

f.i. $+\infty - \infty, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}$

$$\frac{2}{m} \xrightarrow{m \rightarrow +\infty} 0$$

ERRORE

Si SCRIVE Così: $\frac{2}{m} \xrightarrow[m \rightarrow +\infty]{} 0$

PROP f: $X \rightarrow \mathbb{R}$ $X \subseteq \mathbb{R}$ f funz. elementare
 $\{a_n\} \subseteq X$ $a_n \rightarrow l \in X$

es: $\frac{m}{2m+1} \rightarrow \frac{1}{2}$ (verifichiamo il perché)

Limiti di succ espanso mediante le funzioni elementari

Succ potenze m^d $d \in \mathbb{R}$

$$d = 0 \quad m^0 = 1 \rightarrow 1$$

$$d > 0 \quad m^d \rightarrow +\infty \text{ infatti } m^d > k \Leftrightarrow m > k^{\frac{1}{d}} \text{ D. serie}$$

$$d < 0 \quad m^d \rightarrow 0 \quad \Rightarrow \quad m^d = \frac{1}{m^{-d}}$$

Polinomio

$$x_n = e_0 n^r + e_1 n^{r-1} + \dots + e_{r-1} n + e_r \quad e_1, e_2, \dots, e_r \in \mathbb{R} \quad r \in \mathbb{N}$$

e seconde dei regni dei coeff si può avere una
f.i. $+\infty - \infty$

$$(es: 3n^4 - 6n^3 + 2n^2 - 5n + 8)$$

$$\begin{matrix} \frac{1}{+\infty} \\ +\infty \end{matrix} \quad \begin{matrix} \frac{1}{+\infty} \\ +\infty \end{matrix} \quad \begin{matrix} \frac{1}{+\infty} \\ +\infty \end{matrix} \quad \begin{matrix} \frac{1}{+\infty} \\ +\infty \end{matrix}$$

$$x_n = n^r \left(e_n + \frac{e_1}{n} + \frac{e_2}{n^2} + \dots + \frac{e_r}{n^r} \right)$$

$$\text{es: } \underset{+\infty}{\lim} \left(3 - \frac{6}{x^3} + \frac{2}{x^2} - \frac{5}{x^3} + \frac{8}{x^4} \right) \rightarrow +\infty$$

$\begin{array}{ccccccc} +\infty & \xrightarrow{x \rightarrow 0} & 0 & \xleftarrow{x \rightarrow 0} 0 & \xleftarrow{x \rightarrow 0} 0 & \xleftarrow{x \rightarrow 0} 0 \\ & & 0 & & 0 & & 0 \end{array}$

$x \rightarrow \begin{cases} +\infty & \text{se } q_0 > 0 \\ -\infty & \text{se } q_0 < 0 \end{cases}$

O polinomi di segno sempre. Il segno delle derivate è il segno del coefficiente del termine di grado massimo

$$\text{es: } \begin{aligned} x^3 - 2x^2 - 6x - 3 &\rightarrow +\infty \\ x^3 + 2x^2 + 6x - 3x^4 &\rightarrow -\infty \end{aligned}$$

Funzioni razionali finite

$$x_n = \frac{c_0 x^n + \dots + c_r x^r}{b_0 x^q + \dots + b_q} \quad \begin{array}{l} c_i, b_j \in \mathbb{R} \quad b_0 \neq 0 \\ n \in \mathbb{N}_0 \quad q \in \mathbb{N} \end{array}$$

è una f.i. $\frac{\infty}{\infty}$

$$x_n = \frac{x^r \left(c_0 + \dots + \frac{c_{r+1}}{x^{r+1}} \right)}{x^q \left(b_0 + \dots + \frac{b_q}{x^q} \right)} = x^{r-q} \frac{c_0 + \dots + \frac{c_r}{x^{r-q}}}{b_0 + \dots + \frac{b_q}{x^q}}$$

↓

$$\begin{cases} 1 \text{ se } r=q \\ +\infty \text{ se } r>q \\ 0 \text{ se } r<q \end{cases}$$

$$\begin{aligned} \text{quindi } & \begin{cases} r=0 & x_n \rightarrow \frac{c_0}{b_0} \\ r>0 & x_n \rightarrow \infty \end{cases} \\ & \text{se } r=0 \quad x_n \rightarrow \frac{c_0}{b_0} \end{aligned}$$

