## Numerical Methods for Partial Differential Equations A.Y. 2022/2023

## Laboratory 04

Finite Element method for the diffusion-reaction equation in 2D: convergence analysis

## Exercise 1.

Let  $\Omega = (0, 1) \times (0, 1)$ , and let us consider the following diffusion-reaction problem with homogeneous Dirichlet boundary conditions:

$$\begin{cases}
-\nabla \cdot (\mu \nabla u) + \sigma u = f & \mathbf{x} \in \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$
(1a)

where  $\mathbf{x} = (x, y)^T$ ,  $\mu(\mathbf{x}) = 1$ ,  $\sigma = 1$  and

$$f(\mathbf{x}) = (20\pi^2 + 1)\sin(2\pi x)\sin(4\pi y) .$$

The exact solution to this problem is

$$u_{\rm ex}(x,y) = \sin(2\pi x)\sin(4\pi y) .$$

1.1. Write the weak formulation, the Galerkin formulation and the finite element formulation of (1).

**Solution.** We proceed as usual: let  $V = H_0^1(\Omega)$  and  $v \in V$ . We multiply v to (1a) and integrate over  $\Omega$ :

$$\int_{\Omega} -\mathbf{\nabla} \cdot (\mu \, \mathbf{\nabla} u) \, v d\mathbf{x} + \int_{\Omega} \sigma u v d\mathbf{x} = \int_{\Omega} f v \,,$$

$$\int_{\Omega} \mu \, \mathbf{\nabla} u \cdot \mathbf{\nabla} v d\mathbf{x} - \underbrace{\int_{\partial \Omega} \mu (\mathbf{\nabla} u \cdot \mathbf{n}) v d\gamma}_{=0} + \int_{\Omega} \sigma u v d\mathbf{x} = \int_{\Omega} f v d\mathbf{x} \,.$$

Let us define

$$a(u,v) = \int_{\Omega} \mu \nabla u \cdot \nabla v d\mathbf{x} + \int_{\Omega} \sigma u v d\mathbf{x} , F(v) \qquad = \int_{\Omega} f v d\mathbf{x} .$$

The weak formulation reads:

find 
$$u \in V$$
 :  $a(u, v) = F(v) \quad \forall v \in V$ . (2)

Let us introduce a mesh in  $\Omega$ , and let  $V_h = X_h^r \cap V$  be the finite element space. Let  $\varphi_i$  be its basis functions, for  $i = 1, 2, ..., N_h$ . After restricting (2) to  $V_h$  and writing the solution as a linear combination of the basis, we obtain

$$\sum_{i=1}^{N_h} U_j \left( \int_{\Omega} \mu \nabla \varphi_i \cdot \nabla \varphi_j d\mathbf{x} + \int_{\Omega} \sigma \varphi_i \varphi_j d\mathbf{x} \right) = F(\varphi_i) \qquad i = 1, 2, \dots, N_h ,$$

which can be written as a linear system:

$$A\mathbf{u} = \mathbf{f} ,$$

$$A_{ij} = \int_{\Omega} \mu \nabla \varphi_i \cdot \nabla \varphi_j d\mathbf{x} + \int_{\Omega} \sigma \varphi_i \varphi_j d\mathbf{x} ,$$

$$\mathbf{f}_i = F(\varphi_i) .$$

1.2. Starting from the code of Laboratory 3, implement a finite element solver for problem (1). The solver should read the mesh from file (four differently refined meshes are provided as mesh/mesh-square-\*.msh).

**Solution.** See the file src/lab-04.cpp for the implementation. An overview of the differences with respect to previous laboratory is provided below.

The major change is the introduction of the reaction term. This requires to change the assembly, adding to the local matrix the contribution of

$$\int_{\Omega} \sigma \varphi_i \varphi_j d\mathbf{x} .$$

Then, we have to import the mesh from file: see the method Poisson2D::setup for the implementation. The boundary of the imported mesh is labelled following the same convention as done by deal.II (and as seen in previous laboratory).

Finally, we need to impose the appropriate boundary conditions (homogeneous Dirichlet, in this case), and change the definition of the forcing term.

The numerical solution is shown in Figure 1.

