Integrali funzioni razionali fratte

$f(x) = \dfrac{A(x)}{B(x)}$ A polinomio di grado n $(n \in \mathbb{N}_0)$

B " " " m $(m \in \mathbb{N})$

se $n \geq m$ funzione razionale fratta non propria

se $n < m$ " " " propria

svilupperemo sempre le proprie perché:

non proprie $\rightarrow$ polinomio + proprie

infatti $f(x) = B(x)\,Q(x) + R(x)$ grado $R <$ grado B

$$\Rightarrow f(x) = \frac{A(x)}{B(x)} = \underbrace{Q(x)}_{\text{polinomio}} + \underbrace{\frac{R(x)}{B(x)}}_{\text{propria}}$$

es: $$\int \frac{x^2+3}{x+1}\,dx = \int \frac{(x^2+2x+1)-2x+2}{x+1}\,dx = \int \frac{(x+1)^2}{x+1}\,dx - 2\int \frac{x-1}{x+1}\,dx$$

$$= \int (x+1)\,dx - 2\int \frac{x+1-2}{x+1}\,dx = \int (x+1)\,dx - 2\int \frac{x+1}{x+1}\,dx + 4\int \frac{dx}{x+1}$$

$$= \frac{x^2}{2} + x - 2x + 4\log|x+1| + K$$

Occupiamoci intanto di alcuni tipi di funz proprie prima di passare al caso generale

$$I_n = \int \frac{dx}{(x-c)^n}$$

$$I_1 = \int \frac{dx}{x-c} = \log|x-c| + K$$

$n>1$ $$I_n = \int (x-c)^{-n}\,dx \cdot \left[\int t^{-n}\,dt\right]_{t=x-c} = \frac{(x+c)^{-n+1}}{-n+1} + K$$

$$I_n = \int \frac{dx}{(x^2+c^2)} =$$

$$D\left(\frac{x}{c}\right) = \frac{1}{c}$$

$$I_1 = \int \frac{dx}{x^2+c^2} = \int \frac{1}{c^2}\,\frac{dx}{\left(\frac{x}{c}\right)^2+1} = \frac{1}{c}\int \frac{1}{c}\cdot\frac{1}{\left(\frac{x}{c}\right)^2+1}\,dx$$

$$= \frac{1}{c}\left[\int \frac{dt}{t^2+1}\right]_{t=\frac{x}{c}} = \frac{1}{c}\arctan\frac{x}{c} + K$$

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\arctan\frac{x}{2} + K \qquad (c=2)$$

$$\int \frac{dx}{x^2+5} = \frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}} + K \qquad (c=\sqrt{5})$$

$$I_2 = \int \frac{dx}{(x^2+c^2)^2} = \frac{1}{c^2}\int \frac{c^2}{(x^2+c^2)^2}\,dx = \frac{1}{c^2}\int \frac{c^2+x^2-x^2}{(x^2+c^2)^2}\,dx =$$

$$\frac{1}{c^2}\int \frac{-x^2}{(x^2+c^2)^2} = \frac{1}{c^2}I_1 + \frac{1}{2c^2}\int \frac{-2x}{(x^2+c^2)^2}\,x\,dx =$$

$$\frac{1}{c^2}I_1 + \frac{1}{2c^2}\,\frac{x}{x^2+c^2} - \frac{1}{2c^2}\underbrace{\int \frac{dx}{x^2+c^2}}_{= I_1} - 1$$

$$D = \frac{1}{x^2+c^2} = \frac{-2x}{(x^2+c^2)^2}$$

$$\int \frac{dx}{(x^2+9)^2} = \frac{1}{9}\int \frac{9}{(x^2+9)^2}\,dx = \frac{1}{9}\int \frac{9+x^2-x^2}{(x^2+9)^2}\,dx =$$

$$= \frac{1}{9}\int \frac{\cancel{x^2+9}}{(x^2+9)^{\cancel{2}}}\,dx + \frac{1}{18}\int \frac{-2x}{(x^2+9)^2}\,dx$$

$$= \frac{1}{9}\,\frac{1}{3}\arctan\frac{x}{3} + \frac{1}{18}\,\frac{x}{x^2+9} - \frac{1}{18}\int \frac{dx}{x^2+9}$$

$$= \frac{1}{27}\arctan\frac{x}{3} + \frac{1}{18}\,\frac{x}{x^2+9} - \frac{1}{54}\arctan\frac{x}{3} + K$$

