Rotating Spirals in segregated reaction-diffusion systems

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joint works with A. Salort, G. Verzini and A. Zilio

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Table of contents

Pettern formation in competition-diffusion systems

In competition-diffusion systems, pattern formation is driven by strongly repulsive forces. Our ultimate goal is to capture the geometry and analysis of the phase segregation, including its asymptotic aspects and the classification the solutions of the related PDE's. We deal with elliptic, parabolic and hyperbolic systems of differential equations with strongly competing interaction terms, modeling both the dynamics of competing populations (Lotka-Volterra systems) and other relevant physical phenomena, among which the phase segregation of solitary waves of Gross-Pitaevskiĭ systems arising in the study of multicomponent Bose-Einstein condensates.

We approach all these different problems with the same basic methodology which relies on the following steps

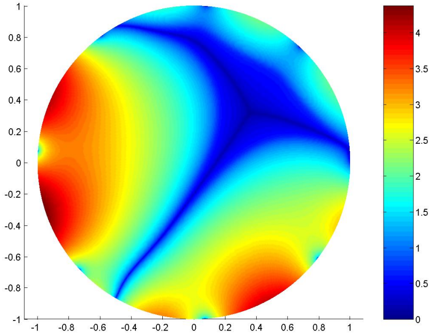
- asymptotic analysis
- analysis of special self-similar simple solutions
- interface analysis
- gluing techniques to build complex solutions.

Table of contents

Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and three populations:

$$\frac{\partial u_i}{\partial t} - \mathrm{div}(d_i \nabla u_i) = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_{ij} u_j \text{ in } \Omega,$$



Competition diffusion systems with Lotka-Volterra interactions : asymmetric competition rates

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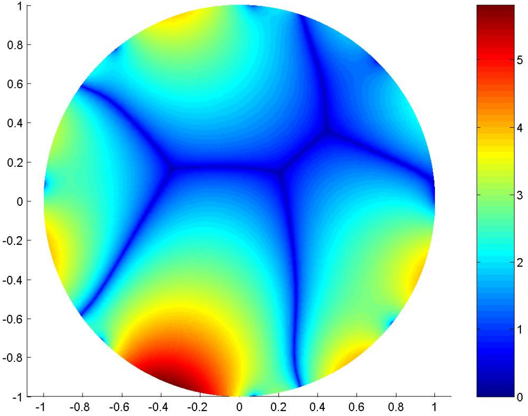
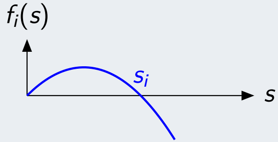


Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

With large and symmetric interspecific competition rates $\beta_{ij} = \beta_{ji}$ and three populations:

$$\overbrace{\frac{\partial u_i}{\partial t}}^{\text{time evolution}} - \overbrace{\mathrm{div}(d_i \nabla u_i)}^{\text{space diffusion}} = \overbrace{f_i(u_i)}^{\text{reaction}} - \overbrace{u_i \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_{ij} u_j}^{\text{competition}} \text{ in } \Omega,$$

u_i is the density of the ith population,
$d_i > 0$ diffusion rates,
$\beta_{i,j}$ interspecific competition rates,
$f_i(s) = u(s_i - u)$ internal forces (s_i, the saturation point)



Diffusion vs strong competition

For the the sake of simplicity, assuming the system be already in equilibrium, we consider only stationary cases, with all equal diffusions, namely we deal with the semilinear ellipitc system:

$$(\mathrm{P}) \quad -\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i}^{k} a_{ij} u_j \qquad \text{in } \Omega, \quad +\text{B.C.}, \quad i = 1, \dots, k,$$

subject to diffusion, reaction and competitive interaction $(a_{ij}, \beta > 0)$.

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Table of contents

Gause's law

Two species competing for the same limiting resources cannot coexist at constant population values. When one species has even the slightest advantage over another, the one with the advantage will dominate in the long term. This leads either to the extinction of the weaker competitor or to an evolutionary or behavioral shift toward a different ecological niche.

Gause, Georgii Frantsevich (1934). The Struggle For Existence (1st ed.). Baltimore: Williams & Wilkins. Archived from the original on 2016-11-28

If similar competing species cannot coexist, then how do we explain the great patterns of diversity that we observe in life? If species living together cannot occupy the same niche indefinitely, then how do competitors coexist?

