$f:(a,b)\to\mathbb{R}$ $\quad c\,(a,b)$ $\quad r(x)=\dfrac{f(x)-f(c)}{x-c}$

$$R(h)=\frac{f(c+h)-f(c)}{h}$$

$$f'(c)=\lim_{x\to c} r(x)=\lim_{h\to 0} R(h)$$

$$r(x)=f(c)+f'(c)(x-c) \qquad y=f(c)+f'(c)(x-c)$$

$$\lim_{x\to c}\frac{f(x)-r(x)}{x-c}=0$$

Regole di derivazione

$f:(a,b)\to\mathbb{R}$ in $c\in(a,b)$ $\quad K\in\mathbb{R}$

$g(x)=Kf(x)$ si ha: $\exists\, g'(c)=Kf'(c)$

$$\frac{kf(x)-kf(c)}{x-c}=k\,\frac{f(x)-f(c)}{x-c}\to kf'(c)$$

2) $f,g:(a,b)\to\mathbb{R}$ in $c\in(a,b)$

$s(x)=f(x)+g(x)$ si ha $s'(c)=f'(c)+g'(c)$

$$\frac{s(x)-s(c)}{x-c}=\frac{f(x)+g(x)-f(c)-g(c)}{x-c}=\underbrace{\frac{f(x)-f(c)}{x-c}}_{f'(c)}+\underbrace{\frac{g(x)-g(c)}{x-c}}_{g'(c)}$$

3) $f,g:(a,b)\to\mathbb{R}$ in $c\in(a,b)$

$r(x)=f(x)g(x)$ si ha $r'(c)=f'(c)g(c)+f(c)g'(c)$

NO DIM

4)

$f:(a,b) \to \mathbb{R}$ in $c \in (a,b)$ $f(c) \neq 0$

($f$ derivabile $\Rightarrow$ $f$ cont in $c$ $f(c) \neq 0 \Rightarrow f(x) \neq 0$ in un intorno di $c$)

$F(x) = \dfrac{1}{f(x)}$ (esiste in un intorno di $c$)

Si ha: $\exists\, F'(c) = \dfrac{-f'(c)}{(f(c))^2}$

NO DIM

5) $f, g: (a,b) \to \mathbb{R}$ der. in $c \in (a,b)$ $g(c) \neq 0$

$q(x) = \dfrac{f(x)}{g(x)}$ Si ha $q'(c) = \dfrac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$

infatti

$q(x) = f(x)\dfrac{1}{g(x)} \Rightarrow q'(c) = f'(c)\dfrac{1}{g(c)} + f(c)\dfrac{-g'(c)}{(g(c))^2} = \dfrac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$

teorema di derivazione delle funzioni composte

IP: $f: (\alpha,\beta) \to \mathbb{R}$ $g: (a,b) \to (\alpha,\beta)$ $c \in (a,b)$

$F(x) = f(g(x))$

$\exists\, g'(c)$, $\exists\, f'(g(c))$

NO DIM

TS: $\exists\, F'(c) = f'(g(c))\,g'(c)$

## teorema di derivazione delle funzioni inverse

$f : (a, b) \to (\alpha, \beta)$ strett crescente (o decres) e continua

$f^{-1} : (\alpha, \beta) \to (a, b)$ sappiamo //

Sia $\gamma \in (\alpha, \beta)$ $\quad \gamma = f(c)$, $c \in (a, b)$

$\exists f'(c) \neq 0$

TS $\exists (f^{-1})'(\gamma) = \dfrac{1}{f'(c)}$ NO DIM

## teorema

Materialmente per trovare $(f^{-1})'(\gamma)$ si deve risolvere l'eq $f(x) = \gamma$ per trovare $c$, calcolare $f'(c)$ se è $\neq 0$ si conclude

## Derivate delle funzioni elementari

$f(x) = K \quad \forall x \in \mathbb{R} \Rightarrow f'(x) = 0 \quad \forall x \in \mathbb{R}$

