Computational Finance and its implementation in Python with applications to option pricing, Green finance and Climate risk

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Main contents

Motivation

- A common problem we face in mathematical finance is the risk neutral valuation of a derivative.
- As you know, the price of a derivative is expressed by the (possibly discounted) expectation of its payoff at maturity, under a pricing measure (also called risk neutral, or martingale measure).
- That is, we have to compute the expectation of a random variable.
- Problem: most often, there is no way to get an analytic formula for the expectation of complex derivates, or even simpler derivatives written on an underlying with non trivial dynamics.
- Broad idea: we can approximate the price by averaging some possible, simulated realizations of the payoff.
- The strong law of large numbers and some other convergence results may help us.

A bit more precisely..

- Consider a random variable $X : \Omega \to \mathbb{R}^N$ defined on a probability space $(\Omega, \mathcal{F}, P)$. The probability measure P may be viewed as a risk neutral measure.
- Also consider a (payoff) function $f : \mathbb{R}^N \to \mathbb{R}$ such that $\mathbb{E}^P[(f(X))^2] < \infty$.
- The aim is to compute the expectation

$$\mu := \mathbb{E}^P[f(X)] = \int_\Omega f(X) dP.$$

- Suppose there is no analytic formula to derive μ above. We have to find an approximation $\hat{\mu}$.

We can define independent drawings of X

- Given $X : \Omega \to \mathbb{R}$ and $(\Omega, \mathcal{F}, P)$ as above, introduce:

$$\tilde{\Omega} := \Omega \times \Omega \times \cdot\cdot \times \Omega = \{\tilde{\omega} = (\omega_1, \dots, \omega_n), \quad \omega_i \in \Omega\},$$
$$\tilde{\mathcal{F}} := \sigma(\mathcal{F} \times \mathcal{F} \times \cdot\cdot \times \mathcal{F}),$$
$$\tilde{P}\left(\prod_{i=1}^{n} A_i\right) := \prod_{i=1}^{n} P(A_i), \quad A_i \in \mathcal{F}.$$

- Also define the random variable $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_n)$ by $\tilde{X}_i(\tilde{\omega}) := X(\omega_i)$.
- Note that $\tilde{X}(\tilde{\omega})$ can be seen as n different realizations $X(\omega_i)$, $i = 1, \dots, n$ of one random variable X, or as one realization of n i.i.d. random variables $\tilde{X}_i(\tilde{\omega})$, $i = 1, \dots, n$.
- This interpretation is at the base of the Monte-Carlo method, as it permits to exploit the Strong Law of Large Numbers.
- A similar construction and interpretation can be given for a N-dimensional random variable X.

Convergence results for sequences of i.i.d. random variables

Theorem: Strong Law of Large Numbers

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. integrable real valued random variables on $(\Omega, \mathcal{F}, P)$, and set

$$\mu := \mathbb{E}^P[X_i], \quad i \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu \quad P-a.s.$$

Theorem: Tschebyscheff Inequality

Let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. square integrable real valued random variables on $(\Omega, \mathcal{F}, P)$, and set

$$\mu := \mathbb{E}^P[X_i], \quad \sigma^2 := \mathbb{E}^P\left[(X_i-\mu)^2\right], \qquad i \in \mathbb{N}.$$

Then for any $\epsilon, \delta > 0$ and any $n \in \mathbb{N}$ we have

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2 n}$$

and

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \frac{\sigma}{\delta^{1/2} n^{1/2}}\right) \leq \delta.$$

Application to Monte-Carlo

Lemma

Let $(X_i)_{i\in\mathbb{N}}$ be a collection of i.i.d. integrable random variables on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^N$, and let $f : \mathbb{R}^N \to \mathbb{R}$. Then the random variables $(f(X_i))_{i\in\mathbb{N}}$ are also i.i.d.

- The lemma above, together with the convergence results of the previous slide, allows us to approximate

$$\mu := \mathbb{E}^P[f(X)] = \int_\Omega f(X)dP$$

by

$$\hat{\mu} := \frac{1}{n}\sum_{i=1}^{n} f(X_i),$$

where $(X_i)_{i=1,\dots,n}$ are independent realizations of X.

- We can generate numerically n realizations of a random variable X with a given distribution P^X, starting from a sequence of (pseudo!) random numbers.
- One must give a *seed*, i.e., a starting point for the pseudo-random numbers sequence.
- The realizations will not be purely random, and not purely independent.

Pro and cons of Monte-Carlo

- Pro:
  - It is very simple to understand and easy to implement.
  - The accuracy does not depend on the domain dimension (i.e., if we simulate N-dimensional random variables the accuracy is the same).
  - The accuracy can be increased by just adding more valuations without loosing the previous estimates.
  - The function f does not need to be continuous, but only square integrable.
- Cons:
  - Look at the Tschebyscheff Inequality: we only have a probabilistic bound. The worst case error is ∞.
  - The estimates depend on the generated random sequence. The sequence is not purely random. First, one has to find a good random number generator.
- There are techniques that can be used to increase the accuracy. In the next slides we will see few of them.

Remark

If X has a cumulative distribution function F which is easy to invert, a realization x_i can be conveniently generated as $x_i = F^{-1}(u_i)$, with u_i realization of $U \sim U((0,1))$.
Therefore, approximating $\mathbb{E}^P[f(X)]$ reduces to approximate

$$\int_0^1 G(x)dx, \tag{1}$$

for $G = f \circ F^{-1}$.

Theorem: Koksma-Hlawka inequality

If G has bounded total variation on $(0,1)$, then for any points $x_1, \dots, x_n \in (0,1)$ it holds

$$\left| \frac{1}{n} \sum_{i=1}^{n} G(x_i) - \int_0^1 G(x)dx \right| \leq V(G) D^*(x_1, \dots, x_n),$$

where

$$V(G) = \sup_S \sum_i |G(y_{i+1}) - G(y_i)|$$

over all partitions $S := \{0 = y_1 < y_2 < \dots < y_n = 1\}$ and $D^*(x_1, \dots, x_n)$ is the star discrepancy

$$D^*(x_1, \dots, x_n) = \sup_{b \in (0,1)} \left| \frac{|\#\{x_i : 0 \leq x_i \leq b\}|}{n} - b \right|.$$

Low-discrepancy sequences

- The result in the previous slide also holds for higher dimensions (here we just wanted to simplify the notation).
- It gives the motivation to look for low discrepancy sequences.
- Most well known low discrepancy sequences: Van der Corput, Halton, Sobol, Hammersley, Sobol, Niederreiter.
- Here we don't focus on Low discrepancy sequences. A bit of references if you want to go deeper on this:
  - J. Dick and F. Pillichshammer, *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration*, Cambridge University Press, Cambridge, 2010
  - M. Drmota and R. F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Math., 1651, Springer, Berlin, 1997.
  - L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Dover Publications, 2005.
  - ...the course *Numerical Methods for Financial Mathematics* at our master!
- We focus instead on variance reduction techniques.

