Computational Finance

Paolo Foschi

Department of Statistics
University of Bologna
paolo.foschi2@unibo.it

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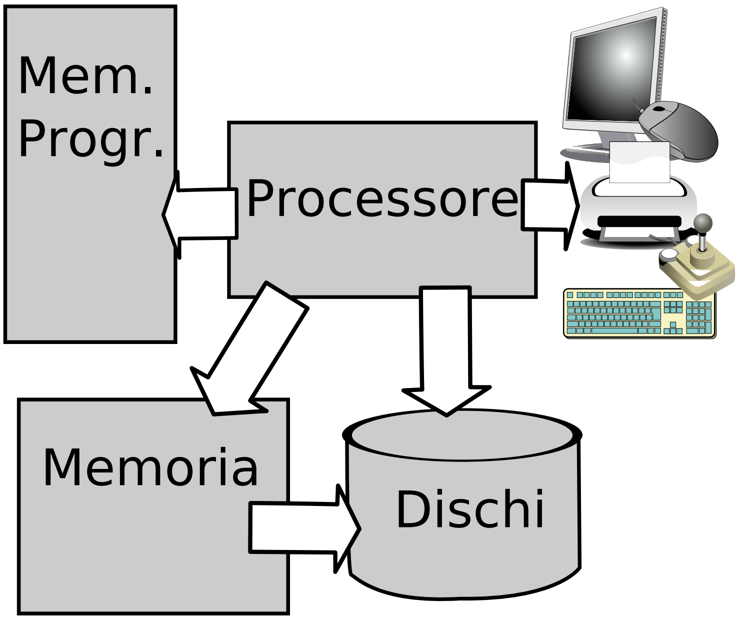
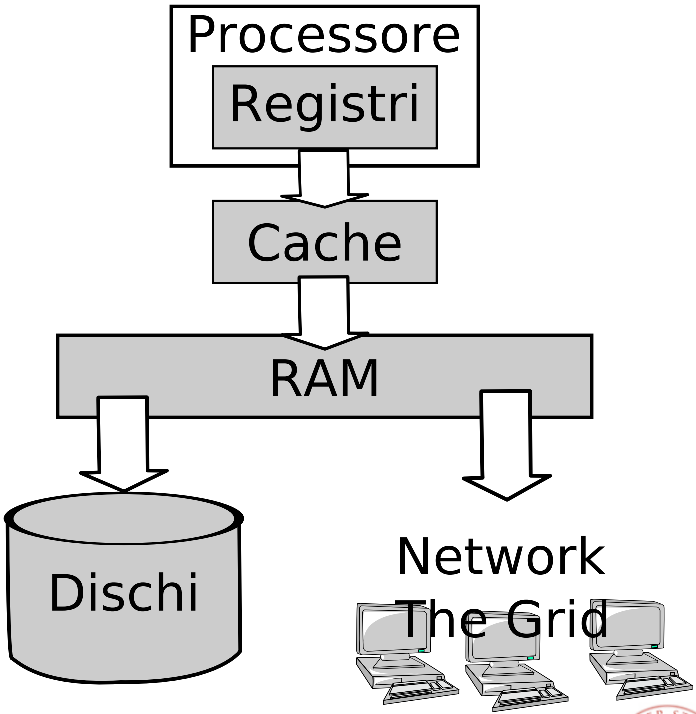
Disclaimer

Struttura degli elaboratori





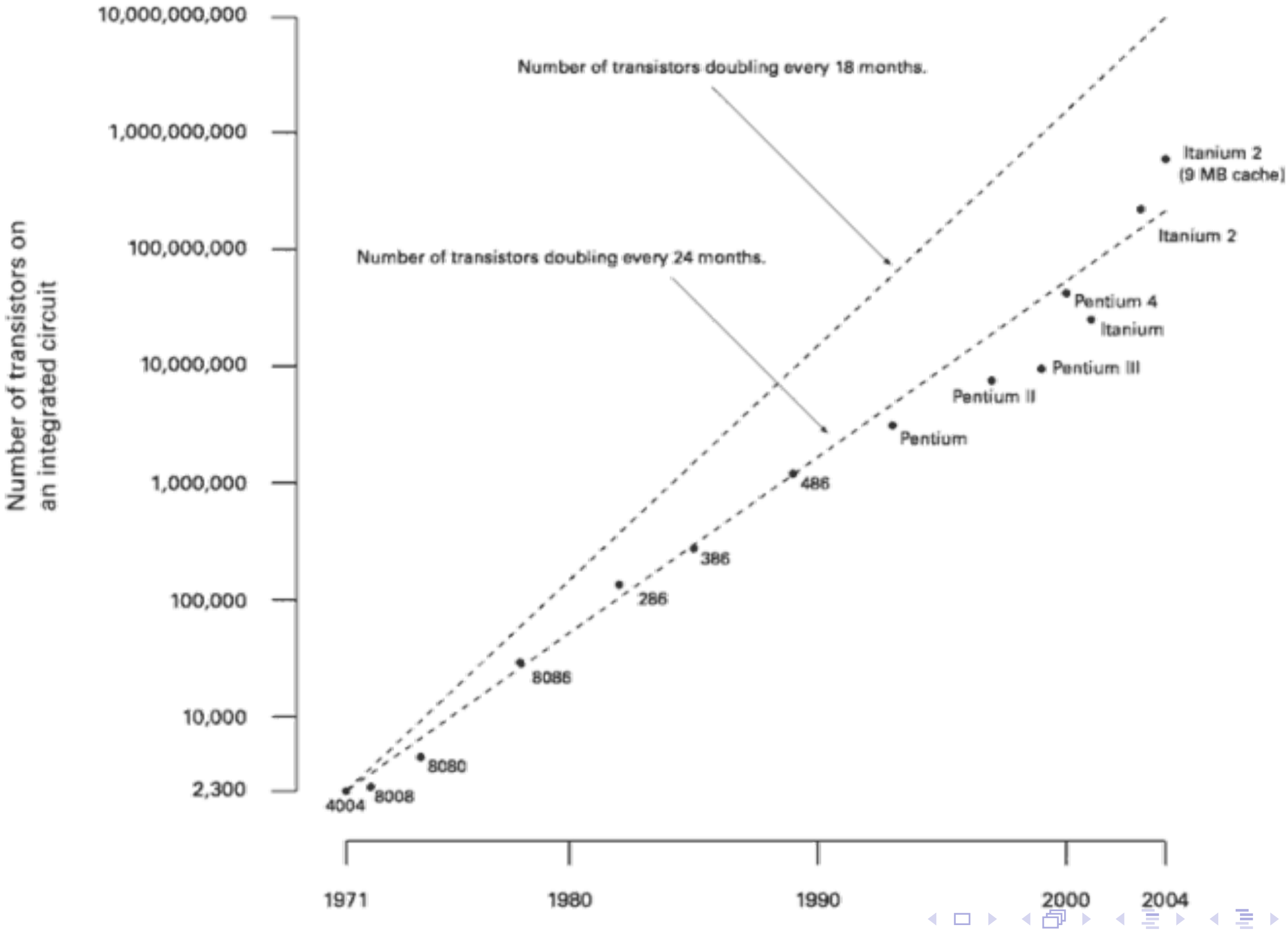
Algoritmo

- **Algoritmo**: descrizione dettagliata e chiara dei passi necessari per trasformare un dato insieme di "input" in particolare insieme di "output".
- Etimologia: Abu Abdullah Muhammad bin Musa "al-Khwarizmi", Astronomo e matematico, nato a Bagdad nel 780AC. Sviluppo' metodi per l'aritmetica nel "nuovo" sistema Indo-Arabico
- **Linguaggio**: Permette di rendere non ambiguo un algoritmo.
- **Sintassi**: insieme di regole che definiscono un particolare linguaggio.
- **Programma**: implementazione di un particolora algoritmo in un particolare linguaggio

Complessità

- Complessità: Dimensione del problema $\longrightarrow$ tempo di esecuzione, (o spazio di memoria utilizzato)
- La complessitá viene espressa con valori asintotici:
  - Ordinamento: complessitá $O(n\log(n))$
  - Risoluzione sistema di eq. lineari: $\leq O(n^3)$
  - Calcolo del prezzo di un'opzione in un modello binomiale: $O(n^2)$
  - Commesso Viaggiatore: $O(2^n)$: non polinomiale
- Complessità non polinomiali:
  - $n = 20$: $2^{20} \simeq 10^6$ sec $\simeq 300$ ore $\simeq 12$ giorni
  - $n = 21$: $2^{21} \simeq 24$ giorni
  - $n = 22$: $2^{22} \simeq 48$ giorni
  - $n = 25$: $2^{23} \simeq 388$ giorni $\simeq 1$ anno
  - E se utilizzassi/aspettassi un computer piú potente.
- Il problema di sapere se un dato programma termina o meno è non decidibile:
  - se termina lo si puó sapere in un tempo finito,

Legge di Moore



Errori di arrotondamento

- Ogni numero reale viene rappresentato come una sequenza di cifre di lunghezza fissa.
- Floating point = numero a virgola fissa + esponente (notazione esponenziale).
- Esempi:

12.3	0.123×10^2	+	1	2	3	0	+	2
-0.005279	-0.5279×10^{-4}	-	5	2	7	9	-	4

- $1 + 0.0005 = 1.0005 \simeq 1$, errore del 0.05%

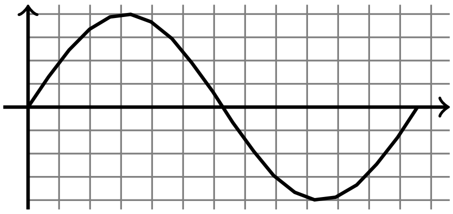
1.0005	$.10005 \times 10^1$	+	1	0	0	0	+	1

Errori di arrotondamento

- Problema: calcolare la somma $1 + 0.0004 - .9999 (= .5)$
  Algoritmo: calcolare la somma $1 + 0.0004$, sommare al risultato $-.9999$
  Risultato: .0001   Errore: 80%
- Double precision: mantissa 52 cifre binarie (16 decimali) + 9 cifre binarie per l'esponente.
- Puó essere un problema se si devono sommare migliaia di numeri.

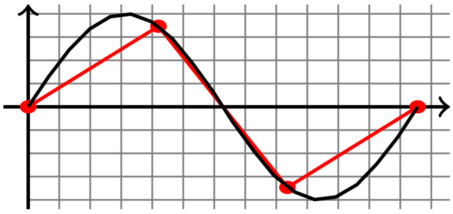
Discretizzazione: errori di approssimazione

- Calcolare la lunghezza della curva $sin(x)$ da $x = 0$ a $x = 2\pi$

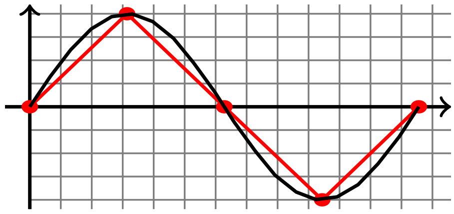


- Approsimazione con una sequenza di corde.

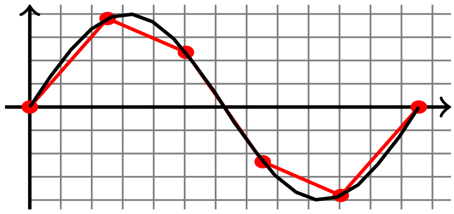
  3 corde $\rightarrow$ 4.742



4 corde $\rightarrow$ 5.086



5 corde $\rightarrow$ 5.064



10 corde $\rightarrow$ 5.215
100 corde $\rightarrow$ 5.2698
1000 corde $\rightarrow$ 5.2704

Algoritmi randomizzati

- Molti algoritmi euristici utilizzano numeri casuali per "sondare" lo spazio delle possibili soluzioni
- Esempi: monte carlo, algoritmi genetici, simulated annealing, ant colonies, etc...
- A parità di input, la soluzione trovata è diversa ad ogni esecuzione del programma
- In realtà i computer non generano numeri casuali, ma pseudo causali:
  - sequenze deterministiche che sembrano casuali (hanno quasi gli stessi momenti)
  - il "seed" è il valore iniziale di una sequenza di numeri (pseudo)-casuali

Cos'è Matlab

Matlab è un ambiente Rapid Application Development (RAD) per il calcolo scientifico, la grafica e visualizzazione.

Matlab fornisce strumenti per:

- Calcolo numerico
- Analisi dei dati e visualizzazione
- Ingegneria e grafica scientifica
- Modellazione e simulazione di processi
- Programmazione e sviluppo di applicazioni
- Altri Toolbox

Perchè Matlab in questo corso

- Pro:
  - ha un linguaggio di programmazione
  - è interpretato ed ha un ambiente per il testing
  - $y = Ax$ si calcola o si risolve scrivendo al più 5 caratteri
  - se dovutamente utilizzato è molto efficiente
  - la finanza computazionale non è altro che calcolo scientifico
- Contro:
  - strutture dati
  - in molte banche italiane si usa Visual Basic/Excel
  - è costoso (per un privato)
- Alternative open source: Scilab, Octave, R, Python, C++, C#
- Per la Finanza: R-metrics e Quantlib

Variabili

- Una variabile associa ad particolare nome un valore.
- Ad ogni variabile è associata una zona di memoria che contiene il valore della variabile.
- Esistono diversi **tipi** di valori: Numeri Interi, Numeri Reali, Numeri Complessi, Vettori di $\mathbb{R}^n$, sequenze di caratteri, funzioni da $\mathbb{R}^n$ a $\mathbb{R}^n$, etc...
- **Dichiarare** o **Definire** una variabile equivale a riservare una particolare zona di memoria per il valore che la variabile conterrà.
- **Assegnare** un valore ad una variabile consiste nel porre tale valore nella corrispondente di memoria.
- In "Pascal", "C/C++" e "Java" le variabili devono essere dichiarate prima di essere utilizzate e non possono cambiare tipo.
- In "Basic", "Fortran", "Matlab" ed "R" la dichiarazione è implicita nel primo assegnamento ed il tipo della variabile può variare in base al dato assegnato.

Istruzioni di Assegnamento

- Sintassi: `<nome variabile> = <espressione>`
- Semantica: metti nella variabile `<nome variabile>` il risultato di `<espressione>`
- Esempi:

script0.m

```
x = 3;
y = x + 2;
y = 2*y;
z = 1/(2*pi)*exp(-x^2/2);
```

- l-values: Variabili a SX dell'assegnamento
- r-values: Variabili a DX dell'=
- Ricorda...
  - `x=expr` è un'azione, non è un equazione.
  - Gli r-values devono essere già stati inizializzati (devono esistere).

Scripts

- Uno “script” è un file `.m` contenente una sequenza di comandi `matlab`
- Equivale a digitare gli stessi comandi nella “command window”
- Esempio:

script1.m

```
a=1; b=-3; c=2;
d  = b^2 - 4*a*c;
x1 = (-b-sqrt(d))/(2*a);
x2 = (-b+sqrt(d))/(2*a);
```

- Viene eseguito digitando il nome del file nella command window
- In `matlab` le istruzioni possono essere separate da “;”, “,” o da capo riga

Input

- Sintassi: `<var.> = input( <stringa messaggio> )`
- Semantica: visualizza il messaggio, attendi che l'utente inserisca un valore numerico e ritorna tale valore.
- Esempio: `c = input('Inserisci la costante');`
- Esempio script:

script2.m
```
% Risolvi eq. di 2^ grado
a  = input('Inserire a: ');
b  = input('Inserire b: ');
c  = input('Inserire c: ');
d  = b^2 - 4*a*c;
x1 = (-b-sqrt(d))/(2*a);
x2 = (-b+sqrt(d))/(2*a);
```

- Input grafico: `x,y = ginput(1);`

Output

- Sintassi: `disp( <expr> )`
- Semantica: visualizza il risultato di <expr> nella command window.
- Sintassi: `fprintf(<fmt>, <expr1>, <expr2>, ..)`
- Semantica: visualizza il risultato delle espressioni <expr1>, <expr2>, etc... secondo quanto prescritto dalla stringa <fmt>
- Esempio:

script3.m

```
% Risolvi eq. di 2^ grado
a  = input('Inserire a: ');
b  = input('Inserire b: ');
c  = input('Inserire c: ');
d  = b^2 - 4*a*c;
x1 = (-b-sqrt(d))/(2*a);
x2 = (-b+sqrt(d))/(2*a);
fprintf('Le radici sono: %g, %g\n',x1, x2);
```

If-then-else

- Sintassi:

ifelse.m

```
if <expr1>
  <statement 1>
else
  <statement 2>
end
```

- Semantica: se <expr1> è vera esegui <statement 1> altrimenti esegui <statement 2>.
- Esempio:

ifexample.m

```
if d<0
  disp('Radici complesse');
else
  x1 = (-d-sqrt(d))/(2*a);
  x2 = (-d+sqrt(d))/(2*a);
```

Operatori e Funzioni Built-in

- Funzioni Built-in
  - Funzioni trigonometriche: `sin`, `cos`, `tan`, `asin`, `acos`, `atan`.
  - Esponenziali: `exp`, `log`, `log2`, `log10`.
  - Arrotondamento: `floor`, `ceil`, `round`, `fix`, `mod`, `rem`
  - `max`, `min`, `abs`, `sign`
  - Numeri casuali: `rand`, `randn`
- Operatori
  - Operatori aritmetici: `+`, `-`, `/`, `\` , `*`
  - Operatori relazionali: `==`, `<`, `>`, `<=`, `>=`, `~=`.
  - Operatori logici: `~` (not), `&` (and), `|` (or).

and	T	F
T	T	F
F	F	F

or	T	F
T	T	T
F	T	F

  - Algebra: `~(a & b)` $\equiv$ `(~a) | (~b)`

Cicli for

- Per ripetere qualcosa `N` volte:

forend.m
```
for i=1:N
  <statement>
end
```

- Semantica:

for2.m
```
i=1, <statement>
i=2, <statement>
...
i=N, <statement>
```

- Il contatore `i` è una variabile a tutti gli effetti

Cicli for

- Esempio: Somma dei primi `N=100` numeri:

for3.m
```
N = 100;   somma = 0;
for j=1:N
  somma = somma+j;
end
```

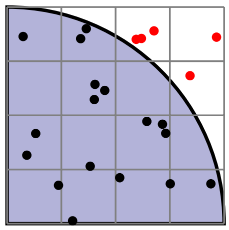
- Ciclo for tipico:

for4.m
```
for i = first:by:last
  <statement>
end
```

- Assegna ad `i` in sequenza `first`, `first+by`, `first+2*by`, etc.. fino al raggiungimento del valore `last`.
- Esempio:
  - `i=1:4:15` equivale alla sequenza `[1 5 9 13]`

Esempio: calcolo di π

- $x, y \sim \text{unif}(0, 1)$ e indipendenti.
- $P[x^2 + y^2 \leq 1] = \pi/4$



pigreco.m

```
dentro=0;
for i=1:N
  x = rand;    y = rand;
  if (x^2+y^2<=1)
    dentro=dentro+1;
  end
end
my_pi = dentro/N*4;
```

Il ciclo While

- Per ripetere finché una tale condizione rimane vera

while1.m
```
<init>
while <expr>
  <statement>
end
```

- Semantica:

while2.m
```
Esegui le istruzioni <init>
Valuta <expr>, se falsa vai ad end
esegui <statement>
valuta <expr>, se falsa vai ad end
esegui <statement>
...
```

- Il `while` cicla per vero

Cicli for e while

- Ogni ciclo `for` può essere espresso con un ciclo `while`

pigreco3.m

```
dentro=0;
i=1;
while (i<=N)
  x = rand;   y = rand;
  if (x^2+y^2<=1)
    dentro=dentro+1;
  end
  i=i+1;
end
pi_greco = dentro/N*4;
```

Analisi top-down

- Spesso conviene un problema complesso in sottoproblemi piú semplici
- Analisi top-down: da un'analisi grossolana si arriva fino ai minimi dettagli
- Problema: ricerca in elenco telefonico.
  - cerca cittá:
    - scegli una pagina - leggi cittá - confronta due parole
  - cerca cognome:
    - scegli pagina e riga - leggi cognome - confronta due parole
  - cerca nome:
    - scegli pagina e riga - leggi nome - confronta due parole

Funzioni

- Alcuni sottoproblemi possono presentarsi piú volte o essere parte dell'analisi di altri problemi
- Ogni sottoproblema è risolto da un sottoprogramma: funzione
- Come i programmi i le funzioni associano a particolari input i dovuti output
- Le funzioni devono essere progettate e utilizzate come scatole nere, di cui si conosce il funzionamento e le caratteristiche (specifiche) ma non il contenuto (implementazione).
- Interagiscono con l'esterno solo tramite le variabili di input e quelle di output.
- In questo modo si potrá cambiare l'implementazione di una funzione senza alterare il comportamento del programma che le utilizza.
- Librerie: insieme di funzioni di utilizzo frequente

Funzioni

- Sintassi:

fun1.m

```
function <output> = <name> ( <in1>, <in2>, ...)
% Post: proprieta'  variabili di output
% Pre:  proprieta' che devono avere le
%       variabili di input
<statements>
```

- Ogni funzione è contenuta in un file `.m` dello stesso nome
- Esempio:

mynormcdf.m

```
function y = mynormcdf(x)
% y = phi(x)
%     calcola la funzione cumulativa della
%     distribuzione normale standard

y = erfc( -x/sqrt(2))/2;
```

Exercise (Prezzo Zero Coupon Bond)

Scrivere una funzione 'ZCB' che calcoli il prezzo P al tempo t di uno Zero Coupon Bond con scadenza T e tasso composto R:

$$P = \frac{100}{(1+R)^{T-t}}$$

Exercise (Prezzo Zero Coupon Bond)

Scrivere una funzione 'ZCB' che calcoli il prezzo P al tempo t di uno Zero Coupon Bond con scadenza T e tasso composto R:

$$P = \frac{100}{(1+R)^{T-t}}$$

zcb.m

```
function p = zcb(t,T,r)
% p = zcb(t,T,r)
%
% Compute the value at t of a ZCB with maturity T and
% rate (composite) r

p = 100/(1+r)^(T-t);
```

Exercise (Prezzo Coupon Bond)

Scrivere una funzione 'CB' che calcoli il prezzo P al tempo t di un Coupon Bond con scadenza T, tasso composto R, che paga una cedola postcipata c ad intervalli dt

$$B = P(t, T) + \frac{c}{100}P(t, T) + \frac{c}{100}P(t, T - dt) + \cdots + \frac{c}{100}P(t, T - ndt)$$

$$n = \lceil \frac{T - t}{dt} \rceil$$

Exercise (Prezzo Coupon Bond)

Scrivere una funzione 'CB' che calcoli il prezzo P al tempo t di un Coupon Bond con scadenza T, tasso composto R, che paga una cedola postcipata c ad intervalli dt

$$B = P(t,T) + \frac{c}{100}P(t,T) + \frac{c}{100}P(t,T-dt) + \cdots + \frac{c}{100}P(t,T-ndt)$$

$$n = \lceil \frac{T-t}{dt} \rceil$$

cb.m

```
function p = cb(t,T,r,c,dt)
% p = cb(t,T,r,c,dt)
%
% calcola il prezzo al tempo t di un coupon bond
% con scadenza T, cedola c, rate ad intervalli dt
% e tasso composto r
n = ceil((T-t)/dt);
p = 0;
for i=0:n-1
    p = p + zcb(t,T-i*dt,r)*c/100;
end
p = p + zcb(t,T,r);
```

Exercise (Black and Scholes Formula)

Scrivere una funzione che calcoli il prezzo C di una call utilizzando la formula di Black and Scholes:

$$C = S_0\Phi(d^+) - e^{-rT}\Phi(d^-), \qquad d^\pm = \frac{\log(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$\Phi(x)$ *funzione di ripartizione di* $N(0,1)$

Exercise (Black and Scholes Formula)

Scrivere una funzione che calcoli il prezzo C di una call utilizzando la formula di Black and Scholes:

$$C = S_0\Phi(d^+) - e^{-rT}\Phi(d^-), \qquad d^\pm = \frac{\log(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

$\Phi(x)$ *funzione di ripartizione di* $N(0,1)$

bs_call.m

```
function C = bs_call(S, K, T, r, sigma )
% C = bs_call(S, K, T, r, sigma )
%   Calcola il valore di una Call vanilla

d1 = (log(S/K) + (r+sigma^2/2)*T)/(sigma*sqrt(T));
d2 = d1 - sigma*sqrt(T);

C = S*mynormcdf(d1) - exp(-r*T)*K*mynormcdf(d2);
```

Scope, Variabili Locali e Globali

- Le funzioni sono scatole nere, le uniche interazioni con l'esterno avvengono tramie gli input/output.
- Nell'esempio precedente sono state definite le variabili `d1` e `d2`.
- Variabili locali: tutte le variabili definite ed utilizzate in una funzione non hanno ripercussioni sull'ambiente (workspace) esterno/globale.
- Esiste uno workspace (locale) per ogni chiamata a funzione

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m

```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m

```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

...

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m
```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

a=3
...

Esempio: Stack e Scope

- Considieriamo la funzione

fun2.m
```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

x=3

a=3

...

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m
```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

t=9
x=3
a=3
...

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m
```matlab
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```matlab
>> a = 3;
>> b = fun2(a)
b =
     12
```

Stack:

y=12
t=9
x=3
a=3
...

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m
```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

b=12
~~y=12~~
~~t=9~~
~~x=3~~
a=3
...

Esempio: Stack e Scope

- Consideriamo la funzione

fun2.m
```
function y = fun2(x)
t = x^2;
y = 3+t;
```

- E lo script

fun3.m
```
>> a = 3;
>> b = fun2(a)
b =
      12
```

Stack:

b=12
a=3
...

Array

- Un array è un collezione di variabili dello stesso tipo organizzate in righe, colonne, strati, etc. . .
- La struttura dati fondamentale in `matlab` è la Matrice: array 2-d
- Un particolare tipo di matrice è il vettore: array 1-d, singola riga o colonna.
- Indice: posizione di un valore in un vettore
- Una posizione in una matrice è definita da due indici

$$\begin{array}{c|cccc|}
1 & 6 & 1 & 3 & -1 \\
2 & 9 & 2 & 2 & -1 \\
3 & 0 & 9 & -3 & 4 \\
4 & 10 & 3 & 2 & 1 \\
\hline
 & 1 & 2 & 3 & 4
\end{array}
\qquad
\begin{array}{c|c|}
1 & 10 \\
2 & 2 \\
3 & 6 \\
4 & 4 \\
5 & 2 \\
\hline
 & 1
\end{array}
\qquad
\begin{array}{c|ccccc|}
1 & 10 & 2 & 6 & 4 & 2 \\
\hline
 & 1 & 2 & 3 & 4 & 5
\end{array}$$

Vettori

Creazione

- Separatori di colonna: spazio e ","; separatore di riga: ";"
- Vettori riga: `vr = [ 2 3.1 -.01]`
- Vettori colonna: `vc = [ 2; 3.1; -.01]`
- Usare funzioni per la creazione di matrici:

  `zeros(m,n)`, Crea una matrice $m \times n$

  `vr = zeros(1,4)`, Crea una riga con 4 zeri

  `vc = zeros(1,4)`, Crea una colonna con 4 zeri
- Funzioni analoghe: `ones`, `nan`, `inf`, `rand`, `randn`, `eye`
- `linspace(a,b,n)` crea un vettore riga con `n` valori equidist. da `a` a `b`

Accesso agli elementi:

- `vect(i)` rappresenta l'`i`-esimo elemento di `vect`.
- `vect(i)` è anche un l-value, cioè puó essere utilizzato a sinistra di istruzioni di assegnamento

Vettori

- Esempio:

vet1.m
```
>> vect = [10 2 6 4 2 6]
vect =
    10     2     6     4     2     6
>> vect(5)
ans =
     2
>> k=3;
>> vect(k)
ans =
     6
>> vect(2) = k
vect =
    10     3     6     4     2     6
>> vect(1) = vect(k+2)
vect =
     2     3     6     4     2     6
```

Exercise (Attualizzazione flussi)

Calcolare il valore attuale (in t=0) di una serie di flussi c_i ai tempi t_i $(i = 1, \ldots, n)$.

Esempio di utilizzo

actualvalue_ex.m

```
t = [.5, 1, 1.5, 2, 2.5];
c = [10, 10, 15, 15, 10];
p = actualvalue( t, c, 0.04 );
```

Exercise (Attualizzazione flussi)

Calcolare il valore attuale (in t=0) di una serie di flussi c_i ai tempi t_i ($i = 1, \ldots, n$).

Esempio di utilizzo

actualvalue_ex.m

```
t = [.5, 1, 1.5, 2, 2.5];
c = [10, 10, 15, 15, 10];
p = actualvalue( t, c, 0.04 );
```

actualvalue.m

```
function p = actualvalue( t, c, r )
% p = actualvalue( t, c, r )
%
% Calcola il valore attuale dei flussi c(i) ai
% tempi t(i) con tasso di interesse composto r
n = length(t);
p = 0;
for i=1:n
    p = p + zcb(0,t(i),r)*c(i)/100;
```

Exercise (Simulazione Random Walk)

Simulare n passi di una realizzazione del processo stocastico a tempo discreto:

$$X_t = X_{t-1} + \Delta W_t, \qquad \Delta W_t \sim N(0,1), \qquad X_0 = x_0$$

(Specifiche: `function x = wiener(n, x0)` *con* $x : 1 \times n$, $x0 : 1 \times 1$*)*

Esempio di utilizzo:

wiener2.m

```
z = wiener( 50, 0 );
t = linspace( 1, 50, 50 );
plot( t, z, '-' );
```

Implementazione:

Exercise (Simulazione Random Walk)

Simulare n passi di una realizzazione del processo stocastico a tempo discreto:

$$X_t = X_{t-1} + \Delta W_t, \qquad \Delta W_t \sim N(0,1), \qquad X_0 = x_0$$

(Specifiche: `function x = wiener(n, x0)` *con* $x : 1 \times n$, $x0 : 1 \times 1$*)*

Esempio di utilizzo:

wiener2.m
```
z = wiener( 50, 0 );
t = linspace( 1, 50, 50 );
plot( t, z, '-' );
```

Implementazione:

wiener.m
```
function x = wiener( n, x0 )

x = zeros(n,1);
x(1) = x0;
for i=1:n-1
```

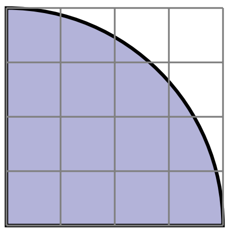
Operazioni su vettori

- Operatori:
  - Aritmetici: `+, -, ./, .*, .^`
  - Relazionali: `==, ~=, <, >, <=, >=`
  - Logici: `&, |, ~`
- Se applicati a:
  - Due scalari: no problem.
  - Due vettori: elemento per elemento
  - Scalare-Vettore: scalare per ogni elemento del vettore
- Esempi (`a=.5; b=2; v=[1 2 3]; w=[4 5 6];`):
  - `a ./ b` ⟶ .25;
  - `v .* w` ⟶ `[4 10 18];`
  - `v .^ b` ⟶ `[1 4 9];`
  - `b .^ v` ⟶ `[2 4 8];`
  - `v > b` ⟶ `[0 0 1];`
  - `v < w/3` ⟶ `[1 0 0];`
  - `(v>b)|(v<w/3)` → `[1 0 1];`
  - `2 .^(v>b)` ⟶ `[1 1 2];`

Operazioni su vettori

- $\mathbb{R}^N \to \mathbb{R}$: `sum`, `prod`, `max`, `min`, `mean`, `std`, `var`, `median`.
  Esempio: `sum([1 2 3 4])` $\longrightarrow$ `10`.
- $\mathbb{R}^N \to \mathbb{R}^m$: `cumsum`, `cumprod`, `diff`, `gradient`
  Esempio: `diff([1 2 4 7 11])` $\to$ `[1 2 3 4]`
- $\mathbb{R}^N \to \mathbb{R}^N$: `abs`, `sign`, `sqrt`, `exp`, `log(10,2)(a)sin(h)`, `(a)cos(h)`, `(a)tan(h)`, etc...
- $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ or $\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$: `max`, `min`;
  - `max(-3:3, 0)` $\longrightarrow$ `[0 0 0 0 1 2 3]`;
  - `max(-3:3, -6:2:6)` $\longrightarrow$ `[-3 -2 -1 0 2 4 6]`.
- Esempio, calcolo di π

pigreco2.m
```
x = rand(N,1);
y = rand(N,1);
dentro = (x.^2 + y.^2 <= 1);
pi_greco = sum(dentro)/N*4;
```



Colon Notation

- L'espressione `1:5` è equivalente a `[1 2 3 4 5]`.
- In generale: `a:b:c` $\equiv$ `[a, a+b, ..., a+m*b]`, con $m = \lfloor \frac{c-a}{b} \rfloor$.
- è possibile accedere a piú di un elemento di un vettore: `a([1,3])` $\equiv$ `[a(1) a(3)]` (se `a` è un vettore riga).
- quindi se `a = [10 2 5 4 2 6]`:
  - `a([1,5])` $\longrightarrow$ `[10 6]`
  - `a(3:6)` $\longrightarrow$ `[5 4 2 6]`
  - `a(1:2:6)` $\longrightarrow$ `[10 5 2]`
  - `a([6:-1:1])` $\longrightarrow$ `[6 2 4 5 2 10]`
- Esiste un indice particolare "`end`" tale che `a` $\equiv$ `a(1:end)`.
  - `a(end)` è l'ultimo elemento di `a`
  - `a(1:2:end)` sono gli elementi di posizione dispari
  - `a(2:2:end)` quelli di posizione pari
  - `a(end:-1:1)` il vettore con posizioni invertite

Matrici

- Ogni elemento è indirizzato da due indici: riga e colonna

	1	2	3	4
1	10	3	2	1
2	0	9	-3	4
3	9	2	2	-1
4	6	4	3	4

`A(3,2)` $\longrightarrow$ `2`

- Colon notation:

	1	2	3	4
1	10	3	2	1
2	0	9	-3	4
3	9	2	2	-1
4	6	4	3	4

`A([1:2:end],[2:3])` $\longrightarrow$ `[3 2; 2 2]`

Struttura fondamentale di Matlab

- 1×1: scalari
- $1 \times N$: vettori riga
- $M \times 1$: vettori colonna
- $M \times N$: matrici

- Comodo per l'algebra lineare.
- Ci sono problemi di coerenza.
- Spesso fanno comodo strutture dati piú sofisticate.

