f.i. $+\infty-\infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$

$\cancel{\frac{2}{n} \to \frac{2}{+\infty} = 0}$ ERRORE GRAVE

SI SCRIVE COSÌ $\frac{2}{\underset{\to +\infty}{n}} \to 0$

PROP. IP $f: X \to \mathbb{R}$ $X \subseteq \mathbb{R}$ f funz. elementare

$\{a_n\} \subseteq X$ $a_n \to \ell \in X$

TS $f(a_n) \to f(\ell)$

es. $\frac{n}{2n+1} \to \frac{1}{2}$ (vedremo il perché)

$\left\{\frac{n}{2n+1}\right\} \subseteq]0,+\infty[$ allora $\log\frac{n}{2n+1} \to \log\frac{1}{2}$

Limiti di succ espresse mediante funz. elementari

Succ. potenza n^α $\alpha \in \mathbb{R}$

$\alpha = 0$ $n^0 = 1 \to 1$

$\alpha > 0$ $n^\alpha \to +\infty$ infatti $n^\alpha > k \iff n > k^{\frac{1}{\alpha}}$ D vera

$\alpha < 0$ $n^\alpha \to 0$ " $n^\alpha = \frac{1}{n^{-\alpha}}$

Polinomio $x_n = a_0 n^p + a_1 n^{p-1} + \dots + a_{p-1} n + a_p$ $a_0, \dots, a_p \in \mathbb{R}$ $p \in \mathbb{N}$

a seconda dei segni dei coeff si può avere una f.i. $+\infty - \infty$

(es. $\underset{+\infty}{3n^4} - \underset{+\infty}{6n^3} + \underset{+\infty}{2n^2} - \underset{+\infty}{5n} + 8$)

$x_n = \underset{+\infty}{n^p}\Big(\underbrace{a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_p}{n^p}}_{a_0}\Big)$

es. $\underset{+\infty}{n^4}\Big(\overbrace{3 - \underset{0}{\frac{6}{n}} + \underset{0}{\frac{2}{n^2}} - \underset{0}{\frac{5}{n^3}} + \underset{0}{\frac{8}{n^4}}}^{\to 3}\Big) \to +\infty$

$\Downarrow$
a_0 $\quad$ $+\infty$ $\quad 0 \quad 0 \quad 0 \quad 0$

$$x_n \to \begin{cases} +\infty & \text{se } a_0 > 0 \\ -\infty & \text{se } a_0 < 0 \end{cases}$$

i polinomi divergono SEMPRE. Il segno della divergenza è il segno del coefficiente del termine di grado massimo.

es. $\quad n^3 - 2n^2 - 6n - 3 \to +\infty$

$$n^3 + 2n^2 + 6n - 3n^4 \to -\infty$$

Funz razionali fratte (ridotte ai minimi termini)

$$x_n = \frac{a_0 n^p + \dots + a_p}{b_0 n^q + \dots + b_q} \qquad a_i, b_j \in \mathbb{R} \quad b_0 \neq 0 \quad p \in \mathbb{N}_0 \quad q \in \mathbb{N}$$

è una f.i. $\frac{\infty}{\infty}$.

$$x_n = \frac{n^p \left(a_0 + \dots + \frac{a_p}{n^p}\right)}{n^q \left(b_0 + \dots + \frac{b_q}{n^q}\right)} = n^{p-q} \; \frac{a_0 + \dots + \frac{a_p}{n^p}}{b_0 + \dots + \frac{b_q}{n^q}} \to \frac{a_0}{b_0}$$

$$n^{p-q} \to \begin{cases} 1 & \text{se } p = q \\ +\infty & \text{se } p > q \\ 0 & \text{se } p < q \end{cases}$$

quindi: se $p = q \quad x_n \to \frac{a_0}{b_0}$

se $p > q \quad x_n \to \infty \quad$ ($+\infty$ se a_0, b_0 concordi, $-\infty$ se a_0, b_0 disc.)

se $p < q \quad x_n \to 0$

es. $\quad \dfrac{3n^2 - 5n + 1}{(n+2)^2} \to 3 \qquad \dfrac{-2n^4 - 6n}{3n^4 + 5n^2 + 7} \to -\dfrac{2}{3}$

$$\frac{3n^2 - 5n + 1}{(n+2)^4} \to 0 \qquad \frac{3n^2 - 5n + 1}{n - 3} \to +\infty$$

$$\frac{3n^2 - 5n^4 + 1}{n^2 + 8} \to -\infty \qquad \frac{3n^2 - 5n^4 + 1}{n^2 - 8n^3 + 3} \to +\infty$$

Successione geometrica

Successione geometrica

$a \in \mathbb{R}$ $\qquad \{a^n\}$

$a = 0 \qquad a^n \to 0$

$a = 1 \qquad a^n \to 1$

$a = -1 \qquad \{a^n\}$ oscill.

