

Hodge theory and stationary Maxwell equations

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Abstract

These notes are the content of my seminar for the final exam of the course *Mathematical Physics – Differential Geometric Methods*, taught by Prof. Enrico Pagani during the Spring semester of 2025 at University of Trento.

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Introduction

The main purposes of these notes are

- to give a quick introduction to Hodge theory on oriented closed Riemannian manifolds;
- to analyze the relations between the topology of a domain and a boundary value problem on it (the stationary Maxwell equations in vacuum with homogeneous Dirichlet boundary condition).

The two goals are very different, because they have different geometric settings. Nevertheless, it's possible to build a theory analogous to the Hodge theory for manifolds with boundary, when imposing homogeneous Dirichlet boundary conditions.

Therefore, in the first section I introduce the Hodge star operator without assuming anything on the manifold, other than it is connected, oriented and

compact (however, one could do everything with compactly supported forms and remove also the latter assumption). In the second section, I assume that the manifold is without boundary and treat Hodge theory as done in [Pet06]. In particular, I prove some result concerning the first Betti number of closed orientable manifolds admitting non negative Ricci curvature. In the third section, I present the analogous of Hodge theory for manifolds with boundary, without proving the main result (Theorem 3.6). A detailed reference for this topic is [Sch95]. In the fourth and last section, I analyze a boundary value problem associated to the stationary Maxwell equation in vacuum depending on the topology of the domain in \mathbb{R}^3 . I find this discussion quite interesting, because it shows that topology can obstruct both existence and uniqueness of the solution to the boundary value problem.

1 Hodge star operator

Let (M, g) be an oriented connected Riemannian n -manifold and denote by $\omega_g \in \Omega^n(M)$ the corresponding Riemannian volume form.

Recall that the metric tensor g induces a scalar product (still denoted by g) on each fiber $(T_p^* M)^{\otimes k}$ by letting

$$g(\alpha, \beta) = g^{i_1 j_1} \cdots g^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k}$$

if $\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$ and $\beta = \beta_{j_1 \dots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k}$ in local coordinates. This scalar product restricts also to the subspace $\Lambda^k T_p^* M$, and actually, since for an orthonormal dual basis $\{e^1, \dots, e^n\}$

$$\begin{aligned} g(e^{i_1} \wedge \cdots \wedge e^{i_k}, e^{i_1} \wedge \cdots \wedge e^{i_k}) \\ &= \frac{1}{(k!)^2} \sum_{\sigma, \tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \delta^{\sigma(i_1) \tau(i_1)} \cdots \delta^{\sigma(i_k) \tau(i_k)} \\ &= \frac{1}{(k!)^2} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 = \frac{1}{k!}, \end{aligned}$$

(and $g(e^I, e^J) = 0$ if $I \neq J$, of course) we have for any $\omega = \omega_{i_1 \dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$ and $\eta_{j_1 \dots j_k} e^{j_1} \wedge \cdots \wedge e^{j_k} \in \Lambda^k T_p^* M$

$$g(\omega, \eta) = \frac{1}{k!} \delta^{i_1 j_1} \cdots \delta^{i_k j_k} \omega_{i_1 \dots i_k} \eta_{j_1 \dots j_k} = \frac{1}{k!} g(\bar{\omega}, \bar{\eta})$$

where

$$\begin{aligned} \bar{\omega} &= \omega_{i_1 \dots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k} \\ \bar{\eta} &= \eta_{j_1 \dots j_k} e^{j_1} \otimes \cdots \otimes e^{j_k} \end{aligned}$$

(and the indices are not ordered in this sum).

We can identify $(\Lambda^k T_p^* M)^* = \Lambda^k T_p M$ with $\Lambda^k T_p^* M$ by lowering every index, that is via the musical isomorphism

$$\sharp: \Lambda^k T_p^* M \rightarrow (\Lambda^k T_p^* M)^* = \Lambda^k T_p M$$

$$\omega \mapsto \omega^\sharp = g(\omega, \cdot).$$

We denote by \flat the inverse of \sharp .

On the other hand, the wedge product induces a pairing

$$\wedge: \Lambda^k T_p^* M \times \Lambda^{n-k} T_p^* M \rightarrow \mathbb{R}$$

$$(\omega, \eta) \mapsto g(\omega \wedge \eta, \omega_g|_p)$$

that allows us to identify $\Lambda^{n-k} T_p^* M \equiv (\Lambda^k T_p^* M)^*$.

Combining these two identifications, we can give the following definition.

Definition 1.1. The *Hodge star operator* is the $C^\infty(M)$ -linear isomorphism

$$\star: \Omega^k(M) \rightarrow \Omega^{n-k}(M)$$

defined by

$$g(\eta, \star\omega) = g(\eta \wedge \omega, \omega_g) \quad \forall \eta \in \Omega^k(M).$$

Lemma 1.2. Let $\omega, \eta \in \Omega^k(M)$.

- (a) $\omega \wedge \star\eta = \eta \wedge \star\omega$.
- (b) $\star\star\omega = (-1)^{k(n-k)}\omega$, i.e., $\star\star = (-1)^{k(n-k)}\text{Id}$.
- (c) $g(\star\omega, \star\eta) = g(\omega, \eta)$, i.e., \star is an isometry.