$+\infty$ se c_0 e b_0 sono concordi, $-\infty$ se discordi)

$$\text{se } r < q \quad x_n \rightarrow 0$$

$$\text{es: } \frac{3m^2 - 5 + 1}{(m+2)^2} \rightarrow 3 \quad \frac{-2m^4 - 6m}{3m^4 + 5m^2 + 7} \rightarrow -\frac{2}{3}$$

$$\frac{3m^2 - 5 + 1}{(m+2)^4} \rightarrow 0 \quad \frac{3m - 5m + 1}{m - 3} \rightarrow \infty$$

$$\frac{3m^2 - 5m^4 + 1}{m^2 + 8} \rightarrow -\infty \quad \frac{3m^2 - 5m^4 + 1}{m^2 - 8m^3 + 3} \rightarrow \infty$$

Succezioane parametriche

$$q \in \mathbb{R} \quad \{q^n\}$$

$$q = 0 \quad q^n \rightarrow 0$$

$$q = 1 \quad q^n \rightarrow 1$$

$$q = -1 \quad q^n \text{ oscill.}$$

$$q > 1 \quad q^n \rightarrow +\infty \text{ infatti } q^n > k \Leftrightarrow n > \log_q k$$

$$q < -1 \quad q^n = ((-1)(-\infty)) = (-1)^n (-e)^n \text{ oscill.}$$

\downarrow
se $e \rightarrow +\infty$
 $(-e) \rightarrow -1$

$-1 < q < 1 \quad q^n \rightarrow 0$ infatti $|q^n| < \varepsilon \Leftrightarrow n > \log_{|q|} \varepsilon$ vero
perché $|q| < 1$ \Rightarrow

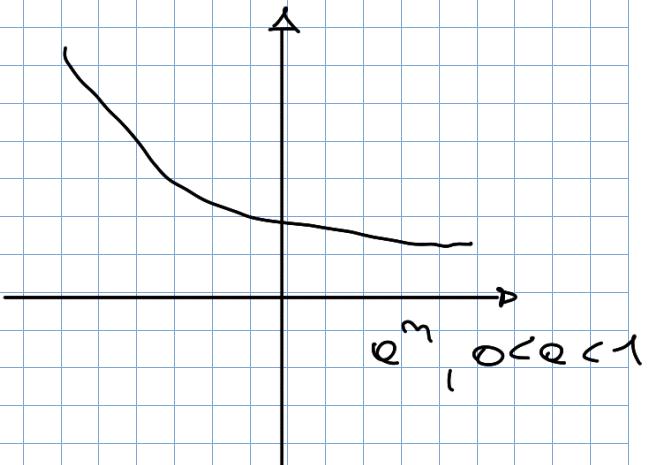
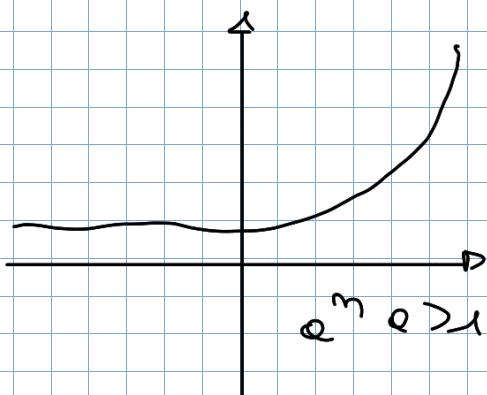
$$\text{es: } \left(-\frac{1}{5}\right)^n \rightarrow 0 \quad (-6)^n \text{ osc} \quad (18)^n \rightarrow +\infty$$

$$\left(\frac{2}{3}\right)^n \rightarrow 0$$

Succinione composta mediante le funzioni esponenziali

$\{x_n\}$ regolare con e^{x_n} con $e > 0, e \neq 1$

$\forall x_n \rightarrow l \in \mathbb{R} \rightarrow e^x \rightarrow e^l$



$\forall e > 1$

$$e^{x_n} \begin{cases} +\infty & \forall x_n \rightarrow +\infty \\ 0 & \forall x_n \rightarrow -\infty \end{cases}$$

$$e^{x_n} \begin{cases} 0 & \forall x_n \rightarrow +\infty \\ +\infty & \forall x_n \rightarrow -\infty \end{cases}$$

es:

