Symbolic dynamics for the anisotropic N-centre problem at negative energies

Susanna Terracini
Università di Torino

Joint works with
Vivina Barutello, Gianmarco Canneori, Nicola Soave and Gianmaria Verzini

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Motivation

- It is part of the mathematical folklore that Dynamical Systems featuring many nonlinear interactions should display chaotic behavior and possess complex dynamics, whatever this means.
- On the other hand, for natural systems, this lacks a rigorous statement and even more, a rigorous proof, specially when we are leaving the perturbative setting.
- Starting from papers by Mather, Séré, Coti-Zelati & Rabinowitz in the early '90s, there has been an attempt to construct complex trajectories by the use of global variational methods.

Periodic trajectories

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable...".

According to Poincaré, periodic orbits catch the complexity of the global dynamics:

"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13[1] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."*

H. Poincaré: Les Méthodes Nouvelles de la Mécanique Céleste (1892)

[1] Formula n. 13 is Hamilton equation.

The N-body problem

We wish to construct complex (periodic/chaotic) trajectories of N heavy bodies which move in $\mathbb{R}^2$ under their mutual gravitational attraction.
Motion equation:

$$m_j \ddot{x}_j(t) = -\sum_{k \neq j} \frac{m_j m_k}{|x_k(t) - x_j(t)|^3} (x_k(t) - x_j(t)),$$

where $x_k(t)$ denotes the position of the k-th body at time t, and $m_k > 0$ its mass.

Generally, global complex dynamics is connected with the existence of periodic solutions featuring a complex behaviour (Poincaré conjecture).

Periodic solutions: a global approach

We may try to take advantage of

- Symmetries: the problem is invariant with respect to
  - the orthogonal group $O(d)$,
  - if $m_i = m_j$, the permutations of x_i and x_j,
  - time shift and reversal.
- Topology: of the loop space over the N-body configuration space.

Due to the difficulty of the general problem, we may also think to some (yet highly nontrivial) simplifications:

- the (circular, planar) restricted N-body problem;
- the N-center problem.

Settings

- N point particles with masses $m_1, m_2, \ldots m_N$ and positions $x_1, x_2, \ldots,$ $x_N \in \mathbb{R}^d$, with $d \geq 2$, which form the configuration space $\mathcal{X} = \mathbb{R}^{Nd}$.
- Interaction potential: $U(x) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$; when $\alpha = 1$ we have the gravitational Newton potentials.
- On collisions ($x_i = x_j$ for some $i \neq j$) potential $U = +\infty$. Admissible configurations: $\tilde{\mathcal{X}} = \mathcal{X} \setminus \{\mathit{collisions}\}$.
- Collisionless T-periodic orbits: solutions of the Newton equations (such that $\forall t : x(t + T) = x(t) \in \tilde{\mathcal{X}}$).

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}.$$

Action and Maupertuis' functionals

- Lagrangian: $L(x,\dot{x}) = \overbrace{\sum_i \frac{1}{2} m_i |\dot{x}_i|^2}^{K} + \overbrace{\sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}}^{U}.$
- Action functional: for $x \in H^1_T$ ($T > 0$ fixed period)
  $\mathcal{A}_T(x) = \int_0^T L(x(t),\dot{x}(t))dt.$
- Maupertuis functional: $x \in H^1_1$ ($h \in \mathbb{R}$ fixed energy)
  $\mathcal{J}_h(x) = \left(\int_0^1 K dt\right)\left(\int_0^1 (h+U)dt\right).$

Variational approach

We seek critical points of the action functional, (or the Maupertuis one) on

$$\mathcal{A}\colon \Lambda \to \mathbb{R} \cup \infty, \quad x \mapsto \int_0^T \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^\alpha}$$

constrained on suitable admissible linear subspaces $\Lambda_0 \subset \Lambda$.
Special case: two bodies and $\alpha = 1$.

- First attempt:

$$\mathcal{A}\colon \Lambda \to \mathbb{R} \cup \infty, \quad x \mapsto \int_0^T \frac{1}{2} m_1 |\dot{x}_1|^2 + \frac{1}{2} m_2 |\dot{x}_2|^2 + \frac{m_1 m_2}{|x_1 - x_2|}$$

  minimize $\mathcal{A}$ over Λ. No minimizer.

- Second attempt: minimize $\mathcal{A}$ over $\{x \in \Lambda \ : \ \deg(x_1 - x_2; 0) \neq 0\}$. A continuous of minimizers connecting the circular trajectory with a degenerate ellipse (collision minimizer). Very degenerate and unstable under small perturbations.

Theorem (W. Gordon 1977)

The keplerian ellipses minimize the action among loops having nontrivial winding number about the origin.

- Third attempt: minimize $\mathcal{A}$ over

$$\{x \in \Lambda \ : \ \deg(x; 0) \neq 0 \ \& \ |x_1(t) - x_2(t)| \geq \varepsilon\}$$

Success (?)

Main problems:

- The action functional is not coercive on Λ. The minimum needs not to be achieved.
- We can seek critical point others than minimizers: e.g.
  - Local minimizers
  - Constrained minimizers
  - Other type of critical points (mountain pass).
- The action functional does not satisfy the Palais-Smale compactness condition on Λ: sequences of almost–critical points may diverge.
- The potential U is singular on collisions, and thus miminizers or other critical points can *a priori* have collisions.