Proof. Point (a) follows from the symmetry of scalar product:

$$\omega \wedge \star\eta = g(\omega, \eta)\omega_g = \eta \wedge \star\omega.$$

To prove point (b), by linearity, it's enough to prove it for elements of the form $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. So let $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$. Then

$$\star(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \frac{\sigma(i_1 \dots i_k j_1 \dots j_{n-k})}{\sqrt{\det(g_{ij})}} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

where

$$\sigma(i_1 \dots i_k j_1 \dots j_{n-k}) = (-1)^{i_1 + \cdots + i_k - (1 + \cdots + k)}$$

is the sign of the corresponding permutation. Thus,

$$\star\star(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \frac{(-1)^{1+\cdots+n-(1+\cdots+k)-(1+\cdots+(n-k))}}{\sqrt{\det(g_{ij})}} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Then observe that

$$1 + \cdots + n - (1 + \cdots + k) - (1 + \cdots + (n - k)) = k(n - k).$$

Finally, point (c) follows from (b):

$$g(\star\omega, \star\eta)\omega_g = \star\omega \wedge \star\star\eta = (-1)^{k(n-k)} \star\omega \wedge \eta = \eta \wedge \star\omega = g(\eta, \omega)\omega_g. \quad \square$$

2 Hodge theory for closed manifolds

2.1 Co-differential and Laplacians

Assume that (M, g) is also closed (compact and without boundary). From now on, we will omit the Riemannian volume form when integrating functions, that is, we write

$$\int_M f := \int_M f\omega_g.$$

We can endow $\Omega^k(M)$ with the scalar product

$$(\omega, \eta)_{L^2} := \int_M g(\omega, \eta) = \int_M \omega \wedge \star\eta.$$

Taking the metric completion with respect to the induced norm, we construct the Hilbert space of L^2 sections of $\Lambda^k T^* M$, which we will denote by $L^2(M; \Lambda^k T^* M)$. Observe that since the Hodge star is a linear isometry pointwise, it's also a linear isometry between the corresponding L^2 -spaces.

Similarly, defining the scalar product

$$(\omega, \eta)_{H^m} := \int_M g(\omega, \eta) + \sum_{j=1}^m \int_M g(\nabla^j \omega, \nabla^j \eta)$$

and taking the metric completion of $\Omega^k(M)$ with respect to the induced norm, we find the Sobolev spaces $H^m(M; \Lambda^k T^* M)$.

Observe that for any $\eta \in \Omega^k(M)$,

$$\|\mathrm{d}\eta\|_{L^2} \leq \|\nabla\eta\|_{L^2} \leq \|\eta\|_{H^1}.$$

Indeed, at the center of normal coordinates,

$$\mathrm{d}\eta = \partial_j \eta_{i_1 \dots i_k} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = dx^j \wedge \nabla_j \eta$$

so,

$$g(\mathrm{d}\eta, \mathrm{d}\eta) \leq g(\nabla\eta, \nabla\eta).$$

Thus, the exterior derivative extends to a bounded linear operator

$$\mathrm{d}: H^1(M; \Lambda^k T^* M) \rightarrow L^2(M, \Lambda^{k-1} T^* M).$$

Moreover, for fixed $\omega \in L^2(M; \Lambda^k T^* M)$, the linear operator

$$\eta \mapsto \int_M g(\omega, \mathrm{d}\eta)$$

is H^1 -bounded, and can therefore be represented uniquely by an element in $H^1(M; \Lambda^* T^{k-1} M)$.

Definition 2.1. The *co-differential* $\delta = d^*$ is the adjoint of the exterior derivative, i.e., the linear mapping

$$\delta: L^2(M; \Lambda^k T^* M) \rightarrow H^1(M; \Lambda^{k-1} T^* M)$$

defined by

$$\int_M g(\delta\omega, \eta) = \int_M g(\omega, d\eta) \quad \forall \eta \in \Omega^{k-1}(M)$$

for every $\omega \in L^2(M; \Lambda^k T^* M)$.

Proposition 2.2. For every $\omega \in L^2(M; \Lambda^k T^* M)$,

$$\delta\omega = (-1)^{kn+n+1} \star d \star \omega.$$

In particular, if $\omega \in \Omega^k(M)$ then $\delta\omega \in \Omega^{k-1}(M)$.

Proof. Let us prove the claim for $\omega \in \Omega^k(M)$ (then the general claim follows by density). By Stokes theorem and the properties of \star , for every $\eta \in \Omega^{k-1}(M)$,

$$\begin{aligned} (\omega, d\eta)_{L^2} &= \int_M d\eta \wedge \star\omega \\ &= \int_M d(\eta \wedge \star\omega) - (-1)^{k-1} \int_M \eta \wedge d\star\omega \\ &= (-1)^k \int_M \eta \wedge d\star\omega \\ &= (-1)^{k+(n-k+1)(k-1)} \int_M \eta \wedge \star\star d\star\omega \\ &= (-1)^{kn+n+1} \int_M \eta \wedge \star\star d\star\omega = ((-1)^{kn+n+1} \star d\star\omega, \eta)_{L^2}. \quad \square \end{aligned}$$

Definition 2.3. The *Hodge Laplacian* is the second order differential operator

$$\Delta = \delta d + d\delta.$$

Definition 2.4. The *connection Laplacian* is the second order differential operator $\nabla^* \nabla$ defined by

$$\int_M g(\nabla^* \nabla \omega, \eta) = \int_M (\nabla \omega, \nabla \eta) \quad \forall \eta \in \Omega^k(M)$$

for any $\omega \in \Omega^k(M)$.

