

Computational Finance and its implementation in Python with applications to option pricing

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- A common problem we face in mathematical finance is the **risk neutral valuation of a derivative**.
- As you know, the **price of a derivative** is expressed by the (possibly discounted) **expectation of its payoff** at maturity, under a pricing measure (also called pricing, or martingale measure).
- That is, **we have to compute the expectation of a random variable**.
- Problem: most often, there is **no way to get an analytic formula** for the expectation of complex derivatives, or even simpler derivatives written on underlying with non trivial dynamics.
- Broad idea: we can **approximate the price by averaging** some possible, **simulated realizations** of the payoff.
- The strong law of large numbers and some other convergence results may help us.

- Consider a random variable $X : \Omega \rightarrow \mathbb{R}^N$ defined on a probability space (Ω, \mathcal{F}, P) . The probability measure P may be viewed as a risk neutral measure.
- Also consider a (payoff) function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\text{Var}[f(X)] < \infty$.
- The aim is to compute the expectation

$$\theta := \mathbb{E}^P[f(X)] = \int_{\Omega} f(X) dP.$$

- Suppose there is no analytic formula to derive θ above. We have to find an **approximation** $\hat{\theta}$.

We can define independent drawings of X

- Given $X : \Omega \rightarrow \mathbb{R}$ and (Ω, \mathcal{F}, P) as above, introduce:

$$\begin{aligned}\tilde{\Omega} &:= \Omega \times \Omega \times \cdots \times \Omega = \{\tilde{\omega} = (\omega_1, \dots, \omega_n), \omega_i \in \Omega\}, \\ \tilde{\mathcal{F}} &:= \sigma(\mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F}), \\ \tilde{P} \left(\prod_{i=1}^n A_i \right) &:= \prod_{i=1}^n P(A_i), \quad A_i \in \mathcal{F}.\end{aligned}$$

- Also define the random variable $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_n)$ by $\tilde{X}_i(\tilde{\omega}) := X(\omega_i)$.
- This is a way to see $\tilde{X}(\tilde{\omega})$ as n different realizations $X(\omega_i)$, $i = 1, \dots, n$ of one random variable X , or as one realization of n i.i.d. random variables $\tilde{X}_i(\tilde{\omega})$, $i = 1, \dots, n$.
- This interpretation is at the base of the Monte-Carlo method, as it permits to exploit the Strong Law of Large Numbers.
- A similar construction and interpretation can be given for a N -dimensional random variable X .

Theorem: Strong Law of Large Numbers

Let $(X_i)_{i \in \mathbb{N}}$ be i.i.d. integrable real valued random variables on (Ω, \mathcal{F}, P) , and set

$$\mu := \mathbb{E}^P[X_i], \quad i \in \mathbb{N}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i = \mu \quad P - a.s.$$

Theorem: Tschebyscheff Inequality

Let $(X_i)_{i \in \mathbb{N}}$ be i.i.d. square integrable real valued random variables on (Ω, \mathcal{F}, P) , and set

$$\mu := \mathbb{E}^P[X_i], \quad \sigma^2 := \mathbb{E}^P[(X_i - \mu)^2], \quad i \in \mathbb{N}.$$

Then for any $\epsilon, \delta > 0$ we have

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2 n}.$$

and

$$P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \frac{\sigma}{\delta^{1/2} n^{1/2}}\right) \leq \delta.$$

Lemma

Let $(X_i)_{i \in \mathbb{N}}$ be a collection of i.i.d. integrable real valued random variables on (Ω, \mathcal{F}, P) , and let $f : \mathbb{R}^N \rightarrow \mathbb{R}$. Then the random variables $(f(X_i))_{i \in \mathbb{N}}$ are also i.i.d.

- The lemma above, together with the convergence results of the previous slide, allows us to approximate

$$\theta := \mathbb{E}^P[f(X)] = \int_{\Omega} f(X) dP.$$

by

$$\hat{\theta} := \frac{1}{n} \sum_{i=1}^n f(X_i),$$

where $(X_i)_{i=1, \dots, n}$ are independent realizations of X .

- We can generate numerically n realizations of a random variable X with a given distribution P^X , starting from a sequence of (pseudo!) random numbers.
- One must give a *seed*, i.e., a starting point for the pseudo-random numbers sequence.
- The realizations will not be purely random, and not purely independent.

- Pro:

- It is very simple to understand and easy to implement.
- The accuracy does not depend on the domain dimension (i.e., if we simulate N -dimensional random variables the accuracy is the same).
- The accuracy can be increased by just adding more valuations without losing the previous estimates.
- The function f does not need to be continuous, but only square integrable.

- Cons:

- Look at the Tschebyscheff Inequality: we only have a probabilistic bound. The worst case error is ∞ .
 - The estimates depend on the generated random sequence. The sequence is not purely random. First, one has to find a good random number generator.
- There are techniques that can be used to increase the accuracy. In the next slides we will see few of them.

Remark

If X has uniform distribution or has a cumulative distribution function F which is easy to invert (in that case a realization x_i can be generated as $x_i = F^{-1}(u_i)$, with u_i realization of $U \sim U((0, 1))$) then approximating $\mathbb{E}[f(X)]$ reduces to approximate

$$\int_0^1 G(x)dx, \quad (1)$$

for $G : [0, 1] \rightarrow \mathbb{R}$.

Theorem: Koksma-Hlawka inequality

If G has bounded total variation on $(0, 1)$, then for any points $x_1, \dots, x_n \in (0, 1)$ it holds

$$\left| \frac{1}{n} \sum_{i=1}^n G(X_i) - \int_0^1 G(x)dx \right| \leq V(G) D^*(x_1, \dots, x_n),$$

where

$$V(G) = \sup_S \sum_i |G(y_{i+1}) - G(y_i)|$$

over all partitions $S := \{0 = y_1 < y_2 < \dots < y_n = 1\}$ and $D^*(x_1, \dots, x_n)$ is the star discrepancy

$$D^*(x_1, \dots, x_n) = \sup_{b \in (0, 1)} \left| \frac{|\#\{x_i : 0 \leq x_i \leq b\}|}{n} - b \right|.$$

- The result in the previous slide also holds for higher dimensions (here we just wanted to simplify the notation).
- It gives the motivation to look for low discrepancy sequences.
- Most well known low discrepancy sequences: Van der Corput, Halton, Sobol, Hammersley, Sobol, Niederreiter.
- Here we don't focus on Low discrepancy sequences. A bit of references if you want to go deeper on this:
 - J. Dick and F. Pillichshammer, *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration*, Cambridge University Press, Cambridge, 2010
 - M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Math., 1651, Springer, Berlin, 1997.
 - L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Dover Publications, 2005.
 - ... the course *Numerical Methods for Financial Mathematics* at our master!
- We focus instead on variance reduction techniques.

