Homework #8
Name: Andrea Livingston
Due: November 11th, 2016
CBE660: Intermediate Problems in Chemical and Biological Engineering - Fall 2016
Department of Chemical and Biological Engineering, University of Wisconsin-Madison

Problem 1. Solve problem 2.5 in the textbook and show that the Fourier series $f(x) = \sum_{k=-K}^{K} c_k e^{ikx}$ can be expressed as $f(x) = a_0 + \sum_{k=1}^{K} (a_k \cos(kx) + b_k \sin(kx))$. How are the coefficients c_k related to a_0, a_k, b_k?

Solution:

We can show that the expressions are equivalent by showing that $c_k = \bar{c}_{-k}$

$$c_k = \frac{(f, e^{ikx})}{e^{ikx}, e^{ikx}} = \frac{(f, e_k)}{(e_k, e_k)}$$

$$c_{-k} = \frac{(f, e^{-ikx})}{e^{-ikx}, e^{-ikx}}$$

Finding the complex conjugate for c_{-k}

$$\bar{c}_{-k} = \frac{(f, e^{ikx})}{e^{ikx}, e^{ikx}}$$

Because this expression for $\bar{c}_{-k}$ is equivalent to the expression for c_k, the expressions for f(x) are equivalent.

Thus,

$$\sum_{k=-K}^{K} c_k e^{ikx} = a_0 + \sum_{k=1}^{K} (a_k \cos(kx) + b_k \sin(kx))$$

From inspection, we can see that $a_0 = c_0$. Now relating a_k and b_k to c_n.
Consider the signals that oscillate once in a period (k=1)

$$c_{-1} e^{-ix} + c_1 e^{ix} = a_1 \cos(x) + b_1 \sin(x)$$

And then for two oscillations in a single period (k=2)

$$c_{-2} e^{-2ix} + c_2 e^{2ix} = a_2 \cos(2x) + b_2 \sin(2x)$$

Generalizing this relationship, we can say

$$c_k e^{kix} + c_{-k} e^{-kix} = a_k \cos(kx) + b_k \sin(kx)$$

Given that f(x) is a real function and that e^{-kix} is the complex conjugate of e^{kix}, we can say that c_{-k} is the complex conjugate of c_k. This allows the complex terms to cancel out from the exponential expression, such that the final expression is entirely real. $c_k = Re(c_k) + Im(c_k)$ and $\bar{c}_k = Re(c_k) - Im(c_k)$

$$c_k e^{kix} + c_{-k} e^{-kix} = a_k \cos(kx) + b_k \sin(kx)$$

$$c_k e^{kix} + c_{-k} e^{-kix} = (Re(c_k) - j * Im(c_k))(\cos(kx) - j\sin(kx)) + (Re(c_k) + j * Im(c_k))(\cos(kx) + j * \sin(kx))$$

Distributing the terms and simplifying yeilds

$$c_k e^{kix} + c_{-k} e^{-kix} = 2 * Re(c_k)\cos(kx) - 2 * Im(c_k)\sin(kx)$$

Substituting back into the exponential side of the original expression

$$2 * Re(c_k)\cos(kx) - 2 * Im(c_k)\sin(kx) = a_k \cos(kx) + b_k \sin(kx)$$

Using this, $2 * Re(c_k) = a_k$ and $-2 * Im(c_k) = b_k$ Relating this back to c_k

$$c_k = \frac{a_k}{2} - j * \frac{b_k}{2}$$

Problem 2. Use a Fourier series to approximate the Gaussian kernel function $f(x) = \exp(-ax^2)$ with $a \geq 0$ and $x \in [-L, L]$. Develop Matlab code to approximate the function for $L = 1$ and $a = 0, 2, 4, 8$. Plot the error of the approximation as a function of K. What order of convergence do you observe?