Both operators are self-adjoint with respect to the L^2 -product: for every $\omega, \eta \in \Omega^k(M)$, we have

$$\begin{aligned} (\Delta\omega, \eta)_{L^2} &= \int_M g(\Delta\omega, \eta) = \int_M g(d\omega, d\eta) + \int_M g(\delta\omega, \delta\eta) \\ (\nabla^* \nabla \omega, \eta)_{L^2} &= \int_M g(\nabla^* \nabla \omega, \eta) = \int_M (\nabla \omega, \nabla \eta) \end{aligned}$$

and both identities are symmetric on the right hand side. Moreover, for functions ($k = 0$), since $\nabla = d$ and $\delta = 0$, they coincide to the usual Laplace-Beltrami operator (up to sign). For order higher than zero they don't coincide. However, there is a relation which involves curvature terms (the proof is just a computation, see [Pet06, Theorem 9.4.1]).

Lemma 2.5 (Weitzenböck identity). *For any $\omega \in \Omega^k(M)$,*

$$\Delta\omega = \nabla^*\nabla\omega + \text{Ric}(\omega),$$

where, for any orthonormal frame $\{e_1, \dots, e_n\}$

$$\text{Ric}(\omega)(X_1, \dots, X_k) := \sum_{i=1}^k \sum_{j=1}^n (R(e_j, X_i)\omega)(X_1, \dots, e_j, \dots, X_k)$$

is the Weitzenböck curvature operator¹.

Remark 2.6. If $\omega \in \Omega^1(M)$ is a 1-form, then

$$\begin{aligned} g(\text{Ric}(\omega), \omega) &= \text{Ric}(\omega)(\omega^\sharp) = \sum_j (R(e_j, \omega^\sharp)\omega)(e_j) = \sum_j g(R(e_j, \omega^\sharp)\omega^\sharp, e_j) \\ &= \text{Ric}(\omega^\sharp, \omega^\sharp). \end{aligned}$$

2.2 The Hodge theorem

The Hodge theorem is about the relation between the topology of M and the presence of (Hodge) harmonic forms.

Definition 2.7. A form $\omega \in \Omega^k(M)$ is *(Hodge) harmonic* if $\Delta\omega = 0$. We will denote by

$$\mathcal{H}^k(M) = \mathcal{H}^k(M, g) := \{\omega \in \Omega^k(M) : \Delta\omega = 0\}$$

the subspace of harmonic k -forms.

Lemma 2.8. *A form $\omega \in \Omega^k(M)$ is harmonic if and only if it is both closed and co-closed. In other words,*

$$\mathcal{H}^k(M) = \ker d|_{\Omega^k(M)} \cap \ker \delta|_{\Omega^k(M)}.$$

Proof. Just observe that for any $\omega \in \Omega^k(M)$,

$$(\Delta\omega, \omega)_{L^2} = (d\omega, d\omega)_{L^2} + (\delta\omega, \delta\omega)_{L^2}. \quad \square$$

¹Following the convention used in [Pet06], here

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Theorem 2.9 (Hodge decomposition). *The space $\mathcal{H}^k(M)$ is finite dimensional and it holds the L^2 -orthogonal decomposition*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).$$

In particular, every $\omega \in \Omega^k(M)$ can be written uniquely as

$$\omega = \alpha + d\beta + \delta\gamma,$$

with $\alpha \in \mathcal{H}^k(M)$ harmonic, $\beta \in \Omega^{k-1}(M)$ and $\gamma \in \Omega^{k+1}(M)$. Moreover, we can choose β to be co-exact and γ to be exact.

Proof. The classical Fredholm alternative for elliptic operators yields that $\mathcal{H}^k(M)$ is finite dimensional and the orthogonal decomposition

$$L^2(M; \Lambda^k T^* M) = \ker(\Delta^*) \oplus \text{Im}(\Delta) = \ker(\Delta) \oplus \text{Im}(\Delta),$$

having used that Δ is self-adjoint. Then, intersecting with $\Omega^k(M)$ and applying elliptic regularity, we get

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \Delta(\Omega^k(M)).$$

Clearly, $\Delta(\Omega^k(M)) \subset d\Omega^{k-1}(M) + \delta\Omega^{k+1}(M)$, as

$$\Delta\eta = d(d\eta) + \delta(d\eta) \quad \forall \eta \in \Omega^k.$$

On the other hand, both $d\Omega^{k-1}(M)$ and $\delta\Omega^{k+1}(M)$ are orthogonal to $\mathcal{H}^k(M) = (\Delta(\Omega^k(M)))^\perp$, so $\Delta(\Omega^k(M)) = d\Omega^{k-1}(M) + \delta\Omega^{k+1}(M)$

Moreover, $d\Omega^{k-1}$ and $\delta\Omega^{k+1}$ are orthogonal to each other. Indeed, for any $\beta \in \Omega^{k-1}$ and $\gamma \in \Omega^{k+1}$,

$$g(d\beta, \delta\gamma) = g(d^2\beta, \gamma) = 0.$$

□

Let us denote

$$H_{dR}^k(M) := \frac{\{\omega \in \Omega^k(M) : d\omega = 0\}}{\{d\eta : \eta \in \Omega^{k-1}(M)\}}$$

the *de Rham cohomology groups* (but they are actually vector spaces, with the operation induced by $\Omega^k(M)$).