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- Consider a random variable $X : \Omega \rightarrow \mathbb{R}^N$ defined on a probability space (Ω, \mathcal{F}, P) and a (payoff) function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\text{Var}[f(X)] < \infty$.
- Monte-Carlo method: choosing $n \in \mathbb{N}$ large enough, we approximate

$$\hat{\mu} := \frac{1}{n} \sum_{k=1}^n f(X_k) \approx \mu := \mathbb{E}^P[f(X)],$$

where $(X_k)_{k=1, \dots, n}$ are realizations of X , i.e., have same distribution as X .

- The estimator is of course *unbiased*, i.e.,

$$\mathbb{E}^P[\hat{\mu}] = \mathbb{E}^P \left[\frac{1}{n} \sum_{k=1}^n f(X_k) \right] = \mathbb{E}^P[f(X)] =: \mu$$

- We are **interested in** the variance of our estimator, i.e., in the quantity

$$\text{Var}(\hat{\mu}) = \mathbb{E}^P \left[\left(\frac{1}{n} \sum_{k=1}^n f(X_k) - \mu \right)^2 \right].$$

- We have seen that if $(X_i)_{i=1,\dots,n}$ are independent, we have convergence results for our estimator. Moreover,

$$\text{Var}(\hat{\mu}) = \mathbb{E}^P \left[\left(\frac{1}{n} \sum_{k=1}^n f(X_k) - \mu \right)^2 \right] = \frac{1}{n} \text{Var}[f(X)].$$

- It makes sense: the larger the number n of simulated realizations of X , the smaller the variance of our estimator.
- In particular, we have to increase the number of simulations by a factor of C to reduce the standard deviation by a factor of \sqrt{C} .
- The question now is: can we do it better?
- **Variance reduction techniques** aim to **reduce the variance of our estimator, without increasing the number of simulations.**

Three well known variance reduction techniques are:

- Antithetic variables
- Control variates
- Importance sampling

We will focus mostly on the first two techniques, together with applied examples. Here some references if you want to deepen Importance sampling:

- A, Bouhari. *Adaptative Monte Carlo Method, A Variance Reduction Technique*. Monte Carlo Methods and Their Applications. 10 (1): 1-24, 2004.
- P. J. Smith, M. Shafi, H. Gao. *Quick simulation: A review of importance sampling techniques in communication systems*. IEEE Journal on Selected Areas in Communications. 15 (4): 597-613, 1997.
- Again, the course *Numerical Methods for Financial Mathematics* at our master!

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Let's start from a simple result..

Lemma

Let $f, h : \mathbb{R} \rightarrow \mathbb{R}$ be two monotone functions, both increasing or both decreasing, and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable defined on a probability space (Ω, \mathcal{F}, P) . Then

$$\mathbb{E}^P[f(X)h(X)] \geq \mathbb{E}^P[f(X)]\mathbb{E}^P[h(X)].$$

Proof

The monotonicity assumption on f and h implies that for any $x, y \in \mathbb{R}$ we have

$$(f(x) - f(y))(h(x) - h(y)) \geq 0.$$

Therefore, for any i.i.d. real valued random variables X and Y on (Ω, \mathcal{F}, P) it holds

$$(f(X) - f(Y))(h(X) - h(Y)) \geq 0$$

and then

$$\mathbb{E}[(f(X) - f(Y))(h(X) - h(Y))] \geq 0,$$

so that

$$\mathbb{E}[f(X)h(X)] + \mathbb{E}[f(Y)h(Y)] \geq \mathbb{E}[f(Y)h(X)] + \mathbb{E}[f(X)h(Y)].$$

Since X and Y are identically distributed, it follows that

$$2\mathbb{E}[f(X)h(X)] \geq 2\mathbb{E}[f(Y)h(X)],$$

and since they are also independent, this implies that

$$\mathbb{E}^P[f(X)h(X)] \geq \mathbb{E}^P[f(X)]\mathbb{E}^P[h(X)].$$

Proposition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function, and $X : \Omega \rightarrow \mathbb{R}$ a random variable defined on a probability space (Ω, \mathcal{F}, P) . Then

$$\text{Cov}[f(X), f(-X)] \leq 0.$$

Proof

We have that

$$\text{Cov}[f(X), f(-X)] = \mathbb{E}^P[f(X)f(-X)] - \mathbb{E}^P[f(X)]\mathbb{E}^P[f(-X)].$$

The result then follows since a direct application of the Lemma of the previous slide with $h(x) := -f(-x)$ implies that

$$\mathbb{E}^P[f(X)]\mathbb{E}^P[f(-X)] \geq \mathbb{E}^P[f(X)f(-X)]$$

Application to Monte-Carlo

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function, and let $X : \Omega \rightarrow \mathbb{R}$ be a **symmetric** random variable defined on a probability space (Ω, \mathcal{F}, P) .
- From the last proposition we know that

$$\text{Cov}[f(X), f(-X)] \leq 0.$$

- Idea: choose n even and generate $n/2$ realizations of X , call them $(X_i)_{i=1, \dots, n/2}$. Then define $X_{n/2+i} := -X_i, i = 1, \dots, n/2$.
- Since X is symmetric, the estimator is unbiased:

$$\mathbb{E}[\hat{\mu}] = \frac{1}{n} \mathbb{E} \left[\sum_{k=1}^{n/2} f(X_i) + \sum_{k=1}^{n/2} f(-X_i) \right] = \frac{1}{n} \left(\sum_{k=1}^{n/2} \mathbb{E}[f(X_i)] + \sum_{k=1}^{n/2} \mathbb{E}[f(-X_i)] \right) = \mu.$$

- What about the variance?

