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Notes and Comments

A discrete-time algorithm for pricing double barrier options

Massimo Costabile

Dipartimento di Organizzazione Aziendale e Amministrazione Pubblica,
Università della Calabria
e-mail: massimo.costabile@unical.it

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1. Introduction

A double knock-out European option is an option that expires if the underlying asset price reaches a lower or an upper barrier before maturity. Otherwise, the option payoff at maturity is the same as that of a standard European option. The problem of pricing these options was tackled by Kunimoto and Ikeda (1992) and Geman and Yor (1996) who proposed accurate algorithms developed in a continuous-time setting under the usual assumptions of the Black–Scholes analysis. Both these models are based on the assumption of a continuous monitoring of the option contract. This means that, at any time in the life of the option, the contract is checked to verify if the underlying asset price has touched or crossed the barriers. However, in many cases this assumption is not realistic and only a discrete monitoring of the contract is possible. In these situations a discrete-time algorithm is needed to give the correct answer to the problem of pricing barrier options.

In this paper we present a discrete-time method for pricing a double knock-out European option. In this framework, the underlying asset process

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is a random walk with two absorbing barriers. The option price is determined as the discounted value of the option payoff at maturity under the risk-neutral probability measure. In order to compute this price we need to consider the number of trajectories that touch one or both of the barriers. This number can be calculated by repeatedly using the reflection principle introduced by Desiré André (see Feller (1968) for a detailed description). After this, it remains to choose a suitable number of steps for the option evaluation. In fact, as pointed out by Boyle and Lau (1994), a naive application of the traditional Cox–Ross–Rubinstein algorithm (1979) can lead to a consistent bias in barrier options evaluation even when the number of steps used in computations is very high. We suggest a way to compute this number in the case of double knock-out European options and, as a result, we calculate option prices and compare them with those obtained using the Kunimoto–Ikeda and Geman–Yor algorithms.

The remainder of the paper is organized as follows. Section 2 illustrates the reflection principle for a random walk with one absorbing barrier. Section 3 shows the Boyle–Lau algorithm for pricing single barrier options. Section 4 considers the reflection principle for random walks with two absorbing barriers. Section 5 extends the Boyle–Lau algorithm to double knock-out European options and illustrates the numerical results of the binomial algorithm proposed in this paper.

2. The reflection principle for random walks with one absorbing barrier

We consider a particle whose value at time t is $S(t) = S$. After one period, at time $t+1$, the particle can take either the value $S(t+1) = S+1$ or $S(t+1) = S-1$. In the first case we say that an up-step has occurred and, in the second case, a down-step. After n periods, at time $t+n$, the particle can take one of the $n+1$ values $S(t+n) = S+(n-2i) \quad \text{for} \quad i = 0, \dots, n$. In this way, we construct a binomial lattice whose nodes represent the values of the particle at the end of each period.

For ease of notation, we set $t = 0$ and $S = 0$ and label by P the point $(0, 0)$. We denote by $P^* \equiv (n, 2j-n)$ the position of the particle after j up-steps and $n-j$ down-steps; the number of paths from P to P^* is the well-known binomial coefficient $N_{n,j} = \binom{n}{j}$.

We recall the reflection principle following the approach developed in Mohanty (1979). We consider the binomial lattice depicted in Figure 1 with a barrier L starting from the point $P_1 \equiv (0, -m)$ (m is a positive integer) and parallel to the x-axis. Moreover, we suppose that the point P^* is located, as in Figure 1, above the barrier. Our goal is to compute the number of paths from P to P^* touching the barrier (in Figure 1, m=1 and $P^* = (7, 1)$).

