Starting from the jj-coupled scheme we go to LS (1)

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LS COUPLED MATRTRIX ELEMENT

recoupling of two-body wavefunction

$$\psi^{J\Pi}_{j_1 j_2} = \sum_{LS} \sqrt{2j_1+1}\sqrt{2j_2+1}\sqrt{2L+1}\sqrt{2S+1}\begin{Bmatrix} l_1 & \frac{1}{2} & j_1\\ l_2 & \frac{1}{2} & j_2 \\ L & S & J\end{Bmatrix}\psi^{LS}_{J\Pi}$$

$$\Rightarrow \psi^{00}_{jj} = \sum_{LS}(2j+1)\sqrt{2L+1}\sqrt{2S+1}\begin{Bmatrix} l & \frac{1}{2} & j\\ l & \frac{1}{2} & j\\ L & S & 0\end{Bmatrix}\psi^{LS}_{00}$$

developing the coupling

$$\to = \frac{\delta_{jj}\,\delta_{LS}}{\sqrt{2L+1}\sqrt{2j+1}}(-1)^{l+\frac{1}{2}+j+S}\begin{Bmatrix} l & \frac{1}{2} & j\\ \frac{1}{2} & l & S\end{Bmatrix}$$

$$= \sum_{\substack{S\\ \pi_S\pi_S'}} \sqrt{2j+1}\sqrt{2S+1}(-1)^{l+\frac{1}{2}+j+S}\begin{Bmatrix} l & \frac{1}{2} & j\\ \frac{1}{2} & l & S\end{Bmatrix}\underbrace{\langle S\pi_S; S\pi_S'|00\rangle}_{(-1)^{S-\pi_S}\frac{\delta_{\pi_S,-\pi_S'}}{\sqrt{2S+1}}}\varphi^S_{l\sigma}\varphi^S_{l\sigma}$$

$\to$ LS coupled one body wavef.

$$= \sum_{S\pi_S}\sqrt{2j+1}(-1)^{l+\frac{1}{2}+j-\pi_S}\begin{Bmatrix} l & \frac{1}{2} & j\\ \frac{1}{2} & l & S\end{Bmatrix}|ll; S-\pi_S\rangle|ss; S\pi_S\rangle$$

$\to$ I want couple l and s separately

$$\psi^{00}_{m_j}(\vec r_1\sigma_1,\vec r_2\sigma_2) = \sum_{\substack{S\pi_S\\ m_{l_1}m_{l_2}\\ \sigma_1\sigma_2}}\Big[\sqrt{2j+1}(-1)^{l+\frac{1}{2}+j-\pi_S}\begin{Bmatrix} l & \frac{1}{2} & j\\ \frac{1}{2} & l & S\end{Bmatrix}\langle l m_{l_1}, l m_{l_2}|S-\pi_S\rangle\langle \tfrac{1}{2}\sigma_1\, \tfrac{1}{2}\sigma_2|S\pi_S\rangle$$
$$Y^l_{m_{l_1}}(\hat r_1)\,Y^l_{m_{l_2}}(\hat r_2)\,R_{m_j}(r_1)\,R'_{m_j}(r_2)\Big]$$

Thus one can separate $S=0$ and $S=1$ contributions

moreover for $\underline{S=0}$ $\Rightarrow \pi_S=0$ $\Rightarrow$ $\begin{Bmatrix} l & \frac{1}{2} & j\\ \frac{1}{2} & l & 0\end{Bmatrix} = \frac{(-1)^{l+\frac{1}{2}+j}}{\sqrt{2}\sqrt{2l+1}}$ $\quad\langle l m_{l_1}, l m_{l_2}|00\rangle = (-1)^{l-m_{l_1}}\frac{\delta_{m_{l_1},-m_{l_2}}}{\sqrt{2l+1}}$

$$\psi^{J=0}_{m_j}(\vec r_1\sigma_1,\vec r_2\sigma_2) = \sum_{m_{l_1},\sigma_1}\frac{\sqrt{2j+1}}{2(2l+1)}(-1)^{l-m_{l_1}}Y^l_{m_{l_1}}(\hat r_1)\,Y^l_{-m_{l_1}}(\hat r_2)\,R(r_1)R(r_2)$$

$\langle \tfrac{1}{2}\sigma_1\, \tfrac{1}{2}\sigma_2|00\rangle = (-1)^{\frac{1}{2}-\sigma_1}\frac{\delta_{\sigma_1,-\sigma_2}}{\sqrt{2}}$

simplifies greatly.

The L.O. term for the gaussian interaction proceeds then as usual considering the multipole expansion of the gaussian

$$g(\vec r_1-\vec r_2) = \frac{e^{-|\vec r_1-\vec r_2|^2/a^2}}{(a\sqrt{\pi})^3} = \frac{4}{a^3\sqrt{\pi}}e^{-\frac{r_1^2}{a^2}-\frac{r_2^2}{a^2}}\sum_{L\Pi}(-1)^L j_L\left(i2\frac{r_1 r_2}{a^2}\right)Y^{L*}_{\Pi}(\hat r_1)\,Y^L_{\Pi}(\hat r_2)$$

and we remember $\langle r_1 r_2|V_{LO}|r_1' r_2'\rangle = \delta(r_1-r_1')\delta(r_2-r_2')\,g(r_1-r_2)\,t_0(1-x_0P^\sigma-y_0P^\tau+z_0P^\sigma P^\tau)$