$$\begin{aligned} \text{Var}[\hat{\mu}] &= \frac{1}{n^2} \text{Var} \left[\sum_{k=1}^{n/2} f(X_i) + \sum_{k=1}^{n/2} f(-X_i) \right] \\ &= \frac{1}{n^2} \left(n \text{Var}[f(X)] + \text{Cov} \left(\sum_{k=1}^{n/2} f(X_i), \sum_{k=1}^{n/2} f(X_i) \right) \right) \\ &= \frac{1}{n} \text{Var}[f(X)] + \frac{1}{n} \text{Cov}[f(X), f(-X)] \leq \frac{1}{n} \text{Var}[f(X)]. \end{aligned}$$

- To recap: if X is symmetric, then setting $X_{n/2+i} := -X_i$ for $i = 1, \dots, n/2$ gives us an unbiased estimator $\hat{\mu}$ such that

$$\text{Var}[\hat{\mu}] \leq \frac{1}{n} \text{Var}[f(X)].$$

- But $\frac{1}{n} \text{Var}[f(X)]$ is the variance of the classical estimator, when we generate n i.i.d. realizations of X !
- In this way, we reduce the variance of the estimator.
- This approach is known as Antithetic variables.

- Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone function, and let $X : \Omega \rightarrow \mathbb{R}$ be a random variable defined on a probability space (Ω, \mathcal{F}, P) .
- Suppose X to be not symmetric. How can we apply Antithetic variables to reduce the variance of our estimator?
- Call F the cumulative distribution function of X . Suppose that we know (at least a good approximation of) F^{-1} .
- Well known result: let $U \sim \text{Unif}(0, 1)$ and define $Y := F^{-1}(U)$. Then X and Y have same distribution.
- Let $U \sim \text{Unif}(0, 1)$. Because of the result above, we have

$$\mathbb{E}^P[f(X)] = \mathbb{E}^P[h(U)]$$

with $h(x) = f \circ F^{-1}$.

- Since U is symmetric, we can approximate $\mathbb{E}^P[h(U)]$ by Antithetic variables.

Example: valuation of a call option under Black-Scholes

- We want to test the benefits of using Antithetic variables in the valuation of a call option under the Black-Scholes model.
- This is indeed a case when we have of course the benchmark of the analytic formula for a call option.
- In particular, we want to approximate the expectation $\mathbb{E}^P[f(X_T)]$ for $T > 0$, in the case when

$$g(x) = (x - K)^+$$

with $K > 0$ and $X = (X_t)_{0 \leq t \leq T}$ is a stochastic process with initial value $X_0 = x_0$ and dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad 0 \leq t \leq T,$$

where $W = (W_t)_{0 \leq t \leq T}$ is P -Brownian motion.

- Interpretation: r is the risk free rate and P is the martingale measure, i.e., the probability measure under which the discounted process $(e^{-rt}X_t)_{0 \leq t \leq T}$ is a martingale.

- The problem reduces to the valuation of the expectation

$$\mathbb{E}^P[g(X - K)^+]$$

where X is the random variable

$$X = X_0 e^{(r - \sigma^2)T + \sigma \sqrt{T}Z},$$

with $Z \sim \mathcal{N}(0, 1)$.

- That is, we have to value

$$\mathbb{E}^P[f(Z)]$$

where

$$f(z) = \left(X_0 e^{(r - \sigma^2)T + \sigma \sqrt{T}z} - K \right)^+.$$

- So, we have a function of a symmetric random variable! We can directly use Antithetic variables.
- We simulate $n/2$ realizations $(z_i)_{i=1, \dots, n/2}$ of a standard normal random variable and then define $z_{i+n/2} = -z_i$, $i = 1, \dots, n/2$.

- In the Python package

```
montecarlovariancereduction.antitheticvariables
```

you can find the code relative to the comparison of Antithetic variables against the standard Monte-Carlo method.

- In particular, in the class `GenerateBlackScholes` we generate the values of

$$X = X_0 e^{(r-\sigma^2)T + \sigma\sqrt{T}Z},$$

starting from the ones of Z . We do this using both the standard Monte-Carlo approach and the Antithetic variables approach illustrated in the previous slide.

- Note that the method

```
numpy.random.standard_normal(n)
```

generates n returns of a standard normal random variable. In this case, we give no seed: it will be different every time this method is called.

In

```
antitheticVariablesTest
```

and

```
compareStandardMCWithAV
```

we do the following experiment:

- We fix the parameters $X_0 = K = 100$, $T = 3$, $r = 0.05$, $\sigma = 0.5$.
- For any number of simulations $n = 10^3, 10^4, 10^5$ and 10^6 , we perform 100 different valuations of the price of the call option, both with the standard and the Antithetic variables Monte-Carlo method.
- We then compute the average percentage error for both the methods.

The following table illustrates the results

	$n = 10^3$	$n = 10^4$	$n = 10^5$	$n = 10^6$
av. % error standard MC	6.25	2.07	0.59	0.20
av. % error AV	5.51	1.77	0.53	0.17

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- Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables defined on a probability space (Ω, \mathcal{F}, P) .
- Suppose you know the analytic value of

$$a := \mathbb{E}^P[X], \quad \sigma_X^2 := \text{Var}[X], \quad \sigma_{XY} := \text{Cov}[X, Y],$$

and also suppose $\sigma_{XY} > 0$.

- Assume you want to approximate

$$b := \mathbb{E}^P[Y]$$

- The goal is to find an unbiased estimator of b which has low variance.

- Idea: simulate n realizations (x_i, y_i) , $i = 1, \dots, n$, of (X, Y) . Compute then

$$\hat{a} := \frac{1}{n} \sum_{i=1}^n x_i, \quad \hat{b} := \frac{1}{n} \sum_{i=1}^n y_i.$$

- Note that

$$\text{Cov}[\hat{a}, \hat{b}] = \frac{1}{n} \sigma_{XY}.$$

- What about an estimator

$$\hat{b}_{CV} := \hat{b} - \beta(\hat{a} - a)$$

for a given $\beta > 0$?

