User Equilibrium Solution

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Abstract

We have given an equivalent formulation, which is a convex optimization problem, of finding user equilibrium solution in the traffic flow assignment, with proof of the equivalence. For the equivalent formulation, we have demonstrated the existence and uniqueness of minimizer. Moreover, the variant of Frank-Wolfe Algorithm is introduced for numerically solving the equivalent formulation.

1 Statement of Problem

1.1 Decision Variables and Parameters

Suppose we have a road network, which can be represented by nodes and directed links. We denote the nodes set as $\mathcal{N}$, and the links set $\mathcal{A}$, whose elements are ordered tuples of form (n_1, n_2), where $n_1, n_2 \in \mathcal{N}$. We also have set of origin nodes $\mathcal{R} \subset \mathcal{N}$ and set of destination nodes $\mathcal{S} \subset \mathcal{N}$. Then, we shall define the Origin-Destination (O-D) pair, which is exactly the ordered tuple of form (r, s), where $r \in \mathcal{R}$ and $s \in \mathcal{S}$.

Moreover, we need the concept of path. We denote the set of paths connecting O-D pair r-s as $\mathcal{K}_{rs}$, whose element can be roughly represented as

$$\{(r, n_1), (n_1, n_2), \cdots, (n_{k-1}, n_k), (n_k, s)\}$$

where $(r, n_1) \in \mathcal{A}$, $(n_k, s) \in \mathcal{A}$, and $(n_j, n_{j+1}) \in \mathcal{A}$ for all $2 \leq j < k$; $n_1, n_2, \cdots, n_k \in \mathcal{N}$, $r \in \mathcal{R}$ and $s \in \mathcal{S}$; $k \in \mathbb{N}$. In particular, we shall introduce function $\delta_{a,k}^{rs}$ to represent the relationship between link and path; that is, if the link $a \in \mathcal{A}$ belongs to $k \in \mathcal{K}_{rs}$, then $\delta_{a,k}^{rs} = 1$, otherwise it equals to 0.

Now, we should introduce the concept of flow, which represents the number of passenger car units passing through specific link or path in some unit time. We denote the flow on the link $a \in \mathcal{A}$ as x_a, denote f_k^{rs} as the flow on path k connecting O-D pair r-s. Notice that f_k^{rs} are exactly our decision variables, and $x_a, f_k^{rs} \in \mathbb{R}^+ := [0, +\infty)$. On the other hand, for each O-D pair r-s, there is a conjugated term: demand q_{rs}, which indicates the traffic demand between origin r and destination s during the observation period. Thus, we apparently have the constraint:

$$\sum_{k \in \mathcal{K}_{rs}} f_k^{rs} = q_{rs} \quad \forall r \in \mathcal{R} \ \ \forall s \in \mathcal{S}$$

Beside, we can also use $\delta_{a,k}^{rs}$ to show the relationship between link flow x_a and path flow f_k^{rs}, which can be written as:

$$x_a = \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{K}_{rs}} f_k^{rs} \delta_{a,k}^{rs} \quad \forall a \in \mathcal{A} \tag{1.1}$$

Associated with each link $a \in \mathcal{A}$, there is link performance function $p_a : \mathbb{R}^+ \to \mathbb{R}^+$, which express the traveling time for each car unit under given traffic flow, with the following assumptions:

1. $p \in \mathcal{C}^1(\mathbb{R}^+)$, i.e. f is continuously differentiable on $\mathbb{R}^+$.

2. p is strictly increasing on $\mathbb{R}^+$.

In applications, one may follow the suggestion of Federal Highway Administration to choose the link performance function as

$$p_a(x_a) = t_0 \left(1 + \alpha \left(\frac{x_a}{c} \right)^{\beta} \right) \tag{1.2}$$

where $t_0 \in \mathbb{R}^+$ is the free time of link, and $c \in \mathbb{R}^+$ is the capacity of link, and usually we set $\alpha = 0.15$ and $\beta = 4$. However, the concrete form of link performance function, like in Equality 1.2, will not enter our proofs.