Mimura'sresult

M. Mimura, Asymptotic behaviors of a parabolic system related to a planktonic prey and predator model, SIAM J. Appl. Math. 37(3) (1979) 499-512

Mimura considered predator-prey system with no flux boundary condition in a bounded set:

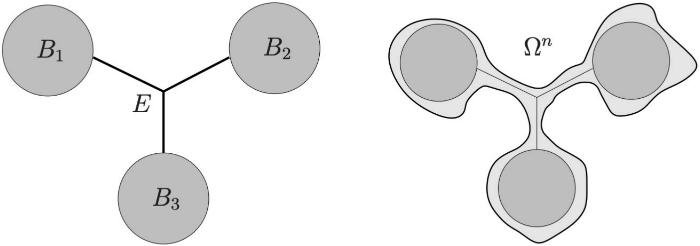
$$\begin{cases} u_t = d_1 \Delta u + f(u)u - uv \\ v_t = d_2 \Delta v + g(u)u + uv \end{cases}$$

Showing that if $f'(u) \leq 0$ and $g'(v) \geq 0$ for $u \geq 0$, $v \geq 0$, and if there is a positive, spatially constant, steady state then every uniformly bounded, nonnegative solution becomes spatially homogeneous as $t \to +\infty$.

Berestycki and A. Zilio, Predators-prey models with competition I-IV

Ecological niches

Coexistence needs complex geometries, that allow the presence of niches, or strongly inhomo-
geneous environments (modeled by strongly varying diffusion functions d_i).



(a) the set $\Omega^0 = B_1 \cup B_2 \cup B_3$ and segments E joining the balls

(b) sets Ω obtained by small perturbation of Ω^0.

Felli, V. and Conti, M. (2008). Coexistence and segregation for strongly competing species in special domains. INTERFACES AND FREE BOUNDARIES, 10(2), 173-195.

Model case: two populations without reaction

The easiest case one can face is

$$\begin{cases} -\Delta u_1 = -\beta a_{12} u_1 u_2 & \text{in } \Omega \\ -\Delta u_2 = -\beta a_{21} u_1 u_2 & \text{in } \Omega \\ u_i = \varphi_i \geq 0, \quad \varphi_1 \varphi_2 \equiv 0 & \text{on } \partial\Omega. \end{cases}$$

Here symmetry can be recovered by scaling: $v_i = a_{ji} u_i$ satisfies

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Subtracting the two equations, we obtain $v_1 - v_2 = \Phi$, where

$$\begin{cases} -\Delta \Phi = 0 & \text{in } \Omega \\ \Phi = a_{21} \varphi_1 - a_{12} \varphi_2 & \text{on } \partial\Omega. \end{cases}$$

With this notation, writing $u = u_1$, the system becomes equivalent to

$$-\Delta u = -\beta u (u - \Phi), \quad u > \Phi^+ \text{ on } \Omega.$$

Segregation limit for two populations

Theorem (Conti-S.T.-Verzini 2005)

- *For every positive β the elliptic equation has a unique solution u_β with*
$$\|u_\beta - \Phi^+\|_{H^1(\Omega)} = O(\beta^{-1/6}) \quad \text{as } \beta \to \infty.$$
- *If the boundary data are Lipschitz continuous, then $\{u_\beta\}_\beta$ is equi-lipschitz.*

Related results by [Dancer-Du, Dancer-Hilhorst-Mimura-Peletier].

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We deduce that, as $\beta \to +\infty$,

$$u_1 \to \Phi^+, \qquad u_2 = u_1 - \Phi \to \Phi^- \qquad \text{in } C^{0,\alpha}.$$

The segregation limit (Φ^+, Φ^-) is Lipschitz; the free boundary (interface) has the regularity of the zero set of an harmonic function.

Table of contents

The symmetric case for $k \geq 3$ populations

We assume $a_{ij} = a_{ji} (= 1$ w.l.o.g.$)$. The system becomes

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k,$$

Theorem (Conti-S.T.-Verzini 2005, Soave-Zilio 2015)

Let U_β be a family of H^1–bounded solutions. There exists $L_\alpha > 0$ such that

$$\sup_{x,y \in \Omega} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x - y|} \leq L$$

for all $i = 1, \dots, k$ and for all $\beta > 0$.

This allows to pass to the limit as $\beta \to +\infty$.

