Numerical Methods for Partial Differential Equations
A.Y. 2022/2023

# Laboratory 02

# Finite Element method for the Poisson equation in 1D: convergence analysis

## Exercise 1.

Let $\Omega = (0, 1)$. Let us consider the Poisson problem

$$\begin{cases} -\left(\mu(x)\, u'(x)\right)' = f(x) \quad x \in \Omega = (0,1) & \text{(1a)} \\ u(0) = u(1) = 0 & \text{(1b)} \end{cases}$$

$\mu(x) = 1$ and $f(x) = 4\,\pi^2 \sin(2\pi x)$ for $x \in \Omega$.

**1.1.** Show that $u_{\text{ex}}(x) = \sin(2\pi x)$ is the exact solution to (1).

**Solution.** There holds $u_{\text{ex}}(0) = u_{\text{ex}}(1) = 0$, so that (1b) is satisfied. Moreover,

$$u'_{\text{ex}}(x) = 2\pi \cos(2\pi x) ,$$
$$(\mu(x)\, u'_{\text{ex}}(x))' = -4\pi^2 \sin(2\pi x) = -f(x) ,$$

so that also (1a) is satisfied.

**1.2.** Starting from the solution of the first laboratory, implement a method `double Poisson1D::compute_error(const VectorTools::NormType &norm_type) const` that computes the $L^2(\Omega)$ or $H^1(\Omega)$ norm (depending on the input argument) of the error between the computed solution and the exact solution:

$$e_{L^2} = \|u_h - u_{\text{ex}}\|_{L^2} = \sqrt{\int_0^1 |u_h - u_{\text{ex}}|^2 \, dx} ,$$
$$e_{H^1} = \|u_h - u_{\text{ex}}\|_{H^1} = \sqrt{\int_0^1 |u_h - u_{\text{ex}}|^2 \, dx + \int_0^1 |\boldsymbol{\nabla} u_h - \boldsymbol{\nabla} u_{\text{ex}}|^2 \, dx} .$$

**Solution.** See the file `src/Poisson1D.cpp`.

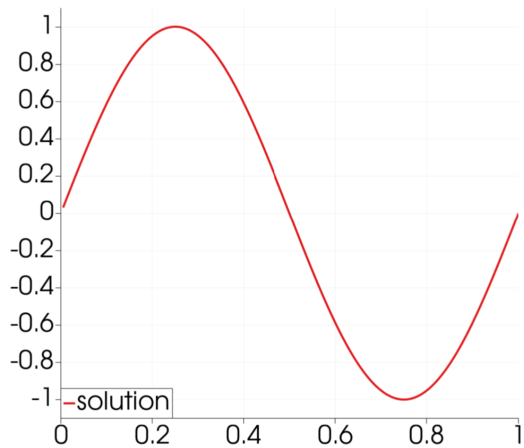



Figure 1: Solution $u_h$ of Exercise 1.3, computed using $r = 1$ and $N + 1 = 160$.

**1.3.** With polynomial degree $r = 1$, solve the problem (1) with finite elements, setting $N + 1 = 10, 20, 40, 80, 160$. Compute the error in both the $L^2(\Omega)$ and $H^1(\Omega)$ norms as a function of the mesh size $h$, and compare the results with the theory.

**Solution.** See the file `src/lab-02.cpp` for the implementation. The solution for this test is plotted in Figure 1.

From the convergence theory, we expect that, if $u_{\text{ex}} \in H^{r+1}(\Omega)$ (in this case, $u_{\text{ex}} \in H^p$ for all $p = 1, 2, 3, \dots$),

$$e_{L^2} \leq C_1 h^{r+1} |u_{\text{ex}}|_{H^{r+1}} ,$$
$$e_{H^1} \leq C_2 h^{r} |u_{\text{ex}}|_{H^{r+1}} ,$$

where $h = 1/(N + 1)$ is the size of mesh elements. Therefore, we expect the error in the $L^2(\Omega)$ norm to converge to zero with order $r + 1 = 2$, and the error in the $H^1(\Omega)$ norm to converge to zero with order $r = 1$.

To verify this, we use the `deal.II` class `ConvergenceTable` to compute the estimated convergence order. Notice that the class works as expected as long as the mesh size between subsequent solutions is halved. We obtain the following results:

| $h$ | $e_{L^2}$ | convergence order $e_{L^2}$ | $e_{H^1}$ | convergence order $e_{H^1}$ |
|---|---|---|---|---|
| 0.1 | 2.5199e-02 | - | 8.0096e-01 | - |
| 0.05 | 6.3529e-03 | 1.99 | 4.0231e-01 | 0.99 |
| 0.025 | 1.5916e-03 | 2.00 | 2.0139e-01 | 1.00 |
| 0.0125 | 3.9811e-04 | 2.00 | 1.0072e-01 | 1.00 |
| 0.00625 | 9.9539e-05 | 2.00 | 5.0364e-02 | 1.00 |

The estimated convergence orders for the two errors are in agreement with the theory.