Operazioni fra Matrici

- Tutti gli operatori “elemento per elemento” visti finora vengono utilizzati in maniera analoga fra matrici.
- Gli operatori prodotto e divisione, `*`,`\`,`/`, rappresentano operazioni fra matrici:
  - `X=A*B`: moltiplicazione riga-per-colonna;
  - `X=A/B`: divisione a destra, X soluzione di $XB = A$;
  - `X=A\B`: divisione a sinistra, X soluzione di $AX = B$;

  - Le dimensioni di `A` e `B` devono essere compatibili.
  - $XB = A$ e $AX = B$ possono essere anche sotto/sovra determinati o non-crameriani (soluzione minimi quadrati)
- Trasposizione: `A'` è la trasposta di `A`
- Se `v` e `w` sono due vettori colonna di uguali dimensioni: `v'*w` è il loro prodotto scalare

Funzioni su matrici

- $\mathbb{R}^{M\times N} \rightarrow \mathbb{R}^{M\times N}$: (es. `abs`, `sin`) applicate ad ogni elemento;
- $\mathbb{R}^{M\times N} \rightarrow \mathbb{R}^{N}$: (es. `sum`, `prod`, `mean`) applicate ad ogni colonna;
- $\mathbb{R}^{M\times N} \rightarrow \mathbb{R}^{m\times N}$: (es. `cumsum`) applicate ad ogni colonna;
- `size(A)` restituisce un vettore di due elementi con le dimensioni di `A`
- `length(v)` restituisce il numero di elementi del vettore `v`
- Esempi:
    - `A = ones(3,2); v=1:5;`
    - `size(A)` $\longrightarrow$ `[3,2]`
    - `size(v)` $\longrightarrow$ `[1,5]`
    - `size(8)` $\longrightarrow$ `[1,1]`
    - `length(v)` $\longrightarrow$ `5`

Concatenazione di Matrici

- Matrici a blocchi: se $A : m \times n$, $B : m \times q$, $C : p \times n$ e $D : p \times q$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1q} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_{m1} & \cdots & b_{mq} \\ c_{11} & \cdots & c_{1n} & d_{11} & \cdots & d_{1q} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{p1} & \cdots & c_{pn} & d_{p1} & \cdots & d_{pq} \end{pmatrix}$$

- In Matlab: `M = [ A, B; C,D ];`
- Esempi: dati `v =[1 2 3]; w=[4 5 6]; x=[3;4]; y=[2,3; 4,5];`
  - `[v,w]` $\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$
  - `[v;w]` $\longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
  - `[x,y]` $\longrightarrow \begin{pmatrix} 3 & 2 & 3 \\ 4 & 4 & 5 \end{pmatrix}$
  - `[v',w']` $\longrightarrow \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$
  - `[[v;w],x]` $\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 4 \end{pmatrix}$

Indicizzazione con booleani

- Vettori booleani (Vero/Falso) ottenuti da confronti possono essere utilizzare per estrarre elementi da un array.
- Esempio: si vogliono estrarre tutti i valori positivi del vettore `x = [2.1 3.4 -0.4 -1.2 3.2 -2.5 2.1]`
  1. `pos = (x>=0);` $\longrightarrow$ `pos = [1 1 0 0 1 0 1]`
  2. `y = x(pos);` $\longrightarrow$ `y = [2.1 3.4 3.2 2.1]`
- In generale se `x` e `pos` sono due matrici di pari dimensioni e `pos` e' di tipo booleano, allora `y = x(pos)` è il vettore che contiene solo i valori di `x` corrispondenti ai valori "true" (non zero) del vettore `pos`.

La funzione find

- Un metodo alternativo per estrarre valori che soddisfano una data condizione consiste nell'utilizzare la funzione `find`.
- `find(pos)` resituisce gli indici di tutti i valori "true" nel vettore/matrice `pos`
- Se `x` e `pos` sono vettori/matrici di pari dimensioni, allora `x(find(pos))` è equivalente a `x(pos)`.
- Esempio: dato il vettore `x = [2.1 3.4 -0.4 -1.2 3.2 -2.5 2.1]`
  1. `x >= 0` $\longrightarrow$ `[1 1 0 0 1 0 1]`
  2. `find(x>=0)` $\longrightarrow$ `[1 2 5 7]`
  3. `x(find(x>=0))` $\longrightarrow$ `[2.1 3.4 3.2 2.1]`

Exercise (Vettorializzazione)

Modificare la funzione `zcb` *in modo che accetti matrici come argomenti per* t, T *e* r

Exercise (Vettorializzazione)

Modificare la funzione `zcb` *in modo che accetti matrici come argomenti per* t, T *e* r

zcb2.m

```
function p = zcb2(t,T,r)
% p = zcb(t,T,r)
%
% Compute the value at t of a ZCB with maturity T and
% rate (composite) r

p = 100./(1+r).^(T-t);
```

Exercise (Vettorializzazione)

Modificare la funzione `bs_call` *in modo che accetti matrici come argomenti per* K *e* T

Exercise (Vettorializzazione)

Modificare la funzione `bs_call` *in modo che accetti matrici come argomenti per* K *e* T

bs_call2.m

```
function C = bs_call2(S, K, T, r, sigma )
% C = bs_call(S, K, T, r, sigma )
%   Calcola il valore di una Call vanilla

d1 = (log(S./K) + (r+sigma^2/2).*T)./(sigma.*sqrt(T));
d2 = d1 - sigma.*sqrt(T);

C = S.*mynormcdf(d1) - exp(-r.*T).*K.*mynormcdf(d2);
```

Ciclo For

- Ciclo for: `for i=fist:by:last; ...; end`
- Caso generale: `for i=A; <expr>; end`,
  - se `A` è una matrice,
    ripeti le istruzioni `<expr>` assegnando ad `i` ogni colonna di `A`
  - se `A` è un vettore riga,
    ripeti le istruzioni `<expr>` assegnando ad `i` ogni valore di `A`

Creare un grafico

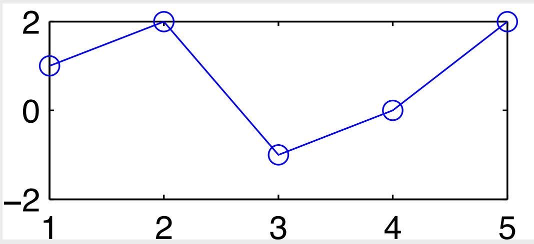
Siano `x`, `y` vettori di lunghezza n

La funzione `plot` ha le seguenti specifiche:

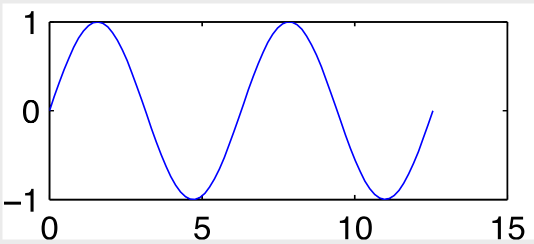
- `plot(y)` produce il grafico della spezzata $(1, y_1), (2, y_2), \dots, (n, y_n)$;
- `plot(x,y)` produce la spezzata $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$;
- `plot(x,y,'o')` disegna i punti $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$;

Example

```
y = [1 2 -1 0 2];
plot( y );
```




```
x = linspace(0,4*pi,80);
y = sin(x);
plot(x,y);
```

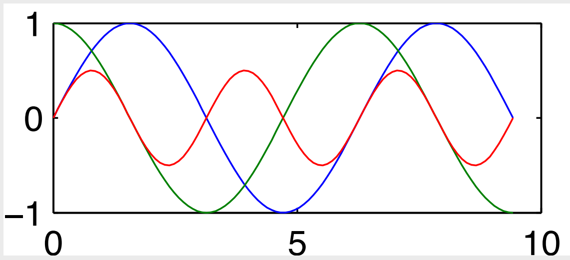


Diverse serie in unico grafico

`plot(x1,y1, x2,y2, x3,y3)`: visualizza in unico grafico tre spezzate.

Example

```
x  = linspace(0,3*pi,100);
y1 = sin(x);
y2 = cos(x);
y3 = y1.*y2;
plot( x,y1, x,y2, x,y3 );
```



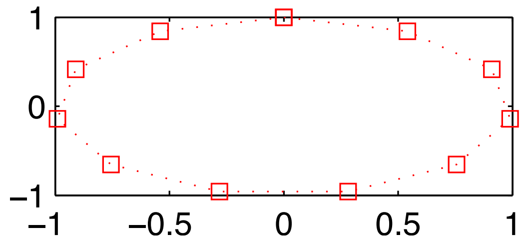
- `hold on` permette di aggiungere grafici alla figura corrente
- `hold off` fa in modo che la figura venga cancellata prima di ogni plot
- Il comando `subplot(m,n,i)` divide la figura in $m \times n$ sottofigure e si posiziona sulla i-esima.

Specificare il tipo di linea

- plot(x,y,'linestyle') dove linestyle é una stringa composta dai seg. caratteri:

Colori		Linee		Punti	
c	azzurro	-	continua	+	+
m	magenta	--	tratteggiata	o	cerchietto
y	giallo	:	punteggiata	*	punto
r	rosso	-.	linea-punto	x	x
g	verde			s	quadrato
b	blue			^v<>	triangoli
w	bianco			p	stella a 5 punte
k	nero			h	stella a 6 punte

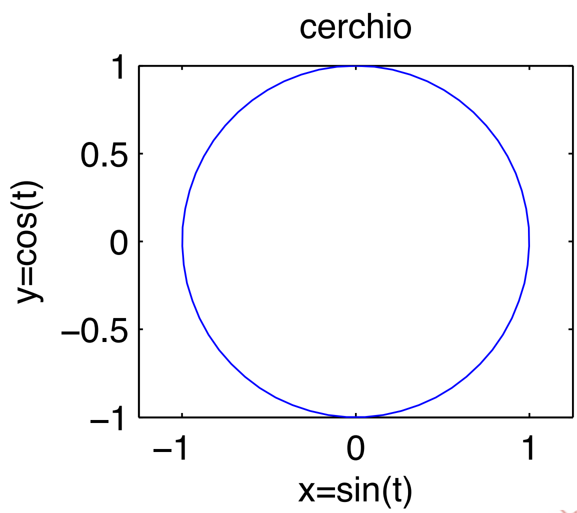
```
t = linspace(0,2*pi,12);
plot( sin(t), cos(t), 'r:s');
```



Annotare i grafici

- `title('titolo')` imposta il titolo della figura
- `xlabel` e `ylabel` impostano le etichette degli assi
- `legend` descrive ogni serie visualizzata
- `axis` imposta gli estremi degli assi

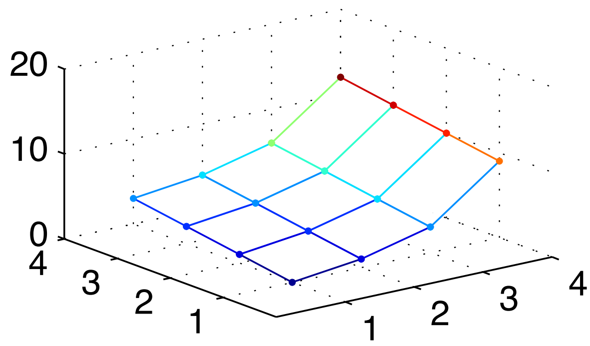
```
t = linspace(0,2*pi,60);
plot( sin(t), cos(t) );
axis([-1.25 1.25 -1 1])
title('cerchio')
xlabel('x=sin(t)')
ylabel('y=cos(t)');
```



Visualizzare Matrici e Superfici

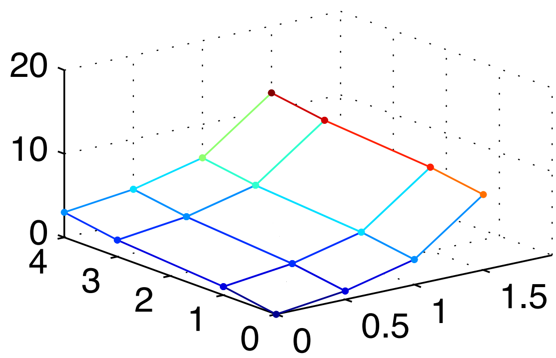
- `mesh(A)` visualizza una mesh sui punti (i, j, A_{ij})

```
A = [0 1 3 9
     1 2 4 10
     2 3 5 11
     3 4 6 12 ];
mesh(A)
```



- `mesh(x,y,A)` visualizza una mesh con punti (x_j, y_i, A_{ij})

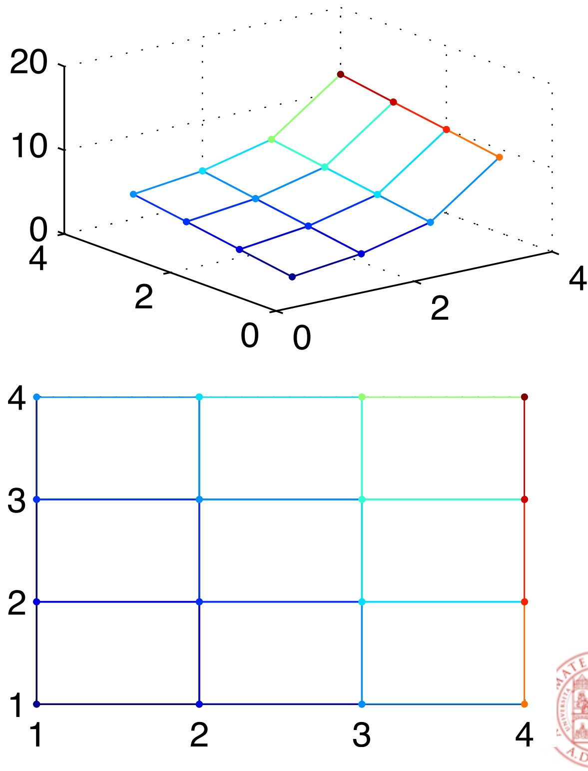
```
x = [0 .5 1 1.5];
y = [0 1 3 4];
mesh(x,y,A)
```



Visualizzare Matrici

- `mesh(X,Y,A)` crea una con punti (X_{ij}, Y_{ij}, A_{ij})

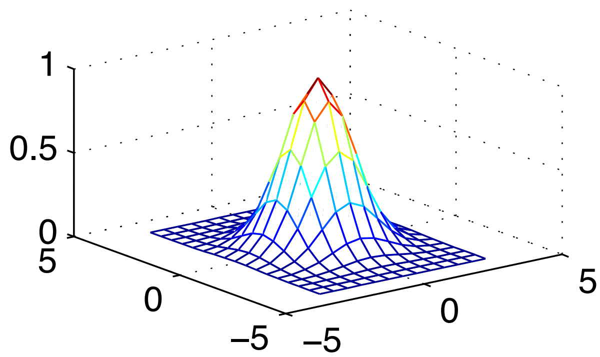
```
A = [0 1 3 9
     1 2 4 10
     2 3 5 11
     3 4 6 12 ];
X = [1 2 3 4
     1 2 3 4
     1 2 3 4
     1 2 3 4];
Y = [1 1 1 1
     2 2 2 2
     3 3 3 3
     4 4 4 4];
mesh(X,Y,A)
```



Funzioni in 2 variabili

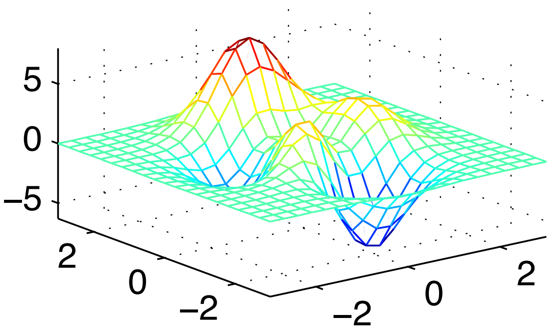
- Per visualizzare funzioni $z = f(x, y)$
  - generare una griglia di punti (x, y), cioé due matrici `X, Y (m x n)` contenenti le coordinate dei punti
  - generare la matrice `Z` in modo che `z(i,j) = f(x(i,j),y(i,j))`.
- Esempio: visualizzare la funzione $z = \exp(-(x^2 + y^2)/2)$

```
xpts = linspace(-3,3,13);
ypts = linspace(-4,4,17);
[X,Y] = meshgrid(xpts,ypts);
Z = exp(-(X.^2+Y.^2)/2);
mesh(X,Y,Z);
```

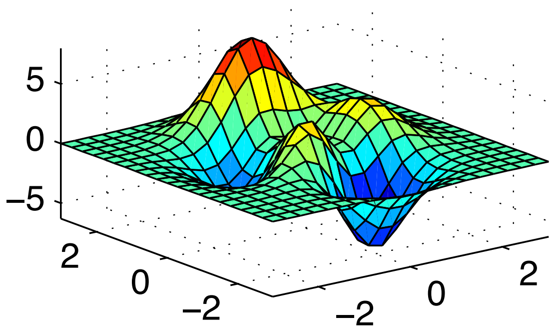


Visualizzazione di superfici

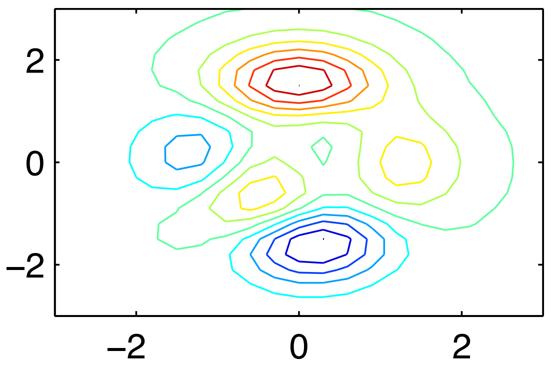
`mesh(X,Y,Z);`



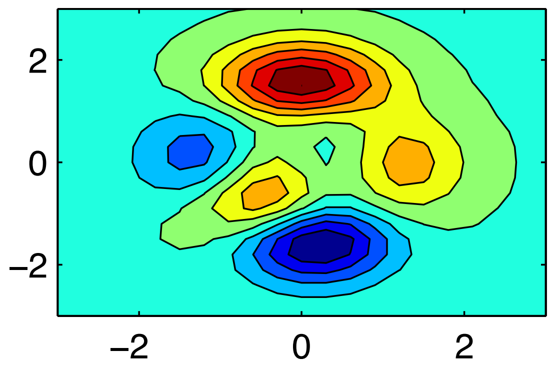
`surf(X,Y,Z);`



`contour(X,Y,Z);`



`contourf(X,Y,Z);`



Payoff: valore di un derivato in funzione di Strike (K) e Scadenza (T)

Exercise (Grafico payoff opzioni europee)

Utilizzare la funzione `bs_call` *per visualizzare il Payoff di un opzione europea*

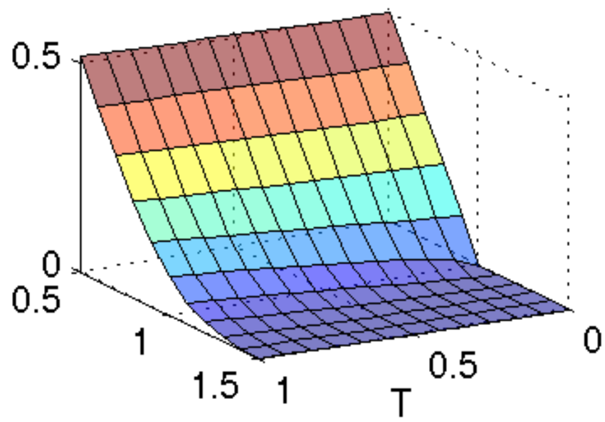
Dati: $S_0 = 1$, $\sigma = 0.2$, $r = 0.03$, $K \in [0.5, 1.4]$ *e* $T \in [0, 1]$

```
S=1; r=.03; sigma=.2;
m=11;  K=linspace(.5,1.5,m);
n=13;  T=linspace(0,1,n);
U = zeros(m,n);
for i=1:m
  for j=1:n
    if (T(j)==0)
      U(i,j)=max(S - K(i),0);
    else
      U(i,j)=bs_call(S,K(i),T(j),r,sigma);
    end
  end
end
surf(T,K,U);
```

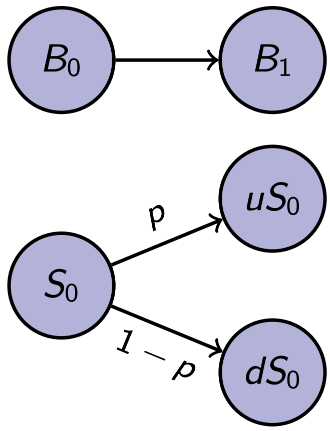
Esercizio: payoff di opzioni europee

- Oppure...

```
S=1; r=.03; sigma=.2;

m=11;  K=linspace(.5,1.5,m);
n=13;  T=linspace(0,1,n);
[K,T] = meshgrid(K,T);

U = bs_call2(S,K,T,r,sigma);
U(1,:) = max(S-K(1,:),0);

surf(T,K,U);
xlabel('T'); ylabel('K');
alpha(0.5);  view(150,25);
```



Binomial model



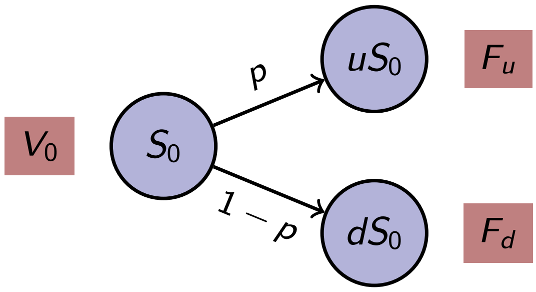
- Discrete time: $t_0 = 0$, $t_1 = T$.
- Risk-free asset: $B_1 = B_0(1+\rho)$.
- Risky asset: $S_1 = \begin{cases} uS_0, & \text{prob. } p, \\ dS_0, & \text{prob. } 1-p. \end{cases}$
- Portfolio:
  - $V_n = \alpha_n S_n + \beta_n B_n$.
  - Self-financing: $\alpha_{n-1} S_n + \beta_{n-1} B_n = V_n$.
  - Predictable: α_n and β_n only depend on the past.
- The binomial model is:
  - arbitrage free;
  - complete: every european derivative can be replicated by means of a self-financing portfolio.

Arbitrage

- Arbitrage portfolio V
  1. V is self-financing and predictable
  2. $P[V_0 = 0] = 1$,
  3. $P[V_n \geq 0] = 1$ and $P[V_n > 0] > 0$ for some n
- No arbitrage implies $d < 1 + \rho < u$.
  - Suppose $1 + \rho \leq d < u$.
    Let consider the self-financing portfolio $V_n = S_n - \frac{S_0}{B_0} B_n$.
    Then, $V_0 = 0$ and $V_1 = S_1 - \frac{S_0}{B_0} B_1 = S_1 - S_0(1 + \rho) \geq 0$.
    Furthermore, with probablity $p > 0$, $S_1 = uS_0$ and $V_1 > 0$.
  - Assume $d < u \leq 1 + \rho$.
    and consider the self-financing portfolio $V_n = \frac{S_0}{B_0} B_n - S_n$.
    Then, $V_0 = 0$ and $V_1 = S_0(1 + \rho) - S_1 \geq 0$ and $V_1 > 0$ with probability $(1 - p) > 0$.
- When $d < 1 + \rho < u$ the binomial model is arbitrage-free
  - Consider the self-financing portfolio: $V_n = \gamma S_n - \gamma \frac{S_0}{B_0} B_n$ $(V_0 = 0)$.

Replicating portfolio

- Portfolio: $V_n = \alpha_n S_n + b_n$, $n = 0, 1$
- Should replicate the payoff $V_1 = F(S_1)$:
  - $\alpha_1 u S_0 + b_1 = F_u \equiv F(uS_0)$
  - $\alpha_1 d S_0 + b_1 = F_d \equiv F(dS_0)$
- the solution of this linear system is



$$\alpha_1 = S_0^{-1} \frac{F_u - F_d}{u - d} \qquad b_1 = \frac{uF_d - dF_u}{u - d}$$

- Single period (no rebalancing): $\alpha_0 = \alpha_1 = S_0^{-1} \dfrac{F_u - F_d}{u - d}$, $b_0 = \frac{1}{1+\rho} b_1$
- $V_0 = \dfrac{F_u - F_d}{u - d} + \dfrac{uF_d - dF_u}{(1+\rho)(u-d)} = \dfrac{(1+\rho-d)F_u - (1+\rho-u)F_d}{(1+\rho)(u-d)}$
- Delta Hedging: $\alpha_0 = \dfrac{F_u - F_d}{S_u - S_d} = \dfrac{\Delta F}{\Delta S}$

Equivalent Martingale Measure

- $V_0 = \dfrac{(1+\rho-d)F_u - (1+\rho-u)F_d}{(1+\rho)(u-d)} = \dfrac{1}{1+\rho}\left(\dfrac{1+\rho-d}{u-d}F_u + \dfrac{u-(1+\rho)}{u-d}F_d\right)$
- $q = \dfrac{1+\rho-d}{u-d}, \quad 1-q = -\dfrac{1+\rho-u}{u-d},$
- if $d < 1+\rho < u$ then $0 < q < 1$

$\Rightarrow$ q defines a probability measure, let call it Q

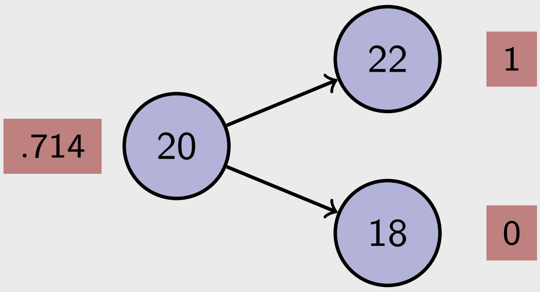
- $V_0 = \dfrac{1}{1+\rho}(qF_u + (1-q)F_d) = \dfrac{1}{1+\rho}\,\mathrm{E}^Q[F(S_1)] = \dfrac{1}{1+\rho}\,\mathrm{E}^Q[V_1]$

  here V_1 is the value of the contingent claim at time $t = 1$.

The value of the contingent claim is given by the discounted expected value (w.r.t. the measure Q) of its payoff:

$$V_0 = \frac{1}{1+\rho}\,\mathrm{E}^Q[V_1]$$

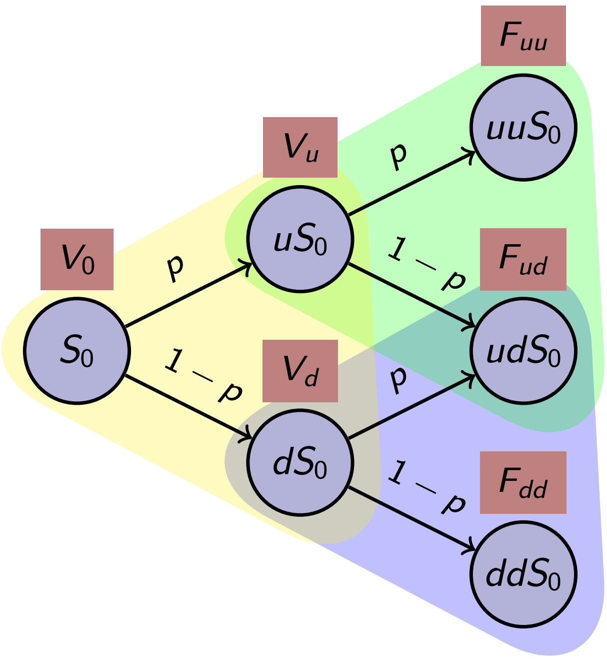
Example



- $\rho = 0.05$
- $S_0 = 20,\ u = 1.1,\ d = .9$
- $F(S_1) = \max(S_1 - 21, 0)$

- The replication portfolio is $V_1 = \alpha_1 S_1 + b_1 \equiv F(S_1)$
$$\begin{cases} \alpha_1 18 + b_1 = 1 \\ \alpha_1 22 + b_1 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha_1 = \frac{1-0}{22-18} = 1/4 \\ b_1 = -22\alpha_1 = -22/4 \end{cases}$$
so that
$$V_0 = \alpha_1 S_0 + b_0 = 1/4 \cdot 20 - \tfrac{1}{1.05} 22/4 = 0.714$$
- Alternatively (martingale approach):
  - $q = \dfrac{1+\rho - d}{u - d} = \dfrac{1 + .05 - .9}{1.1 - .9} = \dfrac{.15}{.2} = 3/4$
  - $V_0 = \dfrac{1}{1+\rho} \mathrm{E}^Q[V_1] = \dfrac{1}{1.05}\left(\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0\right) = \dfrac{3/4}{1.05} = 0.714$

A two-steps binomial model



Dynamics:

- $S_{n+1} = \begin{cases} uS_n, & p \\ dS_n, & 1-p \end{cases}$
- $B_{n+1} = (1+\rho)B_n$

At time $t = 1$

The payoff F_{uu}, F_{ud} is replicable by a portfolio with value V_u

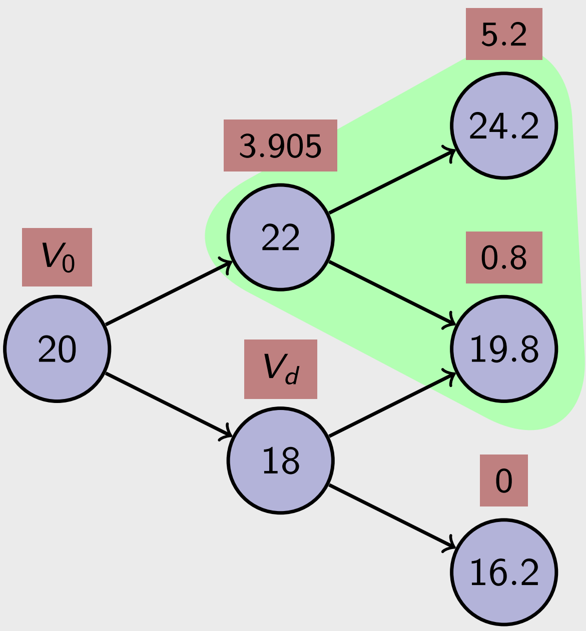
The payoff F_{ud}, F_{dd} is replicable by portfolio with value V_d

At time $t = 0$

The payoff V_u, V_d is replicable by a portfolio with value V_0

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$



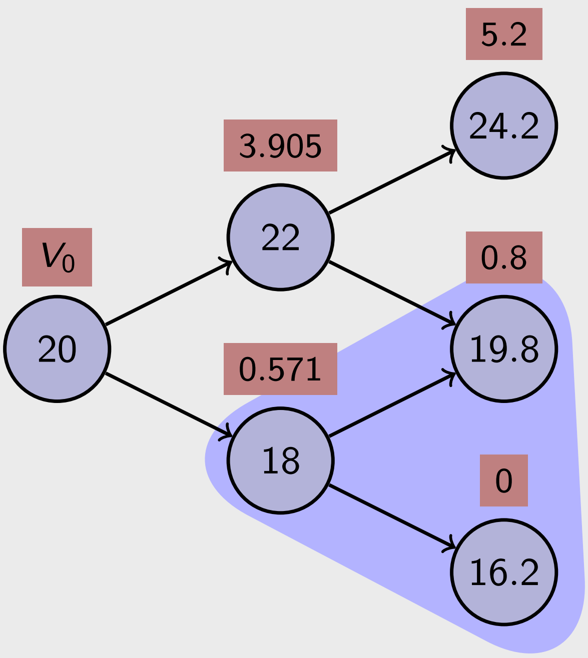
$$\alpha_u = \frac{1}{22} \cdot \frac{5.2 - .8}{.2} = 1$$

$$b_u = \frac{1.1 \cdot 0.8 - 0.9 \cdot 5.2}{.2 \cdot 1.05} = -18.095$$

$$V_u = 22 \cdot 1 - 18.095 = 3.905$$

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$



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$$V_u = 22 \cdot 1 - 18.095 = 3.905$$

