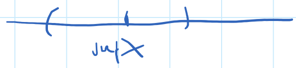
17 marzo 2025
lunedì 17 marzo 2025 13:04

$\sup X = \max X$ $\qquad X = [a, b]$

$\nexists \max X$ ma $\sup X \in \mathbb{R}$ $\qquad X = ]a, b[$

$\sup X \in \text{int}(X)$ non esiste




$\sup X \in D(X)$ $\qquad X = [a, b]$ opp. $X = ]a, b[$

$\sup X \notin D(X)$ $\qquad X = \{1; 2; 3\}$ opp $X = [0, 1] \cup \{3\}$

$X$ : $X$ ha sia p. di acc. che p. isolati $\qquad X = [0, 1] \cup \{3\}$

$\mathbb{R}$ ha la seguente proprietà di separazione

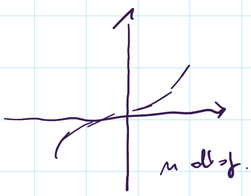
$a, b \in \mathbb{R}$ con $a \neq b$ $\quad \exists\, r, s > 0 : B(a, r) \cap B(b, s) = \emptyset$

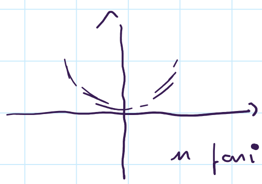

$a + r < b - s$

FUNZIONI ELEMENTARI

1) funz. potenza

$n \in \mathbb{N}$ $\qquad f(x) = x^n$ $\qquad \forall x \in ]-\infty, +\infty[$

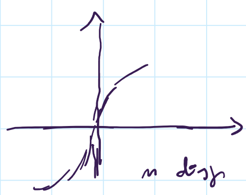


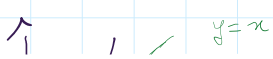

n disp $\Rightarrow$ invert. in $]-\infty, +\infty[$

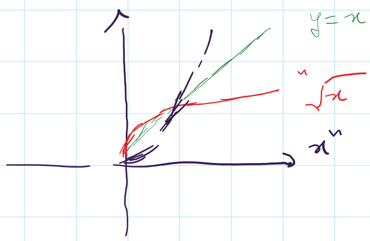
n pari $\Rightarrow$ " in $[0, +\infty[$ $\big($ e in $]-\infty, 0]\big)$ $\qquad$ inversa $\sqrt[n]{x}$

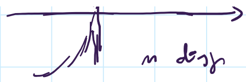
$f(x) = \sqrt[n]{x}$ $\qquad \forall x \geq 0$ se n pari
$\qquad\qquad\qquad \forall x \in \mathbb{R}$ se n disp.

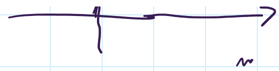




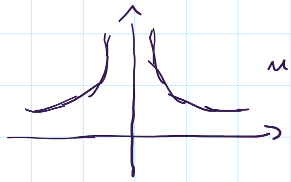


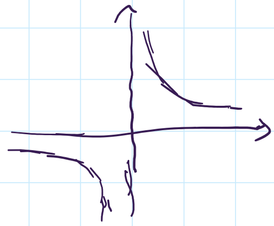



$n \in \mathbb{N}$ $\quad f(x) = x^{-n} \quad \forall x \neq 0 \qquad x^{-n} = \dfrac{1}{x^n}$




$n \in \mathbb{N} \quad f(x) = x^{\frac{1}{n}} \quad \left( = \sqrt[n]{x} \quad \text{se } x \geq 0 \right)$

$r = \frac{m}{n} \in \mathbb{Q} \quad f(x) = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m \qquad \forall x \geq 0 \quad \text{se } r > 0$

$\forall x > 0 \quad \text{se } r < 0$

$s \in \mathbb{R} \setminus \mathbb{Q} \quad f(x) = x^s \qquad \forall x \geq 0 \quad \text{se } s > 0$