Motivation - 1

- Consider a random variable $X : \Omega \to \mathbb{R}^N$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and a (payoff) function $f : \mathbb{R}^N \to \mathbb{R}$ such that $\mathbb{E}^P[(f(X))^2] < \infty$.
- Monte-Carlo method: choosing $n \in \mathbb{N}$ large enough, we approximate

$$\hat{\mu} := \frac{1}{n}\sum_{i=1}^{n} f(X_i) \approx \mu := \mathbb{E}^P[f(X)],$$

  where $(X_i)_{i=1,\dots,n}$ are realizations of X, i.e., have same distribution as X.
- The estimator is of course *unbiased*, i.e.,

$$\mathbb{E}^P\left[\hat{\mu}\right] = \mathbb{E}^P\left[\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right] = \mathbb{E}^P\left[f(X)\right] =: \mu$$

- We are interested in the variance of our estimator, i.e., in the quantity

$$\text{Var}\left(\hat{\mu}\right) = \mathbb{E}^P\left[\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mu\right)^2\right].$$

Motivation - 2

- We have seen that if $(X_i)_{i=1,\dots,n}$ are independent, we have convergence results for our estimator. Moreover,

$$\text{Var}(\hat{\mu}) = \mathbb{E}^P\left[\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mu\right)^2\right] = \frac{1}{n}\text{Var}[f(X)].$$

- It makes sense: the larger the number n of simulated realizations of X, the smaller the variance of our estimator.
- In particular, we have to increase the number of simulations by a factor of C to reduce the standard deviation by a factor of $\sqrt{C}$.
- The question now is: can we do it better?
- Variance reduction techniques aim to reduce the variance of our estimator, without increasing the number of simulations.

Three well known variance reduction techniques are:

- Antithetic variables
- Control variates
- Importance sampling

We will focus mostly on the first two techniques, together with applied examples. Here some references if you want to deepen Importance sampling:

- A, Bouhari. *Adaptative Monte Carlo Method, A Variance Reduction Technique*. Monte Carlo Methods and Their Applications. 10 (1): 1-24, 2004.
- P. J. Smith, M. Shafi, H. Gao. *Quick simulation: A review of importance sampling techniques in communication systems*. IEEE Journal on Selected Areas in Communications. 15 (4): 597-613, 1997.
- Again, the course *Numerical Methods for Financial Mathematics* at our master!

Let's start from a simple result..

Lemma

Let $f, h : \mathbb{R} \to \mathbb{R}$ be two monotone functions, both increasing or both decreasing, and let $X : \Omega \to \mathbb{R}$ be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Then

$$\mathbb{E}^P[f(X)h(X)] \geq \mathbb{E}^P[f(X)]\mathbb{E}^P[h(X)].$$

Proof

The monotonicity assumption on f and h implies that for any $x, y \in \mathbb{R}$ we have

$$(f(x) - f(y))\,(h(x) - h(y)) \geq 0.$$

Therefore, for any i.i.d. real valued random variables X and Y on $(\Omega, \mathcal{F}, P)$ it holds

$$(f(X) - f(Y))\,(h(X) - h(Y)) \geq 0$$

and then

$$\mathbb{E}^P\left[(f(X) - f(Y))\,(h(X) - h(Y))\right] \geq 0,$$

so that

$$\mathbb{E}^P\left[f(X)h(X)\right] + \mathbb{E}^P\left[f(Y)h(Y)\right] \geq \mathbb{E}^P\left[f(Y)h(X)\right] + \mathbb{E}^P\left[f(X)h(Y)\right].$$

Since X and Y are identically distributed, it follows that

$$2\mathbb{E}^P\left[f(X)h(X)\right] \geq 2\mathbb{E}^P\left[f(Y)h(X)\right],$$

and since they are also independent, this implies that

$$\mathbb{E}^P[f(X)h(X)] \geq \mathbb{E}^P[f(X)]\mathbb{E}^P[h(X)].$$

An interesting consequence

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and $X : \Omega \to \mathbb{R}$ a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Fix $a \in \mathbb{R}$. Then

$$\text{Cov}[f(X), f(a-2X)] \leq 0.$$

Proof

We have that

$$\text{Cov}[f(X), f(a-2X)] = \mathbb{E}^P[f(X)f(a-2X)] - \mathbb{E}^P[f(X)]\mathbb{E}^P[f(a-2X)].$$

The result then follows since a direct application of the Lemma of the previous slide with $h(x) := -f(a-2x)$ implies that

$$\mathbb{E}^P[f(X)]\mathbb{E}^P[f(a-2X)] \geq \mathbb{E}^P[f(X)f(a-2X)].$$

Application to Monte-Carlo

- Let $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and let $X : \Omega \to \mathbb{R}$ be a symmetric random variable around a defined on a probability space $(\Omega, \mathcal{F}, P)$.
- From the last proposition we know that
$$\text{Cov}[f(X), f(a - 2X)] \leq 0.$$
- Idea: choose n even and generate $n/2$ realizations of X, call them $(X_i)_{i=1,\dots,n/2}$. Consider then the estimator
$$\hat{\mu} := \frac{1}{n}\left(\sum_{i=1}^{n/2} f(X_i) + \sum_{i=1}^{n/2} f(2a - X_i)\right)$$
- Since X is symmetric, the estimator is unbiased:
$$\mathbb{E}^P[\hat{\mu}] = \frac{1}{n}\left(\sum_{i=1}^{n/2}\mathbb{E}^P[f(X_i)] + \sum_{i=1}^{n/2}\mathbb{E}^P[f(2a - X_i)]\right) = \frac{1}{n}\left(\sum_{i=1}^{n/2}\mathbb{E}^P[f(X_i)] + \sum_{i=1}^{n/2}\mathbb{E}^P[f(X_i)]\right) = \mu.$$
- What about the variance?
$$\begin{aligned}\text{Var}[\hat{\mu}] &= \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n/2} f(X_i) + \sum_{i=1}^{n/2} f(2a - X_i)\right]\\ &= \frac{1}{n^2}\left(n\text{Var}[f(X)] + \text{Cov}\left(\sum_{i=1}^{n/2} f(X_i), \sum_{i=1}^{n/2} f(2a - X_i)\right)\right)\\ &= \frac{1}{n}\text{Var}[f(X)] + \frac{1}{2n}\text{Cov}[f(X), f(2a - X)] \leq \frac{1}{n}\text{Var}[f(X)].\end{aligned}$$

Application to Monte-Carlo

- To recap: if X is symmetric around $a \in \mathbb{R}$, then $\hat{\mu}$ defined as

$$\hat{\mu} := \frac{1}{n}\left(\sum_{i=1}^{n/2} f(X_i) + \sum_{i=1}^{n/2} f(2a - X_i)\right)$$

  in unbiased and satisfies

$$\text{Var}[\hat{\mu}] \leq \frac{1}{n}\text{Var}[f(X)].$$

- But $\frac{1}{n}\text{Var}[f(X)]$ is the variance of the classical estimator, when we generate n i.i.d. realizations of X!
- In this way, we reduce the variance of the estimator.
- This approach is known as Antithetic variables.