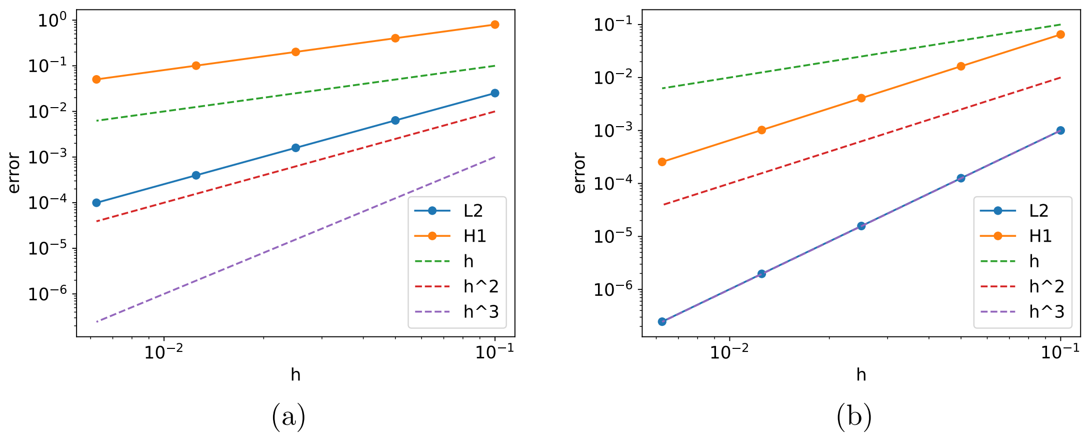



Figure 2: Exercises 1.3 and 1.4. Error in the $L^2$ and $H^1$ norms against $h$ for $r = 1$ (a) and $r = 2$ (b).

We can draw the same conclusions by plotting the error against $h$ in the log-log plane. Indeed, we want to assess whether

$$e_{L^2} \approx C_1 |u_{\text{ex}}|_{H^{r+1}} h^{r+1}$$

holds. By taking the logarithm of both sides, we get

$$\log e \approx \log \left( C_1 |u_{\text{ex}}|_{H^{r+1}} \right) + (r + 1) \log h \ ,$$

so that the logarithm of the $L^2$ error is a linear function of the logarithm of $h$, with slope given by the convergence rate $r + 1 = 2$. Therefore, in the log-log plane, the plot of the $L^2$ error should be a line parallel to the one obtained by plotting the function $h^{r+1}$. Similar considerations hold for the $H^1$ error, which should be represented by a line of slope $r = 1$.

The errors against $h$ are written to a file `convergence.csv`, which can be opened in any 2D graphing software (MATLAB, Python, ...). For convenience, you can find a Python script at `scripts/plot-convergence.py`, that can be called as `./plot-convergence.py convergence.csv` and produces a PDF file `convergence.pdf` containing the plot. The script requires that the package `matplotlib` is available: you can download and install it by running `pip install matplotlib`.

The result is displayed in Figure 2a. We can see how the error in the $L^2$ norm is represented by a straight line parallel to the one representing $h^2$, as expected, confirming that it tends to zero with order $r + 1 = 2$. Similarly, the $H^1$ error tends to zero with order $r = 1$ as expected.

**1.4.** Repeat the previous point setting $r = 2$.

**Solution.** By changing the polynomial degree to $r = 2$, we obtain the following convergence table:

| $h$ | $e_{L^2}$ | convergence order $e_{L^2}$ | $e_{H^1}$ | convergence order $e_{H^1}$ |
|---|---|---|---|---|
| 0.1 | 1.0028e-03 | - | 6.5007e-02 | - |
| 0.05 | 1.2590e-04 | 2.99 | 1.6319e-02 | 1.99 |
| 0.025 | 1.5754e-05 | 3.00 | 4.0840e-03 | 2.00 |
| 0.0125 | 1.9698e-06 | 3.00 | 1.0213e-03 | 2.00 |
| 0.00625 | 2.4624e-07 | 3.00 | 2.5533e-04 | 2.00 |

The corresponding error plots are displayed in Figure 2b. From, both, we can observe the expected convergence rates: the $L^2$ error tends to zero with rate $r + 1 = 3$, and the $H^1$ error tends to zero with rate $r = 2$.

**1.5.** Let us now redefine the forcing term as

$$f(x) = \begin{cases} 0 & \text{if } x \leq \dfrac{1}{2}\,, \\ -\sqrt{x - \dfrac{1}{2}} & \text{if } x > \dfrac{1}{2}\,. \end{cases}$$

The exact solution in this case is

$$u_{\text{ex}}(x) = \begin{cases} Ax & \text{if } x \leq \dfrac{1}{2}\,, \\ Ax + \dfrac{4}{15}\left(x - \dfrac{1}{2}\right)^{\frac{5}{2}} & \text{if } x > \dfrac{1}{2}\,, \end{cases}$$

$$A = -\frac{4}{15}\left(\frac{1}{2}\right)^{\frac{5}{2}}\,.$$

Check the convergence order of the finite element method in this case, with polynomial degrees $r = 1$ and $r = 2$. What can you observe?

