$f: (a,b) \to \mathbb{R} \quad c \in (a,b) \qquad r(x) = \dfrac{f(x)-f(c)}{x-c}$

$R(h) = \dfrac{f(c+h)-f(c)}{h}$

$f'(c) = \lim_{x \to c} r(x) = \lim_{h \to 0} R(h)$

$\lim_{x \to c} \dfrac{f(x)-p(x)}{x-c} = 0 \qquad p(x) = f(c) + f'(c)(x-c)$

$y = f(c) + f'(c)(x-c)$ eq. tang.

## Regole di derivazione

1) $f: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$ $\quad k \in \mathbb{R}$

$g(x) = k f(x)$ si ha: $\exists\, g'(c) = k f'(c)$

$$\frac{kf(x) - kf(c)}{x-c} = k \frac{f(x)-f(c)}{x-c} \to k f'(c)$$

2) $f, g: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$

$s(x) = f(x) + g(x)$ si ha $s'(c) = f'(c) + g'(c)$

$$\frac{s(x)-s(c)}{x-c} = \frac{f(x)+g(x)-f(c)-g(c)}{x-c} = \underbrace{\frac{f(x)-f(c)}{x-c}}_{\downarrow\, f'(c)} + \underbrace{\frac{g(x)-g(c)}{x-c}}_{\downarrow\, g'(c)}$$

3) $f, g: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$

$p(x) = f(x)g(x)$ si ha $p'(c) = f'(c)g(c) + f(c)g'(c)$
(NO DIM)

4) $f: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$ $\quad f(c) \neq 0$

($f$ der $\Rightarrow$ $f$ cont in $c$, $f(c) \neq 0 \Rightarrow f(x) \neq 0$ in un intorno di $c$)

$F(x) = \dfrac{1}{f(x)}$ (esiste in un intorno di $c$)

Si ha: $\exists\, F'(c) = -\dfrac{f'(c)}{(f(c))^2}$ (NO DIM.)

5) $f, g: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$ $\quad g(c) \neq 0$

$q(x) = \dfrac{f(x)}{g(x)}$ si ha $q'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$

infatti $q(x) = f(x)\dfrac{1}{g(x)} \Rightarrow q'(c) = f'(c)\dfrac{1}{g(c)} + f(c)\dfrac{-g'(c)}{(g(c))^2} =$

$$= \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$$

Teorema di derivazione delle funzioni composte

IP $f: (\alpha,\beta) \to \mathbb{R} \quad g: (a,b) \to (\alpha,\beta) \quad c \in (a,b) \quad F(x) = f(g(x))$

$\exists\, g'(c)$, $\exists\, f'(g(c))$

TS $\exists\, F'(c) = f'(g(c))\, g'(c)$ NO DIM.

Teorema di derivazione delle funzioni inverse

IP $f: (a,b) \to (\alpha,\beta)$ strett. cresc. (o decr.) e continua

$f^{-1}: (\alpha,\beta) \to (a,b)$ sappiamo che è strett cresc (o decr) e continua

Sia $\gamma \in (\alpha,\beta)$ $\gamma = f(c)$ , $c \in (a,b)$

$\exists f'(c) \neq 0$

TS $\exists (f^{-1})'(\gamma) = \frac{1}{f'(c)}$ NO DIM.

materialmente per trovare $(f^{-1})'(\gamma)$ si deve risolvere l'eq. $f(x) = \gamma$ per trovare c , calcolare $f'(c)$ , se è $\neq 0$ si conclude

Derivate delle funz. elementari

$f(x) = k \quad \forall x \in \mathbb{R} \Rightarrow f'(x) = 0 \quad \forall x \in \mathbb{R}$

$f(x) = x \quad '' \Rightarrow f'(x) = 1 \quad ''$

$f(x) = x^2 \qquad r(x) = \frac{x^2 - c^2}{x-c} = \frac{\cancel{(x-c)}(x+c)}{\cancel{x-c}} \to 2c \Rightarrow f'(x) = 2x \quad \forall x \in \mathbb{R}$

$f(x) = x^3 \qquad r(x) = \frac{x^3 - c^3}{x-c} = \frac{\cancel{(x-c)}(x^2+xc+c^2)}{\cancel{x-c}} \to 3c^2 \Rightarrow f'(x) = 3x^2 \quad \forall x \in \mathbb{R}$

...

