The sharp isoperimetric inequality

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Abstract

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1 Introduction

In these notes we are going to prove the isoperimetric inequality in $\mathbb{R}^n$, characterizing the equality case. The main reference is [Mag12, §13.2 and Chapter 14].

Theorem 1.1 (Isoperimetric inequality)**.** *For every Lebesgue measurable set $E \subset \mathbb{R}^n$,*

$$P(E) \geq n\omega_n^{1/n}|E|^{\frac{n-1}{n}},$$

where $\omega_n = |B_1|$ is the n-volume of the standard unit ball of $\mathbb{R}^n$. Moreover, the equality holds if and only if E is equivalent to a ball, meaning that there exist $x \in \mathbb{R}^n$ and $R > 0$ (that necessarily satisfies $|E| = \omega_n R^n$) such that $|E \triangle B_R(x)| = 0$.

In other words, for every $R > 0$

$$\inf\{P(E) \colon E \subset \mathbb{R}^n, |E| = \omega_n R^n\} = n\omega_n R^{n-1}$$

and the infimum is realized only by sets equivalent to balls of radius R.

When $n = 1$, this is quite easy.

Theorem 1.2 (1-dimensional isoperimetric inequality)**.** *If $E \subset \mathbb{R}$ is a non empty Lebesgue measurable set with $|E| < \infty$, then*

$$P(E) \geq 2,$$

with equality if and only if E is equivalent to an interval of length $|E|$.

Proof. Since $E \neq \varnothing$ and $E \neq \mathbb{R}$, χ_E is not constant and we have that $P(E) = |D\chi_E|(\mathbb{R}) > 0$. Assume also that $P(E) < \infty$. Then E is equivalent to a finite union of disjoint open intervals, so $P(E) = 2N$ where N is the number of intervals. $\square$

For $n > 1$, the proof of the inequality is not hard: on bounded sets it's just the direct method (cf. Lemma 4.1), and the unbounded case can be obtained by approximating sets of finite perimeter with bounded sets of finite perimeter.

The actual work is to characterize the equality, i.e., minimizers of the variational problem

$$\alpha(n) = \inf \left\{ \frac{P(E)}{|E|} : E \subset \mathbb{R}^n \text{ Lebesgue measurable, } 0 < |E| < \infty \right\}, \tag{1}$$

for $m > 0$ fixed. A way to proceed is by using the Steiner inequality (cf. Theorem 3.1), which states that the Steiner symmetrization does not increase the perimeter. This implies that minimizers must have many symmetries, and it's not hard to deduce that they are balls. Therefore, a sketch of the proof is to show, in order

- finite perimeter sets can be approximated by bounded sets with polyhedral boundary (cf. Theorem 2.5);
- the Steiner inequality, proving it first on sets with polyhedral boundary and then approximating;
- minimizers of (1) must be equivalent to balls, using the Steiner inequality.

Another approach that the authors find interesting is considering the problem from the point of view of differential geometry, i.e., considering the class of smooth compact hypersurfaces that are the boundary of some open bounded set Ω. Assuming regularity, we can derive formulae for the first and second variation of the perimeter and use much finer tools. However, this point of view has a structural limit in high dimension, as *a priori* it's not clear if the class of smooth hypersurfaces contains minimizers of the isoperimetric problem. We discuss briefly this point of view in the last section of these notes.

2 Polyhedral approximation

Given a function $u\colon \mathbb{R}^{n-1} \to \mathbb{R}$ and a subset $G \subset \mathbb{R}^{n-1}$, we denote the *graph of u over G* by

$$\Gamma(u, G) := \{(x, u(x)) \in \mathbb{R}^n : x \in G\}.$$

Definition 2.1. A *polyhedral set* is a finite, non empty intersection of closed half-spaces.

A continuous function $u\colon G \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *finitely piecewise affine* if there exists a finite partition of polyhedral sets $\{G_i\}_{i=1}^k$ of G such that u is affine on each G_i.

A set $E \subset \mathbb{R}^n$ is said to have *polyhedral boundary* if for every $x \in \partial E$ there exist $r > 0$, a finitely piecewise affine function $u\colon (-r, r)^{n-1} \to \mathbb{R}$ and an direct isometry $T\colon \mathbb{R}^n \to \mathbb{R}^n$ with $T(x) = 0$ (so $T(u) = M(y-x)$ for some $M \in SO(n)$) such that

$$T(\partial E) \cap (-r, r)^n = \Gamma(u, (-r, r)^{n-1})$$
$$T(E) \cap (-r, r)^n = \{(z, t) \in (-r, r)^{n-1} \times (-r, r) : t < u(z)\}.$$

Remark 2.2. Let E be a bounded set with polyhedral boundary. By compactness of ∂E, there exists a finite number N of positive real numbers $r_1, \ldots, r_N$, affine isometries $T_1, \ldots, T_N$ and affine functions $u_1, \ldots, u_n$ as in the definition above such that

$$E = \bigcup_{i=1}^{N} \Gamma_i, \qquad \Gamma_i \coloneqq T_i^{-1}(\Gamma(u_i, (-r_i, r_i)^{n-1}).$$

The *outer unit normal* $\mathbf{n}_{\partial E}$ of E can be defined $\mathscr{H}^{n-1}$-a.e. on ∂E as the finitely piecewise constant function

$$\mathbf{n}_{\partial E} \circ T_i = M_i^{-1}(-\nabla u_i, 1)$$

on Γ_i, where $M_i = \mathrm{d}T_i \in SO(n)$. In particular, $\mathbf{n}_{\partial E}$ assumes finitely many values of $\mathbb{S}^{n-1} = \{v \in \mathbb{R}^n \colon |v| = 1\}$. This property is one of the key features of bounded sets with polyhedral boundary that we are going to employ in the proof of the Steiner inequality.

Remark 2.3. It's not hard to prove that any function $u \in C^1_c(\mathbb{R}^n)$ can be approximated by *compactly supported finitely piecewise affine functions* in the C^1-norm. Indeed, one could proceed as follows: take $\varepsilon > 0$ and subdivide $\mathbb{R}^n$ in n-simplexes of side ε. Then interpolate the values assumed by u on the vertices of those simplexes with an affine function on each simplex, obtaining a finitely piecewise affine function $u_\varepsilon \in C^0_c(\mathbb{R}^n)$, whose gradient ∇u_ε is defined away from the boundaries of those simplexes and constant in the interior of each simplex. It's easy to check that $u_\varepsilon \to u$ and $\nabla u_\varepsilon \to \nabla u$ in $L^\infty(\mathbb{R}^n)$.