Ooptimal uniform Lipschitz bounds have been obtained by [Soave-Zilio, *ARMA* 2015]

Segregation limit in the symmetric case

Theorem

Let $U_\beta = (u_{1,\beta}, ..., u_{k,\beta})$ be a solution of the system at fixed β, and $\beta \to \infty$. There exists U such that, for all $i = 1, \dots, k$:

1. *up to subsequences, $u_{i,\beta} \to u_i$ strongly in H^1 and in $\mathcal{C}^\alpha$, for any $\alpha \in (0,1)$*
2. *if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω*
3. $-\Delta u_i \leq f(x, u_i)$
4. $$-\Delta\left(u_i - \sum_{j\neq i} u_j\right) \geq f(x,u_i) - \sum_{j\neq i} f(x,u_j)$$
5. *the segregated limiting profiles are Lipschitz.*

This agrees with the case $k = 2$, which reads

$$-\Delta(u_1 - u_2) \geq 0, \qquad -\Delta(u_2 - u_1) \geq 0.$$

The class $\mathcal{S}$

Define

$$\widehat{u}_i = u_i - \sum_{j \neq i} u_j$$

and similarly

$$\widehat{f}(x, \widehat{u}_i) = \begin{cases} f_i(x, u_i) & \text{if } x \in \mathrm{supp}(u_i) \\ -f_j(x, u_j) & \text{if } x \in \mathrm{supp}(u_j),\ j \neq i. \end{cases}$$

Then the segregation limits belong to the class

$$\mathcal{S} = \left\{ (u_1, \cdots, u_k) : \begin{array}{l} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq f(x, u_i) \\ -\Delta \widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i),\ \forall i \end{array} \right\}$$

(+ boundary conditions)

Basic properties in $\mathcal{S}$

The multiplicity of a point $x \in \Omega$ is

$$m(x) = \sharp\{i : meas(\{u_i > 0\} \cap B(x,r)) > 0 \ \forall\, r > 0\} \ .$$

Proposition

Let $x_0 \in \Omega$:

(a) If $m(x_0) = 0$, then there is $r > 0$ such that $u_i \equiv 0$ on $B(x,r)$, for every i.

(b) If $m(x_0) = 1$, then there are i and $r > 0$ such that $u_i > 0$ and

$$-\Delta u_i = f_i(x, u_i) \quad \text{ on } B(x,r).$$

(c) If $m(x_0) = 2$, then are i, j and $r > 0$ such that $u_k \equiv 0$ for $k \neq i, j$ and

$$-\Delta(u_i - u_j) = g_{ij}(x, u_i - u_j) \quad \text{ on } B(x,r),$$

where $g_{i,j}(x,s) = f_i(x, s^+) - f_j(x, s^-)$.

Structure of the nodal set

Theorem (Conti-S.T.-Verzini 2005, Caffarelli-Karakayan-Lin 2008, Tavares-Terracini 2012)

Let U be in the class $\mathcal{S}$, and let $\Gamma_U = \{x \in \Omega \ : \ U(x) = 0\}$. Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ the regular part, relatively open in Γ_U, such that

- *Σ_U is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \Sigma_U$*

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

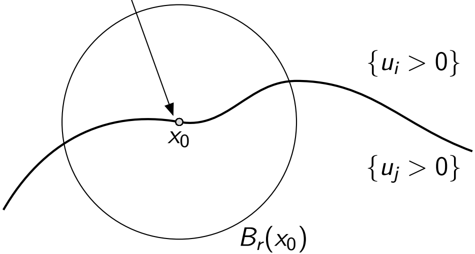
  where the limits as $x \to x_0^\pm$ are taken from the opposite sides of the hyper-surface;

- *$\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N-2$, and $\lim_{x \to x_0} |\nabla U(x)| = 0$.*

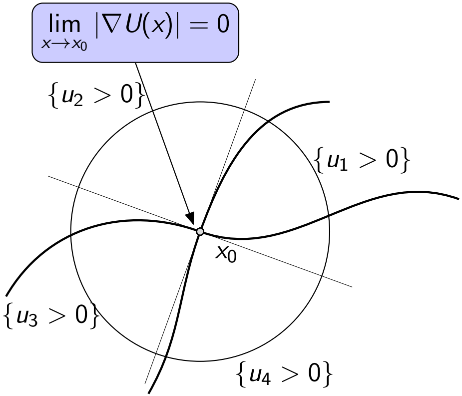
Furthermore, if $N = 2$ then Σ_U consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points.