$f(x) = x^n \qquad f'(x) = n x^{n-1} \quad \forall x \in \mathbb{R}$

$$\frac{x^n - c^n}{x-c} = \frac{(x-c)}{x-c}\underbrace{(x^{n-1} + x^{n-2}c + \dots + xc^{n-2} + c^{n-1})}_{n} \to nc^{n-1}$$

$f(x) = x^{-n} = \frac{1}{x^n} \qquad f'(x) = \frac{-n\,x^{n-1}}{x^{2n}} = -n\,x^{n-1-2n} = -n\,x^{-n-1}$

$(x \neq 0)$

$f(x) = e^x \qquad r(x) = \frac{e^x - e^c}{x-c} = e^c \, \frac{e^{x-c} - 1}{x-c} \to e^c \qquad \lim_{t\to 0} \frac{e^t - 1}{t} = 1$

$\downarrow$ 1

$f'(x) = e^x \quad \forall x \in \mathbb{R}$

$\lim_{t\to 0} \frac{a^t - 1}{t} = \log a$

$f(x) = a^x \quad (a>0, a \neq 1) \qquad f'(x) = a^x \log a$

$f(x) = \log x \quad c>0 \qquad r(x) = \frac{\log x - \log c}{x-c} = \frac{\log\left(\frac{x}{c} - 1 + 1\right)}{c\left(\frac{x}{c} - 1\right)} = \frac{1}{c} \qquad \lim_{t\to 0} \frac{\log(t+1)}{t} = 1$

$\to 1$, $\frac{x}{c}-1 \to 0$

$f'(x) = \frac{1}{x} \quad \forall x>0$

possiamo vederla anche come funzione inversa

$f(x) = e^x : ]-\infty, +\infty[ \to ]0, +\infty[$

$f^{-1}(y) = \log y : ]0, +\infty[ \to ]-\infty, +\infty[$

$\gamma \in ]0, +\infty[$ cerchiamo c: $f(c) = \gamma \qquad e^c = \gamma \Rightarrow c = \log \gamma$

$f'(c) = \log\gamma \neq 0 \qquad (f^{-1})'(\gamma) = \frac{1}{\log\gamma} = \frac{1}{c}$

Se $a \neq e \qquad f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x} \log_a e$

$f(x) = |x| \qquad \nexists f'(0) \quad$ ma $f'(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases} = \frac{|x|}{x}$ ("segno di x")

$-1 \quad x<0$

Sia $f$ deriv. e $\neq 0$ e cons $f(x) = |f(x)|$ $\qquad f'(x) = \dfrac{|f(x)|}{f(x)} f'(x)$

Cons ora $f(x) = \log|f(x)|$ $\qquad f'(x) = \dfrac{1}{\cancel{|f(x)|}} \dfrac{\cancel{|f(x)|}}{f(x)} f'(x) = \dfrac{f'(x)}{f(x)}$

es. $F(x) = \log|x^3 - 3x|$ in tutti i p. in cui $x^3 - 3x \neq 0$ si ha

$$f'(x) = \frac{3x^2 - 3}{x^3 - 3x}$$

$f(x) = x^\alpha$ ($\alpha$ non intero) è def. in $[0, +\infty[$ se $\alpha > 0$
in $]0, +\infty[$ se $\alpha < 0$

$c > 0$ $\qquad x^\alpha = e^{\log x^\alpha} = e^{\alpha \log x}$ $\qquad t > 0 \quad t = e^{\log t}$

$$f'(x) = e^{\alpha \log x} \alpha \frac{1}{x} = x^\alpha \alpha \frac{1}{x} = \alpha x^{\alpha - 1}$$

es. $\alpha = \frac{1}{2}$ $\quad f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2} x^{\frac{1}{2} - 1} = \dfrac{1}{2\sqrt{x}}$ $\quad x > 0$

se $0 < \alpha < 1$ $\quad \nexists f'(0)$ $\quad \dfrac{x^\alpha - c^\alpha}{x - c}$ diverge per $x \to c$

invece di $f'(c)$ si può scrivere $(D f(x))_{x=c}$ $\quad$ es $D(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

cons $f(x) = x^n$ $\qquad n \in \mathbb{N}$

$f'(x) = n x^{n-1}$

$f''(x) = n(n-1) x^{n-2}$

$f'''(x) = n(n-1)(n-2) x^{n-3}$

...

