5 novembre 2025_MZ

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$y' + a(x)y = f(x)$ (1)

$y' + a(x)y = 0$ (2)

y, z sol di (1) $\Rightarrow$ $\forall x \in (\alpha, \beta)$ $y'(x) + a(x)y(x) = f(x)$

$z'(x) + a(x)z(x) = f(x)$

Sia $w = y - z$

$$w'(x) + a(x)w(x) = y'(x) - z'(x) + a(x)(y(x) - z(x)) =$$

$$= \underbrace{\left(y'(x) + a(x)y(x)\right)}_{= f(x)} - \underbrace{\left(z'(x) + a(x)z(x)\right)}_{= f(x)} = 0$$

$y'' + a(x)y' + b(x)y = 0$ Siano y, z due sol., $h, k \in \mathbb{R}$

dim. che $w = hy + kz$ è sol

$w'(x) = hy'(x) + kz'(x)$

$w''(x) = hy''(x) + kz''(x)$

$$w''(x) + a(x)w'(x) + b(x)w(x) = hy''(x) + kz''(x) + a(x)hy'(x) + a(x)kz'(x) +$$

$$+ b(x)hy(x) + b(x)kz(x) = h\left(y''(x) + a(x)y'(x) + b(x)y(x)\right) +$$

$$+ k\left(z''(x) + a(x)z'(x) + b(x)z(x)\right) = h \cdot 0 + k \cdot 0 = 0$$

ESERCIZI

1. $y'' + 5y = 0$

eq. caratt. $\alpha^2 + 5 = 0$ $\alpha = \pm\sqrt{5}\,i$

int gen. $y(x) = k_1 \cos\sqrt{5}x + k_2 \sin\sqrt{5}x$

$y'' + 5 = 0$

$y'' = -5$

$y' = -5x$

$y = -\frac{5}{2}x^2$

2. $y'' - 5y = 0$

eq. caratt. $\alpha^2 - 5 = 0$ $\alpha = \pm\sqrt{5}$

int gen. $y(x) = k_1 e^{\sqrt{5}x} + k_2 e^{-\sqrt{5}x}$

3. $y'' - 5y' = 0$

eq caratt $\alpha^2 - 5\alpha = 0$ $\alpha = 0$, $\alpha = 5$

int gen. $y(x) = k_1 + k_2 e^{5x}$

4. 1) $y^{IV} - 3xy'' + \log y = x^2 e^x$

2) $y''' - \frac{y''}{\cos x} + (\log x)y + \frac{y}{x} = \sin x$

3) $y' = \frac{y}{x}$

4) $y' = \frac{x}{y}$

5) $y'' - \frac{x+1}{y'} + 3y = 0$

6) $y'' - \sqrt{xy} = 3x$

7) $y'' + xy' + y^2 = 0$

8) $y' + (\log x)y = x^2$

9) $y' + x(\log y) = x^2$

10) $y'' + x^2y' + xy^2 = 3$

QUALI SONO LINEARI ?

2 3 8

4. $y^{IV} + y''' - 2y'' = xe^x$

EQ OMOG. $y^{IV} + y''' - 2y'' = 0$

EQ CARATT $\alpha^4 + \alpha^3 - 2\alpha^2 = 0$

$\alpha^2(\alpha^2 + \alpha - 2) = 0$ $\alpha = 0$ $s = 2$

$\alpha = \frac{-1 \pm 3}{2} \begin{cases} -2 \\ 1 \end{cases}$

int gen. omog. $y(x) = k_1 + k_2 x + k_3 e^{-2x} + k_4 e^x$

$f(x) = xe^x$ $h = 1$ $s = 1$ $m = 1$ $p(x)\,e^{hx}$

$y(x) = e^{x}\, x\,(ax+b) = e^{x}(ax^2+bx)$

$y'(x) = e^{x}(ax^2+bx+2ax+b)$

$y''(x) = e^{x}(ax^2+bx+4ax+2b+2a)$

$y'''(x) = e^{x}(ax^2+bx+4ax+2b+2a+2ax+b+4a) =$

$= e^{x}(ax^2+6ax+bx+6a+3b)$

$y^{IV}(x) = e^{x}(ax^2+6ax+bx+6a+3b+2ax+6a+b)$

$y^{IV}+y'''-2y'' = x\,e^{x}$

sost nell' eq.