Solution:

The Fourier series for the Gaussian Kernel Function $f(x) = \exp(-ax^2)$ with $a \geq 0$ and $x\epsilon[-L, L]$

$$f(x) = \exp(-ax^2) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

To change the boundaries from π to L, now $x = x\frac{\pi}{L}$

$$c_k = \frac{1}{2L} \int_{-L}^{L} e^{-a(x\frac{\pi}{L})^2} e^{-ikx\frac{\pi}{L}}$$

Substituting $e^{-ikx} = \cos(kx) - i * \sin(kx)$

$$c_k = \frac{1}{2L} \int_{-L}^{L} e^{-a(x\frac{\pi}{L})^2} \left(\cos(-kx\frac{\pi}{L}) - i * \sin(kx\frac{\pi}{L}) \right)$$

$$c_k = \frac{1}{2L} \int_{-L}^{L} e^{-a(x\frac{\pi}{L})^2} \cos(kx\frac{\pi}{L}) - \frac{1}{2L} \int_{-L}^{L} e^{-a(x\frac{\pi}{L})^2} i * \sin(kx\frac{\pi}{L})$$

The sin function is an odd function such that the integral of sin will be zero. The positive area of the sin curve plus the negative area of the sin curve will go to zero when summed. Although the Gaussian function is an even function, an even function times an odd function will still yield an odd function. Thus, the second term goes to zero when integrated.

$$c_k = \frac{1}{2L} \int_{-L}^{L} e^{-a(x\frac{\pi}{L})^2} \cos(kx\frac{\pi}{L})$$

While this integral can be analytically determined, it is easiest to approximate the integral as $x\epsilon[-\infty, \infty]$ so that it represents a standard integral.

$$\int_{-\infty}^{\infty} e^{\frac{-x^2}{\sigma^2}} \cos(\beta x)dx = \sqrt{\pi}\sigma e^{\frac{-\beta^2\sigma^2}{4}}$$

Using this standard integral approximation $\frac{1}{\sigma^2} = a(\frac{\pi}{L})^2$ and $\beta = k\frac{\pi}{L}$.

$$\sigma = \frac{L}{\pi}\sqrt{\frac{1}{a}}$$

$$c_k = \frac{1}{2L}\sqrt{\pi}\sigma \exp\left[-(\frac{\beta\sigma}{2})^2\right]$$

$$c_k = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left[\frac{-k^2\pi^2}{4a}\right]$$

Returning to the original series

$$f(x) = e^{-ax^2} = \sum_{k=-\infty}^{\infty} c_k e^{\frac{ikx\pi}{L}}$$

$$f(x) = \exp(-ax^2) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left[\frac{-k^2\pi^2}{4a}\right]\right) e^{\frac{ikx\pi}{L}}$$

