15 ottobre 2025_MZ

mercoledì 15 ottobre 2025 14:04

ESERCIZI SULLA SECONDA FORMULA DI INTEGR PER SOSTIT

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$$I = \int \frac{x + \sqrt{x-1}}{x+2}\, dx \qquad \begin{matrix} x \geq 1 \\ x \neq -2 \end{matrix} \qquad (a,b) = [1, +\infty[$$

pongo $\sqrt{x-1} = t \qquad t \geq 0$

ricavo x $\quad x - 1 = t^2 \Rightarrow x = t^2 + 1 = g(t)$

$t \geq 0 \Rightarrow t^2 + 1 \geq 1$? si quindi $(c,d) = [1, +\infty[$

$g'(t) = 2t \geq 0$, $=0 \Leftrightarrow t = 0 \Rightarrow g$ strett cresc $\Downarrow$ invert

$$I = \left[\int \frac{t^2+1+t}{t^2+3}\, 2t\, dt\right]_{t=\sqrt{x-1}} = 2\left[\underbrace{\int \frac{t^3+t^2+t}{t^2+3}\, dt}_{J}\right]_{t=\sqrt{x-1}}$$

$$\begin{array}{rrrr|l} t^3 & +t^2 & +t & & t^2+3 \\ -t^3 & & -3t & & \overline{t+1} \\ \hline & t^2 & -2t & & \\ & -t^2 & & -3 & \\ \hline & & -2t & -3 & \end{array}$$

$$J = \int \left(t + 1 - \frac{2t+3}{t^2+3}\right) dt =$$

$$= \frac{1}{2}t^2 + t - \log(t^2+3) - \frac{3}{\sqrt{3}} \operatorname{arctg} \frac{t}{\sqrt{3}} + k$$

$$I = 2\left(\frac{1}{2}(x-1) + \sqrt{x-1} - \log(x-1+3) - \sqrt{3} \operatorname{arctg} \sqrt{\frac{x-1}{3}}\right) + k$$

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$$I = \int \frac{x - \sqrt{x+3}}{x-4}\, dx \qquad \begin{matrix} x \geq -3 \\ x \neq 4 \end{matrix} \qquad (a,b) =]4, +\infty[\text{ opp. } (a,b) = [-3, 4[$$

$(a,b) =]4, +\infty[$ $\quad$ pongo $\sqrt{x+3} = t \geq 0$

ricavo $x = t^2 - 3 = g(t)$

$t^2 - 3 > 4$ se $t > \sqrt{7} \Rightarrow (c,d) =]\sqrt{7}, +\infty[$

se $(a,b) = [-3, 4[$ $\quad t^2 - 3 \geq -3 \Leftrightarrow t^2 \geq 0$ vera

$t^2 - 3 < 4 \Rightarrow t^2 < 7 \Rightarrow (c,d) = [0, \sqrt{7}[$

$g'(t) = 2t > 0 \quad \forall t \in]\sqrt{7}, +\infty[\Rightarrow g$ invert.

$\geq 0 \quad \forall t \in [0, \sqrt{7}[$, $=0 \Leftrightarrow t = 0$

$$I = \left[\int \frac{t^2-3-t}{t^2-3-4}\, 2t\, dt\right]_{t=\sqrt{x+3}} = 2\left[\underbrace{\int \frac{t^3-t^2-3t}{t^2-7}\, dt}_{J}\right]_{t=\sqrt{x+3}}$$

$$\begin{array}{rrrr|l} t^3 & -t^2 & -3t & & t^2-7 \\ -t^3 & & +7t & & \overline{t-1} \\ \hline & -t^2 & +4t & & \\ & t^2 & & -7 & \\ \hline & & 4t & -7 & \end{array}$$

$$J = \int \left(t - 1 + \frac{4t-7}{t^2-7}\right) dt =$$

$$= \frac{1}{2}t^2 - t + 2\log|t^2-7| - 7\int \frac{dt}{t^2-7}$$

$$\frac{1}{t^2-7} = \frac{A}{t-\sqrt{7}} + \frac{B}{t+\sqrt{7}} = \frac{At + \sqrt{7}A + Bt - \sqrt{7}B}{t^2-7}$$

$$\begin{cases} A + B = 0 \\ \sqrt{7}A - \sqrt{7}B = 1 \end{cases} \qquad \begin{matrix} B = -A \\ A = \frac{1}{2\sqrt{7}} \end{matrix}$$

$$J = \frac{1}{2}t^2 - t + 2\log|t^2-7| - 7\left(\frac{1}{2\sqrt{7}} \log|t-\sqrt{7}| - \frac{1}{2\sqrt{7}} \log|t+\sqrt{7}|\right) + k$$