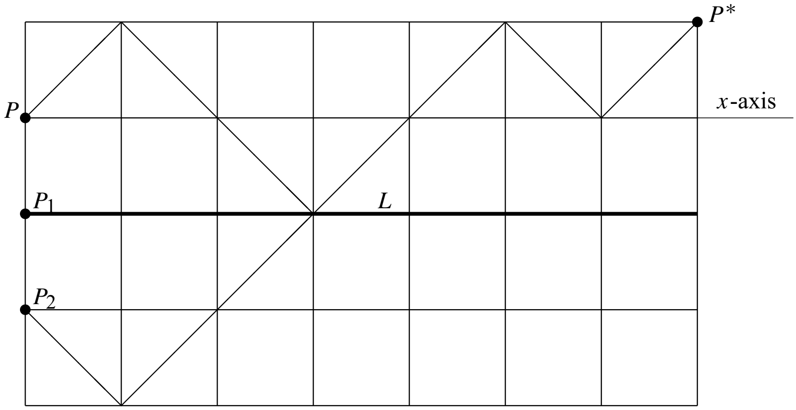

Fig. 1. The reflection principle for random walks with one absorbing barrier

Let $\mathcal{A}$ denote the set of paths from P to P^* touching the barrier and let $N^t_{n,j}$ be the number of elements of $\mathcal{A}$. The reflection principle allows us to compute $N^t_{n,j}$; in fact, every path from P to P^* that touches the barrier can be divided into two different parts:

a) the portion from P to the barrier;
b) the portion from the barrier to P^*.

Let $P_2 \equiv (0, -2m)$ be the point symmetric to P with respect to the barrier. It is easy to verify that for every trajectory from P to one point of the barrier, there exists one and only one trajectory from P_2 to the same point of the barrier, obtained by changing each up-step to a down-step and vice versa. In other words, there is a one-to-one correspondence between the paths from P to the barrier and the paths from P_2 to the barrier. The problem of computing the number of trajectories from P to P^* which touch the barrier can be reduced to the problem of counting the number of trajectories from P_2 to P^*. We can conclude that the number of paths from P to the point P^* that touch the barrier is equal to the number of paths from P_2 to P^*. These are paths with n steps and $j + m$ up-steps. As a consequence, the number of paths from P_2 to P^* is equal to the binomial coefficient $\binom{n}{j+m}$, whence

$$N^t_{n,j} = \binom{n}{j+m}.$$

Finally, the number of trajectories from P to P^* not touching the barrier, $N^{nt}_{n,j}$, can be obtained as the difference between $N_{n,j}$ and $N^t_{n,j}$,

$$N^{nt}_{n,j} = \binom{n}{j} - \binom{n}{j+m}.$$

3. The Boyle–Lau method for valuing barrier options

We next consider a down-and-out European call option with a barrier set at level L. This contract is like a standard European call except for the fact that the option vanishes if the underlying asset price reaches the barrier. Let $S(t) = S$ be the underlying asset value at time t, K the strike price, T the maturity, r the continuously compounded risk-free interest rate and σ the volatility. Boyle and Lau (1994) emphasized that the evaluation of this kind of option by the standard binomial method of Cox, Ross and Rubinstein (1979) provides a good level of accuracy only if a suitable number of steps is used in price computations. Indeed, even when a large number of steps is used, a consistent bias may be observed with respect to the continuous-time values computed with the Merton model (see Merton (1990)). The reason for this unsatisfactory level of accuracy is that, when the barrier is set between two layers of horizontal nodes in the binomial tree, the further the distance from the layers to the barrier, the less precise is the evaluation. Boyle and Lau suggest a way to solve this problem. The trick is to divide the binomial grid into a number of steps such that "the barrier is close to but just above a layer of horizontal nodes in the tree" (Boyle and Lau (1994)). In order to compute this number they argued as follows. Let d be the magnitude of a downward step in the binomial tree. It is well-known that convergence to the continuous-time values requires

$$d = e^{-\sigma\sqrt{h}}, \quad \text{where } h = \frac{T-t}{n},$$

and n is the number of steps used in price computations. In the binomial tree the barrier L is located between two sets of horizontal nodes, the upper one located m successive down-steps below the underlying asset price at inception and the lower $m+1$ down-steps. In order to reduce the distance from the barrier to a set of horizontal nodes, we need to construct a binomial tree with a number of steps n equal to the largest integer smaller than $f(m)$, where