2

$$f(x) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \sum_{k=-\infty}^{\infty} exp\left[\frac{-k^2\pi^2}{4a}\right] e^{\frac{ikx\pi}{L}}$$
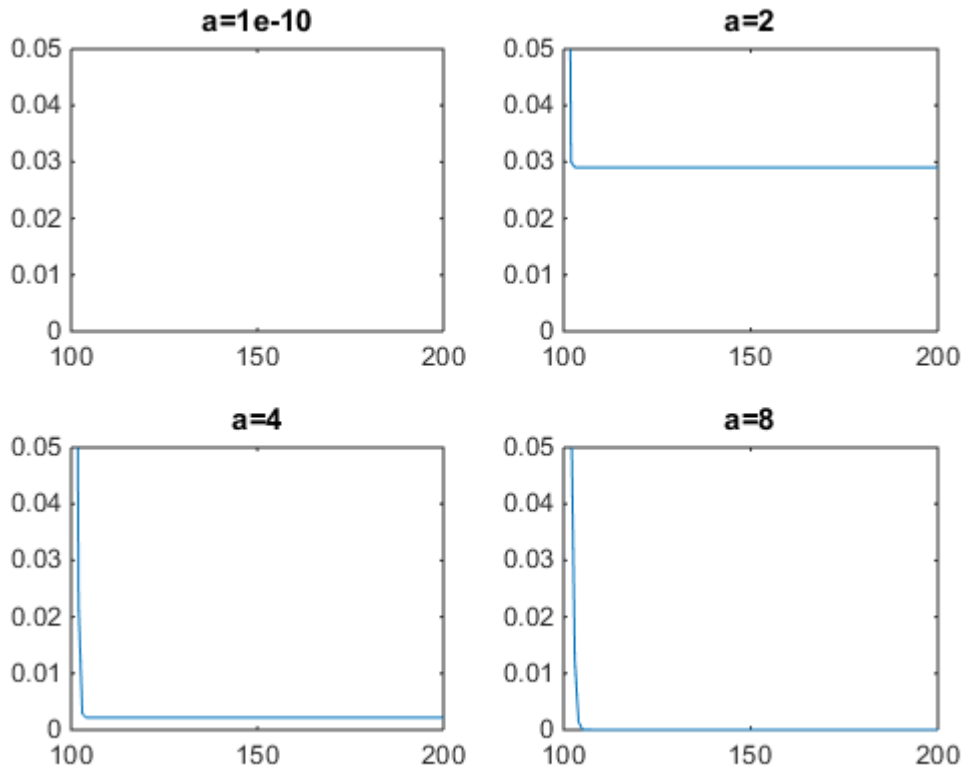
This can be approximated by truncating the summation from $[-\infty, \infty]$ to $[-c, c]$

$$f(x) \approx \frac{1}{2}\sqrt{\frac{\pi}{a}} \sum_{k=-c}^{c} exp\left[\frac{-k^2\pi^2}{4a}\right] e^{\frac{ikx\pi}{L}}$$