Collisions

Theorem (Marchal, Chenciner, Ferrario-Terracini)

For the N-body problem in $\mathbb{R}^d$, with $d \geq 2$, a trajectory which is locally minimizing in the sense of Morse (for either the action or the Maupertuis functional) is free of collisions, for every choice of isotropic α-homogeneous potentials.

Locally minimizing here means with respect to small compactly supported variations. The proof relies upon an averaged variation technique.

- Averaging destroys topology.

For topologically constrained minimizers Marchal's argument does not work, and other devices have to be designed to avoid the occurrence of collisions.

Absence of collisions for minimizers of Bolza problems in the isotropic case

In the planar case, we use polar coordinates (r, θ).

Definition

We say that $x = (r, \vartheta) \in \mathcal{AC}(t_1, t_2)$ is a fixed-time Bolza minimizer associated to the ends $x_1 = r_1 e^{i\varphi_1}$, $x_2 = r_2 e^{i\varphi_2}$, in the sector $(\vartheta^-, \vartheta^+)$, if

- $\vartheta^- \leq \vartheta(t) \leq \vartheta^+ \ \forall t \in [t_1, t_2]$;
- $r(t_i) = r_i$ and $\vartheta(t_i) = \varphi_i$, $i = 1, 2$;
- for every $z = (\rho, \zeta) \in \mathcal{AC}(t_1, t_2)$ taking values in the sector $(\vartheta^-, \vartheta^+)$, there holds

$$\rho(t_i) = r_i,\ \zeta(t_i) = \varphi_i, \quad \Longrightarrow \quad ([t_1, t_2]; x) \leq ([t_1, t_2]; z).$$

If $\min_{t \in [t_1, t_2]} r(t) > 0$ we say that the Bolza minimizer is collisionless.

Theorem (Soave-T., 2013)

Consider a perturbed Kepler potential $V = \dfrac{1}{r^\alpha} + W$*, with* $\alpha > \alpha'$ *and*

$$\lim_{r\to 0} r^{\alpha'}\left(W(x) + r|\nabla W(x)|\right) = 0\ .$$

Given any pair of points x_1 *and* x_2 *in the sector* $(\vartheta^-, \vartheta^+)$*, if* $\vartheta^+ - \vartheta^- < 2\pi/(2-\alpha)$ *then all fixed-time Bolza minimizers associated to* x_1*,* x_2 *within the sector* $(\vartheta^-, \vartheta^+)$ *are free of collisions.*

The restriction that the minimizing path stays in the sector $(\vartheta^-, \vartheta^+)$ can be removed, when $W \equiv 0$, as it is implied by the conservation of the angular momentum. The theorem easily extends to $\mathbb{R}^3 \setminus \{x_1 = x_2 = 0\}$ for potentials with cylindrical symmetry (Hip-hop, T.- Venturelli)

Some remarks

- If conversely $\vartheta^+ - \vartheta^- \geq 2\pi/(2-\alpha)$, then there are always some Bolza problems which admit only collision minimizers. It is enough to chose $x_1 = \vartheta^-$ and $x_2 = \vartheta^+$ and T as the natural time of the free time minimizer.
- If $\vartheta^+ - \vartheta^- = 2\pi/(2-\alpha)$ and $W \equiv 0$, then the following alternative holds:
  - either the minimizer is collisionless,
  - or $(x_1, x_2) = (\vartheta^-, \vartheta^+)$ and the minimizer is a collision–ejection homothetic trajectory.
- The latter statement is a generalization of the Marchal's statement about the existence of direct and inverse action–minimizing keplerian arcs.

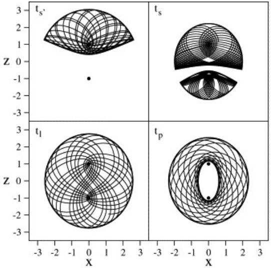
The fixed N-centre problem

***N*-centre problem**: the determination of the trajectory of a test particle, moving under the gravitational attraction of $c_1, \ldots, c_N \in \mathbb{R}^d$ $(d = 2, 3)$ fixed heavy bodies.

The equation of motion is

$$\ddot{x}(t) = -\sum_{j=1}^{N} \frac{m_j}{|x(t) - c_j|^3}(x(t) - c_j)$$

Like the one Kepler center, also the planar 2-center problem can be integrated through the use of elliptic-hyperbolic coordinates.
Trajectories can be classified according to their topological type.



Some references

When $N \geq 3$, in positive energy shells, the system starts to be chaotic.

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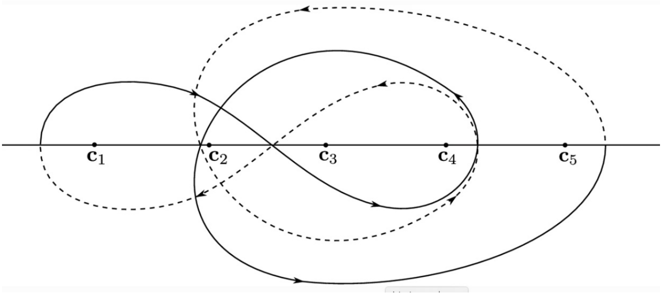
Positive energy case

Theorem (Klein-Knauf, Bolotin-Kozlov, Castelli, Chen-Yu)

In the planar N-centre here are infinitely many noncontractible collision-free periodic trajectories with non negative energy, which are minimizers of the Jacobi length

$$J(x) = \int_0^1 \|\dot{x}(t)\| \sqrt{V(x(t)) + h}\, dt$$

in suitable homotopy classes. Furthermore, the phase flow on the energy level h admits a compact chaotic invariant set with positive topological entropy.

