Homework #9

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CBE660: Intermediate Problems in Chemical and Biological Engineering - Fall 2016

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Problem 1. Solve Exercise 4.15 in the textbook and reproduce Figures 4.2 and 4.3 in page 360.

Solution:

Rearranging $x^T A x = b$ gives $f(x) = x^T A x - b$. To find the side length, c, we must minimize $f(x)$ s.t. $c = e_i^T x$ where e_i is the i^{th} column (or row) of the identity matrix.

$$f(x) = x^T A x - b$$

Define the Lagrange problem

$$L = x^T A x - \lambda(e_i^T x - c)$$

At the minimum $\frac{dL}{dx} = 0$ and $\frac{dL}{d\lambda} = 0$

$$\frac{dL}{dx} = 0 = Ax + A^T x - \lambda e_i = 2Ax - \lambda e_i$$

$$\frac{dL}{d\lambda} = 0 = e_i^T x - c = 0$$

$$\begin{bmatrix} 2a & -e_i \\ e_i & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -c \end{bmatrix}$$

Solve $\frac{dL}{dx}$ for x.

$$2Ax = \lambda e_i$$

$$x = \frac{1}{2} A^{-1} e_i \lambda$$

Substitute x into $\frac{dL}{d\lambda}$ and solve for λ

$$e_i^T x = c$$

$$e_i^T \left(\frac{1}{2} A^{-1} e_i \lambda \right) = c$$

$$\lambda = \frac{2c}{e_i^T A^{-1} e_i}$$

Let $e_i^T A^{-1} e_i = \widetilde{A}_{ii}$

$$\lambda = \frac{2c}{\widetilde{A}_{ii}}$$

Substitute the expression for λ into x

$$x = \frac{1}{2} A^{-1} e_i \lambda$$

$$x = \frac{1}{2} A^{-1} e_i \frac{2c}{\widetilde{A}_{ii}}$$

$$x = A^{-1} e_i \frac{c}{\widetilde{A}_{ii}}$$

substitute x into the original expression $x^T A x = b$ and solve for c

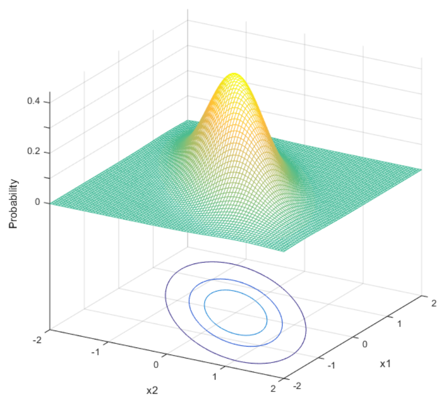
$$b = x^T A x$$

$$b = \frac{c}{\widetilde{A}_{ii}} e_i^T A^{-1} A A^{-1} e_i \frac{c}{\widetilde{A}_{ii}}$$

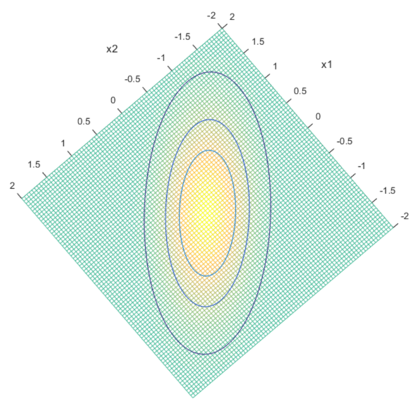
$$b = \left(\frac{c}{\widetilde{A}_{ii}}\right)^2 e_i^T A^{-1} e_i = \left(\frac{c}{\widetilde{A}_{ii}}\right)^2 \widetilde{A}_{ii} = \frac{c^2}{\widetilde{A}_{ii}}$$

$$c = \sqrt{b\widetilde{A}_{ii}}$$

Figure 4.2 was recreated by creating a 2-dimensional normal probability distribution with an expected value of zero for both $x1$ and $x2$. The Matlab function "Contour" was used to create 'slices' of the probability distribution at given values. Figure 4.3 was recreated by taking one of those 'slices', given by a representative matrix A and applying the results to show the minimum rectangle and the radii.