The traveling time on the path $k \in \mathcal{K}_{rs}$ is denoted as c_k^{rs}, which can be written as:

$$c_k^{rs} = \sum_a t_a \delta_{a,k}^{rs} \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \tag{1.3}$$

where $t_a := p_a(x_a)$ is the traveling time on the link $a \in \mathcal{A}$.

1.2   Objective and Definition of User Equilibrium

The definition of user equilibrium can be concisely expressed as [2]: no traveler can improve his travel time by unilaterally changing routes.

Meanwhile, Wardrop [1] has provided an equivalent definition of user equilibrium, which is call *Wardrop's First Principle*: For each O-D pair, at user equilibrium, the travel time on all used paths is equal, and (also) less than or equal to the travel time that would be experienced by a single vehicle on any unused path. Therefore, we can state our problem in the following way:

Problème 1 *Given road network* $\{\mathcal{N}, \mathcal{A}, \mathcal{R}, \mathcal{S}\}$, *traffic demands* $(q_{rs})_{r \in \mathcal{R}, s \in \mathcal{S}}$ *and link performance functions* $(p_a)_{a \in \mathcal{A}}$. *Determine all* $f_k^{rs} \in \mathbb{R}^+$ *such that the Wardrop First Principle holds.*

In other words, for determined flow on paths $\boldsymbol{f} := \left(f_k^{rs} \right)_{k \in \mathcal{K}_{rs}}^{r \in \mathcal{R}, s \in \mathcal{S}}$, by Equality 1.1 we can calculate flow on links $\boldsymbol{x} := (x_a)_{a \in \mathcal{A}}$. Then, using link performance functions $(p_a)_{a \in \mathcal{A}}$ we can obtain the traveling time of links $\boldsymbol{t} := (t_a)_{a \in \mathcal{A}}$. According to Equality 1.3 we can compute the traveling time of paths $\boldsymbol{c} := \left(c_k^{rs} \right)_{k \in \mathcal{K}_{rs}}^{r \in \mathcal{R}, s \in \mathcal{S}}$. As we said, the result $\boldsymbol{c}$ must satisfies the *Wardrop First Principle*, that is: for each fixed $r \in \mathcal{R}$ and $s \in \mathcal{S}$, there exists a constant $c_{rs} \in \mathbb{R}^+$ such that: $c_k^{rs} \equiv c_{rs}$ for all $k \in \mathcal{K}_{rs}$ such that $f_k^{rs} > 0$; and $c_k^{rs} \geq c_{rs}$ for all $k \in \mathcal{K}_{rs}$ such that $f_k^{rs} = 0$.

2   Equivalent Mathematical Formulation

2.1   Statement of Equivalent Formulation

The most famous mathematical formulation for the user equilibrium is as follows:

Problème 2 *Find the minimizer, if it exists, of the optimization problem*

$$\min_{\boldsymbol{f} \geq \boldsymbol{0}} z(\boldsymbol{f}) = \sum_{a \in \mathcal{A}} \int_0^{x_a(\boldsymbol{f})} p_a(\omega) d\omega \tag{2.1}$$

$$\sum_{k \in \mathcal{K}_{rs}} f_k^{rs} = q_{rs} \quad \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \tag{2.2}$$

in which

$$x_a(\boldsymbol{f}) = \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{K}_{rs}} f_k^{rs} \delta_{a,k}^{rs}$$

for all $a \in \mathcal{A}$, *according to Equality 1.1.*

We shall notice that the objective function does not have any intuitive interpretation.

2.2 Existence of Minimizer

Let us denote $N := \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} |\mathcal{K}_{rs}|$, $M := |\mathcal{A}|$, and the the feasible set of Problem 2.1 - 2.2 as $F \subset \mathbb{R}^N$, which exactly is

$$F = \{ \boldsymbol{f} \in \mathbb{R}^N : \boldsymbol{f} \geq 0; \sum_{k \in \mathcal{K}_{rs}} f_k^{rs} = q_{rs}, \ \forall r \in \mathcal{R}, \ \forall s \in \mathcal{S} \}$$

Obviously F is a closed subset of $\mathbb{R}^N$.