This implies the LO term being proportional to

$S=0$

$$\langle \psi_{J'} | V_{LO} | \psi_{J'} \rangle = \sum_{\substack{m_{l_1}, m_{l_1}' \\ \sigma_1, \sigma_1'}} \frac{2J+1}{4(2l+1)^2} \int d^3 r_1 d^3 r_2\, d^3 r_3\, d^3 r_4\, R(r_1) R(r_2) R(r_3) R(r_4)$$

$$(-1)^{2l - m_{l_1} - m_{l_1}'} Y^{l*}_{m_{l_1}}(\hat{r}_1) Y^{l*}_{-m_{l_1}}(\hat{r}_2) Y^{l}_{m_{l_1}'}(\hat{r}_3) Y^{l}_{-m_{l_1}'}(\hat{r}_4)$$

$$\delta(\vec{r}_1 - \vec{r}_1')\,\delta(\vec{r}_2 - \vec{r}_2')\, g_a(\vec{r}_1 - \vec{r}_2)$$

$$= \sum_{\substack{m_l m_l' \pi \\ \sigma_1 \sigma_1'}} \frac{2J+1}{4(2l+1)^2} \left[\int dr_1\, dr_2\, r_1^2 r_2^2 R^2(r_1) R^2(r_2) \frac{1}{a^3 \sqrt{\pi}^3} e^{\frac{-r_1^2 - r_2^2}{a^2}}\, i_L\!\left(2\frac{r_1 r_2}{a^2}\right)\right] \equiv R_L$$

$$L\pi \quad \int d^2\hat{r}_1 d^2\hat{r}_2\, Y^{l*}_{m_{l_1}}(\hat{r}_1) Y^{l}_{m_{l_1}'}(\hat{r}_1) Y^{L*}_{\pi}(\hat{r}_1)$$

$$Y^{l*}_{-m_{l_1}}(\hat{r}_2) Y^{l}_{-m_{l_1}'}(\hat{r}_2) Y^{L}_{\pi}(\hat{r}_2)\, (-1)^{-m_{l_1} - m_{l_1}'}$$

$$= \sum_{\pi} \frac{2J+1}{\cancel{(2l+1)^2}} R_L\, (-1)^{\pi}\, \cancel{(2l+1)^2}\, \frac{2L+1}{4\cancel{\pi}} \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2 \underline{\begin{pmatrix} l & l & L \\ -m_{l_1} & m_{l_1}' & -\pi \end{pmatrix} \begin{pmatrix} l & l & L \\ m_{l_1} & -m_{l_1}' & \pi \end{pmatrix}}$$

$$(*)\ (-\sum_{m_l} \begin{pmatrix} l & l & L \\ m_l & -m_l' & \pi \end{pmatrix} \begin{pmatrix} l & l & L \\ m_l & -m_l' & \pi \end{pmatrix} = \underline{= \frac{\delta_{m_l m_l'}}{2l+1}}$$

$$= \frac{2J+1}{2} \sum_L (2L+1)\, R_L \begin{pmatrix} l & l & L \\ 0 & 0 & 0 \end{pmatrix}^2$$

(*) Insert in here to consider $S \neq 0$. For $S=1$ could still be beneficial to consider the coupling cases as c-numbers.

NLO interaction considers second order in momenta (derivatives) to account for the plane-wave exponential expansion, in the Skyrme Interaction fashion. The terms, following Raimondi et al 2014 arXiv paper, are divided into $\hat{T}_1, \hat{T}_2, \hat{T}_3$ terms, $\hat{T}_3$ being non hermitian coming out only on NNLO, $\hat{T}_1$ and $\hat{T}_2$ can be moreover divided into local and non-local part of the interaction (cf. sec. 5). The local part of the gradients, $\propto (k'^{*}_{12} - k_{12})^{2m}$ commutes with the locality deltas, and thus acts only (in a rather trivial way) with the radial part of the Gaussian

$$V^{NLO}_{loc} \propto (k'^{*}_{12} - k_{12})^2 \, \delta(r_1 - r_1') \, \delta(r_2 - r_2') \, g_a(\vec{r}_1 - \vec{r}_2) = \delta(r_1 - r_1') \, \delta(r_2 - r_2') \left[(\cancel{k'^{*}_{12}} - k_{12})^2 g_a(\vec{r}_1 - \vec{r}_2) \right]$$

$$\Rightarrow (\nabla_1 - \nabla_2)^2 g_a(\vec{r}_1 - \vec{r}_2) = \Delta g_a(r) = \frac{2}{a^2}\left(2\frac{r^2}{a^2} - 3 \right) g_a(r)$$

$$\Rightarrow \langle V^{NLO}_{loc} \rangle = -\frac{6}{a^2} \langle V^{LO} \rangle + \frac{4}{a^4} \langle \tilde{V}^{LO} \rangle$$

↳ when $\tilde{V}^{LO}$ has the same angular part and $|\vec{r}_1 - \vec{r}_2|^2$ factor in the integral

In general the laplacian is given by the derivative respect to a

$$\frac{\partial g_a(r)}{\partial a} = \frac{(2r^2 - 3a^2)}{a^3} g_a(r) \Rightarrow \underline{\frac{\partial^2}{\partial r^2} g_a(r)} = \frac{2}{a} \frac{\partial g_a(r)}{\partial a}$$

And then the local part of the interaction

N^mLO order $\propto (k'^{*}_{12} - k_{12})^{2m} \propto \left(\frac{2}{a}\right)^m \frac{\partial^m}{\partial a^m} g_a(r)$