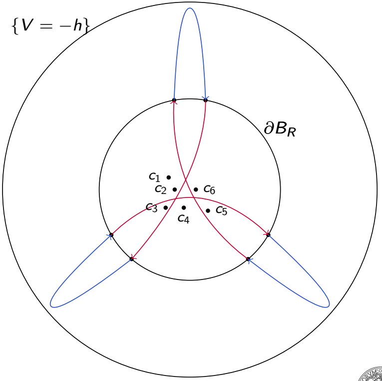


Negative energy shells

N. Soave, S. Terracini, Symbolic dynamics for the N-centre problem, (2012): at slightly negative energy, show existence of non-collision periodic trajectories separating the centres according to a sequence of prescribed partitions, as critical points of the Jacobi length.

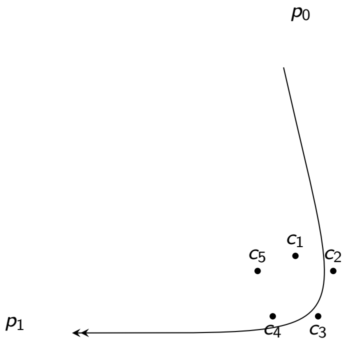
$$J(x) = \int_0^1 \|\dot{x}(t)\| \sqrt{V(x(t)) + h}\, dt$$

1. The metric degenerates at the boundary of the Hill's region $\{V = -h\}$
2. Geodesic arcs approaching the boundary cease to be minimal geodesics for J
3. How to avoid collisions.



Partitions of the centers and parabolic trajectories

If $\alpha \in [1, 2)$, given a partition of the centers and two asymptotic directions, there is a collisionless minimizing parabolic trajectory dividing the centers according with this particular partition (with a few exceptions).



Next:

- We scale to normalize the energy;
- introduce as parameter $\varepsilon > 0$ the maximal distance of the centers from their barycenter.

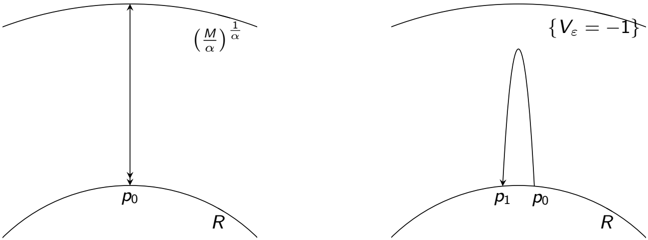
Remark

Under a suitable rescaling $y(t)$ of a classical solution $x(t)$, we have

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h. \end{cases} \quad \Leftrightarrow \begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1, \end{cases}$$

and viceversa, with $\varepsilon = (-h)^{1/\alpha_1}$.

Outer dynamics



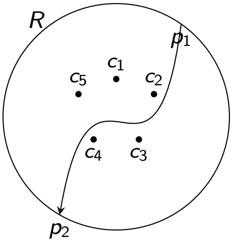
Theorem (outer arcs)

There exist $\delta > 0$ and $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$ we find $T > 0$ and a unique solution $y_{\mathrm{ext}}(\cdot\,; p_0, p_1; \varepsilon)$ of

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) & t \in (0, T), \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1 & t \in (0, T), \\ |y(t)| > R & t \in (0, T), \\ y(0) = p_0, \quad y(T) = p_1. \end{cases}$$

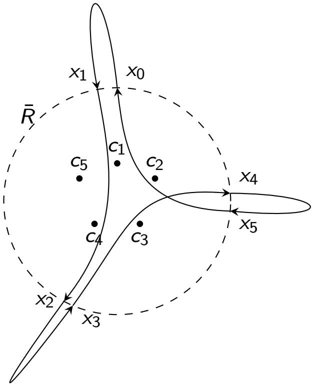
Inner dynamics

We will consider $\varepsilon \in (0, \bar{\varepsilon})$ small. Given two points $p_1, p_2 \in \partial B_R(0)$ and a partition of the centers in two nonempty sets, there exists a non collision, self-intersection free Morse-minimizing arc which separates the centers according with the prescribed partition.



Periodic solutions

Given a sequence of partitions, we alternate outer and inner arcs realizing those partitions. Now, the only variables are the common endpoints on the circle. Extremals of the total length will match the derivatives at the junctions.