Moreover, notice that the linear, therefore continuous, map $\boldsymbol{x} : F \subset \mathbb{R}^N \to \mathbb{R}^M$ can be represented as a matrix with binary entries with respect to canonical bases in $\mathbb{R}^N$ and $\mathbb{R}^M$, i.e. matrix contains only 0 or 1; but it cannot be zero matrix, otherwise it will be meaningless. We shall call this property of $\boldsymbol{x}$ as positivity, that is: $\boldsymbol{x}(\boldsymbol{f}) > 0$ for all $\boldsymbol{f} > 0$. Next, We affirm the properties of $z : \mathbb{R}^N \to \mathbb{R}$ as:

1. z is coercive with respect to $\boldsymbol{f}$. In fact, if $\boldsymbol{f} \to +\infty$, then there exists $a_0 \in \mathcal{A}$ such that $x_{a_0}(\boldsymbol{f}) \to +\infty$; by non-negativity, monotonicity and continuity of p_a,

$$z(\boldsymbol{f}) = \sum_{a \in \mathcal{A}} \int_0^{x_a(\boldsymbol{f})} p_a(\omega) d\omega \geq \int_0^{x_{a_0}(\boldsymbol{f})} p_a(\omega) d\omega \to \infty$$

2. $z \in \mathcal{C}^0(F)$, as the composition of continuous map $\boldsymbol{x} \mapsto \sum_{a \in \mathcal{A}} \int_0^{x_a} p_a(\omega) d\omega$ and linear map $\boldsymbol{x} : F \subset \mathbb{R}^N \to \mathbb{R}^M$.

The coercivity of z tells us: if the minimizer exists, it cannot appear at the infinity, then no matter where it is, we can always constrict it in a bounded and closed (since the feasible set is closed) set $F_0 \subset F$; in other words, as a result of Heine-Borel Theorem, in a compact set F_0. Moreover, The continuity of z implies the lower semi-continuity.

Théorème 1 (Weierstrass) *Suppose $F_0 \subset \mathbb{R}^N$ is a compact set, and $z : F_0 \to \mathbb{R}$ is lower semi-continuous, then there exists $\tilde{f} \in F_0$ such that*

$$z(\tilde{f}) = \min_{f \in F_0} z(f)$$

Theorem 1 shows the existence of minimizer of Problem 2.1 - 2.2.

2.3 Convexity of Equivalent Formulation

Let us recall the definition of convex optimization problem:

Définition 1 (Convex optimization Problem) *A convex optimization problem is an optimization problem in which the objective function is a convex function and the feasible set is a convex set.*

As we know, in convex optimization problems, every local minimum is a global minimum; moreover, if the objective function is strictly convex, then we have also the uniqueness of minimizer. However, convex optimization problem does not always admit the minimizer; but luckily in the last section we have already shown the existence of minimizer of Problem 2.1 - 2.2.

Observe that, choose arbitrary $\boldsymbol{f}_1, \boldsymbol{f}_2 \in F$, their convex combination $t\boldsymbol{f}_1 + (1-t)\boldsymbol{f}_2 \in F$, where $t \in [0,1]$, thus F is convex. For demonstrating the convexity of z, we need the following lemma:

Lemme 1 *Suppose $p : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable and strictly increasing, then for given $x > 0$, we have*

$$\int_0^{tx} p(\omega) d\omega < t \int_0^x p(\omega) d\omega$$

for all $t \in (0,1)$.

