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Bananas in a Civilized Jungle

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1 Introduction

This dissertation aims to extend the setting in which the civilized jungle concept is presented. I will describe the basic features of a civilized jungle as presented by A. Rubinstein and K. Yildiz [7] and show some of their results. Then I formally present a brief review of the welfare theorems, which are a core topic of the essay. Then I enlarge the setting. At first, I analyze a civilized jungle with divisible goods, which is done by introducing consumption bundles for the commodities and consumption set for the agents. This first part has fundamental insights from the paper by M. Piccione and A. Rubinstein [5]. Secondly I introduce production. My ultimate goal is to study an analogous of the second welfare theorem for a civilized jungle with divisible goods and production.

2 Civilized Jungle

Rubinstein and Yildiz [7] present a civilized jungle as a tuple $\langle N, X, (\succeq^i)_{i\in N}, \trianglerighteq, \mathcal{L}\rangle$ where N are the agents, X are the n object, $(\succeq^i)_{i\in N}$ are the preferences of each agents over the objects, $\vartriangleright$ is a strict and complete ordering over N which represents the power relation between the agents and $\mathcal{L} = \{\geqslant_\lambda\}_{\lambda\in\Lambda}$ is a set of complete and transitive (possibly not antisymmetric) binary relations which stands for different criteria that rank agents.
A distribution of the resources over the agents is represented by an assignment $\mathbf{x}$ that is a map that maps every agent to an object, that is $\mathbf{x} : N \to X : i \mapsto x^i$. We will use the notation $\mathbf{x} = (x^1, \dots, x^2)$. How do we define an equilibrium? Given an assignment, every agent has to be the strongest among those who envy her and can justify themself. How can someone justify herself among a group of agents? We say that agent $i \in N$ is justified by $\geqslant_\lambda$ over $I \subseteq N$ if $i \geqslant_\lambda j \;\; \forall\, j \in N$, then:

i is justified among I if $\exists\, \lambda \in \Lambda$ such that she is justified by that criteria over I.

That is, if you are the best in a group for at least one criteria then you are justified in that group. The justification proposed by R-Y is exclusively meritocratic, and therefore far from reality. Nevertheless, that is the concept of a *civilized* jungle. Let $J_{\mathcal{L}}(I)$ be the set of justifiable agents in I. Formally, $J_{\mathcal{L}} : \mathcal{P}(N) \to \mathcal{P}(N)$ such that:

$$\tilde{J}_{\mathcal{L}}(I) = \left\{ i \in N \mid \exists\, \lambda \in \Lambda \text{ s.t. } i >^\lambda j \;\; \forall\, j \in I \right\}$$

Actually, the definition of justified agents in a group is given by:

$$J_{\mathcal{L}}(I) = \tilde{J}_{\mathcal{L}}(I) \cap I$$

We restrain the justification only to those who are in the group because we will use this notion just for those who envy or dream about someone, as we will see. Therefore those who are not envious are not interested in being justified.
The last core concept is *envy*. Trivially, an agent i envies another agent j if $x^j \succeq^i x^i$. We define $E(\mathbf{x}, i)$ as the set of agents that envies i, given the assignment $\mathbf{x}$.
We are now able to define the equilibrium concept in a civilized jungle.

Definition 2.1. *An assignment $\boldsymbol{x}$ is a **civilized equilibrium** if $\forall\, i \in N$ then $i \in J_{\mathcal{L}}(E(\boldsymbol{x}, i))$ and:*

$$i \vartriangleright j \qquad \forall\, j \in J_{\mathcal{L}}(E(\boldsymbol{x}, i))$$

A civilized equilibrium is also called a C equilibrium. The existence of such an equilibrium will be discussed in different instances, and it will help us understand how civilization embroils jungle's nature. An interesting kind of language are those that partition agents into two indifferent sets, for each criterion. Such languages are called dichotomous languages.

Definition 2.2. *A language $\mathcal{L}$ is dichotomous if for each criterion $\lambda \in \Lambda$ there exist $i, j \in N$ such that $x >_\lambda y$ and $\forall\, k \in N$ either $k =_\lambda i$ or $k =_\lambda j$.*

Given a dichotomous language $\mathcal{L}$ we can therefore partition N by $\sim_\lambda$ for all $\lambda \in \Lambda$, where:

$$x \sim_\lambda y \Leftrightarrow x =_\lambda y$$

And get $N/\sim_\lambda = \{0, 1\}$. Another handy way of representing a dichotomous language is to define a set of properties ϕ^i for each agent $i \in N$. These sets are defined to be the collection of criteria with respect to which the agent is in the upper equivalence class. In economic terms, a dichotomous language provides a distinction into strong and weak agents for each criterion.

Example 2.1 (Collegio Carlo Alberto). *Let the Collegio Carlo Alberto be a Jungle, civilized. The agents are all the Professors and the Allievi. The commodities are the different speeds at which an agent can be served at a buffet in the common room. Of course, everyone has the same preference relation: before is better. We summarize by letting N finite, $X \subset \mathbb{N}$ finite, where each number corresponds to a different speed and $k \succeq^i j$ if and only if $k \leqslant j$, where $\leqslant$ is the usual ordering over the natural numbers for all $i \in N$. We assume that there is a power relation $\trianglerighteq$ over N defined by each agent's capability of winning a buffet race. Nevertheless, one can exercise her power only if she is justified by the one criterion that governs the Collegio: being a scholar. Formally, $\Lambda = \{\lambda\}$ is a singleton and $x >_\lambda y$ if and only if x is a Professor and y an Allievo. The only civilized equilibrium is straightforward: the serial dictatorships over the two equivalence classes induced by $\sim_\lambda$. Therefore, the fastest and mostly skilled Professor will eat instantly and all the others will follow, based on the power criteria that govern their world. Only after, when so-called civilization has played its role, will the Allievi fight for the remaining food, based on the power relation that reigns among them.*

A language truly brings civilization to a jungle if it has more than one criterion. In fact as [7, RY] show, if $\mathcal{L}$ consists of just one strict ordering $\geqslant$, then the unique "civilized" equilibrium is a serial dictatorship according to $\geqslant$. Then the C equilibrium does not depend on the power relation over the agents and the civilized jungle is just a normal jungle with another power relation.

