R ha la seguente **proprietà di separazione**:

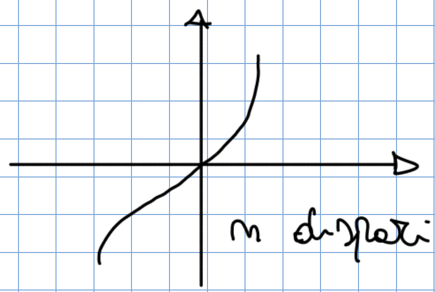
$a, b \in \mathbb{R}$ con $a \neq b$ $\exists\, r, s > 0$ $B(a,r) \cap B(b,s) = \emptyset$

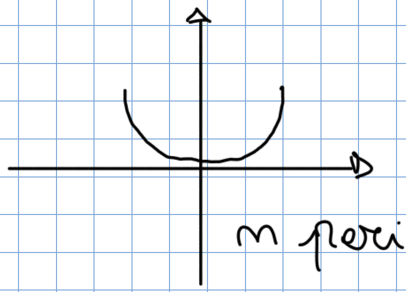
$a + r < b - s$

Funzioni elementari:

1) Funzione potenza

$n \in \mathbb{N}$ $\quad f(x) = x^n \quad \forall x \in]-\infty; +\infty[$





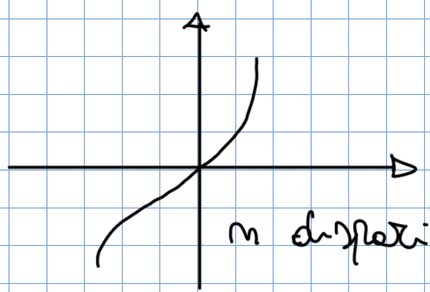
n dispari $\Rightarrow$ invertibile in $]-\infty; +\infty[$

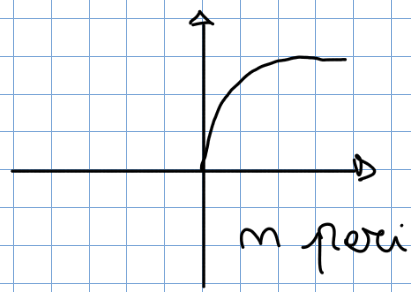
n pari $\Rightarrow$ // in $[0, +\infty[$ e $]-\infty, 0]$

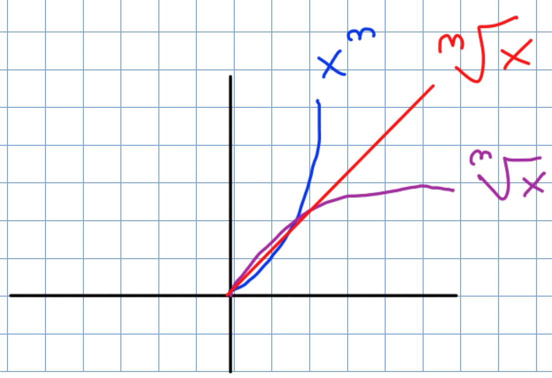
inversa: $\sqrt[n]{x}$

$f(x) = \sqrt[n]{x}$ $\quad \forall x \geq 0$ se n pari

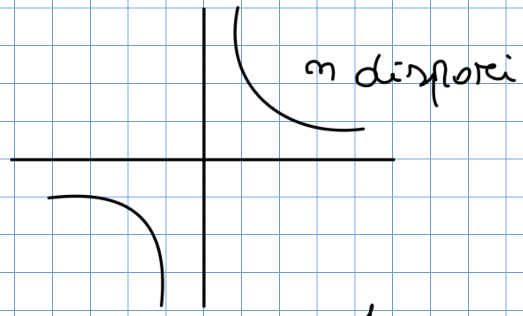
$\forall x \in \mathbb{R}$ se n dispari

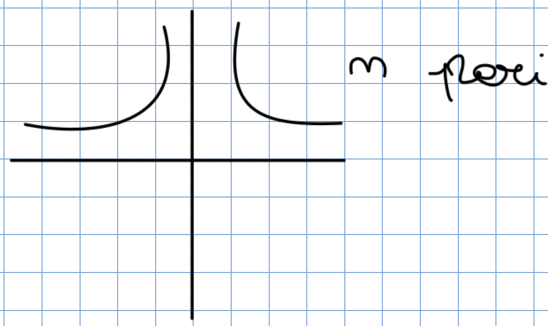






$n \in \mathbb{N}$ $\quad f(x) = x^{-n}$ $\quad \forall x \neq 0$ $\quad x^{-n} = \frac{1}{x^n}$