$f(x) = x \quad$ " $\quad \Rightarrow f'(x) = 1$

$f(x) = x^2 \qquad r(x) = \dfrac{x^2 - c^2}{x - c} = \dfrac{\cancel{(x-c)}(x+c)}{\cancel{(x-c)}} \to 2c$

$\Rightarrow f'(x) = 2x \quad \forall x \in \mathbb{R}$

$f(x) = x^3 \qquad r(x) = \dfrac{x^3 - c^3}{x - c} = \dfrac{\cancel{(x-c)}(x^2 + xc + c^2)}{\cancel{(x-c)}} \to 3c^2$

$\Rightarrow f'(x) = 3x^2 \quad \forall x \in \mathbb{R}$

$\vdots$

$f(x) = x^n \qquad f'(x) = nx^{n-1} \quad \forall x \in \mathbb{R}$

$$\frac{x^n - c^n}{x - c} = \frac{(x-c)\left(x^{n-1} + x^{n-2}c + \ldots + x c^{n-2} + c^{n-1}\right)}{(x-c)} \to n c^{n-1}$$

$$f(x) = x^{-n} = \frac{1}{x^n} \qquad f'(x) = \frac{-n x^{n-1}}{x^{2n}} = -n x^{n-1-2n} =$$

$$-n x^{-n-1} \quad (x \neq 0)$$

$$f(x) = e^x \qquad r(x) = \frac{e^x - e^c}{x-c} = e^c \underbrace{\frac{e^{x-c} - 1}{x-c}}_{\downarrow\, 1} \to e^c$$

$$f'(x) = e^x \quad \forall x \in \mathbb{R}$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} = \log e$$

$$f(x) = a^x \quad (a > 0,\ a \neq 1)$$

$$f'(x) = a^x \log a$$

$$\lim_{t \to 0} \frac{\log(t+1)}{t} = 1$$

$$f(x) = \log x \qquad c > 0 \qquad r(x) = \frac{\log x - \log c}{x - c} = \underbrace{\frac{\log\left(\frac{x}{c} - 1 + 1\right)}{c\left(\frac{x}{c} - 1\right)}}_{1}$$

$$f'(x) = \frac{1}{x} \quad \forall x > 0$$

possiamo vederla anche come funzione inversa

$$f(x) = e^x : \ ]-\infty; +\infty[ \ \to \ ]0, +\infty[$$

$$f^{-1}(y) = \log y : \ ]0, +\infty[ \ \to \ ]-\infty, +\infty[$$

$$y \in \ ]0, +\infty[ \qquad c : f(c) = y \quad e^c = y \Rightarrow c = \log y$$

$$f'(c) = \log y \neq 0 \qquad (f^{-1})'(y) = \frac{1}{\log y} = \frac{1}{c}$$

Se $e \neq c$ $f(x) = \log_e x \Rightarrow f'(x) = \frac{1}{x} \log_e e$

$f(x) = |x|$ $\nexists f'(0)$ ma $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} = \frac{|x|}{x}$

("segno di $x$")

Sia $f$ deriv e $\neq 0$ e cons $F(x) = |f(x)|$

$$F' = \frac{|f(x)|}{f(x)} f'(x)$$

Consideriamo ora $F(x) = \log|f(x)|$ $F'(x) = \frac{1}{\cancel{|f(x)|}} \frac{\cancel{|f(x)|}}{f(x)}$

$$\downarrow$$

$$F'(x) = \frac{f'(x)}{f(x)}$$

es: $F(x) = \log|x^3 - 3x|$ in tutti i punti in cui $x^3 - 3x \neq 0$ si ha

$$F'(x) = \frac{3x^2 - 3}{x^3 - 3x}$$

$f(x) = x^\alpha$ ($\alpha$ non intero) def in $[0, +\infty[$ se $\alpha > 0$

in $]0, +\infty[$ se $\alpha < 0$

$c > 0$ $t > 0$ $t = e^{\log t}$

$$x^\alpha = e^{\log x^\alpha} = e^{\alpha \log x}$$

$$f'(x) = e^{\alpha \log x} \cdot \alpha \cdot \frac{1}{x} = x^\alpha \alpha \frac{1}{x} = \alpha x^{\alpha - 1}$$