$\forall x > 0 \quad \text{se } s < 0$

$x^r$ $(r \in \mathbb{Q})$, $x^s$ $(s \in \mathbb{R} \setminus \mathbb{Q})$ sono strett. cresc. in $]0, +\infty[$

l'inversa è $x^{\frac{1}{r}}$, $x^{\frac{1}{s}}$

2) POLINOMI $\quad f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n \qquad a_0, \dots, a_n \in \mathbb{R}$

$n \in \mathbb{N}_0$

$\forall x \in \mathbb{R}$

$a_0, \dots, a_n$ coefficienti $\qquad n$ grado

$n = 0$ polinomio costante $\quad f(x) = a_0 \quad \forall x \in \mathbb{R}$

se $a_0 = 0$ pol. nullo

$f(x) = a_0 x^n + \dots + a_n$

$g(x) = b_0 x^p + \dots + b_p$

sono uguali (cioè $f(x) = g(x)$ $\forall x$) $\Leftrightarrow$ sono due polinomi identici cioè $n = p$, $a_0 = b_0, \dots, a_n = b_n$

(principio di identità dei polinomi)

dati due pol. $f, g$ $\exists$ $q(x)$, $r(x)$ con grado di $r$ < grado di $g$, tali che

$$f(x) = g(x)\,q(x) + r(x)$$ (se $r(x) = 0$ $f$ è divisibile per $g$)

teorema di Ruffini: il polin. $f(x)$ è divisibile per $x - c \iff f(c) = 0$

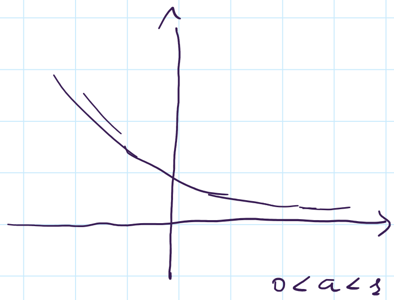
3) funz. razionale fratta $f(x) = \dfrac{a_0 x^n + \dots + a_n}{b_0 x^p + \dots + b_p}$ $a_0, \dots, a_n, b_0, \dots, b_p \in \mathbb{R}$, $n \in \mathbb{N}_0$, $p \in \mathbb{N}$

$f$ è def in $\mathbb{R} \setminus \{c \in \mathbb{R} : b_0 c^p + \dots + b_p = 0\}$

4) funz. esponenziale $a \in \mathbb{R}$ $a > 0, a \neq 1$ $f(x) = a^x$ $\forall x \in \mathbb{R}$

$a^x > 0$ $\forall x \in \mathbb{R}$

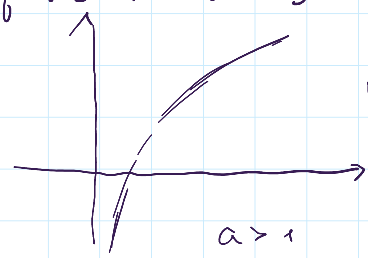
$a > 1$ strett. cresc



$0 < a < 1$ strett decr

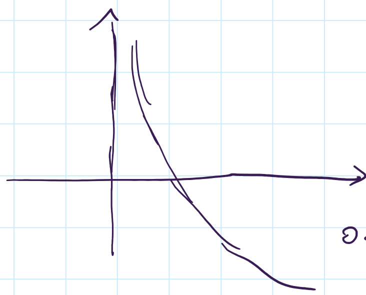
$f : ]-\infty, +\infty[ \to ]0, +\infty[$