$n \in \mathbb{N}$ $\quad f(x) = x^{\frac{1}{n}}$ $\left(= \sqrt[n]{x} \text{ se } x \geq 0\right)$

$r = \frac{m}{n} \in \mathbb{Q}$ $\quad f(x) = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m$

$s \in \mathbb{R} - \mathbb{Q}$ $\quad f(x) = x^s$ $\quad \forall x \geq 0$ se $s > 0$

$\forall x > 0$ se $s < 0$

x^r $(r \in \mathbb{Q})$, x^s $(s \in \mathbb{R} - \mathbb{Q})$ all' inverse e $x^{\frac{1}{r}}$, $x^{\frac{1}{s}}$ sono strett cresc in $]0, +\infty[$

2) Polinomi

$f(x) = a_0 x^n + a_1 n^{n-1} + \dots + a_{n-1} x + a_n$

$a_0, \dots a_n \in \mathbb{R}$ $\quad n \in \mathbb{N}_0$ $\quad a_1 \dots a_n$ coefficienti

$\forall x \in \mathbb{R}$

$n = 0$ polinomio costante $f(x) = a_0$ $\forall x \in \mathbb{R}$

se $a_0 = 0$ polinomio nullo

$f(x) = a_0 x^n + \dots + a_n$

$g(x) = b_0 x^p + \dots + a_p$

sono uguali (cioè $f(x) = g(x)$ $\forall x$) se e solo se sono due polinomi identici cioè $n = p$ (grado)

(principio di identità del polinomio)

Dati 2 polinomi f, g $\exists\ q(x), r(x)$ con grado di r $<$ grado di g tali che $f(x) = g(x)\,q(x) + r(x)$

teorema di ruffini

il polinomio $f(x)$ è divisibile per $x - c \iff f(c) = 0$

3) Funzione razionale fratta

$$f(x) = \frac{a_0 x^n + \dots + a_n}{b_0 x^r + \dots + b_r}$$

$a_0, \dots, a_n, b_0 \dots b_r \in \mathbb{R}$

$n \in \mathbb{N}_0$

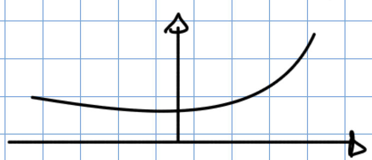
$r \in \mathbb{N}$

f è definita in $\mathbb{R} \setminus \{ c \in \mathbb{R} : b_0 c^r + \dots + b_r = 0 \}$

4) Funzione esponenziale

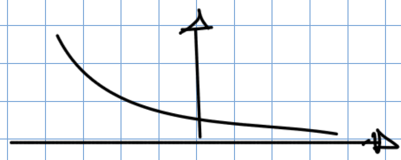
$a \in \mathbb{R}$ $\quad a > 0,\ a \neq 1$ $\quad f(x) = a^x$ $\quad \forall x \in \mathbb{R}$