$a > 1 \qquad a^n \to +\infty$ $\qquad$ infatti $a^n > k \iff n > \log_a k$ vera D

$a < -1 \qquad a^n = \big((-1)(-a)\big)^n = \underbrace{(-1)^n}_{\text{osc.}} \underbrace{(-a)^n}_{\substack{\Downarrow \\ +\infty \\ (-a)>1}}$ $\qquad$ oscill.

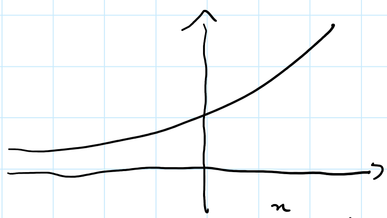
$-1 < a < 1 \qquad a^n \to 0$ $\qquad$ infatti $|a^n| < \varepsilon \iff \underset{\substack{\uparrow \\ \text{perché } |a|<1}}{n >} \log_{|a|} \varepsilon$ vera D

es. $\left(-\frac{1}{5}\right)^n \to 0 \qquad (-6)^n$ oscill. $\qquad (18)^n \to +\infty \qquad \left(\frac{2}{3}\right)^n \to 0$

Succ composte mediante la funz esponenziale

$\{x_n\}$ regolare $\qquad$ cons. a^{x_n} $\qquad$ con $a > 0$, $a \neq 1$

se $x_n \to l \in \mathbb{R} \implies a^{x_n} \to a^l$



se $a > 1$ $\qquad a^{x_n} \to \begin{cases} +\infty & \text{se } x_n \to +\infty \\ 0 & \text{se } x_n \to -\infty \end{cases}$

se $0 < a < 1$ $\qquad a^{x_n} \to \begin{cases} 0 & \text{se } x_n \to +\infty \\ +\infty & \text{se } x_n \to -\infty \end{cases}$

es. $\qquad 2^{\overbrace{\frac{3n^2-1}{n+4}}^{\to +\infty}} \to +\infty$ $\qquad \cancel{2^{+\infty} = +\infty}$ $\qquad$ ERRORE

$3^{\overbrace{\frac{n^6-6}{2-n^4}}^{\to -\infty}} \to 0$

$\left(\frac{1}{\pi}\right)^{\overbrace{\frac{n^4-2}{3-n}}^{\to -\infty}} \to +\infty$

$\left(\frac{1}{\ }\right)^{\overbrace{\frac{3-n^6}{2-n^2}}^{\to +\infty}} \to 0$

$$\left(\frac{1}{3}\right)^{\frac{3-n^6}{2-n^2} \to +\infty} \to 0$$

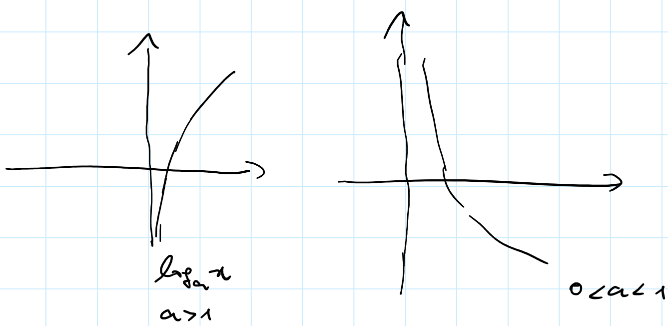
$$4^{\frac{n^2+1}{(2n+3)^2}} \to 4^{\frac{1}{4}}$$