Proposition 2.1. *If $\mathcal{L}$ consists of just one strict ordering $\geqslant$, then the unique civilized equilibrium is a serial dictatorship according to $\geqslant$.*

Proof. It is an equilibrium since the strongest with respect to $\geqslant$ he's going to be the only one justified among those who dream about him. We rule him out. Proceeding iteratively we get the thesis. This argument almost shows uniqueness of the equilibrium. Let us consider another equilibrium. In this equilibrium there must be some agent j dreaming about i and being stronger, civilization wise, that him and weaker to no one in the set of those who dream about i. If so, he would be the only one justified, absurd. $\square$

The reasoning used in Proposition 2.1 allows us to state also the following property, whose proof is immediate and left to the reader.

Corollary 2.2. *If $\mathcal{L}$ is a singleton then the serial dictatorships over all the equivalence classes induced by equivalent power is a C equilibrium.*

We can think of an uncivilized jungle. Piccione and Rubinstein [5, PR] defined the concept of jungle, as a civilized jungle without a language and a related equilibrium.

Definition 2.3. *An assignment is a **jungle equilibrium** if:*

$$\forall\ i \in N\ \ \forall\ j \in E(\boldsymbol{x}, i)\ \text{then}\ i \rhd j$$

Therefore, the unique jungle equilibrium is obtained through a serial dictatorship governed by the power relation.

Proposition 2.3. *Let $\langle N, X, (\succeq^i)_{i\in N}, \unrhd\rangle$ be a jungle, a jungle equilibrium is given by the assignment obtained through the serial dictatorship ruled by $\unrhd$.*

Proof. We define a serial dictatorship ruled by $\unrhd$ recursively as $x^1 = \max_{\succeq^1} X$ and:

$$x^i = \max_{\succeq^i} X \backslash \{x^1, \ldots, x^{i-1}\}$$

Where $\max_{\succeq^j} Y$ stands for maximize Y with respect to the preference relation $\succeq^j$. We assume that N is ordered following $\unrhd$, if not, a permutation suffices to extend the definition. This is a jungle equilibrium. Let y □

2.1 Comparison between equilibria

Which relation do C equilibrium and jungle equilibrium have? Let us state a property of the power relation in order to answer this question.

Definition 2.4. *A strict ordering $\unrhd$ is **weakly $\mathcal{L}$-concave** if $\forall\ i, j \in N$ and $\forall\ \lambda \in \Lambda$:*

$$\exists\ i_\lambda \in N\backslash\{j\}\ \text{s.t.}\ i_\lambda \geqslant_\lambda j\ \text{and}\ i \rhd i_\lambda \Rightarrow i \rhd j$$

Weak convexity is an extension of the basic notion of convexity. It was proposed by Richter and Rubinstein in [6, RR], and it allows the definition of the property for any space without the need of an algebraic structure. In our setting it is quite intuitive if seen as: if *for all* the criteria that we are using in our language if I (i) can find some one (i_λ) weaker than me and best suited than you (j) then I am stronger than you. Under this condition the jungle equilibrium is a C equilibrium.

Proposition 2.4. *Let $\langle N, X, (\succeq^i)_{i\in N}, \unrhd, \mathcal{L}\rangle$ be a civilized jungle with a weakly $\mathcal{L}$-concave power relation, then the jungle equilibrium is a civilized equilibrium.*

Proof. Let $\mathbf{x}$ be the jungle equlibrium. Of course in a serial dictatorship $E(\mathbf{x}, i) \subseteq \{j \mid i \rhd j\}$, then i is the strongest among $J_{\mathcal{L}}(E(\mathbf{x}, i))$ if i belongs to it. In fact, recall that justified agents in a group are part of that group. If by contradiction i is not the strongest in $J_{\mathcal{L}}(E(\mathbf{x}, i))$ then $i \notin J_{\mathcal{L}}(E(\mathbf{x}, i))$, but then $\forall\ \lambda \in \Lambda$ there exists $i_\lambda \neq i$ such that $i_\lambda \geqslant_\lambda i$ but also $i \rhd i_\lambda$, and so by weak concavity $i \rhd i$. □

Furthermore, if the notion of concavity is empowered the jungle equilibrium becomes the only C equilibrium.