$a^x > 0 \quad \forall x \in \mathbb{R}$



$a > 1$

strett crescente



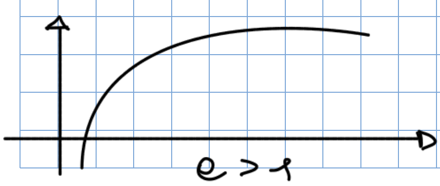
$0 < a < 1$

strett decresc

$f :]-\infty ; +\infty[\ \to\ ]0, +\infty[$

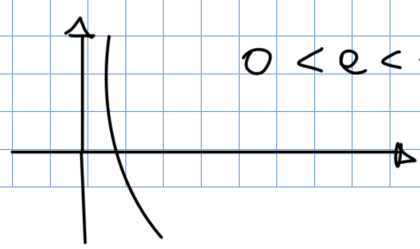
Funzione logaritmica

$f^{-1} :]0 ; +\infty[\ \to\ ]-\infty ; +\infty[$



$a > 1$

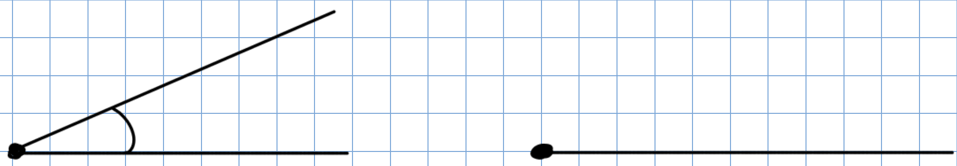
f^{-1} è la funzione logaritmo

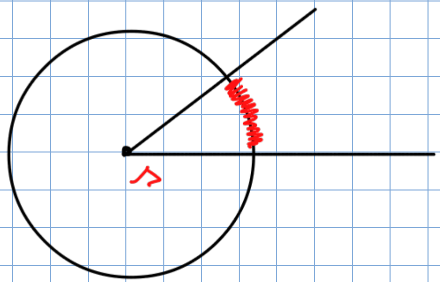


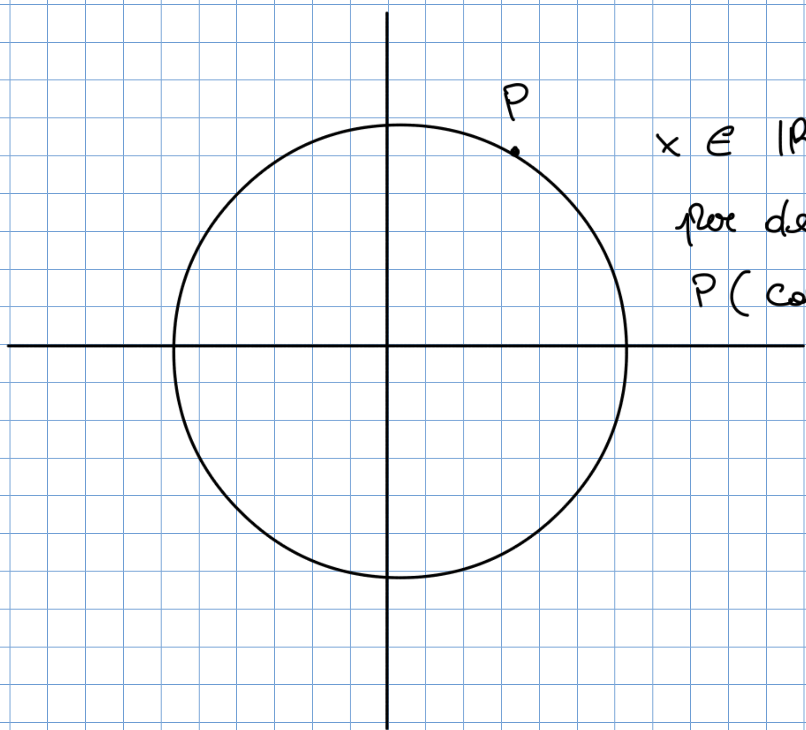
$0 < a < 1$

$f^{-1}(f(x)) = x$ $\quad f(f^{-1}(y)) = y$

5) funzioni trigonometriche







$x \in \mathbb{R}$

per definizione

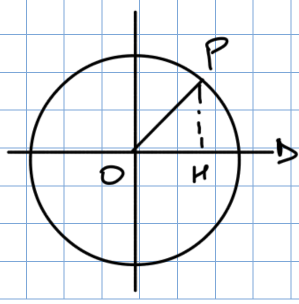
$P(\cos x, \sin x)$

Se $x \in \mathbb{R}$, $\cos x$ ($\text{sen}\, x$) è l'ascisse (l'ordinate) del secondo estremo di un arco della circonferenza trigonometrica avente primo estremo $A(1,0)$ e misura in radianti in x.