Comparing the approximation to Gaussian distribution we can see that the fit improves as a increases (the lines are stacked). The fit with a=0 was approximated with a very small value for a to avoid dividing by a zero. However, dividing by a small number gives a very large result for the approximation.
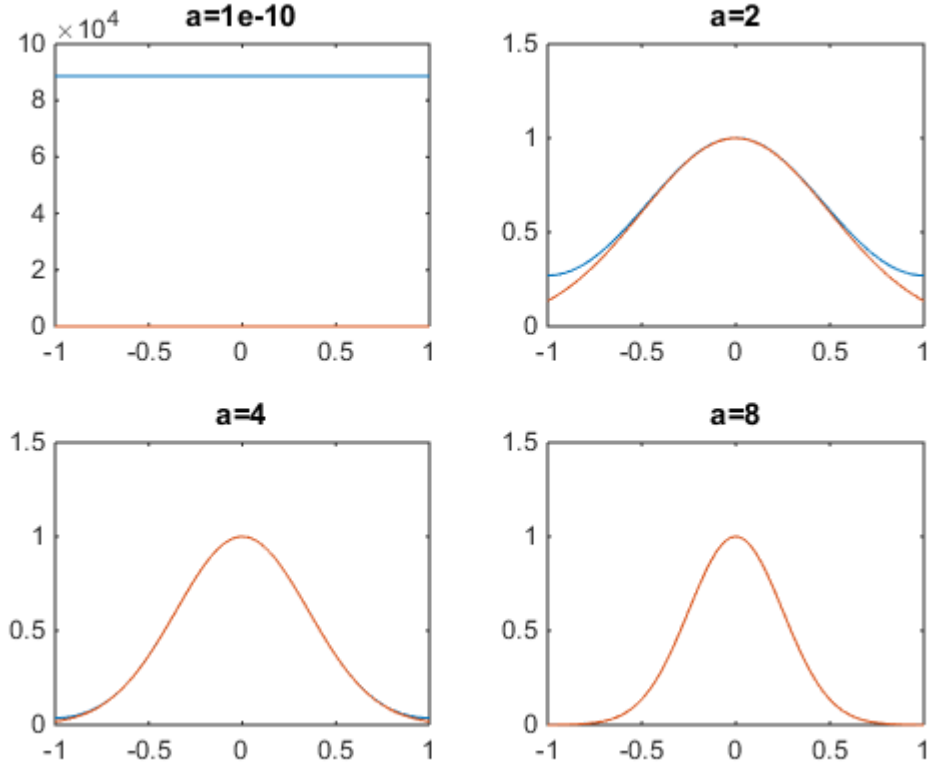
As a result of the approximation for a=0, the error results were not meaningful. For the others, it can be seen that the error decreases as a increases.

```matlab
clear all
L=1 ;
a0=[1e-10,2,4,8] ;
x=-1:0.01:1 ;
K=-100:1:100 ;
f=zeros(length(a0),length(x)) ;
Error=zeros(length(a0),length(x)) ;

for i=1:length(a0)
    a=a0(i) ;
    G(i,:)=exp(-a*x.^2) ;
  for j=1:length(K)
    k=K(j) ;

    f(i,:)=f(i,:)+1/2*(a/pi)^-0.5 .*exp(-k^2*pi^2/4/a).*exp(1i*k*x*pi) ;

    Error(i,j)=mean(abs(G(i,:)-real(f(i,:)))) ;

  end

end



figure (1)
    subplot(2,2,1)
    plot(Error(1,:))
```

```matlab
28        axis([100,200,0,0.05])
29        title('a=1e-10')

30

31        subplot(2,2,2)
32        plot(Error(2,:))
33        axis([100,200,0,0.05])
34        title('a=2')

35

36        subplot(2,2,3)
37        plot(Error(3,:))
38        axis([100,200,0,0.05])
39        title('a=4')

40

41        subplot(2,2,4)
42        plot(Error(4,:))
43        axis([100,200,0,0.05])
44        title('a=8')

45

46    figure(2)
47        subplot(2,2,1)
48        plot(x,real(f(1,:)),x,G(1,:))
49        title('a=1e-10')

50

51

52        subplot(2,2,2)
53        plot(x,real(f(2,:)),x,G(2,:))
54        title('a=2')

55

56        subplot(2,2,3)
57        plot(x,real(f(3,:)),x,G(3,:))
58        title('a=4')

59

60        subplot(2,2,4)
61        plot(x,real(f(4,:)),x,G(4,:))
62        title('a=8')
```
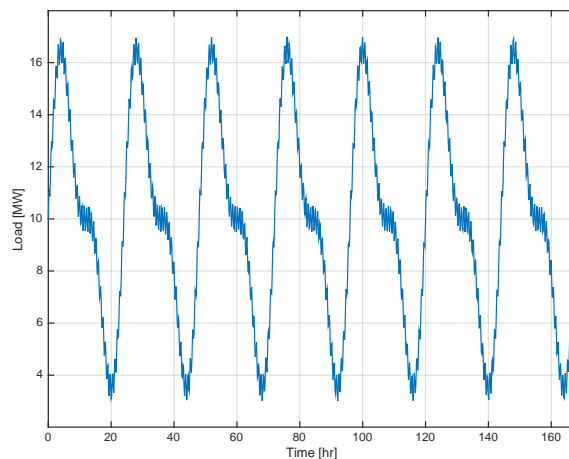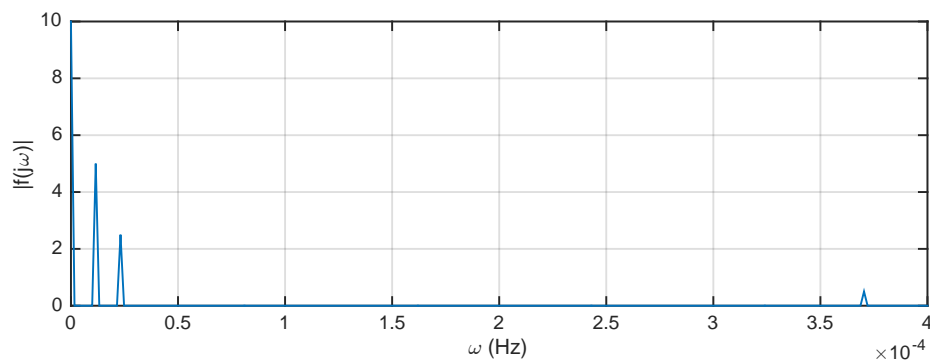
Figure 1: Electrical load.