$$f(m) = \frac{m^2\sigma^2(T-t)}{\log^2(L/S)}.$$

Once this number has been calculated, the reflection principle helps us in computing the option price. Recalling the results illustrated in Section 2, the number of trajectories with n steps and j upward moves, not touching the barrier, is

$$N_{n,j}^{nt} = \binom{n}{j} - \binom{n}{j+m},$$

where m is the number of successive down-steps for the underlying asset price at inception to touch or cross the barrier. The value of the down-and-out European call option is

$$\mathrm{DOC}(t) = e^{-r(T-t)} \sum_{j=0}^{n} N_{n,j}^{nt} p^j q^{n-j} \max[Su^j d^{n-j} - K, 0],$$

where $p = (e^{rh} - d)/(u - d)$ is the risk-neutral probability of an up-step, $q = 1 - p$ and $u = 1/d$.

For a down-and-in European call with $K > L$ we can use the same approach and find

$$\mathrm{DIC}(t) = e^{-r(T-t)} \sum_{j=0}^{n} N_{n,j}^{t} p^j q^{n-j} \max[Su^j d^{n-j} - K, 0],$$

where

$$N_{n,j}^t = \binom{n}{j+m}.$$

4. The reflection principle for random walks with two absorbing barriers

We consider a particle that starts from $P \equiv (0, 0)$ and moves, as in the previous sections, in a binomial lattice. Now we consider the case of two absorbing barriers parallel to the x-axis; the first one, which we label U, begins from the point $P_1 \equiv (0, m_1)$ and the second, L, starts from the point $P_2 \equiv (0, -m_2)$ (m_1 and m_2 are positive integers). Set $m = m_1 + m_2$; in Figure 2 below, $m_1 = 2$ and $m_2 = 1$.

Let P^* be the position of the particle after n steps with j up-moves and let us suppose that P^* is located between L and U. Our goal is to count the number of paths from P to P^* touching one or both of the barriers. Denote this number by $N_{n,j}^t$ again.

Let $\mathcal{A}_1$ be the set of paths from P to P^* that touch the lower barrier, let $\mathcal{A}_2$ be the set of paths that touch first L and then U and, in general, let $\mathcal{A}_i$ be the set of trajectories from P to P^* that touch $L, U, L, \ldots$ i times in the specified order ($i \geq 1$). In the same way, denote by $\mathcal{B}_i$, $i \geq 1$, the set of trajectories from P to P^* touching the barriers i times in the order $U, L, U, \ldots$. Note that $\mathcal{A}_i \subseteq \mathcal{A}_j$ and $\mathcal{B}_i \subseteq \mathcal{B}_j$ for $i > j$. Denote by $|\mathcal{A}_i|$ and by $|\mathcal{B}_i|$ the number of elements of $\mathcal{A}_i$ and $\mathcal{B}_i$, respectively. In order to compute $N_{n,j}^t$ one might be tempted to sum the number of paths $|\mathcal{A}_1|$ from P to P^* touching the lower barrier and the number of paths touching the upper barrier $|\mathcal{B}_1|$. This is not the correct answer to our problem; indeed,

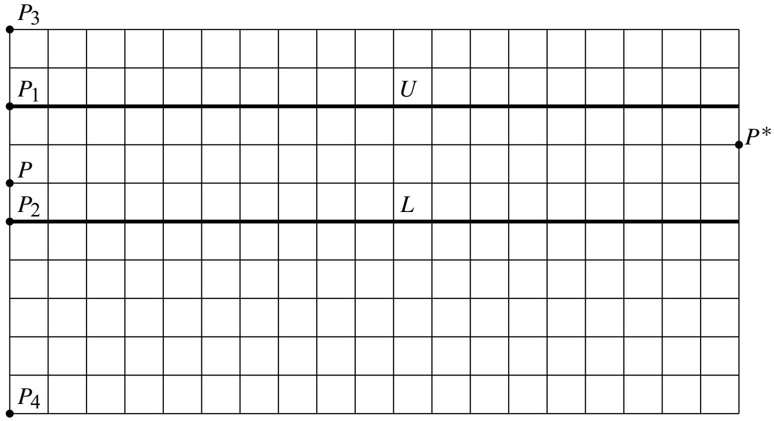

Fig. 2. The reflection principle for random walks with two absorbing barriers

both the sets $\mathcal{A}_1$ and $\mathcal{B}_1$ could contain the same trajectories from P to P^* which repeatedly touch the barriers. In order to calculate $N^t_{n,j}$ we need to use the inclusion-exclusion principle (cf. Mohanty (1979)). It follows that