Antithetic variables for non symmetric X

- Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and let $X : \Omega \to \mathbb{R}$ be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$.
- Suppose X to be not symmetric. How can we apply Antithetic variables to reduce the variance of our estimator?

Remark

Let $U \sim$ Unif$(0,1)$. Then U is symmetric around $\frac{1}{2}$ and the estimator $\hat{\mu}$ defined by

$$\hat{\mu} = \frac{1}{n}\left(\sum_{i=1}^{n/2} h(U_i) + \sum_{i=1}^{n/2} h(1-U_i)\right),$$

where $U_i \sim$ Unif$(0,1)$, $i = 1, \dots, n$, is unbiased and satisfies Var$[\hat{\mu}] \leq \frac{1}{n}Var[h(U)]$.

Antithetic variables for non symmetric X

- Call F the cumulative distribution function of X. Suppose that we know (at least a good approximation of) F^{-1}.
- Let $U \sim$ Unif$(0,1)$ and define $Y := F^{-1}(U)$. Then X and Y have same distribution.
- Let $U \sim$ Unif$(0,1)$. Because of the result above, we have

$$\mathbb{E}^P[f(X)] = \mathbb{E}^P[h(U)]$$

  with $h(x) = f \circ F^{-1}$.
- Simulate independent realizations $(U_i)_{i=1,\dots,n/2}$ and define

$$\hat{\mu} = \frac{1}{n}\left(\sum_{i=1}^{n/2} h(U_i) + \sum_{i=1}^{n/2} h(1-U_i)\right).$$

- By the remark above, this also an Antithetic variables approach which gives a reduction of the variance.

Example: valuation of a call option under Black-Scholes

- We want to test the benefits of using Antithetic variables in the valuation of a call option under the Black-Scholes model.
- This is indeed a case when we have of course the benchmark of the analytic formula for a call option.
- In particular, we want to approximate the expectation $\mathbb{E}^P[g(X_T)]$ for $T > 0$, in the case when

  $$g(x) = (x - K)^+$$

  with $K > 0$ and $X = (X_t)_{0 \le t \le T}$ is a stochastic process with initial value $X_0 = x_0$ and dynamics

  $$dX_t = rX_t dt + \sigma X_t dW_t, \quad 0 \le t \le T,$$

  where $W = (W_t)_{0 \le t \le T}$ is P-Brownian motion.
- Interpretation: r is the risk free rate and P is the martingale measure, i.e., the probability measure under which the discounted process $(e^{-rt}X_t)_{0 \le t \le T}$ is a martingale.

Valuation with Antithetic variables

- The problem reduces to the valuation of the expectation

$$\mathbb{E}^P[(X-K)^+]$$

  where X is the random variable

$$X = x_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z},$$

  with $Z \sim \mathcal{N}(0,1)$.
- That is, we have to valuate

$$\mathbb{E}^P\left[f(Z)\right]$$

  where

$$f(z) = \left(x_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}z} - K\right)^+.$$

- So, we have a function of a random variable which is symmetric around zero! We can directly use Antithetic variables.
- We simulate $n/2$ realizations $(z_i)_{i=1,\dots,n/2}$ of a standard normal random variable and then define $z_{i+n/2} = -z_i$, $i = 1,\dots,n/2$.

- In the Python package

  `montecarlovariancereduction.antitheticvariables`

  you can find the code relative to the comparison of Antithetic variables against the standard Monte-Carlo method.
- In particular, in the class `GenerateBlackScholes` we generate the values of

$$X = x_0 e^{(r-\sigma^2)T+\sigma\sqrt{T}Z},$$

  starting from the ones of Z. We do this using both the standard Monte-Carlo approach and the Antithetic variables approach illustrated in the previous slide.
- Note that the method

  `numpy.random.standard_normal(n)`

  generates n returns of a standard normal random variable. In this case, we give no seed: it will be different every time this method is called.

Experiment and results

In

`antitheticVariablesTest`

and

`compareStandardMCWithAV`

we do the following experiment:

- We fix the parameters $x_0 = K = 100$, $T = 3$, $r = 0.05$, $\sigma = 0.5$.
- For any number of simulations $n = 10^3$, 10^4, 10^5 and 10^6, we perform 100 different valuations of the price of the call option, both with the standard and the Antithetic variables Monte-Carlo method.
- We then compute the average percentage error for both the methods.

The following table illustrates the results:

	$n = 10^3$	$n = 10^4$	$n = 10^5$	$n = 10^6$
av. $\%$ error standard MC	6.11	1.97	0.61	0.18
av. $\%$ error AV	5.70	1.77	0.55	0.17

Setting and motivation

- Let $X, Y : \Omega \to \mathbb{R}$ be two random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.
- Suppose you know the analytic value of

$$\mu_X := \mathbb{E}^P[X], \qquad \sigma_X^2 := \text{Var}[X], \qquad \sigma_{XY} := \text{Cov}[X, Y],$$

  and also suppose $\sigma_{XY} > 0$.
- Assume you want to approximate

$$\mu_Y := \mathbb{E}^P[Y].$$

- The goal is to find an unbiased estimator of μ_Y which has low variance.

Control variates - 1

- Consider n independent realizations (X_i, Y_i) of (X, Y), $i = 1, \ldots, n$, and define

$$\hat{\mu}_X := \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \hat{\mu}_Y := \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

- Note that

$$\text{Cov}[\hat{\mu}_X, \hat{\mu}_Y] = \frac{1}{n}\sigma_{XY}.$$

- What about an estimator

$$\hat{\mu}_Y^{CV} := \hat{\mu}_Y - \beta(\hat{\mu}_X - \mu_X)$$

  for a given $\beta > 0$?
- It is unbiased:

$$\mathbb{E}^P[\hat{\mu}_Y^{CV}] = \mathbb{E}^P[\hat{\mu}_Y] - \beta\mathbb{E}^P[\hat{\mu}_X - \mu_X] = \mu_Y.$$

- What about the variance?

$$\text{Var}[\hat{\mu}_Y^{CV}] = \frac{1}{n}\sigma_Y^2 + \beta^2\frac{1}{n}\sigma_X^2 - 2\beta\frac{1}{n}\sigma_{XY}.$$

- It is minimized by $\beta = \frac{\sigma_{XY}}{\sigma_X^2}$. For such a value of β, we find

$$\text{Var}[\hat{\mu}_Y^{CV}] = \text{Var}[\hat{\mu}_Y] - \frac{1}{n}\frac{\sigma_{XY}^2}{\sigma_X^2}.$$