$$\frac{3m^2 - 1}{m+4} \rightarrow +\infty$$

$$\left(\frac{1}{3}\right) \frac{m^6 - 2}{3 - m} \rightarrow -\infty$$

$$\frac{m^6 - 6}{2 - m^5} \rightarrow -\infty$$

$$\rightarrow 0$$

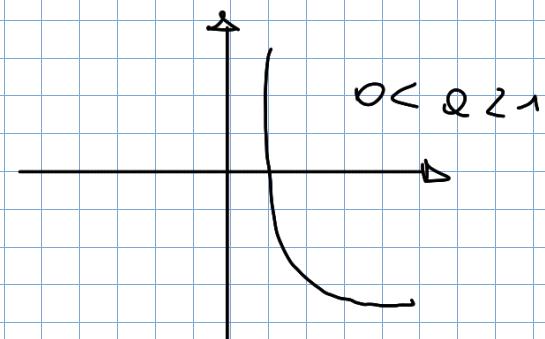
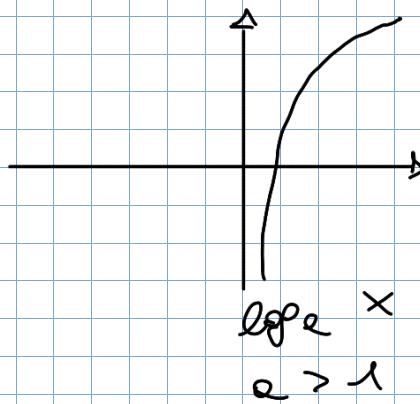
Funzione logoritmo

$\{b_m\}$ succ regolare $b_m > 0 \quad \forall m \quad \varrho > 0, \varrho \neq 1$

$$x_m = \log_{\varrho} b_m$$

$\Rightarrow b_m \rightarrow \infty \Rightarrow x_m \rightarrow \log_{\varrho} \infty$

$$\text{es: } \log_2 \frac{4m+1}{m+3} \rightarrow \log_2 4 = 2$$



$\Rightarrow \varrho > 1 \Rightarrow x_m \nearrow \infty \text{ se } b_m \rightarrow +\infty$
 $\Rightarrow \varrho > 1 \Rightarrow x_m \searrow -\infty \text{ se } b_m \rightarrow 0$

$\Rightarrow 0 < \varrho < 1 \Rightarrow x_m \begin{cases} \nearrow -\infty & \text{se } b_m \rightarrow +\infty \\ \nearrow +\infty & \text{se } b_m \rightarrow 0 \end{cases}$

$$\text{es: } \log_3 \frac{2m+1}{m^2+8} \rightarrow -\infty \quad \log_2 \frac{2m^6+1}{m^4+8} \rightarrow +\infty$$

$$\log_{\frac{1}{2}} \frac{1-3m^4}{2-m^2} \rightarrow +\infty \rightarrow -\infty$$

$$\log_{\frac{1}{4}} \frac{3m+1}{2m^4+6} \rightarrow 0 \rightarrow +\infty$$

Se $x \rightarrow +\infty$, $e^x \rightarrow +\infty$

Si deve avere $e^{x_m} > k \Leftrightarrow x_m > \log_k h$ visto D

$$\text{se } 0 < q < 1 \quad e^{x_m} = \frac{1}{\left(\frac{1}{q}\right)^{x_m}} \rightarrow 0$$

$\Rightarrow \infty \quad \frac{1}{q} > 1$

se $b_m \rightarrow +\infty$, $e > 1 \Rightarrow \log_e b_m \rightarrow +\infty$

Si deve avere $\log_e b_m > k \Leftrightarrow b_m > e^k$ visto D

se $0 < q < 1 \quad \log_q b = (\log_q \frac{1}{q})(\log_{\frac{1}{q}} b_m) \rightarrow -\infty$

se $a_m \rightarrow \infty$ allora $\log a_m, \log a_m + \log b_m$ sono non regolari

Consideriamo ora le successive successione del tipo $a_m b^m$ con $a_m > 0 \quad a_m \neq 1 \quad \forall m \quad \{a_m\}, \{b_m\}$ regolari

Ricordiammo che se $x > 0 \quad x = e^{\log x}$ quindi:

$$a_m b^m = e^{\log a_m + \log b^m} = e^{\log a_m + b_m \log b}$$

$$\Rightarrow a_m \rightarrow 1, \quad b_m \rightarrow 3 \quad e^{3 \log a_m} \rightarrow e^{3 \log 2} = e^{\log 8} = 8$$

