f.i. $+\infty-\infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$

~~$\frac{2}{n} \to \frac{2}{+\infty} = 0$~~ ERRORE

Si SCRIVE COSì: $\frac{2}{n} \xrightarrow[n \to +\infty]{} 0$

PROP $f: X \to \mathbb{R}$ $X \subseteq \mathbb{R}$ f funz elementare

$\{a_n\} \subseteq X$ $a_n \to l \in X$

es: $\frac{n}{2n+1} \to \frac{1}{2}$ (vedremo il perché)

Limiti di succ espresse mediante le funzione elementari

Succ potenze n^α $\alpha \in \mathbb{R}$

$\alpha = 0$ $n^0 = 1 \to 1$

$\alpha > 0$ $n^\alpha \to +\infty$ infatti $n^\alpha > K \Leftrightarrow n > K^{\frac{1}{\alpha}}$ D. vera

$\alpha < 0$ $n^\alpha \to 0$ // $n^\alpha = \frac{1}{n^{-\alpha}}$

Polinomio

$x_n = a_0 n^p + a_1 n^{p-1} + a_{p-1} n + a_p$ $a_1 a_2 \dots a_p \in \mathbb{R}$, $p \in \mathbb{N}$

e seconda dei segni dei coeff si può avere una

f.i. $+\infty - \infty$

(es: $\underset{+\infty}{3n^4} - \underset{+\infty}{6n^3} + \underset{+\infty}{2n^2} - \underset{+\infty}{5n} + 8$)

$x_n = n^p \left(a_n + \frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_p}{n^p}\right)$

es: $n^4\left(3-\frac{6}{n}+\frac{2}{n^2}-\frac{5}{n^3}+\frac{8}{n^4}\right)\to+\infty$

$n^4 \to +\infty$, $\frac{6}{n}\to 0$, $\frac{2}{n^2}\to 0$, $\frac{5}{n^3}\to 0$, $\frac{8}{n^4}\to 0$

$$x_n \to \begin{cases} +\infty & \text{se } a_0>0 \\ -\infty & \text{se } a_0<0 \end{cases}$$

I polinomi divergono sempre. Il segno della divergenza è il segno del coefficiente del termine di grado massimo

es: $n^3-2n^2-6n-3\to+\infty$

$n^3+2n^2+6n-3n^4\to-\infty$

Funzioni razionali fratte

$$x_n=\frac{a_0 n^p+\ldots+a_p}{b_0 n^q\ldots+b_q}$$ $a_i, b_j\in\mathbb{R}$ $b_0\neq 0$, $p\in\mathbb{N}_0$ $q\in\mathbb{N}$

è una f.i. $\frac{\infty}{\infty}$

$$x_n=\frac{n^p\left(a_0+\ldots+\frac{a_p}{n^p}\right)}{n^q\left(b_0+\ldots+\frac{b_q}{n^q}\right)}=n^{p-q}\,\frac{a_0+\ldots\frac{a_p}{n^p}}{b_0+\ldots+\frac{b_q}{n^q}}$$

$$n^{p-q}\to\begin{cases}1 & \text{se } p=q\\ +\infty & \text{se } p>q\\ 0 & \text{se } p<q\end{cases}$$

quindi se $p=q$ $x_n\to\frac{a_0}{b_0}$

se $p>q$ $x_n\to\infty$

$+\infty$ se a_0 e b_0 sono concordi, $-\infty$ se discordi)

se $p<q$ $x_n\to 0$

es: $\dfrac{3n^2-5+1}{(n+2)^2} \to 3$ $\qquad$ $\dfrac{-2n^4-6n}{3n^4+5n^2+7} \to -\dfrac{2}{3}$

$\dfrac{3n^2-5+1}{(n+2)^4} \to 0$ $\qquad$ $\dfrac{3n-5n+1}{n-3} \to \infty$

$\dfrac{3n^2-5n^4+1}{n^2+8} \to -\infty$ $\qquad$ $\dfrac{3n^2-5n^4+1}{n^2-8n^3+3} \to \infty$

Successione geometrica

$q \in \mathbb{R}$ $\qquad$ $\{q^n\}$

$q=0$ $\qquad$ $q^n \to 0$

$q=1$ $\qquad$ $q^n \to 1$

$q=-1$ $\qquad$ q^n oscill.