Control variates - 2

- We have seen that taking
$$\hat{\mu}_Y^{CV} := \hat{\mu}_Y - \beta(\hat{\mu}_X - \mu_X), \qquad \beta = \frac{\sigma_{XY}}{\sigma_X^2}$$
gives an optimal variance
$$\text{Var}[\hat{\mu}_Y^{CV}] = \text{Var}[\hat{\mu}_Y] - \frac{1}{n}\frac{\sigma_{XY}}{\sigma_X^2}.$$
- Note that the gain of the new estimator with respect to the old one only depends on the correlation of X and Y:
$$\frac{\text{Var}[\hat{\mu}_Y^{CV}]}{\text{Var}[\hat{\mu}_Y]} = 1 - \frac{\sigma_{XY}}{n\sigma_X^2 \text{Var}[\hat{\mu}_Y]} = 1 - \frac{\sigma_{XY}}{\sigma_X^2 \sigma_Y^2} = 1 - \rho_{XY}^2.$$
- Problem: we have to compute $\beta = \frac{\sigma_{XY}}{\sigma_X^2}$, but often we don't know σ_X^2 and σ_{XY}.
- Solution: estimate σ_X^2 and σ_{XY} from the generated sample, i.e., set
$$\hat{\sigma}_X^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \hat{\mu}_X)^2, \qquad \hat{\sigma}_{XY} = \frac{1}{n-1}\sum_{i=1}^n (X_i - \hat{\mu}_X)(Y_i - \hat{\mu}_Y)$$
and choose
$$\beta = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X^2}.$$
- Note that this last choice of β actually depends on the generated sample.
- The associated estimator $\hat{\mu}_Y^{CV} := \hat{\mu}_Y - \beta(\hat{\mu}_X - \mu_X)$ is thus unbiased only asymptotically.

- Cliquet options are an example of exotic, path dependent options. In particular, their payoff depends on the returns of the underlying.
- Let $X = (X_t)_{t\ \in[0,T]}$ be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
- Fix a partition

$$0 = t_0 < t_1 < \cdots < t_N := T$$

  of the interval $[0, T]$.
- For any $n = 1, \ldots, N$ define $R_n^* := (R_n)_{[F_\ell, C_\ell]}$ for $F_\ell < C_\ell$, where

$$R_n := \frac{X_{t_n}}{X_{t_{n-1}}} - 1$$

  is the n-th return and $(x)_{[a,b]} := \min\left(\max(x, a), b\right)$, $a < b$, is the truncation of x.
- The payoff of the Cliquet option with local floor and cap F_ℓ, C_ℓ, global floor and cap $F_g < C_g$ and monitoring dates $0 < t_1 < \cdots < t_N := T$ is then

$$R_g^* := (R_g)_{[F_g, C_g]}$$

  where

$$R_g = R_1^* + R_2^* + \cdots + R_N^*.$$

Control variates for Cliquet options: motivation

- There is no analytic formula for the expectation of the payoff of a Cliquet option, not even under the Black-Scholes model.
- Observation: there is of course a positive correlation between $R_g^* := (R_g)_{[F_g, C_g]}$ and R_g, and also between R_g^* and R_k^*, $k = 1, \dots, N$, since

$$R_g = R_1^* + R_2^* + \cdots + R_N^*.$$

- Can we find an analytic formula for the expectation of R_g and R_n^*, at least under a suitable model as Black-Scholes?

Control variates for Cliquet options - 1

Lemma

Let $b > a$. The truncating function $(x)_{[a,b]} := \min\left(\max(x,a), b\right)$ can be rewritten as

$$(x)_{[a,b]} = a + (x-a)^+ - (x-b)^+.$$

Proof

We have that

$$\begin{aligned} a + (x-a)^+ - (x-b)^+ &= a + \max(x-a, 0) + \min(b-x, 0) \\ &= \max(x,a) + \min(b-x, 0). \end{aligned}$$

We then easily see that both $\min\left(\max(x,a), b\right)$ and the function above are equal to a when $x < a$, x if $a \le x \le b$ and b if $x > b$.

Control variates for Cliquet options - 2

- The lemma in the previous slide tells us that, defining $Y_n := \frac{X_{t_n}}{X_{t_{n-1}}}$, the quantity R_n^* can be seen as the difference between two payoffs of call options, plus a constant:

$$R_n^* = F_\ell + (Y_n - (F_\ell + 1))^+ - (Y_n - (C_\ell + 1))^+.$$

- That is, we have an analytic formula for the expectation of R_n^*, at least if Y_n is log-normal or normal.
- It is it reasonable to expect that R_g^* and R_g are more correlated than R_g^* and R_n^*.
- So, what about an analytic formula for the expectation of

$$R_g = R_1^* + R_2^* + \cdots + R_N^*?$$

This comes directly from the one for R_n^*.

- We assume that our underlying X follows dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad 0 \leq t \leq T$$

  under the martingale measure P.
- Then

$$Y_n := \frac{X_{t_n}}{X_{t_{n-1}}} = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(t_n - t_{n-1}) + \sigma(W_{t_n} - W_{t_{n-1}})\right\},$$

  for any $n = 1, \dots, N$.
- The random variables Y_n, $n = 1, \dots, N$, are independent and log-normally distributed.
- Since

$$R_n^* = F_\ell + (Y_n - (F_\ell + 1))^+ - (Y_n - (C_\ell + 1))^+,$$

  we can get $\mathbb{E}^P[R_n^*]$ via Black-Scholes formula, for any $n = 1, \dots, N$.
- Moreover, we get

$$\mathbb{E}^P[R_g] = \mathbb{E}^P[R_1^*] + \dots + \mathbb{E}^P[R_N^*].$$

Application in Python

- In

  `montecarlovariancereduction.controlvariates`

  you can find the code for the application of Control variates in the case of Cliquet options under the Black-Scholes model. We assume $T_k - T_{k-1}$ constant.
- In `cliquetOptionTest` we compare the classical Monte-Carlo approach, Monte-Carlo with Antithetic variables and Monte-Carlo with control variates on two aspects, for 30 tests with 10^4 simulations:
  - variance of the estimates
  - time (in seconds) needed for a single estimate.
- The results are shown in the following table.

	classical MC	MC with AV	MC with CV
variance	$3.94 \cdot 10^{-6}$	$1.32 \cdot 10^{-6}$	$4.79 \cdot 10^{-7}$
time	0.21	0.23	0.48

- You can see that Control variates effectively reduce the variance. However, as it is now, it is slower. Exercise: change the implementation (also of the class `CliquetOption` if needed) in order to make the Control variates application faster without loosing accuracy.

Motivation

The multi-period Binomial model for option pricing is widely used by practitioners in financial applications mainly because:

- It is very easy to understand and simulate.
- It is particularly convenient to price options involving a choice of the holder, like American and Bermudan options.
- It approximates the Black-Scholes model when the length of the periods tends to zero.
- Option pricing is not based on pure Monte-Carlo techniques but relies on weighting the payoff relative to any scenario by the (analytic!) probability of the scenario.

