Integrazione per razionalizzazione

1. $\int \frac{e^x}{e^x+3}\,dx = \left[\int \frac{dt}{t+3}\right]_{t=e^x} = \log(e^x+2)+k$

perché $e^x = D(e^x)$

2. $\int \frac{e^x+2}{e^x+1}\,dx = \int \underset{D(e^x)}{\underset{\uparrow}{e^x}} \cdot \frac{e^x+2}{e^x(e^x+1)}\,dx = \Big[\underbrace{\int \frac{t+2}{t(t+1)}\,dt}_{J}\Big]_{t=e^x}$

$\frac{t+2}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} = \frac{(A+B)t+A}{t(t+1)}$ $\quad \begin{cases} A+B=1 \\ A=2 \end{cases} \Rightarrow \begin{cases} B=-1 \\ A=2 \end{cases}$

ln

$J = 2\log|t| - \log|t+1| + K \Rightarrow I = 2x - \boxed{\log}(e^x+1)+K$

$D(tg\,x) = tg^2x+1$

3.

$\int \frac{tg\,x+2}{tg^2x+tg\,x-2}\,dx = \int (tg^2x+1)\,\frac{tg\,x+2}{(tg^2x+1)(tg^2x+tg\,x-2)}\,dx =$

$= \left[\int \frac{t+2}{(t^2+1)(t^2+t-2)}\,dt\right]_{t=tg\,x}$

$t^2+t+2=0 \qquad t = \frac{1\pm3}{2} \begin{matrix} \nearrow -2 \\ \searrow 1 \end{matrix} \qquad \frac{t+2}{(t^2+1)(t^2+t-2)} = \frac{At+B}{t^2+1} + \frac{C}{t+2} + \frac{D}{t-1}$

$= \frac{At^3+At^2-2At+Bt^2+Bt-2B+Ct^3+Ct-Ct^2-C+Dt^3+Dt+2Dt^2+2D}{\dots}$

$\begin{cases} A+C+D=0 \\ A+B-C+2D=0 \\ -2A+B+C+D=1 \\ -2B-C+2D=2 \end{cases} \Rightarrow \begin{cases} A=-C-D \\ B-2C+D=0 \\ B+3C+3D=1 \\ 2B+C-2D=2 \end{cases} \Rightarrow \begin{cases} A=-C-D \\ B=2C-D \\ 5C+2D=1 \\ 5C-4D=-2 \end{cases}$

$\begin{cases} 1-2D=-2+4D \Rightarrow D=\frac{1}{2} \\ 5C=1-2D \Rightarrow C=0 \\ B=-\frac{1}{2} \\ A=-\frac{1}{2} \end{cases}$

$$J = \int \frac{-\frac{1}{2}t + \frac{1}{2}}{t^2+1} dt + \int \frac{\frac{1}{2}}{t-1} dt = -\frac{1}{2} \int \frac{t+1}{t^2+1} dt + \frac{1}{2} \log|t-1| =$$

$$= -\frac{1}{4} \log(t^2+1) - \frac{1}{2} \arctan t + \frac{1}{2} \log|t-1| + K$$

$$I = -\frac{1}{4} \log(tg^2 x + 1) - \frac{1}{2} x + \frac{1}{2} \log|tg x - 1| + K$$

Non è razionalizzazione ma sostituzione se abbiamo:

$$\int \frac{\sin 2x}{\sin^2 x + 3} dx = 2 \int \frac{\sin x}{\sin^2 x + 3} \cos x \, dx = 2 \left[\int \frac{t}{t^2+3} dt \right]_{t = \sin x}$$

$$= \log(\sin^2 x + 3) + K$$

$$I = \int \frac{\log x + 4}{x(\log^2 x + 2)} dx = \left[\int \frac{t+4}{t^2+2} dt \right]_{t = \log x}$$

$$J = \int \frac{t}{t^2+2} dt + 4 \int \frac{dt}{t^2+2} = \frac{1}{2} \log(t^2+2) + \frac{4}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + K$$

$$c = \sqrt{2}$$

$$I = [J]_{t = \log x}$$

Seconda formula di integrazione per sostituzione

IP: $f(a, b) \Rightarrow \mathbb{R}$ dotata di prim

$g(c, d) \to (a, b)$ suriettiva
derivabile
invertibile

TS

$$\boxed{\int f(x) dx = \left[\int f(g(t)) \, g'(t) \, dt \right]_{t = g^{-1}(x)}}$$

Dim:

dalla prima formula si ha che

$$\int f(g(t)) g'(t) dt = \left[\int f(x) dx \right]_{x = g(t)}$$

Computando ambo i membri con $t = \varphi^{-1}(x)$

$$\left[\int f(\varphi(t))\,\varphi'(t)\,dt\right]_{t=\varphi^{-1}(x)} = \left[\int f(x)\,dx\right]_{x=\varphi(\varphi^{-1}(x))=x}$$

Queste formule si utilizzano quando ci sono radici con sostituzioni del tipo: $\sqrt{ax+b} = t \quad t \geq 0$

$$ax+b = t^2 \Rightarrow x = \frac{t^2-b}{a}$$

$$\sqrt{\frac{ax+b}{cx+d}} = t \quad t \geq 0$$

$$x = \dots$$

Esempi:

1. $\displaystyle\int \frac{x+\sqrt{x-2}}{x+1}\,dx$ $\quad x \neq 1$, $x \geq 2$ $\quad (a,b) = [2;+\infty[$

pongo $\sqrt{x-2} = t \quad t \geq 0$

$(c,d) = [0,+\infty[$?