$q>1$ $\qquad$ $q^n \to +\infty$ infatti $q^n > k \Leftrightarrow n > \log_q k$

$q<-1$ $\qquad$ $q^n = ((-1)(-\infty)) = \underbrace{(-1)^n}_{\text{osc}} \underbrace{(-q)^n}_{+\infty}$ oscill

$(-q)>1$

$-1<q<1$ $\qquad$ $q^n \to 0$ infatti $|q^n| < \varepsilon \Leftrightarrow n > \log_{|q|} \varepsilon$ vera

perché $|q|<1$

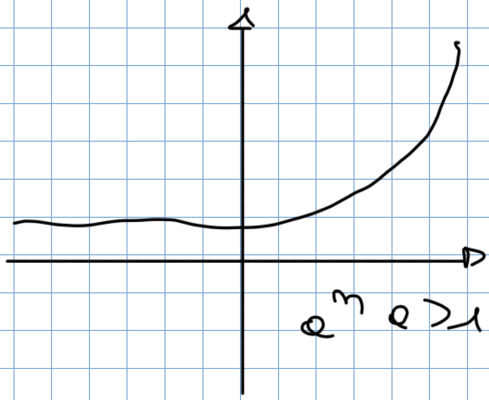
es: $\left(-\dfrac{1}{5}\right)^n \to 0$ $\qquad$ $(-6)^n$ osc $\qquad$ $(18)^n \to +\infty$

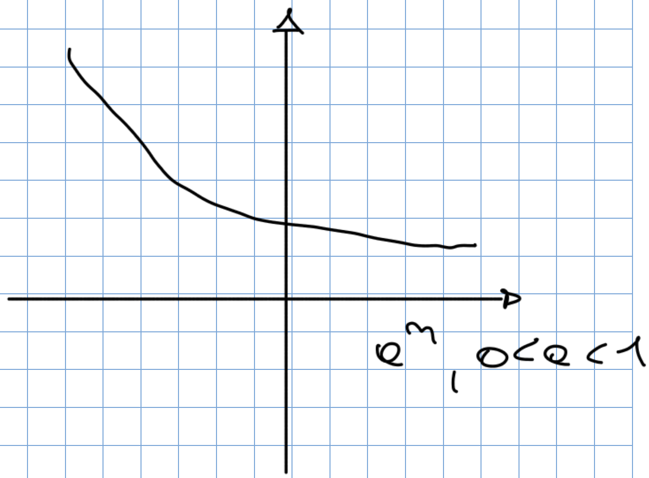
$\left(\dfrac{2}{3}\right)^n \to 0$

Successione composta mediante la funzione esponenziale

$\{x_n\}$ regolare con a^{x_n} con $a>0,\ a\neq 1$

se $x_n \to l \in \mathbb{R} \to a^x \to a^l$





se $a>1$

$$a^{x_n} \begin{cases} \to +\infty & \text{se } x_n \to +\infty \\ \to 0 & \text{se } x_n \to -\infty \end{cases}$$

se $0<a<1$

$$a^{x_n} \to \begin{cases} 0 & \text{se } x_n \to +\infty \\ +\infty & \text{se } x_n \to -\infty \end{cases}$$

es:

$$2^{\frac{3n^2-1}{n+4}} \qquad \frac{3n^2-1}{n+4} \to +\infty \qquad \to +\infty$$

$$3^{\frac{n^6-6}{2-n^4}} \qquad \frac{n^6-6}{2-n^4} \to -\infty \qquad \to 0$$

$$\left(\frac{1}{3}\right)^{\frac{n^6-2}{3-n}} \qquad \frac{n^6-2}{3-n} \to -\infty \qquad \to +\infty$$