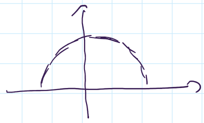
$f^{(n)}(x) = n(n-1)(n-2) \dots 1 \cdot x^0 = n!$

$f^{(k)}(x) = 0 \quad \forall k > n$

$f(x) = \cos x$ $\qquad R(h) = \dfrac{\cos(c+h) - \cos c}{h} = \dfrac{\cos c \cos h - \sin c \sin h - \cos c}{h} =$

$= \cos c \underbrace{\dfrac{\cos h - 1}{h}}_{\to 0} - \sin c \underbrace{\dfrac{\sin h}{h}}_{\to 1} \to -\sin c$ $\qquad f'(x) = -\sin x \ \forall x$

$f(x) = \cos x : [0, \pi] \to [-1, 1]$



$f^{-1}(x) = \arccos x : [-1, 1] \to [0, \pi]$

$y \in [-1, 1]$ $\quad y = \cos c \Downarrow c = \arccos y$ $\quad c \in [0, \pi]$ $\quad f'(c) = -\sin c \neq 0$ per $c \in ]0, \pi[ \Downarrow y \in ]-1, 1[$

$(f^{-1})'(y) = \dfrac{1}{-\sin c} = -\dfrac{1}{\sin c} = -\dfrac{1}{\sqrt{1 - \cos^2 c}} =$ $\qquad$ perché $\sin^2 c = 1 - \cos^2 c$ e $\sin c > 0$

$= -\dfrac{1}{\sqrt{1 - \cos^2(\arccos y)}} = -\dfrac{1}{\sqrt{1 - y^2}}$ $\quad \forall y \in ]-1, 1[$ $\qquad$ proveremo che $\nexists$ la deriv. in $-1$ e $1$

$$f(x) = \sin x \qquad R(h) = \frac{\sin(c+h) - \sin c}{h} = \frac{\sin c \cos h + \cos c \sin h - \sin c}{h} =$$

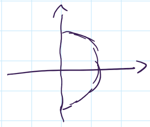
$$= \sin c \underbrace{\frac{\cos h - 1}{h}}_{\downarrow\, 0} + \cos c \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} \to \cos c \qquad f'(x) = \cos x \quad \forall x$$

$$f(x) = \sin x : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \to [-1, 1]$$

$$f^{-1}(x) = \arcsin x : [-1, 1] \to \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \qquad y \in [-1, 1] \qquad c \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] : \sin c = y$$

$$c = \arcsin y$$

$$f'(c) = \cos c \neq 0 \quad \text{se} \quad c \neq \pm \tfrac{\pi}{2} \Leftrightarrow y \neq \pm 1$$



$$(f^{-1})'(y) = \frac{1}{\cos c} = \frac{1}{\sqrt{1 - \sin^2 c}} = \frac{1}{\sqrt{1 - \sin^2(\arcsin y)}} = \frac{1}{\sqrt{1 - y^2}}$$

non esiste la der. in 1 e −1

$$f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x} \qquad x \neq \tfrac{\pi}{2} + k\pi$$

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \operatorname{tg}^2 x$$

$$f(x) = \operatorname{tg} x : \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[ \to \left]-\infty, +\infty\right[$$

$$f^{-1}(y) = \operatorname{arctg} y : \left]-\infty, +\infty\right[ \to \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[$$

$$y \in \left]-\infty, +\infty\right[ \qquad y = \operatorname{tg} c \qquad c \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[ \qquad f'(c) = 1 + \operatorname{tg}^2 c$$

$$\Downarrow$$

$$c = \operatorname{arctg} y$$

$$(f^{-1})'(y) = \frac{1}{1 + \operatorname{tg}^2 c} = \frac{1}{1 + \operatorname{tg}^2(\operatorname{arctg} y)} = \frac{1}{1 + y^2}$$