$x-2 = t^2 \Rightarrow t^2+2 = \varphi(t)$

$t \geq 0 \Rightarrow t^2+2 \geq 2$? Si

$\Rightarrow (c,d) = [0,+\infty[$

$\varphi'(t) = 2t \geq 0$, $= 0 \Leftrightarrow t = 0$ $\quad \Rightarrow \varphi$ inv

$\varphi^{-1}(x) = \sqrt{x-2}$

$$I = \left[\underbrace{\int \frac{t^2+2+t}{t^2+3}\,2t\,dt}_{J}\right]_{t=\sqrt{x-2}} = \left[2\int \frac{t^3+t^2+2t}{t^2+3}\,dt\right]_{t=\sqrt{x-2}}$$

$$\begin{array}{l|l} t^3+t^2+2t & t^2+3 \\ \hline -t^3 \quad\quad -3t & t+1 \\ \hline \quad t^2 - t & \\ \quad -t^2 - 3 & \\ \hline \quad\quad -t-3 & \end{array}$$

$$J = \int \left(t+1 - \frac{t+3}{t^2+3} \right) dt = \frac{1}{2} t^2 + t - \frac{1}{2} \log (t^2+3) + \sqrt{3} \arctan \frac{t}{\sqrt{3}}$$

$$I = \frac{1}{2}(x-2) + \sqrt{x-2} - \frac{1}{2} \log (x+1) + \sqrt{3} \arctan \frac{\sqrt{x-2}}{\sqrt{3}} + K$$

2.

$$\int \frac{x+4}{x^2+\sqrt{x-1}} \, dx$$

$(a,b) = [1; +\infty[$

$\sqrt{x-1} = t \qquad t \geq 0$

$x-1 = t^2 \Rightarrow x = t^2+1 = f(t)$

$t \geq 0 \Rightarrow t^2+1 \geq 1$? Si $\Rightarrow (c,d) = [0, +\infty[$

$f'(t) = 2t \geq 0$ $\qquad \Rightarrow f$ invert

$= 0 \Leftrightarrow t = 0$

$f^{-1}(x) = \sqrt{x-1}$

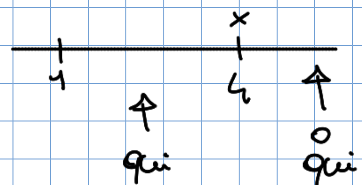
$$I = \left[\int \frac{t^2+1+4}{(t^2+1)^2+K} \, 2t \, dt \right]_{t=\sqrt{x-2}}$$

3

$$\int \frac{x+\sqrt{x-1}}{x-4}$$

$x \geq 1$

$x \neq 4$



1° caso $(a,b) = [1, 4[$ $\qquad \sqrt{x-1} = t \quad t \geq 0$

2° caso $(a,b) =]4; +\infty[$ $\qquad x-1 = t^2 \Rightarrow x = t^2+1 = f(t)$

1° CASO $\quad t \geq 0 \Rightarrow 1 \leq t^2+1 \leq 4$? $\qquad t^2+1 < 4$ se $t^2 < 3$

allora $(c,d) = [0, \sqrt{3}[$

2° CASO $\quad t \geq 0 \Rightarrow t^2+1 > 4$? $\qquad t^2+1 > 4$ se $t^2 > 3$

allora $(c,d) =]\sqrt{3}, +\infty[$

$f'(t) = 2t$ 1° CASO $2t \geq 0$ $\Rightarrow f$ inv
$=0 \iff t=0$

2° CASO $2t > 0 \Rightarrow f$ inv

$f^{-1}(x) = \sqrt{x-1}$

$$I = \left[\int \frac{t^2+1+t}{t^2+1-4} \, 2t \, dt\right]_{t=\sqrt{x-1}} = \left[2\int \frac{t^3+t^2+t}{t^2-3} \, dt\right]_{t=\sqrt{x-1}}$$

$$\begin{array}{l|l} t^3+t^2+t & t^2-3 \\ \cline{2-2} -t^3 \quad\quad +3t & t+1 \\ \hline \quad t^2+4t & \\ \quad -t^2+3 & \\ \hline \quad\quad 4t & \end{array}$$

$$J = 2\int\left(t+1+\frac{4t}{t^2-3}\right) dt =$$

$$= t^2+2t+4\log|t^2-3| + k$$

$$I = x-1+2\sqrt{x-1} + 4\log|x-4| + K$$

nel caso 1 sarà $\log(4-x)$
nel caso 2 sarà $\log(x-4)$

4.

