User Equilibrium Solution

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1 INSTRUCTIONS

1.1 Create Template

Use the *create_template.py* to build a template Excel file. Before running it, you're supposed to revise the file location as where you want to save it. Remember that all the data will be read from this template later.

1.2 Import Data

Fill in the template with your own data. Please follow the given format, or the program may not function well. Moreover, the road network cannot contain a loop.

1.3 Revise Parameters

Revise the data file (the template you have imported data in) location and accuracy parameters in the main.py as your prefer. However, do NOT let the accuracy of convex program problem's solution too small, or the Franke-Wolfe Algorithm cannot converge.

1.4 Process

Run the *main.py*, then analyze the output.

2 PRINCIPLES

In this section, some principles of traffic volume assignment will be introduced. If you do not care about them, you can skip this section.

At first Wardrop's Principle, i.e. the definition of user-equilibrium will be explained.

For each O-D pair, at user equilibrium, the travel time on all used paths is equal, and (also) less than or equal to the travel time that would be experienced by a single vehicle on any unused path. This definition means that at equilibrium, the paths connecting each O-D pair can be divided into two groups. The first *group includes paths that carry flow. The travel time on all these paths will be the same. The other group includes paths that do not carry any flow. The travel time on each of these paths will be at least as large as the travel time on the paths in the first group.*

A stable condition is reached only when no traveler can improve his travel time by unilaterally changing routes. This is the characterization of the user-equilibrium (UE) condition.

This program is aimed to obtain the user-equilibrium solution of given traffic demand and road network. The mathematical model is demonstrated as follows:

$$\min z(x) = \sum_a \int_0^{x_a} t_a(\omega) d\omega$$

subject to:

$$\sum_k f^{rs}k = qrs \quad \forall r, s$$

$$f_k^{rs} \geq 0 \quad \forall k, r, s$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_a^{rs}, k \quad \forall a$$

Where x_a is the flow on link a, f_k is the flow on path k connecting origin r with destination s, and δ is the Link-Path Incidence Matrix.

the function t is the performance function, which indicates the relationship between flows (traffic volume) and travel time on the same link. According to the suggestion from the Federal Highway Administration (FHWA), we could use the following function:

$$t(x) = t_0(1 + \alpha(\frac{x}{c})^{\beta})$$

Where x is the link flow, c is the capacity, t_0 is the free flow travel time. As usual, we can set the α as 0.15 and the β as 4.

We can notice that the objective function is physically meaningless, but it was proved that the optimal solution of this program problem exactly is the user-equilibrium solution. If you're interested, please read the book *Urban Transportation Networks: Equilibrium Analysis with Mathematical Models* by Yosef Sheffi, Professor at Massachusetts Institute of Technology.

Meanwhile, this program is a convex program, we can use Frank-Wolfe Algorithm to solve it, the steps of the algorithm are displayed as follows:

Step 0: Initialization. Perform the all-or-nothing assignment based on the zero link flow. This yields x_a^n.

Step 1: Update. Set:

$$t_a^n = t_a(x_a^n) \quad \forall a$$

Step 2: Direction finding. Perform the all-or-nothing assignment based on the t_a^n. This yields a set of auxiliary link flows y_a^n.

Step 3: Line search. Find θ that solves:

$$\min_{0 \leq \theta \leq 1} \sum_a \int_0^{x_a^n + \theta(y_a^n - x_a^n)} t_a(\omega) d\omega$$

let:

$$f(\theta) = \sum_a \int_0^{x_a^n + \theta(y_a^n - x_a^n)} t_a(\omega) d\omega$$

thus:

$$f^{(1)}(\theta) = \sum_a t_a(x_a^n + \theta(y_a^n - x_a^n))(y_a^n - x_a^n)$$

$$f^{(2)}(\theta) = \sum_a t'_a(x_a^n + \theta(y_a^n - x_a^n))(y_a^n - x_a^n)^2$$

Because the function t is strictly increasing, so we can know:

$$f^{(2)}(\theta) \geq 0$$

then, if $f^{(1)}(\theta) = 0$, we can get a $\hat{\theta}$, which minimizes the function above. In other word, in this problem, we can solve it analytically. However, some numerical methods are often implied, also in this project.

Step 4: Move. Set:

$$x_a^{n+1} = x_a^n + \theta(y_a^n - x_a^n)$$

Step 5: Convergence test. If a convergence criterion is met, stop, the current solution is the set of equilibrium link flows; otherwise, go to step 1. In my program, the convergence test is as follows:

$$\frac{|x^{n+1} - x^n|}{|x^n|} \leq \epsilon$$

3 EXAMPLE

Please read this section in markdown file, because I do not want to enclose the images in this PDF file, for which the space will be wasted.