Esercizi

$$f(x) = 3x^4 - \cos x + e^x \qquad f'(x) = 12x^3 + \sin x + e^x$$

$$f(x) = (\sin x)\sqrt{x} \qquad f'(x) = (\cos x)\sqrt{x} + (\sin x)\frac{1}{2\sqrt{x}}$$

$$f(x) = \frac{\operatorname{tg} x}{x^6} \qquad f'(x) = \frac{(1 + \operatorname{tg}^2 x)x^6 - \operatorname{tg} x\,(6x^5)}{x^{12}}$$

$$f(x) = \frac{x \arcsin x}{\arccos x} \qquad f'(x) = \frac{\left(\arcsin x + \frac{x}{\sqrt{1-x^2}}\right)\arccos x + \frac{1}{\sqrt{1-x^2}}\, x \arcsin x}{\arccos^2 x}$$

$$f(x) = \sin(4x^2 - 1) \qquad f'(x) = \cos(4x^2 - 1) \cdot 8x$$

$$f(x) = e^{\sqrt{x}} \qquad f'(x) = e^{\sqrt{x}}\,\frac{1}{2\sqrt{x}}$$

$$f(x) = \operatorname{arctg} \frac{2x}{x^2 - 1} \qquad f'(x) = \frac{1}{1 + \frac{4x^2}{(x^2-1)^2}} \cdot \frac{2(x^2 - 1) - 2x \cdot 2x}{(x^2 - 1)^2}$$

$$f(x) = \sqrt[3]{x - \sqrt{\sin x}} \qquad f'(x) = \frac{1}{3}\left(x - \sqrt{\sin x}\right)^{\frac{1}{3} - 1}\left(1 - \frac{\cos x}{2\sqrt{\sin x}}\right)$$

$$D\left(\frac{1}{f}\right) = -\frac{f'}{f^2}$$

$$f(x) = \frac{1}{\operatorname{arctg} \frac{2x}{\sqrt{x+1}}} \qquad f'(x) = -\frac{\frac{1}{1 + \left(\frac{2x}{\sqrt{x+1}}\right)^2} \cdot \frac{2\sqrt{x+1} - 2x\,\frac{1}{2\sqrt{x+1}}}{x+1}}{\operatorname{arctg}^2 \frac{2x}{\sqrt{x+1}}}$$

$$f(x) = e^{\operatorname{arctg} x} \qquad f'(x) = e^{\operatorname{arctg} x} \frac{1}{1+x^2}$$

$$f(x) = e^{\sin \frac{1}{x}} \qquad f'(x) = e^{\sin \frac{1}{x}} \left(\cos \frac{1}{x}\right)\left(\frac{-1}{x^2}\right) \qquad x^{-1}$$

$$f(x) = \cos \sqrt{x^2 - \frac{1}{x}} \qquad f'(x) = -\sin \sqrt{x^2 - \frac{1}{x}} \; \frac{1}{2\sqrt{x^2 - \frac{1}{x}}} \left(2x + \frac{1}{x^2}\right)$$

$$f(x) = \operatorname{arctg} \sqrt{\frac{1+x}{1-x}} \qquad f'(x) = \frac{1}{1 + \frac{1+x}{1-x}} \; \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \; \frac{1\cdot(1-x) - (-1)(1+x)}{(1-x)^2}$$

$$f(x) = \log 2x \qquad f'(x) = \frac{\cancel{2}}{2x}$$

$$f(x) = \log \sqrt{\frac{x}{x-1}} \qquad f'(x) = \sqrt{\frac{x-1}{x}} \; \frac{1}{2\sqrt{\frac{x}{x-1}}} \; \frac{1\cdot(x-1) - x\cdot 1}{x^2+1-2x}$$

$$D\left(\log|f|\right) = \frac{f'}{f}$$

$$f(x) = \log|x^4 + 3x| \qquad f'(x) = \frac{4x^3+3}{x^4+3x}$$

$$f(x) = \log|\sqrt[4]{x} - x^3| \qquad f'(x) = \frac{\frac{1}{4}\frac{1}{\sqrt[4]{x^3}} - 3x^2}{\sqrt[4]{x} - x^3}$$