es: $\alpha = \frac{1}{2}$ $f(x) = \sqrt{x}$ $f'(x) = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2\sqrt{x}}$ $x > 0$

se $\alpha < 1$ $\quad \nexists f'(0)$ $\quad \dfrac{x^\alpha - c^\alpha}{x-c}$ diverge per $x \to c$

invece di $f'(c)$ si può scrivere $\left(D(f(x)\right)_{x=c}$

es: $D(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

con $f(x) = x^n \quad n \in \mathbb{N}$

$f'(x) = n x^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f'''(x) = n(n-1)(n-2)x^{n-3}$

$\vdots$

$f^{(n)}(x) = n(n-1)(n-2)\dots 1 \cdot x^0 = n!$

$f^{(k)}(x) = 0 \quad \forall k > n$

Funzioni trigonometriche

$f(x) = \cos x \qquad R(h) = \dfrac{\cos(c+h) - \cos c}{h} =$

$= \dfrac{\cos c \cos h - \sin c \sin h - \cos c}{h} = \cos c \ \dfrac{\cos h - 1}{h} -$

$\sin c \ \dfrac{\sin h}{h} \to -\sin c$

$f(x) = \cos x : [0, \pi] \to [-1, 1]$

$f^{-1}(x) = \arccos [-1,1] \to [0,\pi]$

$\gamma \in [-1, 1] \quad \gamma = \cos c \quad c \in [0, \pi] \quad f'(c) = \underset{\neq 0}{-\sin c}$

per $c \in ]0, \pi[$

$\Downarrow$

$\gamma \in ]-1, 1[$

$(f^{-1})'(\gamma) = \dfrac{1}{-\sin c} = -\dfrac{1}{\sin c} = -\dfrac{1}{\sqrt{1-\cos^2 c}} =$

perchè $\sin^2 c = 1 - \cos^2 c$ e $\sin c > 0$

$$= -\frac{1}{\sqrt{1-\cos^2(\arccos y)}} = \frac{1}{\sqrt{1-y^2}} \quad \forall y \in ]-1,1[$$

$$f(x) = \sin x \qquad R(h) = \frac{\sin(c+h) - \sin c}{h} =$$

$$\frac{\sin c \cos h + \cos c \sin h - \sin c}{h} = \sin c \, \frac{\cos h - 1}{h} +$$

$$\cos c \, \frac{\sin h}{h} \to \cos c \qquad f'(x) = \cos x \quad \forall x$$

$$f(x) = \sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$$

$$f^{-1}(y) = \arcsin y : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad y \in [-1,1]$$

$$c \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] : \sin c = y \qquad c = \arcsin y$$

$$f'(c) = \cos c \neq 0 \text{ se } c \neq \pm\frac{\pi}{2} \Leftrightarrow y \neq \pm 1$$

$$(f^{-1})'(y) = \frac{1}{\cos c} = \frac{1}{\sqrt{1-\sin^2 c}} = \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} =$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$f(x) = \operatorname{tg} x = \frac{\sin x}{\cos x} \qquad x \neq \frac{\pi}{2} + k\pi$$

$$f'(x) = \frac{\cos x \cos x - \sin(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \operatorname{tg}^2$$

$$f(x) = \operatorname{tg} x : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \to ]-\infty, +\infty[$$

$$f^{-1}(y) = \arctan y : ]-\infty; +\infty[ \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$y \in ]-\infty; +\infty[ \qquad y = \tan c \qquad c \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$

$f'(c) = 1 + tg^2 c$

$$(f^{-1})'(y) = \frac{1}{1+tg^2 c} = \frac{1}{1+tg^2(\arctan y)} = \frac{1}{1+y^2}$$