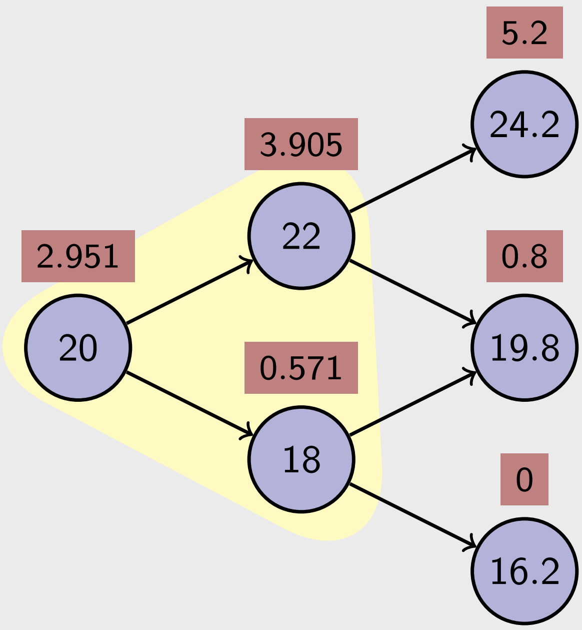
$$\alpha_d = 18^{-1} \frac{0.8 - 0}{0.2} = 0.222$$

$$b_d = \frac{1.1 \cdot 0 - 0.9 \cdot 0.8}{0.2 \cdot 1.05} = -3.429$$

$$V_d = 0.2222 \cdot 18 - 3.429 = 0.571$$

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$



$$\alpha_u = \frac{1}{22} \cdot \frac{5.2 - .8}{.2} = 1$$

$$b_u = \frac{1.1 \cdot 0.8 - 0.9 \cdot 5.2}{.2 \cdot 1.05} = -18.095$$

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$$b_d = \frac{1.1 \cdot 0 - 0.9 \cdot 0.8}{0.2 \cdot 1.05} = -3.429$$

$$V_d = 0.2222 \cdot 18 - 3.429 = 0.571$$

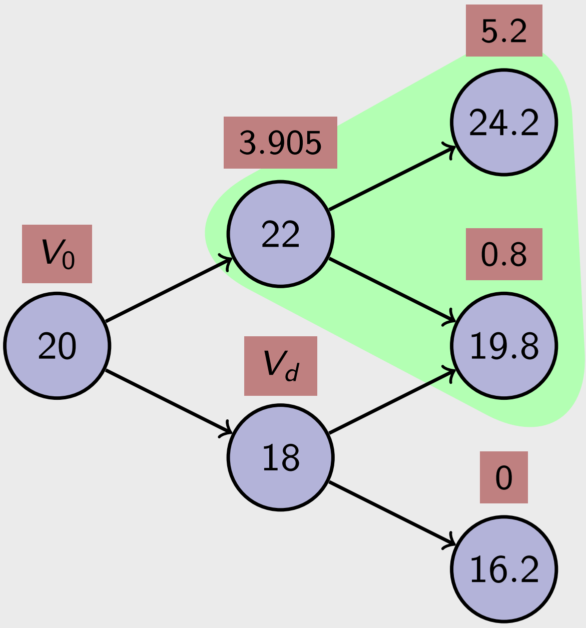
$$\alpha = 20^{-1} \frac{3.905 - .571}{.2} = 0.834$$

$$b = \frac{1.1 \cdot 0.571 - .9 \cdot 3.905}{.2 \cdot 1.05} = -13.729$$

$$V_0 = 20 \cdot 0.834 - 13.729 = 2.951$$

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$

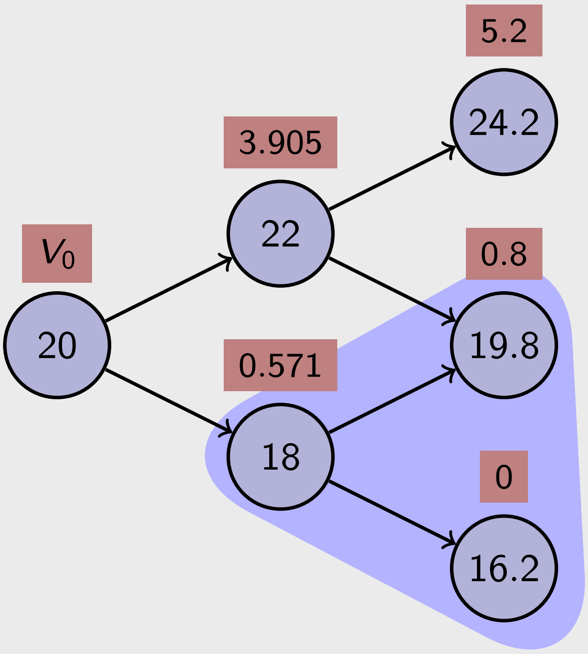


$$q = \frac{1+\rho-d}{u-d} = \frac{1.05-0.9}{1.1-0.9} = 0.75$$

$$V_u = \frac{0.75 \cdot 5.2 + 0.25 \cdot 0.8}{1.05} = 3.905$$

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$



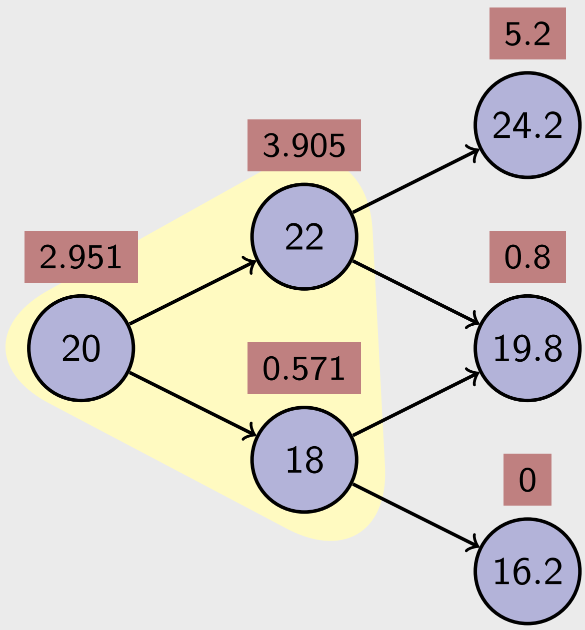
$$q = \frac{1+\rho-d}{u-d} = \frac{1.05-0.9}{1.1-0.9} = 0.75$$

$$V_u = \frac{0.75 \cdot 5.2 + 0.25 \cdot 0.8}{1.05} = 3.905$$

$$V_d = \frac{0.75 \cdot 0.8 + 0.25 \cdot 0}{1.05} = 0.571$$

Example

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F_T = \max(S_T - 19, 0)$



$$q = \frac{1+\rho-d}{u-d} = \frac{1.05-0.9}{1.1-0.9} = 0.75$$

$$V_u = \frac{0.75 \cdot 5.2 + 0.25 \cdot 0.8}{1.05} = 3.905$$

$$V_d = \frac{0.75 \cdot 0.8 + 0.25 \cdot 0}{1.05} = 0.571$$

$$V = \frac{0.75 \cdot 3.905 + 0.25 \cdot 0.571}{1.05} = 2.951$$

Multiperiod Binomial Model

- Each node corresponds to a specific time (n) and a specific scenario (j)
- At time n in the scenario with j up-movements ($n-j$ down-moves) the underlying is given by
  - $S_{nj} = u^j d^{n-j} S_0, \qquad n = 1, \dots, N,\ j = 0, \dots, n$
- Analogously the values for claim and the portfolio composition are:
  - $V_{nj}, \alpha_{nj}, b_{nj}$, for $n = 1, \dots, N,\ j = 0, \dots, n$

Algorithm

- $S_{0,0} = S_0, \qquad S_{n+1,j} = dS_{n,j}$ and $S_{n+1,j+1} = uS_{n,j}$ (forward)
- $V_{Nj} = F(S_{Nj}), \qquad V_{nj} = \dfrac{qV_{n+1,j+1} + (1-q)V_{n+1,j}}{1+\rho}$ (backward)
- $\alpha_{nj} = \dfrac{V_{n+1,j+1} - V_{n+1,j}}{S_{n+1,j+1} - S_{n+1,j}}, \qquad b_{nj} = V_{nj} - \alpha_{nj} S_{nj}$ (backward)

Matlab implementation

binomial1.m

```
function [C,V,S] = binomial1(S0,K,u,d,rho,N)

S=nan(N,N);  V=nan(N,N);

% Scenario Generation
S(1,1)=S0;
for n=1:N-1
  for j=1:n; S(n+1,j) = d*S(n,j); end
  S(n+1,n+1) = u*S(n,n);
end

% Compute replicating portfolio
q = (1+rho-d)/(u-d);
V(N,:) = max(S(N,:)-K,0);
for n=N-1:-1:1
  for j=1:n
    V(n,j) = (q*V(n+1,j+1) + (1-q)*V(n+1,j))/(1+rho);
  end
end
C = V(1,1);
```

Matlab implementation

binomial2.m

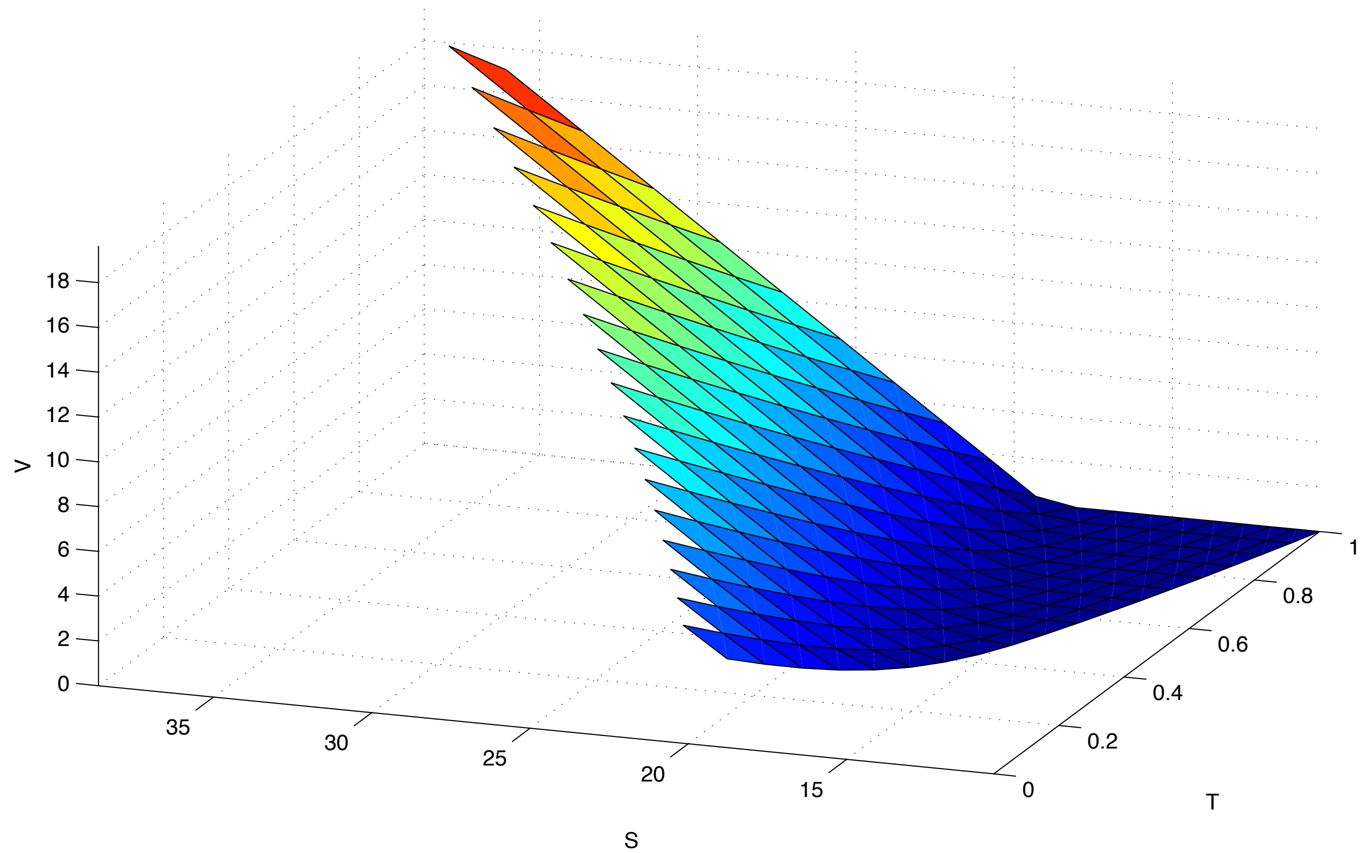
```
function [C,V,S] = binomial2(S0,K,u,d,rho,N)

S=nan(N,N);  V=nan(N,N);

% Scenario Generation
S(1,1)=S0;
for n=1:N-1
  S(n+1,1:n) = d*S(n,1:n);
  S(n+1,n+1) = u*S(n,n);
end

% Compute replicating portfolio
q = (1+rho-d)/(u-d);
V(N,:) = max(S(N,:)-K,0);
for n=N-1:-1:1
  V(n,1:n) = (q*V(n+1,2:n+1) + (1-q)*V(n+1,1:n))/(1+rho);
end
C = V(1,1);
```

Payoff computed by the Binomial method



Calibration

Consider the r.v. S_{n+1}: $\quad S_{n+1} = \begin{cases} uS_n, & \text{prob. } p \\ dS_n, & \text{prob.} 1-p \end{cases}$

Equivalently: $S_{n+1} = S_n \xi_n$ with $\xi_n = \begin{cases} u, & \text{prob. } p \\ d, & \text{prob. } 1-p \end{cases}$ (independent)

Thus,

$$\log(S_N/S_0) = \sum_{n=1}^{N} \log(\xi_n) \sim N \log(\xi_n) \qquad \text{(the } \xi_n \text{ are i.i.d.)}$$

so that

$$\mathrm{E}[\log(S_N/S_0)] = N\,\mathrm{E}[\log(\xi_n)]$$

and

$$\mathrm{Var}[\log(S_N/S_0)] = N^2\,\mathrm{Var}[\log(\xi_n)]$$

Theorem (Moments of the binomial distribution)

Let $\xi = \begin{cases} u, & \textit{prob. } p, \\ d, & \textit{prob. } 1-p. \end{cases}$ *then*

$\mathrm{E}[\log(\xi)] = p\log(u) + (1-p)\log(d)$ *and* $\mathrm{Var}[\log(\xi)] = p(1-p)\log(u/d)^2$

Proof.

$$
\begin{aligned}
\mathrm{Var}[\log(\xi)] &= \mathrm{E}[\log(\xi)^2] - \mathrm{E}[\log(\xi)]^2 \\
&= p\log(u)^2 + (1-p)\log(d)^2 \\
&\quad - p^2\log(u)^2 - (1-p)^2\log(d)^2 - 2p(1-p)\log(u)\log(d) \\
&= p(1-p)\log(u)^2 + p(1-p)\log(d)^2 - 2p(1-p)\log(u)\log(d) \\
&= p(1-p)(\log(u) - \log(d))^2 \\
&= p(1-p)\log(u/d)^2
\end{aligned}
$$

□

Binomial model: calibration

Given T (expiration) and N (number of steps) $\Rightarrow$ $\Delta_t = T/N$.

From the continuous risk-free rate r: $\rho = e^{r\Delta_t} - 1$.

The risky asset log-return: $\mu = \frac{1}{T}\log(S_T/S_0) \sim \frac{N}{T}\log(\xi_n) = \frac{1}{\Delta_t}\log(\xi_n)$.

Want to match market expected return m and volatility σ^2:

$$\begin{cases} \frac{1}{\Delta_t}\mathrm{E}[\log\xi_n] & = m \\ \frac{1}{\Delta_t^2}\mathrm{Var}[\log\xi_n] & = \sigma^2 \end{cases} \Rightarrow \begin{cases} p\log(u/d) + \log(d) & = m\Delta_t \\ \sqrt{p(1-p)}\log(u/d) & = \sigma\Delta_t \end{cases}$$

$p = 1/2$

$\log(u/d) = 2\sigma\Delta_t$
$\log(d) = m\Delta_t - \sigma\Delta_t$
$\xi_n = \exp((m \pm \sigma)\Delta_t)$

$ud = 1$

$\log(u/d) = \log(u^2) = 2\log(u)$
...

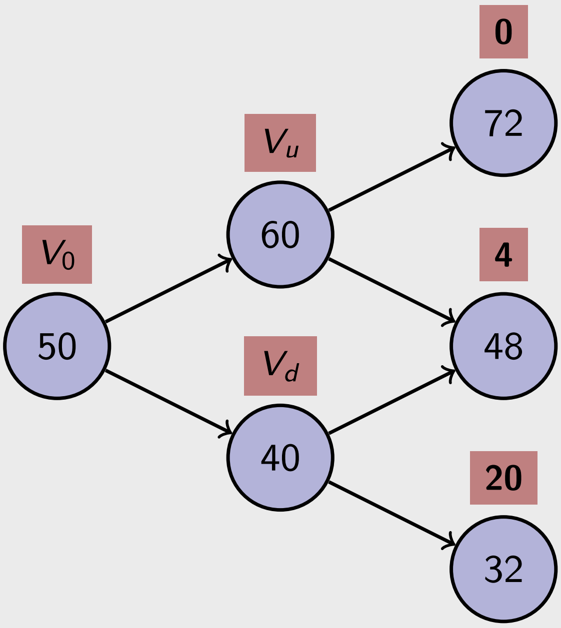
Working in a risk-neutral measure, better to use $m = r$.

American options

- European options: exercise only at expiration
- Opzioni Americane: can be exercised before or at expiration
- At each time the holder of the contract can choose if
  - exercise the option right
  - hold the contract
- Will choose the maximum between
  - the current payoff
  - the fair value of the option at time t (discounted expected value of the future payoff)
- $V_n = \max(\, F(S_n),\, (1+\rho)^{-1}\, \mathrm{E}^Q[V_{n+1}|S_n]\,)$

Example (American Options)

$S_0 = 50,\ u = 1.2,\ d = .8,\ \rho = 0.05,\ F = \max(52 - S, 0) \quad \Rightarrow \quad q = .625$



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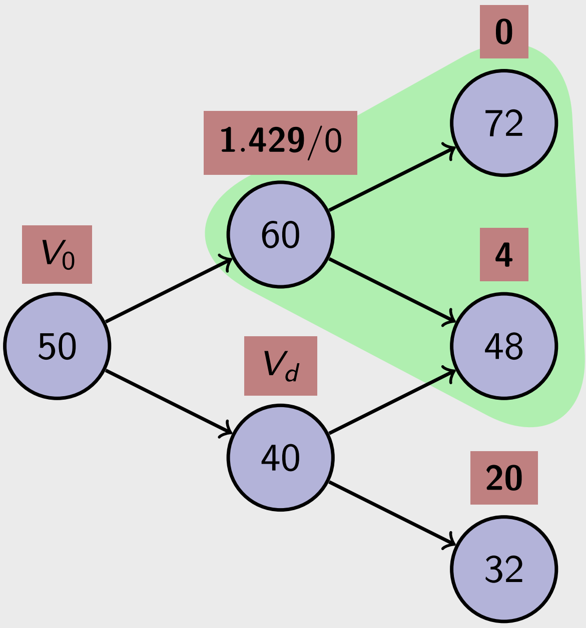
$S_0 = 50,\ u = 1.2,\ d = .8,\ \rho = 0.05,\ F = \max(52 - S, 0) \quad \Rightarrow \quad q = .625$

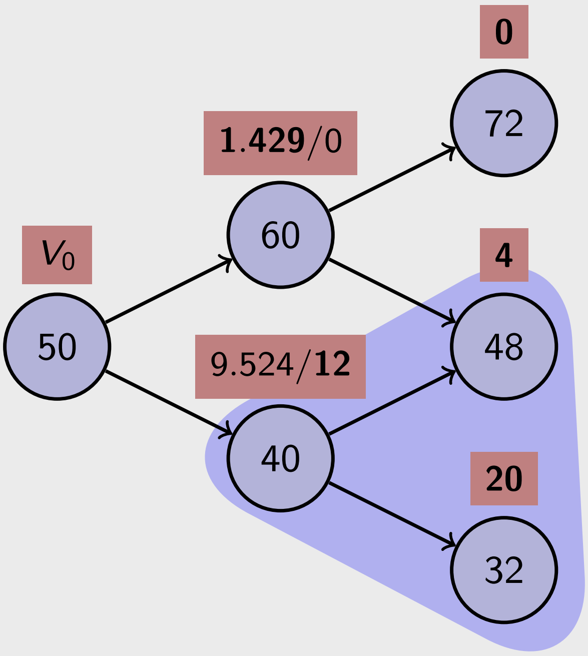


$$V_u^* = \frac{0.625 \cdot 0 + 0.375 \cdot 4}{1.05} = \mathbf{1.429}$$

$$F_u = \max(52 - 60, 0) = 0$$

Example (American Options)

$S_0 = 50,\ u = 1.2,\ d = .8,\ \rho = 0.05,\ F = \max(52 - S, 0) \quad \Rightarrow \quad q = .625$



$$V_u^* = \frac{0.625 \cdot 0 + 0.375 \cdot 4}{1.05} = \mathbf{1.429}$$

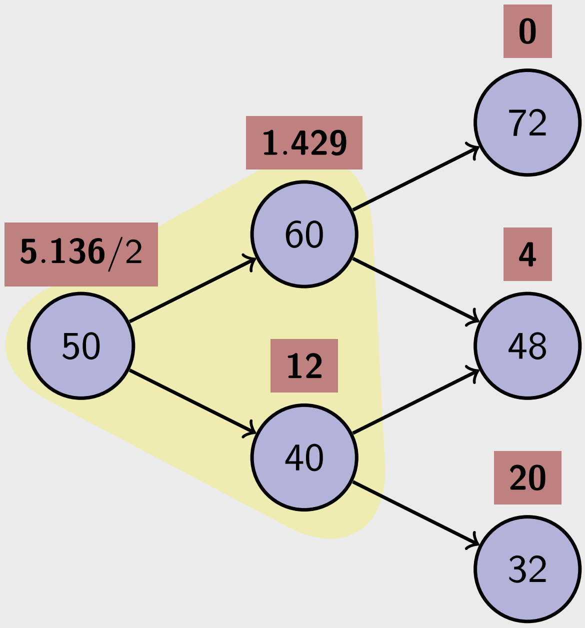
$$F_u = \max(52 - 60, 0) = 0$$

$$V_d = \frac{0.625 \cdot 4 + 0.375 \cdot 20}{1.05} = 9.524$$

$$F_d = \max(52 - 40, 0) = \mathbf{12}$$

Example (American Options)

$S_0 = 50,\ u = 1.2,\ d = .8,\ \rho = 0.05,\ F = \max(52 - S, 0) \quad \Rightarrow \quad q = .625$



$$V_u^* = \frac{0.625 \cdot 0 + 0.375 \cdot 4}{1.05} = \mathbf{1.429}$$

$$F_u = \max(52 - 60, 0) = 0$$

$$V_d = \frac{0.625 \cdot 4 + 0.375 \cdot 20}{1.05} = 9.524$$

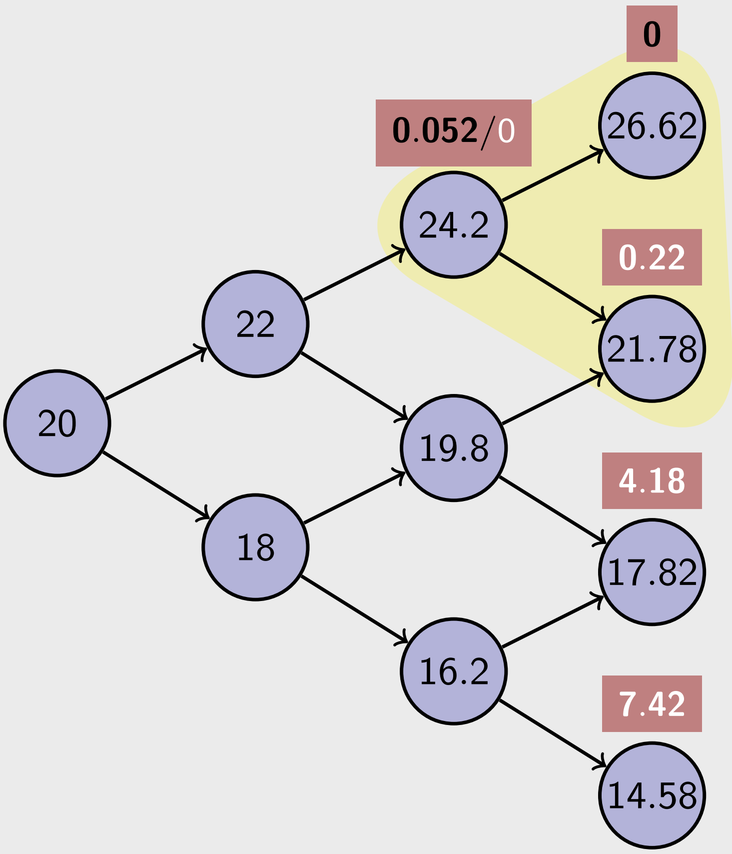
$$F_d = \max(52 - 40, 0) = \mathbf{12}$$

$$V = \frac{0.625 \cdot 1.429 + 0.375 \cdot 12}{1.05} = \mathbf{5.136}$$

$$F = \max(52 - 50, 0) = 2$$

Example (American Options)

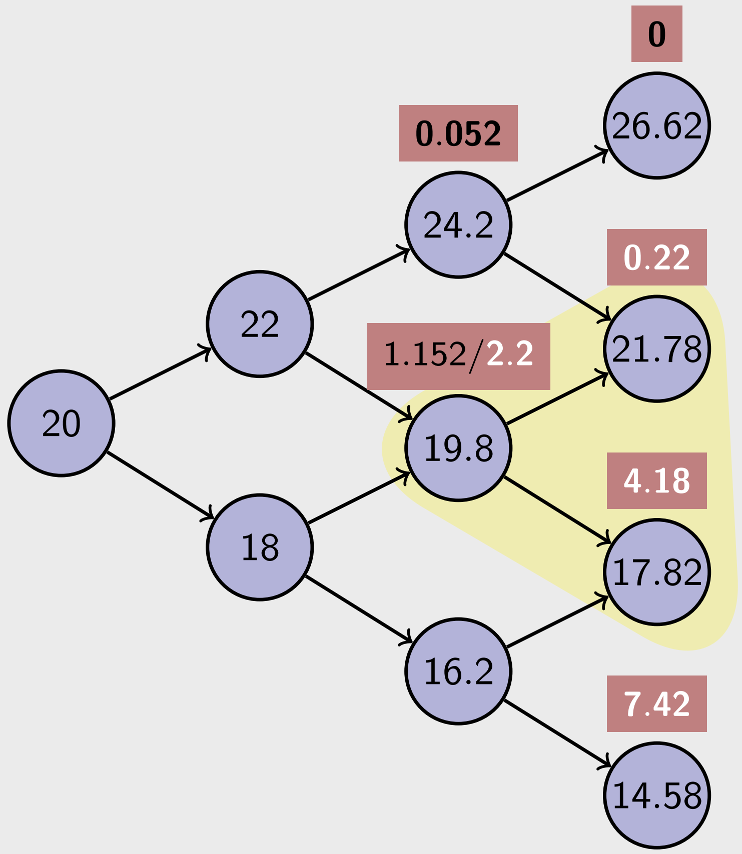
$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$

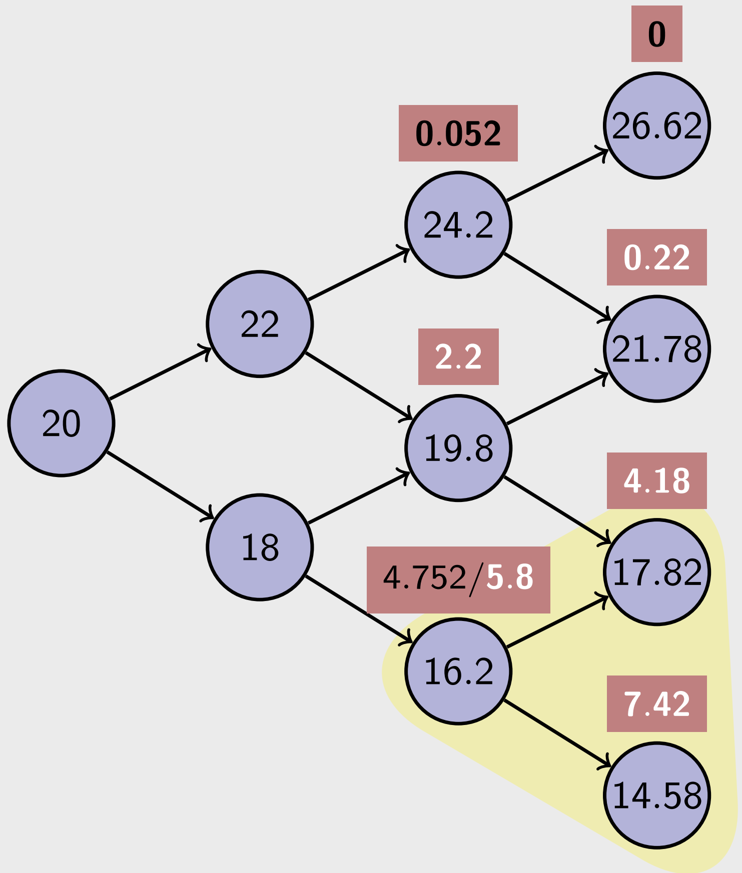


$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

$V^*_{ud} = 1.152,\ F_{ud} = \mathbf{2.2}$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



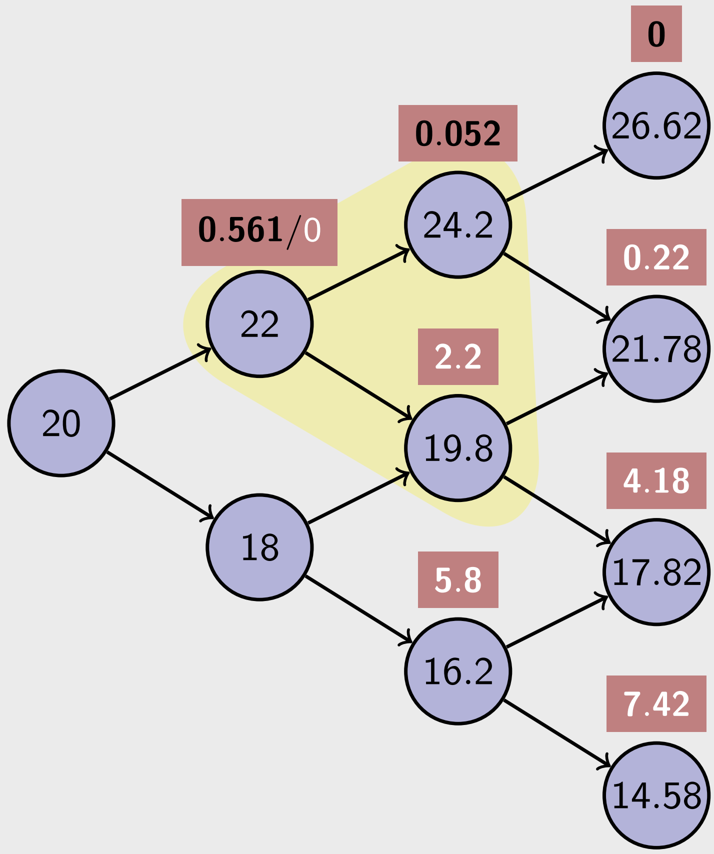
$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

$V^*_{ud} = 1.152,\ F_{ud} = \mathbf{2.2}$

$V^*_{dd} = 4.752,\ F_{dd} = \mathbf{5.8}$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

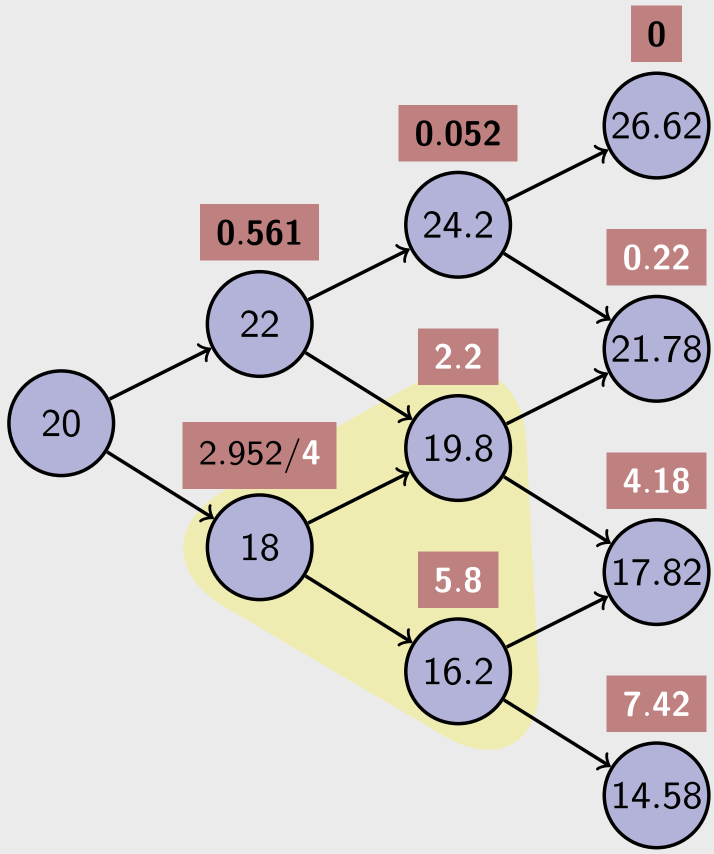
$V^*_{ud} = 1.152,\ F_{ud} = \mathbf{2.2}$

$V^*_{dd} = 4.752,\ F_{dd} = \mathbf{5.8}$

$V^*_{u} = \mathbf{0.561},\ F_{u} = 0$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



$V_{uu}^* = \mathbf{0.052},\ F_{uu} = 0$

$V_{ud}^* = 1.152,\ F_{ud} = \mathbf{2.2}$

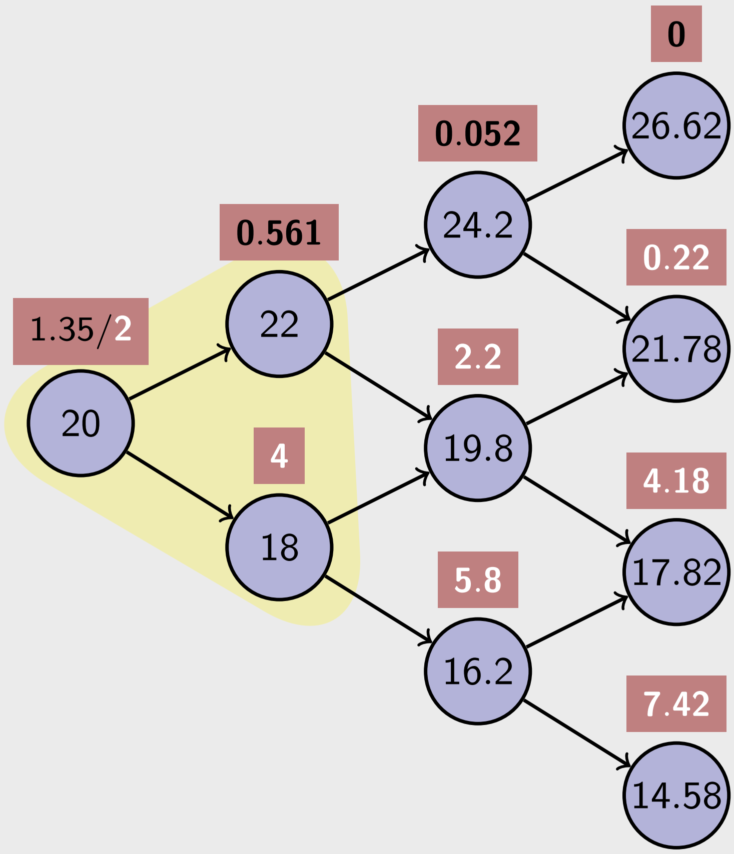
$V_{dd}^* = 4.752,\ F_{dd} = \mathbf{5.8}$

$V_u^* = \mathbf{0.561},\ F_u = 0$

$V_d^* = 2.952,\ F_d = \mathbf{4}$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