Si osserva che se c'è una f.i. nel prodotto $b_m \log a_m$

$$b_m \rightarrow$$

$$\text{f.i. } 0^0 (+\infty)$$

$\log a_m \rightarrow \infty$ cioè $a_m \rightarrow 0$ oppure $b_m \rightarrow +\infty$

$$B_m \rightarrow 0$$

$$\log \varrho_m \rightarrow 0 \quad \text{cioè } \varrho_m \rightarrow 1$$

f.i. \downarrow^∞

Le f.i. ollere sono: $+\infty - \infty, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}, 1^0$
 $(+\infty)^0, 1^\infty$

$\log e_m b_m$ $b_m > 0 \forall m, \varrho_m > 0 \forall m, e_m \neq 1 \forall m,$

$\{\varrho_m\}$ reg, $\{b_m\}$ reg

$$\log e_m b_m = (\log e_m e) (\log b_m) \frac{\log b_m}{\log e_m}$$

$$\begin{matrix} & \infty \\ 1 & \end{matrix}$$

$$e_m = \left(1 + \frac{1}{n}\right)^n \quad \text{ri presente nelle f.i. } 1^\infty$$

Si può dim che $\{e_m\}$ è strett crescente e $e_m < 3 \forall m$

quindi converge ad un numero < 3 che chiamiamo e

Def $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad \text{si } \text{he } e \in \mathbb{Q}$

$$e > 1$$

Si può dimostrare che se $x_m \rightarrow \infty$ allora $\left(1 + \frac{1}{x_m}\right)^{x_m} \rightarrow e$

es: $\left(1 + \frac{1}{n}\right)^n = \left[\left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^2 \rightarrow e^2$
se $d \in \mathbb{R}$

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e^3$$

$$\left(1 - \frac{1}{n}\right)^n \rightarrow \frac{1}{e}$$

$$\left(\frac{m+9}{m+6}\right)^{2m+1} =$$

$$\left(\frac{2m+9}{m+6}\right)^{2m+1} \rightarrow +\infty$$

que non si applica
perché $x \neq 0$

$$= \left[\left(1 + \frac{1}{x_m} \right)^{x_m} \right] = \left(\frac{m+4+2-2}{m+6} \right) = \left(\frac{m+6-2}{m+6} \right) = \left(1 + \frac{-2}{m+6} \right)^{2m+1} =$$

$$\left(1 + \frac{1}{\frac{m+6}{-2}} \right)^{2m+1} = \left[\left(1 + \frac{1}{\frac{m+6}{-2}} \right)^{\frac{m+6}{-2}} \right]^{\frac{-2(2m+1)}{m+6}} \xrightarrow[e]{+} e^{-4} \rightarrow e^{-4}$$

$$\left(\frac{m^2+2}{(m+1)^2} \right)^{m+3} = \left(\frac{m^2+2}{m^2+2m+1} \right)^{m+3} = \left(\frac{m^2+2m+1-2m-1+2}{m^2+2m+1} \right)^{m+3} =$$

$$= \left(1 + \frac{1-2m}{m^2+2m+1} \right)^{m+3} = \left[\left(1 + \frac{1}{\frac{m^2+2m+1}{1-2m} + 1} \right)^{\frac{m^2+2m+1}{1-2m}} \right]^{\frac{(1-2m)(m+3)}{m^2+2m+1}} \xrightarrow[e]{+} e^{-2}$$

Altri limiti dedotti dal numero e

$$1) \lim_{n \rightarrow \infty} \frac{e^{a_n} - 1}{a_n} \rightarrow 1$$

$$2) \lim_{n \rightarrow \infty} \frac{\log(1+a_n)}{a_n} \rightarrow 1$$

$$3) \lim_{n \rightarrow \infty} \frac{(1+a_n)^d - 1}{a_n} \rightarrow d \quad (d \in \mathbb{R})$$

Esempio:

$$m \cdot \log\left(\frac{m+1}{m}\right) = m \cdot \log\left(1 + \frac{1}{m}\right) = \frac{\log\left(1 + \frac{1}{m}\right)}{\frac{1}{m}} \rightarrow 1$$

