $\quad$ $I_n = \int (x-c)^{-n}\,dx = \left[\int t^{-n}\,dt\right]_{t=x-c} = \frac{(x-c)^{-n+1}}{-n+1} + k$

es. $\int \frac{dx}{(x+3)^4} = -\frac{1}{3}\frac{1}{(x-3)^3} + k$

$I_n \int \frac{dx}{(x^2+c^2)^n}$

$I_1 = \int \frac{dx}{x^2+c^2} = \int \frac{1}{c^2}\frac{dx}{\left(\frac{x}{c}\right)^2+1} = \frac{1}{c}\int \underset{\substack{\uparrow \\ D\left(\frac{x}{c}\right)}}{\frac{1}{c}} \frac{1}{\left(\frac{x}{c}\right)^2+1}\,dx =$ $\qquad$ $D\left(\frac{x}{c}\right) = \frac{1}{c}$

$= \frac{1}{c}\left[\int \frac{dt}{t^2+1}\right]_{t=\frac{x}{c}} = \frac{1}{c}\,\text{arctg}\,\frac{x}{c} + k$

es. $\int \frac{dx}{x^2+4} = \frac{1}{2}\,\text{arctg}\,\frac{x}{2} + k$ $\qquad (c=2)$

$\int \frac{dx}{x^2+5} = \frac{1}{\sqrt{5}}\,\text{arctg}\,\frac{x}{\sqrt{5}} + k$ $\qquad (c=\sqrt{5})$

$I_2 = \int \frac{dx}{(x^2+c^2)^2} = \frac{1}{c^2}\int \frac{c^2}{(x^2+c^2)^2}\,dx = \frac{1}{c^2}\int \frac{c^2+x^2-x^2}{(x^2+c^2)^2}\,dx = \frac{1}{c^2}\int \underset{\uparrow}{\frac{x^2+c^2}{(x^2+c^2)^2}} + \frac{1}{c^2}\int \frac{-x^2}{(x^2+c^2)^2} =$

$$D\left(\frac{1}{x^2+c^2}\right) = \frac{-2x}{(x^2+c^2)^2}$$

$$= \frac{1}{c^2}\underbrace{I_1}_{} + \frac{1}{2c^2}\int \frac{-2x}{(x^2+c^2)^2}\, x\, dx =$$

PD

$$= \frac{1}{c^2} I_1 + \frac{1}{2c^2}\frac{x}{x^2+c^2} - \frac{1}{2c^2}\int \frac{dx}{x^2+c^2}\cdot 1$$

$$\int \frac{dx}{x^2+c^2} = I_1$$

es. $$\int \frac{dx}{(x^2+9)^2} = \frac{1}{9}\int \frac{9}{(x^2+9)^2}\,dx = \frac{1}{9}\int \frac{9+x^2-x^2}{(x^2+9)^2}\,dx = \frac{1}{9}\int \frac{\cancel{x^2+9}}{(x^2+9)^{\cancel{2}}}\,dx + \frac{1}{18}\int \frac{-2x}{(x^2+9)^2}\,x\,dx$$

$$D\left(\frac{1}{x^2+9}\right) = \frac{-2x}{(x^2+9)^2}$$

$$= \frac{1}{9}\,\frac{1}{3}\operatorname{arctg}\frac{x}{3} + \frac{1}{18}\,\frac{x}{x^2+9} - \frac{1}{18}\int \frac{dx}{x^2+9} =$$

$$= \frac{1}{27}\operatorname{arctg}\frac{x}{3} + \frac{1}{18}\,\frac{x}{x^2+9} - \frac{1}{54}\operatorname{arctg}\frac{x}{3} + k$$

se al denom. c'è un polinomio di II grado con $\Delta < 0$ si cerca di ottenere al denom. una somma del tipo $(x+a)^2+b^2$
(COMPLETAMENTO DEI QUADRATI)

$$\int \frac{dx}{x^2+2x+4} = \int \frac{dx}{(x+1)^2+3} = \left[\int \frac{dt}{t^2+3}\right]_{t=x+1} =$$