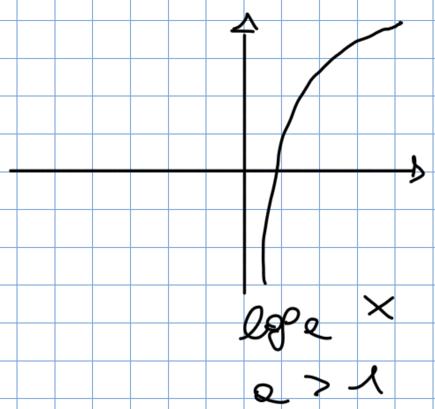
Funzione logaritmo

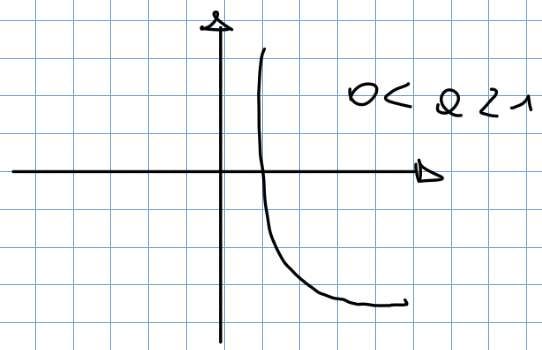
$\{b_n\}$ succ regolare $b_n > 0 \quad \forall n \quad a>0, a \neq 1$

$x_n = \log_a b_n$

se $b_n \to l > 0 \Rightarrow x_n \to \log_a l$

es: $\log_2 \frac{4n+1}{n+3} \to \log_2 4 = 2$





se $a>1 \Rightarrow x_n \begin{cases} +\infty & \text{se } b_n \to +\infty \\ -\infty & \text{se } b_n \to 0 \end{cases}$

se $0<a<1 \Rightarrow x_n \begin{cases} -\infty & \text{se } b_n \to +\infty \\ +\infty & \text{se } b_n \to 0 \end{cases}$

es: $\log_3 \frac{2n+1}{n^2+8} \to -\infty \qquad \log_2 \frac{2n^6+1}{n^4+8} \to +\infty$

$\log_{\frac{1}{2}} \underbrace{\frac{1-3n^4}{2-n^2}}_{\to +\infty} \longrightarrow -\infty$

$\log_{\frac{1}{4}} \underbrace{\frac{3n+1}{2n^4+6}}_{\to 0} \longrightarrow +\infty$

per es se $x_n \to +\infty$, $a>1 \Rightarrow a^{x_n} \to +\infty$

si deve avere $a^{x_n} > K \Leftrightarrow x^n > \log_a K$ vera $\Rightarrow$

se $0<a<1$ $a^{x_n} = \dfrac{1}{\underbrace{\left(\frac{1}{a}\right)^{x_n}}_{\to \infty}} \to 0$ $\frac{1}{a}>1$

se $b_n \to +\infty$, $a>1 \Rightarrow \log_a b_n \to +\infty$

si deve avere $\log_a b_n > K \Leftrightarrow b_n > a^K$ vera $\Rightarrow$

se $0<a<1$ $\log_a b = \left(\log_a \frac{1}{a}\right)\left(\log_{\frac{1}{a}} b_n\right) \to -\infty$

se $a_n \to \infty$ allora $\cos a_n$, $\sin a_n$, $\operatorname{tg} a_n$ sono non regolari

Consideriamo ancora una successione del tipo con $a_n b^n$
con $a_n>0$ $a_n \neq 1$ $\forall n$ $\{a_n\}, \{b_n\}$ regolare

Ricordiamo che se $x>0$ $x = e^{\log x}$ quindi

$$a_n^{b_n} = e^{\log a_n^{b_n}} = e^{b_n \log a_n}$$

es: $a_n \to$, $b_n \to 3$ $e^{3\log a_n} \to e^{3\log 2} = e^{\log 8} = 8$

Si avrà una f.i. se c'è una f.i nel prodotto
$b_n \log a_n$

$b_n \to$ f.i $0^0 (+\infty)^0$

$\log a_n \to \infty$ cioè $a_n \to 0$ opp. $a_n \to +\infty$

$b_n \to \infty$

$\log a_n \to 0$ cioè $a_n \to 1$ f.i 1^{∞}

Le f.i. allora sono: $+\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0

$(+\infty)^0$, 1^{∞}

$\log_{a_n} b_n$ $b_n > 0\ \forall n$, $a_n > 0\ \forall n$, $a_n \neq 1\ \forall n$,

