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INTEGRAZIONE PER RAZIONALIZZAZIONE

1. $\displaystyle\int \frac{e^x}{e^x+3}\,dx = \left[\int \frac{dt}{t+3}\right]_{t=e^x} = \log(e^x+3) + c$

perchè $e^x = D(e^x)$

2. $\displaystyle I = \int \frac{e^x+2}{e^x+1}\,dx = \int \underset{\substack{\uparrow \\ D(e^x)}}{e^x} \frac{e^x+2}{e^x(e^x+1)}\,dx = \Big[\underbrace{\int \frac{t+2}{t(t+1)}\,dt}_{J}\Big]_{t=e^x}$

$$\frac{t+2}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{(A+B)t+A}{t(t+1)} \qquad \begin{cases} A+B=1 \\ A=2 \end{cases} \Rightarrow \begin{cases} B=-1 \\ A=2 \end{cases}$$

$J = 2\log|t| - \log|t+1| + c \Rightarrow I = 2x - \log(e^x+1) + c$

$D(\operatorname{tg} x) = \operatorname{tg}^2 x + 1$

3. $\displaystyle I = \int \frac{\operatorname{tg} x+2}{\operatorname{tg}^2 x + \operatorname{tg} x - 2}\,dx =$

$$= \int \underset{\substack{\uparrow \\ D(\operatorname{tg} x)}}{(\operatorname{tg}^2 x+1)} \frac{\operatorname{tg} x+2}{(\operatorname{tg}^2 x+1)(\operatorname{tg}^2 x+\operatorname{tg} x-2)}\,dx = \Big[\underbrace{\int \frac{t+2}{(t^2+1)(t^2+t-2)}\,dt}_{J}\Big]_{t=\operatorname{tg} x}$$

$t^2+t-2=0 \qquad t = \frac{-1\pm 3}{2} \begin{matrix} \nearrow -2 \\ \searrow 1 \end{matrix} \qquad \frac{t+2}{(t^2+1)(t^2+t-2)} = \frac{At+B}{t^2+1} + \frac{C}{t+2} + \frac{D}{t-1}$

$= \frac{At^3+At^2-2At+Bt^2+Bt-2B+Ct^3+Ct-Ct^2-C+Dt^3+Dt+2Dt^2+2D}{\dots\dots}$

$$\begin{cases} A + C + D = 0 \\ A + B - C + 2D = 0 \\ -2A + B + C + D = 1 \\ -2B - C + 2D = 2 \end{cases} \Rightarrow \begin{cases} A = -C-D \\ B - 2C + D = 0 \\ B + 3C + 3D = 1 \\ 2B + C - 2D = -2 \end{cases} \Rightarrow \begin{cases} A = -C-D \\ B = 2C - D \\ 5C + 2D = 1 \\ 5C - 4D = -2 \end{cases}$$

$1 - 2D = -2 + 4D \Rightarrow D = \frac{1}{2}$

$5C = 1 - 2D \Rightarrow C = 0$

$B = -\frac{1}{2}$

$A = -\frac{1}{2}$

$$J = \int \frac{-\frac{1}{2}t - \frac{1}{2}}{t^2+1}\,dt + \int \frac{\frac{1}{2}}{t-1}\,dt = -\frac{1}{2}\int \frac{t+1}{t^2+1}\,dt + \frac{1}{2}\log|t-1| =$$

$$= -\frac{1}{4}\log(t^2+1) - \frac{1}{2}\operatorname{arctg} t + \frac{1}{2}\log|t-1| + c$$

$I = -\frac{1}{4}\log(\operatorname{tg}^2 x+1) - \frac{1}{2}x + \frac{1}{2}\log|\operatorname{tg} x - 1| + c$

Non è razionalizzazione ma sostituzione se abbiamo.

$$\int \frac{\sin 2x}{\sin^2 x+3}\,dx = 2\int \frac{\sin x}{\sin^2 x+3}\cos x\,dx = 2\left[\int \frac{t}{t^2+3}\,dt\right]_{t=\sin x}$$

$\log(\sin^2 x+3) + c$

$$I = \int \frac{\log x + 4}{x(\log^2 x+2)}\,dx = \Big[\underbrace{\int \frac{t+4}{t^2+2}\,dt}_{J}\Big]_{t=\log x}$$

$$J = \int \frac{t}{t^2+2}\,dt + 4\underbrace{\int \frac{dt}{t^2+2}}_{c=\sqrt{2}} = \frac{1}{2}\log(t^2+2) + \frac{4}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}} + c$$

$I = [J]_{\dots}$

$\int \;\; t^2+2 \qquad \int \;\; t^2+2 \qquad 2 \qquad \sqrt{2} \qquad \sqrt{2}$

$c=\sqrt{2}$

$$I = [J]_{t=\log x}$$

Seconda formula di integrazione per sostituzione

IP $\quad f:(a,b)\to\mathbb{R}$ dotata di prim.