Esercizi

$$f(x) = 3x^4 - \cos x + e^x$$

$$f'(x) = 12x^3 + \operatorname{sen} x + e^x$$

$$f(x) = (\sin x)\sqrt{x}$$

$$f'(x) = (\cos x)\sqrt{x} + \sin x \frac{1}{2\sqrt{x}}$$

$$f(x) = \frac{\tan x}{x^6}$$

$$f'(x) = \frac{(1+tg^2 x)x^6 - tg x(6x^5)}{x^{12}}$$

$$f(x) = \frac{\arcsin x}{\arccos x}$$

$$f'(x) \frac{\left(\arcsin x + \frac{x}{\sqrt{1+x^2}}\right)\arccos x + \frac{1}{\sqrt{1-x^2}} \times \arcsin x}{\arccos^2 x}$$

$f(x) = \sin(4x^2-1) \qquad f'(x) = \cos(4x^2-1)\cdot 8x$

$f(x) = e^{\sqrt{x}} \qquad f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$

$f(x) = \operatorname{arctg} \frac{2x}{x^2-1} \qquad f'(x) = \frac{1}{1+\frac{4x^2}{(x^2-1)^2}} \; \frac{2(x^2-1)-2x\cdot 2x}{(x^2-1)^2}$

$$f(x) = \sqrt[3]{x - \sqrt{\sin x}}$$

$$f'(x) = \frac{1}{3}\left(x - \sqrt{\sin x}\right)^{\frac{1}{3}-1} \left(1 - \frac{\cos x}{2\sqrt{\sin x}}\right)$$

$$f(x) = e^{\arctan x}$$

$$f'(x) = e^{\arctan x} \cdot \frac{1}{1+x^2}$$

$$f(x) = e^{\sin \frac{1}{x}}$$

$$f'(x) = e^{\sin \frac{1}{x}} \cdot \left(\cos \frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$$

$$f(x) = \cos \sqrt{x^2 - \frac{1}{x}}$$

$$f'(x) = -\operatorname{sen} \sqrt{x^2 - \frac{1}{x}} \cdot \frac{1}{2\sqrt{x^2 - \frac{1}{x}}} \cdot \left(2x + \frac{1}{x^2}\right)$$

$$f(x) = \arctan \sqrt{\frac{1+x}{1-x}}$$

$$f'(x) = \frac{1}{1 + \frac{1+x}{1-x}} \cdot \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \cdot \frac{1(1-x) \cdot (-1)(1+x)}{(1-x)^2}$$

$$f(x) = \log 2x \qquad f'(x) = \frac{1}{2x} \cdot 2 = \frac{2}{2x}$$

$$f(x) = \log \sqrt{\frac{x}{x-1}}$$

$$f'(x) = \frac{1}{\sqrt{\frac{x}{x-1}}} \cdot \frac{1}{2\sqrt{\frac{x}{x-1}}} \cdot \frac{1(x-1) - x \cdot 1}{x^2+1-2x}$$

$$D(\log|f|) = \frac{f'}{f}$$

$$f(x) = \log|x^4+3x|$$

$$f'(x) = \frac{4x^3+3}{x^4+3x}$$

$$f(x) = \log|\sqrt[4]{x} - x^3$$

$$f'(x) = \frac{\frac{1}{4} \cdot \frac{1}{\sqrt[4]{x^3}} - 3x^2}{\sqrt[4]{x} - x^3}$$

$$f(x) = \frac{\log \frac{3x}{x-1}}{\arcsin x^2}$$

$$f'(x) = \frac{\frac{x-1}{3x} \cdot \frac{3(x-1)-3x}{(x-1)^2} \cdot \arcsin x^2 - \log \frac{3x}{x-1} \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x}{\arcsin^2 x^2}$$

Imparare le regole di derivazione

$$f(x) = \sin(\sin x)$$

$$f'(x) = \cos(\sin x) \cdot \cos x$$

$$f(x) = \cos(\cos(\cos x))$$

$$f'(x) = -\sin(\cos(\cos x))\,(-\sin(\cos x)(-\sin x)$$

$$f(x) = x^x = e^{\log x^x} = e^{x \log x}$$

$$f'(x) = e^{x\log x} \cdot \left[1 \cdot \log + x \, \frac{1}{x}\right] = x^x(\log x + 1)$$