- It is unbiased:

$$\mathbb{E}^P[\hat{b}_{CV}] = \mathbb{E}^P[\hat{b}] + \beta \mathbb{E}^P[\hat{a} - a] = b.$$

- What about the variance?

$$\text{Var}[\hat{b}_{CV}] = \frac{1}{n} \sigma_Y^2 + \beta^2 \frac{1}{n} \sigma_X^2 - 2\beta \frac{1}{n} \sigma_{XY}.$$

- It is minimized by $\beta = \frac{\sigma_{XY}}{\sigma_X^2}$. For such a value of β , we find

$$\text{Var}[\hat{b}_{CV}] = \text{Var}[\hat{b}] - \frac{1}{n} \frac{\sigma_{XY}^2}{\sigma_X^2}.$$

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The multi-period Binomial model for option pricing is widely used by practitioners in financial applications mainly because:

- It is very easy to understand and simulate.
- It is particularly convenient to price options involving a choice of the holder, like American and Bermudan options.
- It approximates the Black-Scholes model when the length of the periods tends to zero.
- Option pricing is not based on pure Monte-Carlo techniques but relies on weighting the payoff relative to any scenario by the (analytic!) probability of the scenario.

- Consider a multi-period model with times $t = 0, 1, \dots, T$, and consider a **probability space** $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t=0, \dots, T}$ is a filtration representing information.
- Suppose there exist:
 - A **risk free asset** defined by $S_t^0 = (1 + \rho)^t$, $t = 0, \dots, T$, with a deterministic interest rate $\rho > 0$.
 - A **risky asset** adapted to \mathbb{F} defined by

$$S_t = S_0 \cdot Y_1 \cdots Y_t, \quad t = 1, \dots, T,$$

where Y_t can take the two values d, u with $0 < d < 1 + \rho < u$, for any $t = 1, \dots, T$, and $(Y_t)_{t=1, \dots, T}$ are i.i.d. and such that Y_{t+1} is independent of \mathcal{F}_t .

- It then holds

$$S_t^0 = S_{t-1}^0(1 + \rho), \quad t = 1, \dots, T$$

and

$$S_t = S_{t-1}Y_t, \quad t = 1, \dots, T.$$

At every time $t = 0, \dots, T - 1$, an investor can construct a **portfolio of value V_t** , trading on the risk-free asset S^0 and on the risky asset S .

- The value of the portfolio is given by

$$V_t = \alpha_t S_t + \beta_t S_t^0, \quad t = 1, \dots, T,$$

where $(\alpha_t)_{t=1, \dots, T}$ and $(\beta_t)_{t=1, \dots, T}$ are \mathbb{F} -predictable, discrete process.

- The strategy (α, β) must be **self-financing**: it must hold

$$V_t = \alpha_t S_t + \beta_t S_t^0 = \alpha_{t-1} S_t + \beta_{t-1} S_t^0, \quad t = 1, \dots, T.$$

Definition

A portfolio V is an **arbitrage** if:

- V is obtained by a self-financing and predictable strategy;
- $P(V_0 = 0) = 1$;
- $P(V_t \geq 0) = 1$ and $P(V_t > 0) > 0$ for some t .

Proposition

The market is **arbitrage free** only if $d < 1 + \rho < u$.

- Suppose $1 + \rho \leq d < u$, and consider the self-financing portfolio defined by

$$V_t = S_t - \frac{S_0}{S_0^0} S_t^0, \quad t = 0, 1, \dots, T.$$

Then we have $V_0 = 0$ and

$$V_1 = S_1 - \frac{S_0}{S_0^0} S_1^0 \geq S_0 d - S_0(1 + \rho) \geq 0.$$

- If $d < u \leq 1 + \rho$, changing the signs to the strategy above leads to an arbitrage

Equivalent martingale measure

In order for the market to be arbitrage-free and complete, **there must exist a unique measure $Q \sim P$ such that $\frac{S}{S^0}$ is a martingale**, i.e., such that

$$\mathbb{E}^Q \left[\frac{S_{t+1}}{S_{t+1}^0} \middle| \mathcal{F}_t \right] = \frac{S_t}{S_t^0}, \quad t = 0, \dots, T-1. \quad (2)$$

Note that the measure Q is identified by the probability $q := P(Y_t = u)$. Since

$$\mathbb{E}^Q \left[\frac{S_{t+1}}{S_{t+1}^0} \middle| \mathcal{F}_t \right] = \frac{(qu + (1-q)d)S_t}{S_t^0(1+\rho)}, \quad t = 0, \dots, T-1,$$

equation (2) holds if and only if $qu + (1-q)d = 1 + \rho$, that is,

$$q = \frac{1 + \rho - d}{u - d}.$$

Such Q exists and is unique as we have supposed $0 < d < 1 + \rho < u$, and

$$\frac{dQ}{dP}(\omega) = \left(\frac{q}{p} \right)^{n(\omega)} \left(\frac{1-q}{1-p} \right)^{T-n(\omega)},$$

where $p := P(Y_t = u)$ and $n(\omega)$ is the number of times $t = 1, \dots, T$ when $Y_t(\omega) = u$.

- Assume we want to find an admissible strategy (α_t, β_t) , $t = 0, \dots, T$, such that the value of the portfolio

$$\alpha_t S_t + \beta_t (1 + \rho)^t$$

equals the value V_t of an option at every time $t = 0, \dots, T$.

- From now on, fix $t = 1, \dots, T$, and suppose we know S_{t-1} .
- Call V_t^u the value of the option at time t when $Y_t = u$ and V_t^d the value of the option at time t when $Y_t = d$.
- It must hold

$$\begin{cases} \alpha_t u S_{t-1} + \beta_t (1 + \rho)^t = V_t^u, \\ \alpha_t d S_{t-1} + \beta_t (1 + \rho)^t = V_t^d. \end{cases}$$

- The solution to the system above is

$$\alpha_t = \frac{V_t^u - V_t^d}{S_{t-1}(u - d)},$$
$$\beta_t = \frac{u V_t^d - d V_t^u}{(1 + \rho)^t (u - d)}.$$

and gives the right replicating strategy.

- Remember that our strategy (α_t, β_t) , $t = 0, \dots, T$, has to be admissible!
- This means that we must have that

$$\begin{aligned} V_{t-1} &= \alpha_{t-1} S_{t-1} + \beta_{t-1} (1 + \rho)^{t-1} \\ &= \alpha_t S_{t-1} + \beta_t (1 + \rho)^{t-1} \\ &= \frac{V_t^u - V_t^d}{u - d} + \frac{u V_t^d - d V_t^u}{(1 + \rho)(u - d)} \\ &= \frac{(1 + \rho)(V_t^u - V_t^d) + u V_t^d - d V_t^u}{(1 + \rho)(u - d)} \\ &= \frac{(1 + \rho - d) V_t^u + (u - 1 - \rho) V_t^d}{(1 + \rho)(u - d)} \\ &= \frac{q V_t^u + (1 - q) V_t^d}{1 + \rho} \\ &= \frac{1}{1 + \rho} \mathbb{E}^Q[V_t | \mathcal{F}_{t-1}]. \end{aligned}$$

- Then we have that the value $(V_t)_{t=0, \dots, T}$ of the option is a martingale under Q .
- This gives us a pricing theorem.