The setting

- Consider a multi-period model with times $t = 0, 1, \dots, T$, and consider a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t=0,\dots,T}$ is a filtration representing information.
- Suppose there exist:
  - A risk free asset defined by $S_t^0 = (1+\rho)^t$, $t = 0, \dots, T$, with a deterministic interest rate $\rho > 0$.
  - A risky asset adapted to $\mathbb{F}$ defined by

    $$S_t = S_0 \cdot Y_1 \cdot \dots \cdot Y_t, \quad t = 1, \dots, T,$$

    where Y_t can take the two values d, u with $0 < d < 1 + \rho < u$, for any $t = 1, \dots, T$, and $(Y_t)_{t=1,\dots,T}$ are i.i.d. and such that Y_{t+1} is independent of $\mathcal{F}_t$.
- Then it holds

$$S_t^0 = S_{t-1}^0 (1+\rho), \quad t = 1, \dots, T$$

and

$$S_t = S_{t-1} Y_t, \quad t = 1, \dots, T.$$

Admissible strategies

At every time $t = 0, \dots, T-1$, an investor can construct a portfolio of value V_t, trading on the risk-free asset S^0 and on the risky asset S.

- The value of the portfolio is given by

$$V_t = \alpha_t S_t + \beta_t S_t^0, \quad t = 1, \dots, T,$$

  where $(\alpha_t)_{t=1,\dots,T}$ and $(\beta_t)_{t=1,\dots,T}$ are $\mathbb{F}$-predictable, discrete processes.
- The strategy (α, β) must be self-financing: it must hold

$$V_t = \alpha_t S_t + \beta_t S_t^0 = \alpha_{t+1} S_t + \beta_{t+1} S_t^0, \quad t = 1, \dots, T.$$

Definition

A portfolio V is an arbitrage if:

- V is obtained by a self-financing strategy;
- $P(V_0 = 0) = 1$;
- $P(V_t \geq 0) = 1$ and $P(V_t > 0) > 0$ for some t.

Proposition

The market is arbitrage free only if $d < 1 + \rho < u$.

- Suppose $1 + \rho \leq d < u$, and consider the self-financing portfolio defined by

$$V_t = S_t - \frac{S_0}{S_0^0} S_t^0, \quad t = 0, 1, \ldots, T.$$

Then we have $V_0 = 0$ and

$$V_1 = S_1 - \frac{S_0}{S_0^0} S_1^0 \geq S_0 d - S_0(1 + \rho) > 0.$$

- If $d < u \leq 1 + \rho$, changing the signs to the strategy above leads to an arbitrage.

Equivalent martingale measure

In order for the market to be arbitrage-free and complete, there must exist a unique measure $Q \sim P$ such that $\frac{S}{S^0}$ is a martingale, i.e., such that

$$\mathbb{E}^Q\left[\frac{S_{t+1}}{S^0_{t+1}}\big|\mathcal{F}_t\right] = \frac{S_t}{S^0_t}, \quad t = 0, \dots, T-1. \tag{2}$$

Note that the measure Q is identified by the probability $q := Q(Y_t = u)$.
Since

$$\mathbb{E}^Q\left[\frac{S_{t+1}}{S^0_{t+1}}\big|\mathcal{F}_t\right] = \frac{(qu + (1-q)d)S_t}{S^0_t(1+\rho)}, \quad t = 0, \dots, T-1,$$

equation (2) holds if and only if $qu + (1-q)d = 1 + \rho$, that is,

$$q = \frac{1 + \rho - d}{u - d}.$$

Such Q exists and is unique as we have supposed $0 < d < 1 + \rho < u$, and

$$\frac{dQ}{dP}(\omega) = \left(\frac{q}{p}\right)^{n(\omega)}\left(\frac{1-q}{1-p}\right)^{T-n(\omega)},$$

where $p := P(Y_t = u)$ and $n(\omega)$ is the number of times $t = 1, \dots, T$ when $Y_t(\omega) = u$.

Replicating strategy

- Assume we want to find an admissible strategy (α_t, β_t), $t = 1, \dots, T$, such that the value of the portfolio
$$\alpha_t S_t + \beta_t (1+\rho)^t$$
equals the value V_t of an option at every time $t = 1, \dots, T$.
- From now on, fix $t = 1, \dots, T$, and suppose we know S_{t-1}.
- Call V_t^u the value of the option at time t when $Y_t = u$ and V_t^d the value of the option at time t when $Y_t = d$.
- It must hold
$$\begin{cases} \alpha_t u S_{t-1} + \beta_t (1+\rho)^t = V_t^u, \\ \alpha_t d S_{t-1} + \beta_t (1+\rho)^t = V_t^d. \end{cases}$$
- The solution to the system above is
$$\alpha_t = \frac{V_t^u - V_t^d}{S_{t-1}(u-d)},$$
$$\beta_t = \frac{u V_t^d - d V_t^u}{(1+\rho)^t (u-d)}.$$

Option pricing from admissibility

- Remember that our strategy (α_t, β_t), $t = 1, \dots, T$, has to be admissible!
- This means that we must have that

$$
\begin{aligned}
V_{t-1} &= \alpha_{t-1} S_{t-1} + \beta_{t-1}(1+\rho)^{t-1} \\
&= \alpha_t S_{t-1} + \beta_t (1+\rho)^{t-1} \\
&= \frac{V_t^u - V_t^d}{u-d} + \frac{uV_t^d - dV_t^u}{(1+\rho)(u-d)} \\
&= \frac{(1+\rho)(V_t^u - V_t^d) + uV_t^d - dV_t^u}{(1+\rho)(u-d)} \\
&= \frac{(1+\rho-d)V_t^u + (u-1-\rho)V_t^d}{(1+\rho)(u-d)} \\
&= \frac{qV_t^u + (1-q)V_t^d}{1+\rho} \\
&= \frac{1}{1+\rho}\mathbb{E}^Q[V_t | \mathcal{F}_{t-1}].
\end{aligned}
$$

- Then we have that the value $(V_t)_{t=0,\dots,T}$ of the option is a martingale under Q.
- This gives us a pricing theorem.

Theorem

The value V_0 of a contingent claim with maturity T and payoff V_T depending on the realizations of S until time T, is given by

$$V_0 = \frac{1}{(1+\rho)^T}\mathbb{E}^Q[V_T].$$

Remark

Because of the theorem above, we always generate our Binomial model under the risk neutral measure Q.

Motivation

- Our main goal here is to get the price of European and (most importantly) American options written on an underlying Binomial model.
- This valuation will approximate the price of the options written on an underlying log-normal model.
- We then generate the realizations of the underlying model in Python, and get the payoff on the realizations, along with its expectation.
- Remember we have to price under the risk neutral measure Q: then we generate the realizations of the process under Q.
- The most naive way we can imagine to do this is a brute force Monte-Carlo approximation..