$-1 \leq \cos x \leq 1$

$-1 \leq \sin x \leq 1$ $\quad \forall x \in \mathbb{R}$

$\cos^2 x + \sin^2 x = 1 \quad \forall x \in \mathbb{R}$

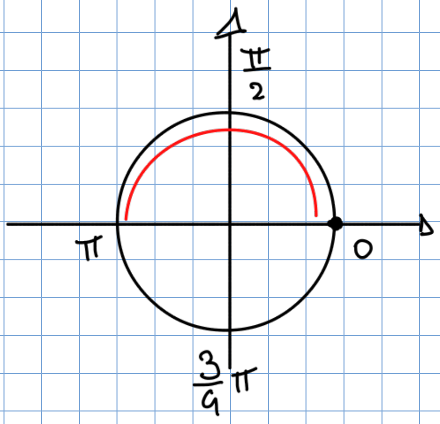


$PH = \sin x$

$OH = \cos x$

$OP = 1$

$$\left.\begin{aligned}\cos(x+2k\pi) &= \cos x\\ \sin(\quad // \quad) &= \sin x\end{aligned}\right\}\ \forall x\in\mathbb{R},\ \forall k\in\mathbb{R}$$



$\cos 0 = 1$ in $[0,\pi]$ $f(x)=\cos x$ è

$\cos\pi = -1$ strett. decr. e assume

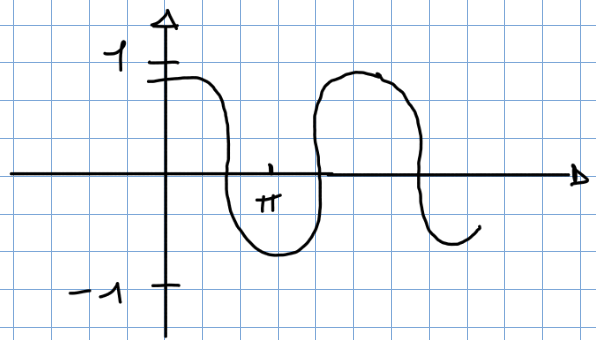
tutti i valori tra -1 e 1 quindi è

invertibile e abbiamo

$\cos: [0,\pi] \to [-1,1]$

$\arccos: [-1,1] \to [0,\pi]$

$\forall \alpha \in [-1;1]$ def $\arccos\alpha =$ l'unico $x\in[0,\pi]$: $\cos x = \alpha$



$\cos x = \alpha$ se $\alpha\in[-1;1]$ sol $\pm\arccos\alpha + 2k\pi$

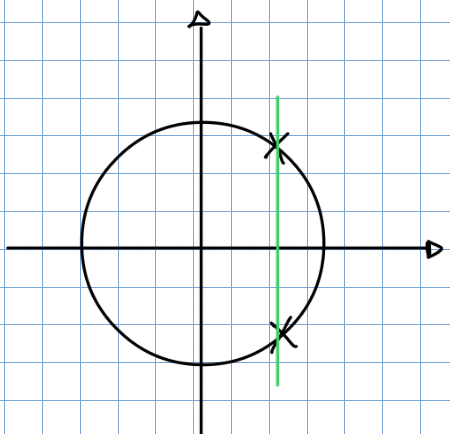
se $\alpha>1$ oppure $\alpha<-1$ non sol.

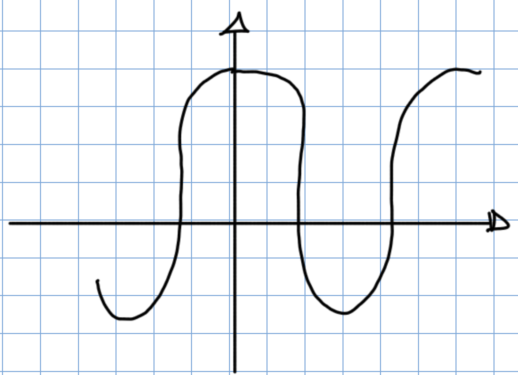
$\cos x > \alpha$ se $-1\le\alpha<1$ sol: $-\arccos\alpha + 2k\pi < x < \arccos\alpha + 2k\pi$