Lemma 2.4 (Approximation by finitely piecewise affine functions)**.** *Let $n \geq 2$. For every $u \in BV(\mathbb{R}^n)$ there exists a sequence $u_h \colon \mathbb{R}^n \to \mathbb{R}$ of compactly supported finitely piecewise affine functions such that $u_h \to u$ in the BV-strict sense, that is, $u_h \to u$ in $L^1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |\nabla u_h| \mathrm{d}z \to |Du|(\mathbb{R}^n).$$

Proof. Step 1. Assume that $u \in C^1(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$. Take a sequence of cutoff functions $\varphi_h \colon \mathbb{R}^n \to \mathbb{R}$, that is, for some sequence $R_h \nearrow \infty$, $\varphi_h \in C^\infty_c(B_{R_h+1/h})$ with $0 \leq \varphi_h \leq 1$ on $\mathbb{R}^n$, $\varphi_h = 1$ on B_{R_h} and $|\nabla \varphi_h| \leq Ch$ on $B_{R_h+1/h} \setminus B_{R_h}$, for a constant $C > 0$ not depending on h. In particular,

$$\int_{B_{R_h+\frac{1}{h}} \setminus B_{R_h}} |\nabla \varphi_h| \mathrm{d}x \leq \frac{C'}{h^{n-1}} \to 0.$$

Then $v_h \coloneqq u\varphi_h \in C^1_c(\mathbb{R}^n)$, $v_h \to u$ in $L^1(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} |\nabla v_h| \mathrm{d}x = \int_{B_{R_h}} |\nabla u| \mathrm{d}x + \int_{B_{R_h+\frac{1}{h}} \setminus B_{R_h}} |\nabla v_h| \mathrm{d}x \to \int_{\mathbb{R}^n} |\nabla u| \mathrm{d}x$$

as $h \to \infty$. For every $h \in \mathbb{N}$, take a sequence $v_h^k \colon \mathbb{R}^n \to \mathbb{R}$ of compactly supported finitely piecewise affine functions approximating each v_h in $C_c^1(\mathbb{R}^n)$ with respect to the C^1-norm. Then in particular $v_h^k \to v_h$ in $L^1(\mathbb{R}^n)$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\nabla v_h^k| \mathrm{d}x = \int_{\mathbb{R}^n} |\nabla v_h| \mathrm{d}x.$$

With a diagonal argument, we can extract a diagonal subsequence $u_h := v_h^{k(h)}$ that satisfies the required properties.

Step 2. Now take any $u \in BV(\mathbb{R}^n)$, and fix a standard mollifier $\rho \in C_c^\infty(\mathbb{R}^n)$. Let $v_h = u \star \rho_{\varepsilon_h}$, with $\varepsilon_h \to 0$, and for each $h \in \mathbb{N}$ take a sequence $v_h^k \colon \mathbb{R}^n \to \mathbb{R}$ of finitely piecewise affine functions approximating each v_h, i.e., such that $v_h^k \to v_h$ in $L^1(\mathbb{R}^n)$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |\nabla v_h^k| \mathrm{d}x = \int_{\mathbb{R}^n} |\nabla v_h| \mathrm{d}x.$$

Since

$$\lim_{h\to\infty} \int_{\mathbb{R}^n} |\nabla v_h| \mathrm{d}x = |Du|(\mathbb{R}^n),$$

we can extract as before a diagonal sequence $u_h := v_h^{k(h)}$ that satisfies the required properties. □

Theorem 2.5 (Polyhedral approximation)**.** *Let $E \subset \mathbb{R}^n$ be a set with $|E| < \infty$ and $P(E) < \infty$. Then there exists a sequence of bounded sets $E_h \subset \mathbb{R}^n$ with polyhedral boundary such that*

$$E_h \to E, \qquad P(E_h) \to P(E).$$

Remark 2.6. In the proof, we are going to use the following formula: for every measurable set $E \subset \mathbb{R}^n$ and $u, v \in L^1_{loc}(\mathbb{R}^n)$,

$$\int_E |u - v| \mathrm{d}x = \int_{\mathbb{R}} \mathscr{L}^n(E \cap \{u > t\} \triangle \{v > t\}) \mathrm{d}t, \tag{2}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. This is follows from the usual interpretation of $\int_E |u - v|$ as the area between the graph of u and the graph of v over E. Indeed, consider separately $E^+ = E \cap \{u > v\}$ and $E^- = E \cap \{u < v\}$. By Fubini's theorem,

$$\begin{aligned}
\int_{E^+} |u - v| \mathrm{d}x &= \int_{E^+} \mathscr{L}^1((v(x), u(x))) \mathrm{d}x \\
&= \mathscr{L}^{n+1}(\{(x,t) \in E \times \mathbb{R} \colon v(x) < t < u(x)\}) \\
&= \int_{\mathbb{R}} \mathscr{L}^n(\{x \in E \colon v(x) < t < u(x)\}) \mathrm{d}t \\
&= \int_{\mathbb{R}} \mathscr{L}^n(E \cap \{u > t\} \setminus \{v > t\}) \mathrm{d}t
\end{aligned}$$

and similarly for E^-.

Proof of Theorem 2.5. Since $|E| < \infty$ and $P(E) < \infty$, $\chi_E \in BV(\mathbb{R}^n)$. By Lemma 2.4, there is a sequence of compactly supported finitely piecewise affine functions $u_h \colon \mathbb{R}^n \to \mathbb{R}$ such that $u_h \to \chi_E$ in $L^1(\mathbb{R}^n)$ and

$$\lim_{h\to\infty} \int_{\mathbb{R}^n} |\nabla u_h| \mathrm{d}x = |D\chi_E|(\mathbb{R}^n) = P(E). \tag{3}$$

For every $h \in \mathbb{N}$ and $t \in (0,1)$ consider the bounded set

$$E_h^t := \{u_h > t\}.$$

Since u_h is compactly supported and finitely piecewise affine, E_h^t is bounded and has polyhedral boundary. We are going to choose a suitable $t \in (0,1)$ (see Figure 1) so that E_h^t approximates E in the sense of the statement.

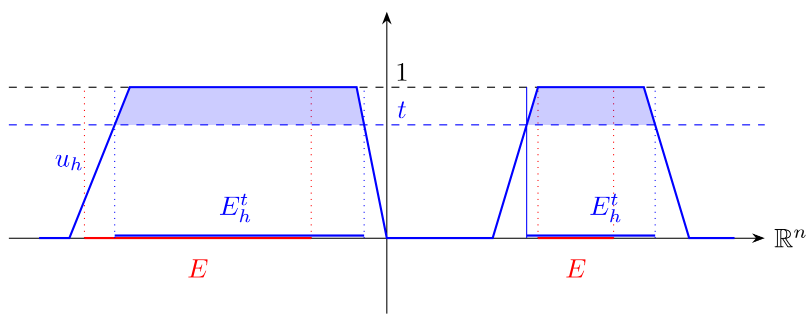

Figure 1: Construction of E_h^t.