$f^{-1} : ]0, +\infty[ \to ]-\infty, +\infty[$



$a > 1$

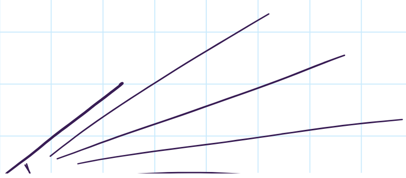
$f^{-1}$ è la funz. logaritmo

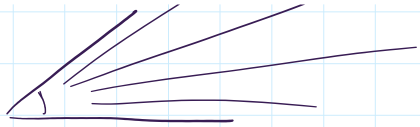


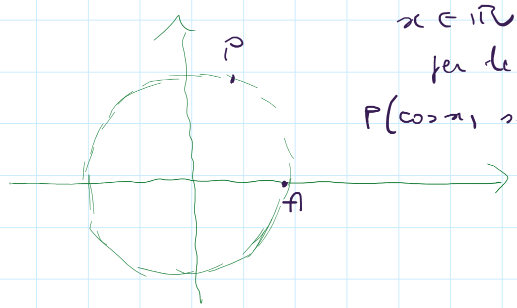
$0 < a < 1$

$$f^{-1}(f(x)) = x \qquad f(f^{-1}(y)) = y$$

5) funzioni trigonometriche



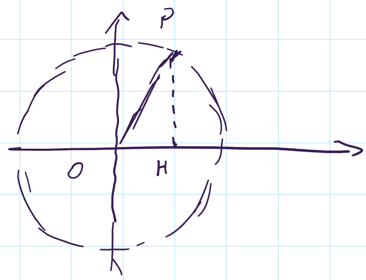





$x \in \mathbb{R}$
per def
$P(\cos x, \sin x)$

Se $x \in \mathbb{R}$, $\cos x$ ($\sin x$) è l'ascissa (l'ordinata) del secondo estremo di un arco della circonf. trigonom. avente primo estremo $A(1,0)$ e misura in radianti $x$.

$-1 \le \cos x \le 1$ $\quad \forall x \in \mathbb{R}$
$-1 \le \sin x \le 1$
$\cos^2 x + \sin^2 x = 1 \quad \forall x \in \mathbb{R}$

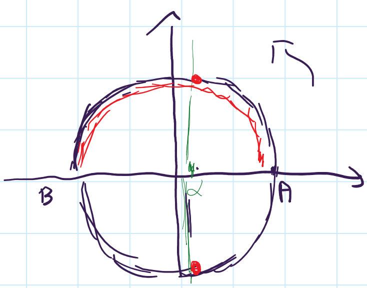



$\left.\begin{array}{l} PH = \sin x \\ OH = \cos x \\ OP = 1 \end{array}\right\} \Rightarrow T.S.$

$\cos(x + 2k\pi) = \cos x$
$\sin(\qquad) = \sin x \qquad \forall x \in \mathbb{R}, \quad \forall k \in \mathbb{Z}$

$T = 2\pi$

$\cos(-x) = \cos x \quad \forall x$
(funz pari)

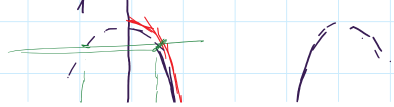


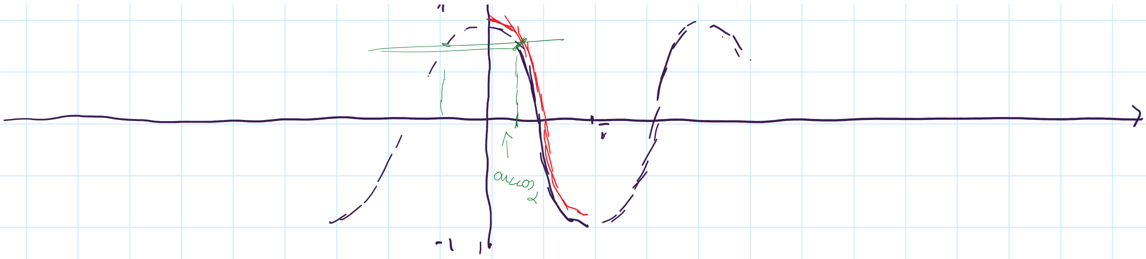

$\cos 0 = 1$
$\cos \pi = -1$

in $[0, \pi]$ $f(x) = \cos x$ è strett. decr. e assume tutti i valori compresi tra $-1$ e $1$, quindi è invert. e abbiamo