$c = \sqrt{3}$

$$= \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + k$$

$$4 - \frac{9}{4} = \frac{7}{4}$$

$$c = \frac{\sqrt{7}}{2}$$

$$\int \frac{dx}{x^2+3x+4} = \int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{7}{4}} = \left[\int \frac{dt}{t^2+\frac{7}{4}}\right]_{t=x+\frac{3}{2}} =$$

$$= \frac{2}{\sqrt{7}}\operatorname{arctg}\frac{x+\frac{3}{2}}{\frac{\sqrt{7}}{2}} + k$$

$$\frac{1}{2} - \frac{1}{16} = \frac{7}{16}$$

$$c = \frac{\sqrt{7}}{4}$$

$$\int \frac{dx}{2x^2+x+1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}} =$$

$$= \frac{1}{2}\,\frac{4}{\sqrt{7}}\operatorname{arctg}\frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}} + k$$

$$\frac{\text{POLIN DI I GRADO}}{\text{POLIN DI II GRADO}}$$

I caso denom. con $\Delta < 0$

es. $$\int \frac{2x+3}{x^2+x+1}\,dx =$$

I passo ottenere al numer. la der. del denom.

$$= \int \frac{2x+1+2}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + 2\int \frac{dx}{x^2+x+1} = \log(x^2+x+1) + 2\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \log(x^2+x+1) + 2\,\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + k$$

$c = \frac{\sqrt{3}}{2}$

LOG + ARCTG

$$\int \frac{ax+b}{x^2+px+q}\,dx \qquad (p^2-4q<0)$$

$$\int \frac{x+4}{x^2+3x+5} = \frac{1}{2}\int \frac{2x+8}{x^2+3x+5}dx = \frac{1}{2}\int \frac{2x+3+5}{x^2+3x+5}dx = \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}dx + \frac{5}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{11}{4}} =$$

$$= \frac{1}{2}\log(x^2+3x+5) + \frac{5}{2}\frac{2}{\sqrt{11}} \operatorname{arctg} \frac{x+\frac{3}{2}}{\frac{\sqrt{11}}{2}} + k \qquad c = \frac{\sqrt{11}}{2}$$

II caso denom. con $\Delta > 0$ Si decompone la frazione

$$I = \int \frac{x+1}{x^2-x-6}dx$$

$x^2 - x - 6 = 0$ per $x = \frac{1 \pm 5}{2} = \begin{cases} -2 \\ 3 \end{cases}$

$x^2 - x - 6 = (x+2)(x-3)$

si può far vedere che $\frac{x+1}{x^2-x-6} = \frac{A}{x+2} + \frac{B}{x-3}$

$$= \frac{Ax - 3A + Bx + 2B}{(x+2)(x-3)} \qquad \begin{cases} A + B = 1 \\ -3A + 2B = 1 \end{cases} \qquad \begin{cases} B = 1 - A \\ -3A + 2 - 2A = 1 \end{cases} \qquad \begin{matrix} B = \frac{4}{5} \\ A = \frac{1}{5} \end{matrix}$$

$$I = \int \frac{\frac{1}{5}}{x+2}dx + \int \frac{\frac{4}{5}}{x-3}dx = \frac{1}{5}\log|x+2| + \frac{4}{5}\log|x-3| + k$$

$$I = \int \frac{3x+1}{x^2+4x}dx = \int \frac{3x+1}{x(x+4)}dx \qquad \frac{3x+1}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4} = \frac{Ax + 4A + Bx}{x(x+4)}$$

$$\begin{cases} A + B = 3 \\ 4A = 1 \end{cases} \qquad \begin{cases} B = \frac{11}{4} \\ A = \frac{1}{4} \end{cases}$$

$$I = \frac{1}{4}\int \frac{dx}{x} + \frac{11}{4}\int \frac{dx}{x+4} = \frac{1}{4}\log|x| + \frac{11}{4}\log|x+4| + k$$