Anisotropic potentials

M.C. Gutzwiller introduced the Anisotropic Kepler problem as a classical mechanical approximation to certain quantum mechanical systems (J. Math. Phys. 1973-77)

Take the potential

$$V_\mu(x) = \frac{1}{\sqrt{\mu x_1^2 + x_2^2}}, \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}, \;\; \mu > 1$$

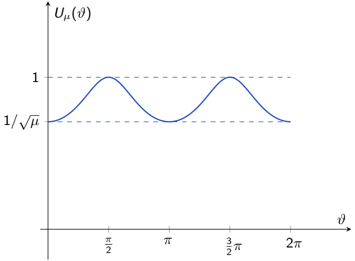
and study the system

$$\begin{cases} \ddot{x}(t) = \nabla V_\mu(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V_\mu(x) = h \end{cases}$$

- $V_\mu(x) = r^{-1} U_\mu(\vartheta)$, where

  $U_\mu(\vartheta) = V_\mu(\cos\vartheta, \sin\vartheta)$

- non-radial force field



The anisotropic N-centre problem

We consider the sum of N-attracting anisotropic and $-\alpha$-homogenous potentials.

$$V_j(x) = |x|^{-\alpha_j} V_j\left(\frac{x}{|x|}\right) = r^{-\alpha_j} U_j(\theta) \ ,$$

where (r, θ) are polar coordinates and $U_j \doteq V_j|_{\mathbb{S}^1}$. Denoting by $c_1, \ldots, c_N \in \mathbb{R}^2$ the positions of the $N \geq 2$ centres, we introduce the total potential

$$V(x) = \sum_{j=1}^{N} V_j(x - c_j) = \sum_{j=1}^{N} |x - c_j|^{-\alpha_j} V_j\left(\frac{x - c_j}{|x - c_j|}\right) .$$

so that the equation of motion reads as

$$\ddot{x}(t) = \nabla V(x(t)),$$

where $x = x(t)$ represents the position of the moving particle at time $t \in \mathbb{R}$.

The anisotropic N-centre problem

At first, without loss of generality, we can assume

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N.$$

Clearly, the smallest degree of homogeneity α_1 leads the overall potential at infinity. Hence, assuming that $\alpha_1 = \alpha_2 = \ldots = \alpha_k$ for some $1 \leq k < N$, and denoting $\alpha \doteq \alpha_1$, it is convenient to gather all the $-\alpha$-homogeneous potentials in this way

$$V(x) = W(x) + \sum_{j=k+1}^{N} |x - c_j|^{-\alpha_j} V_j \left(\frac{x - c_j}{|x - c_j|} \right) ,$$

where

$$W(x) \doteq \sum_{i=1}^{k} V_i(x - c_i) = \sum_{i=1}^{k} |x - c_i|^{-\alpha} V_i \left(\frac{x - c_i}{|x - c_i|} \right),$$

so that $W \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{c_1, \ldots, c_k\})$. We set:

$$U(\vartheta) \doteq \sum_{i=1}^{k} U_i(\vartheta), \quad \text{for } \vartheta \in \mathbb{S}^1,$$

so that $W(x) \simeq |x|^{-\alpha} U\left(\frac{x}{|x|}\right)$ when $|x| >> 1$.

The planar case

In the planar case, we use polar coordinates about the singularity.

Definition

Given $U = U(\theta)$ and two absolute minimizers $[\vartheta^-, \vartheta^+]$ for U, we define $\bar{\alpha}(U, \vartheta^-, \vartheta^+)$ as the unique exponent for which the potential $V = U(\theta)/r^\alpha$ admits a global parabolic trajectory with asymptotic directions ϑ^- and ϑ^+.

Theorem (Barutello-T-Verzini 2014)

Let $\vartheta^- < \vartheta^+$, and consider a perturbed potential $V = \dfrac{U(\vartheta)}{r^\alpha} + W$, with $V \in \mathcal{C}^1(\mathbb{R}^2 \setminus 0)$, $\alpha > \alpha'$ and

$$\lim_{r \to 0} r^{\alpha'} \left(W(x) + r|\nabla W(x)| \right) = 0 \,.$$

If $\alpha > \bar{\alpha}(U, \vartheta^-, \vartheta^+)$ then all fixed-time Bolza minimizers associated to $x_1 = (r_1, \varphi_1)$ and $x_2 = (r_2, \varphi_2)$ within the sector $[\vartheta^-, \vartheta^+]$ are collisionless.

It is worthwhile noticing that, if conversely $\alpha \le \bar{\alpha}(U, \vartheta^-, \vartheta^+)$, then there are always some Bolza problems which admit only colliding minimizers.

Theorem (Barutello-T-Verzini 2014)

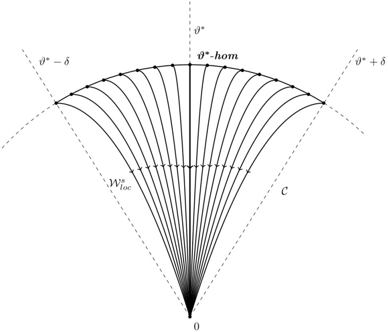
Let U, $\vartheta^- < \vartheta^+$, and V be a perturbed potential as in the previous theorem, with $\alpha = \bar{\alpha}(U, \vartheta^-, \vartheta^+)$. Given any pair of points x_1 and x_2 in the sector $(\vartheta^-, \vartheta^+)$, all fixed-time Bolza minimizers associated to x_1, x_2 within the sector $[\vartheta^- + \varepsilon, \vartheta^+ - \varepsilon]$, for some $\varepsilon > 0$, are collisionless.

Some further interesting consequences can be drawn, in the special case when $\vartheta^+ = \vartheta^- + 2k\pi$, which connect the parabolic threshold with the existence of non-collision periodic orbits having a prescribed winding number (this is connected with the minimizing property of Kepler ellipses, highlighted by Gordon).

What happens at threshold?