By (2) we have that

$$\begin{aligned}
\int_0^1 \int_{\mathbb{R}^n} |\chi_{E_h^t} - \chi_E| \mathrm{d}x\mathrm{d}t &= \int_0^1 \int_{\mathbb{R}^n} |\chi_{\{u_h>t\}} - \chi_{\{\chi_E>t\}}| \mathrm{d}x\mathrm{d}t \\
&= \int_0^1 \mathscr{L}^n(\{u_h > t\} \triangle \{\chi_E > t\})\mathrm{d}t \\
&\le \int_{\mathbb{R}} \mathscr{L}^n(\mathbb{R}^n \cap \{u_h > t\} \triangle \{\chi_E > t\})\mathrm{d}t \\
&= \int_{\mathbb{R}^n} |u_h - \chi_E| \mathrm{d}x \to 0,
\end{aligned}$$

so $E_h^t \to E$ for a.e. $t \in (0,1)$, and in particular, by the lower semicontinuity of the perimeter,

$$P(E) \le \liminf_{h\to\infty} P(E_h^t) \text{ for a.e. } t \in (0,1). \tag{4}$$

By (3),

$$\begin{aligned}
P(E) &= \lim_{h\to\infty} \int_{\mathbb{R}^n} |\nabla u_h| \mathrm{d}x \\
&= \lim_{h\to\infty} \int_{\mathbb{R}} P(\{u_h > t\}) \mathrm{d}t \qquad \text{(coarea formula)} \\
&\geq \int_{\mathbb{R}} \liminf_{h\to\infty} P(\{u_h > t\}) \mathrm{d}t \quad \text{(Fatou's lemma)} \\
&\geq \int_0^1 \liminf_{h\to\infty} P(E_h^t) \mathrm{d}t \\
&\geq P(E),
\end{aligned}$$

having used (4) in the last inequality. Therefore,

$$P(E) = \int_0^1 \liminf_{h\to\infty} P(E_h^t) \mathrm{d}t,$$

and again by (4) it must be

$$P(E) = \liminf_{h\to\infty} P(E_h^t) \qquad \text{for a.e. } t \in (0,1).$$

Up to passing to a subsequence (which we don't relabel),

$$P(E) = \lim_{h\to\infty} P(E_h^t) \qquad \text{for a.e. } t \in (0,1). \tag{5}$$

Thus we can find $t \in (0,1)$ such that $E_h \coloneqq E_h^t$ is as required. □

3 Steiner inequality

Let us recall the definition of Steiner symmetrization. Let $E \subset \mathbb{R}^n$ and fix a unit vector $v \in \mathbb{R}^n$. For every $z \in \langle v\rangle^\perp = \mathbb{R}^{n-1}$, consider the *$v$-vertical z-slice*

$$E_z^v \coloneqq \{t \in \mathbb{R} \colon z + tv \in E\} \subset \mathbb{R}.$$

The *Steiner symmetrization of E with respect to the direction v* is the set

$$S_v(E) \coloneqq \left\{ z + tv \in \mathbb{R}^n = \langle v\rangle^\perp \oplus \langle v\rangle \colon |t| < \frac{1}{2}\mathscr{L}^1(E_z^v) \right\}.$$

By Fubini's theorem, $|S_v(E)| = |E|$. It is also quite easy to prove that

$$\mathrm{diam}(S_v(E)) \leq \mathrm{diam}(E).$$

In what follows, we will always identify $\langle v\rangle^\perp = \mathbb{R}^{n-1}$ with a suitable change of coordinates, that is, we take $v = e_n$. The usual notation will then be to write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ with coordinates $x = (z,t)$, $z = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $t = x_n \in \mathbb{R}$.

Theorem 3.1 (Steiner inequality)**.** *Let $E \subset \mathbb{R}^n$ be a set with $|E| < \infty$ and finite perimeter $P(E) < \infty$. Then for every unit vector $v \in \mathbb{R}^n$*

$$P(S_v(E)) \leq P(E),$$

and the equality holds if and only the z-slice E_z^v is an interval for a.e. $z \in \mathbb{R}^{n-1}$.

Proof. Without loss of generality, assume that $v = e_n$ and denote $E_z = E_z^v$, $E^s \coloneqq S_{e_n}(E)$. We divide the proof into three steps.

Step 1. Assume that E is a bounded set with polyhedral boundary ∂E, and that the outer unit normal $\mathbf{n}_{\partial E}$ of E is never orthogonal to e_n. In particular, $P(E) < \infty$ and $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial E$. Let G be the *essential projection* of E on $\mathbb{R}^{n-1}$, that is,

$$G \coloneqq \{z \in \mathbb{R}^{n-1} \colon \mathscr{L}^1(E_z) > 0\}.$$

Since $\mathbf{n}_{\partial E}$ is never orthogonal to e_n, in the definition of set with polyhedral boundary, we can always take T to be a translation, so there exist a finite partition $\{G_h\}_{h=1}^M$ of G made of $(n-1)$-polyhedral sets and affine functions

$$v_h^1, u_h^1, \ldots, v_h^{N(h)}, u_h^{N(h)} \colon \mathbb{R}^{n-1} \to \mathbb{R},$$

such that

$$\begin{aligned}
\partial E &= \bigcup_{h=1}^{M} \bigcup_{k=1}^{N(h)} \Gamma(u_h^k, G_h) \cup \Gamma(v_h^k, G_h) \\
E &= \bigcup_{h=1}^{M} \left\{ (z,t) \in G_h \times \mathbb{R} \colon t \in \bigcup_{k=1}^{N(h)} (v_h^k(z), u_h^k(z)) \right\}
\end{aligned}$$

(see Figure 2). With this notation,

$$\begin{aligned}
P(E) = \mathscr{H}^{n-1}(\partial E) &= \sum_{h=1}^{M} \mathscr{H}^{n-1}(\partial E \cap (G_h \times \mathbb{R})) \\
&= \sum_{h=1}^{M} \int_{G_h} \sum_{k=1}^{N(h)} \left(\sqrt{1 + |\nabla v_h^k|^2} + \sqrt{1 + |\nabla u_h^k|^2} \right) \mathrm{d}z.
\end{aligned} \tag{6}$$