$V^*_{ud} = 1.152,\ F_{ud} = \mathbf{2.2}$

$V^*_{dd} = 4.752,\ F_{dd} = \mathbf{5.8}$

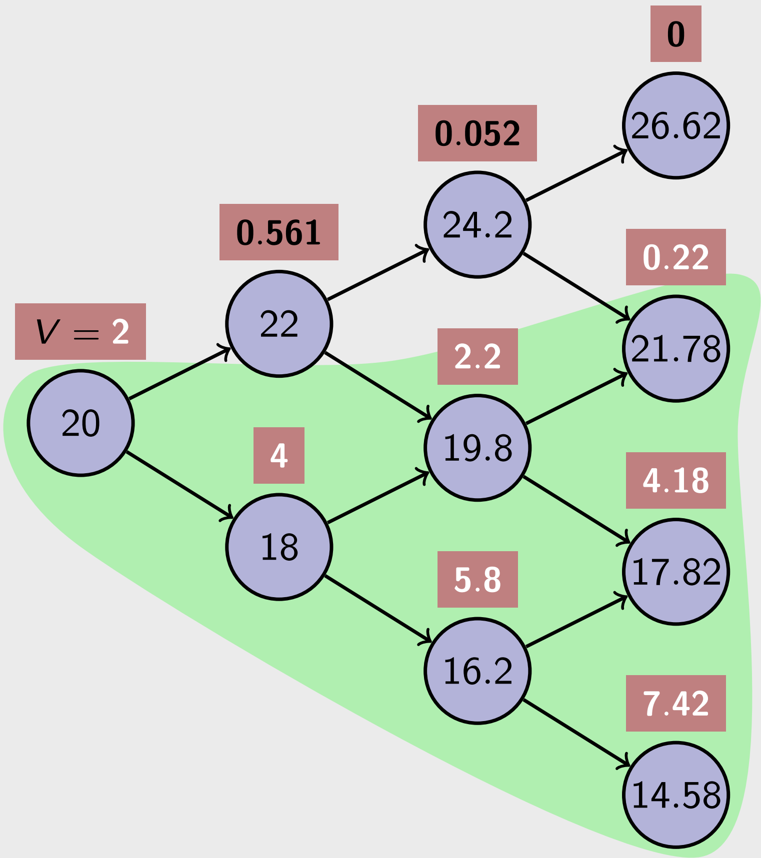
$V^*_{u} = \mathbf{0.561},\ F_{u} = 0$

$V^*_{d} = 2.952,\ F_{d} = \mathbf{4}$

$V^* = 1.35,\ F = \mathbf{2}$

Example (American Options)

$S_0 = 20,\ u = 1.1,\ d = .9,\ \rho = 0.05,\ F = \max(22 - S, 0) \quad \Rightarrow \quad q = .75,$



$V^*_{uu} = \mathbf{0.052},\ F_{uu} = 0$

$V^*_{ud} = 1.152,\ F_{ud} = \mathbf{2.2}$

$V^*_{dd} = 4.752,\ F_{dd} = \mathbf{5.8}$

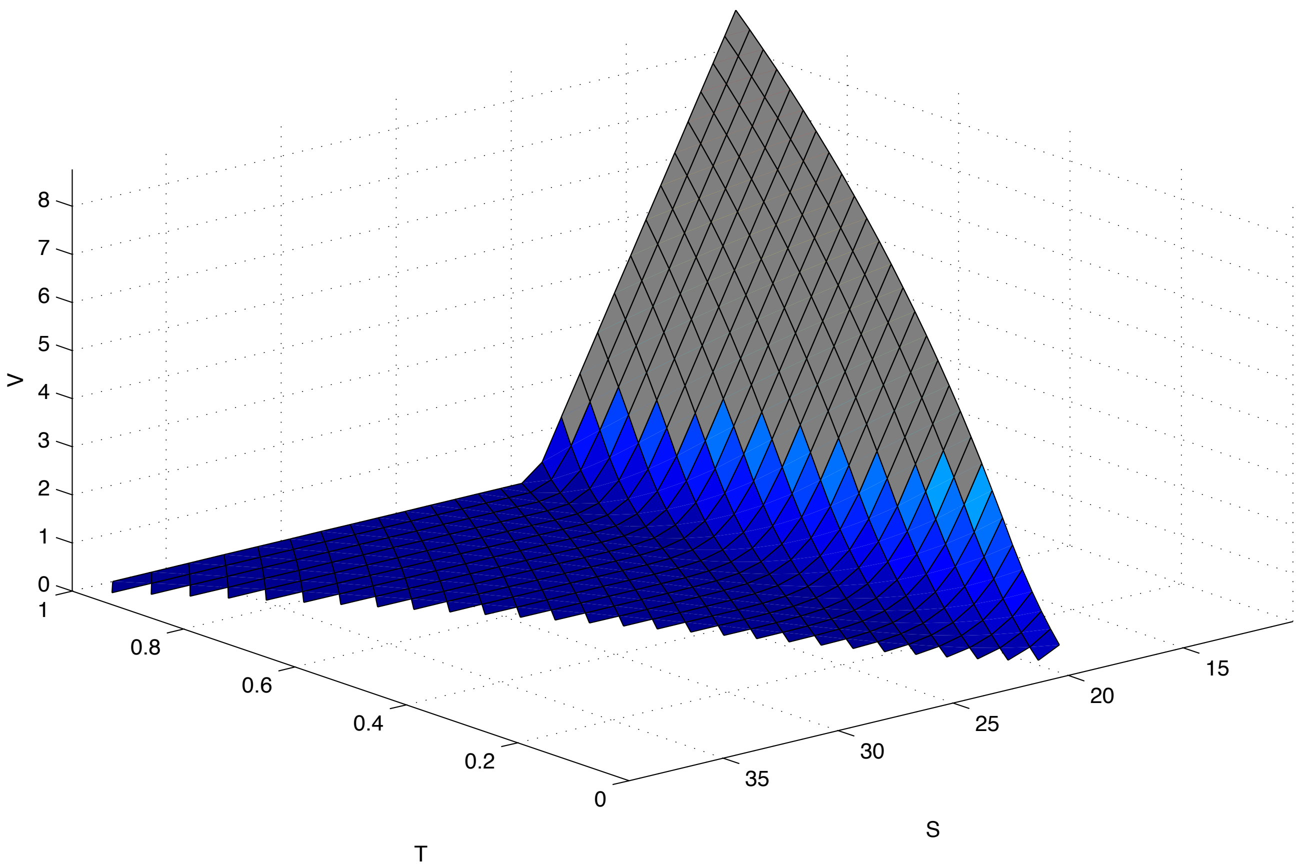
$V^*_{u} = \mathbf{0.561},\ F_{u} = 0$

$V^*_{d} = 2.952,\ F_{d} = \mathbf{4}$

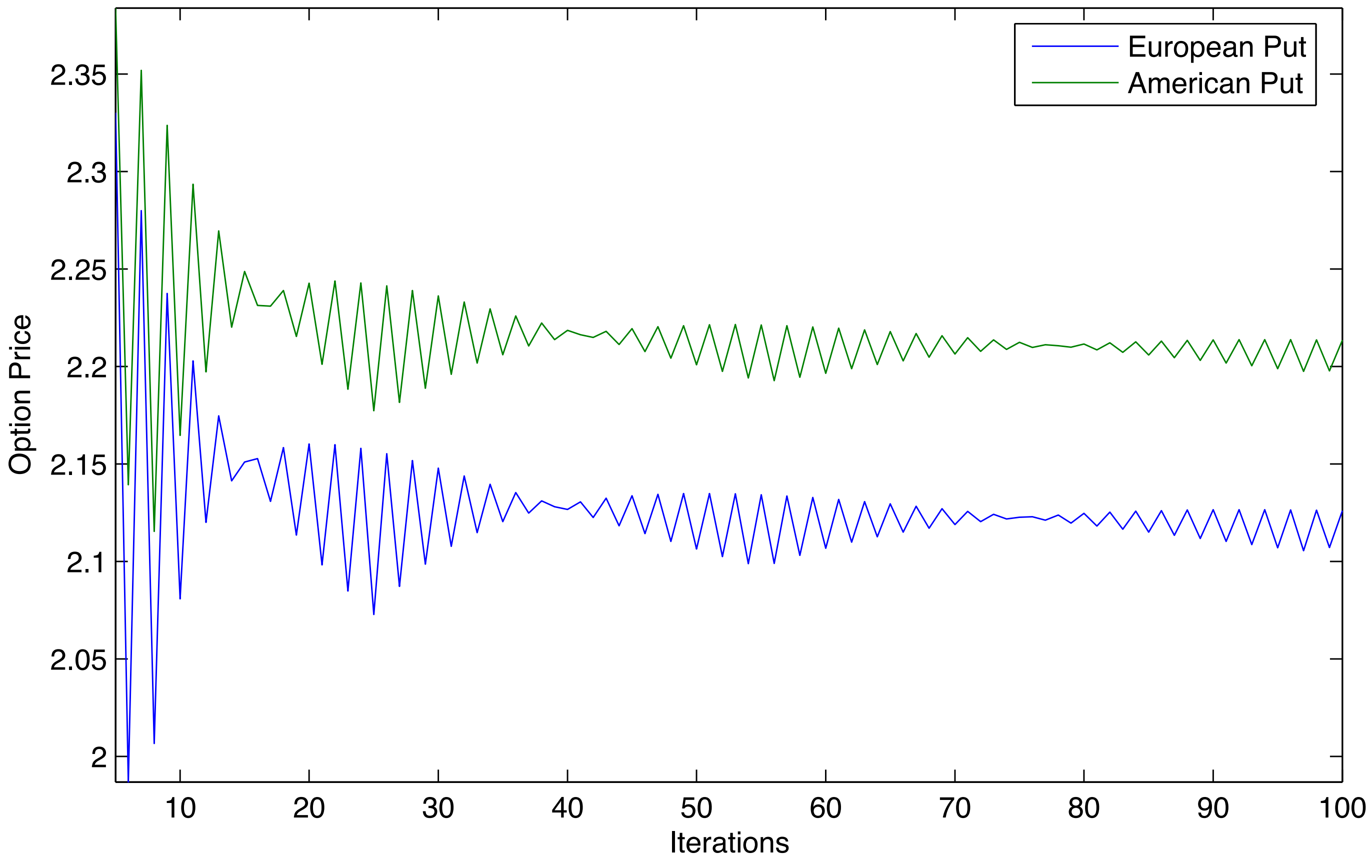
$V^* = 1.35,\ F = \mathbf{2}$

$\Rightarrow$ early exercise region

Payoff of an American Put



Convergence



Computational efficiency

To save memory: vectors instead of matrices (more in [?])

binomial4.m

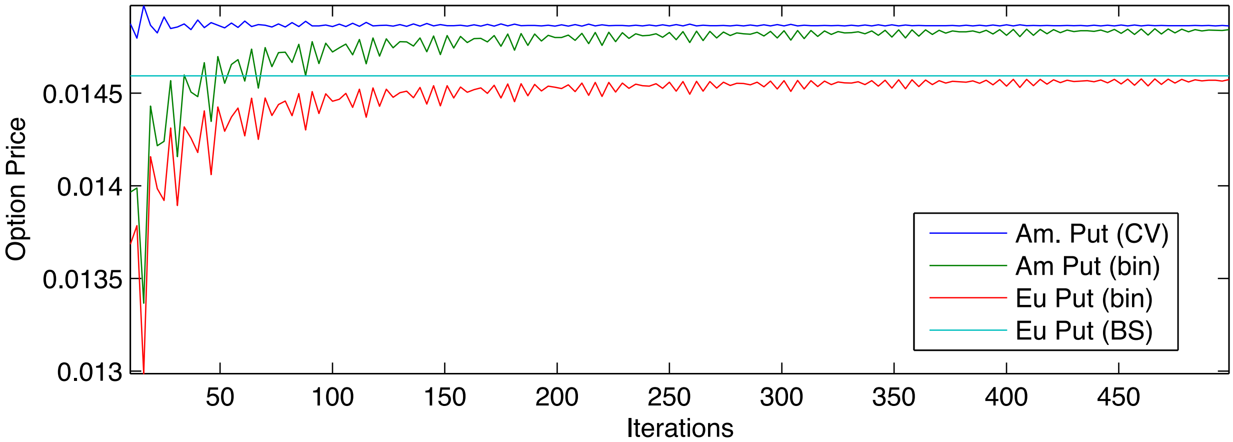
```
function P = binomial4(S0,K,u,d,rho,N)

S=nan(N,1);  V=nan(N,1);
% Scenario Generation
S(1) = S0*d^(N-1);
for j=2:N
  S(j) = S(j-1)*u/d;
end

% Claim price computation
q = (1+rho-d)/(u-d);
V(:) = max(K-S(:),0);
for n=N-1:-1:1
  S(1:n) = S(1:n)/d;
  V(1:n) = max( (q*V(2:n+1) + (1-q)*V(1:n))/(1+rho), K-S(1:n) );
end
P = V(1,1);
```

Control Variates

Let

- P^A: Amercan Put price (unknown)
- P^E: European Put price (known by B&S)
- P_n^A: American Put price computed by an n-steps binomial method
- P_n^E: European Put price computed by an n-steps binomial method

Euristics: $P^A - P_n^A \simeq P^E - P_n^E$ da cui $P^A \simeq P_n^A - (P^E - P_n^E) =: P_n^{CV}$



`binomial_cv.m` `bin_convergence2.m`

Control Variates

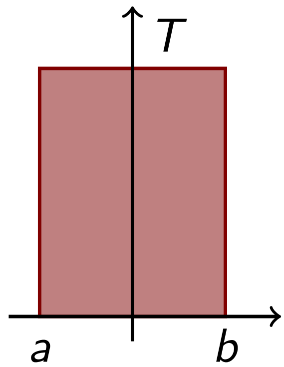
binomial_cv.m

```
function [P,PA,PE,PBS] = binomial_cv(S0,K,T,r,sigma,N)

u=exp(sigma*sqrt(T/N));  d=1/u;
rho=exp(r*T/N)-1;
q = (1+rho-d)/(u-d);

% Scenario Generation
S=nan(N,1);  S(1)=S0*d^(N-1);
for j=2:N;   S(j)=S(j-1)*u/d;  end

% Option pricing
VE = max(K-S(:),0);  V=VE;
for n=N-1:-1:1
  S(1:n) = S(1:n)/d;
  VE(1:n) = (q*VE(2:n+1) + (1-q)*VE(1:n))/(1+rho);
  V(1:n) = max( (q*V(2:n+1) + (1-q)*V(1:n))/(1+rho), K-S(1:n) );
end
PE=VE(1,1);  PA=V(1,1);  PBS = bs_put(S0,K,T,r,sigma);
P = PA + PBS-PE;
```

The Heat Equation

- Usually, partial differential equations arising in option pricing are
  - of second order
  - parabolic (or semi-parabolic)
- A prototype equation is the Heat Equation:

  $\partial_{xx} u(x,t) - \partial_t u(x,t) = 0$ on $[a,b] \times [0,T]$



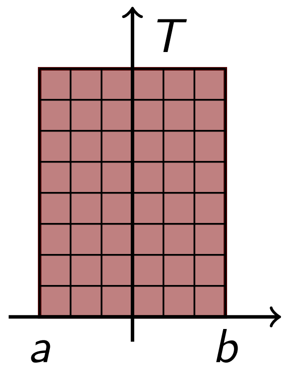
- Evolution equation with initial conditions

  $u(x,0) = \phi(x), \quad x \in [a,b]$
- A finite domain $a \neq -\infty$ e $b \neq \infty$ needs border conditions:

  $u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t) \quad t \in]0,T]$
- The Black & Scholes equations can be reformulated as a Heat Eqn.

Discretization

- Continuous domain.
- Memory and computing time are finite.
- Discretization: approximate u on a finite grid
- Uniform grid:
  - $(i, j) \longrightarrow (x_i, t_j)$,
  - $x_i - x_{i-1} = \Delta_x, \quad x_1 = a, \; x_M = b,$
  - $t_n - t_{n-1} = \Delta_t, \quad t_1 = 0, \; t_N = T,$
  - $u_{i,n} = u(x_i, t_n)$
- thus
  - $\Delta_x = \dfrac{b-a}{M-1}$ and $x_i = a + (i-1)\Delta_x$
  - $\Delta_t = \dfrac{T}{N-1}$ and $t_n = (n-1)\Delta_t$



Problem

Approximate $\partial_{xx} u$ and $\partial_t u = 0$

Taylor Expansion

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + \frac{1}{24}h^4 f^{IV}(x) + O(h^5)$$

First order finite differences

Forward: $\delta^+_{x,h} f \equiv \dfrac{f(x+h) - f(x)}{h} = f'(x) + O(h)$

Backward: $\delta^-_{x,h} f \equiv \dfrac{f(x) - f(x-h)}{h} = f'(x) + O(h)$

Centered: $\delta^o_{x,h} f \equiv \dfrac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$

$$f(x+h) = f(x) + hf'(x) + h^2 f''(x)/2 + h^3 f'''(x)/6 + o(h^4)$$

$$f(x-h) = f(x) - hf'(x) + h^2 f''(x)/2 - h^3 f'''(x)/6 + o(h^4)$$

Second order finite differences

Centered: $\delta^o_{xx,h} f \equiv \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + O(h^2)$

Proof:

$$\begin{array}{rcl}
f(x+h) - f(x) & = & + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}f'''(x) + O(h^4) \\
- f(x) \quad + f(x-h) & = & - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}f'''(x) + O(h^4) \\
\hline
f(x+h) - 2f(x) + f(x-h) & = & + h^2 f''(x) \qquad\qquad + O(h^4)
\end{array}$$

Note that: $\delta^o_{xx,h} f = \dfrac{1}{h}(\delta^+_{x,h} f - \delta^-_{x,h} f)$

Explicit Method

Solve $\delta^o_{xx,h} u - \delta^+_{t,k} u = 0$ with initial condition $u(0,x) = \phi(x)$

- Approximate ∂_{xx} with $\delta^o_{xx,h}$ and ∂_t with $\delta^+_{t,k}$
- Using $h = \Delta_x$ and $k = \Delta_t$ the method can be computed on the grid points:

$$\delta^o_{xx,\Delta_x} u(x_i, t_n) = \frac{u(x_{i+1}, t_n) - 2u(x_i, t_n) + u(x_{i-1}, t_n)}{\Delta_x^2}$$

$$\delta^+_{t,\Delta_j} u(x_i, t_n) = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta_t}$$

- or, equivalently,

$$\delta^o_{xx} u_{i,n} = \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{\Delta_x^2} \qquad \delta^+_t u_{i,n} = \frac{u_{i,n+1} - u_{i,n}}{\Delta_t}$$

$$u_{i,n+1} = u_{i,n} + \frac{\Delta_t}{\Delta_x^2}(u_{i+1,n} - 2u_{i,n} + u_{i-1,n})$$

Border conditions

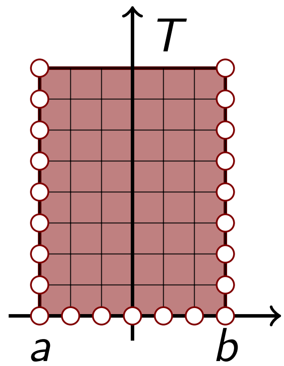
$$u_{i,n+1} = u_{i,n} + \frac{\Delta_t}{\Delta_x^2}(u_{i+1,n} - 2u_{i,n} + u_{i-1,n}) \qquad (*)$$

- Eq. (*) is not defined for $i = 1$ or $i = M$ and for $n = 0$

- Initial Conditions: $u_{i,1} = \phi(x_i)$, $\; i = 1, \ldots, M$
- Border Conditions:

$$u_{1,n+1} = \phi_a(t_{n+1}),$$
$$u_{M,n+1} = \phi_b(t_{n+1}), \qquad n = 1, \ldots, N-1$$

- Note: (*) has to be computed in the correct order: Firstly, all the u in the second row ($n = 1$), then in the third ($n = 2$) and so on.



Exercise (Forward Euler FD method)

Using the explicit method approximate the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_{xx} u - \partial_t u &= 0, & (x,t) &\in (-a,a) \times (0,T] & (*) \\ u(x,0) &= (e^x - 1)^+, & x &\in [-a,a] \\ u(x,t) &= (e^x - 1)^+, & (x,t) &\in -a, a \times [0,T] \end{aligned}$$

and write a matlab function

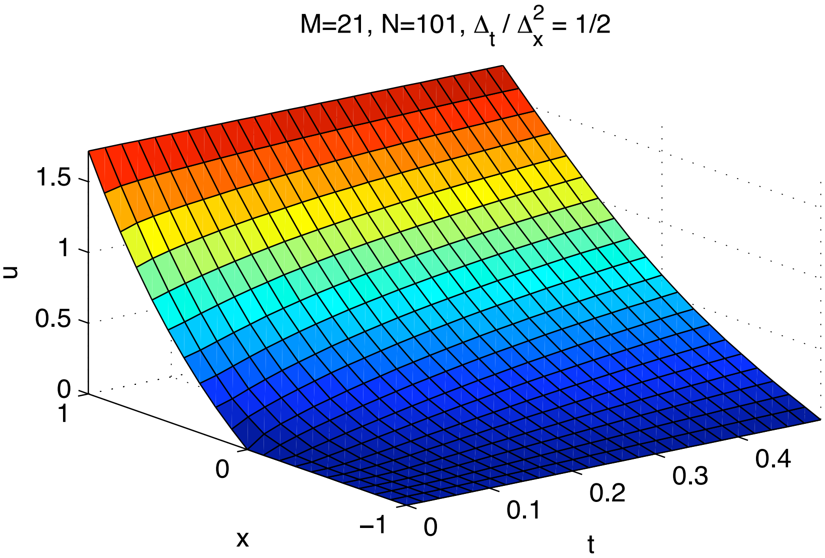
```
function U=heat(a,T,M,N)
```

where $U_{i,n} \simeq u(x_i, t_n)$ *and*

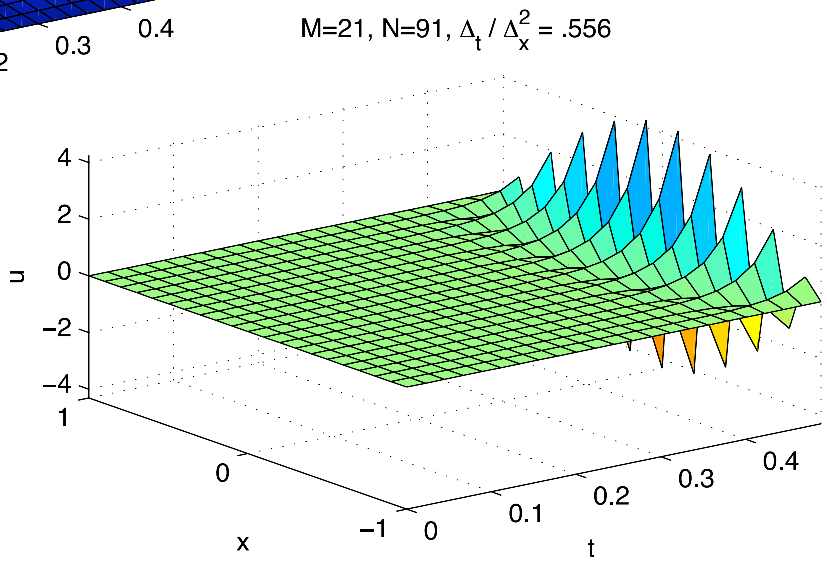
$$\begin{aligned} x_i &= -a + (i-1)\Delta_x, & \Delta_x &= \frac{2a}{M-1}, & i &= 1, \dots, M \\ t_n &= (n-1)\Delta_t, & \Delta_t &= \frac{T}{N-1}, & n &= 1, \dots, N \end{aligned}$$

Test the function with $a = 1$, $T = 0.5$, $M = 21$, $N = 151, 101, 91$ and draw U as a surface (see `mesh` and `surf`) heat.m

```
function V = heat( a,T, M,N )

%% Initializations
dx = (2*a)/(M-1);     dt = T/(N-1);
V = zeros(M,N);
nu = dt/(dx*dx);

x = linspace(-a,a,M);
V(:,1) = max(exp(x)-1,0);

%% Compute V(:,2), ..., V(:,N)
for n=2:N
  V(1,n) = V(1,1);
  V(M,n) = V(M,1);
  V(2:M-1,n) = (1-2*nu)*V(2:M-1,n-1) + nu*V(1:M-2,n-1) ...
                + nu*V(3:M,n-1);
end
```



M=21, N=101, Δ_t / Δ_x^2 = 1/2
u
x
t



M=21, N=91, Δ_t / Δ_x^2 = .556
u
x
t

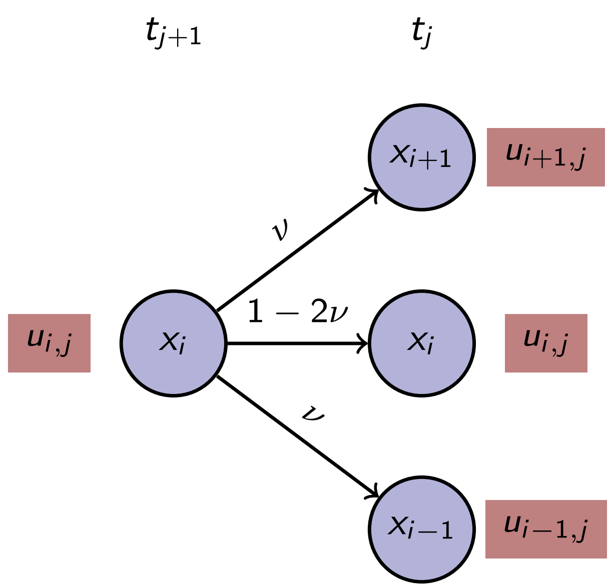
Finite differences - trinomial connection

Finite Differences ($\nu = \Delta_t/\Delta_x^2$) :

- $u_{i,n+1} = \nu u_{i-1,n} + (1-2\nu) u_{i,n} + \nu u_{i+1,n}$
- $x_i = a + (i-1)\Delta_x, \; t_n = (n-1)\Delta_t$
- Usually domain is a rectangle
- Border conditions can be imposed
- $\{\nu, 1-2\nu, \nu\}$ probability when $2\nu < 1$

Trinomial Tree:

- $v_{i,m-1} = q_1 v_{i-1,m} + q_2 v_{i,m} + q_3 v_{i+1,m}$
- Probability: $0 \leq q_i \leq 1$ e $\sum_i q_i = 1$
- Backward: $m = N + 1 - n$
- Domain is triangular



Consistency

What is the error at each step?

- Let U be the solution to the PDE $\partial_{xx}U - \partial_t U = 0$ with the appropriate border conditions
- $\delta^o_{xx} - \delta^+_t$ discrete operator
- Truncation error: $\mathcal{T}(x,t) = \delta^o_{xx}U(x,t) - \delta^+_t U(x,t)$
- Facts: $\delta^o_{xx}U = \partial_{xx}U + O(\Delta_x^2)$ and $\delta^+_t U = \partial_t U + O(\Delta_t)$
- Thus, $\mathcal{T}(x,t) = O(\Delta_t) + O(\Delta_x^2)$

Consistency: $\mathcal{T}(x,t) \to 0$ for $\Delta_t \to 0$ and $\Delta_x \to 0$

- But $\Delta_t \to 0$ implies $N \to \infty$: consistency is not sufficient, errors may accumulate

Stability

- U and u solve $(\partial_{xx} - \partial_t)U = 0$ and $(\delta_t^+ - \delta_{xx}^o)u = 0$, resp.
- Error: $e_{i,n} = u_{i,n} - U(x_i, t_n)$

Convergence: $e_{i,n} \to 0$ when $\Delta_t \to 0$ and $\Delta_x \to 0$.

- Recall $\mathcal{T} = (\delta_t^+ - \delta_{xx}^o)U(x_i, t_n) = O(\Delta_t + \Delta_x^2)$
- Thus

$$
\begin{aligned}
(\delta_t^+ - \delta_{xx}^o)e_{i,n} &= (\delta_t^+ - \delta_{xx}^o)u_{i,n} + (\delta_t^+ - \delta_{xx}^o)U(x_i, t_n) = \mathcal{T} \\
\frac{e_{i,n+1} - e_{i,n}}{\Delta_t} &= \frac{e_{i+1,n} - 2e_{i,n} + e_{i-1,n}}{\Delta_x^2} + \mathcal{T} \\
e_{i,n+1} &= \nu e_{i-1,n} + (1 - 2\nu)e_{i,n} + \nu e_{i-1,n} + \mathcal{T} \\
|e_{i,n+1}| &\leq \nu|e_{i-1,n}| + |(1 - 2\nu)e_{i,n}| + \nu|e_{i-1,n}| + \Delta_t\mathcal{T}
\end{aligned}
$$

$$|e_{i,n+1}| \leq \nu|e_{i-1,n}| + |(1-2\nu)e_{i,n}| + \nu|e_{i-1,n}| + \Delta_t \mathcal{T}$$

- If $\nu \leq 1/2$ then $|(1-2\nu)e_{i,n}| = (1-2\nu)|e_{i,n}|$ so that

$$|e_{i,n+1}| \leq \nu|e_{i-1,n}| + (1-2\nu)|e_{i,n}| + \nu|e_{i-1,n}| + \Delta_t \mathcal{T}$$

- Define $E_n = \max_i |e_{i,n}|$, thus

$$|e_{i,n+1}| \leq \nu E_n + (1-2\nu)E_n + \nu E_n + \Delta_t \mathcal{T} = E_n + \Delta_t \mathcal{T}$$
$$E_{n+1} \leq E_n + \Delta_t \mathcal{T}$$

- Starting $E_0 = 0$, after n steps $E_n \leq n\Delta_t \mathcal{T}$
- Fixing the time t^*, $n = t^*/\Delta_t$ and

$$E_{t^*} \leq t^* \mathcal{T} = t^* O(\Delta_t + \Delta_x^2) \to 0 \quad \text{for } \Delta_t \to 0 \text{ and } \Delta_x \to 0$$

- Remark: Δ_t and Δ_x converge to 0 conditionally on $\Delta_t/\Delta_x^2 < 1/2$.