$$f(x) = \frac{\log \frac{3x}{x-1}}{\arcsin x^2} \qquad f'(x) = \frac{\frac{x-1}{3x}\frac{3(x-1)-3x}{(x-1)^2} \arcsin x^2 - \log\frac{3x}{x-1} \frac{1}{\sqrt{1-x^4}} 2x}{\arcsin^2 x^2}$$

$$f(x) = \sin(\sin x) \qquad f'(x) = \left(\cos(\sin x)\right)\cos x$$

$$f(x) = \cos(\cos(\cos x)) \qquad f'(x) = -\sin(\cos(\cos x))\left(-\sin(\cos x)\right)\left(-\sin x\right)$$

$$f(x) = x^x = e^{\log x^x} = e^{x \log x} \qquad f'(x) = e^{x\log x}\left[1\cdot \log x + x\,\frac{1}{x}\right] = x^x(\log x + 1)$$

$$f(x) = (\sin x)^{\sin x} = e^{\sin x(\log(\sin x))} \qquad f'(x) = (\sin x)^{\sin x}\left[\cos x \log(\sin x) + \sin x\,\frac{\cos x}{\sin x}\right]$$

$$f(x) = \log\frac{1}{\operatorname{tg} x} \qquad f(x) = \sqrt{\cos^3 x - \sin x} \qquad f(x) = e^{\frac{1}{\operatorname{arctg} x}}$$

ESERC. PROPOSTI

$$f(x) = \log|\sqrt[3]{x^2 - x}| \qquad f(x) = \operatorname{tg}(\log(\cos x)) \qquad f(x) = \sqrt{x^2-1}\,\sqrt[3]{3x^4-2}$$

Applicazioni del calcolo differenziale allo studio delle funzioni

Criterio di monotonia locale.

Se $f'(c) > 0$ allora $f$ è cresc. in $c$

infatti $f'(c) > 0 \Rightarrow r(x) > 0$ in un intorno di $c \Rightarrow f$ cresc. in $c$ (v. cap. 1)

Criterio di stretta monotonia.

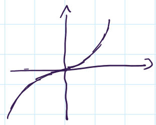
Se $f'(x) > 0 \ \forall x \in (a,b) \Rightarrow f$ è strett. cresc in $(a,b)$

infatti è cresc. in ogni punto per il teor. prec.

Il viceversa non vale, nemmeno nel criterio di monot. locale

$f$ cresc. in $c \Rightarrow f'(c) \geq 0$ (se fosse $< 0$ $f$ sarebbe decresc.)

Cons $f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad f'(0) = 0$ ma $f$ è cresc. strett.

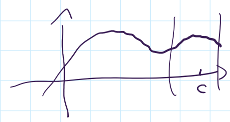


Ci serve allora un criterio di stretta monotonia in cui $f'$ possa anche essere zero. Per arrivarci dobbiamo studiare i teoremi fondamentali sul calcolo differenziale

TEOREMA DI FERMAT

IP $f : (a,b) \to \mathbb{R}$ $\quad c \in \,]a,b[\,$ p. di estr. rel.
$\exists f'(c)$

TS $f'(c) = 0$

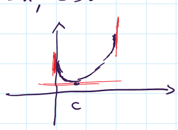


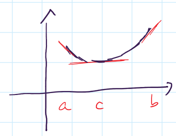

Ricordiamo che $c \in (a,b)$ è p. di max rel se $\exists I(c)$ : $f(x) \leq f(c)$ $\forall x \in I(c)$

OSSERV.

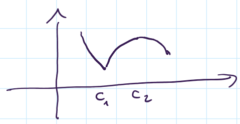
1) non vale il viceversa es. $f(x) = x^3$, $c = 0$

2) non vale se $c = a$ opp. $c = b$




3) cercheremo i punti di estr. rel. fra i punti interni in cui la derivata è zero (PUNTI STAZIONARI) e quelli in cui non c'è derivata




$c_1$, $c_2$ p. di estr rel
$c_2$ stazionario