For every $z \in \mathbb{R}^{n-1}$,

$$E_z = \begin{cases} \displaystyle\bigcup_{k=1}^{N(h)} (v_h^k(z), u_h^k(z)) & \text{if } z \in G_h, \\ \varnothing & \text{if } z \in \mathbb{R}^{n-1} \setminus G \end{cases}$$

and, if we set $m(z) = \mathscr{L}^1(E_z)$ for every $z \in \mathbb{R}^{n-1}$,

$$m(z) \coloneqq \mathscr{L}^1(E_z) = \begin{cases} \displaystyle\sum_{k=1}^{N(h)} u_h^k(z) - v_h^k(z) & \text{if } z \in G_h, \\ 0 & \text{if } z \in \mathbb{R}^{n-1} \setminus G. \end{cases} \tag{7}$$

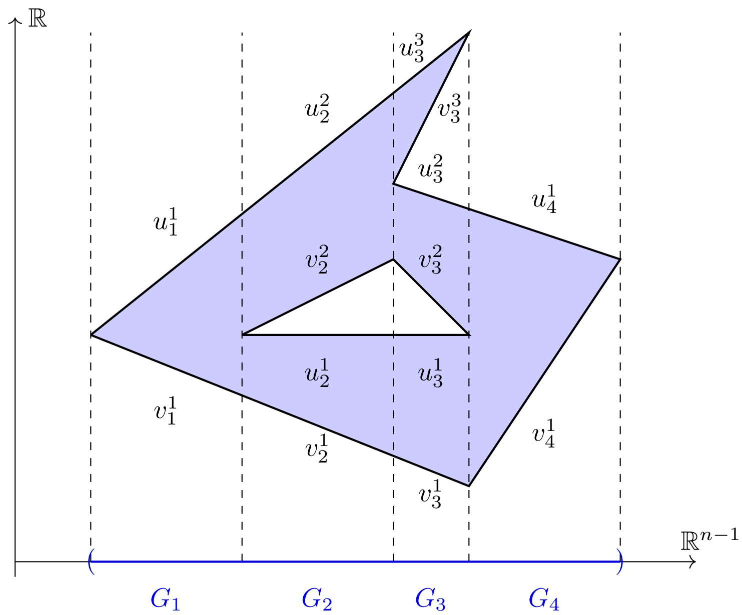

Figure 2: Notation adopted in step 1 of the proof of Theorem 3.1.

Observe that $m\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is affine on each G_h. Therefore,

$$E^s = \{(z,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |t| < m(z)/2\}$$

is a bounded set with polyhedral boundary, and in particular it has finite perimeter and $|D\chi_{E^s}| = \mathscr{H}^{n-1} \llcorner (\Gamma(m/2, G) \cup \Gamma(-m/2, G))$ given by

$$\begin{aligned} P(E^s) = \mathscr{H}^{n-1}(\partial E^s) &= 2\int_G \sqrt{1 + \left|\frac{1}{2}\nabla m\right|^2}\mathrm{d}z \\ &= \sum_{h=1}^{M} \int_{G_h} \sqrt{4 + |\nabla m|^2}\mathrm{d}z \end{aligned} \tag{8}$$

By (7) and the convexity of the function $z \mapsto \sqrt{1+|z|^2}$,

$$\begin{aligned} \sum_{k=1}^{N(h)} \sqrt{1+|\nabla v_h^k|^2} + \sqrt{1+|\nabla u_h^k|^2} &\geq 2N(h)\sqrt{1 + \left|\sum_{k=1}^{N(h)} \frac{\nabla u_h^k - \nabla v_h^k}{2N(h)}\right|^2} \\ &= \sqrt{4N(h)^2 + |\nabla m|^2} \end{aligned}$$

for every $h \in \{1, \ldots, M\}$. Therefore, by (6) and (8),

$$P(E^s) \leq \sum_{h=1}^{M} \int_{G_h} \sqrt{4N(h)^2 + |\nabla m|^2}\mathrm{d}z \leq P(E). \tag{9}$$

We have just found that

$$\begin{aligned}P(E)-P(E^s) &\geq \sum_{N(h)\geq 2}\int_{G_h}\left(\sqrt{4N(h)^2+|\nabla m|^2}-\sqrt{4+|\nabla m|^2}\right)\mathrm{d}z\\ &= \sum_{N(h)\geq 2}\int_{G_h}\frac{4(N(h)^2-1)}{\sqrt{4N(h)^2+|\nabla m|^2}+\sqrt{4+|\nabla m|^2}}\mathrm{d}z\\ &\geq \sum_{N(h)\geq 2}\int_{G_h}2\frac{1}{\sqrt{4N(h)^2+|\nabla m|^2}}\mathrm{d}z. \end{aligned}\tag{10}$$

If

$$D := \{z\in\mathbb{R}^{n-1}\colon E_z \text{ is not an interval}\} = \bigcup_{N(h)\geq 2} G_h,$$

then by Cauchy-Schwarz inequality, (9) and (10),

$$\begin{aligned}2\mathscr{L}^{n-1}(D)^2 &= 2\left(\sum_{N(h)\geq 2}\int_{G_h}\frac{(4N(h)^2+|\nabla m|^2)^{1/4}}{(4N(h)^2+|\nabla m|^2)^{1/4}}\right)^2\\ &\leq P(E)(P(E)-P(E^s)).\end{aligned}\tag{11}$$

Step 2. Now let $E\subset\mathbb{R}^n$ with $|E|<\infty$ and $P(E)<\infty$. By Theorem 2.5 there exists a sequence of bounded sets E_h with polyhedral boundary such that

$$E_h\to E,\qquad P(E_h)\to P(E).$$

Note that we can assume that the outer unit normals $\mathbf{n}_{\partial E_h}$ of E_h are never orthogonal to e_n. Indeed, as explained in Remark 2.2, each $\mathbf{n}_{\partial E_h}$ assumes finitely many values of $\mathbb{S}^{n-1}$, so if it happens that $\mathbf{n}_{\partial E_h}\perp e_n$ somewhere, we only need to rotate slightly ∂E. Doing smaller and smaller rotations as $h\to\infty$, we still have that $E_h\to E$ (while $P(E_h)\to P(E)$ isn't affected by rotations).