Limite notevoli con funz. trigonometriche

$$\lim_{n \rightarrow \infty} \theta_n \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \sin \theta_n \rightarrow 0$$

$$\frac{\sin \theta_n}{\theta_n}$$

$$f.o.i = 0$$

$$\theta_n \neq 0$$

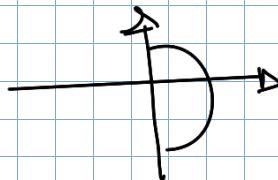
$$\text{Si } f(x) \quad \forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\quad |\sin x| \leq |x| \leq |\tan x|$$

$$\text{Se } \theta_n \rightarrow 0 \Rightarrow D \quad -\frac{\pi}{2} < \theta_n < \frac{\pi}{2} \Rightarrow |\sin \theta_n| \leq |\theta_n| \leq |\tan \theta_n|$$

$$\Rightarrow 1 \leq \left| \frac{\sin \theta_n}{\theta_n} \right| \leq \frac{1}{|\cos \theta_n|}$$

||

$$\frac{\theta_n}{\sin \theta_n} \leq \frac{1}{\cos \theta_n}$$



$$\Rightarrow \cos \theta_n \leq \frac{\sin \theta_n}{\theta_n} \leq 1$$

↓ ↓

$$\boxed{\theta_n \rightarrow 0}$$

$\Rightarrow \frac{\sin \theta_n}{\theta_n} \rightarrow 1$

$$\frac{\sin \theta_n}{\theta_n} \rightarrow 1 \quad \frac{\theta_n}{\sin \theta_n} \rightarrow 1 \quad \frac{\tan \theta_n}{\sin \theta_n} \quad \frac{1}{\cos \theta_n} \quad \frac{\theta_n}{\tan \theta_n} \rightarrow 1$$

↓ ↓

$$\frac{\operatorname{sen} \theta_n}{\theta_n} \rightarrow 1 \quad \frac{\theta_n + \operatorname{sen} \theta_n}{\theta_n} \rightarrow 1$$

$$\frac{1 - \cos \theta_n}{\theta_n} = \frac{(1 - \cos \theta_n)(1 + \cos \theta_n)}{\theta_n(1 + \cos \theta_n)} = \frac{1 - \cos^2 \theta_n}{\theta_n(1 + \cos \theta_n)} = \frac{1}{1 + \cos \theta_n}$$

$$= \frac{\sin \theta_n}{\theta_n} \quad \frac{\sin \theta_n}{1 + \cos \theta_n} \rightarrow 0$$

$$\frac{1 - \cos \theta_n}{\theta_n^2} = \frac{\sin^2 \theta_n}{\theta_n^2} \quad \frac{1}{1 + \cos \theta_n} \rightarrow \frac{1}{2}$$

↓ ↓

Exemplo:

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2n+1}}{\frac{2}{3n}} = \frac{\lim_{n \rightarrow \infty} \frac{1}{2n+1}}{\frac{1}{2n+1}} \left(\frac{1}{2n+1} \cdot \frac{3n}{2} \right)$$

↓
n/3

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n+3}}{\frac{n+1}{n^2+4}} = \frac{\lim_{n \rightarrow \infty} \frac{n-1}{n^2+4}}{\lim_{n \rightarrow \infty} \frac{n+1}{n^2+4}} \left(\frac{n^2+4}{n+1} \cdot \frac{1}{n+3} \right) \rightarrow 1$$

↓
1

$$(2n^2+3) \lim_{n \rightarrow \infty} \frac{n}{n^3+1} = \underbrace{\frac{\lim_{n \rightarrow \infty} \frac{n}{n^3+1}}{\frac{n}{n^3+1}}}_{\downarrow 1} \cdot \left(\frac{n}{n^3+1} (2n^2+3) \right) \rightarrow 2$$

↓
2

$$(3n-1) + \frac{2n}{n^2+4} = \frac{+8 \frac{2n}{n^2+4}}{\frac{2n}{n^2+4}} \left(\frac{2n}{n^2+4} (3n-1) \right) \rightarrow 6$$

↓
1 ↓
6

$$\frac{\lim^2 \frac{2}{n+3}}{+\frac{3}{3n^3+4}} = \frac{\lim^2 \frac{2}{n+3}}{\left(\frac{2}{n+3}\right)^2} - \frac{\frac{3}{3n^3+4}}{+\frac{3}{3n^3+4}} \left(\left(\frac{2}{n+3}\right)^2 - \frac{3n^3+4}{3n^3+4} \right) \rightarrow 12$$

↓
1 ↓
1 ↓
12

$$\frac{1 - \cos \frac{2}{n+1}}{\lim_{n \rightarrow \infty} \frac{n+1}{3n^3+2}} = \frac{1 - \cos \frac{2}{n+1}}{\frac{1}{(n+1)^2}} - \frac{\frac{n+1}{n^3+2}}{\lim_{n \rightarrow \infty} \frac{n+1}{3n^3+2}} \left(\frac{1}{(n+1)^2} - \frac{3n^3+2}{3n^3+2} \right)$$

↓
1 ↓
1 ↓
1 ↓
1