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## Metodo di integrazione indefinita per parti

IP $f, g : (a,b) \to \mathbb{R}$ derivabili
$fg'$ sia dotata di primitive

TS $f'g$ è dotata di primitive e

$$\int \underset{\substack{\uparrow \\ FD}}{f'(x)}\, \underset{\substack{\uparrow \\ FF}}{g(x)}\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

(fattore differenziale) (fattore finito)

DIM. $\int f'(x)\, g(x)\, dx = \int \left( f'(x) g(x) + \underline{f(x) g'(x)} - f(x) g'(x) \right) dx$

$$= \int \underbrace{\left( f'(x) g(x) + f(x) g'(x) \right)}_{= D(fg)} dx - \int f(x) g'(x)\, dx = f(x) g(x) - \int f(x) g'(x)\, dx$$

ESEMPI

$\int x \cos x\, dx$

I modo $x = FD$ $\quad \int x \cos x\, dx = \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x\, dx$ NON CONVIEN

II modo $\cos x = FD$ $\quad \int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + h$

$\int e^x x^3\, dx$

FD $x^3$ $\to$ $e^x \frac{x^4}{4} - \int e^x \frac{x^4}{4}\, dx$ NON CONVIENE

FD $e^x$ $\quad \int e^x x^3\, dx = e^x x^3 - 3 \int e^x x^2\, dx =$

$= e^x x^3 - 3 e^x x^2 + 6 \int e^x x\, dx =$

$= e^x x^3 - 3 e^x x^2 + 6 e^x x - 6 \int e^x\, dx =$

$= e^x (x^3 - 3x^2 + 6x - 6) + h$

**eserc. $\int e^x x^5\, dx$**

$\int \log x\, dx = \int \underset{\substack{\uparrow \\ FD}}{1} \cdot \log x\, dx = x \log x - \int x \frac{1}{x}\, dx =$

$= x (\log x - 1) + h$

$\int \underset{\substack{\uparrow \\ FD}}{x^2} \log x\, dx = \frac{x^3}{3} \log x - \frac{1}{3} \int x^3 \frac{1}{x}\, dx = \frac{x^3}{3} \left( \log x - \frac{1}{3} \right) + h$

**eserc. $\int x^6 \log x\, dx$**

$\int \operatorname{arctg} x\, dx = \int \underset{\substack{\uparrow \\ FD}}{1} \cdot \operatorname{arctg} x\, dx = x \operatorname{arctg} x - \int \frac{x}{1+x^2}\, dx =$

$= x \operatorname{arctg} x - \frac{1}{2} \log(1+x^2) + h$

$I_n = \int \frac{dx}{(1+x^2)^n}$ $\quad n \in \mathbb{N}$ li risolviamo per ricorrenza

$I_1 = \operatorname{arctg} x + h$

$I_2 = \int \frac{dx}{(x^2+1)^2} = \int \frac{1+x^2-x^2}{(x^2+1)^2}\, dx = \underbrace{\int \frac{\cancel{1+x^2}}{(x^2+1)^{\cancel{2}}}\, dx}_{= I_1} - \int \frac{x^2}{(x^2+1)^2}\, dx =$

$D\left( \frac{1}{x^2+1} \right) = \frac{-2x}{(x^2+1)^2}$

$= I_1 + \frac{1}{2} \int \underset{\substack{\uparrow \\ FD}}{\frac{-2x}{(x^2+1)^2}}\, x\, dx =$

$= I_1 + \frac{1}{2} \frac{1}{1+x^2} x - \frac{1}{2} \underbrace{\int \frac{1}{1+x^2} \cdot 1\, dx}_{= I_1} = \ldots$

$= \frac{1}{2} \operatorname{arctg} x + \frac{1}{2} \frac{x}{x^2+1} + h$

$I_3 = \int \frac{dx}{(x^2+1)^3} = \int \frac{1+x^2-x^2}{(x^2+1)^3}\, dx = \underbrace{\int \frac{\cancel{1+x^2}}{(x^2+1)^{\cancel{3}\,2}}\, dx}_{= I_2} - \int \frac{x^2}{(x^2+1)^3}\, dx =$

$D\left( \frac{1}{(x^2+1)^2} \right) = \frac{-2\cancel{(x^2+1)}\, 2x}{(x^2+1)^{\cancel{4}\,3}} = \frac{-4x}{(x^2+1)^3}$

$$= I_2 + \frac{1}{4}\int \underset{\uparrow \, FD}{\frac{-4x}{(x^2+1)^3}}\, x\,dx = I_2 + \frac{1}{4}\frac{1}{(x^2+1)^2} - \frac{1}{4}\underbrace{\int \frac{dx}{(x^2+1)^2}}_{I_2}$$

esec. $I_4$

Vediamo ora questo esempio

$$\int \underset{\uparrow\, FD}{e^x} \sin x\,dx = e^x \sin x - \int \underset{\uparrow\, FD}{e^x}\cos x\,dx =$$

$$X \qquad = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

idea $\rightarrow$ $X = e^x(\sin x - \cos x) - X \Rightarrow X = \frac{1}{2}e^x(\sin x - \cos x)$

Si ha un teorema (senza dim.)