$$= \frac{1}{2}t^2 - t + 2\log|t^2-7| - \frac{\sqrt{7}}{2} \log\left|\frac{t-\sqrt{7}}{t+\sqrt{7}}\right| + k$$

$$I = 2\left(\frac{1}{2}(x+3) - \sqrt{x+3} + 2\log|x-4| - \frac{\sqrt{7}}{2} \log\left|\frac{\sqrt{x+3}-\sqrt{7}}{\sqrt{x+3}+\sqrt{7}}\right|\right) + k$$

$$\sqrt{\frac{ax+b}{cx+d}} \qquad ad - bc \neq 0$$ (se fosse $=0$ non sarebbe una frazione)

Esempi:

$$\int \sqrt{\frac{x-1}{x+2}}\, dx \qquad x < -2 \ \vee\ x \geq 1$$

Esempi.

$$\int \sqrt{\frac{x-1}{x+2}}\, dx \qquad x<-2 \ \vee\ x\geq 1$$

$$(a,b) =]-\infty,-2[\quad \vee \quad (a,b) = [1,+\infty[$$

pongo $\sqrt{\frac{x-1}{x+2}} = t \geq 0$

ricavo x $\quad \frac{x-1}{x+2} = t^2 \Rightarrow x-1 = t^2 x + 2t^2 \Rightarrow x = \frac{2t^2+1}{1-t^2} = g(t)$

se $(a,b) =]-\infty,-2[$ $\quad g(t) < -2 \Rightarrow \frac{2t^2+1}{1-t^2} + 2 < 0 \Rightarrow \frac{2t^2+1+2-2t^2}{1-t^2} < 0$

$\Rightarrow 1-t^2 < 0 \Rightarrow t \in]1,+\infty[\quad (c,d) =]1,+\infty[$

se $(a,b) = [1,+\infty[$ $\quad g(t) \geq 1 \Rightarrow \frac{2t^2+1}{1-t^2} - 1 \geq 0 \Rightarrow \frac{2t^2+1-1+t^2}{1-t^2} \geq 0$

$\Rightarrow (c,d) = [0,1[$

$$g(t) = \frac{2t^2+1}{1-t^2} \qquad g'(t) = \frac{4t(1-t^2)+2t(2t^2+1)}{(t^2-1)^2} = \frac{4t-4t^3+4t^3+2t}{(t^2-1)^2}$$

$$= \frac{6t}{(t^2-1)^2} > 0 \quad \text{se } (a,b) =]-\infty,-2[$$

≥ 0, $=0 \Leftrightarrow t=0$ $\quad$ se $(a,b) = [1,+\infty[$

g è invertibile.

$$I = \left[\int t\, \frac{6t}{(t^2-1)^2}\, dt\right]_{t=\sqrt{\frac{x-1}{x+2}}} = 6\left[\underbrace{\int \frac{t^2}{(t^2-1)^2}\, dt}_{J}\right]_{t=\sqrt{\ }}$$

$$\frac{t^2}{(t^2-1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2} =$$

$$= \frac{A(t-1)(t+1)^2 + B(t+1)^2 + C(t+1)(t-1)^2 + D(t-1)^2}{(t^2-1)^2} =$$

$$= \frac{At^3+2At^2+At-At^2-2At-A+Bt^2+2Bt+B+Ct^3-2Ct^2+Ct+Ct^2-2Ct+C+Dt^2-2Dt+D}{(t^2-1)^2}$$

$$\begin{cases} A + C = 0 \\ A + B - C + D = 1 \\ A + 2B - C - 2D = 0 \\ -A + B + C + D = 0 \end{cases} \Rightarrow \begin{cases} C = -A \\ 2A + B + D = 1 \\ B = D \\ -2A + B + D = 0 \end{cases} \Rightarrow \begin{cases} C = -A \\ 2A + 2B = 1 \\ D = B \\ -2A + 2B = 0 \end{cases}$$

$$\Rightarrow \begin{cases} C = -A \\ D = B \\ B = A \\ A = \frac{1}{4} \end{cases} \qquad J = \frac{1}{4}\int \frac{dt}{t-1} + \frac{1}{4}\int \frac{dt}{(t-1)^2} - \frac{1}{4}\int \frac{dt}{t+1} + \frac{1}{4}\int \frac{dt}{(t+1)^2} =$$

$$= \frac{1}{4}\log|t-1| - \frac{1}{4}\frac{1}{t-1} - \frac{1}{4}\log|t+1| - \frac{1}{4}\frac{1}{t+1} + k$$

$$I \quad 6[J]_{t=\sqrt{\ }}$$

per eserc.