Theorem

The value V_0 of a contingent claim with maturity T and payoff V_T depending on the realizations of S until time T , is given by

$$V_0 = \frac{1}{(1 + \rho)^T} \mathbb{E}^Q[V_T].$$

Remark

Because of the theorem above, we always simulate our Binomial model under the risk neutral measure Q .

1 Monte-Carlo method for option pricing and variance reduction techniques

- The Monte-Carlo method: motivation and a brief overview
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2 Option pricing under the Binomial model

- Motivation and setting
- **Simulation of the Binomial model**
- Calibration of the Binomial model
- American options valuation

- Our main goal here is to get the price of European and (most importantly) American options written on an underlying Binomial model.
- This valuation will approximate the price of the options written on an underlying log-normal model.
- We then simulate the realizations of the underlying model in Python, and get the payoff on the realizations, along with its expectation.
- Remember we have to price under the risk neutral measure Q : then we simulate the realizations of the process under Q .
- The most naive way we can imagine to do this is a brute force Monte-Carlo approximation..

- Imagine we want to value the discounted price of an European option with a given payoff function $f : \mathbb{R} \rightarrow \mathbb{R}$, written on the process S , with maturity T .
- Suppose we don't know any analytic formula in order to derive the price as

$$V_0 = \frac{1}{(1 + \rho)^T} \mathbb{E}^Q[f(S_T)].$$

- We consider N *states of the world* $\omega_1, \omega_2, \dots, \omega_N \in \Omega$.
- To any $\omega_1, \omega_2, \dots, \omega_N$, we associate a given trajectory of the process $(S_t)_{t=0, \dots, T}$, with dynamics given under the measure Q .
- In particular, we suppose that the trajectories $(S_t(\omega_k))_{t=0, \dots, T}$, $k = 1, 2, \dots, N$ are *independent* of each other.
- Strong law of large numbers:

$$\frac{1}{n} \sum_{k=1}^n f(S_T(\omega_k)) \rightarrow \mathbb{E}^Q[f(S_T)] \quad \text{a.s., when } n \rightarrow \infty.$$

- The idea is to simulate such trajectories and approximate

$$\mathbb{E}^Q[f(S_T)] \approx \frac{1}{N} \sum_{k=1}^N f(S_T(\omega_k)).$$

- Our first goal is then to generate a sequence of random numbers in order to simulate N independent trajectories $(S_t(\omega_k))_{t=0,\dots,T}$, $k = 1, 2, \dots, N$ of S under the risk neutral measure Q , and store them in a $(T+1) \times N$ matrix.
- First (not that) bad news: it is not possible to generate a sequence of perfectly random numbers, the best we can get is a sequence of *pseudo*-random numbers.
- Idea: **generate** (with the help of Python in our case) a sequence of $T \cdot N$ **uniformly distributed, pseudo-random numbers** $0 < x_{i,j} < 1$, $i = 1, \dots, T$, $j = 1, \dots, N$.
- Fix $\rho > 0$, $u > 1 + \rho$, $d < 1$, $q = \frac{1+\rho-d}{u-d}$.
- For every $i = 1, \dots, T$, $j = 1, \dots, N$, define

$$Y_i(\omega_j) = \begin{cases} u & \text{if } x_{i,j} < q \\ d & \text{if } x_{i,j} \geq q \end{cases}$$

and

$$S_{i+1}(\omega_j) = Y_i(\omega_j)S_i(\omega_j).$$

- You can find the code relative to the simulation of the Binomial model with the pure Monte-Carlo approach described above in

```
binomialmodel.creationandcalibration.binomialModelMonteCarlo
```

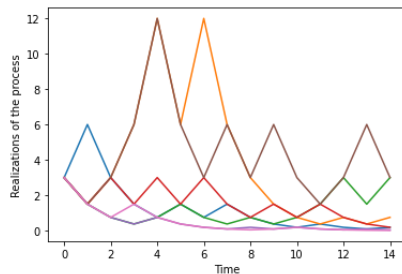
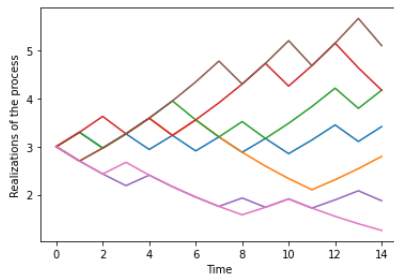
- Note that the class you find there extends the one in

```
binomialmodel.creationandcalibration.binomialModel.
```

- This is done in order to implement in the parent class some methods that do not strictly depend on the way in which we simulate the process.
- In this way, we don't have to copy and paste these methods in every class where we simulate the model in some way: object oriented programming feature.

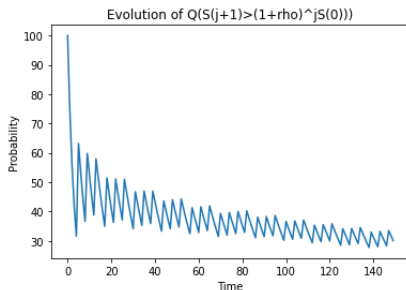
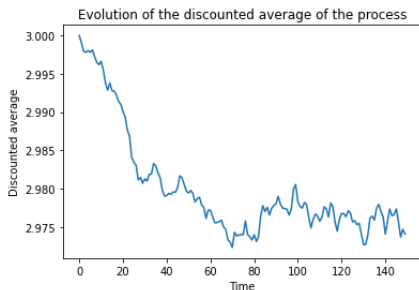
Some paths

- We plot below some paths of the Binomial model.
- In the figure at the left we take $S_0 = 3$, $u = 1.1$, $d = 0.9$, $r = 0.05$, $T = 150$, having then $q = \frac{1+\rho-d}{u-d} = 0.75$.
- On the right, $S_0 = 3$, $u = 2$, $d = 0.5$, $r = 0.1$, $T = 150$, $q = \frac{1+\rho-d}{u-d} = 0.4$.