Theorem (Barutello-Canneori-T. 2020)

Let $\vartheta^* \in \mathbb{S}^1$ be a minimal non-degenerate central configuration for U_j. There exists $r^* > 0$ and $\delta > 0$ such that, for every $q = re^{i\vartheta}$ with $r < r^*$ and $\vartheta \in (\vartheta^* - \delta, \vartheta^* + \delta)$ there exists a unique minimizer of the Maupertuis' functional in the set of colliding paths H^q_{coll}. In particular, this path cannot leave the cone emanating from the origin and bounded by the arc-neighbourhood $(\vartheta^* - \delta, \vartheta^* + \delta)$.



Central configurations

Critical point of the potential U on the spere will be termed a central configuration. Our basic assumption on U is about the number of its non-degenerate minimal central configurations.

$$\begin{cases} \alpha < 2; \\ \exists\, (\vartheta_l^*)_{l=1}^m \subseteq \mathbb{S}^1 \ : \ \ \forall\ l = 1, \dots, m, \\ U''(\vartheta_l^*) > 0,\ U(\vartheta) \geq U(\vartheta_l^*) > 0,\ \forall\ \vartheta \in \mathbb{S}^1. \end{cases} \tag{V}$$

Energy shells

The associated Hamiltonian being

$$H(x, v) = \frac{1}{2}|v|^2 - V(x),$$

every solution of the motion equation verifies the energy conservation law

$$\frac{1}{2}|\dot{x}|^2 - V(x) = h. \tag{1}$$

For a fixed energy $h < 0$ we seek critical points of the Jacobi length

$$J(x) = \int_0^1 \|\dot{x}(t)\| \sqrt{V(x(t)) + h}\, dt$$

in the bounded Hill's region

$$\mathcal{R}_h = \{x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \ : \ V(x) \geq -h\}.$$

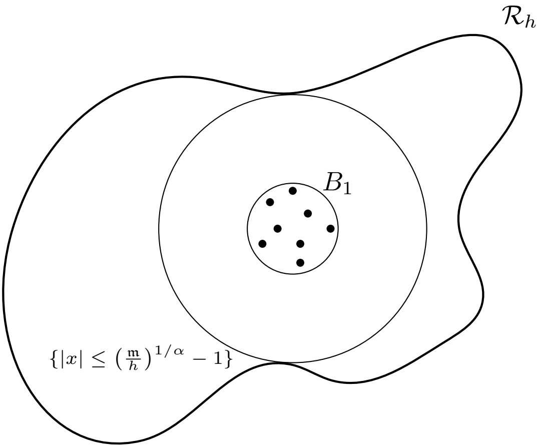
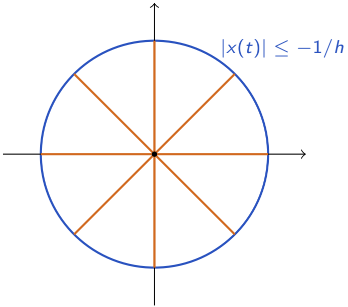
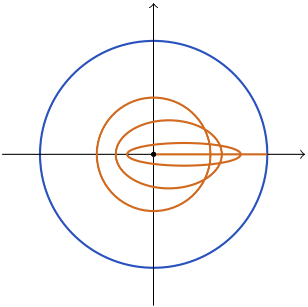


Figure: An example of Hill's region for the anisotropic N-centre problem that includes a ball of radius greater than 1.

The anisotropy changes the structure of the phase portrait

Kepler problem at negative energies $\rightarrow$ $h < 0$, $\mu = 1$

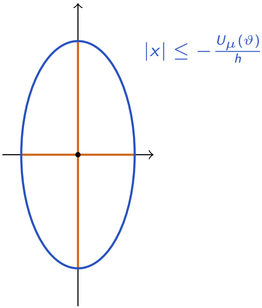




- circular Hill's region
- all configurations are central and allow for homothetic collision motions

- near-collision orbits remain *close* to the 1-dimensional collision orbit
- the problem is integrable and collisions can be regularized

Anisotropic Kepler problem at negative energies $\rightarrow h < 0,\ \mu > 1$



- non-radial Hill's region
- 1-dimensional collision orbits occur only along critical points of U_μ: the central configurations
- no conservation of the angular momentum
- the integrability of the system may be destroyed by the anisotropy, together with collisions regularization

Main results about one anisotropic center: R.L. Devaney (1978,..): non-integrability, collision orbits analysis, McGehee coordinates, symbolic dynamics..

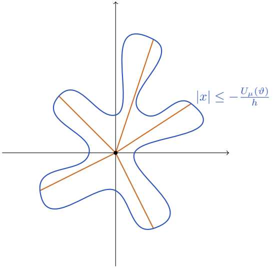
More general anisotropic potentials

V. Barutello, S. T., G. Verzini - *Entire minimal parabolic trajectories, the planar anisotropic Kepler problem* (ARMA 2013)

The authors considered a larger class of homogeneous potentials

$$\begin{cases} V \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\}) \\ V(x) = |x|^{-\alpha} V(x/|x|), \quad \alpha \in (0,2) \end{cases}$$

- the Hill's region can be very corrugated
- define $U(\vartheta) \doteq V(\cos\vartheta, \sin\vartheta)$
- assume that $\exists\, \vartheta^* \in \mathbb{S}^1$ such that $U(\vartheta) \geq U(\vartheta^*) > 0$ for every $\vartheta \in \mathbb{S}^1$ and $U''(\vartheta^*) > 0$



Collisional symbolic dynamics

To state our first result, we need to take into account different definitions of solutions, allowing for collisions.