$e^{x}(\cancel{ax^2}+8ax+\cancel{bx}+12a+4b+\cancel{ax^2}+6ax+\cancel{bx}+6a+3b-\cancel{2ax^2}-\cancel{2bx}-8ax-$

$-4b-4a) = x\,e^{x}$ $\begin{cases} 6a = 1 \\ 14a+3b=0 \end{cases}$ $a=\frac{1}{6}$ $b=-\frac{7}{9}$

INT GEN $y(x) = k_1+k_2x+k_3e^{-2x}+k_4e^{x}+e^{x}\left(\frac{1}{6}x^2-\frac{7}{9}x\right)$

6. PC $\begin{cases} y''+y = x^2+3 \\ y(0)=1 \\ y'(0)=0 \end{cases}$

eq. omog. $y''+y=0$

eq. caratt. $\alpha^2+1=0$ $\alpha=\pm i$ int. gen. omog. $y(x)=k_1\cos x+k_2\sin x$

$f(x)=e^{0}(x^2+3)$ $m=2$ $h=0$ $s=0$

cerco $y(x) = ax^2+bx+c$

$y'(x) = 2ax+b$

$y''(x) = 2a$

sost. nell' eq. $2a+ax^2+bx+c = x^2+3$ $\begin{cases} a=1 \\ b=0 \\ 2a+c=3 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \\ c=1 \end{cases}$

int. gen. $y(x)=k_1\cos x+k_2\sin x+x^2+1$

$y'(x) = -k_1\sin x+k_2\cos x+2x$

$\begin{cases} y(0)=1 \\ y'(0)=0 \end{cases} \Rightarrow \begin{cases} k_1+1=1 \\ k_2=0 \end{cases}$ SOL $y(x)=x^2+1$

7. $y''+4y'+3y = \frac{x+3}{e^{x}}+2x\sin x$ (1)

$y''+4y'+3y = e^{-x}(x+3)$ (2)

$y''+4y'+3y = 2x\sin x$ (3) $\Leftarrow y''+4y'+3y = 2x\,e^{ix}$ (4)

un integr. partic. della (1) si ottiene sommando uno della (2) e uno della (3) (che è la parte immag. di uno della (4))

eq. omog. $y''+4y'+3y=0$

eq. caratt. $\alpha^2+4\alpha+3=0$ $\alpha=-2\pm1$ $\begin{matrix} -1 \\ -3 \end{matrix}$

int gen omog $y(x) = k_1e^{-x}+k_2e^{-3x}$

troviamo un int. partic. della (2) $f(x)=e^{-x}(x+3)$

$m=1$ $h=-1$ $s=1$

cerco $y(x)=e^{-x}(ax^2+bx)$

$y'(x)=e^{-x}(-ax^2-bx+2ax+b)$

$y''(x)=e^{-x}(ax^2+bx-4ax-2b+2a)$

sost nell'eq.

$e^{-x}(\cancel{ax^2}+\cancel{bx}-4ax-2b+2a-\cancel{4ax^2}-\cancel{4bx}+8ax+4b+\cancel{3ax^2}+\cancel{3bx}) = \cancel{e^{-x}}(x+3)$

$4ax+2a+2b=x+3$ $\begin{cases} 4a=1 \\ 2a+2b=3 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{4} \\ b=\frac{5}{4} \end{cases}$

$y(x)=e^{-x}\left(\frac{1}{4}x^2+\frac{5}{4}x\right)$

Troviamo un int partic. della (4) $f(x)=2x\,e^{ix}$

$h=i$ $s=0$ $m=1$

cerco $y(x)=e^{ix}(ax+b)$

$y'(x)=e^{ix}(iax+ib+a)$

$y''(x)=e^{ix}(-ax-b+2ia)$

sost. nell' eq.

