General Equilibrium in Representative- and Heterogeneous-Agents Models with Explicit Prices

Macroeconomics 3: TA class #4

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Plan for Today

Objective: **Solve for the equilibrium when explicit prices are involved**

Two (sub-) goals:

- Learn how to solve GE macro models when prices are involved
- Learn how to solve GE macro models when agents are heterogeneous

Working Example

Consider this simple exchange economy with exogenous endowments

$$\max_{C_t, A_{t+1}} \mathbf{E}_0 \left(\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right)$$

$$\text{s.t.} \begin{cases} C_t + A_{t+1} \leq Y_t + (1+r_t)A_t \\ A_{t+1} \geq \underline{A} \end{cases}$$

Today we look at two versions:

- $Y_t = Y$ deterministically (representative-agent economy)
- Y_t is stochastic and idiosyncratic (heterogeneous-agents economy)

When Y_t is stochastic, we assume $Y_t \in \{Y^l, Y^h\}$, with

$$P(Y_{t+1}|Y_t) = \Pi = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$$

RA Economy: Market Clearing

The model is numerically uninteresting, the closed form solution for the price r is given by imposing $A_t^* = 0$ for all periods t, because we have one representative agent, therefore the net financial position must be zero

$$\begin{cases} r_t^* = 1/\beta - 1 & \forall\, t \\ C_t^* = Y & \forall\, t \\ A_t^* = 0 & \forall\, t \end{cases}$$

The Bellman equation

$$V(A) = \max_{C,A'} \frac{C^{1-\gamma}}{1-\gamma} + \beta\, V(A')$$
$$\text{s.t. } \begin{cases} C + A' \leq Y + (1+r)A \\ A' \geq \underline{A} \end{cases}$$

The market clearing condition $A_t^* = 0$ translates into this condition on the policy function: $A'(0) = 0$

- If $A'(0) > 0$, excess demand: RA wants to save, but nobody is there to sell assets
- If $A'(0) < 0$, excess supply: RA wants to borrow, but nobody is there to buy assets

RA Economy: Strategy for Numerical Solution

New element relative to past TA classes: price r

- Solve VFI/PFI given a numerical a value for r
- Check market clearing condition for asset holdings
  - If net excess demand > 0 (i.e., excess demand), r was too low: do it all again with higher r
  - If net excess demand < 0 (i.e., excess supply), r was too high: do it all again with lower r
  - If there is zero excess supply/demand, r was just right: model solved!

The net excess demand in this context is exactly $A'(0)$

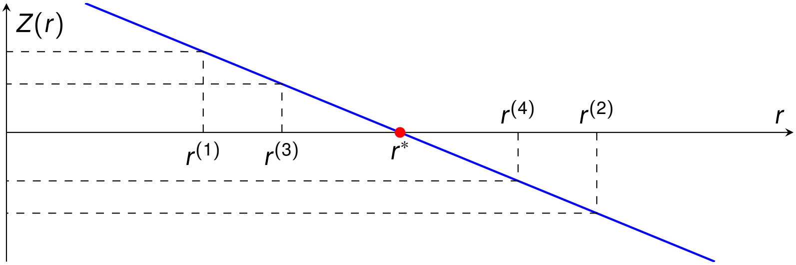
You can see why VFI/PFI must be fast: need to solve for policy functions over and over again

RA Economy: Intuition on Why/How This Works

- Net excess demand: $Z(r) \equiv D(r) - S(r)$
- From theory, $Z(r)$ is decreasing
- From theory, $\exists\, r^* : Z(r^*) = 0$

Algorithm: Given a guess $r^{(j)}$

- if $Z\left(r^{(j)}\right) > 0$, then $r^{(j)} < r^*$
- if $Z\left(r^{(j)}\right) < 0$, then $r^{(j)} > r^*$
- Set $r^{(j+1)}$ accordingly and repeat



RA Economy: Coding Approach

Objective: write a function that takes a price and returns the net excess demand at that price
Objective: use a zero-finding routine that finds the zero of the aforementioned function