Theorem 2.10. *The quotient map induces an isomorphism*

$$\mathcal{H}^k(M) \xrightarrow{\sim} H_{dR}^k(M).$$

In other words, every de Rham cohomology class is represented by a unique harmonic form.

Proof. Let's prove that $\mathcal{H}^k(M) \rightarrow H_{dR}^k(M)$ is injective. Suppose $\omega = d\theta$ is harmonic. In particular, it's co-closed, that is, $\delta d\theta = 0$. Then

$$(\omega, \omega)_{L^2} = (\theta, \delta d\theta)_{L^2} = 0,$$

so $\omega = 0$.

Now let's prove that $\mathcal{H}^k(M) \rightarrow H_{dR}^k(M)$ is surjective. Fix a class $[\omega] \in H_{dR}^k(M)$ and write

$$\omega = \alpha + d\beta + \delta\gamma$$

with $\alpha \in \mathcal{H}^k(M)$. Since $0 = d\omega = d\delta\gamma$,

$$(\delta\gamma, \delta\gamma)_{L^2} = (\gamma, d\delta\gamma)_{L^2} = 0,$$

so actually $\omega = \alpha + d\beta$. Thus $[\omega] = [\alpha]$. \square

Proposition 2.11 (Bochner). *If (M, g) has non negative Ricci curvature, then every harmonic 1-form is parallel.*

Proof. For every $\omega \in \mathcal{H}^1(M)$, by Lemma 2.5 and the computation in Remark 2.6,

$$0 = g(\Delta\omega, \omega) = g(\nabla^*\nabla\omega, \omega) + \text{Ric}(\omega^\sharp, \omega^\sharp).$$

Integrating,

$$0 = \int_M g(\nabla\omega, \nabla\omega) + \int_M \text{Ric}(\omega^\sharp, \omega^\sharp) \geq 0,$$

thus $\nabla\omega = 0$. \square

Corollary 2.12. *If $\text{Ric} \geq 0$ and it is positive at one point, then $b_1(M) = 0$.*

Proof. Let $\omega \in \mathcal{H}^1(M)$. By Proposition 2.11, ω is parallel. But with the same computation made in the proof of Proposition 2.11,

$$0 = \int_M g(\nabla\omega, \nabla\omega) + \int_M \text{Ric}(\omega^\sharp, \omega^\sharp) \geq 0,$$

so $\text{Ric}(\omega, \omega) \equiv 0$. In particular, $\omega^\sharp|_p = 0$, and since ω is parallel, $\omega = 0$. \square

Corollary 2.13. *If $\text{Ric} = 0$, then $b_1(M) = \dim H_{dR}^1(M) \leq n$, with equality if and only if (M, g) is a flat torus.*

Proof. Since $\text{Ric} \geq 0$, by Proposition 2.11 every harmonic 1-form is parallel, so the evaluation $\mathcal{H}^1(M) \rightarrow T_p^*M$ is an injective linear homomorphism. In particular, by Theorem 2.10,

$$b_1(M) = \dim \mathcal{H}^1 \leq n.$$

If equality holds, then a basis of $\mathcal{H}^1(M)$ whose evaluation at a point gives an orthonormal frame gives rise to a global parallel orthonormal frame $\{\omega_1, \dots, \omega_n\}$ of T^*M . This implies that (M, g) is flat. Then, the universal cover is $\tilde{M} = \mathbb{R}^n$ with flat metric. Pulling back the orthonormal frame given by $e_i = (\omega^i)^\sharp$ to $\tilde{M} = \mathbb{R}^n$, the frame $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ remains parallel, hence the \tilde{e}_i 's are the usual cartesian coordinate vector fields. Moreover, they are invariant under the action of $\pi_1(M)$ on \mathbb{R}^n , which means that $\pi_1(M)$ acts by translations. Since $\pi_1(M)$ is finitely generated, $\pi_1(M) = \mathbb{Z}^m$ for some m . Since the quotient $M = \mathbb{R}^n/\pi_1(M)$ is compact and the action is free, it must be $m = n$, so M is a torus. \square

3 Hodge theory for compact manifolds with boundary

In this section, we see the analogous of the Hodge theory, but for compact manifold with boundary. For a detailed exposition, see [Sch95].

Let (M, g) be a compact connected oriented Riemannian n -manifold with non empty boundary ∂M . A possible definition of co-boundary as the L^2 -adjoint of d would not give rise to a self-adjoint Laplacian, because boundary terms would appear. For this reason, we give the definition of co-boundary using Proposition 2.2:

$$\delta := (-1)^{kn+n+1} \star d \star.$$

Lemma 3.1 (Green). *For any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$,*

$$\int_M g(\omega, d\eta) = \int_M g(\delta\omega, \eta) + \int_{\partial M} (\eta \wedge \star\omega)|_{\partial M}.$$

Proof. Compute as in the proof of Proposition 2.2, but without canceling the boundary term given by Stokes theorem. \square

Definition 3.2. We say that $\omega \in \Omega^k(M)$ satisfies

- the (*homogeneous*) *Dirichlet boundary condition* if

$$\omega|_{\partial M} = 0;$$

- the (*homogeneous*) *Neumann boundary condition* if

$$(\star\omega)|_{\partial M} = 0.$$

We denote

$$\begin{aligned}\Omega_D^k(M) &:= \{\omega \in \Omega^k(M) : \omega|_{\partial M} = 0\}, \\ \Omega_N^k(M) &:= \{\omega \in \Omega^k(M) : (\star\omega)|_{\partial M} = 0\}\end{aligned}$$

Lemma 3.3. $d\Omega_D^{k-1}(M) \subset \Omega_D^k(M)$ and $\delta\Omega_N^{k+1}(M) \subset \Omega_N^k(M)$.