Definition 2.5. *A power relation is **strongly $\mathcal{L}$-concave** if* $\forall\, i, j \in N$:

$$\exists\, i_\lambda \in N \backslash \{j\} \ \textit{s.t.}\ i_\lambda \geqslant_\lambda j \ \textit{and}\ i \trianglerighteq i_\lambda \Rightarrow i \triangleright j$$

Proposition 2.5. *Let $\langle N, X, (\succeq^i)_{i \in N}, \trianglerighteq, \mathcal{L}\rangle$ be a civilized jungle of strict orderings criteria with a strongly $\mathcal{L}$-concave power relation, then the jungle equilibrium is the unique civilized equilibrium.*

Proof. Let $\mathbf{y}$ be another C equilibrium. Then not being the serial dictatorship implies the existence of $i, j \in N$ such that $i \triangleright j$ and $x^j \succeq^i x^i$. Because $\mathbf{y}$ is a C equilibrium $i \notin J_{\mathcal{L}}(E(\mathbf{x}, j))$, then $\forall\, \lambda \in \Lambda\ \exists\, i_\lambda \in N \backslash \{i\}$ such that $i_\lambda \geqslant_\lambda i$. Since $\mathbf{y}$ is a C equilibrium $j \trianglerighteq j_\lambda\ \forall\, \lambda$. Then by strong concavity $j \triangleright i$. □

3 Civilized Jungle with divisible commodities

I now extend the setting of a civilized jungle and the relative equilibrium concept for divisible commodities. Relying on the paper by Piccione and Rubinstein [5, PR] I define for each agent $i \in N$ a consumption set $X^i \subseteq \mathbb{R}_+^K$ and $X = (X^1, \dots, X^N)$. I also have to specify an aggregate bundle $w = (w_1, \dots, w_K)$, where $w_k \in \mathbb{R}_+ \ \forall\, k$. Therefore a **civilized jungle** is a tuple:

$$\langle N, K, (X^i)_{i \in N}, w, (\succeq^i)_{i \in N}, \trianglerighteq, \mathcal{L} \rangle$$

Because agents can consume more than just one commodity in different quantities, economic scenarios cannot be represented by an assignment, instead they are modeled through allocations. An allocation is a vector $\mathbf{z} \in \mathbb{R}_+^K \times X$, where the first coordinate stands for the unused goods, therefore $\sum_{i=0}^N z_i = w$. I now adapt the civilized equilibrium concept by redefining the notion of "envy". In this context, agent are not envious, because their economic situation can be, and realistically is always, composed of divisible commodities. Given an allocation, an agent can "dream" about another agent if she sees in the dreamed one's property a more preferred allocation. We enrich the envious relationship as agents have more complex behaviors in this setting. For instance, a monkey can dream about another primate not because she has a certain variety of bananas, but because she has a certain quantity of a certain type of bananas. This situation requires a broader concept of envy, as the different good is not its only driver. To keep it simpler, we do not allow coercion on multiple agents. I define formally the concept of a dreamer.

Definition 3.1. *Given an allocation* $\mathbf{z}$ *and* $i, j \in N$ *we say that* i ***dreams*** *about* j *if:*

$$\exists\, y^i \in X^i \text{ such that } y^i \succeq^i z^i \text{ and } y^i \leqslant z^i + z^j$$

Why do we ask for this broader notion instead of envy? This necessity comes from the larger set of economic situations arising from goods' divisibility. If commodities are indivisible agents' preferences must rank them in function of their unitary value; there is no difference between 2 bananas and 1 stick, you either prefer the banana or the stick. If commodities are divisible quantity plays a fundamental role, it may be the case that you prefer one stick more than one banana and prefer two bananas over one stick. This wider range of possibilities suggests the necessity of enlarging the notion of envy. An agent can build a more preferred allocation by taking a part of another agent's property instead of switching the whole allocation. Since we allow for this instance to take place we named it as *dreaming* about another agent and define the notion of equilibrium not allowing for them to happen.
We define the new set of those who dream about someone in a given allocation:

$$D(\mathbf{z}, i) = \left\{ j \in N \mid \exists\, y^i \in X^i \text{ s.t. } y^i \succeq^i z^i,\, y^i \leqslant z^i + z^j \right\}.$$

We can now define the jungle equilibrium concept, initially defined by PR[5].

Definition 3.2. *An allocation $\boldsymbol{z}$ is a **jungle equilibrium** if $\nexists i, j \in N$ s.t. $i \rhd j$ and $\exists\, y^i \in X^i$ s.t. $y^i \succeq^i z^i$ and:*

$$y^i \leqslant z^i + z^j \text{ or } y^i \leqslant z^i + z^0$$

We can reformulate the definition 3.2 as follows, in terms of dreamers.

Definition 3.3. *An allocation $\boldsymbol{z}$ is a **jungle equilibrium** if:*

- $\forall\, i \in N\ : i \rhd j\ \ \forall\, j \in D(\boldsymbol{z}, i)$
- $D(\boldsymbol{z}, 0) = \varnothing$

In a jungle equilibrium, every agent has to be the strongest among all those who dream of her. In a *civilized* jungle, agents have to justify themselves through at least one criterion to "fight" for commodities. Once agents are justified, jungle law prevails.

Definition 3.4. *An allocation $\boldsymbol{z}$ is a **civilized equilibrium** if $\forall\ i \in N$ the following hold:*

1. $i \in J_{\mathcal{L}}(D(\boldsymbol{z}, i))$
2. $i \rhd j\ \ \forall\, j \in J_{\mathcal{L}}(D(\boldsymbol{z}, i))$
3. $D(\boldsymbol{z}, 0) = \varnothing$

The first two conditions were previously explained. The third one assures that the unused goods aren't useful for anyone, if they were the agents would take them. We will also call it *C equilibrium.* We now present some examples that help to give a better understanding of this setting and concept.
It is clear the further step required by civilization: in equilibrium, agents have to be the strongest among the justified dreamers, not just among those who dream. We recall that agents justified among a group are defined to be part of that group. The definition could be also given without this condition, letting agents be justified in groups of which they are not part, but it seems unreasonable in our setting, given that only dreamers are interested in being justified.