Se al denom c'è un polinomio di II grado con $\Delta<0$ si cerca di ottenere una somma del tipo $(x-a)^2+b^2$

(completamento dei quadrati)

$$\int \frac{dx}{x^2+2x+4} = \int \frac{dx}{(x+1)^2+3} = \left[\int \frac{dt}{t^2+3}\right]_{t=x+1}$$

$a=\sqrt{3}$

$$= \frac{1}{\sqrt{3}} \arctan \frac{x+1}{\sqrt{3}} + k$$

$$\int \frac{dx}{x^2+3x+4} = \int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{7}{4}} = \left[\int \frac{dt}{t^2+\frac{3}{4}}\right]_{t=x+\frac{3}{2}}$$

$4-\frac{3}{4}=\frac{7}{4}$

$$= \frac{2}{\sqrt{3}} \arctan \frac{x+\frac{3}{2}}{\frac{\sqrt{7}}{2}} + k$$

$a=\frac{\sqrt{3}}{2}$

$$\int \frac{dx}{2x^2+x+1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}$$

$\frac{1}{2}-\frac{1}{16}=\frac{7}{16}$

$a=\frac{\sqrt{7}}{4}$

$$= \frac{1}{2}\,\frac{4}{\sqrt{7}} \arctan \frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}} + k$$

Caso: $\dfrac{\text{Polinomio di I grado}}{\text{Polinomio di II grado}}$

denom con $\Delta<0$

I passo: ottenere al num la derivata del den

es: $$\int \frac{2x+3}{x^2+x+1}\,dx = \int \frac{2x+1+2}{x^2+x+1}\,dx = \int \frac{2x+1}{x^2+x+1}\,dx + 2\int \frac{dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \log(x^2+x+1) + 2\,\frac{2}{\sqrt{3}} \arctan \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + k$$

$$\int \frac{ax+b}{x^2+px+q}\,dx \qquad (p^2-4q<0) \qquad \text{LOG + ARCTAN}$$

$$\int \frac{x+4}{x^2+3x+5} = \frac{1}{2}\int \frac{2x+8}{x^3+3x+5}\,dx = \frac{1}{2}\int \frac{2x+3+5}{x^2+3x+5}\,dx =$$

$$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}\,dx + \frac{5}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\frac{11}{2}} =$$

$$\frac{1}{2}\log\left(x^3+3x+5\right) + \frac{5}{2}\,\frac{8}{\sqrt{11}}\,\arctan\frac{x+\frac{3}{2}}{\frac{\sqrt{11}}{2}} + K$$

denom > 0 Si decompone la funzione

$$\int \frac{x+1}{x^2-x-6}\,dx \qquad x^2-x-6=0 \text{ per } x=\frac{1\pm 5}{2} < \begin{matrix}-2\\ 3\end{matrix}$$

$$x^2-x-6=(x+2)(x-3)$$

si può fare vedere che $\frac{x+1}{x^2-x-6} = \frac{A}{x+2} + \frac{B}{x-3}$

$$= \frac{Ax+3A+Bx+2B}{(x+2)(x-3)} \qquad \begin{cases} A+B=1 \\ -3A+2B=1 \end{cases} \qquad \begin{cases} B=1-A \\ -3A+2-2A=1 \end{cases} \qquad \begin{matrix} B=\frac{4}{3} \\ A=\frac{1}{5} \end{matrix}$$

$$I = \int \frac{\frac{1}{5}}{x+2}\,dx + \int \frac{\frac{4}{5}}{x-3}\,dx = \frac{1}{5}\log|x+2| + \frac{4}{5}\log|x-3| + K$$

$$\int \frac{3x+1}{x^2+4x}\,dx = \int \frac{3x+1}{x(x+4)}\,dx$$

$$\frac{3x+1}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4} = \frac{Ax+4A+Bx}{x(x+4)}$$

$$\begin{cases} A+B=3 \\ 4A=1 \end{cases} \qquad \begin{cases} B=\frac{11}{4} \\ A=\frac{1}{4} \end{cases}$$

$$I = \frac{1}{4}\int dt + \frac{11}{4}\int \frac{dx}{x+4} = \frac{1}{4}\log(x) + \frac{11}{4}\log|x+4| + k$$