Funz. logaritmo

$\{b_n\}$ succ. regolare, $b_n > 0 \ \forall n$ $\quad a > 0,\ a \neq 1$ $\quad x_n = \log_a b_n$

se $b_n \to \ell > 0 \Rightarrow x_n \to \log_a \ell$

es. $\log_2 \underbrace{\frac{4n+1}{n+3}}_{\to 4} \to \log_2 4 = 2$



se $a>1 \Rightarrow x_n \to \begin{cases} +\infty & \text{se } b_n \to +\infty \\ -\infty & \text{se } b_n \to 0 \end{cases}$

se $0<a<1 \Rightarrow x_n \to \begin{cases} -\infty & \text{se } b_n \to +\infty \\ +\infty & \text{se } b_n \to 0 \end{cases}$

es. $\log_3 \underbrace{\frac{2n+1}{n^4+8}}_{\to 0} \to -\infty$ $\qquad \log_2 \overbrace{\frac{2n^6+1}{n^4+8}}^{\to +\infty} \to +\infty$

$\log_{\frac{1}{2}} \overbrace{\frac{1-3n^4}{2-n^2}}^{\to +\infty} \to -\infty$ $\qquad \log_{\frac{1}{4}} \overbrace{\frac{3n+1}{2n^4+6}}^{\to 0} \to +\infty$

per esec. se $x_n \to +\infty$, $a>1 \Rightarrow a^{x_n} \to +\infty$

si deve avere $a^{x_n} > k \iff x^n > \log_a k$ vera D

se $0<a<1$ $\quad a^{x_n} = \frac{1}{\underbrace{\left(\frac{1}{a}\right)^{x_n}}_{\to +\infty}} \to 0$ $\quad \left(\frac{1}{a} > 1\right)$

se $b_n \to +\infty$, $a>1 \Rightarrow \log_a b_n \to +\infty$

si deve avere $\log_a b_n > k \iff b_n > a^k$ vera D se $0<a<1$ $\quad \log_a b_n = \underbrace{\left(\log_a \frac{1}{a}\right)}_{=-1} \underbrace{\left(\log_{\frac{1}{a}} b_n\right)}_{\to +\infty} \to \infty$ (perché $\frac{1}{a}>1$)

se $a_n \to \infty$ allora $\{\cos a_n\}$, $\{\sin a_n\}$, $\{\operatorname{tg} a_n\}$ sono non regolari

Caso ora una succ. del tipo $a_n^{b_n}$ con $a_n>0$, $a_n \neq 1$ $\forall n$

$\{a_n\}$, $\{b_n\}$ regolari

Ricordiamo che se $x>0$ $\quad x = e^{\log x}$ quindi

$a_n^{b_n} = e^{\log a_n^{b_n}} = e^{b_n \log a_n}$ $\qquad$ es. se $a_n \to 2$, $b_n \to 3$ $\quad e^{b_n \log a_n} \to e^{3\log 2} =$

$= e^{\log 8} = 8$

Si avrà una f.I. se c'è una f.I. nel prodotto $b_n \log a_n$

$b_n \to 0$

$\log a_n \to \infty$ cioè $a_n \to 0$ opp. $a_n \to +\infty$ $\qquad$ f.I. 0^0, $(+\infty)^0$

opp.

$b_n \to \infty$

$\log a_n \to 0$ cioè $a_n \to 1$ $\qquad$ f.I. 1^∞

Le f.I. allora sono: $+\infty-\infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, $(+\infty)^0$, 1^∞

$\log_{a_n} b_n$ $\qquad$ $b_n>0$ $\forall n$, $a_n>0$ $\forall n$, $a_n \neq 1$ $\forall n$, $\{a_n\}$ reg., $\{b_n\}$ reg.

$\log_{a_n} b_n = (\log_{a_n} e)(\log b_n) = \dfrac{\log b_n}{\log a_n}$

1^∞ $\qquad$ $a_n = \left(1+\frac{1}{n}\right)^n$ si presenta nella f.I. 1^∞

Si può dim. che $\{a_n\}$ è strett. crescente e $a_n < 3$ $\forall n$

quindi converge, ad un numero <3, che chiamiamo e

DEF. $\quad e = \lim \left(1+\frac{1}{n}\right)^n$ $\qquad$ Si ha $e \notin \mathbb{Q}$

$\boxed{e > 1}$

Si può dim. che se $x_n \to \infty$ allora $\left(1 + \frac{1}{x_n}\right)^{x_n} \to e$

es. $\left(1 + \frac{\alpha}{n}\right)^n = \Bigg[\underbrace{\left(1 + \frac{1}{\frac{n}{\alpha}}\right)^{\frac{n}{\alpha}}}_{\to e}\Bigg]^{\alpha} \to e^{\alpha}$

se $\alpha \in \mathbb{R}$

$\left(1 + \frac{3}{n}\right)^n \to e^3$ $\qquad$ $\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}$