Observe that, since $(E_h\triangle E)_z = ((E_h)_z\triangle E_z)$, by Fubini

$$\int_{\mathbb{R}^{n-1}}\mathscr{L}^1((E_h)_z\triangle E_z)\mathrm{d}z = |E_h\triangle E|\to 0,$$

so $(E_h)_z\to E_z$ for a.e. $z\in\mathbb{R}^{n-1}$, and by lower semicontinuity of the perimeter

$$P(E_z)\leq\liminf_{h\to\infty}P((E_h)_z).\tag{12}$$

For every $h\in\mathbb{N}$, denote

$$\begin{aligned}m_h(z) &:= \mathscr{L}^1((E_h)_z),\quad m(z):=\mathscr{L}^1(E_z)\qquad \forall z\in\mathbb{R}^{n-1}\\ G_h &:= \{z\in\mathbb{R}^{n-1}\colon m_h(z)>0\},\qquad G:=\{z\in\mathbb{R}^{n-1}\colon m(z)>0\}\\ D_h &:= \{z\in\mathbb{R}^{n-1}\colon (E_h)_z \text{ is not an interval}\}.\end{aligned}$$

Applying step 1 to each E_h, by (9) and (11) we get

$$P(E_h^s)\leq P(E_h),\qquad 2\mathscr{L}^{n-1}(D_h)^2\leq P(E_h)(P(E_h)-P(E_h^s)).\tag{13}$$

Since $E_h \to E$, we have that

$$G_h \to G. \tag{14}$$

Indeed, $G_h \triangle G$ is equivalent to

$$\{z \in \mathbb{R}^{n-1} \colon \mathscr{L}^1((E_h)_z \triangle E_z) > 0\} = \bigsqcup_{k\in\mathbb{Z}} G_h^k,$$

where

$$G_h^k := \left\{ z \in \mathbb{R}^{n-1} \colon \left(1+\frac{1}{h}\right)^k \leq \mathscr{L}^1((E_h)_z \triangle E_z) < \left(1+\frac{1}{h}\right)^{k+1} \right\}.$$

Then,

$$\sum_{k\in\mathbb{Z}} \left(1+\frac{1}{h}\right)^k \mathscr{L}^{n-1}(G_h^k) \leq \sum_{k\in\mathbb{Z}} \int_{G_h^k} \mathscr{L}^1((E_h)_z \triangle E_z) \mathrm{d}z = |E_h \triangle E|,$$

and taking the limsup as $h \to 0$, we get $\limsup_{h\to\infty} \mathscr{L}^{n-1}(G_h \triangle G) = 0$.

Let us prove that $P(E^s) \leq P(E)$. By Fubini's theorem,

$$\begin{aligned} |E_h^s \triangle E^s| &= \int_{\mathbb{R}^{n-1}} |m_h(z) - m(z)| \mathrm{d}z \\ &\leq \int_{\mathbb{R}^{n-1}} \mathscr{L}^1((E_h)_z \triangle E_z) \mathrm{d}z = |E \triangle E_h| \end{aligned}$$

having used that $|\mathscr{L}^1(A) - \mathscr{L}^1(B)| \leq \mathscr{L}^1(A \triangle B)$ for every $A, B \subset \mathbb{R}$. In particular $E_h^s \to E^s$ and by lower semicontinuity

$$P(E^s) \leq \liminf_{h\to\infty} P(E_h^s) \leq \lim_{h\to\infty} P(E_h) = P(E).$$

Now, assume that $P(E) = P(E^s)$ and let us prove that E_z is equivalent to an interval for a.e. $z \in \mathbb{R}^{n-1}$. By (13) and (14), $\chi_{D_h} \to 0$ and $\chi_{G_h} \to \chi_G$ in $L^1(\mathbb{R}^{n-1})$, so, up to passing to a subsequence, we have the pointwise limit

$$\chi_G(z) = \lim_{h\to\infty} \chi_{G_h \setminus D_h}(z)$$

for a.e. $z \in \mathbb{R}^{n-1}$. Combining this with (12),

$$\chi_G(z) P(E_z) \leq \liminf \chi_{G_h \setminus D_h}(z) P((E_h)_z) \qquad \text{for a.e. } z \in \mathbb{R}^{n-1},$$

and, by Fatou's lemma,

$$\begin{aligned} \int_G P(E_z) \mathrm{d}z &\leq \int_{\mathbb{R}^{n-1}} \liminf \chi_{G_h \setminus D_h}(z) P((E_h)_z) \mathrm{d}z \\ &\leq \liminf_{h\to\infty} \int_{G_h \setminus D_h} P((E_h)_z) \mathrm{d}z \\ &= 2 \liminf_{h\to\infty} \mathscr{L}^{n-1}(G_h \setminus D_h) \\ &= 2\mathscr{L}^{n-1}(G) \end{aligned}$$

having used that for every $z \in G_h \setminus D_h$, the slice $(E_h)_z$ is equivalent to a bounded interval, so $P(E_z) = 2$. But by the 1-dimensional isoperimetric inequality, $P(E_z) \geq 2$ for a.e. $z \in \mathbb{R}^{n-1}$, so

$$\int_G P(E_z)\mathrm{d}z \leq 2\mathscr{L}^{n-1}(G) \leq \int_G P(E_z),$$

and this implies that $P(E_z) = 2$ for a.e. $z \in G$. By the characterization of the equality in the 1-dimensional isoperimetric inequality, E_z is an interval for a.e. $z \in G$. If $z \in \mathbb{R}^n \setminus G$, $\mathscr{L}^1(E_z) = 0$, so E_z is equivalent to the empty set (which is trivially an interval). □

Lemma 3.2. *Let $E \subset \mathbb{R}^n$ be a convex set with $|E| < \infty$ and $P(E) < \infty$, and let $v \in \mathbb{R}^{n-1}$ a unit vector such that $P(S_v(E)) = P(E)$. Then there exists $c \in \mathbb{R}$ such that E is equivalent to the translated $S_v(E) + cv$.*

Proof. Assume without loss of generality that $v = e_n$, and denote $S_v(E) = E^s$. Since $\mathscr{L}^n(\partial E) = 0$[1], we may assume that E is open. Then the projection $G \subset \mathbb{R}^{n-1}$ is a convex set, and for every $z \in G$, the vertical z-slice is equivalent to the open interval $(-v(z), u(z))$, where $v, u \colon G \to \mathbb{R}$ are concave non negative functions, so in particular they are locally Lipschitz and E has Lipschitz boundary. By Rademacher's theorem, ∇u and ∇v are defined at a.e. $z \in G$, so

$$P(E) = \int_G \left(\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}\right) \mathrm{d}z$$
$$P(E^s) = 2\int_G \sqrt{1 + \left|\frac{\nabla u + \nabla v}{2}\right|^2} \mathrm{d}z$$

since

$$E^s = \left\{(z,t) \in G \times \mathbb{R} \colon |t| < \frac{u(z) + v(z)}{2}\right\}.$$