Proof. Let's prove that $d\Omega_D^{k-1}(M) \subset \Omega_D^k(M)$. For any $\omega \in \Omega_D^{k-1}$, $(d\eta)|_{\partial M} = d(\eta|_{\partial M}) = 0$.

To prove $\delta\Omega_N^{k+1}(M) \subset \Omega_N^k(M)$, just observe that $\omega \in \Omega_N^{k+1}(M)$ if and only if $\star\omega \in \Omega_D^{n-k-1}(M)$, so $0 = (d\star\omega)|_{\partial M} = \pm(\star\delta\omega)|_{\partial M}$. \square

Denote also

$$\mathcal{H}^k(M) := \ker d|_{\Omega^k(M)} \cap \ker \delta|_{\Omega^k(M)},$$

and

$$\mathcal{H}_D^k(M) := \mathcal{H}^k(M) \cap \Omega_D^k(M), \quad \mathcal{H}_N^k(M) := \mathcal{H}^k(M) \cap \Omega_N^k(M).$$

Clearly, the Hodge star gives an isomorphism between $\mathcal{H}_D^k(M)$ and $\mathcal{H}_N^{n-k}(M)$.

We can define the Hodge Laplacian as before by

$$\Delta := d\delta + \delta d.$$

However, we did not defined an harmonic form ω by the formula $\Delta\omega = 0$, but rather by requiring that ω is both closed and co-closed, in virtue of Lemma 2.8. The problem is that Δ is not a self-adjoint operator anymore, so the requirement $\Delta\omega = 0$ is weaker.

However, by Lemma 3.1, Δ is self-adjoint on the subspace

$$\mathcal{X}^k := \{\omega \in \Omega^k(M) : \omega|_{\partial M} = 0, (\delta\omega)|_{\partial M} = 0\}.$$

Therefore, the Fredholm alternative yields the following.

Proposition 3.4. *For any $\theta \in \Omega^k(M)$, the problem*

$$\Delta\omega = \theta, \quad \omega|_{\partial M} = 0, (\delta\omega)|_{\partial M} = 0 \tag{1}$$

has a solution if and only if θ is L^2 -orthogonal to

$$\mathcal{H}_{\Delta}^k(M) := \{\eta \in \Omega^k(M) : \Delta\eta = 0, \eta|_{\partial M} = 0, (\delta\eta)|_{\partial M} = 0\}.$$

In that case, the solution is unique.

Remark 3.5. Observe that $\mathcal{H}_D^k(M) \subset \mathcal{H}_{\Delta}^k(M)$. Thus a necessary condition for the existence of a solution of the problem (1) is that $\theta \in \mathcal{H}_D^k(M)^{\perp}$.

It holds the analogous result of Theorem 2.9 (see [Sch95, Theorem 2.2.2, Theorem 2.2.7, Theorem 2.4.2, Theorem 2.4.8, Corollary 2.4.9]).

Theorem 3.6 (Hodge-Morrey-Friedrichs). *The spaces \mathcal{H}_D^k and \mathcal{H}_N^k are finite dimensional. Moreover, there hold the following L^2 -orthogonal decompositions:*

$$\begin{aligned} \Omega^k(M) &= \mathcal{H}^k(M) \oplus d\Omega_D^{k-1}(M) \oplus \delta\Omega_N^{k+1}(M) && \text{(Hodge-Morrey)} \\ \mathcal{H}^k(M) &= \mathcal{H}_D^k(M) \oplus (\mathcal{H}^k(M) \cap \delta\Omega^{k+1}(M)) && \text{(Friedrichs)} \\ \mathcal{H}^k(M) &= \mathcal{H}_N^k(M) \oplus (\mathcal{H}^k(M) \cap d\Omega^{k-1}(M)) && \text{(Friedrichs).} \end{aligned}$$

Consider the *de Rham cohomology groups relative to the boundary*

$$H_{dR}^k(M, \partial M) := \frac{\{\omega \in \Omega_D^k(M) : d\omega = 0\}}{\{d\eta : \eta \in \Omega_D^{k-1}(M)\}}$$

(by Lemma 3.3, the definition is well posed).

Theorem 3.7. *The quotient map induces an isomorphism*

$$\mathcal{H}_D^k(M) \xrightarrow{\sim} H_{dR}^k(M, \partial M).$$

In other words, every de Rham cohomology class relative to the boundary is represented by a unique harmonic form satisfying the homogeneous Dirichlet boundary condition.

Proof. Let's prove that $\mathcal{H}_D^k(M) \rightarrow H_{dR}^k(M, \partial M)$ is injective. Suppose $\omega = d\theta \in \mathcal{H}_D^k(M)$ for some $\theta \in \Omega_D^{k-1}(M)$. By definition, ω is co-closed. Then, using Lemma 3.1,

$$(\omega, \omega)_{L^2} = (d\theta, \omega)_{L^2} = (\theta, d\omega)_{L^2} + \int_{\partial M} \theta \wedge \star \omega = 0,$$

so $\omega = 0$.