Preuve. Define

$$y(t) := \int_0^{tx} p(\omega) d\omega - t \int_0^x p(\omega) d\omega$$

where $t \in [0,1]$. Obviously $y(0) = y(1) = 0$, and we have $y''(t) = p'(tx) \cdot x^2$, which tells us $y''(t) > 0$ for all $t > 0$ on $[0,1]$, since $x > 0$ and p is strictly increasing on $\mathbb{R}^+$. Therefore $y(t) < 0$ on $(0,1)$, the inequality holds. $\qquad\square$

Now, choose arbitrary $f_1, f_2 \in F$ such that $f_1 \neq f_2$, and let $t \in (0, 1)$. With the help of Lemma 1, there is

$$z(tf_1 + (1-t)f_2) = \sum_{a \in \mathcal{A}} \int_0^{x_a(tf_1 + (1-t)f_2)} p_a(\omega)d\omega \tag{2.3}$$

$$= \sum_{a \in \mathcal{A}} \int_0^{tx_a(f_1) + (1-t)x_a(f_2)} p_a(\omega)d\omega \tag{2.4}$$

$$= \sum_{a \in \mathcal{A}} \int_0^{tx_a(f_1)} p_a(\omega)d\omega + \sum_{a \in \mathcal{A}} \int_0^{(1-t)x_a(f_2)} p_a(\omega)d\omega \tag{2.5}$$

$$< t \sum_{a \in \mathcal{A}} \int_0^{x_a(f_1)} p_a(\omega)d\omega + (1-t) \sum_{a \in \mathcal{A}} \int_0^{x_a(f_2)} p_a(\omega)d\omega \tag{2.6}$$

$$= tz(f_1) + (1-t)z(f_2) \tag{2.7}$$

Notice that, $f_1 \neq f_2$ implies they cannot be zero at the same time, by positivity of x, Lemma 1 can be properly applied.

We have proved F is convex and z is strictly convex. Therefore, the Problem 2.1-2.2 is convex optimization problem and it admits a unique minimizer.

2.4   Review on Constrained Problems

We have to refresh ourselves with an important result in constrained optimization problem. Consider

$$\min_{x \in X \subset \mathbb{R}^n} f(x) \tag{2.8}$$

$$\text{subject to } h_i(x) = 0 \quad \forall i \in I := \{1, 2, \cdots, m\} \tag{2.9}$$

$$g_j(x) \leq 0 \quad \forall j \in J := \{1, 2, \cdots, p\} \tag{2.10}$$

where $m \leq n$ and the functions $f, h_i, g_j \in \mathcal{C}^1(X)$ for all $i \in I$ and for all $j \in J$.

Before introducing the important results, we have to define:

Définition 2 (Regular Point [3]) *Let $x^* \in X$ be a point satisfying the constraints $h_i(x^*) = 0$ for all $i \in I$ and $g(x^*) \leq 0$ for all $j \in J$. Moreover, denote $J_a \subset J$ as the set of indices for which $g_j(x^*) = 0$, that is, active constraint. Then x^* is said to be a regular point if the gradient vectors $\nabla h_i(x^*)$ and $\nabla g_j(x^*)$, for all $i \in I$ and for all $j \in J_a$, are linearly independent.*

Then we have the following results:

Théorème 2 (Karush-Kuhn-Tucker Conditions [3]) *Let x^* be a local minimum point for the Problem 2.8 - 2.10, and suppose x^* is a regular point for the constraints. Then there is a vector $\lambda \in \mathbb{R}^m$ and a vector $\mu \in \mathbb{R}^p$ with $\mu \geq 0$ such that*

$$\nabla f(x^*) + \sum_{i \in I} \lambda_i \nabla h_i(x^*) + \sum_{j \in J} \mu_j \nabla g_j(x^*) = 0$$

$$\mu_j g_j(x^*) = 0 \quad \forall j \in J$$

Théorème 3 (KKT Conditions for Convex Optimization Problems) *When the optimization is convex, as in Definition 1, $x^* \in X$ is an optimal solution if and only if x^* satisfies the KKT conditions.*

Above introduced theorems will play vital role in the proof of equivalence between Problem 1 and Problem 2 in the next section.