Example 3.1 (Uncivilized languages)**.** *Not all languages provide actual civilization. Let us consider a singleton language $\mathcal{L} = \{\lambda\}$, how is it going to affect the economy? Let us consider an allocation z and look at the C equilibrium definition 3.4. For every agent i, the set of justified dreamers has to be a singleton $\mathcal{L}(D(z,i) = \{j\}$ where $j > k$ for every $k \in D(z,i)$. If we want z to be a C equilibrium then $i \rhd j$. Therefore the unique C equilibrium in this economy is the serial dictatorship according to the language. We call such a language an uncivilized language, since it just changes the power relation, without introducing any friction between dreams and power. What if the language takes the opposite form? Let us consider a language $\mathcal{L} = \mathcal{S}_N$, that is the set of all permutation over agents. Under this language, every agent is going to be justified in every subset since there will always be a criterion with respect to which it is the strongest. Because of this, the language*

does not change anything on the equilibrium side, its presence is neutral and agents are going to construct equilibria thinking about being justified and the economy falls back into a standard Jungle.
These two examples show that real civilization has to come at a price, they have to be exclusionary to provide useful justifying criteria.

Rubinstein and Yildiz define the "I am how I am" criterion as dichotomous language $\mathcal{L} = \{m_i : i = 1, \ldots, N\}$ where $\{m_i\}$ defines agent i as strictly stronger than the others, who are equally strong. This language provides the same result as $\mathcal{S}_N$.

Example 3.2 (Strictly monotonic preferences)**.** *If we deal with strictly monotone preferences there is a unique C equilibrium. It will be the allocation that gives the whole endowment to the strongest ($\rhd$) among those justified by the language within the set of all agents. That is, $\mathcal{L} = \{\lambda_\lambda,\ \lambda \in \Lambda\}$ selects $\leqslant n$ agents who are the strongest for a criterium, formally $J_{\mathcal{L}}(N) = \{j : \exists\ \lambda \in \Lambda\, j >^\lambda k\ \ \forall\ k \in N\}$. Among those, the strongest with respect to the power relation $\rhd$ is the one who'll get w. That is gonna be the only C equilibrium. Indeed, if z is a C equilibrium then for every agent i who's got positive consumption is going to have everyone dreaming about him, therefore*

$$J_{\mathcal{L}}(D(z,i)) = J_{\mathcal{L}}(N) = \{j : \exists\ \lambda \in \Lambda\, j >^\lambda k\ \ \forall\ k \in N\},$$

but $i \rhd j$ for every $j \in J_{\mathcal{L}}(N)$, which implies that the only agent who can have positive consumption is the strongest among $J_{\mathcal{L}}(N)$. Since he is the only one, he is going to consume the whole aggregate endowment.

The last example we want to show is proposed by Rubinstein and Yildiz in [7, RY], and it combines multiple dichotomous languages in a nested organization.

Example 3.3 (Nested dichotomous languages)**.** *Let us imagine a civilized equilibrium with a dichotomous language such that the set of properties are nested. That is, we imagine an ordering of the agents $i_1, \ldots, i_N$ such that $\phi^{i_N} \subset \cdots \subset \phi^{i_1}$. This civilized jungle has a unique equilibrium, the serial dictatorship according to $i_1, \ldots, i_N$. Again, essentially we can apply the same reasoning used in the proof of Proposition 2.1 to show that it is indeed an equilibrium and is unique.*

3.1 Comparison between equilibria

What is the relation between a civilized and an uncivilized equilibrium? Does their relation for indivisible commodities still hold?

Proposition 3.1. *In a weakly concave jungle, a jungle equilibrium is civilized.*

Proof. The second condition is fulfilled. $J_{\mathcal{L}}(D(\mathbf{z}, \cdot)) \subseteq D(\mathbf{z}, \cdot)$, therefore being the strongest in the latter implies being stronger in the justified envious, if part of it. Then by contradiction as before we prove the belonging. $\square$

Of course, in this setting uniqueness under strict concavity does not hold. The proof by R-Y[7] breaks instantly when the existence of another C equilibrium implies that there is at least one powerful envious, meaning that she is stronger than the envied one. It is not true when goods are more than just units, possibly there are multiple best bundles for each agent.

4 Civilized Welfare Theorems

I now investigate whether or not the welfare theorems hold in a Civilized Jungle. As a first step, we have to adapt the statement of the theorems to this setting. Evidently, a competitive allocation is a civilized equilibrium while the concept of *Pareto efficiency* is intended as follows.

Definition 4.1. *An allocation* $\boldsymbol{z}$ *is* ***Pareto efficient*** *if does not exist another* $\boldsymbol{y}$ *allocation s.t:*

$$y^i \succeq^i z^i \ \forall\, i \in N \ \textit{and} \ \exists\, j \in N \,:\, y^j \succ^j z^j$$

That is, an allocation is Pareto efficient if no one can improve her situation without making anyone else worse off.
In further sections, production will be introduced in the jungle setting, until then the welfare theorems are stated in their simplified versions.

4.1 First civilized theorem

The first theorem can be interpreted as:

A civilized equilibrium is Pareto efficient.

We know from [5, PR] that unique jungle equilibria are indeed Pareto efficient, from now on just efficient. Rubenstein and Yildiz [7, RY] show that a strong result holds for civilized jungles.