Now let's prove that $\mathcal{H}_D^k(M) \rightarrow H_{dR}^k(M, \partial M)$ is surjective. Fix a class $[\omega] \in H_{dR}^k(M, \partial M)$ and, using Theorem 3.6, write

$$\omega = \theta + d\eta + \delta(\alpha + \beta)$$

with $\theta \in \mathcal{H}_D^k(M)$, $\eta \in \Omega_D^{k-1}(M)$, $\alpha \in \Omega_N^{k+1}(M)$ and $\beta \in \Omega^{k+1}(M)$ such that $\delta\beta \in \mathcal{H}^k(M)$. Since $0 = d\omega = d\delta\alpha$ and $\alpha \in \Omega_N^{k+1}(M)$, by Lemma 3.1

$$(\delta\alpha, \delta\alpha)_{L^2} = (\alpha, d\delta\alpha)_{L^2} - \int_{\partial M} \delta\alpha \wedge \star \alpha = 0,$$

so actually $\omega = \theta + d\eta + \delta\beta$. But then $\delta\beta \in \mathcal{H}_D^k(M)$, therefore $\delta\beta = 0$. Thus, $\omega = \theta + d\eta$ with $\eta \in \Omega_D^{k-1}(M)$, i.e., $[\omega] = [\theta]$. \square

4 Stationary Maxwell equations

Consider a connected domain $M \subset \mathbb{R}^3$ with smooth boundary ∂M , having outer unit normal $\nu: \partial M \rightarrow \mathbb{R}^3$. The stationary Maxwell equations in vacuum are

$\operatorname{curl} \mathbf{E} = 0$	(Faraday)
$\operatorname{div} \mathbf{E} = \frac{\rho}{4\pi\varepsilon_0}$	(Gauss)
$\operatorname{div} \mathbf{B} = 0$	(no magnetic charges)
$\operatorname{curl} \mathbf{B} = \frac{\mu_0}{4\pi} \mathbf{j}$	(Ampère)

where the electric field $\mathbf{E}: M \rightarrow \mathbb{R}^3$ and the magnetic field $\mathbf{B}: M \rightarrow \mathbb{R}^3$ are the unknowns, while the charge density $\rho \in C_c^\infty(M)$, the current density $\mathbf{j} \in C_c^\infty(M; \mathbb{R}^3)$, and the electric and magnetic permittivity constants in vacuum $\varepsilon_0, \mu_0 > 0$ (that from now on we assume to be equal to $1/4\pi$ and 4π , after having chosen a suitable unit system) are the data. We will impose the following boundary conditions:

$$\mathbf{E} \times \nu = 0, \quad \mathbf{B} \cdot \nu = 0 \quad \text{on } \partial M. \tag{2}$$

The first one follows from the discontinuity of the electric field at the boundary and the property of perfect conductors in electrostatic [Gri13, §2.3.5 and §2.5.1]. The second one is more artificial (we're interested in the math behind it), but one can think of an experiment where coils have been arranged around ∂M so that produced magnetic field is everywhere tangent to ∂M .

Let's reformulate everything using differential forms instead of vector fields. Consider

$$\begin{aligned} E &:= E_1 dx^1 + E_2 dx^2 + E_3 dx^3, \\ B &:= B_1 dx^2 \wedge dx^3 - B_2 dx^2 \wedge dx^3 + B_3 dx^1 \wedge dx^2 \\ J &:= j_1 dx^1 + j_2 dx^2 + j_3 dx^3 \end{aligned}$$

a simple computation shows that the stationary Maxwell equations become

$$\begin{aligned} dE &= 0 & \delta E &= \rho \\ dB &= 0 & \delta B &= J \end{aligned}$$

with boundary conditions

$$E|_{\partial M} = 0, \quad B|_{\partial M} = 0. \quad (3)$$

In fact, if we consider the simple case where $\partial M \approx \{x^3 = 0\}$ and $M \approx \{x^3 < 0\}$, so that $\nu \approx e_3$, then the boundary conditions (2) become

$$E_1 = E_2 = 0, \quad B_3 = 0.$$

On the other hand,

$$E|_{\partial M} = E_1 dx^2 + E_2 dx^3, \quad B|_{\partial M} = B_3 dx^1 \wedge dx^2,$$

so the two sets of boundary conditions coincide.

Now we apply the results of the previous section to analyze topologically the system of Maxwell equations. By Theorem 3.7, since E and B are closed

$$E = \alpha_E + d\phi, \quad B = \beta_E + dA$$

where $\alpha_E \in \mathcal{H}_D^1(M)$, $\beta_E \in \mathcal{H}_D^2(M)$ are the harmonic representative of the relative cohomology classes and $\phi \in \Omega_D^0(M)$, $A \in \Omega_D^1(M)$. Actually, we can assume that ϕ and A are co-closed. In fact, ϕ is a 0-form, so it's trivially co-closed. To fix A , do a gauge transform $A' = A + d\psi$, for some $\psi \in \Omega_D^0(M)$, so that $dA' = dA$ and $A' \in \Omega_D^1(M)$. Then,

$$\delta A' = \delta A + \Delta\psi,$$

so we just need to take ψ as the solution of

$$\Delta\psi = -\delta A, \quad \psi|_{\partial M} = 0,$$

which exists by standard elliptic theory. By replacing A with the co-closed form A' , we can assume that A is co-closed.