Nodal set: regular points

$$\lim_{\substack{x\to x_0\\ x\in\{u_i>0\}}} \nabla u_i(x) = -\lim_{\substack{x\to x_0\\ x\in\{u_j>0\}}} \nabla u_j(x)$$

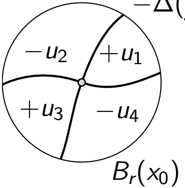


Nodal set: singular points ($N = 2$)



Asymptotic expansion near multiple points

An euristic argument without reactions:



$$-\Delta(\underbrace{u_1 - u_2 + u_3 - u_4}_{w}) = f_1 - f_2 + f_3 - f_4 =$$

Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} \left[a_k \cos(k\vartheta) + b_k \sin(k\vartheta)\right] r^k$

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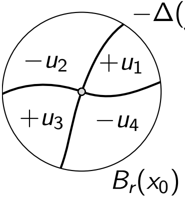

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Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$ and

- $a_k^2 + b_k^2 = 0$ for $k < 0$ as w is not singular in x_0,
- $a_k^2 + b_k^2 = 0$ for $k = 0, 1$ as $m(x_0) = 4$,

$$w(r,\vartheta) = r^2 \cos(2\vartheta + \vartheta_0) + o(r^2) \text{ as } r \to 0.$$

In general, $w \sim r^{m(x_0)/2}$, also in the odd case.

Table of contents

The limiting profiles

Back to the original problem

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k,$$

assume now $a_{ij} \neq a_{ji}$

- **Passing to the limit as $\beta \to \infty$ we find a new class $\mathcal{S}$:**

Define, for every $i = 1, \dots, k$,

$$\widehat{u}_i := u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j,$$

and $\widehat{f}_i$ accordingly. The variational inequalities take the usual form

$$-\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i) \qquad \text{in } \Omega.$$

Asymptotics and nodal set

- What doesn't changes:
  - equi–hölderianity w.r.t. β
  - proportional gradients at points x_0 with $m(x_0) = 2$:

$$\lim_{\substack{x \to x_0 \\ x \in \{u_i > 0\}}} a_{ji} \nabla u_i(x) = - \lim_{\substack{x \to x_0 \\ x \in \{u_j > 0\}}} a_{ij} \nabla u_j(x)$$

  - vanishing of the gradient at points x_0 with $m(x_0) \geq 3$
- What changes:
  - local expansion at multiple points (in dimension $N = 2$).

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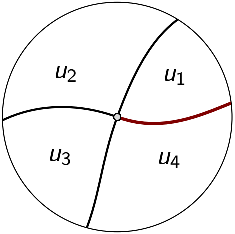
Near an isolated point x_0 with (e.g.) $m(x_0) = 4$
we have that

$$w = u_1 - \frac{a_{12}}{a_{21}} u_2 + \frac{a_{12}a_{23}}{a_{21}a_{32}} u_3 - \frac{a_{12}a_{23}a_{34}}{a_{21}a_{32}a_{43}} u_4$$

satisfies

$$-\Delta w = 0 \qquad \text{in } B_{r_0}(x_0) \setminus \underbrace{\left(\overline{\{u_1 > 0\}} \cap \overline{\{u_4 > 0\}} \right)}_{\tilde{\Gamma}}.$$

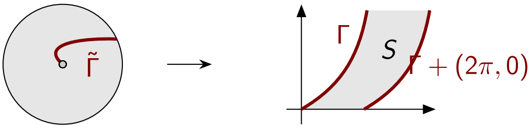


Going to the universal covering

The conformal map

$$(r\cos\vartheta, r\sin\vartheta) \mapsto (x,y) = (\vartheta, -\log(r/r_0))$$

allows to map $B_{r_0}(x_0) \setminus \tilde{\Gamma}$ to the strip $S \subset \mathbb{R}^2_+$.



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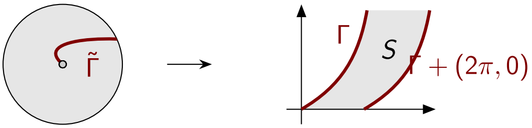

Then a suitably weighted sum of the components u_i corresponds to v, which can be extended from S to $\mathbb{R}^2_+$, with

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ v = 0 & \text{on } \Gamma \\ v(x+2\pi, y) = \lambda v(x,y) \end{cases}$$

where $\lambda = \dfrac{\prod_i a_{i,i+1}}{\prod_i a_{i+1,i}}$.