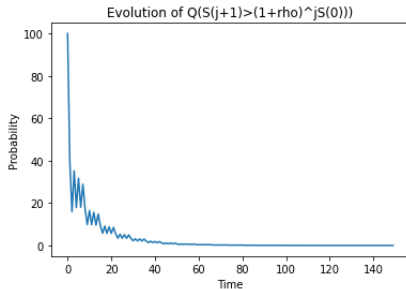
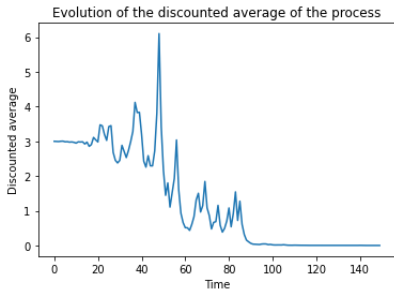
A first test

We show here the evolution of the discounted average of the process and of the probability $Q(S_{t_j} > (1 + \rho)^{t_j} S_0)$, computed by using the Monte-Carlo method with 10^5 simulations, for $S_0 = 3$, $u = 1.1$, $d = 0.9$, $r = 0.05$, $T = 150$. In this case, we have $q = \frac{1+\rho-d}{u-d} = 0.75$.



But something can go wrong..

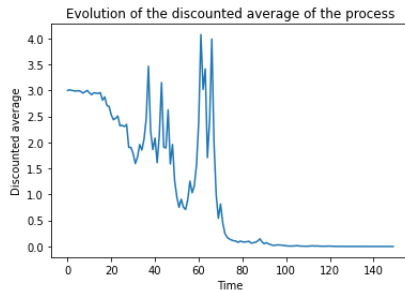
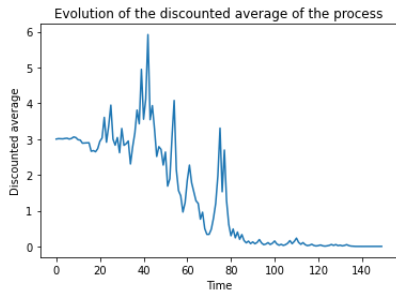
Look at the evolution of the same quantities, again computed by using the Monte-Carlo method, choosing now $S_0 = 3$, $u = 2$, $d = 0.5$, $r = 0.1$, $T = 150$, $q = \frac{1+\rho-d}{u-d} = 0.4$.



Why is the estimate of the average that inaccurate?

- The analytic average of the discounted process is equal to S_0 , due to many realizations such that $S_{t_j} < (1 + \rho)^{t_j} S_0$ and few, extremely high realizations.
- If you buy S at time $t = 0$, and you hold it for 150 time steps, you make a positive gain with a very low probability, but the gain can be extremely high.
- **Problem:** The approximated average is strongly impacted by whether or not those paths leading to high gains are simulated or not.

Let's choose two different seeds, for the same parameters



Maybe a pure Monte-Carlo approach is not the best solution..

- We have seen that, if the volatility is high, the Monte-Carlo approach can be very inaccurate for many time steps.
- Moreover, it is time consuming (this is a problem common to all brute-force Monte-Carlo approaches)
- **Idea:** let us exploit some analytic properties of the Binomial model..

- At the n -th time step, $n + 1$ realizations of the process are possible: $S_0 u^n, S_0 u^{n-1} d, \dots, S_0 u d^{n-1}, S_0 d^n$.
- The number of ups and downs is given by a Bernoulli distribution:

$$P(S_n = S_0 u^k d^{n-k}) = \binom{n}{k} q^k (1 - p)^{n-k}.$$

- Using the expression above, we can compute

$$\begin{aligned} \mathbb{E}^Q[f(S_n)] &= \sum_{k=0}^n Q(S_n = S_0 u^k d^{n-k}) f(S_0 u^k d^{n-k}) \\ &= \sum_{k=0}^n \binom{n}{k} q^k (1 - q)^{n-k} f(S_0 u^k d^{n-k}). \end{aligned}$$

- The idea is then to generate all the possible realizations of the process up to a given time, and to weight them by their probability.
- You can find the code relative to the this approach in

```
binomialmodel.creationandcalibration.binomialModelSmart,  
whose class also extends the one in
```

```
binomialmodel.creationandcalibration.binomialModel.
```

- Doing some tests in

```
binomialmodel.creationandcalibration.binomialModelSmartTest.
```

you can observe that, in this way, the average of the discounted process is stable.
- Moreover, this approach is of course much faster.

Computation of $Q(S_n > S_0(1 + \rho)^n)$, $n = 1, \dots, T$

- Note that for any $k = 0, \dots, n$ it holds

$$\begin{aligned} S_n = S_0 u^k d^{n-k} > S_0(1 + \rho)^n &\iff u^k d^{n-k} > (1 + \rho)^n \\ &\iff \left(\frac{u}{d}\right)^k > \left(\frac{1 + \rho}{d}\right)^n \\ &\iff k > n \log_{\frac{u}{d}} \left(\frac{1 + \rho}{d}\right). \end{aligned}$$

- Then we have

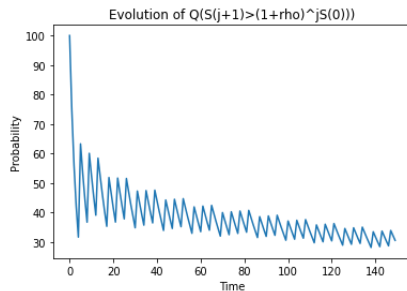
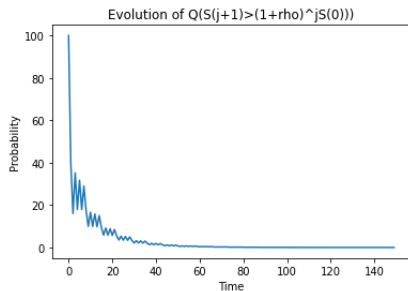
$$\begin{aligned} Q(S_n > S_0(1 + \rho)^n) &= \sum_{k=\bar{k}}^n Q(S_n = S_0 u^k d^{n-k}) \\ &= \sum_{k=\bar{k}}^n \binom{n}{k} q^k (1 - q)^{n-k} f(S_0 u^k d^{n-k}), \end{aligned}$$

where

$$\bar{k} = \min \left\{ k \in \mathbb{N} : k > n \log_{\frac{u}{d}} \left(\frac{1 + \rho}{d} \right) \right\} \leq n.$$

Evolution of the probability plotted with Python

We show here the evolution of the probability computed above, over 150 time steps. On the left, we have parameters $S_0 = 3$, $u = 1.1$, $d = 0.9$, $\rho = 0.1$, $q = \frac{1+\rho-d}{u-d} = 0.75$. On the right, $S_0 = 3$, $u = 2$, $d = 0.5$, $\rho = 0.05$, $q = \frac{1+\rho-d}{u-d} = 0.4$.