By the strict convexity of the function $z \mapsto \sqrt{1 + |z|^2}$,

$$\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2} \geq 2\sqrt{1 + \left|\frac{\nabla u + \nabla v}{2}\right|^2}$$

with equality if and only if $\nabla u = \nabla v$. Hence, if $P(E^s) = P(E)$, the equality holds almost everywhere and there must be that $\nabla u = \nabla v$ a.e. in G. Thus, there exists $c \in \mathbb{R}$ such that $u(z) - v(z) = c$ for a.e. $z \in G$, which implies that

$$E = \{(z,t) \in G \times \mathbb{R} \colon -u(z) - c < t < u(z)\},$$
$$E^s = \{(z,t) \in G \times \mathbb{R} \colon -u(z) - c/2 < t < u(z) + c/2\},$$

that is $E = E^s + \frac{c}{2}e_n$. □

[1] Indeed, without loss of generality assume that $0 \in \mathring{C}$. Then by convexity, for every $\varepsilon > 0$

$$\partial C \subset \left(\frac{1}{1-\varepsilon}\mathring{C}\right) \setminus C.$$

Recall that the density of E at a point $x \in \mathbb{R}^n$ is

$$\Theta(E,x) \coloneqq \lim_{r\to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n}$$

whenever it exists. It's easy to show that, whenever this limit exists, it's actually equal to

$$\Theta(E,x) = \lim_{r\to 0} \frac{|E \cap Q^n(x,r)|}{2^n r^n} \tag{15}$$

where

$$Q^n(x,r) = x + (-r,r)^n$$

is the cube centered at x with side $2r$.

Observe that $\mathscr{L}^n \llcorner E = \chi_E \mathscr{L}^n$, so by Besicovitch differentiation theorem,

$$\chi_E = \frac{\mathrm{d}(\mathscr{L}^n \llcorner E)}{\mathrm{d}\mathscr{L}^n} = \Theta(E,\cdot)$$

a.e. on $\mathbb{R}^n$, that is,

$$\Theta(E,x) = \begin{cases} 1 & \text{a.e. } x \in E \\ 0 & \text{a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

In particular, E is equivalent to

$$E^{(1)} \coloneqq \{x \in \mathbb{R}^n \colon \Theta(E,x) = 1\},$$

the set of with density one points of E. This will be the "good set" to replace E with.

Lemma 3.3. *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter and suppose that E_z is equivalent to an interval for a.e. $z \in \mathbb{R}^{n-1}$. Then $(E^{(1)})_z$ is an interval for every $z \in \mathbb{R}^{n-1}$.*

Proof. Let $z_0 \in \mathbb{R}^{n-1}$, and suppose that $x_1 = (z_0,t_1), x_2 = (z_0,t_2) \in E^{(1)}$ for some $t_1 < t_2$, and fix $t_0 \in (a,b)$. We have to prove that $x_0 = (z_0,t_0) \in E^{(1)}$, that is $\Theta(E,x_0) = 1$.

Fix $0 < \varepsilon < 1$, and for every $t \in \mathbb{R}$, $r > 0$ denote $I_r(t) = B_r(t) = (t-r,t+r)$. Since $x_1, x_2 \in E^{(1)}$, $\Theta(E,x) = \Theta(E,y) = 1$, so by (15) there exists $\overline{r} > 0$ such that for every $r < \overline{r}$

$$|E \cap Q^n(x_k,r)| \geq \left(1 - \frac{\varepsilon}{2}\right) 2^n r^n, \quad I_r(t_0) \cap I_r(t_k) = \varnothing \qquad k = 1,2. \tag{16}$$

Fix $r < \overline{r}$ and for $k = 1,2$ denote by G_k the set of $z \in Q^{n-1}(z_0,r)$ such that E_z is equivalent to an interval and $\mathscr{L}^1(E_z \cap I_r(t_k)) > 0$. Since E_z is equivalent to an interval for a.e. $z \in \mathbb{R}^{n-1}$, by Fubini's theorem

$$|E \cap Q^n(x_k,r)| = \int_{G_k} \mathscr{L}^1(E_z \cap I_r(t_k))\mathrm{d}z \leq 2r\mathscr{L}^{n-1}(G_k).$$

Combining this with (16),

$$\mathscr{L}^{n-1}(G_k) \geq \Big(1 - \frac{\varepsilon}{2}\Big) 2^{n-1} r^{n-1}, \qquad k = 1, 2.$$

Therefore,

$$\begin{aligned}\mathscr{L}^{n-1}(G_1 \cap G_2) &= \mathscr{L}^{n-1}(G_1) + \mathscr{L}^{n-1}(G_2) - \mathscr{L}^{n-1}(G_1 \cup G_2) \\ &\geq \mathscr{L}^{n-1}(G_1) + \mathscr{L}^{n-1}(G_2) - \mathscr{L}^{n-1}(Q^{n-1}(z_0, r)) \\ &\geq (1-\varepsilon) 2^{n-1} r^{n-1}.\end{aligned}$$

Now let $z \in G_1 \cap G_2$. Since E_z is equivalent to an interval and $\mathscr{L}^1(E_z \cap I_r(t_k)) > 0$ for $k = 1, 2$, by (16) we have

$$\mathscr{L}^1(E_z \cap I_r(t_0)) = 2r.$$

Integrating this over $G_1 \cap G_2$, we find

$$\begin{aligned}|E \cap Q^n(x_0, r)| &\geq \int_{G_1 \cap G_2} \mathscr{L}^1(E_z \cap I_r(t_0)) \mathrm{d}z \\ &= 2r \mathscr{L}^{n-1}(G_1 \cap G_2) \geq (1-\varepsilon) 2^n r^n.\end{aligned}$$

for every $r < \overline{r}$. Since $\varepsilon > 0$ was arbitrary,

$$\lim_{r \to 0} \frac{|E \cap Q^n(x_0, r)|}{2^n r^n} = 1,$$

which proves that $x_0 \in E^{(1)}$. □

4 The isoperimetric inequality

Lemma 4.1 (Constrained isoperimetric inequality)**.** *Let $m, R > 0$ with $m < \omega_n R^n$. Then*

$$\alpha(m, R) \coloneqq \inf\{P(E) \colon E \subset B_R \textit{ measurable}, |E| = m\}$$

is realized, and every minimizer is equivalent to a ball. In particular, $\alpha(m, R)$ is the perimeter of a ball of volume m, that is

$$\alpha(m, R) = n \omega_n^{1/n} m^{\frac{n-1}{n}}.$$

Proof. That $\alpha(m, R)$ is realized is a consequence of the Direct Method, as bounded sets in $BV(B_R)$ are precompact with respect to the weak* convergence, the family of characteristic functions of sets in B_R is weak*-closed, and any limit in the class preserves the volume.