Se $\int f(x)\,dx = g(x) + c\int f(x)\,dx \quad c \neq 1$

allora $\int f(x)\,dx = \frac{1}{1-c}\, g(x) + k$

Nel nostro caso $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + k$

esec. $\int e^x \cos x\,dx$

$$\int \underset{\uparrow\, FD}{\sin x} \cos x\,dx = \sin^2 x - \int \sin x \cos x\,dx \Rightarrow$$

$$\Rightarrow \int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + k$$

oppure: $\int \sin x \cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + c$

$-\frac{1}{4}\cos 2x = -\frac{1}{4}(1 - 2\sin^2 x) = -\frac{1}{4} + \frac{1}{2}\sin^2 x$ differiscono per una costante

Prima formula di integr. indef. per sostituzione

IP $f : (\alpha, \beta) \rightarrow \mathbb{R}$ dot. di primitive
$g : (a,b) \rightarrow (\alpha,\beta)$ derivabile

TS $\int f(g(x))\,g'(x)\,dx = \Big[\int f(y)\,dy\Big]_{y=g(x)}$

ins. delle prim. di $f$ composte con $g$

es. $\int \sin^2 x \cos x\,dx =$ $\qquad f(y) = y^2 \quad g(x) = \sin x$
$g'(x) = \cos x$

$= \Big[\int y^2\,dy\Big]_{y = \sin x} = \frac{1}{3}\sin^3 x + k$

$g'(x)\,dx = dg \qquad \int f(g(x))\,g'(x)\,dx = \int f(g)\,dg$

ERRORE CONCETTUALE

OSSERV. il teorema dice che si deve integrare la funz esterna e poi comporre le primitive con g

DIM Sia $F$ una prim. di $f$, cons $F(g(x))$, la sua derivata è $F'(g(x))g'(x) = f(g(x))\,g'(x)$
quindi il I membro della tesi è $F(g(x)) + k$ $\Rightarrow$ TS
anche il II " " "

$\int e^{\alpha x}\,dx$ si può ricondurre a questa situazione
$f(y) = e^y \quad g(x) = \alpha x \quad g'(x) = \alpha$

$$\int e^{\alpha x}\,dx = \frac{1}{\alpha}\int e^{\alpha x}\alpha\,dx = \frac{1}{\alpha}e^{\alpha x} + k$$

$$\int x \log(x^2+4)\,dx = \frac{1}{2}\int \underset{D(x^2+4)}{2x} \log(x^2+4)\,dx =$$

$$= \frac{1}{2}\Big[\int \log y\,dy\Big] \qquad = \dots$$

$$\int x \log(x^2+4)\,dx = \frac{1}{2}\int \underbrace{2x}_{D(x^2+4)} \log(x^2+4)\,dx =$$

$$= \frac{1}{2}\left[\int \log y\,dy\right]_{y=x^2+4} = \dots$$

$$\int 5x^2 \cos(2x^3+8)\,dx = \frac{5}{6}\int 6x^2\cos(2x^3+8)\,dx =$$

$$= \frac{5}{6}\left[\int \cos y\,dy\right]_{y=2x^3+8} = \frac{5}{6}\sin(2x^3+8)+k$$

$$\int \frac{\log^3 x + 3\log^2 x}{x}\,dx = \qquad y^3+3y^2 \qquad \frac{1}{x} = D(\log x)$$

$$=\left[\int (y^3+3y^2)\,dy\right]_{y=\log x} = \frac{1}{4}\log^4 x + \log^3 x + k$$

$$D(\operatorname{tg} x) = 1+\operatorname{tg}^2 x$$

$$\int (\operatorname{tg}^2 x+1)(\operatorname{tg}^4 x + 2\operatorname{tg}^3 x + \operatorname{tg} x)\,dx =$$

$$=\left[\int (y^4+2y^3+y)\,dy\right]_{y=\operatorname{tg}x} = \frac{1}{5}\operatorname{tg}^5 x + \frac{1}{2}\operatorname{tg}^4 x + \frac{1}{2}\operatorname{tg}^2 x + k$$

$$\int \frac{dx}{x\log x} = \int \frac{\frac{1}{x}}{\log x}\,dx = \log|\log x| + k$$