Now, using the inhomogeneous Maxwell equations and the co-closedness of ϕ and A ,

$$\Delta\phi = \rho, \quad \Delta A = J.$$

Thus, by Proposition 3.4, a solution to the Maxwell equations exists if and only if $\rho \in \mathcal{H}_\Delta^0(M)^\perp$ and $J \in \mathcal{H}_\Delta^1(M)^\perp \subset \mathcal{H}_D^1(M)^\perp$. Since $M \subset \mathbb{R}^3$ is a bounded domain, by the usual elliptic theory the only solution of the boundary value problem

$$\begin{aligned}\Delta u &= 0 && \text{on } M \\ u &= 0 && \text{on } \partial M \\ [(\delta u)|_{\partial M} = 0]\end{aligned}$$

is $u = 0$, so $\mathcal{H}_\Delta^0(M) = 0$. This means that there are no conditions on the charge density.

Moreover, by Theorem 3.7, α and β determine the cohomology class relative to the boundary of the electric and the magnetic field, respectively.

Therefore, since it is the reason of the existence of non trivial harmonic forms, the presence of topology forces

- compatibility conditions on the data
- non uniqueness of the solution to the boundary value problems.

In the rest of this section, I will examine a few examples. To make the interpretation easier, we will use the following deep theorem that relates the cohomology relative to the boundary $H_{dR}^k(M, \partial M)$ with the singular homology with real coefficients $H_k(M; \mathbb{R})$ (see [Hat02, Theorem 3.43]).

Theorem 4.1 (Poincaré-Lefschetz duality). *Let M be a compact² n -manifold with boundary ∂M . Then the map*

$$\begin{aligned}H_{dR}^k(M, \partial M) \times H_{dR}^{n-k}(M) &\rightarrow \mathbb{R} \\ ([\omega], [\eta]) &\mapsto \int_M \omega \wedge \eta\end{aligned}$$

is a well defined non degenerate pairing. In other words, by de Rham theorem³

$$H_{dR}^k(M, \partial M) \equiv H_{n-k}(M; \mathbb{R}).$$

canonically.

Example 4.2 (Solid torus). Consider

$$M = \{(\cos \theta x, \sin \theta x, z) \in \mathbb{R}^3 : (x - 2)^2 + z^2 \leq 1, \theta \in [0, 2\pi]\} \cong \mathbb{D}^2 \times \mathbb{S}^1.$$

²In particular, it is a manifold of finite type, so cohomology and homology groups (with real coefficients) are finite dimensional vector spaces.

³The de Rham theorem states that the de Rham cohomology is equivalent to the singular cohomology with real coefficients, i.e.,

$$H_{dR}^p(M) \equiv H^p(M; \mathbb{R}) = H_p(M; \mathbb{R})^*.$$

See [Lee13, Chapter 18].

Then M is homotopically equivalent to \mathbb{S}^1 , so $H_\bullet(M) = H_\bullet(\mathbb{S}^1) = \mathbb{R}[t]/(t^2)$, and by Theorem 4.1

$$\begin{aligned} H_{dR}^0(M, \partial M) &= 0, & H_{dR}^1(M, \partial M) &= 0, \\ H_{dR}^2(M, \partial M) &= \mathbb{R}, & H_{dR}^3(M, \partial M) &= \mathbb{R}. \end{aligned}$$

The non triviality of $H_{dR}^2(M, \partial M)$ is caused by the presence of the non trivial loop

$$\Gamma = \{(2 \cos \theta, 2 \sin \theta, 0) \in \mathbb{R}^3 : \theta \in [0, 2\pi]\} \cong \mathbb{S}^1,$$

the “soul” of the solid torus.

The electric field is always determined by $E = d\phi$, where the scalar potential $\phi \in \Omega_D^0(M)$ is the unique solution of the Poisson equation

$$\Delta\phi = \rho.$$

The magnetic field is $B = \beta_B + dA$, where $\beta_B \in \mathcal{H}_D^2(M)$ is the harmonic representative of the class of B and $A \in \Omega_D^1(M)$ solves

$$\Delta A = J.$$

Since we don’t know what $\mathcal{H}_\Delta^1(M)$ is (and since $\mathcal{H}_D^1(M) \cong H_{dR}^1(M, \partial M) = 0$), we are not able to write any “concrete” condition on J to make sure that A exists (and it is unique).

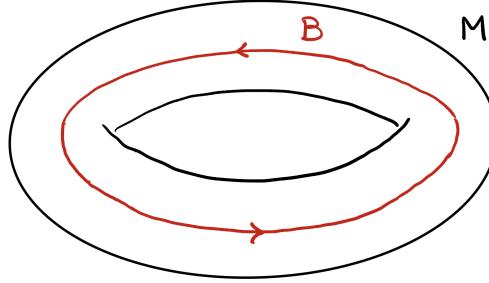


Figure 1: A non trivial magnetic field in the solid torus (no current inside).