Let us now assume that $E \subset B_R$ satisfies $|E| = m$ and $P(E) = \alpha(m, R)$. Since $|E \triangle E^{(1)}| = 0$, we may assume that $E = E^{(1)}$. For every unit vector $v \in \mathbb{R}^n$, $|v| = 1$, by Theorem 3.1 we have that $P(S_v(E)) \leq P(E)$. On the other hand, by the properties of the Steiner symmetrization $|S_v(E)| = m$ and $\operatorname{diam}(S_v(E)) \leq \operatorname{diam} E < R$, so $P(S_v(E)) \geq \alpha(m, R) = P(E)$. Then by Theorem 3.1 and Lemma 3.3

$$E_z^v := \{s \in \mathbb{R} \colon z + sv \in E\}$$

is an interval for every $z \in v^\perp$.

We claim that E is convex. Indeed, let $x, y \in E$ be two distinct points, and set $v = (y - x)/|y - x|$. Then if $z \in v^\perp$ is the projection of x (and y) in $v^\perp$,

$$(1 - \lambda)x + \lambda y = z + [(1 - \lambda)x \cdot v + \lambda x \cdot v]v \in E, \qquad \forall \lambda \in [0, 1]$$

as $[x \cdot v, y \cdot v] \subset E_{v,z}$.

By Theorem 3.1, for every unit vector $v \in \mathbb{R}^n$ there exists $c_v \in \mathbb{R}$ such that $E = S_v(E) + c_v v$. Consider

$$E' := E - (c_{e_1} e_1 + \cdots c_{e_n} e_n).$$

Since E' is still convex and for every unit vector $v \in \mathbb{R}^n$ we still have $P(E') = P(S_v(E'))$, there are constants $c'_v \in \mathbb{R}$ such that $E' = S_v(E') + c'_v v$. But by construction,

$$S_{e_i}(E') = S_{e_i}(E) - \sum_{j \neq i} c_j e_j = E - c_i e_i - \sum_{j \neq i} c_j e_j = E',$$

so $c'_i = 0$ for every $i = 1, \ldots, n$. Hence, E' is invariant under the reflections with respect to the coordinates planes, and therefore invariant under the map $z \mapsto -z$ (the composition of all those reflections). Since for every unit vector v we can write $E' = S_v(E) + c'_v v$, we have

$$S_v(E) + c'_v v = -S_v(E) - c'_v v = S_v(E) - c'_v v. \tag{17}$$

Clearly, $c'_{-v} = -c'_v$ for every unit vector v. On the other hand, combining (17) with the fact that $S_v(E) = -S_v(E) = S_{-v}(E)$, we also have that $c'_v = c'_{-v}$. Therefore, for every unit vector $v \in \mathbb{R}^n$, it must be $c'_v = 0$, that is, $S_v(E') = E'$.

As a consequence, E' is invariant under reflections across hyperplanes through the origin. Since the set of those reflections generates $O(n)$, it follows that E' is invariant under every rotation that fixes the origin. Since $|E'| = m > 0$, $E' \neq \{0\}$. Let $x \in E'$, $x \neq 0$. Since the rotations fixing the origin are a group that acts transitively on the sphere $S_{|x|}$ of radius $|x|$, it follows that $S_{|x|}$ is contained in E'. By the convexity of E', the whole ball $B_{|x|}$ of radius $|x|$ is contained in E'. Since x was arbitrary, then E' is a union of concentric balls centered at the origin and since E' is bounded, it must be a ball. □

We can now prove the isoperimetric inequality.

Proof of Theorem 1.1. Suppose that E is bounded. Then taking $R > 0$ big enough, by Lemma 4.1,

$$P(E) \geq n\omega_n^{1/n} |E|^{\frac{n-1}{n}},$$

and the equality holds if and only if E is equivalent to a ball.

If E is not bounded, by Theorem 2.5 we can approximate it with bounded sets, that is, there are bounded sets $E_h \subset \mathbb{R}^n$ with $E_h \to E$ and $P(E_h) \to P(E)$. Passing to the limit, the inequality is preserved.

Now assume that $E = E^{(1)}$ is unbounded and $|E| < \infty$. Assume by contradiction that

$$P(E) = n\omega_n^{1/n}|E|^{\frac{n-1}{n}}.$$

By (3.1), $P(E) = P(S_v(E))$ for every unit vector $v \in \mathbb{R}^n$. Up to a translation, arguing exactly as in Lemma 4.1 we have that E is a union of concentric balls. Since E is unbounded, we have that $E = \mathbb{R}^n$, a contradiction because we assumed that $|E| < \infty$. $\square$

5 An alternative approach

Adopting the point of view of differential geometry, we could have used a completely different approach. Indeed, fix $m > 0$ and let

$$\alpha(n,m) \coloneqq \inf \left\{ \mathscr{H}^{n-1}(\Sigma) \;\middle|\; \begin{array}{l} \Sigma = \partial\Omega \text{ smooth, connected,} \\ \text{compact hypersurface, } |\Omega| = m \end{array} \right\}. \tag{18}$$

Suppose that $\Sigma = \partial\Omega$ is a minimizer, that is, $\mathscr{H}^{n-1}(\Sigma) = \alpha(m)$ and $|\Omega| = m$. Let $\nu\colon \Sigma \to \mathbb{S}^{n-1}$ be the outer unit normal of Σ, A its second fundamental form, defined by

$$A(X,Y) = \mathrm{d}\nu_x[X] \cdot Y$$

for every tangent vectors $X, Y \in T_x\Sigma = T_{\nu(x)}\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $H = \mathrm{tr}(A)$ its mean curvature. By the inverse function theorem and by the compactness of Σ, there is $\varepsilon > 0$ such that

$$\phi(t,x) = x + t\nu(x),$$

is a diffeomorphism on $(-\varepsilon,\varepsilon) \times \Sigma$. Let $u\colon \Sigma \to \mathbb{R}$ be a smooth function with zero average, i.e., $\int_\Sigma u \mathrm{d}\mathscr{H}^{n-1} = 0$, and define