$$N^t_{n,j} = \sum_{i=1}^{\lceil n/m \rceil} (-1)^{i+1}[|\mathcal{A}_i| + |\mathcal{B}_i|],$$

where $\lceil x \rceil$ denotes the minimum integer greater than or equal to x. To determine $|A_i|$ and $|B_i|$ we need to use the reflection principle repeatedly. For example, $|\mathcal{B}_2|$, which is the number of paths from P to P^* which touch the upper barrier and then the lower barrier, is computed by using the reflection principle twice. At first we calculate the number of paths touching the upper barrier. In Figure 2 these are trajectories from $P_3 \equiv (0, 2m_1)$ to P^*. Using the reflection principle again, we obtain the number of paths from P to P^* which touch the upper barrier and then the lower barrier. This number is equal to the number of paths from the point $P_4 \equiv (0, -2m)$ to P^*, i.e., it is the binomial coefficient $\binom{n}{j+m}$. In general,

$$|\mathcal{B}_{2i}| = \binom{n}{j+im} \quad \text{and } |\mathcal{B}_{2i+1}| = \binom{n}{j-im-m_1}.$$

In the same way we can compute $|\mathcal{A}_{2i}|$ and $|\mathcal{A}_{2i+1}|$, $i \geq 1$, by changing m to $-m$ and m_1 to $-m_2$,

$$|\mathcal{A}_{2i}| = \binom{n}{j-im} \quad \text{and } |\mathcal{A}_{2i+1}| = \binom{n}{j+im+m_2}.$$

Clearly, the number of paths from P to P^* not touching the barriers is

$$N^{nt}_{n,j} = N_{n,j} - N^t_{n,j}.$$

5. Extending the Boyle–Lau method to double barrier options

In Section 3 we illustrated the Boyle–Lau algorithm for pricing single barrier options. In this section we extend the Boyle–Lau algorithm to options with two absorbing barriers.

We consider a double knock-out European call option with a lower barrier set at level L and an upper one set at level U. As before, $S(t) = S$ is the underlying asset price at time t, T is the option maturity, K is the strike price, σ is the volatility and r is the continuously compounded risk-free interest rate. Our goal is to divide the binomial tree into a number of steps such that the lower barrier is close to but just above a set of horizontal nodes and the upper barrier is close to but just below a layer of horizontal nodes. Following the procedure of Section 3, we see that the condition for the upper barrier implies that the number of steps is equal to the largest integer smaller than

$$f(m_1) = \frac{m_1^2 \sigma^2 (T - t)}{\log^2(U/S)}.$$

In the same way, the condition for the lower barrier implies that the number of steps is equal to the largest integer smaller than

$$f(m_2) = \frac{m_2^2 \sigma^2 (T - t)}{\log^2(L/S)}.$$

Let $[x]$ denote the largest integer smaller than x. In general $[f(m_1)]$ and $[f(m_2)]$ can be expressed as functions of m_1 and m_2. Among the different values of m_1 and m_2, we need to select those such that $[f(m_1)] = [f(m_2)]$. The common value $[f(m_1)] = [f(m_2)]$ represents the number of steps, n, to be used in price computations.

As we will see later, it may happens that we cannot find values of m_1 and m_2 such that $[f(m_1)] = [f(m_2)]$. When this happens, we choose m_1 and m_2 in such a way that the absolute difference $|[f(m_1)] - [f(m_2)]|$ is as small as possible. After this, we set the number of steps for options evaluation equal to the minimum of $[f(m_1)]$ and $[f(m_2)]$. The numerical results show that this circumstance does not affect the precision of the evaluation method in a significant way.