Example 4.3 (Ball with spherical hole). Consider

$$M = \{x \in \mathbb{R}^3 : 1 \leq |x| \leq 2\} \cong \mathbb{S}^2 \times [0, 1].$$

Then M is homotopically equivalent to \mathbb{S}^2 , so $H_\bullet(\mathbb{S}^2; \mathbb{R}) = (\mathbb{R}[t]/(t^2), \deg(t) = 2)$, and by Theorem 4.1

$$\begin{aligned} H_{dR}^0(M, \partial M) &= 0, & H_{dR}^1(M, \partial M) &= \mathbb{R}, \\ H_{dR}^2(M, \partial M) &= 0, & H_{dR}^3(M, \partial M) &= \mathbb{R}. \end{aligned}$$

The non triviality of $H_{dR}^1(M, \partial M)$ is caused by the presence of the non trivial “middle sphere”

$$\Sigma = \{x \in \mathbb{R}^3 : |x| = 3/2\} \cong \mathbb{S}^2.$$

The electric field is $E = \alpha_E + d\phi$, where $\alpha_E \in \mathcal{H}_D^1(M)$ is the harmonic representative of the class of E and $\phi \in \Omega_D^0(M)$ is the solution of

$$\Delta\phi = \rho.$$

The magnetic field is $B = dA$, where $A \in \Omega_D^1(M)$ is the solution of

$$\Delta A = J,$$

which exists only if the current density satisfies the compatibility condition

$$\int_M \alpha \wedge \star J = 0,$$

where $\alpha \in \mathcal{H}_D^1(M)$ is the harmonic representative of any non trivial class of $H_{dR}^1(M, \partial M)$, for instance the one associated to the “middle sphere”.

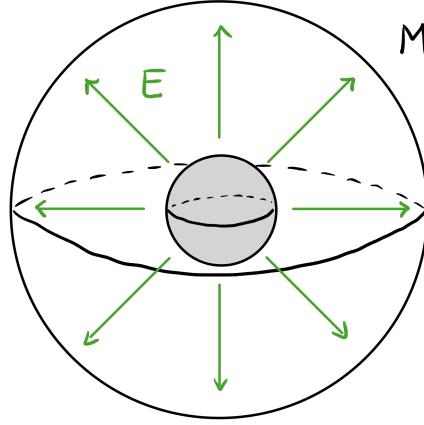


Figure 2: A non trivial electric field in the ball with a spherical hole (no charges inside).

Example 4.4 (Ball with toroidal hole). Let $M = \mathbb{B}^3 \setminus N \subset \mathbb{R}^3$, where

$$\begin{aligned} N &:= \{(\cos \theta x, \sin \theta x, z) \in \mathbb{R}^3 : (x - 1/2)^2 + z^2 < 1/16, \theta \in [0, 2\pi]\} \\ &\cong \mathbb{D}^2 \times \mathbb{S}^1. \end{aligned}$$

A simple computation using the Mayer-Vietoris theorem ($\mathbb{B}^3 = M \cup V$, with $V \sim N$ a small neighborhood of N) shows that

$$\begin{array}{ll} H_0(M) = \mathbb{R}, & H_1(M) = \mathbb{R}, \\ H_2(M) = \mathbb{R}, & H_3(M) = 0. \end{array}$$

By Theorem 4.1,

$$\begin{aligned} H_{dR}^0(M, \partial M) &= 0, & H_{dR}^1(M, \partial M) &= \mathbb{R}, \\ H_{dR}^2(M, \partial M) &= \mathbb{R}, & H_{dR}^3(M, \partial M) &= \mathbb{R}. \end{aligned}$$

The non triviality of H_{dR}^2 is caused by the presence of the loop

$$\Gamma := \{(x, 0, z) \in \mathbb{R}^3 : (x - 1/2)^2 + z^2 = 1/8\} \cong \mathbb{S}^1,$$

while the non triviality of $H_{dR}^1(M)$ is caused by the presence of the “middle torus”

$$T := \{(\cos \theta x, \sin \theta x, z) \in \mathbb{R}^3 : (x, z) \in \Gamma, \theta \in [0, 2\pi]\} \cong \mathbb{S}^1 \times \mathbb{S}^1.$$

The electric field is $E = \alpha_E + d\phi$, where $\alpha_E \in \mathcal{H}_D^1(M)$ is the harmonic representative of the class of E and $\phi \in \Omega_D^0(M)$ is the solution of

$$\Delta\phi = \rho.$$

The magnetic field is $B = \beta_B + dA$, where $\beta_B \in \mathcal{H}_D^2(M)$ is the harmonic representative of the class of B and $A \in \Omega_D^1(M)$ is the solution of

$$\Delta A = J,$$

which exists only if the current density satisfies the compatibility condition

$$\int_M \alpha \wedge \star J = 0,$$

where $\alpha \in \mathcal{H}_D^1(M)$ is the harmonic representative of any non trivial class of $H_{dR}^1(M, \partial M)$, for instance the one associated to the “middle torus” T .

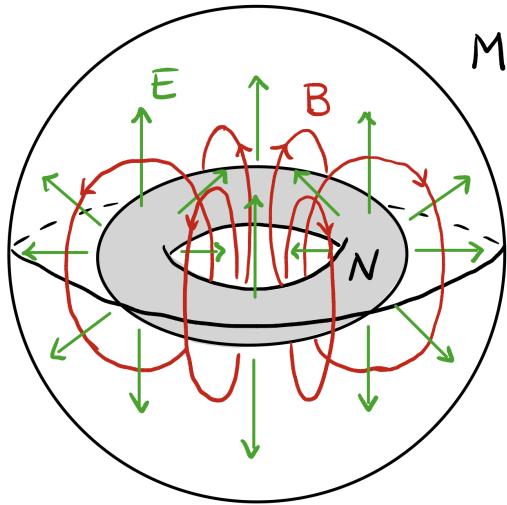


Figure 3: Non trivial electric and magnetic fields in the ball with a toroidal hole (no charges nor current inside).

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