Figure 2: Fourier spectrum of electrical load.

Solution:

The statement begins in the Fourier domain and we need to move back into the time domain. We can

---

do this by taking the Inverse Fourier Transform.

$$f(t) = \int_{-\infty}^{\infty} F(k)e^{2\pi j\omega t}d\omega$$

Developing an expression for $F(k)$ from the provided information

$$F(k) = F(j\omega_0)\delta(\omega - \omega_0) + F(j\omega_1)\delta(\omega - \omega_1) + F(j\omega_2)\delta(\omega - \omega_2) + F(j\omega_3)\delta(\omega - \omega_3) = \sum_{k=0}^{3} F(j\omega_k)\delta(\omega - \omega_k)$$

Substituting this into the inverse Fourier expression

$$f(t) = \int_{-\infty}^{\infty} \sum_{k=0}^{3} F(j2\pi\omega_k)\delta(2\pi\omega - 2\pi\omega_k)e^{2\pi jkt}d\omega$$

Because the summation does not depend on the variable of integration, k, it can be brought outside the integral

$$f(t) = \sum_{k=0}^{3} F(j2\pi\omega_k) \int_{-\infty}^{\infty} \delta(2\pi(\omega - \omega_k))e^{2\pi jkt}d\omega$$

Then the 2π cancels out of the delta function

$$f(t) = \sum_{k=0}^{3} F(j2\pi\omega_k) \int_{-\infty}^{\infty} \delta(\omega - \omega_k)e^{2\pi jkt}d\omega$$

The integral can be evaluated according the the standard integral $\int_{-\infty}^{\infty} f(x)\delta(x - a) = f(a)$ with the integration variable k being replaced by ω_k

$$f(t) = \sum_{k=0}^{3} F(j2\pi\omega_k)e^{2\pi j\omega_k t}$$

Applying the Euler Identity property to break the exponential into trig functions

$$e^{2\pi j\omega_k t} = cos(2\pi\omega_k t) + j * sin(2\pi\omega_k t)$$

$$f(t) = \sum_{k=0}^{3} F(j2\pi\omega_k) \left[cos(2\pi\omega_k t) + j * sin(2\pi\omega_k t)\right]$$

Pulling the first term out of the summation such that the sum runs from k=1 to k=3

$$f(t) = F(2\pi j\omega_0) \left[cos(2\pi\omega_0 t) + j * sin(2\pi\omega_0 t)\right] + \sum_{k=1}^{3} F(j2\pi\omega_k) \left[cos(2\pi\omega_k t) + j * sin(2\pi\omega_k t)\right]$$

Now substituting the value of $\omega_0 = 0$ in to simplify the trigonometric expressions according to $cos(0) = 1$ and $sin(0) = 0$

$$f(t) = F(2\pi j\omega_0) + \sum_{k=1}^{3} F(j2\pi\omega_k) \left[cos(2\pi\omega_k t) + j * sin(2\pi\omega_k t)\right]$$

Consider figure(1) which is an odd function. An odd function can only be created by summing together all odd functions. From our expression, cos is an even function and sin is an odd function because there is no phase shift. Because figure (2) is comprised *only* of odd functions, we can assume that cos()=0 at all locations of interest, thus reducing the overall expression.

$$f(t) = F(2\pi j\omega_0) + \sum_{k=1}^{3} F(j2\pi\omega_k)j * sin(2\pi\omega_k t)$$