se $\alpha \ge 1$ nessuna soluzione

se $\alpha \le -1$ $\forall x\in\mathbb{R}$ è sol

$\cos x < \alpha$

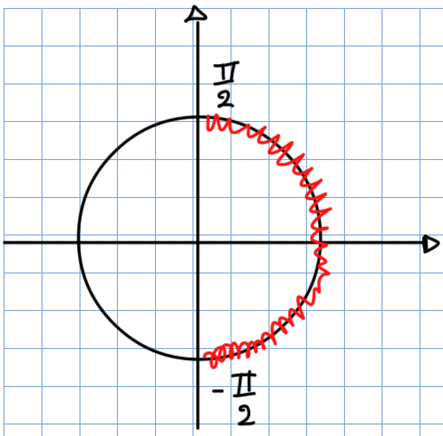




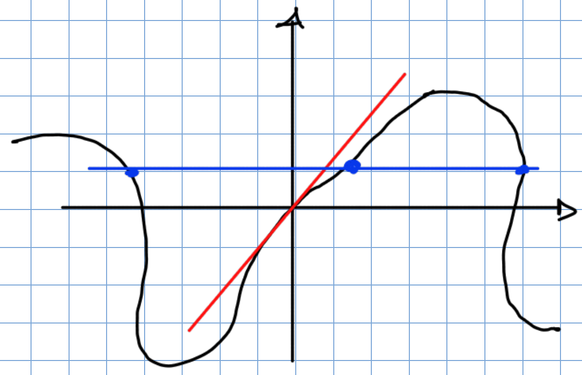
se $-1<\alpha<1$ $\arccos\alpha < x < 2\pi - \arccos\alpha$

se $\alpha \le -1$ nessuna soluzione

se $\alpha > 1$ $\forall x\in\mathbb{R}$ è sol



$\text{sen}(-x) = -\text{sen}\, x \quad \forall x$ (funz dispr)



$\sin\left(-\frac{\pi}{2}\right) = -1 \qquad \text{sen}\left(\frac{\pi}{2}\right) = 1$

in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ è strett cresc e assume tutti valori tra -1 e 1

$\sin: \left[\frac{\pi}{2}, -\frac{\pi}{2}\right] \to [-1, 1]$

$\text{arcsen}: [-1; 1] \to \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$\text{sen}\, x = d \qquad$ se $-1 \leq d \leq 1$

sol $x = \text{arcsen} + 2k\pi$

$\qquad \pi - \text{arcsen} + 2k\pi$

$\text{sen}\, x < d$

$-\pi - \text{arcsen}\, d + 2k < x < \text{arcsen}\, d + 2k\pi \qquad$ se $-1 < d \leq 1$

$\forall x \in R \qquad$ è sol se $d > 1$

non sol. $\qquad$ se $d \leq -1$

$-\pi - \text{arcsen}\, d + 2k\pi < x < \pi - \text{arcsen} + 2k = \qquad$ se $-1 < d < 1$

non sol $\qquad$ se $d \geq 1$

$\forall x \in R$ è sol $\qquad$ se $d < -1$

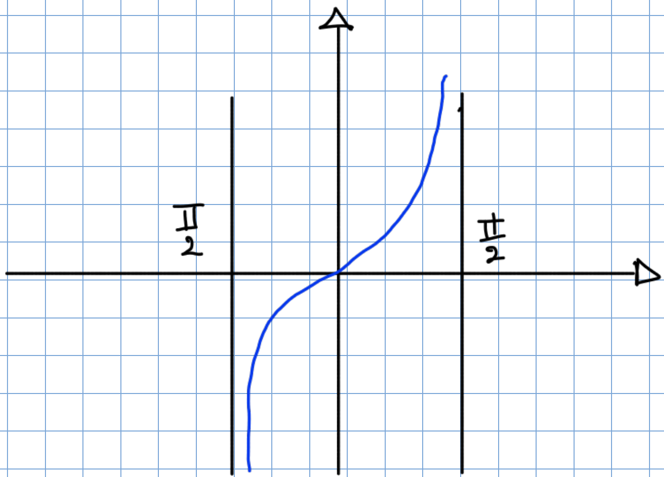
Formule di linearizzazione (Che cose sono?)