Proposition 4.1. *Given a civilized jungle with a language of strict orderings such that there exists no agent* i, j *where one is ranked right above the other and the opposite relation holds for the language. If the power relation is not weakly concave then there exists a preference profile such that there exists no pareto efficient C equilibrium.*

The proof is rather long and not of much interest for our purposes. As the authors point out, Proposition 4.1 together with Proposition 2.4 implies that if a civilized jungle has a language of strict orderings then weak $\mathcal{L}-$concavity of the power relation is almost a necessary and sufficient condition for the existence of a Pareto efficient civilized equilibrium for every preference profile. Since we already lose the first welfare theorem power for indivisible commodities we shift our attention to the second one, which has a civilized version in [5, PR].

4.2 Second civilized theorem

The second welfare theorem, in its standard formulation, guarantees for every Pareto efficient allocation, under suitable assumption, the existence of a price vector and an endowment that sustains that allocation as a competitive one. In a jungle, even if civilized, the only currency is brute force, then prices and personal endowments are substituted by a power relation. Therefore we can restate the theorem as:

For every Pareto efficient allocation there exists a power relation which sustains the allocation as a civilized equilibrium.

If dealing with non-divisible goods, [7, RY] have proved an analogous version of the second welfare theorem. The two key hypotheses are strict orderings as languages and restraining efficient allocation to *J-costrained* efficient allocation. The first one guarantees a clear power relation. The latter is an important and necessary constraint: in a civilized equilibrium each agent has to justify herself among those who envy her, otherwise she cannot use her force against them. This assumption is not necessary for an efficient assignment. It is therefore necessary to introduce the following class of assignments.

Definition 4.2. *An allocation $\boldsymbol{z}$ is J-constrained if $i \in J_{\mathcal{L}}(E(\boldsymbol{z}, i))$ $\forall\, i \in N$.*

Once we constrain an efficient assignment to a language we can inquire under which conditions on $\mathcal{L}$ each efficient assignment is an equilibrium. Turns out that for divisible commodities the adapted proof of Proposition 3.1 from [7, RY] breaks immediately. The idea is to build the power relation as a completion of a non-cyclic binary relation over a subset of N. In particular, the subset over which the (possibly incomplete) binary relation is defined is such that it guarantees the assignment to be C equilibrium. I now present the theorem from [7, RY] and then show why it does not hold in our more general setting.

Theorem 4.2. *Let $\langle N, (X^i)_{i\in N}, (\succeq^i)_{i\in N}, \mathcal{L}\rangle$ be a tuple as above. Then, for every J-constrained efficient assignment $\boldsymbol{x}$ there exists a power relation $\trianglerighteq$ such that $\boldsymbol{x}$ is a C equilibrium for the corresponding civilized jungle.*

Proof. Let $\mathbf{x}$ be a J-constrained efficient assignment. Let P be a binary relation over N such that for each $i, j \in N$ then jPi if i envies j and she is justifiable in $E(\mathbf{x}, j)$. If we show P to be non-cyclic, then it is a pre-order over N. This comes from a standard result in order theory[1], I will talk more about it later in this section. By completing P, we'd get a power relation $\trianglerighteq$, which sustains $\mathbf{x}$ as a C equilibrium. Indeed:

- $i \in J_{\mathcal{L}}(E(\mathbf{z}, i))$ for all $i \in N$ because $\mathbf{x}$ i J-constrained
- If i is justifiable in $E(\mathbf{z}, j)$, then jPi, then $j \triangleright i$

Let us show that P is non-cyclic. Suppose by contradiction that for some $I = \{1, 2, \ldots, m\}$ we have $1P2P\ldots PmP1$. Let us define the allocation $\mathbf{y}$ as $y^i = x^{i-1}$ for $i \in I$ and $y^i = x^i$ for $i \notin I$. The assignment $\mathbf{y}$ is justifiable and pareto dominates $\mathbf{x}$. The latter is obvious by construction. By recalling that the operator $L_{\mathcal{L}}$ is monotone decreasign[2] we prove $\mathbf{y}$ to be justified, indeed for $i \in N$ then:

- If $i \in I$ then $E(\mathbf{y}, i) \subseteq E(\mathbf{x}, i-1)$
- If $i \notin I$ then $E(\mathbf{y}, i) \subseteq E(\mathbf{x}, i)$

[1]Varian (1974) for non cyclic implies completable

[2]Meaning that $A \subseteq B \Rightarrow J_{\mathcal{L}}(A) \supseteq J_{\mathcal{L}}(B)$

But i is justified in both $E(\mathbf{x}, i)$ and $E(\mathbf{x}, i-1)$, because $\mathbf{x}$ is justified and by definition of P. $\square$

The whole proof relies on the possibility of constructing an assignment $\mathbf{y}$ that pareto dominates $\mathbf{x}$. For divisible commodities, two main problems arise. We first think about non-civilized jungles. We call the first one *reciprocal dreaming* the instance in which two agents dream each other. This situation could clearly occur even in the non-divisible goods setting, but only in non-efficient assignments. Indeed, if two monkeys envy each other they can simply switch their bananas and get a more efficient assignment. When bananas are divisible, reciprocal dreaming does not imply inefficiency. But if in an efficient allocation two agents are reciprocal dreamers, then no power relation will sustain the allocation as an equilibrium. Indeed, every power relation would not be strict as for the reciprocal dreamers i, j the equilibrium condition would imply $i \rhd j$ and $j \rhd i$. We found a necessary condition. To formally express it we define reciprocal dreaming.