Definition

We define a non-collision solution in the interval I as a $\mathcal{C}^2$-function $x\colon I \subseteq \mathbb{R} \to \mathbb{R}^2$ such that $x(t) \neq c_j$ for every $t \in I$ and for every $j = 1, \ldots, N$ and that solves the motion equation in the classical sense. We say that x is a collision solution if $x \in H^1(I)$ and there exists a collision instants set $T_c(x) \subseteq J$ such that:

- the set $T_c(x)$ has null measure;
- for any $t \in T_c(x)$, it holds $x(t) = c_j$, for some $j = 1, \ldots, N$;
- for any $(a, b) \subseteq I \setminus T_c(x)$, the restriction $x|_{(a,b)}$ is a non-collision solution;
- for every $t \in J \setminus T_c(x)$, $x(t)$ verifies the energy equation.

It can be shown that the collision set is actually finite. A more stringent definition requires the trajectory to be a limit of noncollision (constrained) minimal arcs.

Symbolic dynamics (possibly collisional) in the anisotropic case

Our first main result states the existence of a (possibly collisional) symbolic dynamics in the presence of at least two centres and two minimal non-degenerate central configurations for W.

Theorem (Barutello,Canneori, T. 2021)

Assume that $N \geq 2$, $m \geq 2$ and (V) holds. There exists $h^* > 0$ and a finite set of symbols $\mathcal{S}$ such that, for every $h \in (0, h^*)$, there exist a subset Π_h of the energy shell $\mathcal{E}_h$, a (possibly collisional) first return map $\mathfrak{R}\colon \Pi_h \to \Pi_h$ and a continuous and surjective map $\pi\colon \Pi_h \to \mathcal{S}^{\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathcal{L} \arrow[r, "T_r"] & \mathcal{L} \end{tikzcd}$$

commutes. In other words, for h sufficiently small, the anisotropic N-centre problem at energy $-h$ admits a symbolic dynamics.

Collisions

We now address the problem of collision exclusion, an issue intimately related to the lack of regularizability of collisions typical of the anisotropic case. Let $W \in \mathcal{C}^1(\mathbb{R}^2 \setminus \{0\})$ be a perturbation for V such that

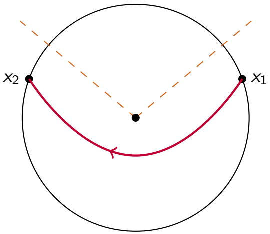
$$\lim_{|x|\to 0} |x|^{-\alpha'} \left(W(x) + r|\nabla W(x)|\right) = 0$$

for some $\alpha' < \alpha$ and define $\tilde{V} \doteq V + W$.
Introduce the Lagrange-action functional

$$\mathcal{A}(x) \doteq \int_a^b \left(\frac{1}{2}|\dot{x}(t)|^2 + \tilde{V}(x(t))\right) dt, \quad x \in H^1([a,b];\mathbb{R}^2)$$

minimizers of $\mathcal{A}$ solve the fixed end (Bolza) problem:

$$\begin{cases} \ddot{x}(t) = \nabla \tilde{V}(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - \tilde{V}(x(t)) = h \\ x(a) = x_1, \; x(b) = x_2 \end{cases}$$

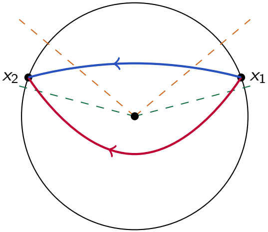


Bolza minimizers

Theorem (Barutello, T., Verzini (ARMA 2014))

There exists $\bar{\alpha} = \bar{\alpha}(U) \in (0,2)$ such that if $\alpha > \bar{\alpha}$ then any fixed-time Bolza minimizer for $\mathcal{A}$ is collision-less.

In fact we have two minimizing arcs joining x_1 and x_2: a long and a short one:



The anisotropic N-centre problem

Assume that every centre is anisotropic (in the Barutello-Terracini-Verzini sense)

$$\frac{m_j}{\alpha|x-c_j|^\alpha} \quad \rightsquigarrow \quad V_j(x-c_j) = |x-c_j|^{-\alpha_j} V_j\left(\frac{x-c_j}{|x-c_j|}\right),$$

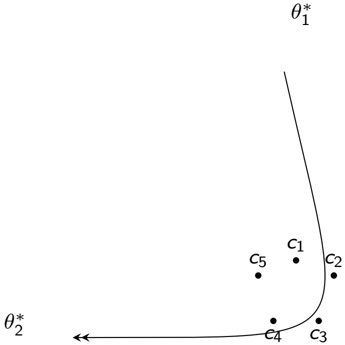
where $V_j \in \mathcal{C}^2(\mathbb{R}^2 \setminus \{0\})$ is a positive $-\alpha_j$-homogeneous function.
Define $U_j \doteq V_j|_{\mathbb{S}^1}$.
Assume that

$$\begin{cases} 0 < \alpha_1 \leq \ldots \leq \alpha_N \\ \forall j \;\; U_j \text{ has at least a global strict minimizer } \vartheta_j^* \in \mathbb{S}^1 \\ \forall j \;\; \alpha_j > \bar{\alpha}_j(U_j) \end{cases} \tag{V_α}$$

From now on we will mainly refer to ϑ_j^* as a minimal non-degenerate central configuration for V_j

Partitions of the centers and parabolic trajectories

If $\alpha_j > \bar{\alpha}_j(U_j)\ \forall j$, given a partition of the centers and two asymptotic directions which are non degenerate minimal central configurations for V_1, there is a collisionless minimizing parabolic trajectory dividing the centers according with this particular partition.