(a)



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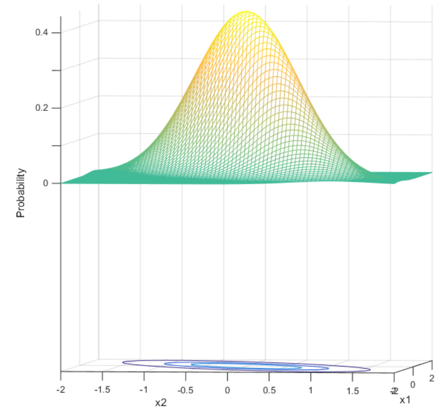

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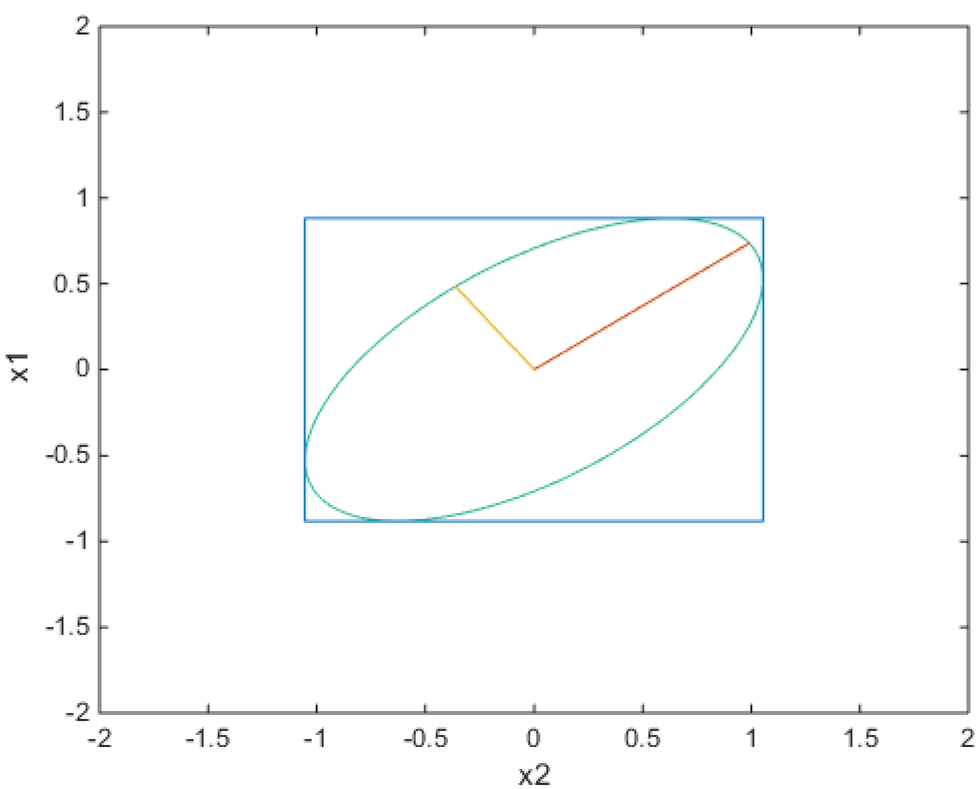

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clc
clear all
close all hidden

% For figure 4.2
x=[-2:0.05:2] ;
y=[-2:0.05:2] ;
sigma=1 ;
m=[0,0] ;
X=zeros(length(x),length(y)) ;
n=2 ;
P=inv([3.5 2.5;2.5 4]) ;

% For figure 4.3
b=zeros(length(x),length(y)) ;
A=[0.7 -0.5;-0.5 1] ;



% build the mesh/grid
for i=1:length(x)
    for j=1:length(y)
        xy(1,1)=x(i) ;
        xy(1,2)=y(j) ;

        % calculate the normal distribution
        prob(j,i)=((2*pi)^(n/2)*(det(P))^0.5)^(-1)*exp(-0.5*(xy-m)*inv(P)*(xy-m)') ;

        % For figure 4.3
        b(j,i)=xy*A*xy' ;
    end
end

figure(1)
mesh(y,x,prob+0.5)
    xlabel('x2')
    ylabel('x1')
    zlabel('Probability')
    set(gca,'ZTick',[0.5:0.1:1])
    z={'0','','0.2','','0.4'} ;
    set(gca,'ZTickLabel',z)
hold on
contour(y,x,prob,[(1-0.95)/2 (1-0.75)/2 (1-0.5)/2] )

v=0.5
AA=inv(A) ;
c1=(AA(1,1)*v).^0.5 ;
c2=(AA(2,2)*v).^0.5 ;
c1Box=[-c1 c1 c1 -c1 -c1];
c2Box=[-c2 -c2 c2 c2 -c2 ] ;