$\quad g:(c,d)\to(a,b)$ suriettiva, derivabile, invertibile

TS $\quad \int f(x)\,dx = \left[\int f(g(t))\,g'(t)\,dt\right]_{t=g^{-1}(x)}$

DIM. dalla I formula si ha

$$\int f(g(t))\,g'(t)\,dt = \left[\int f(x)\,dx\right]_{x=g(t)}$$

compongo ambo i membri con $t=g^{-1}(x)$

$$\left[\int f(g(t))\,g'(t)\,dt\right]_{t=g^{-1}(x)} = \left[\int f(x)\,dx\right]_{x=g(g^{-1}(x))=x}$$

Questa formula si utilizza quando ci sono radici

con sostituzioni $\sqrt{ax+b}=t \qquad t\geq 0$

$\qquad ax+b=t^2 \Rightarrow x=\frac{t^2-b}{a}=g(t)\dots$

$\qquad \sqrt{\frac{ax+b}{cx+d}}=t \qquad t\geq 0$

$\qquad x=\dots$

ESEMPI

1. $I=\int \frac{x+\sqrt{x-2}}{x+1}\,dx$

$x\neq -1$
$x\geq 2 \qquad (a,b)=[2,+\infty[$

pongo $\sqrt{x-2}=t \qquad t\geq 0$

$(c,d)=[0,+\infty[\;?$

ricavo x

$x-2=t^2 \Rightarrow x=t^2+2=g(t)$

$t\geq 0 \Rightarrow t^2+2\geq 2\;?\;$ SI

$\Rightarrow (c,d)=[0,+\infty[$

$g'(t)=2t \;\begin{matrix}\geq 0\\ =0 \Leftrightarrow t=0\end{matrix} \Rightarrow g$ INV.

$g^{-1}(x)=\sqrt{x-2}$

$$I=\left[\int \frac{t^2+2+t}{t^2+3}\,2t\,dt\right]_{t=\sqrt{x-2}} = \left[2\underbrace{\int \frac{t^3+t^2+2t}{t^2+3}\,dt}_{J}\right]_{t=\sqrt{x-2}}$$

$$\begin{array}{rr|l} t^3+t^2+2t & & t^2+3 \\ -t^3 \quad\;\; -3t & & t+1 \\ \hline t^2-t & & \\ -t^2 \quad -3 & & \\ \hline -t-3 & & \end{array}$$

$$J=\int\left(t+1-\frac{t+3}{t^2+3}\right)dt =$$

$$=\frac{1}{2}t^2+t-\frac{1}{2}\log(t^2+3)+\sqrt{3}\arctan\frac{t}{\sqrt{3}}+k$$

$$I=\left(\frac{1}{2}(x-2)+\sqrt{x-2}-\frac{1}{2}\log(x+1)+\sqrt{3}\arctan\frac{\sqrt{x-2}}{\sqrt{3}}\right)\cdot 2+k$$

2. $I=\int \frac{x+4}{x^2+\sqrt{x-1}}\,dx$

$(a,b)=[1,+\infty[$

$\sqrt{x-1}=t \qquad t\geq 0$

$x-1=t^2 \Rightarrow x=t^2+1=g(t)$

$t\geq 0 \Rightarrow t^2+1\geq 1\;?\;$ SI $\Rightarrow (c,d)=[0,+\infty[$

$g'(t)=2t \;\begin{matrix}\geq 0\\ =0 \Leftrightarrow t=0\end{matrix} \Rightarrow g$ invert