$$\int \frac{x+1}{\sqrt{x}+3}\, dx \qquad \int \frac{\sqrt{x-2}+3x}{x+1}\, dx$$

$$\int \frac{\sqrt{x+4}+2x}{x-3}\, dx \qquad \int \sqrt{\frac{x+3}{x-4}}\, dx \qquad \int \sqrt{\frac{x}{x-1}}\, dx$$

Esercizi

1. $$I = \int \frac{\log(x^2+6x+5)}{(2x+6)^2}\, dx$$

$$= \int \underbrace{\frac{1}{(2x+6)^2}}_{FD} \log(x^2+6x+5)\, dx = -\frac{1}{2x+6}\log(\ \ ) + \int \frac{1}{\cancel{2x+6}} \frac{\cancel{2x+6}}{x^2+6x+5}\, dx$$

$$x^2+6x+5 = 0 \qquad x = -3 \pm 2 \quad \begin{matrix} -5 \\ -1 \end{matrix}$$

$$\frac{1}{x^2+6x-5} = \frac{A}{x+5} + \frac{B}{x+1} \ \ldots$$

$$\frac{1}{x^2+6x-5} = \frac{A}{x+5} + \frac{B}{x+1} \quad \dots$$

II modo $\quad I = \int \cancel{x} \cdot \underline{\qquad} \, dx = x \, \frac{\log(x^2+6x+5)}{(2x+6)^2} - \int x \quad D \quad$ NO

2. $$\int x \left[\log(1+x) + e^{-x^2} \right] dx = \int \underset{\uparrow \, FD}{x} \log(1+x) + \int x \, e^{-x^2} dx =$$

$$= \frac{1}{2}x^2 \log(1+x) - \frac{1}{2}\int \frac{x^2}{x+1} dx - \frac{1}{2}\int -2x \, e^{-x^2} dx =$$

$$= \frac{1}{2}x^2 \log(1+x) - \frac{1}{2}\int \frac{x^2-1+1}{x+1} dx - \frac{1}{2}\left[\int e^t \, dt\right]_{t=-x^2} =$$

$$= \frac{1}{2}x^2 \log(1+x) - \frac{1}{2}\int \left(x-1+\frac{1}{x+1}\right) dx - \frac{1}{2}e^{-x^2} =$$

$$= \frac{1}{2}x^2 \log(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\log|x+1| - \frac{1}{2}e^{-x^2} + k$$

3. $$\int \log\left(\sqrt{x+1}-\sqrt{x-1}\right) dx = \int \underset{\uparrow \, FD}{1} \cdot \log\left(\sqrt{x+1}-\sqrt{x-1}\right) dx =$$

$$= x\log\left(\sqrt{x+1}-\sqrt{x-1}\right) - \int x \, \frac{\frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x-1}}}{\sqrt{x+1}-\sqrt{x-1}} dx =$$

$$= x\log\left(\sqrt{x+1}-\sqrt{x-1}\right) - \frac{1}{2}\int x \, \frac{\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}\,\sqrt{x-1}}}{\sqrt{x+1}-\sqrt{x-1}} dx =$$

$$= x\log\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\int \frac{x}{\sqrt{x+1}\,\sqrt{x-1}} dx =$$

$$= x\log\left(\qquad\right) + \frac{1}{4}\int \frac{2x}{\sqrt{x^2-1}} dx =$$

$$= x\log\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{4}\left[\int \frac{1}{\sqrt{t}} dt\right]_{t=x^2-1} =$$

$$= x\log\left(\sqrt{x+1}-\sqrt{x-1}\right) + \frac{1}{2}\sqrt{x^2-1} + k$$

4. Trovare F prim. di $f(x) = \log(|x-2|+3)$ in $]-\infty,+\infty[$ tale che $F(e^2-1) = e^2$

$$f(x) = \begin{cases} \log(5-x) & x<2 \\ \log(x+1) & x \ge 2 \end{cases}$$

$$\int 1 \cdot \log(5-x)\, dx = x\log(5-x) - \int \frac{-x}{5-x} dx =$$

$$= x\log(5-x) - \int \frac{x-5+5}{x-5} dx =$$

$$= x\log(5-x) - x - 5\log|x-5| + k_1 =$$

$$= (x-5)\log(5-x) - x + k_1$$

$$\int 1 \cdot \log(x+1)\, dx = x\log(x+1) - \int \frac{x+1-1}{x+1} dx =$$

$$= x\log(x+1) - x + \log(x+1) + k_2 =$$

$$= (x+1)\log(x+1) - x + k_2$$

$$F(x) = \begin{cases} (x-5)\log(5-x) - x + k_1 & x<2 \\ (x+1)\log(x+1) - x + k_2 & x \ge 2 \end{cases}$$

$$\lim_{x\to 2^-} F(x) = \lim_{x\to 2^+} F(x) \qquad -3\log 3 - \cancel{2} + k_1 = 3\log 3 - \cancel{2} + k_2 \Rightarrow$$

$$\Rightarrow k_2 = k_1 - 6\log 3$$

$$F(x) = \begin{cases} (x-5)\log(5-x) - x + k & x<2 \\ (x+1)\log(x+1) - x + k - 6\log 3 & x \ge 2 \end{cases} \quad \leftarrow$$