**Solution.** The solution for this test is plotted in Figure 3. For $r = 1$, the convergence table reads

| $h$ | $e_{L^2}$ | convergence order $e_{L^2}$ | $e_{H^1}$ | convergence order $e_{H^1}$ |
|---|---|---|---|---|
| 0.1 | 3.1998e-04 | - | 1.0198e-02 | - |
| 0.05 | 7.9874e-05 | 2.00 | 5.1018e-03 | 1.00 |
| 0.025 | 1.9899e-05 | 2.01 | 2.5514e-03 | 1.00 |
| 0.0125 | 4.9486e-06 | 2.01 | 1.2757e-03 | 1.00 |
| 0.00625 | 1.2281e-06 | 2.01 | 6.3788e-04 | 1.00 |

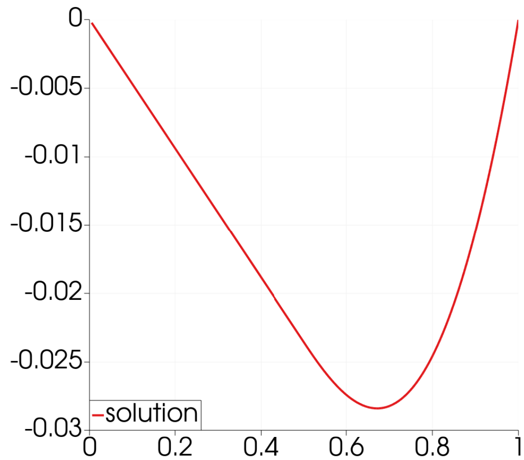


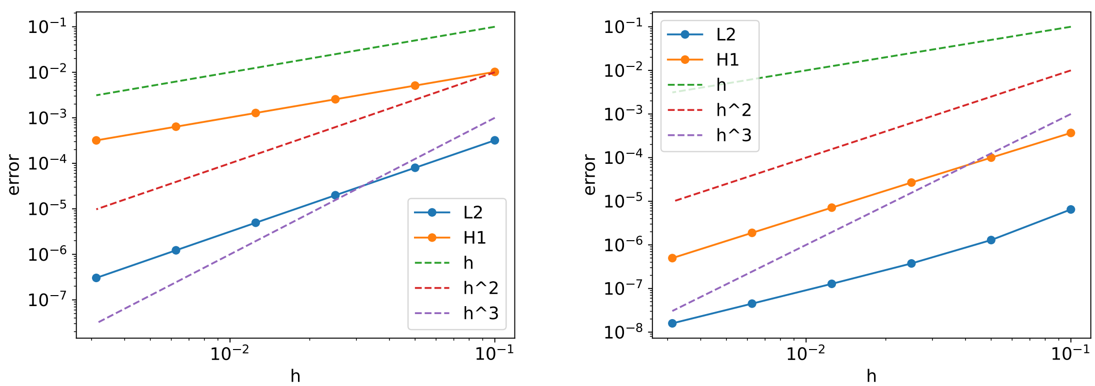

Figure 3: Solution $u_h$ of Exercise 1.5, computed using $r = 1$ and $N + 1 = 160$.

and the associated convergence plot is displayed in Figure 4a. We observe once again the expected convergence rates.

If we set $r = 2$, however, we obtain the following convergence table:

| $h$ | $e_{L^2}$ | convergence order $e_{L^2}$ | $e_{H^1}$ | convergence order $e_{H^1}$ |
|---|---|---|---|---|
| 0.1 | 6.4841e-06 | - | 3.6856e-04 | - |
| 0.05 | 1.2899e-06 | 2.33 | 9.9987e-05 | 1.88 |
| 0.025 | 3.7520e-07 | 1.78 | 2.6826e-05 | 1.90 |
| 0.0125 | 1.2792e-07 | 1.55 | 7.1384e-06 | 1.91 |
| 0.00625 | 4.4981e-08 | 1.51 | 1.8882e-06 | 1.92 |

and the convergence plot of Figure 4b. The convergence rates are not the optimal ones in this case. Indeed, the optimal convergence rates ($r + 1$ in the $L^2$ norm and $r$ in the $H^1$ norm) can be observed only if the solution belongs to the space $H^{r+1}(\Omega)$. In this case, there holds $u \in H^2(\Omega)$ but $u \notin H^3(\Omega)$, so that the observed convergence rates, for the case $r = 2$, are slower than the optimal ones.




(a) (b)

Figure 4: Exercise 1.5. Error in the $L^2$ and $H^1$ norms against $h$, for $r = 1$ (a) and $r = 2$ (b).