2.5   Demonstration of Equivalence

At the beginning we need to do some computation. By Equality 1.1, we can derive

$$\frac{\partial x_a(f)}{\partial f_k^{rs}} = \delta_{a,k}^{rs} \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \tag{2.11}$$

Thanks to Equality 2.11, directly we get

$$\frac{\partial z(\boldsymbol{f})}{\partial f_k^{rs}} = \sum_{a \in \mathcal{A}} \left[p_a \left(x_a(\boldsymbol{f}) \right) \frac{\partial x_a(\boldsymbol{f})}{\partial f_k^{rs}} \right] = \sum_{a \in \mathcal{A}} \left[p_a \left(x_a(\boldsymbol{f}) \right) \delta_{a,k}^{rs} \right] = c_k^{rs}(\boldsymbol{f}) \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.12)$$

Moreover, for the accordance, we transform the constraints in Problem 2.1-2.2 as

$$h_{rs}(\boldsymbol{f}) = \sum_{k \in \mathcal{K}_{rs}} f_k^{rs} - q_{rs} = 0 \quad \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.13)$$

Similarly,

$$g_k^{rs}(\boldsymbol{f}) = -f_{rs}^k \leq 0 \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.14)$$

Obviously, we have

$$\frac{\partial h_{rs}(\boldsymbol{f})}{\partial f_{k'}^{r's'}} = \mathbb{1}_{\{r=r', \ s=s'\}} \quad \forall k' \in \mathcal{K}_{r's'} \ \forall r' \in \mathcal{R} \ \forall s' \in \mathcal{S} \quad (2.15)$$

and

$$\frac{\partial g_k^{rs}(\boldsymbol{f})}{\partial f_{k'}^{r's'}} = -\mathbb{1}_{\{r=r', \ s=s', \ k=k'\}} \quad \forall k' \in \mathcal{K}_{r's'} \ \forall r' \in \mathcal{R} \ \forall s' \in \mathcal{S} \quad (2.16)$$

in which $\mathbb{1}_X(x)$ is indicator function: $\mathbb{1}_X(x) = 1$ if $x \in X$, $\mathbb{1}_X(x) = 0$ if $x \notin X$. Equalities 2.15 and 2.16 also tell us all $f \in F$ are regular points.

Now we suppose that $\tilde{f}$ is a global minimum, and we know it must exist. By Theorem 3, the global minimizer is characterized by the KKT conditions. Apply the KKT conditions to our Problem 2.1 - 2.2, we know there exist $\lambda_{rs} \in \mathbb{R}$ for all $r \in \mathcal{R}$, and for all $s \in \mathcal{S}$; and $\mu_k^{rs} \in \mathbb{R}^+$ for all $k \in \mathcal{K}_{rs}$, for all $r \in \mathcal{R}$, and for all $s \in \mathcal{S}$; such that

$$\nabla z(\tilde{f}) + \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} \lambda_{rs} \nabla h_{rs}(\tilde{f}) + \sum_{r \in \mathcal{R}} \sum_{s \in \mathcal{S}} \sum_{k \in \mathcal{K}_{rs}} \mu_k^{rs} \nabla g_k^{rs}(\tilde{f}) = 0 \quad (2.17)$$

$$\mu_k^{rs} g_k^{rs}(\tilde{f}) = 0 \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.18)$$

Combined with Equalities 2.12, 2.15 and 2.16, there are

$$c_k^{rs}(\tilde{f}) = -\lambda_{rs} + \mu_k^{rs} \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.19)$$

$$\mu_k^{rs} \tilde{f}_k^{rs} = 0 \quad \forall k \in \mathcal{K}_{rs} \ \forall r \in \mathcal{R} \ \forall s \in \mathcal{S} \quad (2.20)$$

which implies: for fixed $r \in \mathcal{R}$ and $s \in \mathcal{S}$, for $k \in \mathcal{K}_{rs}$ such that $\tilde{f}_k^{rs} \neq 0$, there must be $\mu_k^{rs} = 0$, then $\tilde{f}_k^{rs} \equiv -\lambda_{rs}$; for $k \in \mathcal{K}_{rs}$ such that $\tilde{f}_k^{rs} = 0$, since $\mu_k^{rs} \geq 0$, then $\tilde{f}_k^{rs} \geq -\lambda_{rs}$. Notice this is exactly the characterization of user equilibrium solution, thus we have complete the demonstration of equivalence between the Problem 1 and Problem 2.