Now, we can see that $F(2\pi j\omega_k)$ must relate to each of the coefficients c_k in the overall summation. To find their respective values we can consider $|F(2\pi j\omega_k)| = \left[Re()^2 + Im()^2\right]^{1/2}$. From figure(1) we can assume that f(t) has only real components in the domain of interest. To cancel with imaginary term from the sin, $F(2\pi j\omega_k)$ must only contain an imaginary part to make the overall expression real.

$$|F(2\pi j\omega_k)| = [Re()^2 + Im()^2]^{1/2} = |Im()|$$

Let $Im() = -|Im()| * j$ such that $F(2\pi j\omega_k) = -j * Im() = -j * |F(j2\pi\omega_k)|$

$$f(t) = F(2\pi j\omega_0) + \sum_{k=1}^{3} -|F(j2\pi\omega_k)| * j * j * sin(2\pi\omega_k t)$$

Now the imaginary components cancel out $(j * j = -1)$

$$f(t) = F(2\pi j\omega_0) + \sum_{k=1}^{3} |F(2j\pi\omega_k)| * sin(2\pi\omega_k t)$$

$$f(t) = |F(j\omega_0)| + |F(j\omega_1)|sin(2\pi\omega_1 t) + |F(j\omega_2)|sin(2\pi\omega_2 t) + |F(j\omega_3)|sin(2\pi\omega_3 t)$$

$$f(t) = 10 + 5sin(2\pi\omega_1 t) + 2.5sin(2\pi\omega_2 t) + 0.5sin(2\pi\omega_3 t)$$