A representation formula in $\mathbb{R}^2_+$

Thus we have to solve

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathbb{R}^2_+ \\ v = 0 \quad \text{on } \Gamma \\ v(x+2\pi, y) = \lambda v(x,y) \end{cases} \iff \begin{cases} \Delta z + 2\alpha z_x + \alpha^2 z = 0. \quad \text{in } \mathbb{R}^2_+ \\ z = 0 \quad \text{on } \Gamma \\ z(x+2\pi, y) = z(x,y) \\ z(x,y) := e^{-\alpha x} v(x,y), \quad \alpha = \dfrac{\ln \lambda}{2\pi}. \end{cases}$$

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Separating the variables we obtain

$$v(x,y) = \sum_{k \in \mathbb{Z}} \left[a_k \cos\left(kx + \alpha y\right) + b_k \sin\left(kx + \alpha y\right)\right] \exp(\alpha x - ky).$$

From now on, for concreteness, we suppose

$$\lambda = \frac{\prod_i a_{i,i+1}}{\prod_i a_{i+1,i}} > 1, \qquad \text{i.e. } \alpha > 0.$$

Asymptotic spirals

$$v(x,y) = \sum_{k\in\mathbb{Z}} \left[a_k \cos\left(kx+\alpha y\right) + b_k \sin\left(kx+\alpha y\right)\right] \exp(\alpha x - ky).$$

GOAL: show that $a_k^2 + b_k^2 = 0$ for $k < \bar{k} = m(x_0)/2$, $a_{\bar{k}}^2 + b_{\bar{k}}^2 \neq 0$.

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Then

- $v(x,y) \sim \left[a_{\bar{k}} \cos\left(\bar{k}x+\alpha y\right) + b_{\bar{k}} \sin\left(\bar{k}x+\alpha y\right)\right] e^{(\alpha-\bar{k})y}$ as $y \to +\infty$
- S asymptotically lies in the strip $C_1 \leq \bar{k}x + \alpha y \leq C_2$

and finally

$$w(r,\vartheta) = Cr^{\bar{k}} \exp\left(\alpha\vartheta\right) \cos(\bar{k}\vartheta - \alpha \log r + \vartheta_0) + o(r^{\bar{k}}) \qquad \text{as } r \to 0$$

where w is a suitably weighted sum of the components u_i: asymptotic logaritmic spirals!

How to kill the evil part?

Further condition on

$$v(x,y)=\sum_{k\in\mathbb{Z}}\left[a_k\cos\left(kx+\alpha y\right)+b_k\sin\left(kx+\alpha y\right)\right]\exp(\alpha x-ky).$$

By conformality

$$\boxed{\int_S|\nabla v|^2<+\infty.}$$

In terms of z, which is 2π-periodic in y, this reads

$$\int_S e^{2\alpha x}|\nabla z(x,y)|^2\,dxdy<+\infty.$$

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In terms of z, which is 2π-periodic in y, this reads

$$\int_S e^{2\alpha x} |\nabla z(x,y)|^2 \, dxdy < +\infty.$$

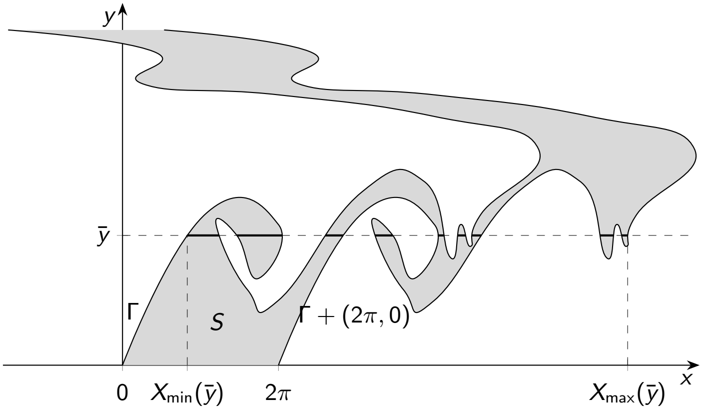
Since we do not know the actual position of S, we can not exclude the integrability on S of quantities of order $e^{2(\alpha x + ky)}$, $k > 0$, even for arbitrarily large k.

Description of the strip S

We denote by

$$S_y := \{(x, y) \in S\}$$

the horizontal sections of S, having endpoints $X_{\min}(y)$ and $X_{\max}(y)$. While $|S_y| \leq 2\pi$, diam S_y may be arbitrarily large.



Conclusion

Let $(u_1, \ldots, u_k)$ be a segregated limiting profile in the asymmetric case.

Theorem

Let $\mathcal{Z}$ be a compact connected component of $\{x : m(x) \geq 3\}$. Then $\mathcal{Z} = \{x_0\}$.