Matrix Form

The explicit method recurs:

$$
\begin{aligned}
u_{1,n+1} &= u_{1,n} + b_1, && b_1 = (\phi_a(t_{n+1}) - \phi_a(t_n)) \\
u_{i,n+1} &= u_{i,n} + \nu u_{i-1,n} - 2\nu u_{i,n} + \nu u_{i+1,n}, && n = 2, \dots, M \\
u_{M,n+1} &= u_{M,n} + b_M, && b_M = (\phi_b(t_{n+1}) - \phi_b(t_n))
\end{aligned}
$$

Example, for $M = 5$:

$$
\begin{aligned}
u_{1,n+1} &= u_{1,n} && && && +b_1 \\
u_{2,n+1} &= u_{2,n} +\nu u_{1,n} - 2\nu u_{2,n} + \nu u_{3,n} \\
u_{3,n+1} &= u_{3,n} + \nu u_{2,n} - 2\nu u_{3,n} + \nu u_{4,n} \\
u_{4,n+1} &= u_{4,n} + \nu u_{3,n} - 2\nu u_{4,n} + \nu u_{5,n} \\
u_{5,n+1} &= u_{5,n} && && && +b_5
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} u_{1,n+1} \\ u_{2,n+1} \\ u_{3,n+1} \\ u_{4,n+1} \\ u_{5,n+1} \end{pmatrix}
=
\begin{pmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ u_{4,n} \\ u_{5,n} \end{pmatrix}
+ \nu
\begin{pmatrix} 0 & 0 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 0 & 0 \end{pmatrix}
\begin{pmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ u_{4,n} \\ u_{5,n} \end{pmatrix}
+
\begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \\ b_5 \end{pmatrix}
$$

$$\begin{pmatrix} u_{1,n+1} \\ u_{2,n+1} \\ u_{3,n+1} \\ u_{4,n+1} \\ u_{5,n+1} \end{pmatrix} = \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ u_{4,n} \\ u_{5,n} \end{pmatrix} + \nu \begin{pmatrix} 0 & 0 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \\ u_{4,n} \\ u_{5,n} \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \\ b_5 \end{pmatrix}$$

with the appropriate definitions it can be written as

$$u_{n+1} = u_n + \nu \tilde{A} u_n + b = (I + \nu \tilde{A}) u_n + b$$

where

- $u_n : M \times 1$ contains the solution at time t_n
- $\tilde{A} : M \times M$ is a tridiagonal matrix
- $b : M \times 1$ contains the contribution from the border conditions
- $A = I + \nu \tilde{A}$ is the tridiagonal matrix:

$$A = \begin{pmatrix} 1 & 0 & & & \\ \nu & 1-2\nu & \nu & & \\ & \nu & 1-2\nu & \nu & \\ & & \nu & 1-2\nu & \nu \\ & & & 0 & 1 \end{pmatrix}$$

Implicit method

- $\partial_{xx}u - \partial_t u = 0$ is approximated by $\partial^o_{xx}u - \partial^-_t u = 0$
- Specifically

$$\frac{u(x,t) - u(x,t-\Delta_t)}{\Delta_t} - \frac{u(x+\Delta_x,t) - 2u(x,t) + u(x-\Delta_x,t)}{\Delta_x^2} =$$

- which on the grid points (x_i, t_n) can be written as

$$\frac{u_{i,n} - u_{i,n-1}}{\Delta_t} - \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{\Delta_x^2} = 0$$

- or

$$-\nu u_{i-1,n} + (1+2\nu)u_{i,n} - \nu u_{i+1,n} = u_{i,n-1}$$

where $\nu = \Delta_t/\Delta_x^2$

- Linear system with M equations and M unknowns $\{u_{i,n}, i = 1, \ldots, M\}$

$$\begin{cases} u_{1,n} & = u_{1,n-1} + b_1 \\ -\nu u_{i-1,n} + (1+2\nu) u_{i,n} - \nu u_{i+1,n} & = u_{i,n-1} \qquad\qquad i = 2, \ldots, \\ u_{M,n} & = u_{M,n-1} + b_M \end{cases}$$

  where $b_1 = \phi_a(t_n) - \phi_a(t_{n-1})$ and $b_M = \phi(t_n) - \phi(t_{n-1})$
- or $Au_n = u_{n-1} + b$
- A is tridiagonal
- Solving an $M \times M$ linear system requires $O(M^3)$
- Solving Tridiagonal systems requires $O(M)$ (see Golub and Van Loan)

- Example ($M = 5$):

$$
\begin{array}{llllll l}
u_{1,n} & & & & & & = u_{1,n-1} + \\
u_{2,n} & -\nu u_{1,n} & +2\nu u_{2,n} & -\nu u_{3,n} & & & = u_{2,n-1} \\
u_{3,n} & & -\nu u_{2,n} & +2\nu u_{3,n} & -\nu u_{4,n} & & = u_{3,n-1} \\
u_{4,n} & & & -\nu u_{3,n} & +2\nu u_{4,n} & -\nu u_{5,n} & = u_{4,n-1} \\
u_{5,n} & & & & & & = u_{5,n-1} +
\end{array}
$$

- $u_n - \nu \tilde{A} u_n = u_{n-1} + b$ where

$$
\tilde{A} = \begin{pmatrix} 0 & 0 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & 0 & 0 \end{pmatrix}
$$

- or: $Au_j = u_{j-1} + b$, where $A = I - \nu \hat{A}$
- Implicit method:
  1. Put initial conditions in u_1
  2. For $j = 2, \ldots, N$ solve $Au_j = u_{j-1} + b$

Stability of implicit method

- Truncation error: $\mathcal{T} = O(\Delta_x^2 + \Delta_t)$
- The error satisfy

$$
\begin{aligned}
(1+2\nu)e_{i,n} &= e_{i,n-1} + \nu e_{i-1,n} + \nu e_{i+1,n} + \Delta_t \mathcal{T} \\
(1+2\nu)|e_{i,n}| &\le |e_{i,n-1}| + \nu |e_{i-1,n}| + \nu |e_{i+1,n}| + \Delta_t \mathcal{T},
\end{aligned}
$$

setting $E_n = \max_i e_{i,n}$,

$$
\begin{aligned}
(1+2\nu)E_j &\le E_{n-1} + \nu E_{n-1} + \nu E_{n-1} + \Delta_t \mathcal{T} \\
E_n &\le E_{n-1} + \mathcal{T} \\
E_n &\le E_1 + (n-1)\Delta_t \mathcal{T}
\end{aligned}
$$

- For a fixed time t^* ($n-1 = t^*/\Delta_t$) and for exact initial conditions

$$E_n \le t^* \mathcal{T} \to 0$$

- No restriction on Δ_t and Δ_x: the method is unconditionally stable heatbw.m

```
function V = heatbw( a,T, M,N )

%% Initializations:
dx = (2*a)/(M-1);     dt = T/(N-1);
V = zeros(M,N);
nu = dt/(dx*dx);

x = linspace(-a,a,M);
V(:,1) = max(exp(x)-1,0);

A = - diag(ones(M-1,1),1) + 2*eye(M) - diag(ones(M-1,1),-1);
A(1,:) = 0;
A(end,:) = 0;

A = eye(M) - nu*A;

%% Compute V(:,2), ..., V(:,N)
for n=2:N
  V(:,n) = A\V(:,n-1);
end
```

heatbw2.m

```
function v = heatbw2( a,T, M,N )

%% Initializations
dx = (2*a)/(M-1);     dt = T/(N-1);
nu = dt/(dx*dx);

x = linspace(-a,a,M);
v = max(exp(x)-1,0);

A = - diag(ones(M-1,1),1) + 2*eye(M) - diag(ones(M-1,1),-1);
A(1,:) = 0;
A(end,:) = 0;

A = eye(M) - nu*A;

%% Compute V(:,2), ..., V(:,N)
for n=2:N
  v = A\v;
end
```

Border conditions

- Until now, only explicit bc have been used: $u(x,t) = \phi(t)$
- Let consider now the bc $\partial_x u(x,t) = c$ on the left border $x = a$
- At the border it holds

$$\begin{aligned}\partial_{xx} u(a,t) &= \frac{\partial_x u(a+\Delta_x, t) - \partial_x u(a,t)}{\Delta_x} + O(\Delta_x)\\ &= \frac{u(a+\Delta_x,t) - u(a,t)}{\Delta_x^2} - \frac{c}{\Delta_x} + O(\Delta_x)\end{aligned}$$

- thus $\partial_t u - \partial_{xx} u$ can be approximated at (a, t_j) as

$$u_{1,j+1} = u_{1,j} + \nu u_{2,j} - \nu u_{1,j} - \frac{\Delta_t}{\Delta_x} c \qquad \text{explicit method}$$

$$u_{1,j} - \nu u_{2,j} + \nu u_{1,j} = u_{1,j-1} - \frac{\Delta_t}{\Delta_x} c \qquad \text{implicit method}$$

Black and Scholes

Black & Scholes: the price of an European option solves

$$\begin{cases} \partial_t V + \dfrac{\sigma^2}{2} S^2 \partial_{SS} V + (r-q) S \partial_S V - rV = 0 \\ V(S,T) = \phi(S) \end{cases}$$

- t time
- T expiry dell'opzione
- S value of the underlying
- σ volatility
- r risk-free interest rate
- q dividend rate
- $\phi(S)$ option payoff $\phi(S) = \begin{cases} \max(S-K,0), & \text{call} \\ \max(K-S,0), & \text{put} \end{cases}$
- K exercise price or strike

B&S and Change of Variables

$$\partial_t V + \frac{\sigma^2}{2} S^2 \partial_{SS} V + (r - q) S \partial_S V - rV = 0$$

First approach: $r = q = 0$

- $x = \log(S)$,
- $\tau = T - t$,
- $V(S, t) = u(\log(S), T - t)$
- $\partial_t V = -\partial_\tau u, \quad \partial_S V = \frac{1}{S} \partial_x u, \quad \partial_{SS} V = \frac{1}{S^2} (\partial_{xx} u - \partial_x u)$

$$\partial_\tau u - \frac{\sigma^2}{2} (\partial_{xx} u - \partial_x u) = 0$$

- Work even when σ is non-constant

B&S and Change of Variables

$$\partial_t V + \frac{\sigma^2}{2} S^2 \partial_{SS} V + (r - q) S \partial_S V - rV = 0$$

Second approach ($r = q = 0$)

- $x = \log(S) - \frac{\sigma^2}{2}(T - t)$,
- $\tau = T - t$,
- $V(S, t) = u(x, \tau)$
- $\partial_t V = -\partial_\tau u + \frac{\sigma^2}{2} \partial_x u, \quad \partial_{SS} V = \frac{1}{S^2}(\partial_{xx} u - \partial_x u)$

$$\partial_\tau u - \frac{\sigma^2}{2} \partial_{xx} u = 0$$

Black and Scholes: Change of variables

$$\partial_t V + \frac{\sigma^2}{2} S^2 \partial_{SS} V + (r - q) S \partial_S V - rV = 0$$

- General case: $r \neq 0$ e $q \neq 0$
  - $x = \log(S) + (r - q - \frac{\sigma^2}{2})(T - t), \quad \tau = T - t, \quad V(S, t) = e^{-r\tau} u(x, \tau)$
  - $\partial_t V = e^{-r\tau}(-\partial_\tau u - (r - q - \frac{\sigma^2}{2})\partial_x u) + re^{-r\tau} u,$
  - $S\partial_S V = e^{-r\tau}(\partial_x u), \qquad S^2 \partial_{SS} V = e^{-r\tau}(\partial_{xx} u - \partial_x u)$

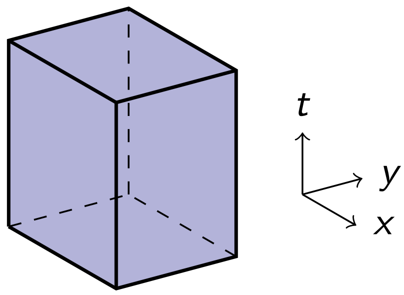
  $\Rightarrow \partial_\tau u - \frac{\sigma^2}{2} \partial_{xx} u = 0$
- Backward/forward prices:

  $x = \log(e^{(r-q)(T-t)} S) + \text{adj}, \qquad V = e^{-r(T-t)} u$
- Homogeneity: using $\log(S/K)$ give same results
- Exercise: by solving a single Cauchy problem compute prices of a set of call options with different exercise prices and different maturities.

Two dimensional heat equation

- 2D: $\partial_t u - \partial_{xx} u - \partial_{yy} u = 0$
- Domain: $[x_a, x_b] \times [y_a, y_b] \times [0, T]$
- Discretization:

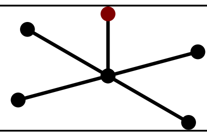
$$
\begin{aligned}
x_i &= x_a + (i-1)\Delta_x, & i &= 1, \dots, I,\\
y_j &= y_a + (j-1)\Delta_y, & j &= 1, \dots, J,\\
t_n &= (n-1)\Delta_t, & n &= 1, \dots, N,\\
u_{ij}^n &= u(x_i, y_j, t_n)
\end{aligned}
$$



- Numerical methods ($\nu = \Delta_t / \Delta_x^2$ and $\Delta_x = \Delta_y$):

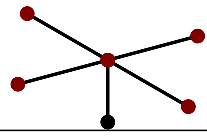
Explicit

$$u_{ij}^{n+1} - u_{ij}^n = \nu(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n)$$



Implicit:

$$u_{ij}^n - u_{ij}^{n-1} = \nu(u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n)$$



2D Heat: matlab implementation

- u_{ij}^n needs an $I \times J \times N$ array
- otherwise, only the current step has to be stored: `U(i,j)` $= u_{ij}^n$
- The algorithm core is:

heat2da.m

```
for n=2:N
  for i=2:I-1
    for j=2:J-1
      Unew(i,j) = (1-4*nu)*U(i,j) + ...
          nu*( U(i-1,j)+U(i+1,j)+U(i,j-1)+U(i,j+1) );
    end
  end
  %% Add border conditions

  U = Unew;
end
```

Exercise

Approximate numerically the Cauchy problem

$$\begin{cases} \partial_t u - \partial_{xx} u - \partial_{yy} u = 0 & \textit{in } \Omega \times [0, T] \\ u(x, y, 0) = \max(x + y - 2, 0), & \textit{per } (x, y) \in \Omega \\ u(x, y, t) = \max(x + y - 2, 0), & \textit{per } (x, y) \in \partial\Omega \textit{ e } t \in [0, T] \end{cases}$$

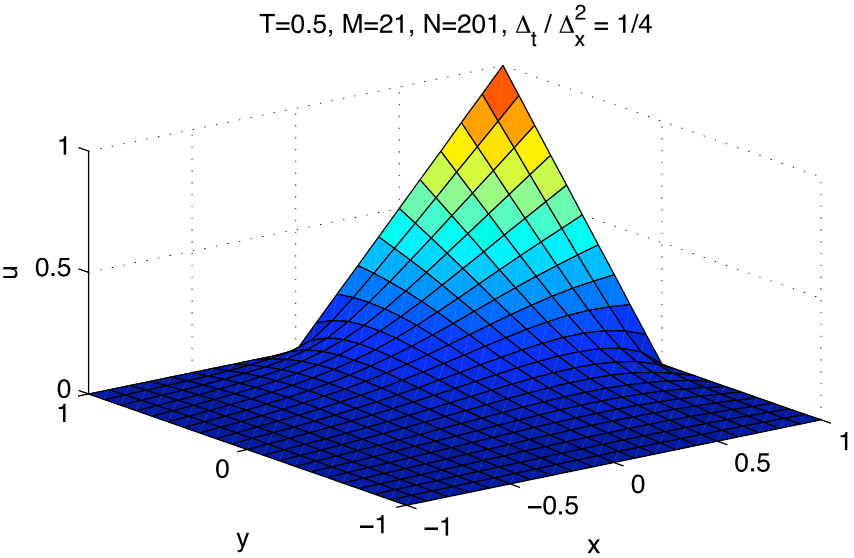
where $\Omega = [-a, a] \times [-a, a]$.

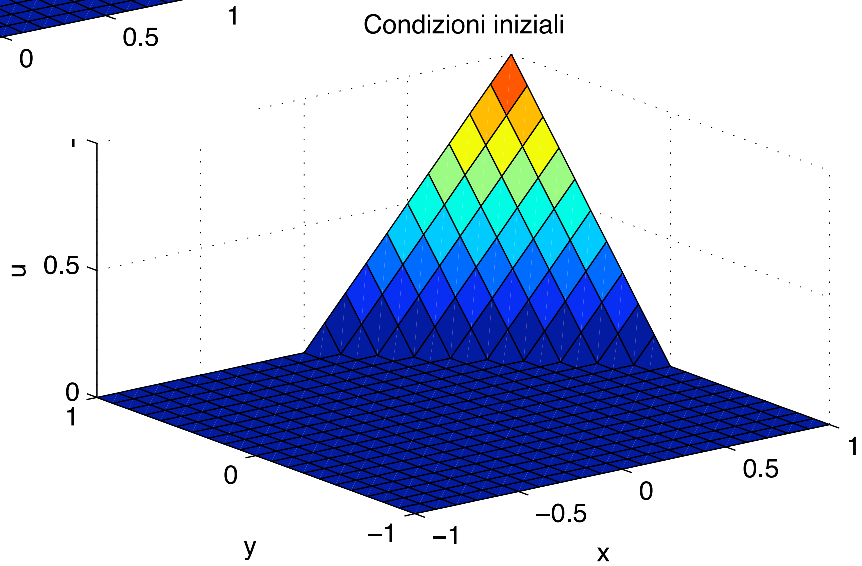
Remark

The stability condition is now $\Delta_t \leq \dfrac{1}{4}\Delta_x^2$

heat2d.m

```
function U = heat2d(a,T, I, N)

dx = 2*a/(I-1);    dt = T/(N-1);   ni = dt/(dx*dx);

x = linspace(-a,a,I);
[X,Y] = meshgrid(x,x);
U = max(X+Y-1,0);
Unew = U;

for n=2:N
  for i=2:I-1
    for j=2:I-1
      Unew(i,j) = (1-4*ni)*U(i,j) + ...
          ni * (U(i-1,j)+U(i+1,j)+U(i,j-1)+U(i,j+1));
    end
  end
  U = Unew;
end
```


T=0.5, M=21, N=201, $\Delta_t / \Delta_x^2 = 1/4$
u
1
0.5
0
y
x
1
0
−1
−0.5
0.5

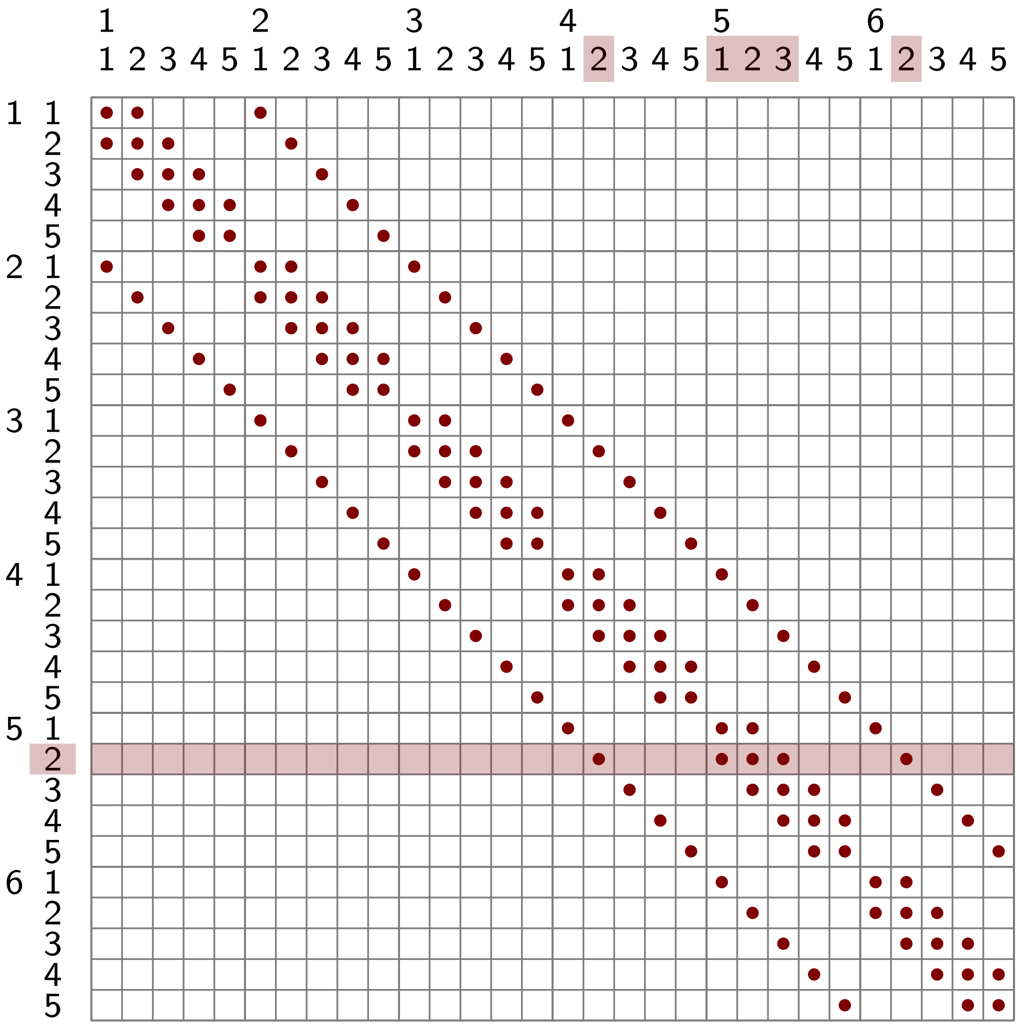

Condizioni iniziali
u
0.5
0
y
x
1
0
−1
−0.5
0.5

2d heat: implicit method

- Problem: rewrite in matrix form the following system

$$u_{ij}^{n} - u_{ij}^{n-1} = \nu(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n})$$

- The $I \times J$ matrix u_{ij} ($i = 1, \ldots, I$, $j = 1, \ldots, J$) needs to be transformed into a $IJ \times 1$ vector


1 2 3 4 5 6
1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5
1 2 3 4 5 6
1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5

Exercise

Approximate and numerically solve the Cauchy problem

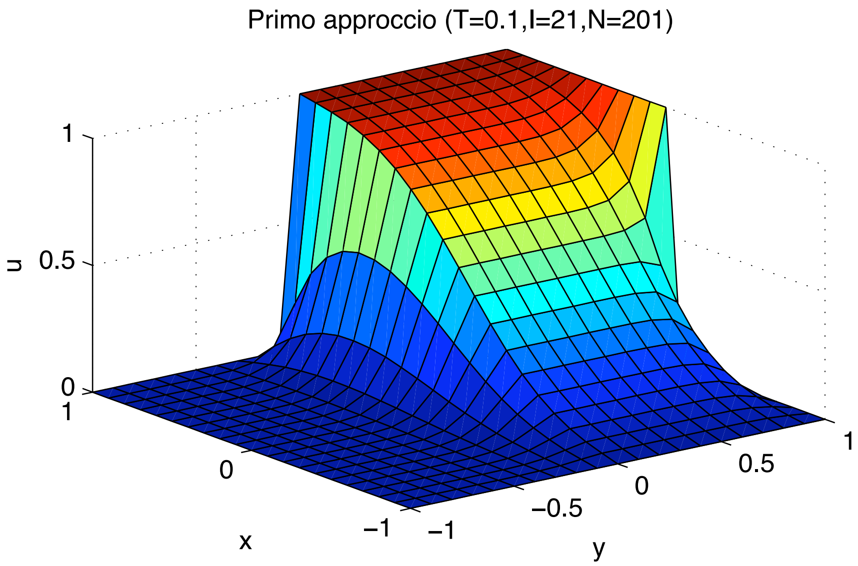
$$\begin{cases} \partial_t u - \partial_{xx} u - \partial_y u = 0 & \text{on } \Omega \times [0, T] \\ u(x, y, 0) = 1 & \text{for } x \geq 0 \text{ and } y \geq 0 \\ u(x, y, 0) = 0 & \text{for } x < 0 \text{ or } y < 0 \\ u(x, y, t) = u(x, y, 0) & \text{for } (x, y) \in \partial\Omega \text{ and } t \in [0, T] \end{cases}$$

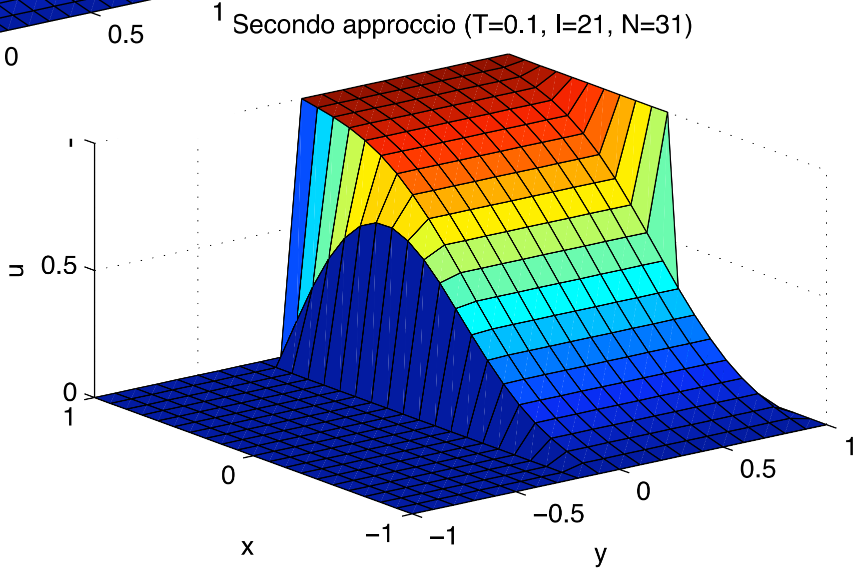
where $\Omega = [-a, a] \times [-a, a]$.

Remark

The terms $\partial_t u - \partial_y u$ *can be approximated:*

- *Independently:* $\partial_t u \simeq \delta_t^{\pm} u$ *and* $\partial_y u \simeq \delta_y^{\pm o} u$
  In that case the direction of the discretization is critical for stability.
- *Jointly:*
  let $f(s) = u(x, y - s, t + s)$ *then* $f'(0) = \partial_t u(x, y, t) - \partial_y u(x, y, t)$,
  so that $\partial_t u - \partial_y u \simeq \Delta_t^{-1}(u(x, y - \Delta_t, t + \Delta_t) - u(x, y, t))$


Primo approccio (T=0.1,I=21,N=201)
u
1
0.5
0
1
0
-1
x
-1
-0.5
0
0.5
1
y


Secondo approccio (T=0.1, I=21, N=31)
u
0.5
0
1
0
-1
x
-1
-0.5
0
0.5
1
y

Expected Values and Numerical Integration

Problem

Compute the expected value

$$\mathrm{E}[f(X)], \qquad \text{where} \qquad X \sim P$$

or equivalently

$$\int_\Omega f(x)dP(x)$$

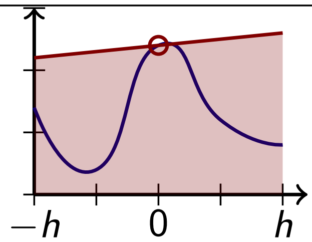
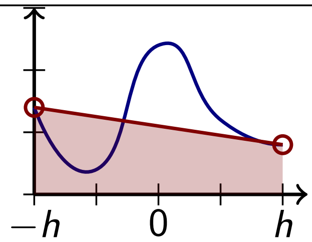
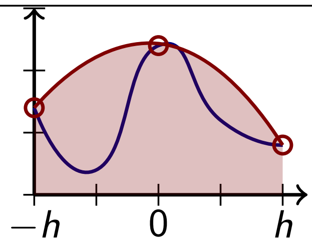
Can be computed

- explicitly
- numerically
- numerically approximated

Problem

Approximate $\quad I(h) = \int_{-h}^{h} f(x) = \tilde{I}(h) + E$

Newton-Côtes approximation methods: polynomial interpolation of $f(x)$

Method		Approximation	Bound $\lvert E \rvert$
Mid Point		$2hf(0)$	$\frac{1}{3}h^3 \Vert f'' \Vert_\infty$
Trapezoidal		$h\big(f(-h) + f(h)\big)$	$\frac{2}{3}h^3 \Vert f'' \Vert_\infty$
Simpson		$\frac{h}{3}\big(f(-h) + 4f(0) + f(h)\big)$	$O(h^5 \Vert f^{(4)} \Vert_\infty)$

- Newton-Cotes: good for small h, but for large h?
- Solution:

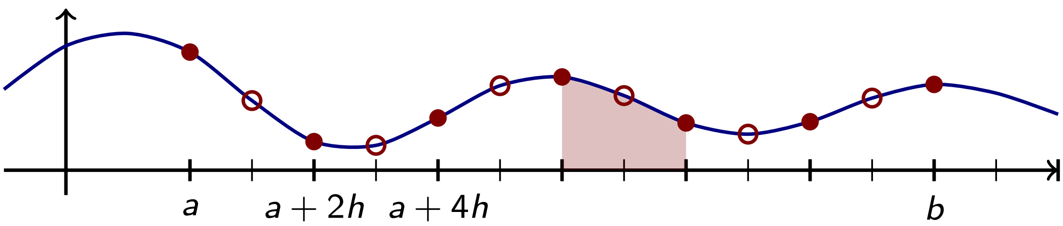
  Uniformly partition the domain:

  $$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{M-1}, x_M],$$

  where $x_i = a + ih, \quad h = (b-a)/M$

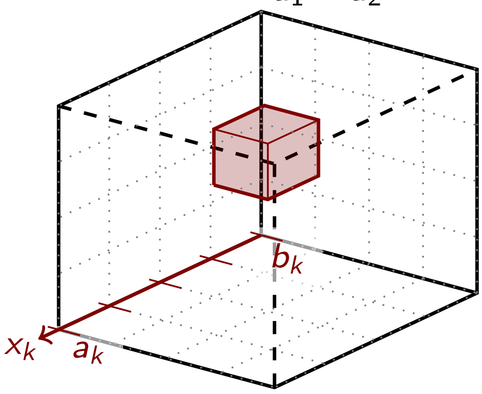
  Then, apply a Newton Cotes integration to each interval
- Notice,
  - there may be more sampling points than intervals (e.g. Simpson)
  - the points may have different weights
- $\int_a^b f(x) = \sum_i \int_{x_{2i}}^{x_{2i+2}} f(x) = \sum_i w_i f_i + o(h^\alpha)$,

$x_i = a + ih$,
$h = (b-a)/M$



Integration in $\mathbb{R}^n$

- Compute $\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(\mathbf{x}) d\mathbf{x}$, with $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$



- Partition each $[a_k, b_k]$ into M intervals:

$\Rightarrow$ $\sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} w_I f(\mathbf{x}_I), \quad I = (i_1, \ldots, i_n).$

- The domains is partitioned into M^n cubes
- Need to compute f for $O(M^n)$ points

- Precision: $O(M^{-\alpha})$
- Computing time: $O(M^n)$
- The precision is $O(t_c^{-\alpha/n})$ for given computing time t_c
- The computational complexity is $O(e^{np/\alpha})$ for a given precision 10^{-p}

Mid point rule

By approximating $f(x)$ with a constant $p(x) = f(0)$, it results

$$f(x) = p(x) + \epsilon, \qquad \epsilon = x\,f'(0) + \frac{1}{2}x^2 f''(\xi)$$

$$I(h) = 2h\,f(0) + E, \qquad |E| \leq \frac{1}{3}h^3 \|f''\|_\infty$$

Trapezoidal rule

Linearly approximate $f(x)$: $p(x) = \dfrac{h-x}{2h} f(-h) + \dfrac{h+x}{2h} f(h)$:

$$f(x) = p(x) + \epsilon, \qquad \epsilon = \frac{h^2 - x^2}{2} x^2 f''(\xi)$$

$$I(h) = h(f(-h) + f(h)) + E, \qquad |E| \le \frac{2}{3} h^3 \|f''\|_\infty$$

Proof.

By Taylor:

$$f(-h) = f(x) - (h+x) f'(x) + \epsilon_1, \quad \epsilon_1 = \tfrac{1}{2}(h+x)^2 f''(\xi_1)$$

$$f(h) = f(x) + (h-x) f'(x) + \epsilon_2, \quad \epsilon_2 = \tfrac{1}{2}(h-x)^2 f''(\xi_2)$$

so that

$$p(x) := \frac{h-x}{2h} f(-h) + \frac{h+x}{2h} f(h) = f(x) + 0 \cdot f'(x) + \epsilon,$$

$$\epsilon = \frac{h^2 - x^2}{2} \left(\frac{h+x}{2h} f''(\xi_1) + \frac{h-x}{2h} f''(\xi_2) \right) = \frac{h^2 - x^2}{2} f''(\xi), \qquad \xi \in [-h, h].$$

Furthermore $|E| = \frac{1}{2} | \int_{-h}^{h} (h^2 - x^2) f''(\xi) | \le \frac{2}{3} h^3 \|f''\|_\infty$ □

Let consider now the Taylor expansion of the $I(h) := \int_{-h}^{h} f(x)$ near $h = 0$

$$I(h) = I(0) + hI'(0) + \frac{1}{2}h^2 I''(0) + \frac{1}{3!}h^3 I^{(3)}(0) + \cdots + \frac{1}{n!}h^n I^{(n)}(0) + R_n$$

where $R_n = \frac{1}{(n+1)!}h^{n+1} I^{(n+1)}(\xi)$, for $\xi \in [0, h]$,
In order The derivatives of $I(h)$ are given by

$$
\begin{aligned}
& & I(0) &= 0 & \\
I'(h) &= f(h) + f(-h) & I'(0) &= 2f(0) & \\
I''(h) &= f'(h) - f'(-h) & I''(0) &= 0 & \\
I^{(3)}(h) &= f''(h) + f''(-h) & I'(0) &= 2f''(0) & \\
I^{(4)}(h) &= f^{(3)}(h) - f^{(3)}(-h) & I''(0) &= 0 & \\
&\vdots & &\vdots & \\
I^{(k)}(h) &= f^{(k-1)}(h) + f^{(k-1)}(-h) & I^{(k)}(0) &= 2f^{(k-1)}(0) & k \text{ dispari} \\
I^{(k)}(h) &= f^{(k-1)}(h) - f^{(k-1)}(-h) & I^{(k)}(0) &= 0 & k \text{ pari}
\end{aligned}
$$

So that the Taylor expansion of $I(h) = \int_{-h}^{h} f(x)$ becomes

$$I(h) = hI'(0) + \frac{h^3}{3!}I^{(3)}(0) + \frac{h^5}{5!}I^{(5)}(0) + \cdots + \frac{h^{2k+1}}{(2k+1)!}I^{(2k+1)}(0) + R_{2k}$$

$$= 2hf(0) + \frac{h^3}{3}f''(0) + \frac{2h^5}{5!}f^{(4)}(0) + \cdots + \frac{2h^{2k+1}}{(2k+1)!}f^{(2k)}(0) + R_{2k-}$$

where $R_{2k+1} = \frac{2h^{2k+1}}{(2k+1)!}f^{2k}(\xi)$ and $\xi \in [0, h]$.

$k = 1$	Mid point rule:	$I(h) = 2hf(0) + \frac{h^3}{3}f''(\xi)$
$k = 2$	Simpson rule:	$I(h) = 2hf(0) + \frac{h^3}{3}f''(0) + \frac{2h^5}{5!}f^{(4)}(\xi)$

- Need $f''(0)$
- Can be approximated by differences

Simpson rule

$$I(h) = 2hf(0) + \frac{h^3}{3} f''(0) + O(h^5 \| f^{(4)} \|_\infty) \tag{1}$$

From Taylor:

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2} f''(0) + \frac{h^3}{3!} f^{(3)}(0) + O(h^4 \| f^{(4)} \|_\infty)$$

$$f(-h) = f(0) - hf'(0) + \frac{h^2}{2} f''(0) - \frac{h^3}{3!} f^{(3)}(0) + O(h^4 \| f^{(4)} \|_\infty)$$

whose sum gives

$$h^2 f''(0) = f(-h) - 2f(0) + f(h) + O(h^4 \| f^{(4)} \|_\infty) \tag{2}$$

By (**??**) and (**??**):

$$I(h) = \frac{h}{3} \Big(f(-h) + 4f(0) + f(h) \Big) + O(h^5 \| f^{(4)} \|_\infty)$$

Trapezoidal Rule

Approximate $f(x)$ by a linear function going through $f(-h)$ e $f(h)$

$$g(x) = \alpha f(-h) + (1-\alpha) f(h), \qquad \alpha = \frac{h-x}{2h} \tag{3}$$

A Taylor expansion of $f(h)$ e $f(-h)$ on $h=0$ gives

$$(h+x)\cdot \quad f(h) = f(x) + (h-x)f'(x) + \frac{1}{2}(h-x)^2 f''(x) + O((h-x)^3\|f'''\|_\infty$$

$$(h-x)\cdot \quad f(-h) = f(x) - (h+x)f'(x) + \frac{1}{2}(h+x)^2 f''(x) + O((h+x)^3\|f'''\|_\infty$$

whose sum gives

$$2h\,g(x) = 2h\,f(x) + 0f'(x) + \frac{1}{2}(h^2-x^2)2h\,f''(x) + O(h^4\|f'''\|_\infty)$$

$$g(x) = f(x) + \frac{1}{2}(h^2-x^2)f''(x) + O(h^3\|f'''\|_\infty)$$

$$g(x) = f(x) + \frac{1}{2}(h^2 - x^2)f''(x) + O(h^3\|f'''\|_\infty)$$

rearranging and integrating

$$I(h) = \int_{-h}^{h} g(x)dx + O(\|f''\|_\infty)\int_{-h}^{h}(h^2 - x^2)dx + \text{smaller terms}$$

and, from

$$\int_{-h}^{h} g(x)dx = h(f(-h) + f(h)) \quad \text{and} \quad \int_{-h}^{h}(h^2 - x^2)dx = 2h^3 - \frac{2}{3}h^3 = \frac{4}{3}h^3$$

it follows that

$$I(h) = h\Big(f(h) + f(-h)\Big) + O(h^3\|f''\|_\infty)$$

Monte Carlo Method

Problem

Given

- $X : \Omega \to \mathbb{R}^N$ r.v. on a probability space $(\Omega, \mathcal{F}, P)$ and
- $g : \mathbb{R}^N \to \mathbb{R}$ such that $\mathrm{Var}[g(X)] = \sigma^2 < \infty$

Compute $\theta = \mathrm{E}[g(X)] = \int_\Omega g(X)dP$

Monte Carlo Method

The Monte Carlo estimate of θ is given by

$$\hat{\theta}_m = \frac{1}{m}\sum_{i=1}^{m} g(x_i), \tag{4}$$

where $x_1, x_2, \ldots, x_m$ are m independent occurrences of X

Monte Carlo

$$\hat{\theta}_m = \frac{1}{m}\sum_{i=1}^{m} g(X_i), \qquad X_i \sim X \text{ and independent} \tag{5}$$

Unbiased: $\mathrm{E}[\hat{\theta}_m] = \frac{1}{m}\sum_i \mathrm{E}[g(X_i)] = \mathrm{E}[g(X)] = \theta$

Efficiency: $\mathrm{Var}[\hat{\theta}_m] = \frac{1}{m^2}\sum_i \mathrm{Var}[g(X_i)] = \frac{1}{m}\mathrm{Var}[g(X)] = \frac{1}{m}\sigma^2$

The error $\hat{\theta}_m - \theta$ can be bounded by

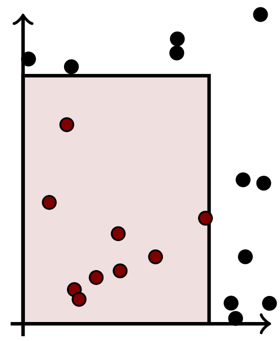
- Chebichev Theorem: $P\left[|\hat{\theta}_m - \theta| \geq \varepsilon\right] \leq \dfrac{\sigma}{\varepsilon^2 m}$!check!
- Central Limit Theorem: $\dfrac{\hat{\theta}_m - \theta}{\sigma/\sqrt{m}} \xrightarrow{d} N(0,1)$

When unknown σ^2 can be estimated as $S^2 = \dfrac{1}{m-1}\sum_i \left(g(x_i) - \hat{\theta}_m\right)^2$

Pro and Cons

- Pro: Require $g \in \mathcal{L}^2$ not necessarily $g \in \mathcal{C}$
- Pro: Error of order $m^{-1/2}$, not dep. on the domain dimension
- Cons: probabilistic bound, the worst case error is ∞
- Cons: the estimates depends on the generated random sequence
  - initial guess
  - "randomness" of the sequence

Low-discrepancy sequences



- Consider $P = \text{unif}([0,1]^N)$.
- Let $Q \subset \mathbb{R}^N$ be a N-dimensional rectangle parallel to the axes and with a vertex on the origin
- Discrepancy: $D_m^* \equiv \sup_Q \left| \frac{\#\{x_i \in Q\}}{m} - P[Q] \right|$
- Low discrepancy: uniformly distributed points
- Koksma-Hlawka:

$$|\theta_m - \theta| \leq V(g) D_m^*, \tag{6}$$

where $V(g) = \sup_s \sum_i |f(x_{i+1}) - f(x_i)|$,
over all the sequences $s = \{0 \equiv x_1 < \cdots < x_n \equiv 1\}$.