Polinomi trigonometrici $(\cos^n x \sin^m x)$ $\quad n, m \in \mathbb{N}_0$

Cominciamo con

$$I_n = \int \cos^n x\,dx$$

$$\cos^2 t = \frac{1+\cos 2t}{2} \qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$I_1 = \int \cos x\,dx = \sin x + k$$

$$I_2 = \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx =$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + k$$

$$I_3 = \int \cos^3 x\,dx = \int \cos^2 x\cdot\cos x\,dx = \int (1-\sin^2 x)\cos x =$$

$$= \int \cos x - \int \sin^2 x \underbrace{\cos x}_{D(\sin x)} = \sin x - \frac{\sin^3 x}{3} + k$$

$$I_4 = \int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx =$$

$$= \frac{1}{4}\int (1+2\cos 2x + \cos^2 2x)\,dx = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\,dx$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + k$$

$$I_5 = \int \cos^5 dx = \int \cos^4 x \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx =$$

$$= \left[\int (1-y^2)^2\right]_{y=\sin x} = \dots$$

si procede così per gli altri valori di $n$

$$J_m = \int \sin^m x\,dx$$

$$J_1 = -\cos x + k$$

$$J_2 = \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \dots$$

$$J_3 = \int \sin^3 x\,dx = \int \sin^2 x\cdot\sin x\,dx = \int (1-\cos^2 x)\sin x\,dx =$$

$$= -\int (1-\cos^2 x)(-\sin x)\,dx = -\left[\int (1-y^2)\,dy\right]_{y=\cos x}$$

$$I = \int \cos^n x \sin^m x\,dx$$

I caso $\quad n, m$ entrambi pari $\quad n = 2p \quad m = 2q$

$$I = \int (\cos^2 x)^p (\sin^2 x)^q\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^p \left(\frac{1-\cos 2x}{2}\right)^q dx$$

si riconduce a potenze del coseno

es. $$\int \cos^4 x \sin^2 x\,dx = \int (\cos^2 x)^2 \sin^2 x\,dx =$$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 \frac{1-\cos 2x}{2}\,dx = \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\,dx =$$

$$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\,dx =$$

$$= \frac{1}{8}\int\left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx =$$

$$= \frac{1}{8}\left(x+\frac{\sin 2x}{2}-\int\frac{1+\cos 4x}{2}\,dx-\frac{1}{2}\int\left(1-\sin^2 2x\right)\cos 2x\,dx\right) =$$

$$= \frac{1}{8}x+\frac{1}{16}\sin 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x-\left[\int\left(1-y^2\right)dy\right]_{y=\sin 2x} =$$

$$= -\frac{3}{8}x+\frac{1}{16}\sin 2x-\frac{1}{8}\sin 4x-\sin 2x+\frac{1}{3}\sin^3 2x+k$$

II caso $n$ disp. ($o$ $m$ disp.) ($o$ entrambi)

se $n$ è dispari si isola un fattore $\cos x$ $\quad n = 2p+1$
$m$ pari $\quad m = 2q$

$$I = \int \cos^{2p}x\,\sin^{2q}x\,\cos x\,dx = \int\left(\cos^2\right)^p x\,\sin^{2q}x\,\cos x\,dx =$$

$$= \int\left(1-\sin^2 x\right)^p \sin^{2q}x\,\underset{D(\sin x)}{\underbrace{\cos x}}\,dx = \left[\int\left(1-y^2\right)^p y^{2q}\,dy\right]_{y=\sin x}$$

se $m$ è dispari si isola un fattore $\sin x$
se sono entrambi dispari se ne sceglie uno da isolare

es. $$\int \cos^3 x\,\sin^2 x\,dx = \int \cos x\left(1-\sin^2 x\right)\sin^2 x\,dx =$$

$$= \int \cos x\left(\sin^2 x-\sin^4 x\right)dx = \left[\int\left(y^2-y^4\right)dy\right]_{y=\sin x} =$$

$$= \frac{1}{3}\sin^3 x-\frac{1}{4}\sin^4 x+k$$

$\sin^4 = (\sin^2)^2$

$$\int \cos^4 x\,\sin^5 x\,dx = \int \cos^4 x\left(1-\cos^2 x\right)^2 \sin x\,dx =$$

$$= -\int\left(\cos^4 x+\cos^8 x-2\cos^6 x\right)\underset{D(\cos x)}{\underbrace{(-\sin x)}}\,dx =$$

$$= -\left(\frac{1}{5}\cos^5 x+\frac{1}{9}\cos^9 x-\frac{2}{7}\cos^7 x\right)+k$$