Noncollision symbolic dynamics

Theorem (Barutello,Canneori, T. 2021)

Assume that $N \geq 2$, $m \geq 1$ and (V)-$(V)_\alpha$ hold. There exists $h^* > 0$ and a finite set of symbols $\mathcal{S}$ such that, for every $h \in (0, h^*)$, there exist a subset Π_h of the energy shell $\mathcal{E}_h$, a (non collision) first return map $\mathfrak{R}\colon \Pi_h \to \Pi_h$ and a continuous and surjective map $\pi\colon \Pi_h \to \mathcal{S}^{\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} \Pi \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Pi \arrow[d, "\pi"] \\ S^{\mathbb{Z}} \arrow[r, "T_r"] & S^{\mathbb{Z}} \end{tikzcd}$$

commutes. In other words, for h sufficiently small, the anisotropic N-centre problem at energy $-h$ admits a non collision symbolic dynamics.

Scaling

For $\varepsilon > 0$, consider the rescaled potential

$$V_\varepsilon(y) = V_1(y - \varepsilon c_1) + \sum_{j=2}^{N} \varepsilon^{\alpha_j - \alpha_1} V_j(y - \varepsilon c_j).$$

Recall that $\alpha_1 \le \alpha_2 \le \ldots \le \alpha_N$

The idea is to put all the centres inside B_ε and then to let $\varepsilon \to 0^+$.

Remark

Under a suitable rescaling $y(t)$ of a classical solution $x(t)$, we have

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h. \end{cases} \quad \Leftrightarrow \quad \begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1, \end{cases}$$

and viceversa, with $\varepsilon = (-h)^{1/\alpha_1}$.

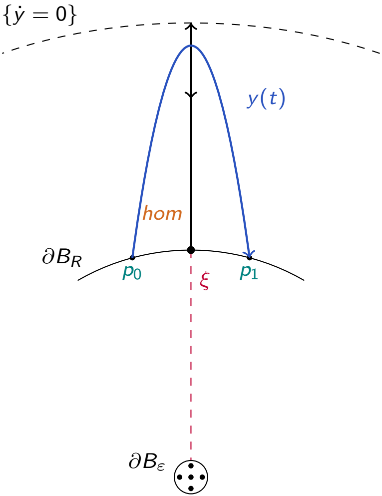
Perturbated arcs

Far enough from the centres, the motion follows a perturbation of an anisotropic Kepler problem driven by potential V_1:

Proposition

Given $\varepsilon > 0$ small and $R >> \varepsilon$, for every $y \in \mathbb{R}^2 \setminus B_R$ we have

$$V_\varepsilon(y) = |y|^{-\alpha_1} V_1\left(\frac{y}{|y|}\right) + O(\varepsilon^{\alpha_2-\alpha_1}) \quad \text{as } \varepsilon \to 0^+.$$



$R >> 1$ and ξ minimum for $V_1|_{\mathbb{S}^1}$.

- $hom(t)$ is a 1-dimensional homothetic trajectory for V_1, which exists only along the direction ϑ_1^* (or along other central configurations)
- when $|y|$ becomes large, close to $\xi = Re^{i\vartheta_1^*}$ it is possible to shadow homothetic trajectories in order to solve a boundary value problem

Shadowing 1-dimensional homothetic orbits

Theorem (Barutello, Canneori, T. (2020))

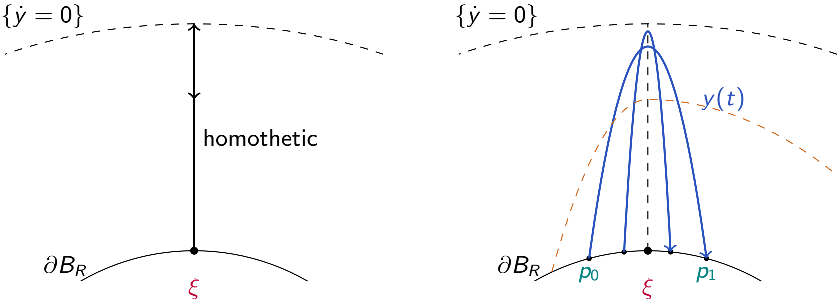
Let $\vartheta_1^* \in \mathbb{S}^1$ be a non-degenerate minimal central configuration for V_1. Then, for small ε there exists a neighbourhood $\mathcal{U} \subseteq \partial B_R$ of $\xi = Re^{i\vartheta_1^*}$ such that, for every $p_0, p_1 \in \mathcal{U}$, problem

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1 \\ |y(t)| > R \\ y(0) = p_0, \quad y(T) = p_1 \end{cases}$$

admits a unique solution $y(t)$.
Moreover, y depends on a $\mathcal{C}^1$-way from the endpoints p_0 and p_1.

For radial potentials, this is true for every $p_0, p_1 \in \partial B_R$ sufficiently close to each other (Soave-Terracini DCDS 2012).