Definizione tangente

$$\tan x = \frac{\operatorname{sen} x}{\cos x} \qquad x \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$$

$$\tan(x + k\pi) = \tan x \quad \forall x, \ \forall k \in \mathbb{Z}$$

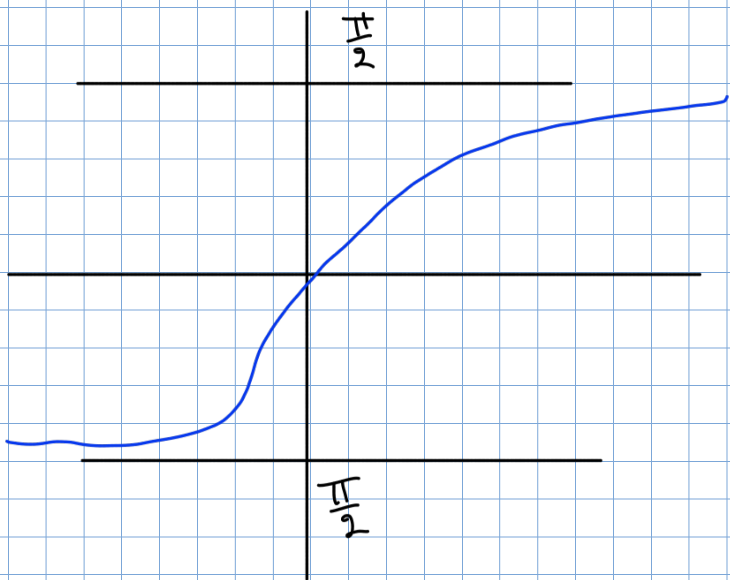
$$\tan(-x) = \frac{\operatorname{sen}(-x)}{\cos(-x)} = \frac{-\operatorname{sen} x}{\cos x} = -\tan x$$



in $]-\frac{\pi}{2}, \frac{\pi}{2}[$ $f(x) = \tan x$ è strett. cresc. e assume tutti i valori reali

$$\tan : \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\to]-\infty; +\infty[$$

$$\operatorname{arctan} :]-\infty; +\infty[\to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$



$\tan x = \alpha \qquad x = \operatorname{arctan} \alpha + k\pi \qquad \forall \alpha \in \mathbb{R}$

$\tan x > \alpha \qquad \operatorname{arctan} \alpha + k\pi < x < \frac{\pi}{2} + k\pi$

$\tan x < \alpha \qquad -\frac{\pi}{2} + k\pi < x < \operatorname{arctan} + k\pi$

$f(x) = \sqrt{\operatorname{arctan} x - \frac{x}{x^2-1}}$ è funz elementare

Esercizi sull'insieme di definizione

$$f(x) = 2^{\frac{\sqrt{x-1}}{|x|-3}}$$

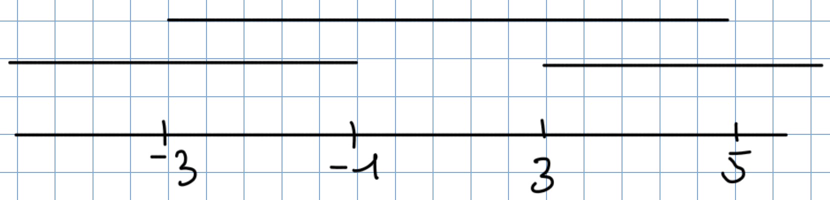
$$\begin{cases} x-1 \geq 0 \\ |x|-3 \neq 0 \end{cases} \quad \begin{cases} x \geq 1 \\ x \neq 3 \,;\; x \neq -3 \end{cases}$$