3 Introduction to Frank-Wolfe Algorithm

The Frank–Wolfe Algorithm is an iterative first-order optimization algorithm for constrained convex optimization. Also known as the conditional gradient method, reduced gradient algorithm and the convex combination algorithm, the method was originally proposed by Marguerite Frank and Philip Wolfe in 1956. In each iteration, the Frank–Wolfe algorithm considers a linear approximation of the objective function, and moves towards a minimizer of this linear function. [4].

Obviously, we need some modifications on original Frank–Wolfe Algorithm to make it adapted to our context, that is, the Problem 2.1-2.2. Therefore, we have the following variant of Frank–Wolfe Algorithm [2]:

Step 0 Initialization. Perform the so-called all-or-nothing assignment based on the zero link flow. All-or-nothing assignment means: given specific traveling time of links $\boldsymbol{t}$, we can directly compute the traveling time of paths $\boldsymbol{c}$. Then, for each traffic demand q_{rs}, corresponding to its O-D pair, we assign all the demand to the path which least costs the traveling time. This yields $(x_a^n)_{a \in \mathcal{A}}$, notice the n at the exponential position means it is n-th iteration.

Step 1 Update. Set

$$t_a^n = p_a(x_a^n) \quad \forall a \in \mathcal{A}$$

Step 2 Direction finding. Perform the all-or-nothing assignment based on $(t_a^n)_{a \in \mathcal{A}}$. This yields a set of auxiliary link flows $(y_a^n)_{a \in \mathcal{A}}$.

Step 3 Line search. Find $\theta \in [0, 1]$ that solves

$$\min_{0 \le \theta \le 1} \tilde{z}(\theta) := \sum_{a \in \mathcal{A}} \int_0^{x_a^n + \theta(y_a^n - x_a^n)} p_a(\omega) d\omega$$

We can also observe that

$$\tilde{z}'(\theta) = \sum_{a \in \mathcal{A}} p_a(x_a^n + \theta(y_a^n - x_a^n))(y_a^n - x_a^n)$$

then

$$\tilde{z}''(\theta) = \sum_a p_a'(x_a^n + \theta(y_a^n - x_a^n))(y_a^n - x_a^n)^2$$

As we mentioned before, the function p_a is strictly increasing, thus we know $\tilde{z}''(\theta) \ge 0$ for all $\theta \in [0, 1]$. Then, some numerical methods can be used to solve this sub-problem, since it is with good property.

Step 4 Move. Set

$$x_a^{n+1} = x_a^n + \theta(y_a^n - x_a^n) \quad \forall a \in \mathcal{A}$$

Step 5 Convergence test. If a convergence criterion is met, stop, the current solution is the set of equilibrium link flows; otherwise, go to back to the first step. One may choose the convergence criterion as

$$\frac{\left\| x^{n+1} - x^n \right\|_{\mathbb{R}^M}}{\left\| x^n \right\|_{\mathbb{R}^M}} < \epsilon$$

for some given small $\epsilon > 0$.

References

[1] John Glen Wardrop. "Road paper. some theoretical aspects of road traffic research." In: *Proceedings of the institution of civil engineers* 1.3 (1952), pp. 325–362.

[2] Yosef Sheffi. *Urban Transportation Networks*. Englewood Cliffs, New Jersey, USA: Prentice-Hall, 1985.

[3] David G Luenberger and Yinyu Ye. *Linear and nonlinear programming, vol. 116*. 2008.

[4] Wikipedia contributors. *Frank–Wolfe algorithm — Wikipedia, The Free Encyclopedia*. [Online; accessed 6-August-2019]. 2019. URL: https://en.wikipedia.org/w/index.php?title=Frank%E2%80%93Wolfe_algorithm&oldid=883838909.