$\{a_n\}$ reg, $\{b_n\}$ reg

$$\log_{a_n} b_n = (\log_{a_n} e)(\log b_n) \quad \frac{\log b_n}{\log a_n}$$

1^{∞}

$a_n = \left(1 + \frac{1}{n}\right)^n$ si presenta nella f.i 1^{∞}

Si può dim che $\{a_n\}$ è strett crescente e $a_n < 3\ \forall n$

quindi converge ad un numero < 3 che chiamiamo e

Def $e = \lim \left(1 + \frac{1}{n}\right)^n$ si ha $e \notin \mathbb{Q}$

$\boxed{e > 1}$

Si può dimostrare che se $x_n \to \infty$ allora $\left(1 + \frac{1}{x_n}\right)^{x_n} \to e$

es: $\left(1 + \frac{\alpha}{n}\right)^n = \left[\underbrace{\left(1 + \frac{1}{\frac{n}{\alpha}}\right)^{\frac{n}{\alpha}}}_{\to e}\right]^{\alpha} \longrightarrow e^{\alpha}$

se $\alpha \in \mathbb{R}$

$\left(1 + \frac{3}{n}\right)^n \to e^3$ $\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}$

$$\left(\frac{n+4}{n+6}\right)^{2n+1} =$$

$$\left(\frac{2n+4}{n+6}\right)^{2n+1} \to +\infty$$

quo non si applica nessuna regola

$$= \left[\left(1+\frac{1}{x_n}\right)^{x_n}\right] = \left(\frac{n+4+2-2}{n+6}\right) = \left(\frac{n+6-2}{n+6}\right) = \left(1+\frac{-2}{n+6}\right)^{2n+1} =$$

$$\left(1+\frac{1}{\frac{n+6}{-2}}\right)^{2n+1} = \left[\left(1+\frac{1}{\frac{n+6}{-2}}\right)^{\frac{n+6}{-2}}\right]^{\frac{-2(2n+1)}{n+6}}$$

$\frac{n+6}{-2} \to n$

$\left[\ \right] \to e$, $\quad \frac{-2(2n+1)}{n+6} \to -4$

$\to e^{-4}$

$$\left(\frac{n^2+2}{(n+1)^2}\right)^{n+3} = \left(\frac{n^2+2}{n^2+2n+1}\right)^{n+3} = \left(\frac{n^2+2n+1-2n-1+2}{n^2+2n+1}\right)^{n+3} =$$

$$= \left(1+\frac{1-2n}{n^2+2n+1}\right)^{n+3} = \left[\left(1+\frac{1}{\frac{n^2+2n+1}{1-2n}}\right)^{\frac{n^2+2n+1}{1-2n}}\right]^{\frac{(1-2n)(n+3)}{n^2+2n+1}}$$

$\left[\ \right] \to e$

$\to e^{-2}$

Altri limiti dedotti dal numero e

1) se $a_n \to$ $\quad \frac{e^{a_n}-1}{a_n} \to 1$ $\qquad$ 2) $\frac{\log(1+a_n)}{a_n} \to 1$

3) $\frac{(1+a_n)^{\alpha}-1}{a_n} \to \alpha$ $\qquad (\alpha \in R)$

esempio:

$$n \cdot \log\left(\frac{n+1}{n}\right) = n \log\left(1+\frac{1}{n}\right) = \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1$$

Limiti notevoli con funz trigonometriche

se $a_n \to 0$ $\quad \sin a_n \to 0 \quad \frac{\sin a_n}{a_n}$ $\quad$ f.i. $= \frac{0}{0}$

$a_n \neq 0$

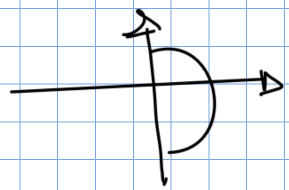
Si ha $\forall x \in \left]-\frac{\pi}{2} ; \frac{\pi}{2}\right[\qquad |\sin x| \le |x| \le |\tg x|$

Se $a_n \to \ \Rightarrow \ -\frac{\pi}{2} < a_n < \frac{\pi}{2} \Rightarrow |\sin a_n| \le |a_n| \le |\tg a_n|$

$$\Rightarrow 1 \le \left|\frac{a_n}{\sin a_n}\right| \le \frac{1}{|\cos a_n|}$$

$$\qquad \| \qquad\qquad \|$$

$$\frac{a_n}{\sin a_n} \qquad \frac{1}{\cos a_n}$$



$$\Rightarrow \underset{\downarrow \atop 1}{\cos a_n} \le \frac{\sin a_n}{a_n} \le \underset{\downarrow \atop 1}{1} \Rightarrow \frac{\sin a_n}{a_n} \to 1$$