$\cos : [0, \pi] \to [-1, 1]$
$\arccos : [-1, 1] \to [0, \pi]$

$\forall \alpha \in [-1, 1]$ def $\arccos \alpha =$ l'unico $x \in [0, \pi]$ : $\cos x = \alpha$






$\cos x = \alpha$ se $\alpha \in [-1, 1]$ sol $\pm \arccos\alpha + 2k\pi$

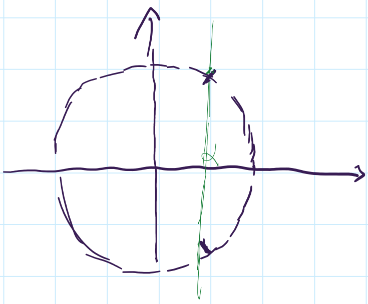
se $\alpha > 1$ opp. $\alpha < -1$ ness. sol.

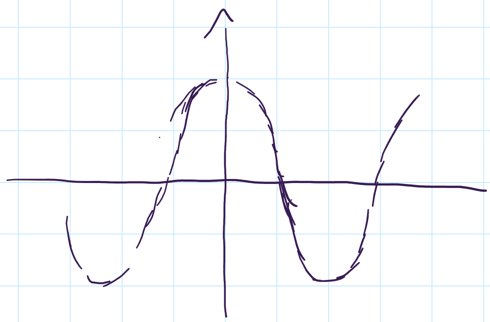
$\cos x > \alpha$ se $-1 \le \alpha < 1$ sol $-\arccos\alpha + 2k\pi < x < \arccos\alpha + 2k\pi$

se $\alpha \ge 1$ ness sol

se $\alpha < -1$ $\forall x \in \mathbb{R}$ è sol.

$\cos x < \alpha$

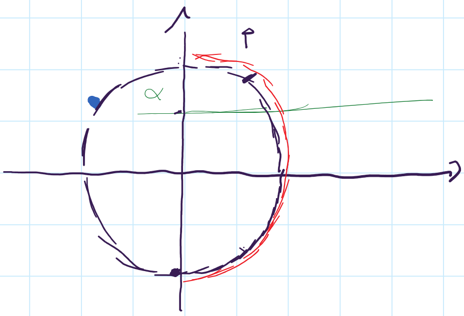


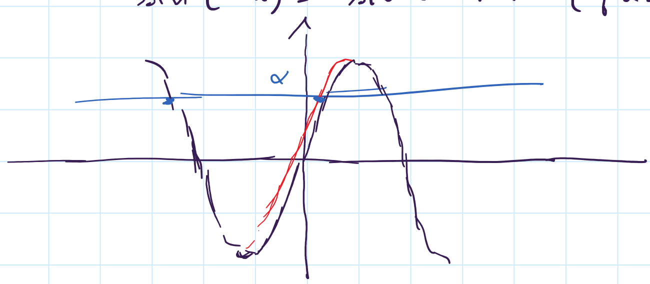
$\cos x < \alpha$ $\arccos\alpha < x < 2\pi - \arccos\alpha$ se $-1 < \alpha \le 1$

ness. sol. se $\alpha \le -1$

$\forall x \in \mathbb{R}$ è sol. se $\alpha > 1$




$\sin(-x) = -\sin x$ $\forall x$ (funz. disp.)