$g^{-1}(x)=\sqrt{x-1}$

$$I=\left[\int\frac{t^2+1+4}{(t^2+1)^2+t}\,2t\,dt\right]_{t=\sqrt{x-1}}$$

$$I = \left[\int \frac{t^2+1+4}{(t^2+1)^2+t} 2t \, dt \right]_{t=\sqrt{x-1}}$$

3. $I = \int \frac{x+\sqrt{x-1}}{x-4}$ $\quad x \geq 1$, $x \neq 4$

1° caso $(a,b) = [1,4[$ $\qquad \sqrt{x-1} = t \quad t \geq 0$

2° " $(a,b) =]4,+\infty[$ $\qquad x-1 = t^2 \Rightarrow x = t^2+1 = g(t)$

1° caso $t \geq 0 \Rightarrow 1 \leq t^2+1 < 4$? $\qquad t^2+1<4$ se $t^2<3$

allora $(c,d) = [0,\sqrt{3}[$

2° caso $t \geq 0 \Rightarrow t^2+1 > 4$? $\qquad t^2+1>4$ se $t^2>3$

allora $(c,d) =]\sqrt{3},+\infty[$

$g'(t) = 2t$ $\quad$ 1° caso $2t \geq 0$, $=0 \Leftrightarrow t=0$ $\Rightarrow g$ INV

2° caso $2t > 0$ $\Rightarrow g$ INV.

$g^{-1}(x) = \sqrt{x-1}$

$$I = \left[\int \frac{t^2+1+t}{t^2+1-4} 2t \, dt \right]_{t=\sqrt{x-1}} = \left[\underbrace{2 \int \frac{t^3+t^2+t}{t^2-3} dt}_{J} \right]_{t=\sqrt{x-1}}$$

$$\begin{array}{l|l} t^3+t^2+t & t^2-3 \\ -t^3 \quad +3t & t+1 \\ \hline t^2+4t & \\ -t^2 \quad +3 & \\ \hline 7t & \end{array}$$

$$J = 2\int \left(t+1+\frac{7t}{t^2-3} \right) dt =$$

$$= t^2+2t+7 \log|t^2-3| + k$$

$$I = x-1+2\sqrt{x-1} + 7 \log|x-4| + k$$

nel caso 1 sarà $\log(4-x)$

" 2 " $\log(x-4)$

4. $\int \sqrt{\frac{x-3}{x+2}} \, dx$ $\qquad$ 1° caso $(a,b) =]-\infty,-2[$

2° - $(a,b) = [3,+\infty[$

pongo $\sqrt{\frac{x-3}{x+2}} = t \qquad t \geq 0$

ricavo x $\qquad \frac{x-3}{x+2} = t^2 \Rightarrow x-3 = t^2x+2t^2 \Rightarrow x = \frac{2t^2+3}{1-t^2} = g(t)$

1° caso $\frac{2t^2+3}{1-t^2} < -2 \Rightarrow \frac{2t^2+3}{1-t^2}+2<0 \Rightarrow \frac{\cancel{2t^2}+3+2-\cancel{2t^2}}{1-t^2} < 0$

$\Rightarrow t^2-1>0 \Rightarrow (c,d) =]1,+\infty[$

2° caso $\frac{2t^2+3}{1-t^2} \geq 3 \Rightarrow \frac{2t^2+3}{1-t^2}-3 \geq 0 \Rightarrow \frac{2t^2+3-3+3t^2}{1-t^2} \geq 0$

$\Rightarrow 1-t^2 \geq 0 \Rightarrow (c,d) = [0,1[$

$g(t) = \frac{2t^2+3}{1-t^2}$ $\qquad g'(t) = \frac{4t(1-t^2)+2t(2t^2+3)}{(t^2-1)^2} =$

$$= \frac{4t-4t^3+4t^3+6t}{(t^2-1)^2} = \frac{10t}{(t^2-1)^2} > 0$$

$$I = \left[\int t \frac{10t}{(t^2-1)^2} dt \right]_{t=\sqrt{\cdots}} = 10 \left[\underbrace{\int \frac{t^2}{(t^2-1)^2} dt}_{J} \right]_{t=\sqrt{\frac{x-3}{x+2}}}$$

$$\frac{t^2}{(t^2-1)^2} = \frac{t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