Using this equation, the load as a function of time was recreated.
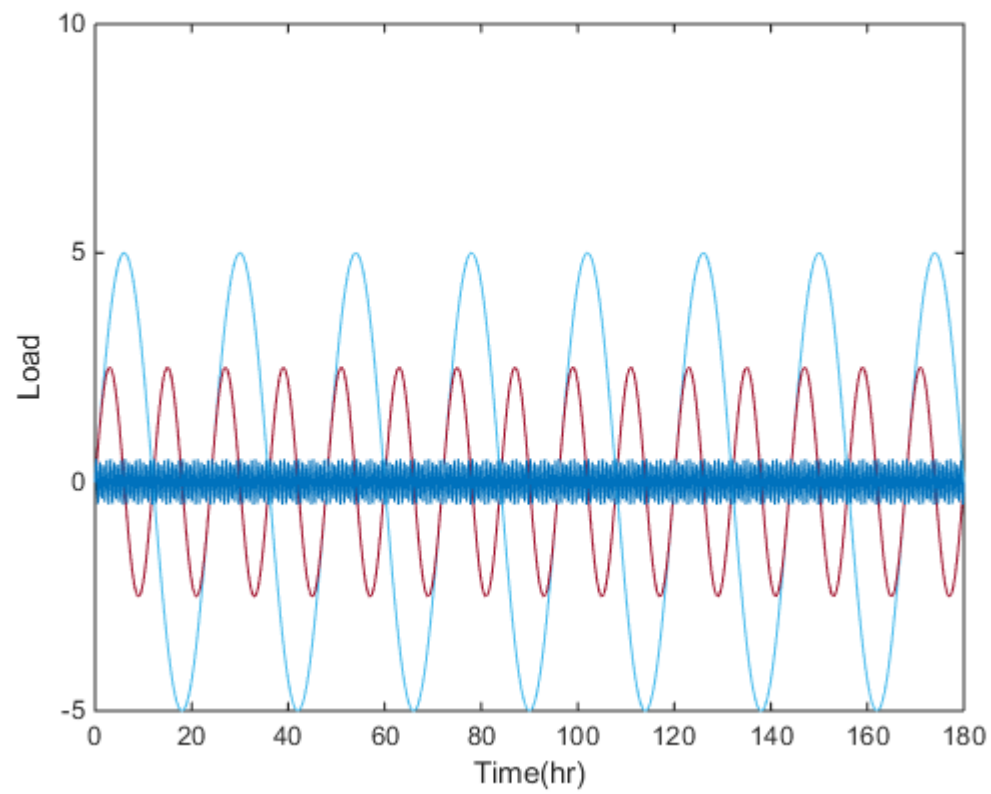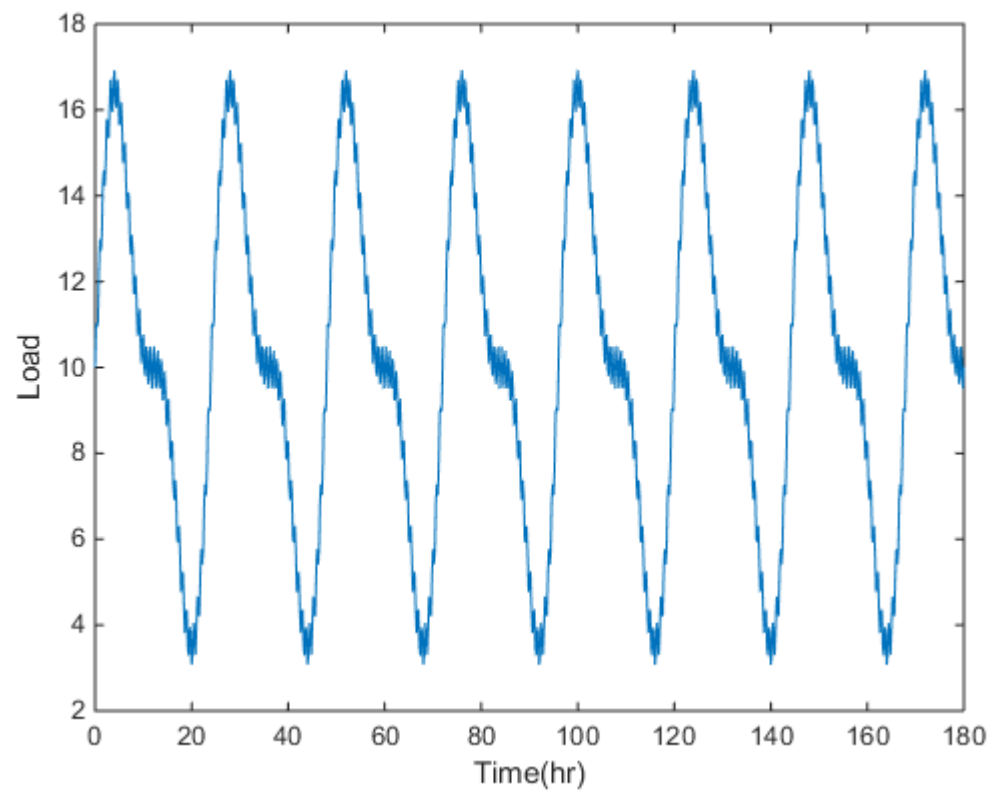The number of cycles per day was calculated using this equation :

$$Number\ of\ Cycles = \omega_k(24hours/day)(60minutes/hour)(60seconds/minute)$$

Cycles, $\omega_0 = 0$
Cycles, $\omega_1 = 2$
Cycles, $\omega_2 = 4$
Cycles, $\omega_3 = 64$

```
1   clear all
2
3   w1=3600*1.1574e−5 ;
4   w2=3600*2.3148e−5 ;
```

```matlab
5  w3=3600*3.7037e−4 ;
6  load=zeros(160,1) ;
7
8  t=0:0.2:180 ;
9
10 f0=10 ;
11 f1=5*sin(2*pi*w1*t) ;
12 f2=2.5*sin(2*pi*w2*t) ;
13 f3=0.5*sin(2*pi*w3*t) ;
14
15 load=f0+f1+f2+f3 ;
16
17 figure(1)
18 plot(t,load)
19 xlabel('Time(hr)')
20 ylabel('Load')
21
22 figure(2)
23 plot(t,f0,t,f1,t,f2,t,f3)
24 xlabel('Time(hr)')
25 ylabel('Load')
```