$\sin\left(-\frac{\pi}{2}\right) = -1$ $\sin\left(\frac{\pi}{2}\right) = 1$

in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ è strett. crescente

e assume tutti i valori compresi fra $-1$ e $1$

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin x = \alpha$ se $-1 \le \alpha \le 1$

sol $x = \arcsin\alpha + 2k\pi$

$\pi - \arcsin\alpha + 2k\pi$

$\sin x < \alpha$

$-\pi - \arcsin\alpha + 2k\pi < x < \arcsin\alpha + 2k\pi$

se $-1 < \alpha \le 1$

$\forall x \in \mathbb{R}$ è sol. se $\alpha > 1$

ness. sol. se $\alpha \le -1$

$\sin x > \alpha$

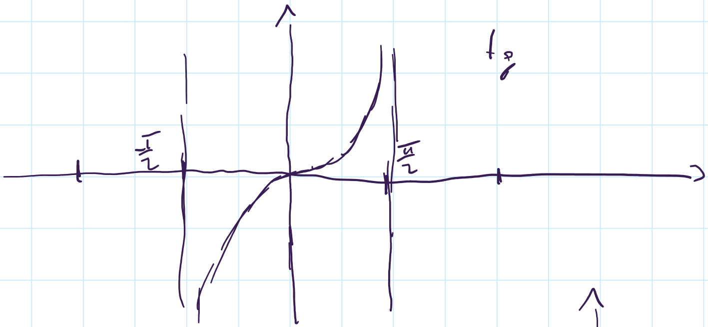
$\arcsin\alpha + 2k\pi < x < \pi - \arcsin\alpha + 2k\pi$ se $-1 \leq \alpha < 1$

ness. sol. se $\alpha \geq 1$

$\forall x \in \mathbb{R}$ è sol se $\alpha < -1$

DEF. $\operatorname{tg} x = \dfrac{\sin x}{\cos x}$ $\quad x \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$ $\qquad \operatorname{tg}(x + k\pi) = \operatorname{tg} x \quad \forall x \ \forall k \in \mathbb{Z}$

$$\operatorname{tg}(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\operatorname{tg} x$$

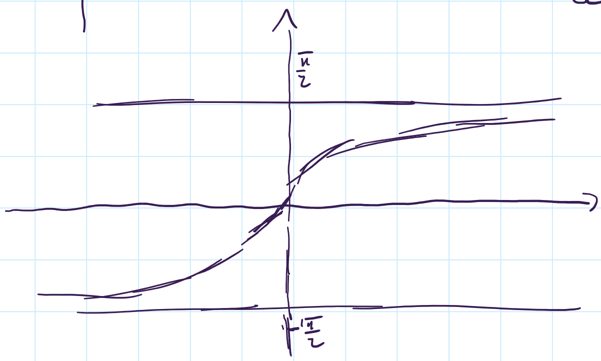


tg

in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ $f(x) = \operatorname{tg} x$ è strett cresc e assume tutti i valori reali

$\operatorname{tg} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \to \left]-\infty, +\infty\right[$

$\operatorname{arctg} : \left]-\infty, +\infty\right[ \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$



arctg

$\operatorname{tg} x = \alpha \qquad x = \operatorname{arctg}\alpha + k\pi \qquad \forall \alpha \in \mathbb{R}$

$\operatorname{tg} x > \alpha \qquad \operatorname{arctg}\alpha + k\pi < x < \frac{\pi}{2} + k\pi$

$\operatorname{tg} x < \alpha \qquad -\frac{\pi}{2} + k\pi < x < \operatorname{arctg}\alpha + k\pi$

$f(x) = \sqrt{\operatorname{arctg} x - \frac{x}{x^2 - 1}}$ è una funz. elementare

Esercizi sull'insieme di definizione

$f(x) = 2^{\frac{\sqrt{x-1}}{|x|-3}}$ $\qquad \begin{cases} x - 1 \geq 0 \\ |x| - 3 \neq 0 \end{cases} \qquad \begin{cases} x \geq 1 \\ x \neq -3, \ x \neq 3 \end{cases}$

$f : \left([1, 3[ \ \cup \ ]3, +\infty[\right) \to \mathbb{R}$

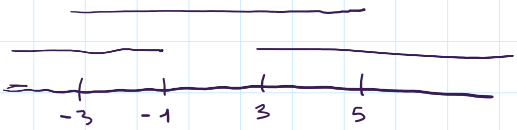
$f(x) = \arcsin\left(|x-1| - 3\right) \qquad -1 \leq |x-1| - 3 \leq 1$