Theorem (S.T.-Verzini-Zilio 2019)

Let $x_0 \in \Omega$ with $m(x_0) = h \geq 3$. Then there exists $\alpha \in \mathbb{R}$ and ϑ_0 such that

$$w(r, \vartheta) = C r^{h/2} \exp(\alpha\theta) \cos\left(\frac{h}{2}\theta - \alpha \log r + \vartheta_0\right) + o(r^{h/2})$$

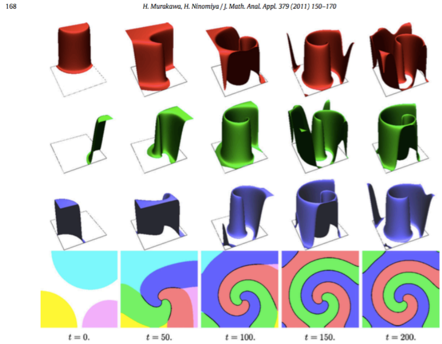
as $r \to 0$, where (r, θ) denotes a system of polar coordinates about x_0 and $\tilde{U}$ is a suitably weighted sum of the components u_i.

Table of contents

The parabolic problem

With asymmetric interspecific competition rates $\beta_{i,j} \neq \beta_{j,i}$ large and three populations:

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



H. Murakawa and H. Ninomiya, *Fast reaction limit of a three-component reaction-diffusion system*. J. Math. Anal. Appl. 379 (2011), no. 1, 150-170,

The parabolic singular limit

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j \neq i}}^{h} a_{i,j} u_j,$$

When we let $\beta \to \infty$ and we obtain the system of parabolic differential inequalities

$$\begin{cases} \partial_t u_i - \Delta u_i \leq f_i(u_i), \\ \partial_t \widehat{u}_i - \Delta \widehat{u}_i \geq \widehat{f}_i(\widehat{u}_i) \\ u_i \cdot u_j \equiv 0\,, \quad i \neq j, \end{cases}$$

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where the differential inequalities are understood in variational sense and

$$\widehat{u}_i = u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j, \qquad \widehat{f}_i(\widehat{u}_i) = f(u_i) - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} f(u_j).$$

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These inequalities incorporate the transmission conditions at the free boundary, that is the closure of the interfaces $\partial\{u_i > 0\} \cap \partial\{u_j > 0\}$, which separate the supports of u_i and u_j at any fixed time t.

Table of contents

The spiralling wave ansatz in two-dimension

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} u_j \text{ in } \mathbb{C},$$

Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right) \ , \qquad x \in \mathbb{C}$$

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Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right)\ , \qquad x \in \mathbb{C}$$

Then $(v_1, \ldots, v_i)$ solve

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}.$$

Spiralling limiting profiles:

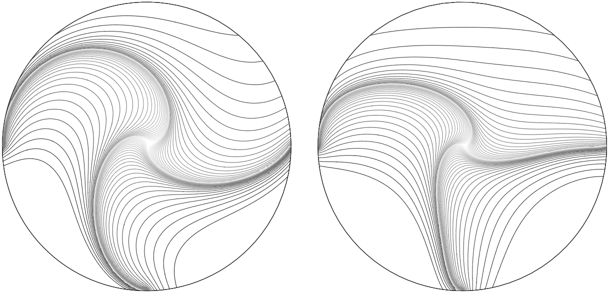
$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j \neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C},$$

Next we pass to the limit as $\beta \to +\infty$. As before, we obtain the system of differential inequalities:

$$(*) \qquad \begin{cases} -\Delta v_i + \omega x^{\perp} \cdot \nabla v_i \leq f(v_i) & \text{in } B \\ -\Delta \widehat{v}_i + \omega x^{\perp} \cdot \nabla \widehat{v}_i \geq \widehat{f}_i(\widehat{v}_i) & \text{in } B \\ \quad v_i \cdot v_j = 0 & \text{for } i \neq j, \end{cases}$$

We are interested in solutions whose nodal set consists in smooth arcs, emanating from ∂B and spiraling towards $\mathbf{0}$, which is the unique singular point of the free boundary.

In this way, each arc is a smooth interface between two adjacent densities, and the origin is the only point with higher multiplicity



Nodal lines of a numerical simulation in the case of $K = 3$ densities, with asymmetric competition such that $\frac{a_{12}}{a_{21}} = \frac{a_{23}}{a_{32}} = \frac{a_{31}}{a_{13}} = 10$, and reaction term $f_i \equiv 0$. The angular velocity is $\omega = 3$ for the picture on the left (counterclockwise spin) and $\omega = -3$ for the picture on the right (clockwise spin). The rotation affects the shape of the spirals, but not their asymptotic behavior close to the center.