$$\int \cos^7 x\,\sin^9 x\,dx = \int \cos^7 x\,\sin^8 x\,\sin x\,dx =$$

$$= -\int \cos^7 x\left(1-\cos^2 x\right)^4\underset{D(\cos x)}{\underbrace{(-\sin x)}}\,dx =$$

$$= -\left[\int y^7\left(1-y^2\right)^4 dy\right]_{y=\cos x} \qquad 1-2y^2+y^4$$

$$= -\left[\int y^7\left(1+4y^4+y^8-4y^2+2y^4-2y^6\right)dy\right]_{y=\cos x} =$$

$$= -\left[\int\left(y^7+4y^{11}+y^{15}-4y^9+2y^{11}-2y^{13}\right)dy\right]_{y=\cos x} =$$

$$= -\left(\frac{1}{8}\cos^8 x+\frac{1}{2}\cos^{12}x+\frac{1}{16}\cos^{16}x-\frac{2}{5}\cos^{10}x-\frac{1}{7}\cos^{14}x\right)+k$$

$$\int \cos^3 x\,\sin^3 x\,dx = \int \cos x\left(1-\sin^2 x\right)\sin^3 x\,dx =$$

$$= \left[\int\left(y^3-y^5\right)dy\right]_{y=\sin x} = \frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+k$$

**eserc.** $\int \cos^2 x\,\sin^4 x\,dx$

$\int \cos^3 x\,\sin^2 x\,dx$

$\int \cos^2 x\,\sin^3 x\,dx$

$\int \cos^5 x\,\sin^3 x\,dx$

Esercizi

1) Trovare le primitive in $]-\infty,+\infty[$ di $f(x) = |x-2|+3x-x^2+1$

$$f(x) = \begin{cases} 3+2x-x^2 & x<2 \\ 4x-x^2-1 & x\geq 2 \end{cases}$$

Se $F$ è prim. di $f$ in $]-\infty,+\infty[$, in particolare lo è in $]-\infty,-2[$ e in $[2,+\infty[$

$$f(x) = \begin{cases} 3x + x^2 - \frac{1}{3}x^3 + k_1 & x < 2 \\ 2x^2 - \frac{1}{3}x^3 - x + k_2 & x \geq 2 \end{cases}$$

ma per essere prim. in $]-\infty, +\infty[$ deve essere derivabile $\Rightarrow$ cont.

Imponiamo la continuità per $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \qquad 10 - \cancel{\frac{8}{3}} + k_1 = 6 - \cancel{\frac{8}{3}} + k_2 \Rightarrow k_2 = k_1 + 4$$

$$f(x) = \begin{cases} 3x + x^2 - \frac{1}{3}x^3 + k & x < 2 \\ 2x^2 - \frac{1}{3}x^3 - x + k + 4 & x \geq 2 \end{cases}$$

2) Trovare $f$ prim in $]-\infty, +\infty[$ di $f(x) = x^2 - 2|x-1| + x + 3$ tale che $f(2) = 6$

$$f(x) = \begin{cases} x^2 + 1 + 3x & x < 1 \\ x^2 - x + 5 & x \geq 1 \end{cases}$$

$$F(x) = \begin{cases} \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + k_1 & x < 1 \\ \frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x + k_2 & x \geq 1 \end{cases}$$

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^+} F(x) \qquad \cancel{\frac{1}{3}} + \frac{3}{2} + 1 + k_1 = \cancel{\frac{1}{3}} - \frac{1}{2} + 5 + k_2$$

$$k_2 = k_1 - 2$$

le prim. sono $F(x) = \begin{cases} \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + k & x < 1 \\ \frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x + k - 2 & x \geq 1 \end{cases}$ ←

$F(2) = 6 \qquad F(2) = \frac{8}{3} - 2 + 10 + k - 2 = 6 \Rightarrow k = 6 - \frac{26}{3} = -\frac{8}{3}$

la prim richiesta è $F(x) = \begin{cases} \frac{1}{3}x^3 + \frac{3}{2}x^2 + x - \frac{8}{3} & x < 1 \\ \frac{1}{3}x^3 - \frac{1}{2}x^2 + 5x - \frac{14}{3} & x \geq 1 \end{cases}$

eserc. 1) trovare tutte le prim. di $|\cos x|$ in $[0, \pi]$

2) trovare $f$ prim. in $]-\infty, +\infty[$ di $f(x) = 3x - |x| + 2x^2 - 1$ tale che $f(-1) = 1$

3) trovare $f$ prim. in $]-\infty, +\infty[$ di $f(x) = 3|x+1| - x + 4$ tale che $f(0) = 2$

4) trovare $f$ prim. in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ di $|\sin x|$ tale che $f(\frac{\pi}{4}) = 0$