Once we have computed the number of steps for option evaluation, it remains to count how many up-steps we need to consider in the binomial algorithm. Indeed, it suffices to consider only the trajectories with terminal underlying asset value greater than L and smaller than U. These paths must have at least a up-steps, where a is the smallest integer greater than

$$\frac{\log(L/Sd^n)}{\log(u/d)},$$

Table 1. Double knock-out European calls with the revised binomial algorithm

Parameters					
$S=2 \quad T-t=1$	$K=2$	$L=1.5$	$U=2.5$	$r=0.02$	$\sigma=0.2$
m_1	$[f(m_1)]$	m_2	$[f(m_2)]$	n	DKOUT
14	157	18	156	156	0.0407
17	232	22	233	232	0.0410
31	771	40	773	771	0.0411
accurate values:		Kunimoto–Ikeda = 0.0411		Geman–Yor = 0.0411	

Table 2. Double knock-out European calls with the revised binomial algorithm

Parameters					
$S=2 \quad T-t=1$	$K=2$	$L=1.5$	$U=3$	$r=0.05$	$\sigma=0.5$
m_1	$[f(m_1)]$	m_2	$[f(m_2)]$	n	DKOUT
7	74	5	75	74	0.0179
24	875	17	872	872	0.0179
31	1461	22	1462	1461	0.0178
accurate values:		Kunimoto–Ikeda = 0.0179		Geman–Yor = 0.0178	

Table 3. Double knock-out European calls with the revised binomial algorithm

Parameters					
$S=2 \quad T-t=1$	$K=1.75$	$L=1$	$U=3$	$r=0.05$	$\sigma=0.5$
m_1	$[f(m_1)]$	m_2	$[f(m_2)]$	n	DKOUT
14	298	24	299	298	0.0756
24	875	41	874	874	0.0761
31	1461	53	1461	1461	0.0761
accurate values:		Kunimoto–Ikeda = 0.0762		Geman–Yor = 0.0762	

and at most b up-steps, where b is the greatest integer smaller than

$$\frac{\log(U/Sd^n)}{\log(u/d)}.$$

The price of a double knock-out European call option is given by

$$\text{DKOUT}(t) = e^{-r(T-t)} \sum_{j=a}^{b} N_{n,j}^{nt} p^j q^{n-j} \max[Su^j d^{n-j} - K, 0],$$

where $N_{n,j}^{nt}$ is the number of trajectories starting from $(t, S(t))$ with n steps and j up-moves not touching either barrier, and is calculated using the reflection principle repeatedly as illustrated in Section 4.

Tables 1, 2 and 3 illustrate the numerical results of pricing double knock-out European call options with the revised binomial method. Parameters are chosen so as to allow comparison with the Kunimoto–Ikeda (1992) and Geman–Yor (1996) algorithms. The computations were performed on a personal computer equipped with a 400-MHz Pentium II processor and 128 Mb of RAM. The computation times of the revised binomial algorithm for all the options considered are below 2 seconds. From this point of view no significant differences are registered with respect to the continuous- time algorithms considered above.

6. Conclusions

The most popular and widespread method for option pricing in a discrete-time setting is that of Cox, Ross and Rubinstein (1979). Boyle and Lau (1994) pointed out that a naive application of this method to single barrier options can lead to very slow convergence to the corresponding continuous-time values. They showed that this drawback can be overcome simply by choosing a suitable number of steps in price computations. In this paper we extend the Boyle–Lau approach to double knock-out European options and we use combinatorial formulae, derived from the reflection principle, to avoid backward recursion in computing option prices. Numerical results show that the binomial algorithm presented here is efficient. Indeed, we obtain very accurate values when compared to those of Kunimoto and Ikeda (1992) and Geman and Yor (1996) and moreover no significant difference is registered between the three algorithms with respect to computation times. However, the revised binomial algorithm is to be preferred in the case of a discrete monitoring of the option contract because of the inadequacy of the continuous-time approach.

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