- Conjecture: $D_m^* \leq C \log(m)^N / m$
- Optimal Sequences:

Faure, Halton, Hammersley, Sobol, Niederreiter and Van der Corput.

Simulating a R.V. from a uniformly distributed R.V.

Lemma

Let $X \sim D$ *(with cdf* D*) and let* $U \sim \mathrm{Unif}(0,1)$,

if $Y = D^{-1}(U)$ *then* $X \sim Y$ *equal in distribution*

Proof.

$P[Y < y] = P[D^{-1}(U) < y] = P[U < D(y)] = \int_0^{D(y)} dx = D(y)$

since $0 \leq D(y) \leq 1$ $\square$

- Work even if D is not continuous
- Often D is difficult to invert

Antithetic Variables

Rationale

1. $\text{Var}[Y_1 + Y_2] = \text{Var}[Y_1] + \text{Var}[Y_2] + 2\,\text{Cov}[Y_1, Y_2]$
2. g monotone then $\text{Cov}[g(X), g(-X)] < 0$

Example

Want to compute $\theta = \text{E}[g(X)]$ where $X \sim N(0, \sigma^2)$ and g is increasing

- Generate $X_1, \dots, X_{m/2} \sim N(0, \omega^2)$ i.i.d.
- Set $X_{m/2+i} = -X_i$ for $i = 1, \dots, m/2$
- Estimate $\hat{\theta}_m = \frac{1}{m} \sum_{i=1}^{m} g(X_i)$

Properties:

- $\hat{\theta}_m$ is unbiased
- $\text{Var}[\hat{\theta}_m] = \frac{1}{m}\text{Var}[g(X)] + \frac{1}{m}\text{Cov}[g(X), g(-X))] \leq \frac{1}{m}\text{Var}[g(X)]$

Lemma

When g and h are increasing then $\mathrm{E}[g(X)h(X)] \geq \mathrm{E}[g(X)]\,\mathrm{E}[h(X)]$

Proof.

For any x, y: $(g(x) - g(y))(h(x) - h(y)) \geq 0$, thus
for X, Y i.i.d.: $(g(X) - g(Y))(h(X) - h(Y)) \geq 0$ and
$\mathrm{E}[(g(X) - g(Y))(h(X) - h(Y))] \geq 0$
$\mathrm{E}[g(X)h(X)] + \mathrm{E}[g(Y)h(Y)] \geq \mathrm{E}[g(Y)h(X)] + \mathrm{E}[g(Y)h(X)]$
$2\,\mathrm{E}[g(X)h(X)] \geq 2\,\mathrm{E}[g(Y)h(X)]$ and from the independence of X and Y,
$\mathrm{E}[g(X)h(X)] \geq \mathrm{E}[g(X)]\,\mathrm{E}[h(X)]$

□

Setting $h(x) = -g(-x)$ and using the property $\mathrm{Cov}[g, h] = \mathrm{E}[gh] - \mathrm{E}[g]\,\mathrm{E}[h]$ gives the following corollary.

Corollary

If g is monotone then $\mathrm{Cov}[g(X), g(-X)] < 0$

Monte Carlo with Antithetic Variables (MC-AV)

- The Antithetic Variable method is a variance reduction technique
- Consists on generating the opposite samples X_i and $-X_i$.
- $\hat{\theta}_m$ is unbiased when X symmetrically distributed (with 0 mean)

But, when X is not symmetrically distributed?

- Let $X \sim P$ and g monotone
- Estimate the expected value of $h(U) = g(P^{-1}(U))$, $U \sim \text{Unif}(0, 1)$
- Generate opposite samples U_i and $1 - U_i$
- The MC-AV estimate is unbiased: $\text{E}[h(1 - U)] = \text{E}[h(U)] = \text{E}[g(X)]$
- When g is monotone so is h and $\text{Cov}[h(U), h(1 - U)] < 0$.
- Analogously: simulate $X_i^+ = P^{-1}(U_i)$ and $X_i^- = P^{-1}(1 - U_i)$

Exercise

Compute the price $C_{0,T}$ *at time* 0 *of an European Call expiring at time* T*:*

$$C_{0,T} = e^{-rT} E[(S_T - K)^+]$$

where $S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right), \quad Z \sim N(0,1).$

Control Variates

Consider the couple of random variables (X, Y)

- Known how to simulate (X, Y)
- $a = \mathrm{E}[X]$, $\mathrm{Var}[X] = \sigma_X^2$ and $\mathrm{Cov}[X, Y] = \sigma_{XY}$ are known

$\Rightarrow$ Want to compute $b = \mathrm{E}[Y]$

Monte Carlo:

- Simulate by MC $(x_i, y_i) \sim (X, Y)$ i.i.d.
- $\hat{a} = \dfrac{1}{m}\sum_i x_i \quad \hat{b} = \dfrac{1}{m}\sum_i y_i \quad \mathrm{Cov}[\hat{a}, \hat{b}] = \dfrac{1}{m}\sigma_{XY}$

The error $\hat{a} - a$ is known, use it to correct $\hat{b}$: $\hat{b}_{CV} = \hat{b} - \beta(\hat{a} - a)$

- $\mathrm{E}[\hat{b}_{CV}] = b$,
- $\mathrm{Var}[\hat{b}_{CV}] = \frac{1}{m}\sigma_Y^2 + \beta^2\frac{1}{m}\sigma_X^2 - 2\beta\frac{1}{m}\sigma_{XY}$
- Variance is minimized by $\beta = \dfrac{\sigma_{XY}}{\sigma_X^2}$: $\quad \mathrm{Var}[\hat{b}_{CV}] = \mathrm{Var}[\hat{b}] - \dfrac{1}{m}\dfrac{\sigma_{XY}^2}{\sigma_X^2}$

$$\hat{b}_{CV} = \hat{b} - \beta(\hat{a} - a), \qquad \beta = \frac{\sigma_{XY}}{\sigma_X^2}, \qquad \mathrm{Var}[\hat{b}_{CV}] = \frac{1}{m}\left(\sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}\right)$$

- Only correlation matters:

$$\frac{\mathrm{Var}[\hat{b}_{CV}]}{\mathrm{Var}[\hat{b}]} = \left(1 - \frac{\sigma_{XY}^2}{\sigma_X^2 \sigma_Y^2}\right) = 1 - \rho_{XY}^2$$

- Usually, σ_X^2 and σ_{XY} are unknown
- Solution: estimate σ_X^2 and σ_{XY} from the generated sample:
  - $\hat{\sigma}_X^2 = \dfrac{1}{m-1} \sum_i (x_i - \hat{a})^2$
  - $\hat{\sigma}_{XY} = \dfrac{1}{m-1} \sum_i (x_i - \hat{a})(y_i - \hat{b})^2$
- β is no longer constant but depends on the sample $\{(x_i, y_i), i = 1, \ldots, m\}$.
- $\hat{b}_{CV}$ is unbiased only asymptotically.

Example (Asian Options)

- Dynamics: $dS_t = rS_t dt + \sigma S_t dW_t$ Geometric Brownian Motion
- n observation dates: $t_i = i\Delta t$, $\Delta t = T/n$

	Average	Fixed Strike	Floating Strike
Arithmetic	$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$	$(A-K)^+$	$(S_T - A)^+$
Geometric	$G = \left(\prod_{i=1}^{n} S_{t_i}\right)^{\frac{1}{n}}$	$(G-K)^+$	$(S_T - G)^+$

- Closed form expression for the fixed strike geometric average call:

$$C^G(S_0, K, r, T, \sigma) = C^{BS}(S_0, K, r, \tilde{T}, \tilde{\sigma}, \tilde{d}), \tag{7}$$

where $\tilde{T} = \frac{n+1}{2n}T$, $\quad \tilde{\sigma}^2 = \frac{2n+1}{3n}\sigma^2$, $\quad \tilde{d} = \frac{n-1}{6n}\sigma^2$.

- Known how to simulate $X = (G_T - K)^+$ and $Y = (A_T - K)^+$
  Known $C^G = e^{-rT}\,\mathrm{E}[X]$ compute $C^A = e^{-rT}\,\mathrm{E}[Y]$

- Problem: simulate $S_{t_1}, S_{t_2}, \dots, S_{t_n}$:

$\Rightarrow$ $S_{t_{i+1}} = S_{t_i} \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma(\Delta W_i)\right), \quad \Delta W_i \sim N(0, \Delta t)$, i.i.d.

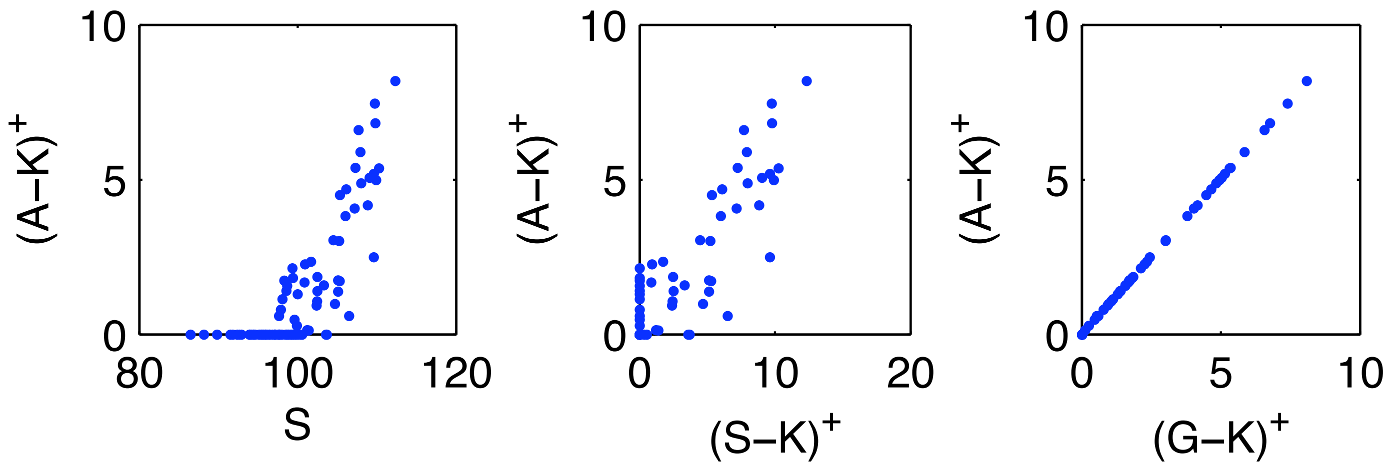
asian_mc.m

```
function [CArith,CGeom] = ...
    asian_mc( S0, K, sigma, r, T, n, m )

dt = T/n;
S = S0 * ones(m,1);  G = ones(m,1);  A = zeros(m,1);
for i=1:n
  dW = randn(m,1)*sqrt(dt);
  S = S .* exp( (r - .5*sigma^2)*dt + sigma *dW );
  G = G .* S;
  A = A + S;
end
G = G.^(1/n);      A = A/n;

CArith = exp(-r*T)*mean( max(A-K,0) );
CGeom  = exp(-r*T)*mean( max(G-K,0) );
```

- Let consider three choices for the control variate:

Underlying	S_T	$e^{-rT}\,\mathrm{E}[S_T] = S_0$
European Call	$(S_T - K)^+$	$e^{-rT}\,\mathrm{E}[(S_T - K)^+]$ Black & Scholes
Geometric Avg.	$(G - K)^+$	$e^{-rT}\,\mathrm{E}[(G - K)^+]$ Closed Form expr.

- Look at the relationships:



- The Geom. Avg. payoff has the strongest dependence

Algorithm:

1. Compute the Geometric Average Call by mean the closed from expression
2. Simulate m replications of S_{t_i}, A and G
3. Compute the Arithmetic and Geometric Average payoffs
4. Compute the two option prices as discounted expected values of the payoffs
5. Compute the variance of the discounted Geometric Avg. payoff
6. Compute the covariance of the two discounted payoffs
7. Correct the Arithmetic option price by using the approximation error of the Geometric Call.

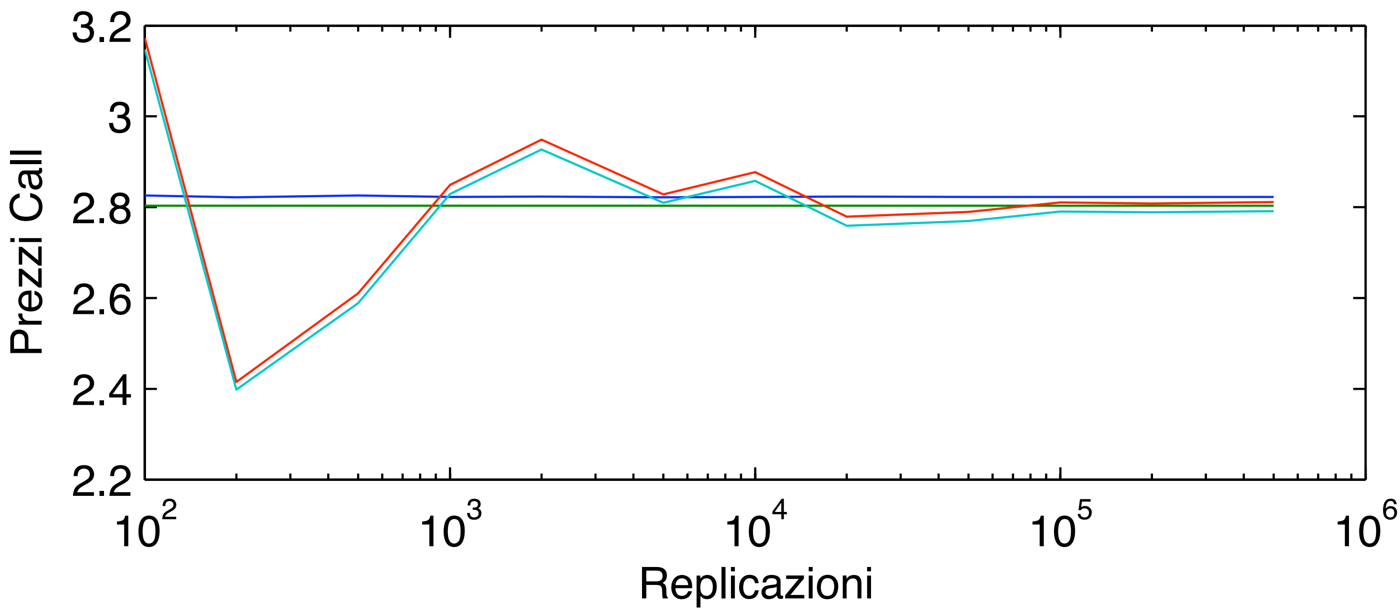
asian_mccv.m

```
function [CArithCV,CGeom,CArithMC,CGeomMC] = ...
    asian_mccv( S0, K, sigma, r, T, n, m )

% Compute exact value of Geometric call
TT = (n+1)/(2*n) * T;
sigmaa = sigma * sqrt( (2*n+1)/(3*n) );
d = (n-1)/(6*n)*sigma^2;
CGeom = blsprice( S0, K, r, TT, sigmaa, d );

% Compute Arith and Geom calls with MC
dt = T/n;
S = S0*ones(m,1);
G = ones(m,1);
A = zeros(m,1);
for i=1:n
  dW = randn(m,1)*sqrt(dt);
  S = S .*exp( (r - .5*sigma^2)*dt + sigma *dW );
  G = G .* S;    A = A + S;
end
```

asian_mccv.m

```
%% Compute Payoffs;
ArithPayoff = exp(-r*T)*max(A-K,0);
GeomPayoff = exp(-r*T)*max(G-K,0);

%% Compute sample mean, variance and covar.
CArithMC = mean(ArithPayoff);
CGeomMC  = mean(GeomPayoff);
CovAG    = cov( ArithPayoff, GeomPayoff );
VarG  = CovAG(2,2);
CovAG = CovAG(1,2);

if (VarG>1e-7)
  CArithCV = CArithMC + CovAG/VarG *(CGeom -  CGeomMC);
else
  CArithCV = CArithMC;
end
```

Control Variates Convergence



Multivariate CV

- X has values in $\mathbb{R}^n$, $\mathrm{E}[X] = a$
- Let $\mathrm{Cov}(X) = \Sigma_X : n \times n$, $\mathrm{Cov}(X, Y) = \sigma_{XY} : n \times 1$

$$\hat{b}_{CV} = \hat{b} - (\hat{a} - a)^T \beta, \qquad \beta = ?$$

- $\mathrm{Var}(\hat{b}_{CV}) = \mathrm{Var}(\hat{b}) + \beta^T \mathrm{Cov}(\hat{a})\beta - 2\beta^T \mathrm{Cov}(\hat{a}, \hat{b})$
- $\mathrm{E}[\hat{a}] = a$,
- $\mathrm{Cov}(\hat{a}) = \mathrm{Cov}(\frac{1}{m}\sum_i x_i) = \frac{1}{m}\mathrm{Cov}(x) = \frac{1}{m}\Sigma_X \qquad (x_i \sim \text{i.i.d. } X\ )$
- $\mathrm{Cov}(\hat{a}, \hat{b}) = \frac{1}{m}\sigma_{XY}$
- We want to find the value β that minimize:
  $\phi(\beta) = \beta^T \Sigma_X \beta - 2\beta^T \sigma_{XY}$

Multivariate CV

Problem

Miminize $\phi(\beta) = \beta^T \Sigma_X \beta - 2\beta^T \sigma_{XY}$

- Σ_X is a covariance matrix, it is positive semidefinite and thus ϕ is convex
- $\partial_\beta \phi = 2\Sigma_X \beta - \sigma_{XY}$
- $\partial_\beta \phi = 0 \qquad \Rightarrow \qquad \beta = \Sigma_X^{-1} \sigma_{XY}$ minimize ϕ
- With this choice of β,

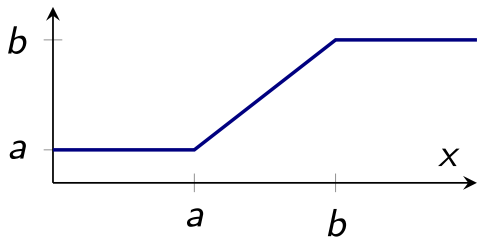
$$\mathrm{Var}(\hat{b}_{CV}) = \mathrm{Var}(\hat{b}) - \frac{1}{m} \sigma_{XY}^T \Sigma_X^{-1} \sigma_{XY}$$

- Let c_{XY} be the vector of correlations between X and Y and C_X be the correlation matrix of X, then

$$\mathrm{Var}(\hat{b}_{CV}) = \mathrm{Var}(\hat{b}) - \frac{1}{m} \mathrm{Var}(y) c_{XY}^T C_X^{-1} c_{XY}$$

- Σ_X need to be positive definite (non-singular).

Exercise: Cliquet Option



- Define the truncating function:

$$(x)_{[a,b]} = \min(\max(x, a), b)$$

- Given the monitoring dates $0 = t_0 < t_1 < \cdots < t_N = T$ consider the returns

$$R_n = \frac{S_{t_n}}{S_{t_{n-1}}} - 1 \qquad\qquad n = 1, \ldots, N$$

- Compute the truncated returns (Cap/Floor)

$$R_n^* = (R_n)_{[F,C]} \qquad F < C, \qquad n = 1, \ldots, N$$

- Accumulate the returns and truncate

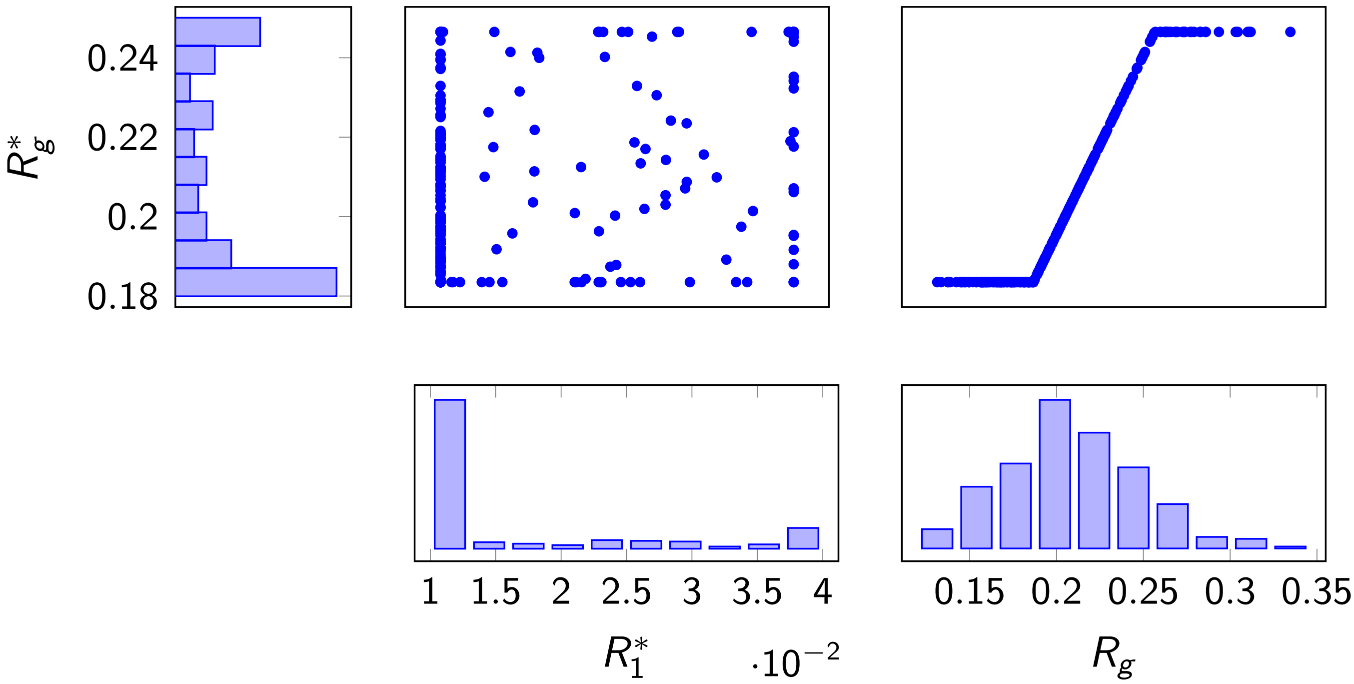
$$R_g = R_1^* + R_2^* + \cdots + R_N^*, \qquad R_g^* = (R_g)_{[F_g, C_g]} \qquad F_g < C_g$$

- The Cliquet payoff is given by R_g^*

cliquet_mc.m

```
function [res,R,Rg] = cliquet_mc( r,vol,t, C,F,Cg,Fg, m )

if( t(1) ~= 0);    t = [0;t(:)];   end
dt = diff(t);
n = length(dt);
R = nan(m,n);
for i=1:n
  z = randn(m,1);
  R(:,i) = exp( (r-.5*vol^2)*dt(i) + vol*sqrt(dt(i))*z) - 1;
end
R = min(max(R,F),C);
Rg = sum(R,2);
Rg = min(max(Rg,Fg),Cg);
res = exp(-r*t(end))*mean(Rg);
```

Cliquet Options (cont.)



- Both R_1^* and R_g are candidates as control variates.
- R_g much better than R_1^*:

$$\mathrm{corr}(R_1^*, R_g^*) = .315 \qquad << \qquad \mathrm{corr}(R_g, R_g^*) = .946$$

Cliquet Options

- It is necessary to analytically compute the expected value of R_1 or R_g.
- To this aim notice that, the truncating function can be rewritten as

$$(x)_{[a,b]} = a + (x-a)^+ - (x-b)^+, \qquad b > a$$

- And R_n^* consists on the difference of two call payoffs on R_n
- Assume a B&S dynamics for S_t:

$$dS_t = rS_t dt + \sigma dW_t$$

then

$$R_n = \exp((r - \frac{1}{2}\sigma^2)(t_n - t_{n-1}) + \sigma(W_{t_n} - W_{t_{n-1}}) - 1$$

- $X_n = R_n + 1$, $(n = 1, \ldots, N)$ are independent and log-normally distributed.
- It follows that R_n^* is given by

$$R_n^* = F + (X_n - (F+1))^+ - (X_n - (C+1))^+, \quad n = 1, \ldots, N$$

- R_n^* can be decomposed on a constant plus the difference of two calls:

$$R_n^* = F + (X_n - (F+1))^+ - (X_n - (C+1))^+,$$
$$X_n = \exp((r - \frac{1}{2}\sigma^2)(t_n - t_{n-1}) + \sigma(W_{t_n} - W_{t_{n-1}}))$$

- Thus, $\mathrm{E}[R_n^*]$ and $\mathrm{E}[R_g] = \sum_n \mathrm{E}[R_n^*]$ are given by B&S formulae:

$$\begin{aligned}\mathrm{E}[R_n^*] = F &+ e^{\Delta t_n}\Phi(d_+^{F,n}) - (F+1)\Phi(d_-^{F,n}) \\ &- e^{\Delta t_n}\Phi(d_+^{C,n}) + (C+1)\Phi(d_-^{C,n})\end{aligned}$$

where $\Delta t_n = t_n - t_{n-1}$,

$$d_\pm^{F,n} = \frac{-\log(F+1) + (r \pm \sigma^2/2)\Delta t_n}{\sqrt{\sigma^2 \Delta t_n}} \qquad \text{and}$$
$$d_\pm^{C,n} = \frac{-\log(C+1) + (r \pm \sigma^2/2)\Delta t_n}{\sqrt{\sigma^2 \Delta t_n}}.$$

- All the R_n^* and the R_g can be used as control variables.

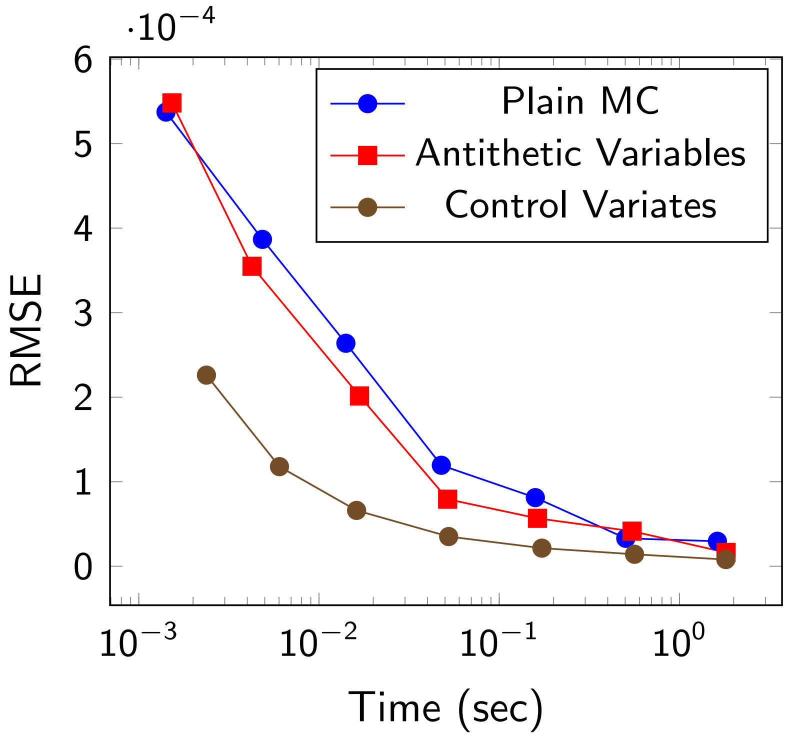
cliquet_mccv.m

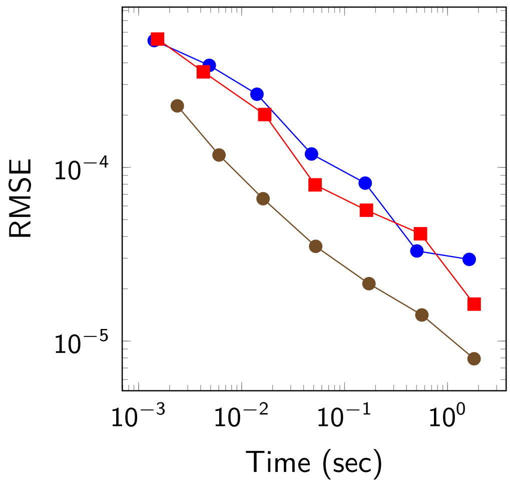
```
function res = cliquet_mccv( r,vol,t, C,F,Cg,Fg, m )

dt = diff(t);
n = length(dt);
R = nan(m,n);
for i=1:n
  z = randn(m,1);
  R(:,i) = exp( (r-.5*vol^2)*dt(i) + vol*sqrt(dt(i))*z) - 1;
end
R = min(max(R,F),C);
Rg = sum(R,2);                    clear R;
RgStar = min(max(Rg,Fg),Cg);

mcRg = exp(-r*t(end))*mean(Rg);
mcRgStar = exp(-r*t(end))*mean(RgStar);
```

cliquet_mccv.m

```
%% Compute the B&S prices
dfplus = (- log(F+1) + (r + .5*vol^2)*dt )./ (vol*sqrt(dt));
dfminus = dfplus - vol*sqrt(dt);
dcplus = (- log(C+1) + (r + .5*vol^2)*dt )./ (vol*sqrt(dt));
dcminus = dcplus - vol*sqrt(dt);

ExpRStar = F + exp(r*dt) .* (normcdf(dfplus) - normcdf(dcplus)
            - (F+1) * normcdf(dfminus) + (C+1) * normcdf(dcmi
ExpR = exp(-r*t(end))*sum(ExpRStar);

%% Compute the CV approximation
covMatrix = cov(Rg,RgStar);
beta = covMatrix(1,2)/covMatrix(1,1);
res = mcRgStar - beta*(mcRg - ExpR);
```

Cliquet Options: Comparison of Methods





- $RMSE = \sqrt{\text{mean}(C_{MC} - C_{best})^2}$
- Here computed over 20 different sets of simulations

Stratified Sampling

- Partition the domain Ω:
  $$\Omega = \bigcup_{k=1}^{K} W_k, \quad W_k \cap W_j = \emptyset, \quad \text{for } k \neq j = 1, \ldots, K.$$
- $p_k = P[W_k]$ is known
- Bayes rule:
  $\theta = E[g(X)] = \sum_k P\left[W_k\right] \mathrm{E}\left[g(X)|W_k\right] = \sum_k p_k \theta_k.$
- For each partition W_k, approximate θ_k by MC:
  - Simulate $x_1^k, x_2^k, \ldots, x_{m_k}^k \sim$ i.i.d. $P_{|W_k}$
  - $\hat{\theta}^k \equiv \frac{1}{m_k} \sum_i g(x_i^k) \sim N(\theta_k, \frac{1}{m_k}\sigma_k^2)$
- $\hat{\theta}^K = \sum_k p_k \hat{\theta}^k \sim N(\theta, \sum_k \frac{p_k}{m_k}\sigma_k^2)$
- Increase m_k where σ_k^2 is large

Example

Add examples

Importance Sampling

- Let consider the change of variables
$$\int_\Omega g(X)dP = \int_\Omega \frac{g(X)}{f(X)} dF, \qquad \mathrm{E}^P[g] = \mathrm{E}^F[g/f],$$
where $\mathrm{E}^P[f] = \int_\Omega f(X)dP = 1$ and $fdP = dF$.
- Use MC to compute the estimate $\hat{\theta}_{IS}$ of $\theta = \mathrm{E}^F[g/f]$
  - Error has variance: $\mathrm{Var}[\hat{\theta}_{IS}] = \frac{1}{m}\mathrm{Var}^F[g/f]$
- The optimal choice is $f(x) = cg(x)$ (utopia)
- In practice f is chosen so that
  - f approximate $|g|$
  - F easy to simulate
  - f should not have a too fast decay

Example

Add examples

Solutions to SDE

- Consider the Stochastic Differential Equation (SDE):

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t, \qquad X_0 = x_0$$

- We know the exact solution if we know a f such that

$$X_t = f(W_t, t), \qquad t \in [0, T]$$

- or if we know the transition density function (TDF)

$$p(y, s; x, t)$$

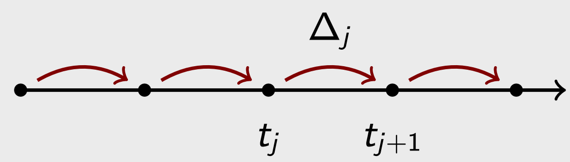
that is, the conditional density of X_s given $X_t = x$.