• $\left(\frac{n+4}{n+6}\right)^{2n+1} =$ $\qquad$ $\left[\left(1 + \frac{1}{x_n}\right)^{x_n}\right]$ $\qquad$ $\left(\frac{2n+4}{n+6}\right)^{2n+1} \to +\infty$

$\frac{2n+4}{n+6} \to 2$, $\quad 2n+1 \to +\infty$

$= \left(\frac{n+4+2-2}{n+6}\right)^{2n+1} = \left(\frac{\cancel{n+6}-2}{\cancel{n+6}}\right)^{2n+1} = \left(1 + \frac{-2}{n+6}\right)^{2n+1} =$

$= \left(1 + \frac{1}{\underbrace{\frac{n+6}{-2}}_{x_n}}\right)^{2n+1} = \Bigg[\underbrace{\left(1 + \frac{1}{\frac{n+6}{-2}}\right)^{\frac{n+6}{-2}}}_{\to e}\Bigg]^{\frac{-2(2n+1)}{n+6} \to -4} \to e^{-4}$

• $\left(\frac{n^2+2}{(n+1)^2}\right)^{n+3} = \left(\frac{n^2+2}{n^2+2n+1}\right)^{n+3} = \left(\frac{\cancel{n^2+2n+1}-2n-1+2}{\cancel{n^2+2n+1}}\right)^{n+3} = \left(1 + \frac{1-2n}{n^2+2n+1}\right)^{n+3} =$

$= \Bigg[\underbrace{\left(1 + \frac{1}{\frac{n^2+2n+1}{1-2n}}\right)^{\frac{n^2+2n+1}{1-2n}}}_{\to e}\Bigg]^{\frac{(1-2n)(n+3)}{n^2+2n+1}} \to e^{-2}$

Altri limiti notevoli ("dedotti dal numero e")

se $a_n \to 0$ $\qquad \frac{e^{a_n}-1}{a_n} \to 1 \qquad \frac{\log(1+a_n)}{a_n} \to 1 \qquad \frac{(1+a_n)^{\alpha}-1}{a_n} \to \alpha$

$(\alpha \in \mathbb{R})$

es. $n \log\left(\frac{n+1}{n}\right) = n \log\left(1 + \frac{1}{n}\right) = \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1$

Limiti notevoli con funz. trigonometriche se $a_n \to 0$ $\quad \sin a_n \to 0 \quad \frac{\sin a_n}{a_n}$ f. i. $\frac{0}{0}$

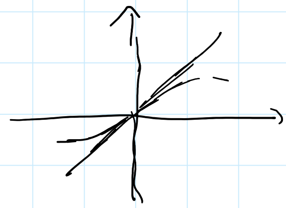
$a_n \neq 0$

Si ha $\forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\qquad |\sin x| \le |x| \le |\text{tg}\, x|$

se $a_n \to 0 \Rightarrow$ D $\; -\frac{\pi}{2} < a_n < \frac{\pi}{2} \Rightarrow |\sin a_n| \le |a_n| \le |\text{tg}\, a_n| \Rightarrow$

$$\Rightarrow 1 \le \left|\frac{a_n}{\sin a_n}\right| \le \frac{1}{|\cos a_n|}$$

$$\left|\frac{a_n}{\sin a_n}\right| = \frac{a_n}{\sin a_n} \qquad \frac{1}{|\cos a_n|} = \frac{1}{\cos a_n}$$



$$\Rightarrow \underset{\substack{\downarrow \\ 1}}{\cos a_n} \le \frac{\sin a_n}{a_n} \le \underset{\substack{\downarrow \\ 1}}{1} \Rightarrow \frac{\sin a_n}{a_n} \to 1$$

$$\boxed{a_n \to 0}$$

$$\frac{\sin a_n}{a_n} \to 1 \qquad \frac{a_n}{\sin a_n} \to 1 \qquad \frac{\text{tg}\, a_n}{a_n} \to 1 \qquad \frac{a_n}{\text{tg}\, a_n} \to 1$$

$$\frac{\text{tg}\, a_n}{a_n} = \underset{\substack{\downarrow \\ 1}}{\frac{\sin a_n}{a_n}} \; \underset{\substack{\downarrow \\ 1}}{\frac{1}{\cos a_n}}$$