$\boxed{a_n \to 0}$

$$\frac{\sin a_n}{a_n} \to 1 \qquad \frac{a_n}{\sin a_n} \to 1 \qquad \underset{\downarrow \atop 1}{\frac{\tan a_n}{\sin a_n}} \quad \underset{\downarrow \atop 1}{\frac{1}{\cos a_n}} \qquad \frac{a_n}{\tan a_n} \to 1$$

$$\frac{\arcsin a_n}{a_n} \to 1 \qquad \frac{\arctan a_n}{a_n} \to 1$$

$$\frac{1-\cos a_n}{a_n} = \frac{(1-\cos a_n)(1+\cos a_n)}{a_n(1+\cos a_n)} = \frac{\overset{\sin^2}{\overbrace{1-\cos^2 a_n}}}{a_n} \frac{1}{1+\cos a_n}$$

$$= \frac{\sin a_n}{a_n} \frac{\sin a_n}{1+\cos a_n} \to 0$$

$$\frac{1-\cos a_n}{a_n^2} = \underset{\downarrow \atop 1}{\frac{\sin^2 a_n}{a_n^2}} \ \underset{\downarrow \atop \frac{1}{2}}{\frac{1}{1+\cos a_n}} \to \frac{1}{2}$$

Esempio:

$$\frac{\sin\frac{1}{2n+1}}{\frac{2}{3n}} = \frac{\sin\frac{1}{2n+1}}{\frac{1}{2n+1}}\underbrace{\left(\frac{1}{2n+1}\cdot\frac{3n}{2}\right)}_{\downarrow\ \frac{3}{4}}$$

$$\frac{\frac{1}{n+3}}{\sin\frac{n+1}{n^2+4}} = \underbrace{\frac{\frac{n+1}{n^2+4}}{\sin\frac{n+1}{n^2+4}}}_{\downarrow\ 1}\left(\frac{n^2+4}{n+1}\cdot\frac{1}{n+3}\right) \to 1$$

$$(2n^2+3)\sin\frac{n}{n^3+1} = \underbrace{\frac{\operatorname{sen}\frac{n}{n^3+1}}{\frac{n}{n^3+1}}}_{\downarrow\ 1}\cdot\underbrace{\left(\frac{n}{n^3+1}(2n^2+3)\right)}_{\downarrow\ 2} \to 2$$

$$(3n-1)\,\operatorname{tg}\frac{2n}{n^2+4} = \underbrace{\frac{\operatorname{tg}\frac{2n}{n^2+4}}{\frac{2n}{n^2+4}}}_{\downarrow\ 1}\underbrace{\left(\frac{2n}{n^2+4}(3n-1)\right)}_{\downarrow\ 6} \to 6$$

$$\frac{\sin^2\frac{2}{n+3}}{\operatorname{tg}\frac{n}{3n^3+4}} = \underbrace{\frac{\sin^2\frac{2}{n+3}}{\left(\frac{2}{n+3}\right)^2}}_{\downarrow\ 1}\ \underbrace{\frac{\frac{n}{3n^3+4}}{\operatorname{tg}\frac{n}{3n^3+4}}}_{\downarrow\ 1}\ \underbrace{\left(\left(\frac{2}{n+3}\right)^2\frac{3n^3+4}{n}\right)}_{\downarrow\ 12} \to 12$$

$$\frac{1-\cos\frac{2}{n+1}}{\sin\frac{n+1}{n^3+2}} = \underbrace{\frac{1-\cos\frac{2}{n+1}}{\frac{4}{(n+1)^2}}}_{\downarrow\ \frac{1}{2}}\ \underbrace{\frac{\frac{n+1}{n^3+2}}{\sin\frac{n+1}{n^3+2}}}_{\downarrow\ 1}\left(\underbrace{\frac{4}{(n+1)^2}}_{\downarrow\ 4}\ \frac{n^3+2}{n+1}\right) \to \frac{1}{8}$$