Monte-Carlo method

- Imagine we want to valuate the discounted price of an European option with a given payoff function $f : \mathbb{R} \to \mathbb{R}$, written on the process S, with maturity T.
- Suppose we don't know any analytic formula in order to derive the price as

$$V_0 = \frac{1}{(1+\rho)^T} \mathbb{E}^Q[f(S_T)].$$

- We consider N *states of the world* $\omega_1, \omega_2, \dots, \omega_N \in \Omega$.
- To any $\omega_1, \omega_2, \dots, \omega_N$, we associate a given trajectory of the process $(S_t)_{t=0,\dots,T}$, with dynamics given under the measure Q.
- In particular, we suppose that the trajectories $(S_t(\omega_k))_{t=0,\dots,T}$, $k = 1, 2, \dots, N$ are independent of each other.
- Strong law of large numbers:

$$\frac{1}{n} \sum_{k=1}^{n} f(S_T(\omega_k)) \to \mathbb{E}^Q[f(S_T)] \quad \text{a.s., when } n \to \infty.$$

- The idea is to simulate such trajectories and approximate

$$\mathbb{E}^Q[f(S_T)] \approx \frac{1}{N} \sum_{k=1}^{N} f(S_T(\omega_k)).$$

- Our first goal is then to generate a sequence of random numbers in order to simulate N independent trajectories $(S_t(\omega_k))_{t=0,\dots,T}$, $k = 1, 2, \dots, N$ of S under the risk neutral measure Q, and store them in a $(T+1) \times N$ matrix (this can be useful for path dependent options).
- Idea: generate (with the help of Python in our case) a sequence of $T \cdot N$ uniformly distributed, pseudo-random numbers $0 < x_{i,j} < 1$, $i = 1, \dots, T$, $j = 1, \dots, N$.
- Choose $\rho > 0$, $d < 1+\rho$, $u > 1+\rho$ and fix $q = \frac{1+\rho-d}{u-d}$.
- For every $i = 1, \dots, T$, $j = 1, \dots, N$, define

$$Y_i(\omega_j) = \begin{cases} u & \text{if } x_{i,j} < q \\ d & \text{if } x_{i,j} \geq q \end{cases}$$

and

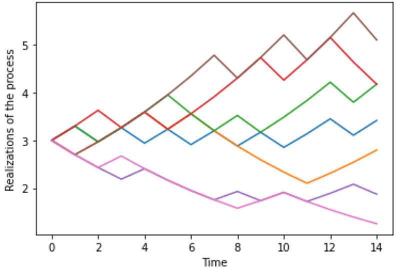
$$S_{i+1}(\omega_j) = Y_i(\omega_j) S_i(\omega_j).$$

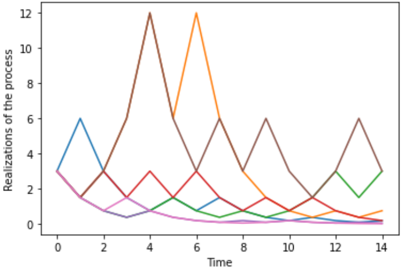
- You can find the code relative to the simulation of the Binomial model with the pure Monte-Carlo approach described above in

  `binomialmodel.creationandcalibration.binomialModelMonteCarlo`
- Note that the class you find there extends the one in

  `binomialmodel.creationandcalibration.binomialModel`.
- This is done in order to implement in the parent class some methods that do not strictly depend on the way in which we simulate the process.
- In this way, we don't have to copy and paste these methods in every class where we simulate the model in some way: object oriented programming feature.

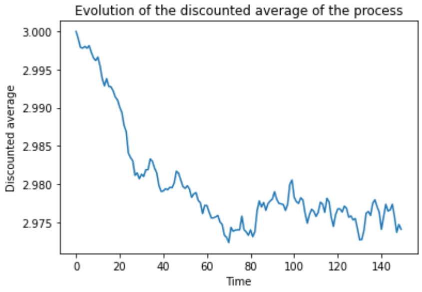
- We plot below some paths of the Binomial model.
- In the figure on the left we take $S_0 = 3$, $u = 1.1$, $d = 0.9$, $r = 0.05$, $T = 150$, having then $q = \frac{1+\rho-d}{u-d} = 0.75$.
- On the right, $S_0 = 3$, $u = 2$, $d = 0.5$, $r = 0.1$, $T = 150$, $q = \frac{1+\rho-d}{u-d} = 0.4$.

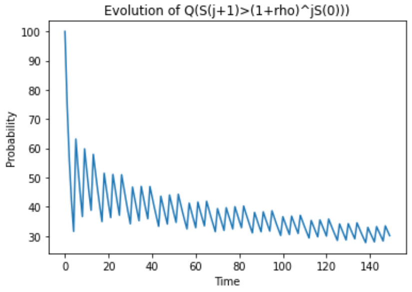




A first test

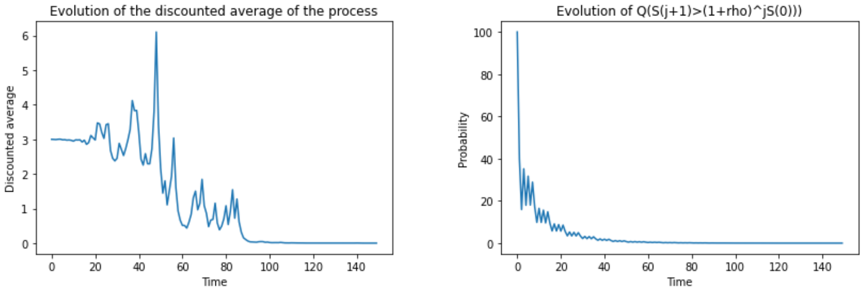
We show here the evolution of the discounted average of the process and of the probability $Q(S_{t_j} > (1+\rho)^{t_j} S_0)$, computed by using the Monte-Carlo method with 10^5 simulations, for $S_0 = 3$, $u = 1.1$, $d = 0.9$, $r = 0.05$, $T = 150$. In this case, we have $q = \frac{1+\rho-d}{u-d} = 0.75$.





But something can go wrong..

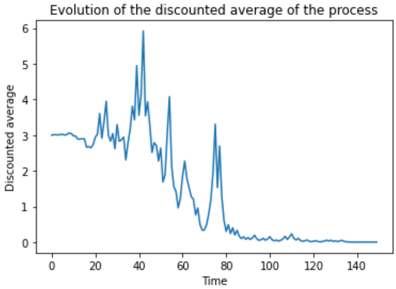
Look at the evolution of the same quantities, again computed by using the Monte-Carlo method, choosing now $S_0 = 3$, $u = 2$, $d = 0.5$, $r = 0.1$, $T = 150$, $q = \frac{1+\rho-d}{u-d} = 0.4$.