ESERCIZI DI RIEPILOGO

1. Trovare f prim in $]-\infty,+\infty[$ di $f(x)=\log(|x-1|+4)$ tale che $F(e^2-3)=3$

$$f(x)=\begin{cases}\log(5-x) & x<1\\ \log(x+3) & x\geq 1\end{cases}$$

$$\int \log(5-x)\,dx = x\log(5-x)-\int\frac{x-5+5}{x-5}dx=$$

$$= x\log(5-x)-x-5\log|x-5|+k$$

$$\int\log(x+3)\,dx = x\log(x+3)-\int\frac{x+3-3}{x+3}dx=$$
$$= x\log(x+3)-x+3\log|x+3|+k$$

$$F(x)=\begin{cases}(x-5)\log(5-x)-x+k_1 & x<1\\ (x+3)\log(x+3)-x+k_2 & x\geq 1\end{cases}$$

$$\lim_{x\to 1^-}F(x)=\lim_{x\to 1^+}F(x) \qquad -4\log 4-1+k_1=4\log 4-1+k_2$$
$$k_1=k_2+8\log 4$$

$$F(x)=\begin{cases}(x-5)\log(5-x)-x+k+8\log 4 & x<1\\ (x+3)\log(x+3)-x+k & x\geq 1\end{cases} \leftarrow$$

$e^2-3>1$ se $e^2>4$ vero

$$F(e^2-3)=e^2\log e^2-e^2+3+k=3 \Rightarrow k=-e^2$$
$$F(x)=\begin{cases}(x-5)\log(5-x)-x-e^2+8\log 4 & x<1\\ (x+3)\log(x+3)-x-e^2 & x\geq 1\end{cases}$$

2. Trovare f prim in $]-\infty,+\infty[$ di $f(x)=x\sin^2x+x^2\cos x$ tale che $F(\frac{\pi}{2})=\frac{5}{16}\pi$

$$\int(x\sin^2x+\underset{\uparrow FD}{x^2\cos x})\,dx=\int\left(x\,\frac{1-\cos 2x}{2}+x^2\cos x\right)dx=$$

$$=\frac{1}{2}\int x\,dx-\frac{1}{2}\int \underset{\uparrow FD}{x\cos 2x}\,dx+x^2\sin x+2\int -x\sin x\,dx=$$

$$=\frac{1}{4}x^2-\frac{1}{2}\left(\frac{1}{2}x\sin 2x-\frac{1}{2}\int\sin 2x\,dx\right)+x^2\sin x+x\cos x-\int\cos x\,dx$$

$$=\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+x^2\sin x+x\cos x-\sin x+k$$

$$F\left(\frac{\pi}{2}\right)=5\frac{\pi^2}{16}$$

$$\frac{\pi^2}{16}+\frac{1}{8}+\frac{\pi^2}{4}-1+k=\frac{5}{16}\pi^2-\frac{7}{8}+k=\frac{5}{16}\pi^2 \text{ se } k=\frac{7}{8}$$

3. Trovare f prim. in $]-\infty,+\infty[$ di $f(x)=e^{|x|}+\log\frac{2|x|+x+1}{|x|+1}$ tale che $F(1)=e$

$$f(x)=\begin{cases}e^{-x} & x<0\\ e^x+\log\frac{3x+1}{x+1} & x\geq 0\end{cases}$$

$$\int e^{-x}\,dx=-e^{-x}+k$$

$$\int\left(e^x+\log\frac{3x+1}{x+1}\right)dx=e^x+x\log\frac{3x+1}{x+1}-\int x\,\frac{x+1}{3x+1}\,\frac{2}{(x+1)^2}\,dx=$$

$$=e^x+x\log\frac{3x+1}{x+1}-2\int\frac{x}{(3x+1)(x+1)}\,dx \quad *$$

$$\frac{x}{(3x+1)(x+1)}=\frac{A}{3x+1}+\frac{B}{x+1}=\frac{Ax+A+3Bx+B}{(3x+1)(x+1)} \qquad \begin{cases}A+3B=1\\ A+B=0\end{cases}$$

$$B=\frac{1}{2},\ A=-\frac{1}{2}$$

$$(*)=e^x+x\log\frac{3x+1}{x+1}+\frac{1}{3}\log|3x+1|-\log|x+1|+k$$

$$f(x)=\begin{cases} -e^{-x}+k_1 & x<0\\ e^x+x\log\frac{3x+1}{x+1}+\frac{1}{3}\log(3x+1)-\log(x+1)+k_2 & x\geq 0\end{cases}$$

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$ $\qquad -1+k_1=1+k_2 \Rightarrow k_1=k_2+2$

$$f(x)=\begin{cases} -e^{-x}+k+2 & x<0\\ \dots\dots+k & x\geq 0 \leftarrow\end{cases}$$