Definition 4.3. *Let z be an allocation. Two agents $i, j \in N$ are **reciprocal dreamers** if:*

$$i \in D(z, j) \text{ and } j \in D(z, i).$$

*An allocation is **non-reciprocal** if no reciprocal dreamers exist.*

Non-reciprocality is a key concept in C-equilibria since it drastically shapes the power relation of the Jungle.

Proposition 4.3. *Let z be a Jungle equilibrium, then z is non-reciprocal.*

Proof. By absurd. Let $i, j \in N$ be reciprocal dreamers. By definition of Jungle equilibrium

$$j \in D(z, i) \Rightarrow i \rhd j$$

and

$$i \in D(z, j) \Rightarrow j \rhd i.$$

Then $\rhd$ is not antisymmetric $\lightning$. $\square$

If we impose non-reciprocal dreaming can we prove the theorem? Not yet, the very essence of the more general setting prevents the allocation $\mathbf{y}$ from being constructed. While for non-divisible commodities monkeys could just switch different bananas, in this setting what is dreamed by two monkeys could be unfeasible together. We therefore should follow a different path, a straightforward extension of the proof is impossible.
Actually, a general result holds: the II welfare theorem does not hold for uncivilized jungles with divisible commodities. Given an allocation, we can sustain it as a C equilibrium only if in the subset of dreamers there are no reciprocal dreamers (we can impose this condition) and those who dream are weaker than the dreamed one. We can easily construct a Pareto efficient allocation where the two previous conditions are fulfilled, but it is not an equilibrium.

Example 4.1. *Let three monkeys fight for one banana, one nut, and one stick. Monkey A prefers one banana and the stick over everything, monkey B prefers the banana and the nut over everything else, and monkey C prefers the nut and the stick. The preference relations are:*

$$\begin{cases} (1,0,1) \succ (1,0,0) \text{ strictly better than all others, over which she is indifferent.} \\ (1,1,0) \succ (0,1,0) \text{ strictly better than all others, over which she is indifferent.} \\ (0,1,1) \succ (0,0,1) \text{ strictly better than all others, over which she is indifferent.} \end{cases}$$

Consider the allocation $z = ((1,0,0),(0,1,0),(0,0,1))$. *It is Pareto efficient (the only allocations where someone is strictly better off are obtained if B takes the banana from B, or if C takes the nut from B or A takes the stick, in all instances, the robbed are worse off). There is no reciprocal dreaming, B dreams about A and is dreamed by C, while A only dreams about C. To have a power relation* $\rhd$ *sustaining z as a jungle equilibrium it has to be the case that*

$$A \rhd B \text{ and } B \rhd C.$$

But then it must be the case that $A \rhd C$*, which is absurd because A dreams about C's stick.*

In the next chapter, we study how to break these circles that prevent an allocation from being sustained as an equilibrium. This will be done by civilization, which will be part of the legislative power of jungle institutions.

5 Jungle policies

In this section, we present how jungle institutions can implement civilized equilibria. We think of a jungle institution as a policy maker who can implement power relations and languages. Such an entity will be able, under certain circumstances, to impose an allocation as an equilibrium. The main concept is **civilized compatibility** (*C-compatibility*) of a language with respect to an allocation. We will prove that every allocation that admits a C-compatible language can, almost, be a C equilibrium under that language.

Let us take a little detour on the basics of order theory, which will help us precisely spot where we can ask for sufficient conditions to be matched for a completion to a preorder. This procedure will be the main idea behind the implementation of a language and a power relation to get a C equilibrium.

I will add a formal introduction to orders and some basic definitions.

We'll need the following modification of the classical extensions theorem from Sziplrajn.

Theorem 5.1. *For a nonempty set N and an irreflexive, transitive, and antisymmetric binary relation $\succ$ there exists a complete extension of $\succ$ that preserves the properties.*

The details of this result are discussed in the appendix.

Having acquired the necessary mathematical tools, we now turn our attention to C equilibria and how they can be sustained by power relations. Let us study when a generic allocation z can be sustained a s C equilibrium. We already noticed in Proposition 4.3 that a Jungle equilibrium implies no reciprocal dreaming in the economy. Since we are now working with civilized equilibria we have to adapt this result. We will use the term equilibrium while referring to C equilibrium from now on. Let us recall that for each pair of agent $i, j \in N$ in a C equilibrium it must hold that if they are reciprocal dreamers then at least one of them is not justified in the set of dreamers of the other. Indeed, to maintain antisymmetry of the power relation if $i \in D(z, j)$ and $j \in D(z, i)$, but z is an equilibrium then

$$j \rhd k \ \ \forall \ k \in J_{\mathcal{L}}(D(z, j)),$$

and

$$i \rhd k \ \ \forall \ k \in J_{\mathcal{L}}(D(z, i)).$$

If both i and j are justified in $D(z, j)$ and $D(z, i)$ respectively, then

$$j \rhd i \text{ and } i \rhd j,$$

which contradicts antisymmetry. It is now clear that whenever reciprocal dreaming occurs in an allocation we entrust the language to break it, to sustain it as an equilibrium. It is crucial to note that, given a generic allocation if we aim to impose it as an equilibrium every power relation $\rhd$ must satisfy

$$i \rhd j \ \ \forall \ j \in J_{\mathcal{L}} D(z, i) \backslash \{i\}. \tag{1}$$