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[V,L]=eig(A)  ;
r1=sqrt(v/L(1,1))*V(:,1)  ;
r2=sqrt(v/L(2,2))*V(:,2)  ;

figure(2)
contour(x,y,b,[v v])
     hold on
plot(c1Box,c2Box)
     hold on
plot(-[0 r1(1)],-[0 r1(2)])
     hold on
plot([0 r2(1)],[0 r2(2)])
     xlabel('x2')
     ylabel('x1')
```

Problem 2. Solve Exercise 4.1 in the textbook.

Solution:

2.a

If B contains some or all of the elements of A ($B \subseteq A$), what is the probability that a given element from set A is not in set B?

$$Pr(x : x \in A \; and \; x \notin B) = Pr(A) - Pr(B)$$

Define $C = A \setminus B$

$$Pr(B \cup C) = Pr(B) + Pr(C) - Pr(B \cap C)$$

Where the $Pr(B \cap C) = 0$ by definition of C, because we know that C only contains elements that are not in B. Therefore, it is impossible for an element to belong to both B and C.

$$Pr(B \cup C) = Pr(B) + Pr(C)$$

We know that $Pr(B \cup C) = Pr(A)$

$$Pr(B \cup C) = Pr(A) = Pr(B) + Pr(C) = Pr(B) + Pr(A \setminus B)$$

$$Pr(A) = Pr(B) + Pr(A \setminus B)$$

$$Pr(A \setminus B) = Pr(A) - Pr(B)$$

2.b

$$Pr(A \cap B) = Pr(A)Pr(B)$$

Using

$$Pr(A \cup \bar{A} = Pr(A) + Pr(\bar{A}) = 1$$

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

We can see that

$$Pr(A \cap B) = (1 - Pr(\bar{A}))(1 - Pr(\bar{B}))$$

$$Pr(A \cap B) = 1 - Pr(\bar{A}) - Pr(\bar{B}) + Pr(\bar{A})Pr(\bar{B})$$

$$Pr(A \cap B) - 1 + Pr(\bar{A}) + Pr(\bar{B}) = Pr(\bar{A})Pr(\bar{B})$$

$$1 - Pr(A \cap B) - \left[Pr(\bar{A}) + Pr(\bar{B})\right] = -Pr(\bar{A})Pr(\bar{B})$$

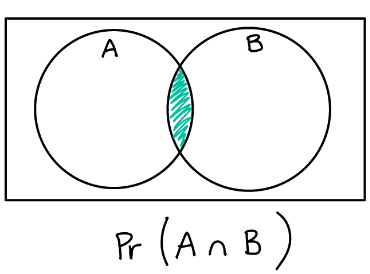
Using the consequence

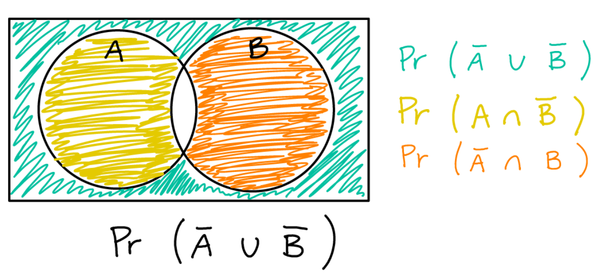
$$Pr(\bar{A}) + Pr(\bar{B} = Pr(\bar{A} \cup \bar{B}) + Pr(\bar{A} \cap \bar{B})$$

We can say that

$$1 - Pr(A \cap B) - Pr(\bar{A} \cup \bar{B}) - Pr(\bar{A} \cap \bar{B}) = -Pr(\bar{A})Pr(\bar{B})$$

$Pr(\bar{A} \cup \bar{B})$ is everything except $Pr(A \cap B)$. Therefore, by inspection, we can say that $Pr(A \cap B) + Pr(\bar{A} \cup \bar{B}) = 1$. Note the mistake in the figure where green should be $Pr(\bar{A} \cap \bar{B})$.





$$1 - 1 - Pr(\bar{A} \cap \bar{B}) = -Pr(\bar{A})Pr(\bar{B})$$
$$Pr(\bar{A} \cap \bar{B}) = Pr(\bar{A})Pr(\bar{B})$$

Thus if A and B are independent $\bar{A}$ and $\bar{B}$ will also be independent.

Problem 3. Solve Exercise 4.2 in the textbook.
Show that ξ and η are statistically independent if and only if

$$p_{\xi,\eta}(x,y) = p_\xi(x) p_\eta(y)$$

Solution:

Beginning with the probability function, assume that it can be separated into $P_{\xi,\eta} = P_\xi(x)P_eta(y)$. Integrating both sides with respect to x and y yields

$$\int_{-\infty}^{y}\int_{-\infty}^{x} P_{\xi,\eta} dxdy = \int_{-\infty}^{y}\int_{-\infty}^{x} P_\xi(x)P_\eta(y)dxdy$$

The integrals can then be separated because $P_\xi(x)$ does not depend on y and $P_\eta(y)$ does not depend on x.

$$\int_{-\infty}^{y}\int_{-\infty}^{x} P_{\xi,\eta} dxdy = \int_{-\infty}^{x} P_\xi(x)dx \int_{-\infty}^{y} P_\eta(y)dy$$