Euler Discretization

SDE: $\quad dX_t = a(X_t)dt + b(X_t)dW_t, \qquad X_0 = x_0$

- Time discretization: $\quad 0 = t_0 < t_1 < \cdots < t_j < \cdots < t_N = T$

$$\Delta_j = t_{j+1} - t_j,$$
$$\Delta W_j = W_{t_{j+1}} - W_{t_j},$$
$$\Delta X_j = X_{t_{j+1}} - X_{t_j}$$



- Then $\Delta W_j \sim N(0, \Delta_j)$ iid.
- Euler method: at each step freeze a and b at (X_{t_j}):

Euler: $\quad \Delta Y_j = a(Y_j, t_j)\Delta_j + b(Y_j, t_j)\Delta W_j \qquad Y_0 = x_0$

or $\quad Y_{j+1} = Y_j + a(Y_j, t_j)\Delta_j + b(Y_j, t_j)\Delta W_j$

where Y_j is the approximation to X_{t_j}.

Example (Geometric Brownian Motion)

$$dS_t = \mu S_t dt + \sigma S_t dW_t \qquad\qquad S_0 = s_0$$

Exact solution by Itô Lemma on $X_t = \log(S_t)$:

$$\begin{aligned} dX_t &= \frac{\partial \log(S_t)}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 \log(S_t)}{\partial S_t^2}\sigma^2 S_t^2 dt \\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt \;=\; (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned}$$

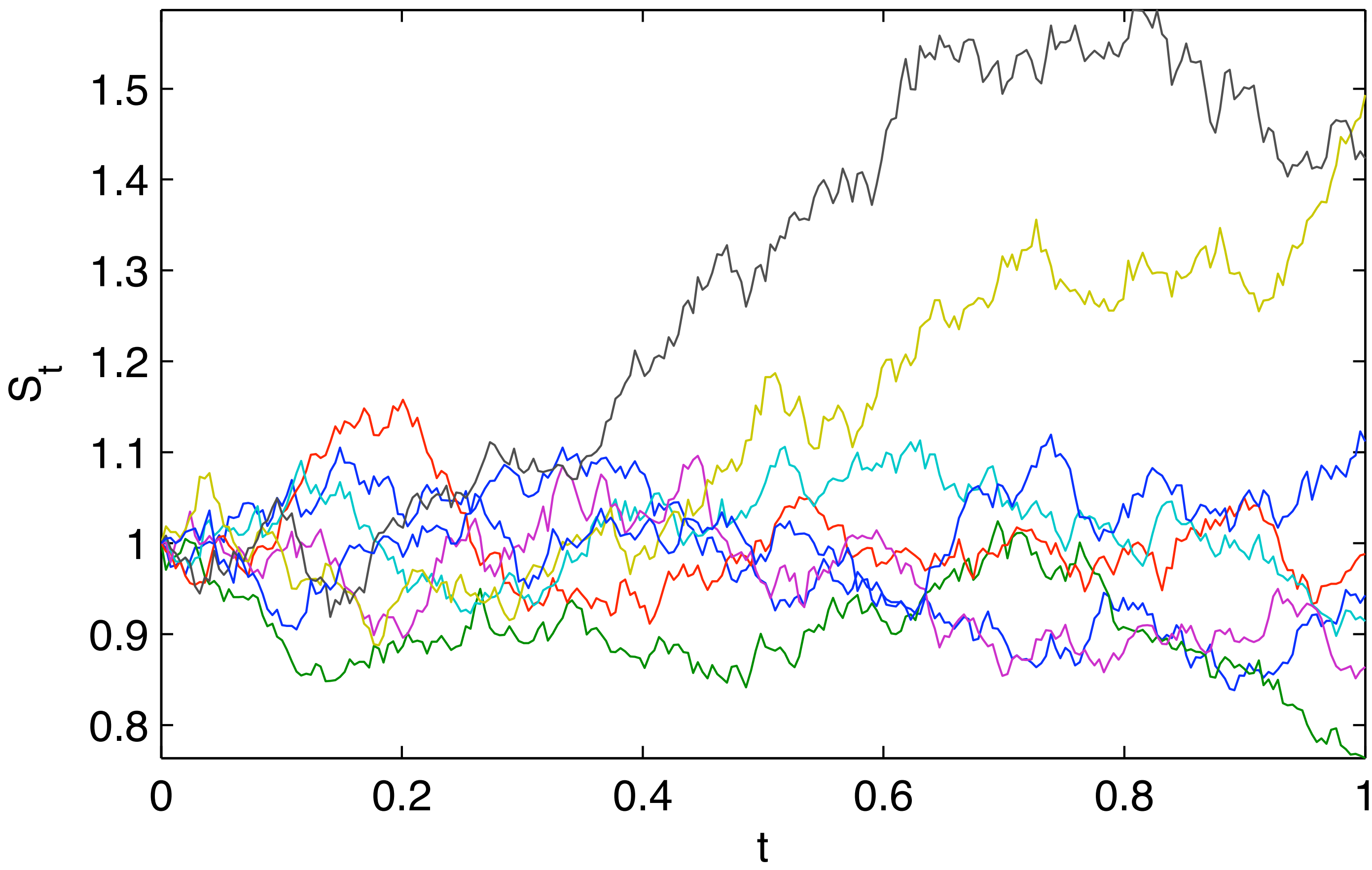
$$X_t - X_0 = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

$$S_t = s_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)$$

Euler discretization:

$$Y_{n+1} - Y_n = \mu Y_n \Delta t + \sigma Y_n \Delta W_t, \qquad Y_0 = s_0$$

Trajectories of Brownian motions



Exercise (European Call)

Compute $C = e^{-rT} E^Q[(S_T - K)^+]$ *where* $dS_t = rS_t dt + \sigma S_t dW_t^Q$ *and* S_0 *is given.*

bs_euler.m

```
function [c,v] = bs_euler( S0, K, T, r, sigma, M, N )

dt = T/(N-1);
C = zeros(M,1);

for i=1:M
  %% simulation of the i-th scenario
  S = S0;
  for n=1:N-1
    dW = randn()*sqrt(dt);
    S = S*exp( (r - .5*sigma^2)*dt + sigma*dW );
  end
  %% payoff in the i-th scenario
  C(i) = max(S-K,0);
end
c = mean(C);    v = var(C);
```

Convergence

- Weak Convergence of order γ:
$$|\,\mathrm{E}[p(Y_n)] - \mathrm{E}[p(X_{t_n})]| \leq O(\Delta_t^\gamma)$$
for all p smooth and with polynomial growth
- Strong Convergence of order γ:
$$\mathrm{E}[\,|Y_n - X_{t_n}|\,] \leq O(\Delta_t^\gamma)$$
- Strong convergence is important if trajectories are of interest
- Weak convergence is important when expected values of functions of X are required

Taylor formula and ODEs

- Consider the ODE: $dX_t = a(X_t)dt$ or $X_t - X_0 = \int_0^t a(X_s)ds$
- Consider $f(X_t)$ with $f \in \mathcal{C}^1(\mathbb{R})$. Thus:

$$df(X_t) = \partial_x f(X_t)dX_t = a(X_t)\partial_x f(X_t)dt$$

or, defining $Lf(x) = a(x)\partial_x f(x)$,

$$f(X_t) = f(X_0) + \int_0^t Lf(X_s)ds \tag{8}$$

- Now, applying (**??**) to the integrand in $f^{(1)} \equiv Lf$, it holds[1]

$$f(X_t) = f(X_0) + \int_0^t f^{(1)}(X_0)ds + \int_0^t \int_0^s Lf^{(1)}(X_z)dzds$$
$$= f(X_0) + tf^{(1)}(X_0) + \int_0^t \int_0^s f^{(2)}(X_z)dzds$$

- and so on:

$$f(X_t) = f(X_0) + tf^{(1)}(X_0) + \frac{t^2}{2}f^{(2)}(X_0) + \int_0^t \int_0^s \int_0^z \dots$$

Milstein method

- Itô formula for $f(X_t)$, where $dX_t = a(X_t)dt + b(X_t)dW_t$:

$$df(X_t) = \underbrace{\left(a_t\partial_x f(X_t) + \frac{1}{2}b_t^2\partial_{xx}f(X_t)\right)}_{L^0 f(X_t)} dt + \underbrace{b\partial_x f(X_t)}_{L^1 f(X_t)} dW_t$$

$$f(X_{t+h}) = f(X_t) + \int_t^{t+h} L^0 f(X_s)ds + \int_t^{t+h} L^1 f(X_s)dW_s \qquad (9)$$

- Consider $\quad X_{t+h} = X_t + \int_t^{t+h} a(X_s)ds + \int_t^{t+h} b(X_s)dW_s$
- Applying Itô to $a_s \equiv a(X_s)$ and $b_s \equiv b(X_s)$, the latter becomes

$$X_{t+h} = X_t + \int_t^{t+h} \underbrace{\left(a_t + \int_t^s L^0 a_z dz + \int_t^s L^1 a_z dW_z \right)}_{a_s} ds$$

$$+ \int_t^{t+h} \underbrace{\left(b_t + \int_t^s L^0 b_z dz + \int_t^s L^1 b_z dW_z \right)} dW_s$$

- Recall

$$X_{t+h} = X_t + \int_t^{t+h} \left(a_t + \int_t^{s} L^0 a_z dz + \int_t^{s} L^1 a_z dW_z \right) ds$$
$$+ \int_t^{t+h} \left(b_t + \int_t^{s} L^0 b_z dz + \int_t^{s} L^1 b_z dW_z \right) dW_s$$

where $L^0 = a\,\partial_x + \frac{1}{2} b^2\,\partial_{xx}$ and $L^1 = b\,\partial_x$

- Thus $X_{t+h} = X_t + a_t h + b_t \Delta W_t + \int_t^{t+h} \int_t^{s} b_z \partial_x b_z dW_z dW_s + R$

$$R = \int_t^{t+h} \int_t^{s} L^0 a_z dz ds + \int_t^{t+h} \int_t^{s} L^1 a_z dW_z ds + \int_t^{t+h} \int_t^{s} L^0 b_z dz dW_s$$

- From Itô expansion of $b_z \partial_x b_z$ and $\int_t^{t+h} \int_t^{s} dW_z dW_s = \frac{1}{2}\left((\Delta W_t)^2 - h\right)$:

$$X_{t+h} = X_t + a_t h + b_t \Delta W_t + \frac{1}{2} b_t \partial_x b_t \left((\Delta W_t)^2 - h\right) + \tilde{R}$$

$$\tilde{R} = R + \int_t^{t+h} \int_t^{s} \left(\int_t^{z} L^0 L^1 b_r dr + \int_t^{z} L^1 L^1 b_r dW_r \right) dz dW_s$$

$$X_{t+h} = X_t + a(X_t)h + b(X_t)\Delta W_t + \frac{1}{2}b(X_t)b'(X_t)\big((\Delta W_t)^2 - h\big) + \tilde{R}$$

- Euler:
  - Strong conv. of order 1/2
  - Weak conv. of order 1
- Milstein:
  - Strong conv. of order 1
  - Weak conv. of order 1

A simple case

- Underlying is S_t and the risk-neutral interest rate is null: $r = 0$
- Choose when to get a payoff $(1 - S_t)^+$: now or at $t = T$
- The contract value is

$$V_T = (1 - S_T)^+ \qquad C_0 = \mathrm{E}[V_T | S_0],$$
$$V_0 = \max((1 - S_0)^+, C_0)$$

- Monte Carlo:

$$V_T^\omega = (1 - S_T^\omega)^+ \qquad \omega = 1, \ldots, m$$
$$\hat{C}_0 = \frac{1}{m} \sum_\omega \hat{V}_T^\omega, \quad \hat{V}_0 = \max((1 - S_0)^+, \hat{C}_0),$$

- $\hat{C}_0$ is unbiased: $\mathrm{E}[\hat{C}_0] = C_0$
- $\hat{V}_0$ is biased high:

$$\begin{aligned}\mathrm{E}[\hat{V}_0] &= \mathrm{E}[\max((1 - S_0)^+, \hat{C}_0)] \\ &\geq \max((1 - S_0)^+, \mathrm{E}[\hat{C}_0]) = \max((1 - S_0)^+, C_0) = V_0\end{aligned}$$

Regression

- Consider the r.vs. $Y \in \mathbb{R}$ and $X \in \mathbb{R}^k$
- We want to approximate Y by means of $X^T\beta$, with $\beta \in \mathbb{R}^k$ constant

$$Y = X^T\beta + U$$

- The Least Squares approximation $\tilde{\beta}$ minimizes $\mathrm{E}[U^2]$:

$$\mathrm{E}[U^2] = \mathrm{E}[Y^2] + \beta^T\,\mathrm{E}[XX^T]\beta - 2\beta^T\,\mathrm{E}[XY]$$

- Thus,

$$\tilde{\beta} = S^{-1}q, \qquad \text{where} \qquad S = \mathrm{E}[XX^T], \qquad q = \mathrm{E}[Xy].$$

- MC: Given $(x_\omega, y_\omega) \sim \mathrm{iid}(X, Y)$, $\omega = 1, \dots, m$

$$\hat{\beta} = \hat{S}^{-1}\hat{q} \quad \text{where} \quad \hat{S} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega x_\omega^T, \qquad \hat{q} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega y_\omega.$$

- $\hat{S} \to S$, $\hat{q} \to q$ and, thus, $\hat{\beta} \to \tilde{\beta}$

Regression (cont.)

- MC approximation:

$$\hat{\beta} = \hat{S}^{-1}\hat{q} \quad \text{where} \quad \hat{S} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega x_\omega^T, \quad \hat{q} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega y_\omega.$$

- With abuse of notation define

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{pmatrix} : m \times k \qquad \text{and} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} : m \times 1$$

- then: $\hat{S} = \frac{1}{m}X^TX$, $\hat{q} = \frac{1}{m}X^Ty$ and $\hat{\beta} = (X^TX)^{-1}X^Ty$
- that is $\hat{\beta}$ is the OLS estimator of the linear regression

$$y = X\beta + u, \qquad u \sim (0, \sigma^2 I_m).$$

- Recall the original model was $Y = X^T\beta + U$

Regression and Approximation

Problem

Approximate $c(\omega) : \Omega \to \mathbb{R}$ *by means of* $\phi_i(\omega) : \Omega \to \mathbb{R}$, $i = 1, \dots, k$

$$c(\omega) = \sum_{i=1}^{k} \phi_i(\omega)\beta_i + \varepsilon(\omega)$$

- ϕ_i basis functions, ε error function
- We need a norm to evaluate the error: $\|\varepsilon\|$
- Given a probability measure $P(\omega)$, define the scalar product

$$\langle u, v \rangle = \mathrm{E}[uv] = \int_\Omega u(\omega)v(\omega)dP(\omega)$$

- $\|\varepsilon\| = \langle \varepsilon, \varepsilon \rangle^{1/2} = \mathrm{E}[\varepsilon^2]^{1/2}$ is a semi-norm
- Back to the regression: $Y = X^T\beta + U$,

$$Y(\omega) = c(\omega), \qquad X(\omega) = \begin{pmatrix} \phi_1(\omega) & \phi_2(\omega) & \cdots & \phi_k(\omega) \end{pmatrix}^T.$$

Exponential R.V.

$$Y \sim \mathrm{Exp}(\lambda)$$

Domain $\mathbb{R}^+$.

CDF $F_Y(y) = 1 - e^{-\lambda y}$, for $y \geq 0$.

PDF $f_Y(y) = \lambda e^{-\lambda y}$, for $y \geq 0$.

Inv. CDF $F_Y^{-1}(u) = -\lambda^{-1} \log(1-u)$.

Moments $\mathrm{E}[Y] = \lambda^{-1}$, $\mathrm{Var}(Y) = \lambda^{-2}$.

Since the inverse CDF is explicitly known an exponential r.v. can be easily simulated:

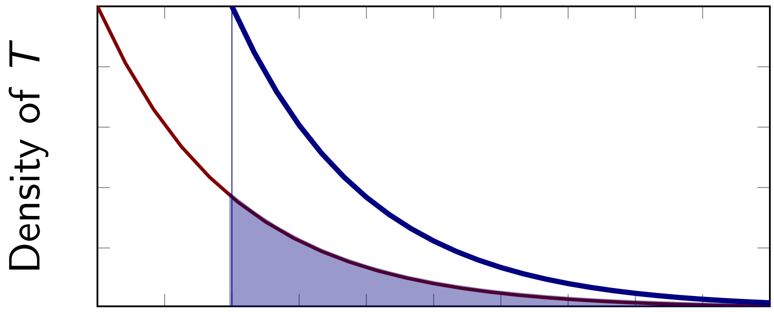
$$U \sim \mathrm{Unif}([0,1]), \qquad \Rightarrow \qquad F_Y^{-1}(U) \sim \mathrm{Exp}(\lambda)$$

Absence of Memory

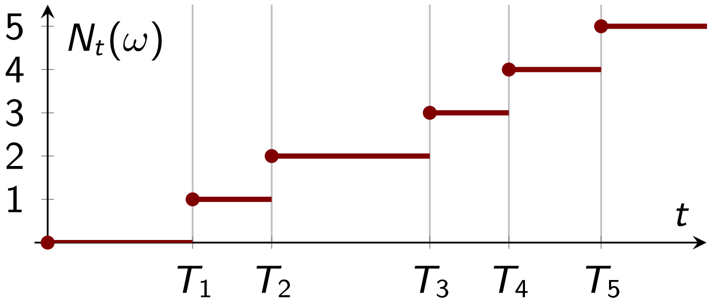
Absence of Memory

$$T \sim \text{Exp}(\lambda) \qquad \Rightarrow \qquad P[T > t + x | T > t] = P[T > x]$$

- Let $T \sim \text{Exp}(\lambda)$ be the time when a specific event occurs. The AoM states states that if we are at time t and that event didn't occur then $T - t$ will have the same density that T had at time 0.
  I.e. the distribution of the waiting time is the same we had at time 0.



The poisson process



For $i = 1, 2, \dots$

- $\tau_i \sim \text{Exp}(\lambda)$, indep.
- $T_i = T_{i-1} + \tau_i$

- T_1 is the time of the first occurence of a specific event
- T_2 that of the second occurence

  ⋮

- T_n the time of the n-th occurence

Poisson Process

N_t counts how many of such events occurs up to time t:

$$N_t(\omega) = \sum_{n=1}^{\infty} 1\{t \geq T_n(\omega)\}, \qquad \omega \in \Omega,\ t \geq 0.$$

Poisson Distribution

$N \sim \text{Pois}(\lambda)$

Domain Non-negative integers: $k = 0, 1, 2, \ldots$

PMF $P_N[k] = \dfrac{\lambda^k}{k!} e^{-\lambda}$

CDF $F_N(k) = \dfrac{\Gamma(k+1, \lambda)}{\Gamma(k+1)} = \displaystyle\sum_{i=1}^{k} P_N[i]$

Moments $E[N] = \lambda$, $\text{Var}(N) = \lambda$

That is,

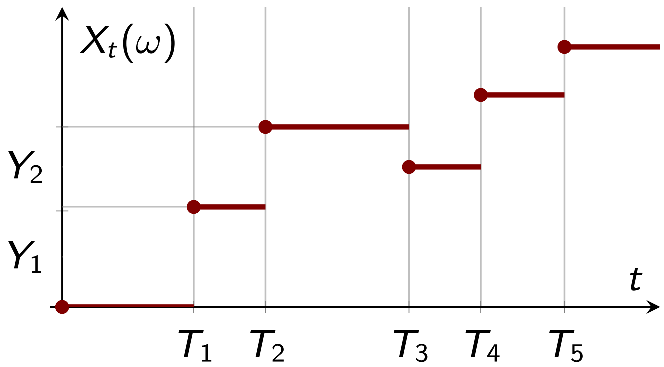
- $N_t \sim \text{Pois}(\lambda t)$
- in an interval $[0, t]$ we expect an average of λt jumps

PS. $\Gamma(x)$ and $\Gamma(x, \lambda)$ are the complete and incomplete Gamma functions.

Compound Poisson Process

Idea

Instead of jumps of unit magnitude, use iid random jump sizes.



- $\tau_i \sim$ i.i.d $\mathrm{Exp}(\lambda)$
- $Y_i \sim$ i.i.d
- $\{Y_i\}$, $\{\tau_i\}$ independent.
- $T_i = T_{i-1} + \tau_i$
- $X_{T_i} = X_{T_i^-} + Y_i$

Compound poisson process:

$$X_t = \sum_{i=1}^{N_t} Y_i = \sum_{i=1}^{\infty} Y_i 1\{t \geq T_i\}.$$

Compound Poisson Processes

- X_t is Compound Poisson Process if and only if it is a Levy process with piecewise constant sample paths (see Cont and Tankov 2004, Prop. 3.3).

The basic scheme

$$X_t = \sum_{i=1}^{N_t} Y_i = \sum_{i=1}^{\infty} Y_i 1\{t > T_i\}, \qquad Y_i \sim \mathrm{N}(\mu, \sigma^2)$$

cpois1.m

```
function x=cpois1(lambda,mu,sigma,t,m)

n = poissrnd(lambda*t,m,1);
x = zeros(m,1);
for w=1:m
    for j=1:n(w)
        y = mu + sigma*randn;
        x(w) = x(w) + y;
    end
end
```

- We can swap the two loops and/or vectorize one of them.

Second Algorithm: foreach sample path

$$X_t = \sum_{i=1}^{N_t} Y_i \qquad Y_i \sim \mathrm{N}(\mu, \sigma^2)$$

cpois2.m

```
function x=cpois2(lambda,mu,sigma,t,m)

n = poissrnd(lambda*t,m,1);
x = zeros(m,1);
for w=1:m
    y = mu + sigma*randn(n(w),1);
    x(w) = sum(y);
end
```

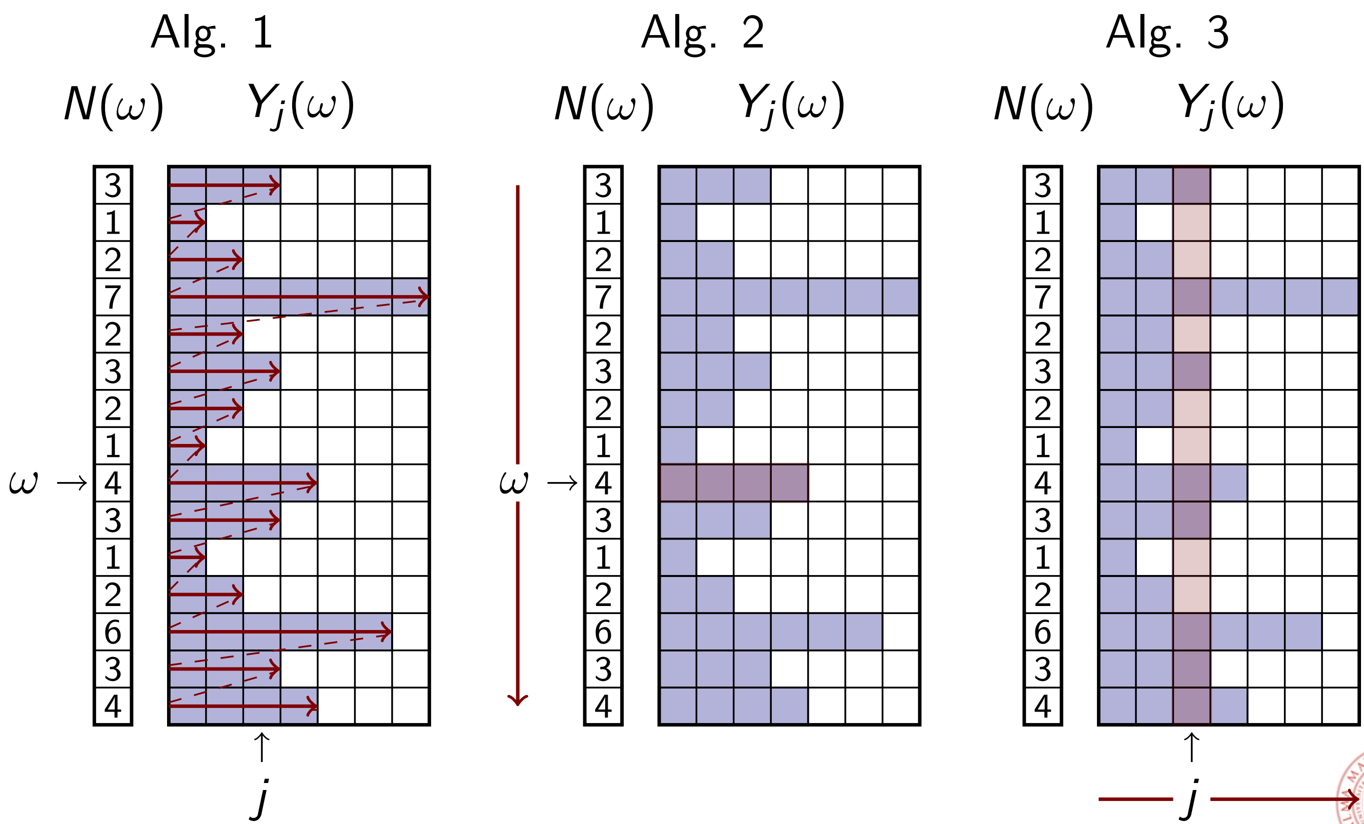
Third Algorithm: for each jump

- $X_t = \sum_{i=1}^{N_t} Y_i \qquad Y_i \sim \mathrm{N}(\mu, \sigma^2).$

cpois3.m

```
function x=cpois3(lambda,mu,sigma,t,m)

n = poissrnd(lambda*t,m,1);
nMax = max(n);
x = zeros(m,1);
for j=1:nMax
    y = mu + sigma*randn(m,1);
    x = x + y .* (j<=n);
end
```

In pictures



Alg. 1
N(ω)
Y_j(ω)
ω →
3 1 2 7 2 3 2 1 4 3 1 2 6 3 4
↑
j
Alg. 2
N(ω)
Y_j(ω)
ω →
3 1 2 7 2 3 2 1 4 3 1 2 6 3 4
Alg. 3
N(ω)
Y_j(ω)
3 1 2 7 2 3 2 1 4 3 1 2 6 3 4
↑
j

Fourth algorithm

Assumption: we know the distribution $\sum_{i=1}^{n} Y_i \sim D_n$.

Example:

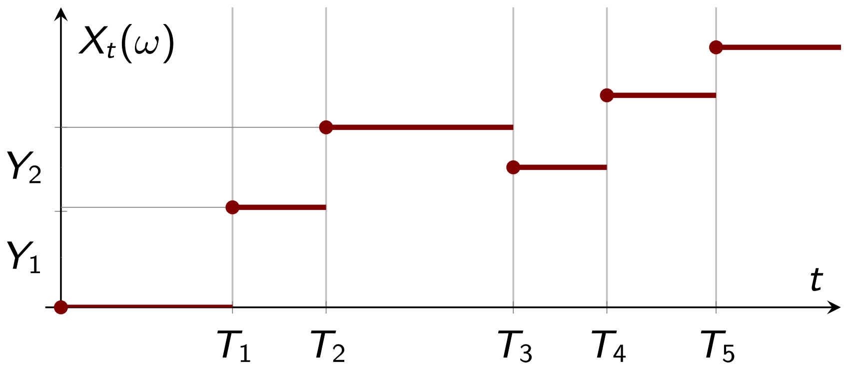
$$\sum_{i=1}^{n} Y_i \sim \mathrm{N}(n\mu, n\sigma^2)$$

cpois4.m

```
function x=cpois4(lambda,mu,sigma,t,m)

n = poissrnd(lambda*t,m,1);
x = n*mu + sigma*sqrt(n).*randn(m,1);
```

Simulating Sample Paths

- Sample paths are described by means of (T_i, X_{T_i}), $i = 1, \ldots, N_T$.



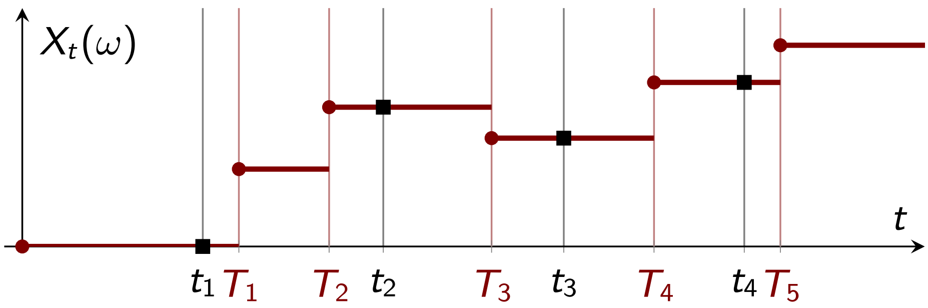
Data Structures:

- Store T_i, Y_i and X_{T_i} in matrices: $m \times nMax$.
  - `X = cumsum(Y,2);`
  - We don't know $nMax$.
- Cell Arrays
- Linearize the data-structure

Sample paths on a grid

$$X_{t_i}(\omega) = \sum_j Y_j(\omega) 1\{t_i \geq T_j(\omega)\}$$

- We may need to evaluate the paths on a grid $(t_1, t_2, \ldots, t_n)$.



- It's an approximation
  - two jump may occur between 2 grid points
  - jump's exact location is lost
- Need to iterate on ω, i, j

Algorithm

cpois5.m

```
function X=cpois5(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n);
for w=1:m                                       % forall (parallel)
    % Simulate the poisson trajectory
    Tmax=0;     T=[];
    while Tmax<t(n)
        Tmax = Tmax + exprnd(lambda);
        T = [T, Tmax];
    end

    % Simulate the comp. poisson
    N = lenth(T);
    y = mu + sigma*randn(1,N);
    for i=1:n
        for j=1:N
            X(w,i) = X(w,i) + y(j) * (t(i) >= T(j) );
        end
    end
end
```

- Vectorize the j loop

cpois6.m

```
function X=cpois6(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n);
for w=1:m                                          % forall
    Tmax=0;     T=[];
    while Tmax<t(n)
        Tmax = Tmax + exprnd(lambda);
        T = [T, Tmax];
    end

    N = lenth(T);
    Y = mu + sigma*randn(1,N);                     % parallel wrt j
    for i=1:n
        X(w,i) = X(w,i) + sum( Y .* (t(i) >= T) );
    end
end
```

- Vectorize the *i* loop

cpois7.m

```
function X=cpois7(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n);

for w=1:m
% forall (parallel)
    Tmax=0;     T=[];
    while Tmax<t(n)
        Tmax = Tmax + exprnd(lambda);
        T = [T, Tmax];
    end

    N = lenth(T);
    y = mu + sigma*randn(1,N);                 % parallel wrt j
    for j=1:N
        X(w,:) = X(w,:) + y(j) * (t >= T(j) );  % parallel wrt i
    end
end
```

- Vectorize the *i* loop and perform some optimisation.

cpois7b.m

```
function X=cpois7b(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n);

for w=1:m                                   % forall
    % Simulate jump times
    T=0;
    while T<t(n)
        T = T +exprnd(lambda);
        if (T>t(n));
            break;
        end
        y = mu + sigma* randn();            % y: scalar
        X(w,:) = X(w,:) + y * (t>=T);       % parallel wrt i
    end
end
```

- Notice that: $X_{t_i} = X_{t_{i-1}} + \sum_j Y_j \times 1\{T_j \in (t_{i-1}, t_i]\}$.

cpois8.m

```
function X=cpois8(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n+1);
t = [0,t];

for w=1:m
  % Simulate jump times
  Tmax=0;     T=[];
  while Tmax<t(n)
    Tmax = Tmax + exprnd(lambda);
    T = [T, Tmax];
  end

  N = lenth(T);
  y = mu + sigma*randn(1,N);
  for i=2:(n+1)
    X(w,i) = X(w,i-1) + sum(y .* (t(i)>=T) .* (t(i-1)<T));
  end
end
```

cpois9.m

```
function X=cpois9(lambda,mu,sigma,t,m)

n=length(t);
X = zeros(m,n);

T = zeros(m,1);
while min(T)<t(n)                                    % for j
    T = T + exprnd(lambda,m,1);
    y = mu + sigma*randn(1,N);
    for i=1:n                                        % forall (parallel)
        X(:,i) = X(:,i) + y .* (t(i)>=T);
    end
end
```

- Truncate the poisson distribution: $X_{t_i} = \sum_{j=1}^{N_{max}} Y_j 1\{t_i \geq T_j\}.$

cpois10.m

```
function X=cpois10(lambda,mu,sigma,t,m,Nmax)

n=length(t);
U = exprnd(lambda,m,Nmax);
T = cumsum(U,2);
Y = mu + sigma*randn(m,Nmax);

X = zeros(m,n);
for i=1:n                                  % forall (parallel)
    for j=1:Nmax
        X(:,i) = X(:,i) + Y(:,j) .* (t(i)>=T(:,j));
    end
end
```

- Truncate the poisson distribution: $X_{t_i} = \sum_{j=1}^{N_{max}} Y_j 1\{t_i \geq T_j\}$.

cpois10b.m

```
function X=cpois10b(lambda,mu,sigma,t,m,Nmax)

n=length(t);
U = exprnd(lambda,m,Nmax);
T = cumsum(U,2);
Y = mu + sigma*randn(m,Nmax);

X = zeros(m,n);
for i=1:n                                % forall (parallel)
    X(:,i) = X(:,i) + sum( Y .* (t(i)>=T), 2 );
end
```

Bermudan Options

Notation

- X_t state variable
- V_t value of the contract at time t
- D_t stoch. discount factor from t to $t+1$
- $t = 1, 2, \ldots, T$ exercise dates

At each exercise date the holder can choose between

- keep the contract: $C_t = \mathrm{E}[D_t V_{t+1} | X_t]$
- exercise and obtain $h_t(X_t)$

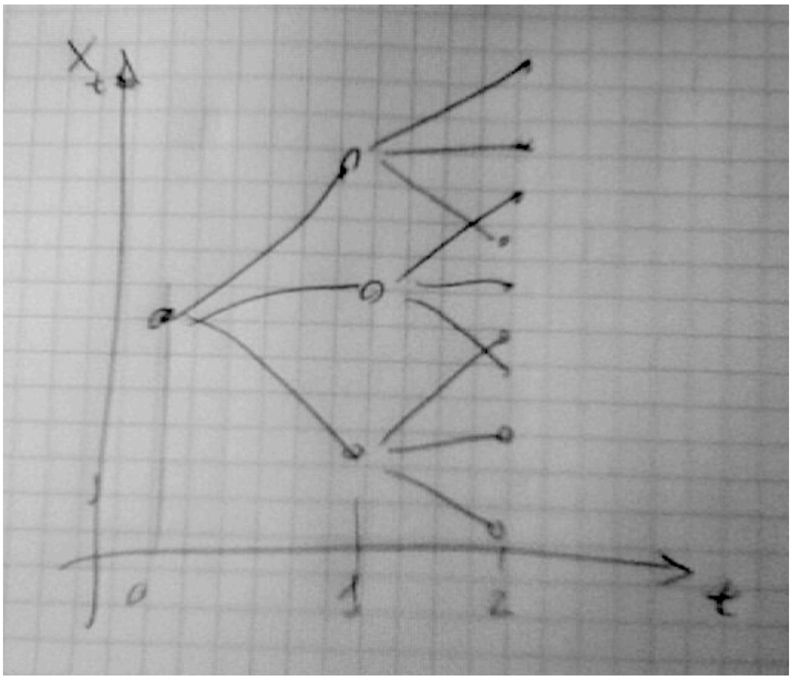
The value of the contract at time t is

$$V_t = \max(h_t(X_t), C_t)$$

Problem: the conditional expectation.

$$V_t = \max\left(h_t(X_t),\ \mathrm{E}[D_t V_{t+1}|X_t]\right)$$

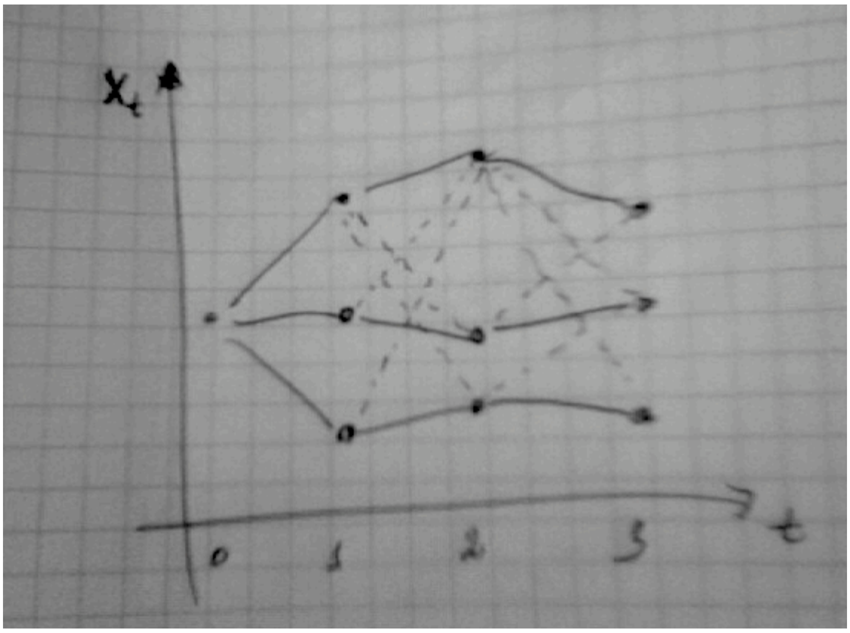
Solution 1: Random Trees



Problem: the conditional expectation.

$$V_t = \max\left(h_t(X_t),\ \mathrm{E}[D_t V_{t+1}|X_t]\right)$$

Solution 2: Stochastic Mesh (recombines)



Problem: the conditional expectation.

$$V_t = \max\left(h_t(X_t),\ \mathrm{E}[D_t V_{t+1} | X_t]\right)$$

Solution 3: Use Regression

Idea

$C_t(x) = \mathrm{E}[D_t V_{t+1} | X_t = x]$ is a function of x.