The setting

We now consider the case of linear reactions:

$$f_i(s) \equiv \mu s\,, \qquad i = 1, \dots, K\,.$$

Let us consider a K-tuple $(\varphi_1, \dots, \varphi_K)$ of segregated boundary traces. Precisely, we assume that, for every $i = 1, \dots, K$,

$$\begin{cases} \varphi_i \in C^{0,1}(\partial B),\ \varphi_i \geq 0, \\ \{x : \varphi_i(x) > 0\} \text{ are connected, non-empty and disjoint arcs,} \\ \bigcup_i \operatorname{supp} \varphi_i = \partial B. \end{cases}$$

We assume that the traces φ_i are labeled in counterclockwise order.
In general, it is not reasonable to expect that any choice of the boundary data provides a solution with a unique singular point at the origin. Indeed, we show that this happens exactly for an explicit subset having codimension $K - 1$ in the space of traces.

Admissible traces and asymmetry parameter

Let $s = (s_1, \dots, s_K) \in \mathbb{R}^K$, with $s_i > 0$ for all i, and let us consider the class of functions

$$\mathcal{S}_{\text{rot}} = \left\{ U = (u_1, \cdots, u_K) \in (H^1(B))^K : \begin{array}{l} u_i \geq 0 \text{ satisfy } (*), \\ u_i = s_i \varphi_i \text{ on } \partial B \end{array} \right\}.$$

To state our main result we introduce the parameter

$$\boxed{\alpha = \frac{1}{2\pi} \ln \left(\frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}} \cdots \frac{a_{K1}}{a_{1K}} \right),}$$

which accounts for the asymmetry of the coefficients a_{ij}.

Theorem (Salort-S.T.-Verzini-Zilio, 2022)

Let $K \geq 3$, $a_{ij} > 0$, $\omega \in \mathbb{R}$. Assume that $\mu < \pi^2$ and $(\varphi_1, \ldots, \varphi_K)$ are admissible traces. There exists

$$\bar{s} = (\bar{s}_1, \ldots, \bar{s}_K) \in \mathbb{R}^K,$$

independent of μ and ω, with $\bar{s}_i > 0$ for all i, such that:

1. *If $s = t\bar{s}$ for some $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains an element with a unique singular point at $\mathbf{0}$. Moreover such element is unique and, denoting with $\mathcal{U}$ a suitable linear combination of its components, we have*

$$\mathcal{U}(r\cos\vartheta, r\sin\vartheta) = Ar^\gamma \cos\left(\frac{K}{2}\vartheta - \alpha \ln r\right) + o(r^\gamma) \qquad \text{as } r \to 0,$$

   where

$$\gamma = \frac{K}{2} + \frac{2\alpha^2}{K} \qquad \text{and} \qquad 0 < A_0 \leq A(x) \leq A_1.$$

2. *If $s \neq t\bar{s}$ for every $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains no such element.*

The symmetric case

Corollary

Under the assumptions of the above theorem, if the problem is invariant under a rotation of $2\pi/K$, i.e.

$$(1) \qquad \varphi_{i+1}(x) = \varphi_1(e^{2\pi i/K}x) \qquad \textit{and} \qquad \frac{a_{i(i+1)}}{a_{(i+1)i}} = \frac{a_{K1}}{a_{1K}},$$

for every i, then

$$\bar{s} = (1, 1, \ldots, 1).$$

Remark

Notice that the asymptotic expansion implies that the free boundary, near the singular point $\mathbf{0}$, is the union of K equi-distributed logarithmic spirals, as long as $\alpha \neq 0$. On the other hand, in case $\alpha = 0$, we obtain that the interfaces enter the origin with a definite angle. In particular, this holds true in the symmetric case $a_{ij} = a_{ji}$ for every $j \neq i$.

We adopt a constructive point of view, building the solution by superposition of fundamental elementary modes. The dependence of such building blocks on the parameter ω and μ shows the presence of resonances at exceptional values. As a byproduct of the analysis of resonances, we have the following results.

Theorem (Homogeneous boundary conditions)

Let $K \geq 3$ and $a_{ij} > 0$. If (μ, ω) belongs to a suitable discrete set then there exists a nontrivial element of $\mathcal{S}_{\mathrm{rot}}$ with null traces. Analogous results hold for homogenous Neumann or Robin boundary conditions.