Therefore, we must address all the instances in which this condition contradicts the very nature of $\rhd$. The first example in which this problem arises is justified reciprocal dreaming, meaning that the reciprocal dreamers are both justified. We aim to impose a condition on the language which excludes this and all the other possibilities. Let us use from scratch the notion of equilibrium to understand where civilization, the language, can help to sustain an allocation as an equilibrium. Let z be an allocation, in a civilized jungle to be constructed, without either power relation or language. We aim at sustaining z as a C equilibrium. A necessary condition on each couple $(\rhd, \mathcal{L})$ of power relation and language is 1. We take such a tuple. We now have a binary relation, probably incomplete, on N, the set of agents. If we can completely extend it to N, showing it to be irreflexive, transitive, and antisymmetric we will have, almost, constructed a civilized power hierarchy, the tuple, that sustains z as an equilibrium. Let us extend the power relation $\rhd$, preserving the necessary properties. We now make use of Theorem 5.1. If we construct an extension of $\rhd$ to a transitive, irreflexive, antisymmetric relation we can apply this slight modification of Sziplrajn's Theorem and get the desired power relation. For transitivity, we take the transitive closure of $\rhd$, denoted by $T(\rhd)$. Is this relation irreflexive and antisymmetric? At this point, civilization itself comes into play; it will prevent $T(\rhd)$ from not being irreflexive and antisymmetric, allowing us to invoke Sziplrajn's Theorem. Actually, we will see that preventing the lack of antisymmetry will also impose irreflexivity. Let us suppose $T(\rhd)$ is no antisymmetric and therefore there exists two agent i, j such that

$$iT(\rhd)j \text{ and } jT(\rhd)i.$$

By definition of $\rhd$ for every agents $k, l \in N$

$$k \rhd l \Leftrightarrow l \in J_{\mathcal{L}}(D(z,k)),$$

therefore, if $iT(\rhd)j$ it must be the case that i and j are connected by a chain of justified dreamers from i to j, that is there exists $k_h \in N$ with $h = 1, \ldots, H$ where $H < N - 1$ such that j dreams about the last k, each k dreams about the previous one, and the first k dreams about i, all justified.

Proposition 5.2. *Let z and allocation and $\rhd$ defined as in 1. Then $iT(\rhd)j$ only if either $i \rhd j$ or $\exists \, \{k_h\}_{h=1}^{H}$ for $k_h \in N$ and $H < N - 1$ such that*

$$j \in J_{\mathcal{L}}(D(z,k_H)),\ k_H \in J_{\mathcal{L}}(D(z,k_{H-1})),\ \ldots,\ k_1 \in J_{\mathcal{L}}(D(z,i)).$$

Proof. Let us suppose that neither $i \rhd j$ nor there exists a chain of justified dreamers connecting i and j. Then we take $\rhd' \overset{def}{=} T(\rhd)\backslash\{i,j\}$. This is an extension of $\rhd$ since $i \rhd j$ does not hold and all other relations are preserved from the transitive closure. It is also transitive. If, by absurd, there exists a chain of k_h such that

$$i \rhd' k_1, k_1 \rhd' k_2 \ldots, k_H \rhd' j,$$

then either all relations are already in $\rhd$, and therefore it contradicts the hypothesis, or there exists a couple of the chain for which the relation only holds in $\rhd'$. If that is the case we apply this reasoning again. Since there are a finite number of agents we end up back with $i \rhd' j$, absurd. $\square$

We can now break via civilization the lack of antisymmetry; we can ask just for one couple in the dreaming loop created by reciprocal dreamers to be broken by civilization. We call this C-compatibility between and allocation and a language.

Definition 5.1. *Let z and allocation and $\mathcal{L}$ a language. They are* ***C-compatible*** *if for all couples of reciprocal dreamers $i, j \in N$ one of them is not justified in the set of dreamers of the other.*

A more formal, less intuitive, but more handy definition is the following.

Definition 5.2. *Let z and allocation and $\mathcal{L}$ a language. They are* ***C-compatible*** *if for all couples of reciprocal dreamers $i, j \in N$ then one of the following holds*

1. $\forall\, \lambda \in \Lambda$ *there exists* $k_\lambda \in N\backslash\{i, j\}$ *such that* $k_\lambda >^\lambda j$,
2. $\forall\, \lambda \in \Lambda$ *there exists* $l_\lambda \in N\backslash\{i, j\}$ *such that* $l_\lambda >^\lambda i$.

C-compatibility breaks those chains that prevented the transitive closure from being antisymmetric. Furthermore, it also prevents a lack of irreflexivity. Indeed, in light of Proposition 5.2, if $iT(\rhd)i$ it must be the case that either $i \rhd i$, which is impossible by definition, or there exists a chain of justified dreamers from i into itself. For a C-compatible language, this chain is necessarily broken somewhere.
We have almost proven a jungle policy theorem, the last ingredient needed is J-constrainment. If we impose it, together with C-compatibility, we can build a power relation that sustains the allocation as an equilibrium. We impose that no unused goods are wanted by anyone since this clearly prevents any jungle policy from sustaining the allocation from being an equilibrium.