III caso denom. con $\Delta = 0$

$$I = \int \frac{x+3}{x^2+2x+1}dx$$

si cercano A, B : $\frac{x+3}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{Ax + A + B}{(x+1)^2}$

$$\begin{cases} A = 1 \\ A + B = 3 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 2 \end{cases}$$

$$I = \int \frac{dx}{x+1} + \int \frac{2}{(x+1)^2}dx = \log|x+1| - \frac{1}{x+1} + k$$

$$I = \int \frac{4x+9}{x^2-4x+4}dx \qquad \frac{4x+9}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax - 2A + B}{(x-2)^2} \qquad \begin{cases} A = 4 \\ -2A + B = 9 \end{cases} \Rightarrow \begin{cases} A = 4 \\ B = 17 \end{cases}$$

$$I = 4\log|x-2| - 17\frac{1}{x-2} + k$$

Ogni funz. raz. fratta propria si decompone nella somma di fratti semplici (tanti quante sono le soluzioni dell'eq. denom. = 0) ogni soluz. dell'eq. denom. = 0 dà luogo a tanti fratti semplici quanto è la sua molteplicità.

sol. ~~reale~~ reale c di molt. p $\rightarrow \frac{A_1}{x-c} \quad \frac{A_2}{(x-c)^2} \quad \cdots \quad \frac{A_p}{(x-c)^p}$

se $a+ib$ è una sol compl. di molt. p anche $a-ib$ lo è

$$[x-(a+ib)][x-(a-ib)] = [(x-a)-ib][(x-a)+ib] = (x-a)^2 - (ib)^2 = (x-a)^2 + b^2$$

la coppia ~~~~ $a \pm ib$ dà luogo ai seguenti fratti semplici

$$\frac{B_1 x + C_1}{(x-a)^2+b^2} \qquad \frac{B_2 x + C_2}{((x-a)^2+b^2)^2} \quad \cdots \quad \frac{B_p x + C_p}{((x-a)^2+b^2)^p}$$ ci fermeremo a $p=2$

$\int \frac{B_1 x + C_1}{(x-a)^2+b^2}\,dx$ lo sappiamo fare

$\int \frac{B_2 x + C_2}{((x-a)^2+b^2)^2}\,dx$ come si fa ?

$$\int \frac{2x+1}{(x-3)^2+4}\,dx = \int \frac{2x-6+7}{(x-3)^2+4}\,dx = \int \frac{2x-6}{(x-3)^2+4}\,dx + 7\int \frac{dx}{(x-3)^2+4} =$$

$$= \log\left((x-3)^2+4\right) + \frac{7}{2}\,\mathrm{arctg}\,\frac{x-3}{2} + k$$

$$\int \frac{x+1}{x^3(x^2+1)(x-2)}\,dx$$

$$\frac{x+1}{x^3(x^2+1)(x-2)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{Bx+C}{x^2+1} + \frac{D}{x-2} =$$

$$= \frac{A_1 x^2(x^2+1)(x-2) + A_2 x(x^2+1)(x-2) + A_3(x^2+1)(x-2) + (Bx+C)x^3(x-2) + Dx^3(x^2+1)}{x^3(x^2+1)(x-2)}$$

$$I = \int \frac{x+3}{x^3+x^2}\,dx = \int \frac{x+3}{x^2(x+1)}\,dx \qquad \frac{x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax^2+Ax+Bx+B+Cx^2}{x^2(x+1)}$$

$$\begin{cases} A + C = 0 \\ A + B = 1 \\ B = 3 \end{cases} \Rightarrow \begin{cases} C = 2 \\ A = -2 \\ B = 3 \end{cases}$$

$$I = -2\log|x| - \frac{3}{x} + 2\log|x+1| + k$$