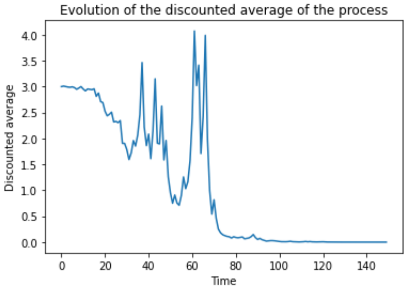


Why is the estimate of the average that inaccurate?

- With the parameters above, the anaytic average of the discounted process is equal to S_0, due to many realizations such that $S_{t_j} < (1+\rho)^{t_j} S_0$ and few, extremely high realizations.
- If you buy S at time $t = 0$, and you hold it for 150 time steps, you make a positive gain with a very low probability, but the gain can be extremely high.
- Problem: The approximated average is strongly impacted by whether or not those paths leading to high gains are simulated or not.

Let’s choose two different seeds, for the same parameters





- We have seen that, if the volatility is high, the Monte-Carlo approach can be very inaccurate for many time steps.
- Moreover, it is time consuming (this is a problem common to all brute-force Monte-Carlo approaches)
- Idea: let us exploit some analytic properties of the Binomial model..

Some simple observations

- At the n-th time step, $n+1$ realizations of the process are possible: $S_0u^n, S_0u^{n-1}d, \dots, S_0ud^{n-1}, S_0d^n$.
- The number of ups and downs is a random variable with Bernoulli distribution:

$$Q(S_n = S_0u^kd^{n-k}) = \binom{n}{k} q^k(1-q)^{n-k}.$$

- Using the expression above, we can compute

$$\begin{aligned}\mathbb{E}^Q[f(S_n)] &= \sum_{k=0}^{n} Q(S_n = S_0u^kd^{n-k})f(S_0u^kd^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} q^k(1-q)^{n-k} f(S_0u^kd^{n-k}).\end{aligned}$$

- The idea is then to generate all the possible realizations of the process up to a given time, and to weight them by their probability.
- You can find the code relative to this approach in

  `binomialmodel.creationandcalibration.binomialModelSmart`,

  whose class also extends the one in

  `binomialmodel.creationandcalibration.binomialModel`.
- Doing some tests in

  `binomialmodel.creationandcalibration.binomialModelSmartTest`.

  you can observe that, in this way, the average of the discounted process is stable.
- Moreover, this approach is of course much faster.

- Note that for any $k = 0, \dots, n$ it holds

$$
\begin{aligned}
S_n = S_0 u^k d^{n-k} \geq S_0(1+\rho)^n &\iff u^k d^{n-k} \geq (1+\rho)^n \\
&\iff \left(\frac{u}{d}\right)^k \geq \left(\frac{1+\rho}{d}\right)^n \\
&\iff k \geq n \log_{\frac{u}{d}} \left(\frac{1+\rho}{d}\right).
\end{aligned}
$$

- Then we have

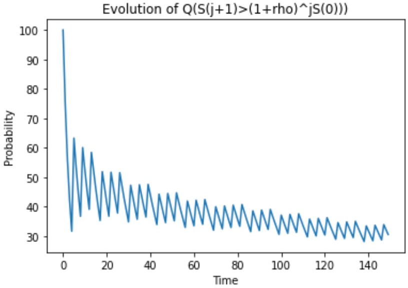
$$
\begin{aligned}
Q\left(S_n \geq S_0(1+\rho)^n\right) &= \sum_{k=\bar{k}}^{n} Q(S_n = S_0 u^k d^{n-k}) \\
&= \sum_{k=\bar{k}}^{n} \binom{n}{k} q^k (1-q)^{n-k},
\end{aligned}
$$

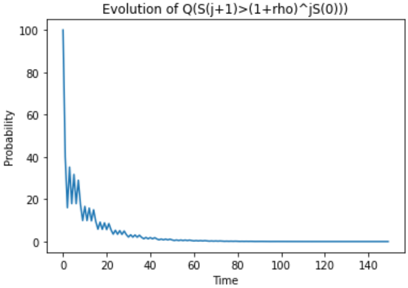
where

$$
\bar{k} = \min\left\{k \in \mathbb{N} : \ k \geq n \log_{\frac{u}{d}} \left(\frac{1+\rho}{d}\right)\right\} \leq n.
$$

Evolution of the probability plotted with Python

We show here the evolution of the probability computed above, over 150 time steps. On the left, we have parameters $S_0 = 3$, $u = 1.1$, $d = 0.9$, $\rho = 0.1$, $q = \frac{1+\rho-d}{u-d} = 0.75$. On the right, $S_0 = 3$, $u = 2$, $d = 0.5$, $\rho = 0.05$, $q = \frac{1+\rho-d}{u-d} = 0.4$.





- As seen before, an application of the simulation of the Binomial model in this way is the valuation of European options, under the pricing measure Q.
- In

  `binomialmodel.optionValuation.europeanOption,`

  you can see some methods relative to this.
- In particular, we compute the expectation of the payoff of European options as

$$\begin{aligned}\mathbb{E}^Q[f(S_n)] &= \sum_{k=0}^{n} Q(S_n = S_0 u^k d^{n-k}) f(S_0 u^k d^{n-k}) \\ &= \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} f(S_0 u^k d^{n-k}).\end{aligned}$$

- We also compute the value of a general option for every time $t = 0, \dots, T-1$, and the corresponding self-financing, replicating strategy (α_t, β_t), $t = 0, \dots, T-1$, described before.
- As an exercise, you can check if the final value of the portfolio given by that strategy equals the payoff, for an option of your choice.

American options

- The holder of an American option with payoff f and maturity T on an underlying X has the right, at any time $t \in [0, T]$, to hold the contract or to exercise the payoff $f(X_t)$.
- The valuation of American options is more complicated than the one of European options, since it involves an optimal exercise problem.
- In order to valuate such an option at time t, indeed, the conditional expectation at time t of the future value of the option has to be computed, and then compared against the present value of the payoff.
- However, the Monte-Carlo computation of a conditional expectation is very time consuming.
- One of the strengths of the Binomial model with respect to other settings is that it permits a favourable pricing of American options.
- Also when dealing with continuous time processes, with suitable dynamics, one may approximate them with a Binomial model in order to get the price.

American options valuation under the Binomial model

- At any time $t = 1, \dots, T$, call $S_t(k)$ and $V_t(k)$ the value of the underlying and of the option, respectively, in the scenario with k ups and $t - k$ downs up to time t.
- Idea: proceed backward.
- First we compute the payoff $f(S_T(k)) = f(S_0 u^k d^{T-k})$, for any $k = 0, \dots, T$.
- We have of course $V_T(k) = f(S_T(k))$, for any $k = 0, \dots, T$.
- At time $T - 1$, for any $k = 0, \dots, T - 1$ we compute

$$
\begin{aligned}
V_{T-1}(k) &= \max\left(f(S_{T-1}(k)), \frac{1}{1+\rho}\left(qV_T(k+1) + (1-q)V_T(k)\right)\right) \\
&= \max\left(f(S_0 u^k d^{T-1-k}), \frac{1}{1+\rho}\left(qV_T(k+1) + (1-q)V_T(k)\right)\right).
\end{aligned}
$$

- For any $t = 1, \dots, T - 2$ we compute with the same argument

$$
V_t(k) = \max\left(f(S_0 u^k d^{t-k}), \frac{1}{1+\rho}\left(qV_{t+1}(k+1) + (1-q)V_{t+1}(k)\right)\right).
$$

- We finally get the value of the option at initial time as

$$
V_0 = \max\left(f(S_0), \frac{1}{1+\rho}\left(qV_1(1) + (1-q)V_1(0)\right)\right).
$$

You can find the code relative to the the valuation of American options in

`binomialmodel.optionValuation.AmericanOption`,

with some tests in

`binomialmodel.optionValuation.AmericanOptionTest`.