$$\frac{\arcsin a_n}{a_n} \to 1 \qquad \frac{\text{arctg}\, a_n}{a_n} \to 1$$

$$\frac{1-\cos a_n}{a_n} = \frac{(1-\cos a_n)(1+\cos a_n)}{a_n\,(1+\cos a_n)} = \frac{\overset{\sin^2}{\overbrace{1-\cos^2 a_n}}}{a_n}\;\frac{1}{1+\cos a_n} = \underset{\substack{\downarrow \\ 1}}{\frac{\sin a_n}{a_n}}\;\underset{\substack{\downarrow \\ 0}}{\frac{\sin a_n}{1+\cos a_n}} \to 0$$

$$\frac{1-\cos a_n}{a_n^2} = \underset{\substack{\downarrow \\ 1}}{\frac{\sin^2 a_n}{a_n^2}}\;\underset{\substack{\downarrow \\ \frac{1}{2}}}{\frac{1}{1+\cos a_n}} \to \frac{1}{2}$$

Esempi

$$\frac{\sin \frac{1}{2n+1}}{\frac{2}{3n}} = \underset{\substack{\downarrow \\ 1}}{\frac{\sin \frac{1}{2n+1}}{\frac{1}{2n+1}}} \underset{\substack{\downarrow \\ \frac{3}{4}}}{\left(\frac{1}{2n+1}\;\frac{3n}{2}\right)} \to \frac{3}{4}$$

$$\frac{\frac{1}{n+3}}{\sin \frac{n+1}{n^2+4}} = \underset{\substack{\downarrow \\ 1}}{\frac{\frac{n+1}{n^2+4}}{\sin \frac{n+1}{n^2+4}}} \underset{\substack{\downarrow \\ 1}}{\left(\frac{n^2+4}{n+1}\;\frac{1}{n+3}\right)} \to 1$$

$$(2n^2+3)\sin\frac{n}{n^3+1} = \frac{\sin\frac{n}{n^3+1}}{\frac{n}{n^3+1}}\left(\frac{n}{n^3+1}(2n^2+3)\right) \to 2$$

$$\frac{\sin\frac{n}{n^3+1}}{\frac{n}{n^3+1}} \to 1 \qquad \frac{n}{n^3+1}(2n^2+3) \to 2$$

$$(3n-1)\,\mathrm{tg}\,\frac{2n}{n^2+4} = \frac{\mathrm{tg}\frac{2n}{n^2+4}}{\frac{2n}{n^2+4}}\left(\frac{2n}{n^2+4}(3n-1)\right) \to 6$$

$$\frac{\mathrm{tg}\frac{2n}{n^2+4}}{\frac{2n}{n^2+4}} \to 1 \qquad \frac{2n}{n^2+4}(3n-1) \to 6$$

$$\frac{\sin^2\frac{2}{n+3}}{\mathrm{tg}\,\frac{n}{3n^3+4}} = \frac{\sin^2\frac{2}{n+3}}{\left(\frac{2}{n+3}\right)^2}\;\frac{\frac{n}{3n^3+4}}{\mathrm{tg}\,\frac{n}{3n^3+4}}\left(\left(\frac{2}{n+3}\right)^2\frac{3n^3+4}{n}\right) \to 12$$

$$\frac{\sin^2\frac{2}{n+3}}{\left(\frac{2}{n+3}\right)^2} \to 1 \qquad \frac{\frac{n}{3n^3+4}}{\mathrm{tg}\,\frac{n}{3n^3+4}} \to 1 \qquad \left(\frac{2}{n+3}\right)^2\frac{3n^3+4}{n} \to 12$$

$$\frac{1-\cos\frac{2}{n+1}}{\sin\frac{n+1}{n^3+2}} = \frac{1-\cos\frac{2}{n+1}}{\frac{4}{(n+1)^2}}\;\frac{\frac{n+1}{n^3+2}}{\sin\frac{n+1}{n^3+2}}\left(\frac{4}{(n+1)^2}\;\frac{n^3+2}{n+1}\right) \to \frac{1}{8}$$

$$\frac{1-\cos\frac{2}{n+1}}{\frac{4}{(n+1)^2}} \to \frac{1}{2} \qquad \frac{\frac{n+1}{n^3+2}}{\sin\frac{n+1}{n^3+2}} \to 1 \qquad \frac{4}{(n+1)^2}\;\frac{n^3+2}{n+1} \to 4$$