Caso denom $= 0$

$$\int \frac{x+3}{x^2+2x+1}\,dx \qquad \text{si cercano } A, B : \frac{x+3}{x^2+2x+1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} =$$

$$= \frac{ax + A + B}{(x+1)^2} \qquad \begin{cases} A = 1 \\ A + B = 3 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 2 \end{cases}$$

$$I = \int \frac{dx}{x+1} + \int \frac{2}{(x+1)^2}\,dx = \log|x+1| - \frac{1}{x+1} + k$$

$$I = \int \frac{4x+3}{x^2-4x+4}\,dx$$

$$\frac{4x+3}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax - 2A + B}{(x-2)^2} \qquad \begin{cases} A = 4 \\ -2A + B = 9 \end{cases} \Rightarrow \begin{cases} A = 4 \\ B = 17 \end{cases}$$

$$I = 4\log(x-2) - 12\,\frac{1}{x-2} + k$$

Ogni funzione fratta propria si decompone nella somma di fratti semplici (tante quante sono le soluzioni dell'equazione del denom $=0$) ogni valore dell'eq denom $=0$ dà luogo a tanti fratti semplici quanto è la sua molteplicità

nel reale c di molteplicità r $\rightarrow$ $\dfrac{A_1}{(x-c)} \dfrac{A_2}{(x-c)^2} \cdots \dfrac{A_r}{(x-c)^r}$

se $a + ib$ è una sol complessa di molt r anche $a - ib$ lo è

$$[x(a+ib)][x-(a-ib)] = [(x-a)-ib][(x-a)+ib]$$

$$= (x-a)^2 - (ib)^2 = (x-a)^2 + b^2$$

la coppia $a \pm ib$ dà luogo ai seguenti fratti semplici

$$\frac{B_1 x + C_1}{(x-a)^2+b^2} \qquad \frac{B_2 x + C_2}{((x-a)^2+b^2)^2} \quad \dots \quad \frac{B_r x + C_r}{((x-a)^2+b^2)^r}$$

$\int \frac{B_1 x + C_1}{(x-a)^2+b^2}\, dx$ lo sappiamo fare

$\int \frac{B_2 x + C_2}{((x-a)^2+b^2)^2}\, dx$ come si fa?

$$\int \frac{2x+1}{(x-3)^2+4}\, dx = \int \frac{2x-6+7}{(x-3)^2+4}\, dx = \int \frac{2x-6}{(x-3)^2+4}\, dx + 7\int \frac{dx}{(x-3)^2+4} =$$

$$\log\left((x-3)^2+4\right) + \frac{7}{2} \arctan \frac{x-3}{2} + k$$

$$\int \frac{x+1}{x^3(x^2+1)(x-2)}\, dx \qquad \frac{x+1}{x^3(x^2+1)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x} + \frac{A_3}{x} + \frac{Bx+C}{x+2} + \frac{D}{x-2} =$$

$$= \frac{A_1 x^2(x^2+1)(x-2) + A_2 x (x^2+1)(x-2) + A_3 x^2(x+1)(x-2) + B(x+C)x^3(x-2) + D x^2(x^2+1)}{x^3(x^2+2)(x-2}$$

PAURA

$$I = \int \frac{x+3}{x^3+x^2}\,dx = \int \frac{x+3}{x^2(x+1)}\,dx$$

$$\frac{x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax^2+Ax+Bx+B+Cx^2}{x^2(x+1)}$$

$$\begin{cases} A+C = 0 \\ A+B = 1 \\ B = 3 \end{cases} \Rightarrow \begin{cases} C = 2 \\ A = -2 \\ B = 3 \end{cases}$$

$$I = -2\log|x| - \frac{3}{x} + 2\log|x+1| + k$$