$$\int \sqrt{\frac{x-3}{x+2}} \, dx$$

1° caso $(a,b) =]-\infty,-2[$
2° caso $(a,b) = [3,+\infty[$

poniamo $\sqrt{\frac{x-3}{x+2}} = t$ $\quad t \geq 0$

ricavo x $\quad \frac{x-3}{x+2} = t^2 \Rightarrow x-3 = t^2x+2t^2 \Rightarrow x = \frac{2t^2+3}{1-t^2} = f(t)$

1° caso: $\frac{2t^2+3}{1-t^2} < -2 \Rightarrow \frac{2t^2+3}{1-t^2} + 2 < 0 \Rightarrow \frac{2t^2+3+2-2t^2}{1-t^2} < 0$

$\Rightarrow t^2-1>0 \Rightarrow (c,d) =]1, +\infty[$

$2°$ CASO: $\frac{2t^2+3}{1-t^2} - 3 \geq 0 \Rightarrow \frac{2t^2+3-3+3t^2}{1-t^2} \geq 0$

$\Rightarrow 1-t^2 \geq 0 \Rightarrow (c,d) = [0,1]$

$f(t) = \frac{2t+3}{1-t^2}$ $\quad f'(t) = \frac{4t(1-t^2)+2t(2t+3)}{(t^2-1)^2} =$

$= \frac{4t-4t^3+4t^3+6t}{(t^2-1)^2} = \frac{10t}{(t^2-1)^2} > 0$

$I = \left[\int t \frac{10t}{(t^2-1)^2} \, dt \right]_{t=\sqrt{\cdots}} = 10 \left[\int \frac{t^2}{(t^2-1)^2} \, dt \right]_{t=\sqrt{\frac{x-3}{x+2}}}$

$\frac{t^2}{(t^2-1)^2} = \frac{t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$

Esercizi di esame

1. trovare la prim in $]-\infty, +\infty[$ di $f(x) = \log(|x-1|+K)$ tale che $f(e^2-3) = 3$

$$f(x) = \begin{cases} \log(5-x) & x<1 \\ \log(x+3) & x \geq 1 \end{cases}$$

$\int \log(5-x)\, dx = x \log(5-x) - \int \frac{x-5+5}{x-5} \, dx$

$= x\log(5-x) - x - 5\log|x-5| + K$

$\int \log(x+3)\, dx = x\log(x+3) - \int \frac{x+3-3}{x+3} \, dx =$

$x \log(x+3) - x + 3 \log|x+3| + K$

$$f(x) \begin{cases} (x-5)\log(5-x) - x + K_1 & x < 1 \\ (x+3)\log(x+3) - x + K_2 & x \geq 1 \end{cases}$$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ $\quad -4\log 4 - 1 + K_1 = 4\log 4 - 1 + K_2$

$K_1 = K_2 + 8\log 4$

$$f(x) = \begin{cases} (x-5)\log(5-x) - x + K + 8\log 4 & x < 1 \\ (x+3)\log(x+3) - x + K & x \geq 1 \end{cases}$$

$e^2 - 3 > 1$ se $e^2 > 4$ vero

$f(e^2-3) = e^2 \log e^2 - e^2 + 3 + K = 3 \Rightarrow K = -e^2$

$$f(x) = \begin{cases} (x-5)\log(5-x) - x - e^2 + 8\log 4 & x < 1 \\ (x+3)\log(x+3) - x - e^2 & x \geq 1 \end{cases}$$

2. trovare f prim in $]-\infty, +\infty[$ di $f(x) = x\sin^2 x + x^2 \cos x$ tale che $f\left(\frac{\pi}{2}\right) = \frac{5}{16}\pi$

$\int (x \sin^2 x + x^2 \cos x)\, dx = \int \left(x \frac{1-\cos 2x}{2} + x^2 \cos x \right) dx =$

$\frac{1}{2}\int x\, dx - \frac{1}{2}\int x\cos 2x + x^2 \sin x - 2\int x \sin x\, dx = \frac{1}{4}x^2$

$= \frac{1}{4}x^2 - \frac{1}{2}\left(\frac{1}{2} x \sin 2x - \frac{1}{2}\int \sin 2x\, dx\right) + x^2 \sin x + x\cos x - \int \cos x\, dx$

$= \frac{1}{4}x^2 - \frac{1}{4} x \operatorname{sen} 2x - \frac{1}{8}\cos 2x + x^2 \sin x + x\cos x - \sin x + K$

$f\left(\frac{\pi}{2}\right) = \frac{5\pi^2}{8}$

$\frac{\pi^2}{16} + \frac{1}{8} + \frac{\pi^2}{4} - 1 + k = \frac{5}{16}\pi^2 - \frac{7}{8} + k = \frac{5}{16}\pi^2$ se $k = \frac{7}{8}$

3. trovare f prim in $]-\infty, +\infty[$ di $f(x) = e^{|x|} + \log \frac{2|x|+x+1}{|x|+1}$
tale che $f(1) = e$