- As seen before, an application of the simulation of the Binomial model in this way is the valuation of European options, under the pricing measure Q .
- In

`binomialmodel.optionValuation.europeanOption`,

you can see some methods relative to this.

- In particular, we compute the expectation of the payoff of European options as

$$\begin{aligned}\mathbb{E}^Q[f(S_n)] &= \sum_{k=0}^n Q(S_n = S_0 u^k d^{n-k}) f(S_0 u^k d^{n-k}) \\ &= \sum_{k=0}^n \binom{n}{k} q^k (1-q)^{n-k} f(S_0 u^k d^{n-k}).\end{aligned}$$

- We also compute the value of a general option for every time $t = 0, \dots, T-1$, and the corresponding self-financing, replicating strategy (α_t, β_t) , $t = 0, \dots, T-1$, described before.
- As an exercise, you can check if the final value of the portfolio given by that strategy equals the payoff, for an option of your choice.

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2 Option pricing under the Binomial model

- Motivation and setting
- Simulation of the Binomial model
- **Calibration of the Binomial model**
- American options valuation

- Recall that we have

$$S_t = S_0 \cdot Y_1 \cdots Y_t, \quad t = 1, \dots, T,$$

where

$$Y_t = \begin{cases} u > 1 + \rho \text{ with (risk-neutral) probability } q = \frac{1+\rho-d}{u-d} \\ d < 1 \text{ with probability } 1 - q \end{cases}, \quad t = 1, \dots, T.$$

- Our goal is to **calibrate** the up and downs parameters u and d , supposing we know the risk neutral probability $q = \frac{1+\rho-d}{u-d}$ and the interest rate $\rho > 0$, and that we can observe

$$\text{Var}[\log(S_T/S_0)] := \mathbb{E}^Q[\log(S_T/S_0)^2] - \mathbb{E}^Q[\log(S_T/S_0)]^2$$

for a given maturity T .

- Observe first that since

$$\log(S_T/S_0) = \sum_{t=1}^T \log(Y_t),$$

and since $(Y_t)_{t=1, \dots, T}$ are equi-distributed, we get

$$\text{Var}[\log(S_T/S_0)] = T^2 \text{Var}[\log(Y_T)].$$

Proposition

Let

$$Y_t = \begin{cases} u > 1 + \rho \text{ with (risk-neutral) probability } q = \frac{1+\rho-d}{u-d} \\ d < 1 \text{ with probability } 1 - q, \end{cases}, \quad t = 1, \dots, T.$$

Then for any $t = 1, \dots, T$ we have

$$\text{Var}[\log Y_t] = q(1 - q) \log(u/d)^2.$$

Proof

$$\begin{aligned} \text{Var}[\log Y_t] &= \mathbb{E}^Q[\log(Y_t)^2] - \mathbb{E}^Q[\log(Y_t)]^2 \\ &= \mathbb{E}^Q[\log(Y_t)^2] - (q \log(u) + (1 - q) \log(d))^2 \\ &= q \log(u)^2 + (1 - q) \log(d)^2 \\ &\quad - q^2 \log(u)^2 - (1 - q)^2 \log(d)^2 - 2q(1 - q) \log(u) \log(d) \\ &= q(1 - q) \log(u)^2 + q(1 - q) \log(d)^2 - 2q(1 - q) \log(u) \log(d) \\ &= q(1 - q) (\log(u) - \log(d))^2 \\ &= q(1 - q) \log(u/d)^2. \end{aligned} \tag{3}$$

- Thanks to

$$q = \frac{1 + \rho - d}{u - d}$$

and to

$$\text{Var}[\log Y_t] = q(1 - q) \log(u/d)^2,$$

along with

$$\sigma_{obs}^2 := \text{Var}[\log(S_T/S_0)] = T^2 \text{Var}[\log(Y_T)],$$

we can **get u and d from q , ρ and σ_{obs}^2** by solving the nonlinear nonsystem

$$\begin{cases} \frac{1+\rho-d}{u-d} = q \\ \log(u/d)^2 = \frac{\sigma_{obs}^2}{q(1-q)} \end{cases} \quad (4)$$

- We can find an approximated solution of (4) by the `fsolve` function of Python.
- Look at

`binomialmodel.creationandcalibration.binomialModelCalibration`
to see an implementation of the calibration of u and d as showed above.

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2 Option pricing under the Binomial model

- Motivation and setting
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- American options valuation

- The holder of an American option with payoff f and maturity T on an underlying X has the right, at any time we $t \in [0, T]$, to hold the contract or to exercise the payoff $f(X_t)$.
- The **valuation of American options is more complicated** than the one of European options, since it involves an optimal exercise problem.
- In order to value such an option at time t , indeed, the conditional expectation of the future value of the option at time t has to be computed, and then compared against the present value of the payoff.
- However, the Monte-Carlo computation of a conditional expectation is very time consuming.
- One of the **strengths of the Binomial model** with respect to other settings is that it **permits a favourable pricing of American options**.
- Also when dealing with continuous time processes, with suitable dynamics, one may approximate them with a Binomial model in order to get the price.