$$F(t,x) = \phi(tu(x), x)$$

for every $x \in \Sigma$ and $t \in \mathbb{R}$ small enough. Denote

$$\Sigma_t = \{F(t,x) \colon x \in \Sigma\}.$$

Then, $\Sigma_t = \partial\Omega_t$ is the boundary of a bounded set Ω_t. For every $x \in \Sigma$, consider the geodesic

$$\gamma_x(s) = \phi(s,x),$$

which has constant velocity $|\dot\gamma_x| = u(x)$. Then, by the coarea formula,

$$\begin{aligned} |\Omega_t| &= |\Omega| + |\{\phi(s,x) \colon 0 < s < tu(x)\}| - |\{\phi(s,x) \colon tu(x) < s < 0\}| \\ &= m + \int_{\{u>0\}} \int_0^t |\dot\gamma_x| \mathrm{d}t \mathrm{d}\mathscr{H}^{n-1}(x) - \int_{\{u<0\}} \int_0^t |\dot\gamma_x| \mathrm{d}t \mathrm{d}\mathscr{H}^{n-1}(x) \\ &= m + t\int_\Sigma u \;\mathrm{d}\mathscr{H}^{n-1} = m, \end{aligned}$$

hence Σ_t is a competitor in (18) for every t. By the minimality of Σ,

$$\mathscr{H}^{n-1}(\Sigma_t) \geq \mathscr{H}^{n-1}(\Sigma),$$

so the function $t \mapsto \mathscr{H}^{n-1}(\Sigma_t)$ has a minimum at $t = 0$. The first variation formula implies that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathscr{H}^{n-1}(\Sigma_t) = \int_\Sigma Hu \,\mathrm{d}\mathscr{H}^{n-1}.$$

Since this hold for every $u : \Sigma \to \mathbb{R}$ with zero average, H is constant, and we say that Σ is *CMC*.

Taking the second derivative with respect to t, the second variation formula implies that

$$0 \leq \int_\Sigma |\nabla_\Sigma u|^2 - |A|^2 u^2 \,\mathrm{d}\mathscr{H}^{n-1} = -\int_\Sigma u L_\Sigma u \,\mathrm{d}\mathscr{H}^{n-1} \tag{19}$$

for every smooth function u with zero average, where

$$L_\Sigma u = \Delta_\Sigma u + |A|^2 u$$

is the *Jacobi operator* of Σ. Any hypersurface Σ that satisfies (19) is called *CMC-stable*.

This approach can give another proof of the isoperimetric inequality. In particular, it holds the following stronger result.

Theorem 5.1 (Barbosa-do Carmo, [BC84]). *Let Σ be a compact, connected, orientable, immersed CMC hypersurface in $\mathbb{R}^n$. Then Σ is CMC-stable if and only if Σ is a round sphere.*

The proof is basically just a computation: test (19) with the function

$$u(x) = H(\nu(x) \cdot x) - n + 1 \qquad \forall x \in \Sigma.$$

A.D. Alexandrov, with the celebrated *moving plane method* (which heavily uses the strong maximum principle), proved that spheres are the only *critical points* of the area functional with respect to variations with fixed volume.

Theorem 5.2 (Alexandrov, [Ale62]). *Let Σ be a compact, connected, embedded CMC hypersurface in $\mathbb{R}^n$. Then Σ is a round sphere.*

A posteriori, it is not restrictive to assume that Σ is connected in the isoperimetric problem (18). In fact assume that $\Sigma = \partial\Omega$ is not connected. Let $\Omega_1, \dots, \Omega_k$ be the connected components of Ω, and $\Sigma_0^j, \dots, \Sigma_{N(j)}^j$ the connected components of Σ such that $\partial\Omega_j = \bigcup_i \Sigma_i^j$. Since each Σ_i^j is CMC-stable, by Theorem 5.1 it's a round sphere, so Σ_i^j a boundary of a ball B_i^j. Hence, every Ω_j must be obtained by removing some disjoint balls $B_1^j, \dots, B_{N(j)}^i$ from an initial big ball B_0^j of radius R_j. Denote $\Omega^* = \bigcup_j B_0^j \supset \Omega$ and $\Sigma^* = \partial\Omega^* \subset \Sigma$. Then it's easy to see (for example, by induction over k) that

$$\begin{aligned}
\mathscr{H}^{n-1}(\Sigma^*)^n &= \left(n\omega_n \sum_{j=1}^k R_j^{n-1}\right)^n = n^n\omega_n^n \left(\sum_{j=1}^k R_j^{n-1}\right)^n \\
&\geq n^n\omega_n^n \left(\sum_{j=1}^k R_j^n\right)^{n-1} = n^n\omega_n^{n-1} \left(\omega_n \sum_{j=1}^k R_j^n\right)^{n-1} \\
&= n^n\omega_n |\Omega^*|^{n-1}
\end{aligned}$$

with strict inequality whenever $k > 1$, which is indeed the case if $\Sigma = \Sigma^*$. If instead $\Sigma \neq \Sigma^*$, then observe that

$$\mathscr{H}^{n-1}(\Sigma) > \mathscr{H}^{n-1}(\Sigma^*) \geq n\omega_n^{1/n}|\Omega^*|^{\frac{n-1}{n}} > n\omega_n^{1/n}|\Omega|^{\frac{n-1}{n}}.$$

However, this approach encounters an insurmountable limit in high dimension: we are implicitly assuming that the category of sets with smooth boundary contains the minimizers of the isoperimetric problem. Although *a posteriori* this is true, it is still is not obvious nor easy to prove *a priori*. Indeed, for dimension $n \geq 8$ there exist CMC-stable hypersurfaces that are the boundary of (unbounded) domains and that present singularities. This phenomena was first discovered by J. Simons in [Sim68], where he proved that the cone (which has been named after him)

$$C \coloneqq \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \colon |x| = |y|\}$$

is a stable minimal hypersurface away from 0, i.e., $H = 0$ and the inequality (19) holds with every compactly supported function u on $C \setminus \{0\}$. One year later, Bombieri, De Giorgi and Giusti proved that C is actually area minimizing [BDG69] and found similar examples for every $n \geq 8$.

In general, an area minimizing "hypersurface" $\Sigma \subset \mathbb{R}^n$ (here, we should actually use the language of currents) must be smooth smooth if $n \leq 7$, while if $n \geq 8$ *a priori* we only know that its singular set has Hausdorff dimension at most $(n-1) - 7$. This result has been proved with the hard work of many great mathematicians such as Fleming and De Giorgi ($n = 3$, [Fle62; De 65]), Almgren ($n = 4$, [Alm66]), Simons ($5 \leq n \leq 7$, [Sim68]) and Federer ($n \geq 8$, [Fed70]).

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