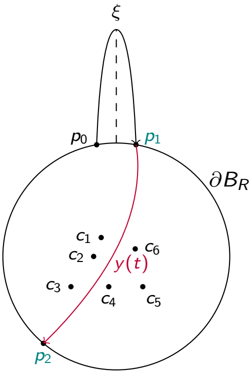
Sketchy proof:



- the homothetic trajectory exists along ξ (brake orbit)
- shoot from a starting point in ∂B_R close enough to ξ (Cauchy problem)
- hit again ∂B_R at the first return instant for transversality of the flow
- local inversion through the implicit function theorem in a neighbourhood of ξ
- Cauchy problem $\rightarrow$ boundary value problem

Inner arcs

The next step consists in building a minimizing collision-less arc which connects $p_1, p_2 \in \partial B_R$.



Find a classical solution of the inner problem

$$
\begin{cases}
\ddot{y}(t) = \nabla V_\varepsilon(y(t)) & t \in [0, T] \\
\frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1 & t \in [0, T] \\
|y(t)| < R & t \in (0, T) \\
y(0) = p_1, \quad y(T) = p_2 &
\end{cases}
$$

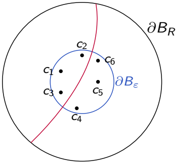
for a prescribed partition of the centres

Direct methods

- Define the Maupertuis' functional

$$\mathcal{M}(u) = \frac{1}{2}\int_0^1 |\dot{u}(t)|^2\,dt \int_0^1 (V_\varepsilon(u(t)) - 1)\,dt \qquad \text{for } u \in H^1([0,1];\mathbb{R}^2);$$

- fix a partition $\mathcal{P}$ of the centres in two non-trivial subsets;



- introduce the set of admissible paths

$$K_{\mathcal{P}} \doteq \left\{ u \in H^1([0,1];\mathbb{R}^2) : \begin{array}{l} u(0) = p_1,\ u(1) = p_2,\ |u| \le R, \\ u \text{ separates the centres } w.r.t.\ \mathcal{P}, \\ u \text{ does not collide with the centres} \end{array} \right\};$$

- allow collisions to w-close $K_{\mathcal{P}}$;
- find a minimizer for $\mathcal{M}$ in the closure of $K_{\mathcal{P}}$ through direct methods.

Classical solutions

Consider again the Lagrange-action functional

$$\mathcal{A}(y) = \int_0^T \left[\frac{1}{2}|\dot{y}(t)|^2 + V_\varepsilon(y(t)) - 1\right] dt \qquad \text{for } y \in H^1([0,T];\mathbb{R}^2)$$

Remark: up to time-reparameterizations

$$\min\left\{\mathcal{A}(y) : y \in H^1(0,T) \ + \ \text{topol. constraint}\right\}$$
$$= \min\left\{\sqrt{2\mathcal{M}(u)} : u \in H^1(0,1) \ + \ \text{topol. constraint}\right\}$$

Lemma (Barutello, Canneori, T.)

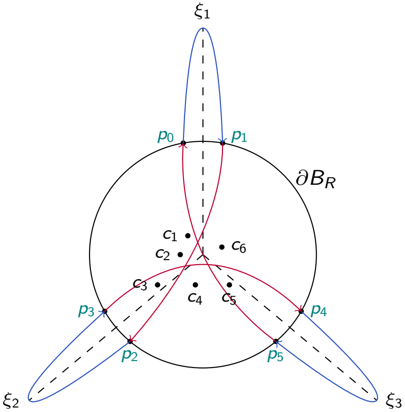
Writing

$$V_\varepsilon(y) = |y|^{-\alpha_j} U_j(\vartheta) + \sum_{i \neq j} V_i(y),$$

if $\alpha_j > \bar{\alpha}_j(U_j)$, then the minimizer of $\mathcal{A}$ does not collide in c_j.

Maupertuis' principle $\rightarrow$ the minimizer represents a classical solution of the inner problem

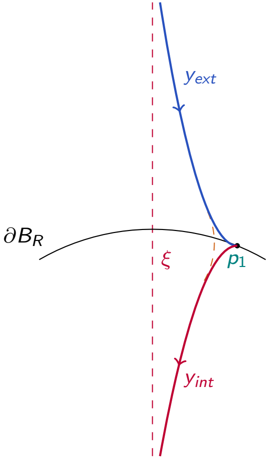
Periodic solutions



Given a sequence of minimal non-degenerate central configurations for V_1 and a sequence of partitions of the centres, we alternate outer and inner arcs.

Glueing arcs

- Every building arc is smooth for classical regularity arguments
- We make use of a broken geodesics argument so that the smoothness is preserved through the junctions
- Extremals of the total length will match the derivatives at junctions



Symbolic dynamics for the N-centre problem

As a corollary of the existence of periodic orbits, we can prove that our dynamical system displays a symbolic dynamics.

- A periodic orbit can be discretized by alternating a minimal central configuration for V_1 (outer arc) and a partition of the centres (inner arc).
- We collect all the possible choices in a finite set S such that

$$\#S = m(2^{N-1} - 1),$$

  for N centres and m non-degenerate minimal central configurations for V_1.
- We prove that the N-centre dynamical system evolving in constant energy shells is topologically semi-conjugate to T_r.
- The presence of a symbolic dynamics strongly suggests the chaoticity of the system.