American options valuation under the Binomial model

- At any time $t = 1, \dots, T$, call $S_t(k)$ and $V_t(k)$ the value of the underlying and of the option, respectively, in the scenario with k ups and $t - k$ downs up to time t .
- Idea: **proceed backward**.
- First we compute the **payoff** $f(S_T(k)) = f(S_0 u^k d^{T-k})$, for any $k = 0, \dots, T$.
- We have of course $V_T(k) = f(S_T(k))$, for any $k = 0, \dots, T$.
- At time $T - 1$, for any $k = 0, \dots, T - 1$ we compute

$$\begin{aligned} V_{T-1}(k) &= \max \left(f(S_{T-1}(k)), \frac{1}{1+\rho} (qV_T(k+1) + (1-q)V_T(k)) \right) \\ &= \max \left(f(S_0 u^k d^{T-1-k}), \frac{1}{1+\rho} (qV_T(k+1) + (1-q)V_T(k)) \right). \end{aligned}$$

- For any $t = 1, \dots, T - 2$ we compute with the same argument

$$V_t(k) = \max \left(f(S_0 u^k d^{t-k}), \frac{1}{1+\rho} (qV_t(k+1) + (1-q)V_t(k)) \right).$$

- We finally get the value of the option at initial time as

$$V_0 = \max \left(f(S_0), \frac{1}{1+\rho} (qV_1(1) + (1-q)V_1(0)) \right).$$

You can find the code relative to the the valuation of American options in

```
binomialmodel.optionValuation.AmericanOption,
```

with some tests in

```
binomialmodel.optionValuation.AmericanOptionTest,
```


Example

We consider a put option with payoff $f(x) = (20 - x)^+$, and choose parameters $T = 3$, $S_0 = 20$, $u = 1.1$, $d = 0.9$, $\rho = 0.05$.

The triangular matrices below show us an analysis of an American option for the such parameters (row 3 shows the values for $t = 3$ and so on).

The upper left and upper right matrices show the amount one would get if exercising the option or holding the contract, respectively; the lower left one the values of the option; the lower right one has 1 in the exercise region and 0 in the hold region

	0	1	2	3
0	0	nan	nan	nan
1	0	2	nan	nan
2	0	0.2	3.8	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0.564464	nan	nan	nan
1	0.123583	1.27551	nan	nan
2	0	0.519048	2.84762	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0	nan	nan	nan
1	0	2	nan	nan
2	0	0.2	3.8	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0	nan	nan	nan
1	0	1	nan	nan
2	0	0	1	nan
3	1	1	1	1

Approximating a Black-Scholes model with a Binomial model

- Consider a continuous, adapted stochastic process $X = (X_t)_{t \geq 0}$ with dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad t \geq 0,$$

where $r, \sigma > 0$ and $W = (W_t)_{t \geq 0}$ is a Brownian motion.

- Suppose you want to price an American option with underlying X and maturity $T > 0$.
- It can be seen that the dynamics of $X = (X_t)_{0 \leq t \leq T}$ can be **approximated by N time steps of a Binomial model with parameters**

$$u = e^{\sigma\sqrt{T/N}}, \quad d = 1/u, \quad \rho = e^{r\sqrt{T/N}}, \quad (5)$$

for N large enough, see for example .

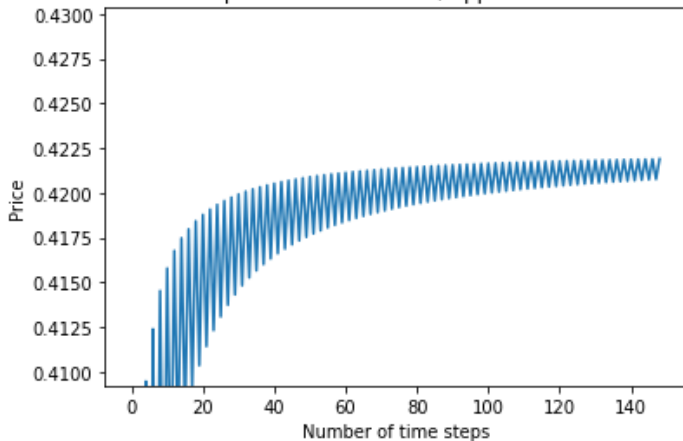
- The idea is to **approximate the price of the American option** of maturity T with the price of an American option with maturity N written a Binomial model with parameters as in (5), for N large enough.
- Indeed, the price of the American option written on the Binomial model can be found as illustrated before.

Example: not such a nice behaviour

We consider an American put option with payoff $f(x) = (1 - x)^+$ and maturity $T = 3$, written on a Black-Scholes model with parameters $r = 0.02$, $\sigma = 0.7$.

The plot below shows the approximated price via the derivation under the Binomial model, for an increasing number of times steps up to $N = 150$.

Price of an American option for a BS model, approximated via binomial model



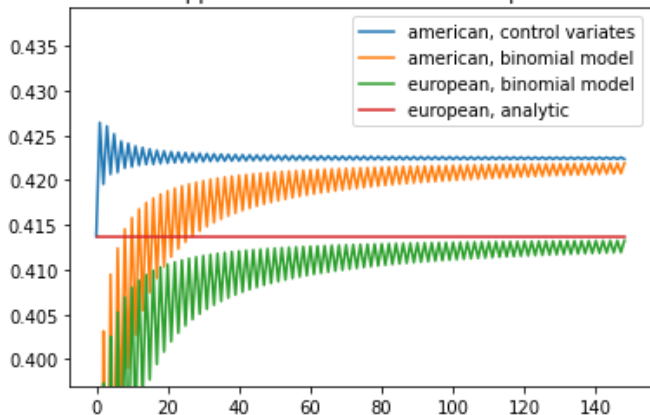
- **First idea:** we know the analytic price of an European put (or call) option under the Black-Scholes model. For example, call P^E the Black-Scholes formula price of an European put option.
- Also call:
 - P_N^E the price of an European put approximated by the Binomial model with N time steps;
 - P^A the analytic price of an American put;
 - P_N^A the price of an American put approximated by the Binomial model with N time steps.
- **Second idea:** we know the euristics $P^A - P_N^A \approx P^E - P_N^E$.
- We then approximate
$$P^A \approx P_N^A + (P^E - P_N^E)$$
- This approximates the price of an American put option via control variates.
- Same thing for a call option.

A nicer behaviour with control variates

We consider again an American put option with payoff $f(x) = (1 - x)^+$ and maturity $T = 3$, written on a Black-Scholes model with parameters $r = 0.02$, $\sigma = 0.7$: same situation as before

The plot below compares the prices introduced in the previous slide, for an increasing number of times steps up to $N = 150$.

Control variates approximation of an American option for a BS model



- You can find some experiments relative to the stability of approximations of prices of American options with the Binomial model in
`binomialmodel.optionValuation.AmericanOptionPriceConvergence,`
- The code performing the control variates approach can be found in
`binomialmodel.optionValuation.controlVariates,`
with some tests in
`binomialmodel.optionValuation.controlVariatesTest.`