$f(x) = \arcsin\left(|x-1| - 3\right)$

$-1 \leq |x-1| - 3 \leq 1$

$2 \leq |x-1| \leq 4$

$\begin{cases} |x-1| \geq 2 \\ |x-1| \leq 4 \end{cases} \Rightarrow \begin{cases} x-1 \leq -2 \ \vee \ x-1 \geq 2 \\ -4 \leq x-1 \leq 4 \end{cases} \Rightarrow$




$\Rightarrow \begin{cases} x \leq -1 \ \vee \ x \geq 3 \\ -3 \leq x \leq 5 \end{cases}$

$f: \left([-3,-1] \cup [3,5]\right) \to \mathbb{R}$

$f(x) = \sqrt[4]{\sqrt[3]{x-2} - 1}$

$\sqrt[3]{x-2} - 1 \geq 0$

$\sqrt[3]{x-2} \geq 1$

$x - 2 \geq 1$

$x \geq 3$ $\qquad f: [3, +\infty[ \to \mathbb{R}$

$$\boxed{e^x \qquad \log x = \log_e x \qquad e > 1}$$

$f(x) = \log \sqrt{\dfrac{|x|-2}{x^2+3}}$

$\begin{cases} \sqrt{\dfrac{|x|-2}{x^2+3}} > 0 \\ \dfrac{|x|-2}{x^2+3} \geq 0 \\ x^2+3 \neq 0 \end{cases} \Rightarrow |x| - 2 > 0$

$\Downarrow$

$x > 2 \ \vee \ x < -2$

$f: \left(]-\infty, -2[ \ \cup \ ]2, +\infty[\right) \to \mathbb{R}$

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Numeri complessi

$\mathbb{C} = \{(a,b) : a, b \in \mathbb{R}\}$ (coppie ordinate)

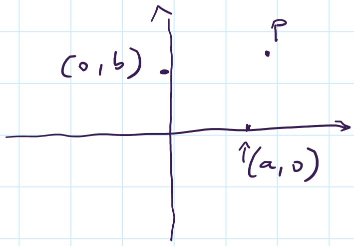
$0 = (0,0)$ zero

$z=(a,b)$ $\quad -z = (-a,-b)$ opposto

$\bar{z} = (a,-b)$ coniugato

$(a,0)$ num. complesso reale (se $b \neq 0$ immaginario)

$(0,b)$ " immaginario puro




$P(a,b) \rightarrow z=(a,b)$

(rappr. geom.)

$z=(a,b)$

$w=(c,d)$

DEF. $z=w$ se $a=c$, $b=d$

se $z \neq w$ non c'è un ordine

$i=(0,1)$ unità immaginaria

$1=(1,0)$ " reale

$z=(a,b)$

$w=(c,d)$

DEF. $z+w=(a+c,\ b+d)$ valgono le prop. comm. e assoc.

$z+0=z$

$z+(-z)=0$

$z+\bar{z} = (a+a,\ b-b) = (2a,0)$ è num. compl. reale

DEF. $zw = (ac-bd,\ ad+bc)$

[ se fosse $zw=(ac,bd)$ non sarebbe la legge di annull. del prod. $(1,0)(0,1)=(0,0)$ ]

$z\bar{z} = (a,b)(a,-b) = (a^2+b^2,\ 0)$ è num. compl. reale

$z=(a,b)$ $\quad |z| = \sqrt{a^2+b^2}$ modulo di $z$

se $z=(a,0)$ $\quad |z| = \sqrt{a^2+0} = |a|$

dim. che prendo $a \rightarrow (a,0)$ si ottiene una corrisp. biun. fra $\mathbb{R}$ e i num. compl. reali, che conserva le operazioni