$e+\cancel{\log 2}+\frac{1}{3}\log 4-\cancel{\log 2}+k=e \Rightarrow k=-\frac{1}{3}\log 4$

4. $I=\int \frac{\log(x+1)}{(x-2)^2}\,dx=\int \underbrace{\frac{1}{(x-2)^2}}_{fD}\log(x+1)\,dx=$

$=-\frac{1}{x-2}\log(x+1)+\int\frac{1}{(x-2)(x+1)}\,dx$

$\frac{1}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}=\frac{(A+B)x+A-2B}{(x-2)(x+1)}$ $\qquad \begin{cases}A+B=0\\ A-2B=1\end{cases}$

$A=\frac{1}{3}$

$B=-\frac{1}{3}$

$I=-\frac{1}{x-2}\log(x+1)+\frac{1}{3}\log|x-2|-\frac{1}{3}\log|x+1|+k$

5. $\int\frac{\log(x^2+4x+3)}{(2x+4)^2}\,dx=$

$=-\frac{1}{2(2x+4)}\log(x^2+4x+3)+\frac{1}{2}\int\frac{1}{\cancel{2x+4}}\,\frac{\cancel{2x+4}}{x^2+4x+3}\,dx$

$x^2+4x+3=0 \qquad x=-2\pm 1 \begin{cases}-3\\ -1\end{cases}$

$\frac{1}{x^2+4x+3}=\frac{A}{x+3}+\frac{B}{x+1}$ $\qquad$ da completare

6. $I=\int\frac{\cos x(\sin x+3)}{\cos^2 x-2\sin^2 x}\,dx=\int\cos x\,\frac{\sin x+3}{1-3\sin^2 x}\,dx=$

$=-\Big[\underbrace{\int\frac{t+3}{3t^2-1}\,dt}_{J}\Big]_{t=\sin x}$

$\frac{t+3}{3t^2-1}=\frac{A}{\sqrt{3}t-1}+\frac{B}{\sqrt{3}t+1}=\frac{(\sqrt{3}A+\sqrt{3}B)t+A-B}{3t^2-1}$

$\begin{cases}\sqrt{3}(A+B)=1\\ A-B=3\end{cases}$ $\qquad \begin{matrix}3+2B=\frac{1}{\sqrt{3}}\\ A=3+B\end{matrix}$ $\qquad \Rightarrow \begin{matrix}B=\frac{1}{2\sqrt{3}}-\frac{3}{2}\\ A=\frac{1}{2\sqrt{3}}+\frac{3}{2}\end{matrix}$

$J=A\int\frac{dt}{\sqrt{3}t-1}+B\int\frac{dt}{\sqrt{3}t+1}=\frac{A}{\sqrt{3}}\log|\sqrt{3}t-1|+\frac{B}{\sqrt{3}}\log|\sqrt{3}t+1|+k$

$I=\left(-\frac{1}{6}+\frac{\sqrt{3}}{2}\right)\log|\sqrt{3}\sin x-1|-\left(\frac{1}{6}+\frac{\sqrt{3}}{2}\right)\log|\sqrt{3}\sin x+1|+k$

7. $\int\frac{x}{\sqrt{1-x^4}}\,dx=\int\frac{x}{\sqrt{1-x^2}\sqrt{1+x^2}}\,dx=\frac{1}{2}\int\frac{2x}{\sqrt{1-x^2}\sqrt{1+x^2}}\,dx=$

$=\frac{1}{2}\left[\int\frac{dt}{\sqrt{1-t}\sqrt{1+t}}\right]_{t=x^2}=\frac{1}{2}\left[\int\frac{dt}{\sqrt{1-t^2}}\right]_{t=x^2}=\frac{1}{2}\arcsin x^2+k$

Trovare f prim in $]-\infty,+\infty[$ di $f(x)=(2|x|+1)\log(x^2+4)$
e tale che $f(2)=\pi$

$\int\frac{x-\sqrt{x+2}}{\dots}\,dx \qquad \int\frac{x-\sqrt{x-1}}{\dots} \qquad \int\sqrt{\frac{x-1}{\dots}}$

e tale che $f(2) = \pi$

$$\int \frac{x - \sqrt{x+2}}{x-1}\,dx \qquad \int \frac{x - \sqrt{x-1}}{x+3} \qquad \int \sqrt{\frac{x-1}{x+4}}$$