Theorem 5.3. *Let $\langle N, K, (X^i)_{i\in N}, w, (\succeq^i)_{i\in N}\rangle$ be a tuple as above. Then, for every $(\mathcal{L}, z)$ where z is J-constrained allocation with no unused-wanted goods, and $\mathcal{L}$ is a C-compatible language there exists a power relation $\trianglerighteq$ such that $\boldsymbol{z}$ is a C equilibrium for the corresponding civilized jungle.*

Proof. Let us consider the civilized jungle given by the tuple and the C-compatible language. Since z is J-constrained we can define $\rhd'$ over a subset of N pairs as follows:

$$i \rhd' j \text{ iff } j \in J_{\mathcal{L}}(D(z, i)).$$

Since $\mathcal{L}$ is C compatible, $T(\rhd')$ is irreflexive and antisymmetric. By Theorem 5.1 there exists a complete extension of $T(\rhd')$, which we call $\rhd$. The allocation z is a C equilibrium in the civilized Jungle defined by $\rhd$ and $\mathcal{L}$ since conditions 1 and 3 of Definition 3.4 are imposed by hypothesis. Condition 2 is imposed by the definition of $\rhd'$.

$\square$

Let us note that the fact that no wanted commodity is unused is, for example, implied by Pareto efficiency.

The interpretation of this theorem is straightforward. We suppose that a jungle institution can distribute power and decide a language. If a jungle institution wants to attain a specific allocation as a civilized equilibrium it has just to check whether the allocation admits a C-compatible language. If so, the C-compatible language and the power relation derived from the C equilibrium definition itself will define the civilized jungle under which the allocation is a civilized equilibrium.

As a consequence, it would be interesting to study sufficient conditions on the existence of a C-compatible language for a given allocation. We could also investigate if the existence of a C-compatible language to an allocation implies some necessary conditions on the allocation itself. Those conditions would, ideally, help jungle institutions understand whether a certain allocation is actually supportable as an equilibrium or not.

6 Conclusive Remarks

In April 2024, Ariel Rubinstein published a book called "…" together with Michael Richter. In it they present four models of non-canonical economies, one of them is the Jungle.

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Appendix

Let us recall some basic facts of order theory.

Definition 6.1. *Let $X \neq \varnothing$, a binary relation R on X is a subset of $X \times X$. We write*

$$xRy \text{ if } \{x, y\} \in R. \tag{2}$$

We define some basic properties of a binary relation.

Definition 6.2. *Let R be a binary relation on X non-empty. We define*

- *Reflexivity:* $\forall\, x \in X : xRx$.
- *Irreflexivity:* $\nexists x \in X : xRx$.
- *Antisymmetry:* $xRy, yRx \Rightarrow x = y$.
- *Symmetry:* $xRy \Rightarrow yRx$.
- *Transitivity:* $xRy, yRz \Rightarrow xRz$.
- *Completness:* $\forall\, x, y \in X \Rightarrow xRy$ *or* yRx.

Typically, a preorder is a binary relation that satisfies reflexivity, transitivity, and symmetry, and a set with a preorder is called a poset. A linear order is a complete preorder, and a set with a linear order is called a loset. Most of order theory is developed on these concepts. Instead, we use antisymmetry instead of symmetry, and irreflexivity instead of reflexivity.
Let us now define the transitive closure of a binary relation.

Definition 6.3. *Let R be a binary relation on X non-empty. The transitive closure of R is the smallest transitive binary relation that contains R. It is denoted with $T(R)$.*

Inclusion of binary relation is intended in the canonical set inclusion, that is given R, S binary relation we say that R is included in S if

$$xRy \Rightarrow xSy,$$

or equivalently $R \subseteq S$.

Proposition 6.1. *Every binary relation R has a transitive closure $T(R)$.*

Proof. The binary relation $S = X \times X$ is transitive and contains R. Furthermore, the transitivity of a binary relation is closed under intersection. Indeed, if V, W are transitive binary relations then $U = V \cap W$ is transitive: if xUy, yUz then xVy, yVz and the same thing for W, then by transitivity of V, W we have xUz and xVz, that is xUz. □

We define the extension of a binary relation.

Definition 6.4. *Let R be a binary relation on X non-empty. An extension R^* of R is a binary relation that contains R and preserves the properties of R.*

We now arrive at the fundamental result used in the jungle policies section.

Theorem. *Let X non-empty and R irreflexive, transitive, and antisymmetric binary relation, then there exists a complete extension of R.*

Before proving the theorem we clarify an unproved result that we will use, the Hausdorff Maximum principle. Let us assume the Axiom of Choice.

Axiom 1. *Let $\mathcal{A}$ be an arbitrary collection of non-empty sets. Then there exists a function $f : \mathcal{A} \to \cup_{A \in \mathcal{A}} A$ such that*

$$f(A) \in A, \ \ \forall \ A \in \mathcal{A}.$$

It can be proven that the axiom of choice is equivalent to the Hausdorff's Maximum principle. A proof of necessity can be found in Rudin **??** (p. 395).

Axiom 2. *There exists an $\supseteq$-maximal loset in every poset.*

Maximality is intended as no other subset with such a binary relation contains Y and is in X. Thanks to this result we can prove the theorem, assuming the Axiom of Chioce.

Proof. Let T_X be the set of all extensions of R. By definition of set inclusion $\supseteq$ then $(T_X, \supseteq)$ is a poset. Let $(A, \supseteq)$ be a maximal loset in T_X. Let us define $R^* = \cup_{S \in A} S$, that is the union of all binary relations in A. Since A is a loset, then union of all its element is its maximal element, therefore R^* is an element of A, and therefore an extension of R. We now show that R^* is complete. If it wasn't there'd be at least two elements $x, y \in X$ unranked by R^*. Let us define $R' = T\left(R^* \cup \{x, y\}\right)$. It is an extension of R[3] and strictly contains R^*, which contradicts the maximality of A. □

[3] It can be directly proven.