$$f: \left([1,3[\;\cup\;]3,+\infty[\right) \to \mathbb{R}$$

$$f(x) = \operatorname{arcsen}(|x-1|-3)$$

$$-1 \leq |x-1|-3 \leq 1 \quad \to \quad 2 \leq |x-1| \leq 4$$

$$\begin{cases} |x-1| \geq 2 \\ |x-1| \leq 4 \end{cases} \quad \begin{cases} x-1 \leq -2 \;\cup\; x-1 \geq 2 \\ -4 \leq x-1 \leq 4 \end{cases} \Rightarrow \begin{cases} x \leq -1 \;\cup\; x \geq 3 \\ -3 \leq x \leq 5 \end{cases}$$



$$f\left([-3,-1] \cup [3,5] \right) \to \mathbb{R}$$

$$f(x) = \sqrt[4]{\sqrt[3]{x-2} - 1}$$

$$\sqrt[3]{x-2} - 1 \geq 0 \quad \to \quad \sqrt[3]{x-2} \geq 1 \quad \to \quad x-2 \geq 1 \to x \geq 3$$

$$f: [3\,; +\infty[\;\to \mathbb{R}$$

e^x $\log x = \log_e x$ $e > 1$

$$f(x) = \log \sqrt{\frac{|x|-2}{x^2+3}}$$

$$\begin{cases} \sqrt{\dfrac{|x|-2}{x^2+3}} > 0 \\ \dfrac{|x|-2}{x^2+3} \geq 0 \quad \Rightarrow \quad |x|-2>0 \\ x^2+3 \neq 0 \end{cases}$$

$$\Downarrow$$

$$x>2 \vee x<-2$$

$$f: (]-\infty,-2[\cup]2,+\infty[\to \mathbb{R}$$

Numeri Complessi

$C = \{(a,b) : a,b \in \mathbb{R}\}$ coppie ordinate

$\boxed{(0,0) = 0 \quad \text{zero}}$

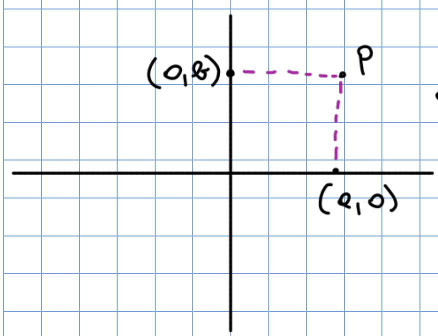
$z = (a,b)$ $\quad -z = (-a,-b)$ opposto

la nostra nuova x

$\bar{z} = (a,-b)$ coniugato

$(a,0) \longrightarrow$ numero complesso reale

$(0,b) \longrightarrow$ numero immaginario puro



$P(a,b) \to z = (a,b)$

2 numeri complessi z, w sono uguali se: $\quad z = (a,b)$, $w = (c,d)$

Definizione: $z = w$ se $a = c$, $b = d$

se $z \neq w$ non c'è un ordine

$i = (0,1)$ unità immaginaria

$1 = (1,0)$ unità reale

$z = (a, b)$
$w = (c, d)$

Definizione somma

$z + w = (a+c \; b+d)$ vale la prop commutativa

$z + 0 = 0$

$z + \bar{z} = (a+a, b-b) = (2a, 0)$ è un numero complesso reale

Definizione moltiplicazione

$z \cdot w = (ac - bd, ad + bc)$

$z\bar{z} = (a, b)(a, -b) = (a^2 + b^2, 0)$ è un numero complesso reale

$z = (a \; b) \quad |z| = \sqrt{a^2+b^2}$ modulo di z

se $z = (a, 0) \quad |z| = \sqrt{a^2+0} = |a|$

Dimostrare che ponendo $a \to (a, 0)$ si ottiene una corrispondenza biunivoca fra R e l'insieme complessi reali che conserva le operazioni

quindi si può scrivere $R \subseteq C \quad a = (a,0)$ identificazione

$$2 + 4 = 6 \quad \Rightarrow \quad (2,0) + (4,0) = (6,0)$$

$$2 \cdot 4 = 8 \quad \Rightarrow \quad (2,0) \cdot (4,0) = (8,0)$$

$$(2,0)\cdot(4,0) = (2 \cdot 4 - 0 \cdot 0, 2 \cdot 0 + 0 \cdot 4) = (8,0)$$

$z \cdot \bar{z} = |z|^2 = |\bar{z}^2| = |-z|^2$