Using equation 4.1 to relate the cumulative probability distribution function (CDF) to the probability density function (PDF) yields equation 4.25 for independent random variables.

$$\int_{-\infty}^{y}\int_{-\infty}^{x} P_{\xi,\eta} dxdy = F_\xi(x)F_\eta(y)$$

$$F_{\xi,\eta}(x,y) = F_\xi(x)F_\eta(y)$$

Thus, if the original statement for the probability density function holds, then ξ and η must be statistically independent.

From equation 4.25 for independent random variables

$$F_{\xi,\eta}(x,y) = F_\xi(x)F_\eta(y)$$

Again using equation 4.1 to relate the CDF's to PFD's

$$F_{\xi,\eta}(x,y) = \int_{-\infty}^{x} P_\xi(x)dx \int_{-\infty}^{y} P_\eta(y)dy$$

$$F_{\xi,\eta}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} P_\xi(x)P_\eta(y)dxdy$$

The relationship only holds for all x and y when the integrands are equal.

$$\int_{-\infty}^{x}\int_{-\infty}^{y} P_\xi(x)P_\eta(y)dxdy = \int_{-\infty}^{y}\int_{-\infty}^{x} P_{\xi,\eta} dxdy$$

$$\int_{-\infty}^{x}\int_{-\infty}^{y} \left(P_{\xi,\eta}(x,y) - P_\xi(x)P_\eta(y)\right) dxdy = 0$$

$$P_{\xi,\eta}(x,y) - P_\xi(x)P_\eta(y) = 0$$

Thus, ξ and η are statistically independent if and only if the original PDF separation is true.

Problem 4. Consider random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ and constants $a, b \in \mathbb{R}$. Use the characteristic function to prove that the random variable $Y = a + bX$ is also normally distributed and find its mean and variance.

Solution:

The Characteristic Function for normal density is

$$\varphi_\xi(t) = E[e^{it\xi}] = \int_{-\infty}^{\infty} e^{itx} p_\xi(x) dX = exp[it\mu - \frac{1}{2}t^2\sigma^2]$$

The characteristic equation for Y then becomes

$$\varphi_Y(t) = E[e^{itY}] = E[e^{it(a+bX)}] = E[e^{ita}e^{itbX}]$$

$$\varphi_Y(t) = \int_{-\infty}^{\infty} e^{ita}e^{itbX} p_X(bX) dX$$

$$\varphi_Y(t) = e^{ita} \int_{-\infty}^{\infty} e^{itbX} p_X(bX) dX$$

Using the transformation into the t-domain from example 4.1 where now t is multiplied by the constant b

$$\varphi_Y(t) = e^{ita} exp[itb\mu_X - \frac{1}{2}(bt)^2\sigma_X^2]$$

$$\varphi_Y(t) = e^{ita} exp[itb\mu_X - \frac{1}{2}b^2t^2\sigma_X^2]$$

Using the property for multiplication by a constant (equation 4.7) where $\eta = a\xi$

$$\varphi_\eta(t) = \varphi_\xi(at)$$

And the property for addition (equation 4.8) where $\eta = \xi_1 + \xi_2$

$$\varphi_\eta(t) = \varphi_{\xi_1}(t)\varphi_{\xi_2}(t)$$

We can conclude that

$$\varphi_Y(t) = \varphi_a(t)\varphi_X(bt) = e^{ita} exp[itb\mu_X - \frac{1}{2}b^2t^2\sigma_X^2]$$

By inspection we can see that $\varphi_a(t) = e^{ita}$ and $\varphi_X(bt) = exp[itb\mu_X - \frac{1}{2}b^2t^2\sigma_X^2]$. Because the expression for $\varphi_Y(t)$ can be separated into the form given by a normal distribution, we can conclude that Y has a normal distribution.

$$\varphi_Y(t) = exp[itb\mu_X + ita - \frac{1}{2}b^2t^2\sigma_X^2]$$

$$\varphi_Y(t) = exp[it(b\mu_X + a) - \frac{1}{2}b^2t^2\sigma_X^2]$$

From this expression, the mean becomes $\mu_Y = b\mu_X - a$ and the variance becomes $b^2\sigma_X^2$. ;

$$E_Y[b\mu_X - a]$$

$$Var(Y) = b^2\sigma_X^2$$