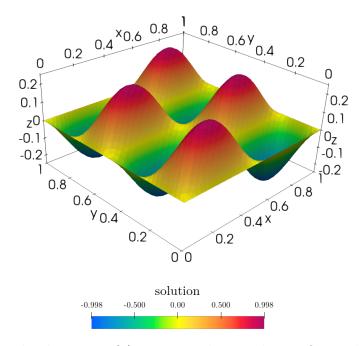


Figure 1: Numerical solution to (1), computed using linear finite elements on mesh mesh-square-40.msh.

**1.3.** Using the four meshes provided, study the convergence of the solver for polynomials of degree r=1 and of degree r=2. Plot the error in the  $L^2$  and  $H^1$  norms against h, knowing that for every mesh file mesh/mesh-square-N.msh, the mesh size equals h=1/N.

**Solution.** See the file src/lab-04.cpp for the implementation.

To study the convergence, we need to define a class ExactSolution to represent  $u_{\rm ex}$ . Moreover, we need to define a method Poisson2D::compute\_error to compute the error. The latter has the same implementation as seen in Laboratory 2, with two significant differences:

- the quadrature formula is defined using the class QGaussSimplex<dim>, so that it works for triangles;
- we need to provide an extra argument to the function integrate\_difference that explicitly describes the mapping  $\phi_c$  from the reference element to the physical elements. This mapping is constructed using the class MappingFE.

Then, the main function is modified to perform the convergence analysis and produce a CSV file convergence.csv, whose content can then be plotted to analyze the convergence results. See Figure 2 for the result. For both r = 1 and r = 2, we observe the expected convergence orders in both the  $L^2$  and  $H^1$  norms.

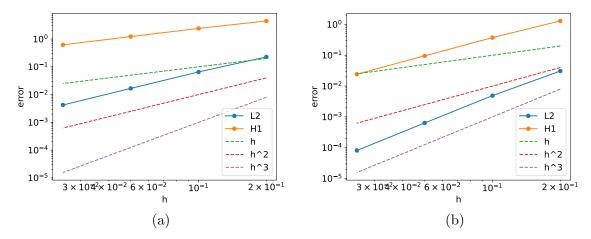


Figure 2: Error in the  $L^2$  and  $H^1$  norms as a function of h for r=1 (a) and r=2 (b).

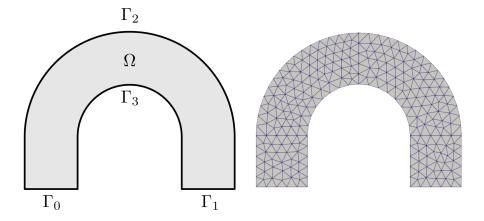


Figure 3: Domain for Exercise 2 (left), and a triangular mesh over it (right), corresponding to the file mesh/mesh-u-5.msh.

## Exercise 2.

Let  $\Omega$  be the domain depicted in Figure 3, contained in the files mesh/mesh-u-\*.msh. The boundaries of the mesh are labelled as shown in Figure 3 (i.e.  $\Gamma_0$  is labelled 0,  $\Gamma_1$  is labelled 1, and so on). Consider the problem:

$$\begin{cases}
-\nabla \cdot (\mu \nabla u) = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma_0, \\
u = 1 & \text{on } \Gamma_1, \\
\mu \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_2 \cup \Gamma_3.
\end{cases}$$

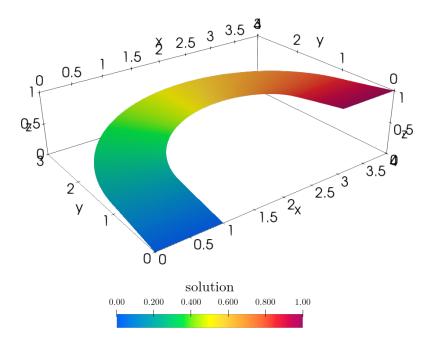


Figure 4: Numerical solution to Exercise 2, computed using linear finite elements and the mesh mesh-u-40.msh.

**2.1.** Starting from the previously implemented code, solve (2) using linear finite elements.

**Solution.** See the file src/lab-04.cpp for the implementation. The resulting solution is depicted in Figure 4.