Example

We consider a put option with payoff $f(x) = (20 - x)^+$, and choose parameters $T = 3$, $S_0 = 20$, $u = 1.1$, $d = 0.9$, $\rho = 0.05$.
The triangular matrices below show us an analysis of the American put option for such parameters (row 3 shows the values for $t = 3$ and so on).
The upper left and upper right matrices show the amount one would get if exercising the option or holding the contract, respectively; the lower left one the values of the option; the lower right one has 1 in the exercise region and 0 in the hold region

	0	1	2	3
0	0	nan	nan	nan
1	0	2	nan	nan
2	0	0.2	3.8	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0.564464	nan	nan	nan
1	0.123583	1.27551	nan	nan
2	0	0.519048	2.84762	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0.564464	nan	nan	nan
1	0.123583	2	nan	nan
2	0	0.519048	3.8	nan
3	0	0	2.18	5.42

	0	1	2	3
0	0	nan	nan	nan
1	0	1	nan	nan
2	0	0	1	nan
3	1	1	1	1

Approximating a Black-Scholes model with a Binomial model

- Consider a continuous, adapted stochastic process $X = (X_t)_{t \geq 0}$ with dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad t \geq 0,$$

where $r, \sigma > 0$ and $W = (W_t)_{t \geq 0}$ is a Brownian motion.

- Suppose you want to price an American option with underlying X and maturity $T > 0.$
- It can be seen that the dynamics of $X = (X_t)_{0 \leq t \leq T}$ can be approximated by N time steps of a Binomial model with parameters

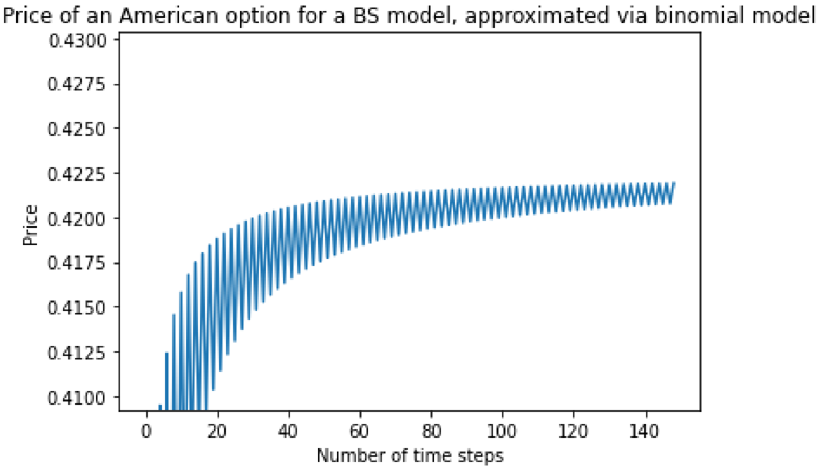
$$u = e^{\sigma\sqrt{T/N}}, \qquad d = 1/u, \qquad \rho = e^{rT/N} - 1, \tag{3}$$

for N large enough, see for example A. A. Dar, and N. Anuradha, *Comparison: binomial model and Black Scholes model.* Quantitative finance and Economics 2.1 (2018): 230-245.

- The idea is to approximate the price of the American option of maturity T with the price of an American option with maturity N written a Binomial model with parameters as in (3), for N large enough.
- Indeed, the price of the American option written on the Binomial model can be computed as illustrated before.

Example: not such a nice behaviour

We consider an American put option with payoff $f(x) = (1-x)^+$ and maturity $T = 3$, written on a Black-Scholes model with parameters $r = 0.02$, $\sigma = 0.7$.
The plot below shows the approximated price via the derivation under the Binomial model, for an increasing number of times steps up to $N = 150$.



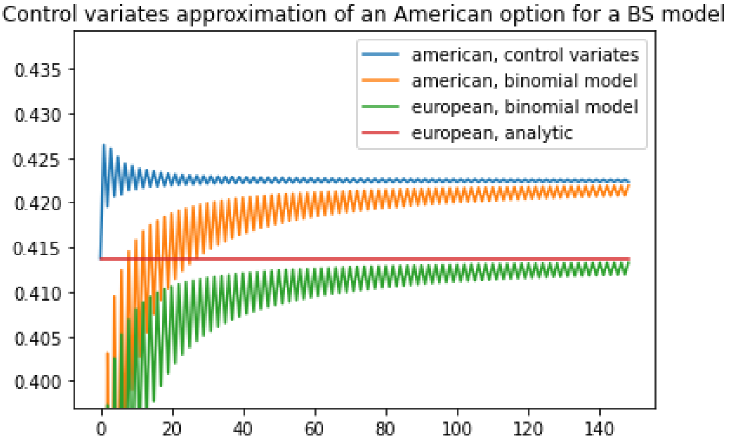
Control variates for American call and put options

- First idea: we know the analytic price of an European put (or call) option under the Black-Scholes model. For example, call P^E the Black-Scholes formula price of an European put option.
- Also call:
  - P_N^E the price of an European put approximated by the Binomial model with N time steps;
  - P^A the analytic price of an American put;
  - P_N^A the price of an American put approximated by the Binomial model with N time steps.
- Second idea: we know the euristics $P^A - P_N^A \approx P^E - P_N^E$.
- We then approximate

$$P^A \approx P_N^A + (P^E - P_N^E)$$

- This approximates the price of an American put option via control variates.
- Remember: the price of an American call equals the price of an European option (i.e., it is always convenient to wait, in expectation).

We consider again an American put option with payoff $f(x) = (1-x)^+$ and maturity $T = 3$, written on a Black-Scholes model with parameters $r = 0.02$, $\sigma = 0.7$: same situation as before.
The plot below compares the prices introduced in the previous slide, for an increasing number of times steps up to $N = 150$.



- You can find some experiments relative to the stability of approximations of prices of American options with the Binomial model in

  `binomialmodel.optionValuation.AmericanOptionPriceConvergence`,

- The code performing the control variates approach can be found in

  `binomialmodel.optionValuation.controlVariates`,

  with some tests in

  `binomialmodel.optionValuation.controlVariatesTest`.