Approximate C_t as a linear combination of functions of x

Regression Based MC

We want to approximate the function:

$$c(x) = \mathrm{E}[D_t V_{t+1} \,|\, X_t = x]$$

Note that $C = c(X_t) = \mathrm{E}[D_t V_{t+1} \,|\, X_t]$ is a r.v.
Use a set of functions of x to approximate c:

$$Z_1 = \phi_1(X_t), Z_2 = \phi_2(X_t), \ldots, Z_k = \phi_k(X_t)$$

Approximation:

$$C = Z_1\beta_1 + Z_2\beta_2 + \cdots + Z_k\beta_k + U$$

In vector form:

$$C = Z^T\beta + U, \qquad \text{where} \qquad Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_k \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}.$$

Here, U is the error term

$$C = Z^T \beta + U, \qquad C = \mathrm{E}[D_t V_{t+1} | X_t]$$

- Find β that minimize $\mathrm{E}[U^2]$

$$\mathrm{E}[U^2] = \mathrm{E}[C^2] - 2\,\mathrm{E}[CZ^T]\beta + \beta^T\,\mathrm{E}[ZZ^T]\beta$$

$$\frac{\partial\,\mathrm{E}[U^2]}{\partial \beta^T} = -2\,\mathrm{E}[CZ^T] + 2\beta^T\,\mathrm{E}[ZZ^T]$$

- The gradient is null for

$$\beta = \mathrm{E}[ZZ^T]^{-1}\,\mathrm{E}[ZC]$$

- $\mathrm{E}[ZZ^T]$ depends on the basis functions
- $\mathrm{E}[ZC]$ depends also on the continuation value C

$$\mathrm{E}[Z\,C] = \mathrm{E}\big[Z\,\mathrm{E}[D_t V_{t+1} | X_t]\big] = \mathrm{E}[Z\,D_t V_{t+1}]$$

- The conditional expectation is no longer necessary!!

$$C = Z^T\beta + U, \qquad \beta = \mathrm{E}[ZZ^T]^{-1}\,\mathrm{E}[ZC]$$

The error is orthogonal to each basis function

$$\mathrm{E}[ZU] = \mathrm{E}[ZZ^T]\beta - \mathrm{E}[ZC] = 0.$$

When $Z_1 = 1$, the approximation does not introduce a bias.
Indeed, since the first component of $\mathrm{E}[ZU]$ is zero,

$$0 = \mathrm{E}[U] = \mathrm{E}[C] - \mathrm{E}[Z^T\beta].$$

$$C = Z^T\beta + U, \qquad \beta = \mathrm{E}[ZZ^T]^{-1}\,\mathrm{E}[ZD_tV_{t+1}]$$

Need to compute $\mathrm{E}[ZZ^T]$ and $\mathrm{E}[ZC]$
Idea: Use MonteCarlo:

$$\mathrm{E}[ZZ^T] = \frac{1}{M}\sum_{\omega} z_\omega z_\omega^T, \qquad \mathrm{E}[ZD_tV_{t+1}] = \frac{1}{M}\sum_{\omega} z_\omega d_{t,\omega} v_{t+1,\omega}$$

where

- z_ω, $d_{t,\omega}$ and $v_{t+1,\omega}$ are the w-th realizations of the r.v. Z_ω, $D_{t,\omega}$, $V_{t+1,\omega}$

Any conditional expectation have been used

A simple case

- Underlying is S_t and the risk-neutral interest rate is null: $r = 0$
- Choose when to get a payoff $(1 - S_t)^+$: now or at $t = T$
- The contract value is

$$V_T = (1 - S_T)^+ \qquad C_0 = \mathrm{E}[V_T | S_0],$$
$$V_0 = \max((1 - S_0)^+, C_0)$$

- Monte Carlo:

$$V_T^\omega = (1 - S_T^\omega)^+ \qquad \omega = 1, \ldots, m$$
$$\hat{C}_0 = \frac{1}{m} \sum_\omega \hat{V}_T^\omega, \qquad \hat{V}_0 = \max((1 - S_0)^+, \hat{C}_0),$$

- $\hat{C}_0$ is unbiased: $\mathrm{E}[\hat{C}_0] = C_0$
- $\hat{V}_0$ is biased high:

$$\mathrm{E}[\hat{V}_0] = \mathrm{E}[\max((1 - S_0)^+, \hat{C}_0)]$$
$$\geq \max((1 - S_0)^+, \mathrm{E}[\hat{C}_0]) = \max((1 - S_0)^+, C_0) = V_0$$

Regression

- Consider the r.vs. $Y \in \mathbb{R}$ and $X \in \mathbb{R}^k$
- We want to approximate Y by means of $X^T\beta$, with $\beta \in \mathbb{R}^k$ constant

$$Y = X^T\beta + U$$

- The Least Squares approximation $\tilde{\beta}$ minimizes $\mathrm{E}[U^2]$:

$$\mathrm{E}[U^2] = \mathrm{E}[Y^2] + \beta^T \mathrm{E}[XX^T]\beta - 2\beta^T \mathrm{E}[XY]$$

- Thus,

$$\tilde{\beta} = S^{-1}q, \qquad \text{where} \qquad S = \mathrm{E}[XX^T], \qquad q = \mathrm{E}[Xy].$$

- MC: Given $(x_\omega, y_\omega) \sim \mathrm{iid}(X, Y)$, $\omega = 1, \ldots, m$

$$\hat{\beta} = \hat{S}^{-1}\hat{q} \quad \text{where} \quad \hat{S} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega x_\omega^T, \qquad \hat{q} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega y_\omega.$$

- $\hat{S} \to S$, $\hat{q} \to q$ and, thus, $\hat{\beta} \to \tilde{\beta}$

Regression (cont.)

- MC approximation:

$$\hat{\beta} = \hat{S}^{-1}\hat{q} \quad \text{where} \quad \hat{S} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega x_\omega^T, \quad \hat{q} = \frac{1}{m}\sum_{\omega=1}^{m} x_\omega y_\omega.$$

- With abuse of notation define

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{pmatrix} : m \times k \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} : m \times 1$$

- then: $\hat{S} = \frac{1}{m}X^T X$, $\hat{q} = \frac{1}{m}X^T y$ and $\hat{\beta} = (X^T X)^{-1} X^T y$
- that is $\hat{\beta}$ is the OLS estimator of the linear regression

$$y = X\beta + u, \qquad u \sim (0, \sigma^2 I_m).$$

- Recall the original model was $Y = X^T\beta + U$

Regression and Approximation

Problem

Approximate $c(\omega) : \Omega \to \mathbb{R}$ by means of $\phi_i(\omega) : \Omega \to \mathbb{R}$, $i = 1, \dots, k$

$$c(\omega) = \sum_{i=1}^{k} \phi_i(\omega)\beta_i + \varepsilon(\omega)$$

- ϕ_i basis functions, ε error function
- We need a norm to evaluate the error: $\|\varepsilon\|$
- Given a probability measure $P(\omega)$, define the scalar product

$$\langle u, v \rangle = \mathrm{E}[uv] = \int_\Omega u(\omega)v(\omega)dP(\omega)$$

- $\|\varepsilon\| = \langle \varepsilon, \varepsilon \rangle^{1/2} = \mathrm{E}[\varepsilon^2]^{1/2}$ is a semi-norm
- Back to the regression: $Y = X^T\beta + U$,

$$Y(\omega) = c(\omega), \qquad X(\omega) = \begin{pmatrix} \phi_1(\omega) & \phi_2(\omega) & \cdots & \phi_k(\omega) \end{pmatrix}^T.$$

American (Bermudan) options

- Assume a flat market: $r = 0$ (i.e. work with forward prices).
- State vector $S_i = S_{t_i}$, $0 = t_0 \leq t_1 \leq \cdots \leq t_n$.
- $h_i(s)$ payoff at the ith step (time t_i)
- $V_i(s)$ contract value at the ith step

Dynamic programming

$$V_n(s) = h_n(s)$$

$$V_i(s) = \max\Big(h_i(s),\ \mathrm{E}[V_{i+1}(S_{i+1})|S_i = s] \Big), \quad i = n-1, n-2, \cdots, 1.$$

- $C_i(s) = \mathrm{E}[V_{i+1}(S_{i+1})|S_i = s]$: continuation value

American Options (cont.)

- Let define the r.vs. $V_i = V_i(S_i)$, $C_i = C_i(S_i)$ and $h_i = h_i(S_i)$, thus

Dynamic programming

$$V_n = h_n, \qquad C_i = \mathrm{E}[V_{i+1}|S_i], \qquad \text{and} \qquad V_i = \max(h_i, C_i(S_i))$$

Regression Based Methods

Linearly approximate C_i by means of the regressors

$$X_i^T = \begin{pmatrix} \phi_1(S_i) & \dots & \phi_k(S_i) \end{pmatrix}$$

That is, find a β_i such that $C_i \simeq X_i^T \beta_i$.

- $\phi_1, \dots, \phi_k$ are a set of properly chosen "basis functions"
- Note: $X_i = \begin{pmatrix} X_{i1} \cdots X_{ik} \end{pmatrix}$ is adapted to the filtration (This is what we really need)

Regression based methods

- Recall $C_i = \mathrm{E}[V_{i+1}|S_i]$
- Regression problem: $C_i = X_i^T \beta_i + U$
- Least Squares: $\mathrm{E}[C_i X_i]$ is needed
- The conditional expectation disappears, indeed

$$\mathrm{E}[C_i X_i] = \mathrm{E}[X_i\, \mathrm{E}[V_{i+1}|S_i]] = \mathrm{E}[X_i V_{i+1}]$$

- Suppose that MC provides a set of scenarios S_i^ω, V_{i+1}^ω, $\omega = 1, \ldots, m$
- Set:

$$v_{i+1} = \begin{pmatrix} V_{i+1}^1 \\ V_{i+1}^2 \\ \vdots \\ V_{i+1}^m \end{pmatrix} \qquad Z_i = \begin{pmatrix} \phi_1(S_i^1) & \phi_2(S_i^1) & \cdots & \phi_k(S_i^1) \\ \phi_1(S_i^2) & \phi_2(S_i^2) & \cdots & \phi_k(S_i^2) \\ \vdots & \vdots & & \vdots \\ \phi_1(S_i^m) & \phi_2(S_i^m) & \cdots & \phi_k(S_i^m) \end{pmatrix}$$

- Compute the OLS $\hat{\beta}_i$ of $v_{i+1} = Z_i \beta_i + u_i$ then $\hat{\beta}_i \to \beta_i$

Algorithm

- Compute the scenarios (forward)

$$S_i^\omega, \qquad i = 1, \dots, n, \omega = 1, \dots, m.$$

- Compute the payoff at T:

$$V_n^\omega = h_n(S_n^\omega), \qquad \omega = 1, \dots, m$$

- Compute the contract value going backward ($i = n-1, n-2, \dots, 1$):
  - Compute Z_i and form v_{i+1}
  - Compute $\hat{\beta}_i = (Z_i^T Z_i)^{-1} Z_i^T v_{i+1}$
  - Compute $V_i^\omega = \max(h_i(S_i^\omega), \beta_i^T X_i^\omega)$

Algorithm (vector-matrix form)

Data structures (each row a scenario):

- $\mathbf{s}_i$: $m \times 1$, all the scenarios at step i
- $\mathbf{v}_i$: $m \times 1$, the value of the contract at step i at all the states
- $\mathbf{h}_i = h_i(\mathbf{s}_i)$: $m \times 1$, exercise values
- $\mathbf{Z}_i = \left(\phi_k(S_i^\omega)\right)_{\omega,k}$: $m \times k$ regressor matrix

Algorithm:

- Compute the scenarios (forward): $\mathbf{s}_{i+1} = \mathbf{s}_i + \dots$
- Compute the payoff at T: $\mathbf{v}_n = \mathbf{h}_n$
- Going Backward:
- Compute $\mathbf{Z}_i$
- Least Squares Approximation: $\hat{\beta}_i = (\mathbf{Z}_i^T \mathbf{Z}_i)^{-1} \mathbf{Z}_i^T \mathbf{v}_{i+1}$
- Compute $\mathbf{v}_i = \max(\mathbf{h}_i, \mathbf{Z}_i \beta_i)$

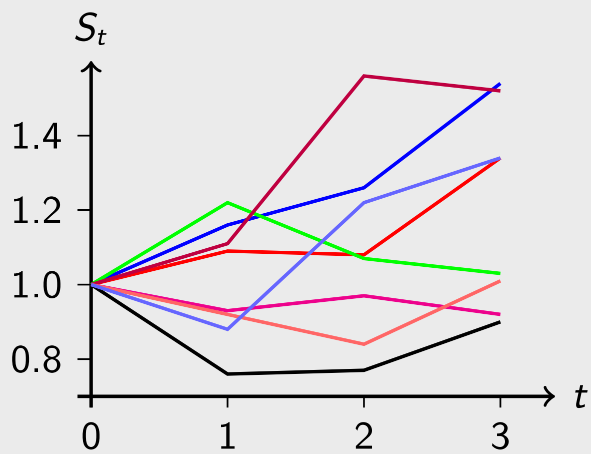
Longstaff & Schwartz Monte Carlo method

Example (Longstaff and Schwartz 2001)

Consider an American Put Option on a non-dividend-paying stock S_t with strike $K = 1.10$. The option is exercisable at times 1,2 and 3. The riskless rate is $r = 6\%$.

Suppose to approximate (by MC) the dynamics with the 8 paths:

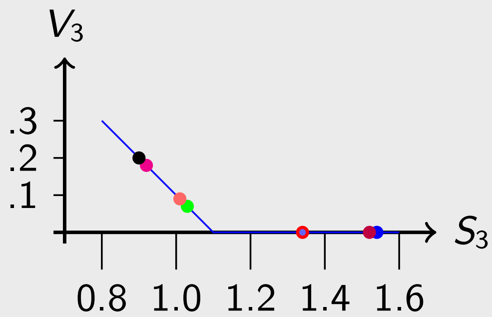
ω	S_0	S_1	S_2	S_3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34



Example (cont.)

At time $t = 3$ the value of the contract is (= European): $V_3 = (K - S_3)^+$

ω	S_0	S_1	S_2	S_3	V_3
1	1.00	1.09	1.08	1.34	0
2	1.00	1.16	1.26	1.54	0
3	1.00	1.22	1.07	1.03	0.07
4	1.00	0.93	0.97	0.92	0.18
5	1.00	1.11	1.56	1.52	0
6	1.00	0.76	0.77	0.90	0.20
7	1.00	0.92	0.84	1.01	0.09
8	1.00	0.88	1.22	1.34	0



- At time $t = 2$: $V_2 = \max((K - S_2)^+, \mathrm{E}^Q[e^{-r} V_3 | S_2])$
- The problem is how to compute $\mathrm{E}^Q[e^{-r} V_3 | S_2]$: continuing to go forward, for each S_2 a new MC is needed
- LSM: $\mathrm{E}^Q[e^{-r} V_3 | S_2] = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k$, with $X_i = f_i(S_2)$ Compute by regressing $e^{-r} V_3$ on $X_1, X_2, \ldots, X_k$

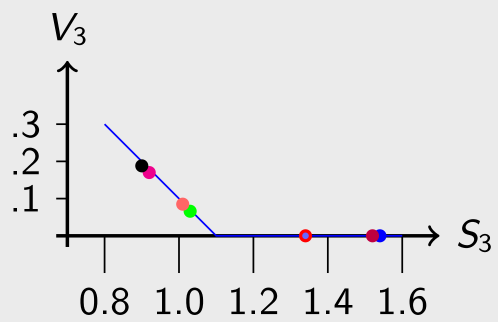
Example (cont.)

At time $t = 2$.
Estimate $\beta_1, \ldots, \beta_k$ in the model

$$\begin{array}{llcccccc} E^Q[e^{-r}V_3 & |S_2] & = & \beta_1 & + & S_2\beta_2 & + & S_2^2\beta_3 \\ E^Q[Y & |X] & = & X_1\beta_1 & + & X_2\beta_2 & + & X_3\beta_3 \end{array}$$

ω	Y	X_1	X_2	X_3
1	0	1	1.08	1.166
2	0	1	1.26	1.588
3	0.066	1	1.07	1.145
4	0.170	1	0.97	0.941
5	0	1	1.56	2.434
6	0.188	1	0.77	0.593
7	0.085	1	0.84	0.706
8	0	1	1.22	1.489

$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
0.821	-1.138	0.389



Fourier transforms and option pricing

European pricing formula is a convolution

European Call:

$$C_T(k) = e^{-rT} \int_{\mathbb{R}} (e^x - e^k)^+ q_T(x) dx = e^{k-rT} \int_{\mathbb{R}} \phi(x-k) q_T(x) dx$$

where $x = \log(S_T)$, $k = \log(K)$, $\phi(m) = (e^m - 1)^+$, q_T risk-neutral pdf

- Often only the characteristic function (cf) ϕ_T is known
- A convolution becomes a product in the Fourier space

$$c(y) = \int_{\mathbb{R}} a(x-y) b(x) dx \qquad \overset{\mathcal{F}}{\Leftrightarrow} \qquad \hat{c}(\omega) = \hat{a}(\omega)\hat{b}(\omega)$$

- Problem: the Fourier transform of common payoffs does not exist

Definition (Fourier Transform)

$$\hat{g} = (\mathcal{F}g)(\omega) = \int_{\mathbb{R}} e^{i\omega x} g(x)dx \qquad g = (\mathcal{F}^{-1}\hat{g})(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega x}\hat{g}(\omega)d\omega$$

- Linearity
- Parseval: $\int_{\mathbb{R}} g(x)\overline{h(x)}dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\omega)\overline{\hat{h}(\omega)}d\omega$
- Convolution: $\mathcal{F}(g * h) = (\mathcal{F}g)(\mathcal{F}h)$, $\quad (g * h)(x) = \int_{\mathbb{R}} g(s)h(x-s)ds$
- Scaling: $(\mathcal{F}g(ax))(\omega) = |a|^{-1}(\mathcal{F}g)(a^{-1}w)$
  - g even $\Leftrightarrow$ $\hat{g}$ even
- Conjugation: $(\mathcal{F}\overline{g})(\omega) = \overline{(\mathcal{F}g)(-\omega)} = \overline{\hat{g}(-\omega)}$
  - g real $\Leftrightarrow$ $\mathrm{Re}(\hat{g})$ even, $\mathrm{Im}(\hat{g})$ odd
  - g real and even $\Leftrightarrow$ $\hat{g}$ real and even

Fourier Transform and Characteristic Functions

Definition (Characteristic Function)

Let X be an $\mathbb{R}$-valued r.v. with pdf $p(x)$, its characteristic function is

$$\phi_X(\omega) = \mathrm{E}[e^{i\omega X}] = \int_{\mathbb{R}} e^{i\omega x} p(x) dx, \qquad \omega \in \mathbb{R}.$$

- Characteristic Function = FT of the pdf: $\phi_X = \mathcal{F}p$
- The expected value of $g(X)$ can be computed as

$$\mathrm{E}[g(X)] = \int_{\mathbb{R}} g(x)p(x)dx = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(-\omega)\phi_X(\omega)d\omega$$

- ♦ g should be square integrable

Option Pricing in Fourier Space

problem

Given ϕ_T the CF of X, compute the price of a Call [a]

$$C_T(k) = \mathrm{E}[(e^x - e^k)^+] = \int_{\mathbb{R}} (e^x - e^k)^+ q_T(x) dx$$

[a] assume for simplicity $r = 0$

Problem: C_T is not L^2, it does not have a FT

Solutions:

- Exponential Damping ([**?**])
  - Numerically unstable when far from the money or near to expiration
- Time Value ([**?**])
- Other approaches: Chen and Scott
- In the following we Follow [**?**]

Exponential Damping

$$C_T(k) = \int_{\mathbb{R}} (e^x - e^k)^+ q_T(x) dx$$

- $C_T(k) \notin L^2$, the FT is not defined
- Damping: consider the FT of $c_T(k) = e^{\alpha k} C_T(k)$

$$\psi_T(\omega) = \int_{\mathbb{R}} e^{i\omega k} c_T(k) dk$$

- so that

$$C_T(k) = e^{-\alpha k} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega k} \psi_T(\omega) d\omega$$

- Let consider $\psi_T(\omega)$

$$
\begin{aligned}
\psi_T(\omega) &= \int_{-\infty}^{+\infty} e^{i\omega k} c_T(k) dk \\
&= \int_{-\infty}^{+\infty} e^{i\omega k} \int_{-\infty}^{+\infty} e^{\alpha k}(e^x - e^k)^+ q_T(x) dx\, dk \\
&= \int_{-\infty}^{+\infty} e^{i\omega k} \int_{k}^{+\infty} (e^{x+\alpha k} - e^{(1+\alpha)k})^+ q_T(x) dx\, dk \\
&= \int_{-\infty}^{+\infty} q_T(x) \int_{-\infty}^{x} e^{i\omega k}(e^{x+\alpha k} - e^{(1+\alpha)k})^+ dk\, dx \\
&= \int_{-\infty}^{+\infty} q_T(x) \left(\frac{e^{(\alpha+1+i\omega)x}}{\alpha + i\omega} - \frac{e^{(\alpha+1+i\omega)x}}{\alpha + 1 + i\omega} \right) dx \\
&= \frac{1}{(\alpha + i\omega)(\alpha + 1 + i\omega)} \int_{-\infty}^{+\infty} q_T(x) e^{(\alpha+1+i\omega)x} dx \\
&= \frac{1}{(\alpha + i\omega)(\alpha + 1 + i\omega)} \phi_T(\omega - i(\alpha + 1))
\end{aligned}
$$

Choosing the dumping factor α

$$\psi_T(\omega) = \frac{1}{(\alpha + i\omega)(\alpha + 1 + i\omega)}\phi_T(\omega - i(\alpha + 1))$$

- $\psi_T(\omega)$ is the FT of $c_T(k)$
- Sufficient condition for $c_T(k)$ to be L^2 is: $\psi_T(0) < \infty$

$$\psi_T(0) = \phi_T(-i(\alpha + 1)) = \mathrm{E}[\, exp(i \overbrace{(-i)(\alpha + 1)}^{\omega} X)\,] = \mathrm{E}[S_T^{\alpha+1}]$$

- Thus, it is sufficient that $\mathrm{E}[S_T^{\alpha+1}] < \infty$
- Choose α to a quarter of the value that guarantee the bound
- Furthermore, the domain needs to be truncated

$$C_T(k) = e^{-\alpha k}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega k}\psi_T(\omega)d\omega$$

more on Carr and Madan (1999).

Time Value Approach

- Assume for simplicity: $r = 0$ and $S_0 = 0$
- Let $C_T = \mathrm{E}[(e^x - e^k)^+]$, $P_T = \mathrm{E}[(e^k - e^x)^+]$ and

$$
\begin{aligned}
z_T(k) &= P_T(k)\chi_{\{k<0\}} + C_T(k)\chi_{\{k\geq 0\}} \\
&= \chi_{\{k<0\}} \int_{-\infty}^{k} (e^k - e^x) q_T(x) dx + \chi_{\{k\geq 0\}} \int_{k}^{\infty} (e^x - e^k) q_T(x) dx
\end{aligned}
$$

- Note that: $z_T = C_T(k) - (e^x - e^k)^+$ is the "time value" of the Call
- Consider the FT of z_T:

$$
\zeta_T(\omega) = \int_{\mathbb{R}} e^{i\omega k} z_T(k) dk, \qquad z_T(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega k} \zeta_T(\omega) d\omega
$$

$$z_T(k) = P_T(k)\chi_{\{k<0\}} + C_T(k)\chi_{\{k\geq 0\}}$$

- Let consider the FT of the first term:

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\omega k} P_T(k)\chi_{\{k<0\}} dk &= \int_{-\infty}^{0} e^{i\omega k} \int_{-\infty}^{k} (e^k - e^x) q_T(x) dx\, dk \\
&= \int_{-\infty}^{0} q_T(x) \int_{x}^{0} (e^{(1+i\omega)k} - e^{x+i\omega k}) dk\, dx \\
&= \int_{-\infty}^{0} q_T(x) \int_{x}^{0} (e^{(1+i\omega)k} - e^{x+i\omega k}) dk\, dx \\
&= \int_{-\infty}^{0} q_T(x) \left(\frac{1}{1+i\omega} - \frac{e^x}{i\omega} + \frac{e^{(1+i\omega)x}}{\omega^2 - i\omega} \right) dx
\end{aligned}$$

analogously

$$\int_{\mathbb{R}} e^{i\omega k} P_T(k)\chi_{\{k<0\}} dk = \int_{0}^{+\infty} q_T(x) \left(\frac{1}{1+i\omega} - \frac{e^x}{i\omega} + \frac{e^{(1+i\omega)x}}{\omega^2 - i\omega} \right) dx$$

- Thus:

$$\zeta_T(\omega) = \int_{-\infty}^{+\infty} q_T(x) \left(\frac{1}{1+i\omega} - \frac{e^x}{i\omega} + \frac{e^{(1+i\omega)x}}{\omega^2 - i\omega} \right) dx$$

since $\mathrm{E}[S_T] = \mathrm{E}[e^x] = 1$

$$\zeta_T(\omega) = \frac{1}{1+i\omega} - \frac{1}{i\omega} + \frac{\phi_T(\omega - i)}{\omega^2 - i\omega} = \frac{1}{\omega^2 - i\omega}(1 + \phi_T(\omega - i))$$

- so that

$$z_T(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega k} \zeta_T(\omega) d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 + \phi_T(\omega - i)}{\omega^2 - i\omega} e^{-i\omega k} d\omega$$

- z_T computed by the FFT
- Unstable at the money (near $k = 0$) when $T \to 0$

Fixing instabilities in Time-Value approach

- $z_T(k)$ unstable for $k \simeq 0$
- Consider the FT of $\sinh(\alpha k) z_T(k)$ ($\sinh(x)$ is null at $x = 0$)

$$
\begin{aligned}
\gamma_T(\omega) &= \int_{\mathbb{R}} e^{i\omega k} \sinh(\alpha k) z_T(k) dk \\
&= \int_{\mathbb{R}} e^{i\omega k} \frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k) dk \\
&= \frac{1}{2} \int_{\mathbb{R}} e^{(i\omega+\alpha)k} z_T(k) dk - \frac{1}{2} \int_{\mathbb{R}} e^{(i\omega-\alpha)k} z_T(k) dk \\
&= \frac{\zeta_T(\omega - i\alpha) - \zeta_T(\omega + i\alpha)}{2}
\end{aligned}
$$

- Then $z_T(k)$ is computed as

$$
z_T(k) = \frac{1}{\sinh(\alpha k)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega k} \gamma_T(\omega) d\omega
$$

Discrete Fourier Transforms

- Consider the sequence u_j, $j = 1, \ldots, N$
- The Discrete Fourier Transform (DFT) of u is the sequence

$$w_s = \sum_{j=1}^{N} e^{-ih(j-1)(s-1)} u_j \qquad\qquad s = 1, \ldots, N$$

  where $h = 2\pi/N$
- Briefly: $w = DFT(u)$
- Typically $N = 2^n$
- FFT computes the whole sequence w in $O(N \log N)$

From FTs to DFTs

- Let consider the inverse FT:

$$g(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega k} \psi(\omega) d\omega$$

- Assuming $g(k) \in \mathbb{R}$ implies $\mathrm{Re}(\psi(\omega))$ even and $\mathrm{Im}(\psi(\omega))$ odd:

$$g(k) = \frac{1}{\pi} \int_0^{\infty} e^{-i\omega k} \psi(\omega) d\omega \simeq \frac{1}{\pi} \int_0^{a} e^{-i\omega k} \psi(\omega) d\omega$$
$$\simeq \frac{1}{\pi} \sum_{j=1}^{N} e^{-i\omega_j k} \psi(\omega_j) \eta$$

  where $\eta = a/N$ and $\omega_j = \eta(j-1)$
- Set $k = -\frac{N}{2}\lambda + \lambda(s-1)$

$$g(k) \simeq g_s = \frac{1}{\pi} \sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(s-1)} e^{i\frac{N}{2}\lambda\eta(j-1)} \psi(\omega_j) \eta$$

- Finally:

$$g(k) \simeq g_s = \frac{1}{\pi}\sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(s-1)}\, e^{i\frac{N}{2}\lambda\eta(j-1)}\psi(\omega_j)\eta$$
$$= \sum_{j=1}^{N} e^{-ih(j-1)(s-1)}\,\frac{\eta}{\pi}\, e^{i\pi(j-1)}\psi(\omega_j)$$

when $\eta\lambda = \frac{2\pi}{N} = h$

- thus

$$g = DFT(u), \quad \text{where} \quad u_j = \frac{\eta}{\pi}e^{i\pi(j-1)}\psi(\omega_j) = \frac{\eta}{\pi}(-1)^{j-1}\psi(\omega_j).$$

- Notice:

$$\begin{array}{llcll} k & \in [-\frac{N}{2}\lambda, (\frac{N}{2}-1)\lambda] & = & [-\frac{\pi}{\eta}, \frac{\pi}{\eta}(1-\frac{2}{N})] & \text{step } \lambda \\ w_j & \in [0, \frac{2\pi}{\lambda}(1-\frac{1}{N})] & = & [0, (N-1)\eta] & \text{step } \eta \end{array}$$

Exercise

Compute the price of a Call under the CGMY or tempered Lèvy model. The CF of log-returns in the CGMY model is given by

$$\frac{1}{T}\log\phi_T(\omega) = -C\,\Gamma(-Y)\Big((M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y\Big) - i\omega\kappa$$

$$\kappa = C\Big(M\big(\Gamma(-Y)Y+\Gamma(1-Y,M)\big) - G\big(\Gamma(-Y)Y+\Gamma(1-Y,G)\big)\Big),$$

where $\Gamma(a,b)=\int_b^\infty x^{a-1}e^{-x}dx$ *and* $\Gamma(a)=\Gamma(a,0)$
Implement it as the matlab function

```
function Call = cgmy_fft(S0,K, C,G,M,Y, T, N)
```

Characteristic exponents for Lèvy models

Model	Characteristic Exponent ($\frac{1}{T} \log \phi_T(\omega)$)
GBM	$i(\mu - \frac{1}{2}\sigma^2)\omega - \frac{1}{2}\sigma^2\omega^2$
Merton JD	$i(\mu - \frac{1}{2}\sigma^2)\omega - \frac{1}{2}\sigma^2\omega^2 + \lambda(e^{i\bar{\mu}\omega - \bar{\sigma}^2\omega^2/2} - 1)$
Kou JD	$i(\mu - \frac{1}{2}\sigma^2)\omega - \frac{1}{2}\sigma^2\omega^2 + i\omega\lambda(\frac{p}{\eta_+ - i\omega} - \frac{1-p}{\eta_- - i\omega})$
VG	$\kappa^{-1}\log(1 - i\mu\kappa\omega + \frac{1}{2}\sigma^2\kappa\omega^2)$
NIG	$\kappa^{-1}(1 - \sqrt{1 - 2i\mu\kappa\omega + \sigma^2\kappa\omega^2})$

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