The function $Z(r)$, given calibrated parameters and relevant grids

1. Solves VFI/PFI and extracts the policy functions
2. Computes the net excess demand
3. Returns the numerical value of the net excess demand

Then, use any of the appropriate functions in `scipy.optimize`:

- `bisect`
- `brentq`
- `ridder`
- `toms748`

Learn more at https://docs.scipy.org/doc/scipy/reference/optimize.html

HA Economy: Market Clearing

The Bellman equation

$$V(A, Y) = \max_{C,A'} \frac{C^{1-\gamma}}{1-\gamma} + \beta\, \mathbf{E}\left(V(A', Y') \middle| A, Y\right)$$

$$\text{s.t.} \begin{cases} C + A' \leq Y + (1+r)A \\ A' \geq \underline{A} \\ P(Y'|Y) = \Pi \end{cases}$$

We have

- An exogenous Markov chain $P(Y'|Y)$
- A policy function $A'(A, Y)$

We obtain

- An endogenous distribution $\lambda_t(A, Y)$
- An ergodic endogenous distr. $\lambda(A, Y)$

Market clearing: total savings = total borrowings

$$\int_A \int_Y \lambda(A, Y)\, A'(A, Y)\, \mathrm{d}Y\, \mathrm{d}A = 0$$

- For the household, r is taken as given (like a parameter)
- For the equilibrium, r depends on the infinite-dimensional object $\lambda(A, Y)$
- We say that $\lambda(A, Y)$ is an infinite-dimensional state variable (for the equilibrium!): infeasible in a computer, must approximate

Objective: approximate the ergodic endogenous distribution $\lambda(A, Y)$

HA Economy: The Endogenous Distribution of Agents

- The exogenous matrix Π maps Y into Y'
- The endogenous policy function $A'(A, Y)$ maps (A, Y) into A'
- Combine them to map (A, Y) into (A', Y')

Formally, let $\lambda_t(A, Y)$ be the endogenous joint distribution of agents at period t

$$\lambda_{t+1}(A', Y') = P(Y'|Y) \cdot A'(A, Y) \cdot \lambda_t(A, Y)$$

- The transition from (A, Y) to (A', Y') is regulated by an endogenous Markov process
- The distribution $\lambda(A, Y)$ is the ergodic distribution associated to such Markov process
- We normally focus on **ergodic recursive equilibria** (else, too much going on)

HA Economy: Strategy for Numerical Solution

- Solve VFI/PFI given a numerical value for r
- Recode the policy function as a set of transition matrices $(\bar{A}^k)_{k=0}^m$ such that

$$\bar{A}^k_{[i,j]} \equiv \begin{cases} 1 & \text{if } A'(A_i, Y_k) = A_j \\ 0 & \text{if } A'(A_i, Y_k) \neq A_j \end{cases}$$

- Combine the matrices $(\bar{A}^k)_{k=0}^m$ in a block diagonal matrix such that

$$\underset{[nm \times nm]}{\bar{A}} \equiv \begin{bmatrix} \bar{A}^1 & 0 & \cdots & 0 \\ 0 & \bar{A}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}^m \end{bmatrix}$$

- Compute the endogenous transition matrix Q as (maps (A, Y) into (A', Y'))

$$\underset{[nm \times nm]}{Q} \equiv (\Pi \otimes I_n) \cdot \bar{A}$$

- Compute the ergodic distribution associated with the transition matrix Q: that is $\lambda(A, Y)$

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Practice Time

Moving to a Jupyter Notebook

Exercises

1. Use the code I have showed for both examples
    1.1 Replace VFI with PFI
    1.2 Report on the speed improvements

2. The second example we saw today is essentially the Huggett model
    2.1 Adapt the code such that it is written as one coherent Python `class`
    2.2 Generalize the code to accept any AR(1) process for the endowment process
    2.3 In what sense the transition matrix Q is obtained the quick-and-dirty way? How could you address the issue?