$$\cancel{e^{ix}}\left(-ax-b+2ia+4iax+4ib+4a+3ax+3b\right)=2x\cancel{e^{ix}}$$

$a(2+4i)x+4a+2b+4ib=2x$ $\qquad$ $a(2+4i)=2 \Rightarrow a=\dfrac{2}{2+4i}=\dfrac{4-8i}{20}=$

$$=\frac{1}{5}-\frac{2}{5}i$$

$\dfrac{4}{5}-\dfrac{8}{5}i+2b+4ib=0 \Rightarrow b\,(2+4i)=-\dfrac{4}{5}+\dfrac{8}{5}i$

$$b\,(5+10i)=-2+4i$$

$$b=\frac{-2+4i}{5+10i}=\frac{(-2+4i)(5-10i)}{125}=$$

$$=\frac{30+40i}{125}=\frac{6}{25}+\frac{8}{25}i$$

$$y(x)=\left(\left(\frac{1}{5}-\frac{2}{25}i\right)x+\frac{6}{25}+\frac{8}{25}i\right)(\cos x+i\sin x)$$

La parte immag. è $\quad v(x)=-\dfrac{2}{25}x\cos x+\dfrac{6}{25}\cos x+\dfrac{1}{5}x\sin x+\dfrac{6}{25}\sin x$

INT GEN $\quad y(x)=k_1e^{-x}+k_2e^{-3x}+e^{-x}\left(\dfrac{1}{4}x^2+\dfrac{5}{4}x\right)+v(x)$

8. $\begin{cases} y''+y=x\cos x & (1)\\ y(\pi)=\frac{5}{4}\pi \\ y'(\pi)=\frac{5}{4}\end{cases}$ $\qquad$ eq. omog. $\;y''+y=0$

" caratt. $\;\alpha^2+1=0 \quad \alpha=\pm i$

int gen omog $\;y(x)=k_1\cos x+k_2\sin x$

troviamo un int parte di $\;y''+y=xe^{ix}\;$ e poi prenderemo la parte reale.

$f(x)=xe^{ix} \qquad m=1 \quad h=i \quad s=1$

cerco $\;y(x)=e^{ix}(ax^2+bx)$

$y'(x)=e^{ix}(iax^2+ibx+2ax+b)$

$y''(x)=e^{ix}(-ax^2-bx+4iax+2ib+2a)$

sost nell' eq

$$\cancel{e^{ix}}\left(-\cancel{ax^2}-\cancel{bx}+4iax+2ib+2a+\cancel{ax^2}+\cancel{bx}\right)=x\cancel{e^{ix}}$$

$\begin{cases} 4ia=1\\ 2a+2ib=0\end{cases} \Rightarrow \quad a=\dfrac{1}{4i}=-\dfrac{1}{4}i$

$\qquad -\dfrac{1}{2}i+2ib=0 \qquad b=\dfrac{1}{4}$

$y(x)=\frac{1}{4}(\cos x+i\sin x)(ix^2+x)$

la parte reale è $\quad u(x)=\frac{1}{4}(x\cos x-x^2\sin x)$

INT GEN $\quad y(x)=k_1\cos x+k_2\sin x+\frac{1}{4}x\cos x-\frac{1}{4}x^2\sin x$

$$y'(x)=-k_1\sin x+k_2\cos x+\frac{1}{4}\cos x-\frac{1}{4}x\sin x-\frac{1}{2}x\sin x-\frac{1}{4}x^2\cos x$$

$\begin{cases} y(\pi)=\frac{5}{4}\pi\\ y'(\pi)=\frac{5}{4}\end{cases}$ $\quad \begin{cases} -k_1-\frac{\pi}{4}=\frac{5}{4}\pi\\ -k_2-\frac{1}{4}+\frac{\pi^2}{4}=\frac{5}{4}\end{cases}$ $\Rightarrow$ $\quad \begin{aligned} k_1&=-\tfrac{3}{2}\pi\\ k_2&=\tfrac{\pi^2}{4}+\tfrac{3}{2}\end{aligned}$

La sol di PC è

$$y(x)=\frac{3}{2}\pi\sin x+\left(\frac{\pi^2}{4}+\frac{3}{2}\right)\cos x+\frac{1}{4}\cos x-\frac{3}{4}x\sin x-\frac{1}{4}x^2\cos x$$

9. $\;y''+8y'=3x+4+x^2e^{x}$

10. $\begin{cases} y''+2y'+y=\dfrac{x^2+3}{e^{x}}\\ y(-1)=\frac{1}{12}e\\ y'(-1)=\frac{1}{12}e\end{cases}$ $\qquad$ eq. omog.