Theorem (Entire solutions)

Let $K \geq 3$ and $a_{ij} > 0$. For almost every (μ, ω) there exists an entire solution of (*) in $\mathbb{R}^2$.

Remark

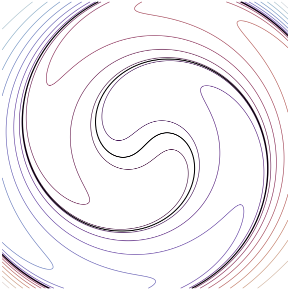
In the particular case $\alpha = \mu = 0$, we obtain that the entire solution found in our main Theorem is related to the nodal components of a smooth rotating solution of the pure heat equation. Let $\omega > 0$, $k \geq 1$ be an integer and let I_k denote the modified Bessel function of the first kind, with parameter k. We have that the function

$$U(re^{i\vartheta}, t) = \mathrm{Re}\left[e^{ik(\vartheta+\omega t)} I_k\left(\frac{\sqrt{2\omega k}}{2}(1+i)r\right)\right]$$

is an entire, eternal rotating solution of the heat equation

$$U_t - \Delta U = 0 \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}$$

having $2k$ nodal regions, which coincide up to rotations multiple of π/k. The equi-distributed nodal lines admit a straight tangent as $r \to 0$, while they behave like arithmetic spirals of equation $\vartheta = \sqrt{\frac{\omega}{2k}}r$ as $r \to +\infty$, see Fig. 1. Notice that, as $\omega \to 0$, a suitable renormalization of U converges to the entire harmonic function $\mathrm{Re}\, z^k$.



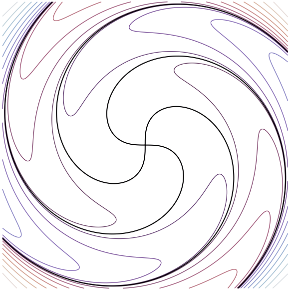
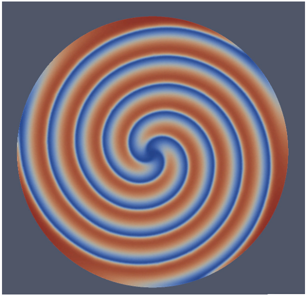
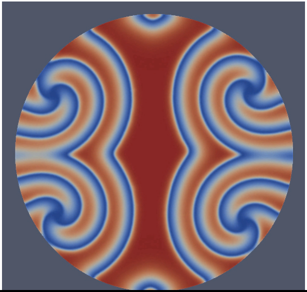
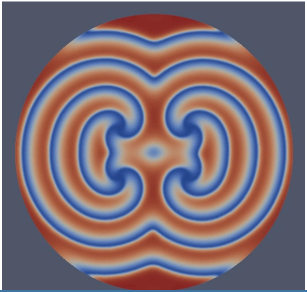


Figure: Contour lines of the rotating caloric functions mentioned in the Remark. Here $\omega = 1$, $k = 1$ and $k = 2$, respectively. In black the nodal lines: the appearance of arithmetic spirals for r large is rather clear in the picture.

Some numerical simulations (by courtesy of Alessandro Zilio), with $f_i(s) = s(1-s)$ (logistic reactions) and Neuman boundary conditions

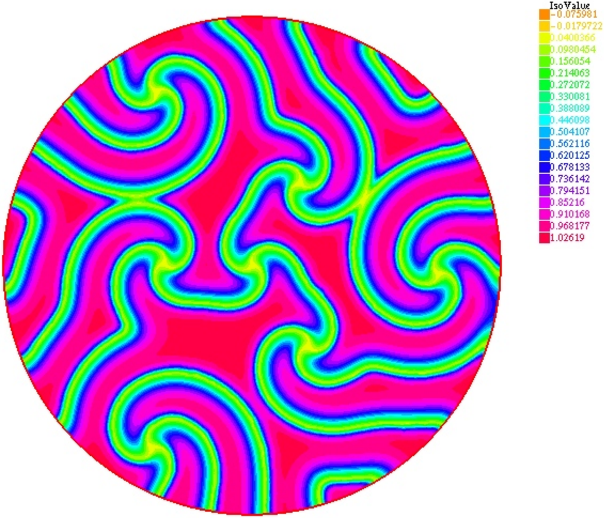






Final remarks

- Rotating spirals appear to exist and being stable for three populations under extreme asymmetric competitive interaction.
- This contradicts Gause's exclusion principle.
- Simulations show that multiple rotating spirals form complex patterns persisting for long times.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