$f(e^2-1)=e^2 \qquad e^2\log e^2-(e^2-1)+k-6\log 3=e^2$

$\cancel{2e^2}-\cancel{e^2}+1+k-6\log 3=\cancel{e^2} \Rightarrow k=6\log 3-1$

$$f(x)=\begin{cases}(x-5)\log(5-x)-x+6\log 3-1 & x<2\\ (x+1)\log(x+1)-x-1 & x\geq 2\end{cases}$$

5. Trovare f prim. di $f(x)=x\cos^2 x+x^2\sin x$ in $]-\infty,+\infty[$ tale che $f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{16}$

$\int x\cos^2 x\,dx=\int x\,\frac{1+\cos 2x}{2}\,dx=\frac{1}{4}x^2+\frac{1}{2}\int x\cos 2x\,dx=$ (FD)

$=\frac{1}{4}x^2+\frac{1}{2}\left(x\,\frac{\sin 2x}{2}-\frac{1}{2}\int\sin 2x\,dx\right)=$

$=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{4}\cos 2x+k$

$\int x^2\sin x\,dx=-x^2\cos x+2\int x\cos x\,dx=$ (FD) (FD)

$=-x^2\cos x+2x\sin x-2\int\sin x\,dx=$

$=-x^2\cos x+2x\sin x+2\cos x+k$

$f(x)=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{4}\cos 2x-x^2\cos x+2x\sin x+2\cos x+k$

$f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{16} \qquad \frac{\pi^2}{16}-\frac{1}{4}+\pi+k=\frac{\pi^2}{16} \Rightarrow k=\frac{1}{4}-\pi$

$f(x)=\dots$

6. Trovare f prim in $]-\infty,+\infty[$ di $f(x)=e^{|x|}+\log\frac{2|x|+x+1}{|x|+1}$ tale che $f(1)=e$

$$f(x)=\begin{cases}e^{-x}+\log\frac{-x+1}{-x+1}=e^{-x} & x<0\\ e^x+\log\frac{3x+1}{x+1} & x\geq 0\end{cases}$$

$\int e^{-x}\,dx=-e^{-x}+k_1$

$I=\int\left(e^x+\log\frac{3x+1}{x+1}\right)dx=e^x+x\log\frac{3x+1}{x+1}-\int x\,\frac{\cancel{x+1}}{3x+1}\,\frac{3\cancel{x}+3-\cancel{3x}-1}{(x+1)^{\cancel{2}}}\,dx=$

$=e^x+x\log\frac{3x+1}{x+1}-2\int\underbrace{\frac{x}{(3x+1)(x+1)}}_{J}\,dx$

$\frac{x}{(3x+1)(x+1)}=\frac{A}{3x+1}+\frac{B}{x+1}=\frac{Ax+A+3Bx+B}{(3x+1)(x+1)} \qquad \begin{cases}A+3B=1\\ A+B=0\end{cases}$

$B=\frac{1}{2}$, $A=-\frac{1}{2}$

$J=-\frac{1}{2}\,\frac{1}{3}\log|3x+1|+\frac{1}{2}\log|x+1|+k$

$I=e^x+x\log\frac{3x+1}{x+1}-\frac{1}{3}\log(3x+1)+\log(x+1)+k_2$

$$f(x)=\begin{cases}e^{-x}+k_1 & x<0\\ e^x+x\log\frac{3x+1}{x+1}-\frac{1}{3}\log(3x+1)+\log(x+1)+k_2 & x\geq 0\end{cases}$$

$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x) \qquad 1+k_1=1+k_2 \Rightarrow k_1=k_2$

$f(1)=e \qquad e+\log 2-\frac{1}{3}\log 4+\log 2+k=e$

$k=-2\log 2+\frac{2}{3}\log 2=-\frac{4}{3}\log 2$

$$f(x)=\begin{cases}e^{-x}-\frac{4}{3}\log 2 & x<0\\ \dots-\frac{4}{3}\log 2 & x\geq 0\end{cases}$$