$y''+2y'+y=0$

eq. caratt.

$\alpha^2+2\alpha+1=0 \qquad (\alpha+1)^2=0$

$\alpha=-1 \qquad s=2$

int gen. omog. $\;y(x)=k_1e^{-x}+k_2xe^{-x}$

$f(x)=e^{-x}(x^2+3) \qquad h=-1 \quad s=2 \quad m=2$

cerco $\;y(x)=e^{-x}x^2(ax^2+bx+c)=e^{-x}(ax^4+bx^3+cx^2)$

$y'(x)=e^{-x}(-ax^4-bx^3-cx^2+4ax^3+3bx^2+2cx)$

$y''(x)=e^{-x}(ax^4+bx^3+cx^2-8ax^3-6bx^2-4cx+12ax^2+6bx+2c)$

sost nell' eq

$$\cancel{e^{-x}}\left(\cancel{ax^4}+\cancel{bx^3}+\cancel{cx^2}-\cancel{8ax^3}-\cancel{6bx^2}-\cancel{4cx}+12ax^2+6bx+2c-\cancel{2ax^4}-\cancel{2bx^3}-\right.$$

$$\left.-\cancel{2cx^2}+\cancel{8ax^3}+\cancel{6bx^2}+\cancel{4cx}+\cancel{ax^4}+\cancel{bx^3}+\cancel{cx^2}\right)=\cancel{e^{-x}}(x^2+3)$$

$\begin{cases} 12a=1\\ 6b=0\end{cases} \Rightarrow \quad y(x)=e^{-x}\left(\dfrac{1}{12}x^4+\dfrac{3}{2}x^2\right)$

$2c = 3$

INT GEN $y(x) = k_1 e^{-x} + k_2 x e^{-x} + e^{-x}\left(\frac{1}{12}x^4 + \frac{3}{2}x^2\right) =$

$$= e^{-x}\left(k_1 + k_2 x + \frac{1}{12}x^4 + \frac{3}{2}x^2\right)$$

$$y'(x) = e^{-x}\left(-k_1 - k_2 x - \frac{1}{12}x^4 - \frac{3}{2}x^2 + k_2 + \frac{1}{3}x^3 + 3x\right)$$

$$\begin{cases} y(-1) = \frac{1}{12}e \\ y'(-1) = \frac{1}{12}e \end{cases} \Rightarrow \begin{cases} e\left(k_1 - k_2 + \frac{1}{12} + \frac{3}{2}\right) = \cancel{e}\left(k_1 - k_2 + \frac{19}{12}\right) = \frac{1}{12}\cancel{e} \Rightarrow \\ \qquad \Rightarrow k_1 - k_2 = -\frac{3}{2} \\ e\left(-k_1 + k_2 - \frac{1}{12} - \frac{3}{2} + k_2 + \frac{1}{3} - 3\right) = \cancel{e}\left(-k_1 + k_2 - \frac{11}{4}\right) = \frac{1}{12}\cancel{e} \\ \qquad \Rightarrow -k_1 + 2k_2 = -\frac{17}{6} \end{cases}$$

$$\begin{cases} k_1 - k_2 = -\frac{3}{2} \\ -k_1 + 2k_2 = -\frac{17}{6} \end{cases}$$

$$k_1 = \begin{bmatrix} -\frac{3}{2} & -1 \\ -\frac{17}{6} & 2 \end{bmatrix} = -3 - \frac{17}{6} = -\frac{35}{6}$$

$$k_2 = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & -\frac{17}{6} \end{bmatrix} = -\frac{17}{6} - \frac{3}{2} = \cancel{} -\frac{13}{3}$$

SOL $y(x) = \left(-\frac{35}{6}e^{-x} - \frac{13}{3}x e^{-x} + \frac{1}{12}x^4 + \frac{3}{2}x^2\right) e^{-x}$