$f(x) = (\sin x)^{\sin x} = e^{\sin \log \sin x}$

$f'(x) = \operatorname{sen} x^{\operatorname{sen} x} \left[ \cos x \log(\sin x) + \sin x \frac{\cos x}{\sin x} \right]$

Per casa

$f(x) = \log \frac{1}{\operatorname{tg} x}$ $\qquad$ $f(x) = \sqrt{\cos^3 x - \sin x}$

$f(x) = e^{\frac{1}{\operatorname{arctg} x}}$ $\qquad$ $f(x) = \log \left| \sqrt[5]{x^2 - x} \right|$

$f(x) = \operatorname{tg}(\log(\cos x))$ $\qquad$ $f(x) = \sqrt{x^2 - 1}\, \sqrt[3]{3x^4 - 2}$

## Applicazioni del calcolo differenziale allo studio delle funzioni

### Criterio di monotonia globale

Se $f'(c) > 0$ allora $f$ è cresc in $c$

infatti $f'(c) > 0 \Rightarrow r(x) > 0$ in un intorno di $c$

$\Rightarrow f$ cresc in $c$

### Criterio di stretta monotonia

Se $f'(x) > 0 \quad \forall x \in (a, b) \Rightarrow f$ è strett. cresc in $(a, b)$

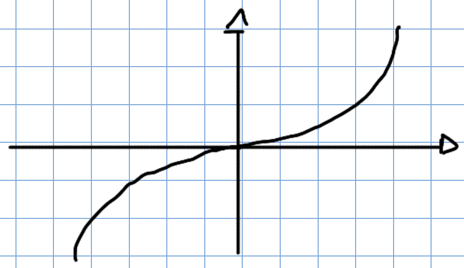
infatti è cresc in ogni punto per il teorema prec.

Il viceversa non vale, nemmeno nella monotonia locale

$f$ cresc in $c \Rightarrow f'(c) \geq 0$ (se fosse $< 0$ $f$ sarebbe decresc.)

consideriamo $f(x) = x^3$ $\quad f'(x) = 3x^2$

$f'(0) = 0$ ma $f$ è cresc strett



Ci serve allora un criterio di stretta monotonia in cui $f'$ possa anche essere 0. Per arrivarci dobbiamo studiare i teoremi fondamentali sul calcolo differenziale

Teorema di Fermat

$f : (a, b) \to \mathbb{R}$ $\quad c \in ]a, b[$ punto di estremo relativo

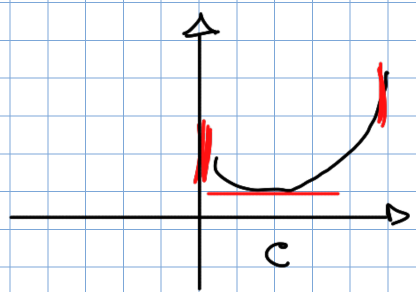
$\exists\, f'(c)$

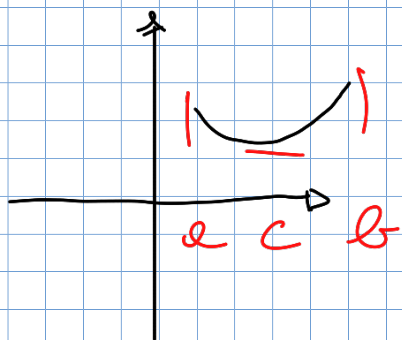
$f'(c) = 0$

Osservazioni

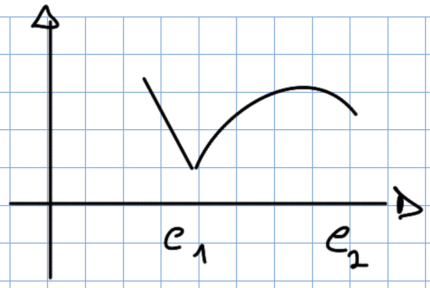
1) non vale il viceversa $f(x) = x^3$, $c = 0$

2) non vale se $c = a$ opp $c = b$




3) Cercheremo i punti di estr rel fra i punti interni in cui la derivata è zero (punti stazionari) e quelli in cui non c'è derivata




$C_2$ punti stazionari