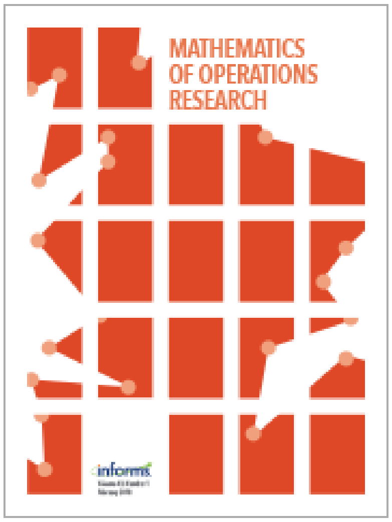

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Unique Tarski Fixed Points

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Abstract. We establish sufficient conditions that ensure the uniqueness of Tarski-type fixed points of monotone operators. A first set of results relies on order concavity, whereas a second one uses subhomogeneity. A few applications that illustrate our results are presented.

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Keywords: fixed points • Tarski fixed point theorem • recursive utilities

1. Introduction

In this paper, we establish sufficient conditions that ensure in ordered vector spaces the uniqueness of fixed points a la Tarski [40], often a highly desirable property in the many applications in economics and operations research in which such fixed points appear (cf. Topkis [42]).

More specifically, our results establish the existence and uniqueness of fixed points of monotone operators that are either order concave or subhomogeneous. Their common feature is to require that no fixed points belong to the lower perimeter of the domain. This novel notion, which we introduce in Section 3, is thus a keystone of our analysis.

We establish our main results in Sections 4 and 5. The results of the latter section rely on a close relation between the subhomogeneous case and the metric introduced by Thompson [41]. This connection permits to prove the uniqueness of fixed points of subhomogeneous operators. Besides the role of lower perimeters, the clarification of the connection between subhomogeneity and this metric is the other main contribution of this paper.

We illustrate our uniqueness results with some applications on recursive utilities and Bellman equations, integral equations, and complementary problems in Section 6. We conclude by discussing the related literature in Section 7 and with an Appendix that further elaborates on the connection between subhomogeneity and the Thompson metric.

In sum, this paper presents a general analysis of the existence and uniqueness of fixed points in ordered vector spaces. In so doing, it extends earlier ad hoc analyses in Montrucchio [31] and Marinacci and Montrucchio [29], as we discuss in Section 5.2.

2. Preliminaries

In this introductory section, we briefly present a few basic notions that we use in the paper (we refer to Davey and Priestley [14], Luxemburg and Zaanen [27], and Ok [35] for comprehensive studies).

Posets. A *poset* $(A, \geq)$ is *chain complete* (resp., *σ-complete*) if it has a minimum element and if every (resp., countable) chain has a supremum.[1] A lattice is *complete* when every nonempty subset has an infimum and supremum element. A lattice is complete if and only if is chain complete.

If $a \leq b$ are two elements of a poset A, then $[a, b] = \{x \in A : a \leq x \leq b\}$ is an *order interval.* A poset is *Dedekind* (*σ-complete*) *complete* if every order interval is chain complete (σ-complete).

An element $a \in A$ is: (1) *dominated* if there is $b \in B$ such that $a < b$, (2) *minimal* if there is no $b \in A$ such that $b < a$, and (3) a *minimum* if $a \leq b$ for all $b \in A$.

Spaces. Throughout this paper, V is a (partially) ordered vector space with order relation $\geq$, and K will always denote its positive cone. If V is Dedekind σ-complete, then it is Archimedean.[2] When V is a lattice, it is called *Riesz space*. In this case, to be Dedekind complete amounts to say that the order intervals $[a,b] \subseteq V$ are complete lattices.

Fixed Points. A self-map $T : A \to A$ is *monotone* (or *order preserving*) if $a \leq b$ implies $T(a) \leq T(b)$ for all $a, b \in A$.

A fixed point theorem due to Tarski [40] says that the set of fixed points of a monotone self-map defined on a complete lattice is a nonempty complete lattice. A generalized version of this result says that the set of fixed points of a monotone self-map defined on a chain complete poset is a nonempty chain complete poset.[3]

Given an increasing sequence (a_n), we write $a_n \uparrow a$ when $a = \sup_n a_n$. The meaning of $a_n \downarrow a$ for a decreasing sequence is similar. A sequence (b_n) *converges in order* (or *order converges*) to some $b \in V$, written $b_n \xrightarrow{\circ} b$, if there is a sequence $a_n \downarrow 0$ such that $-a_n \leq b_n - b \leq a_n$ for each $n \in \mathbb{N}$.

A self-map $T : A \to A$ is *order continuous* if, given any countable chain $\{a_n\} \subseteq A$ for which $\sup a_n$ exists, we have $T(\sup a_n) = \sup T(a_n)$. Clearly, order-continuous self-maps are monotone. A fixed-point theorem, essentially due to Kantorovich [20], says that an order-continuous self-map defined on a chain σ-complete poset has a least fixed point.

Concavity. A subset A of V is *order convex* if $a \leq c \leq b$ and $a, b \in A$ imply $c \in A$. This amounts to say that A contains all order intervals (and so all segments) determined by its elements.

A self-map $T : A \to A$ defined on an order convex subset is *order concave* if

$$T(ta + (1-t)b) \geq tT(a) + (1-t)T(b),$$

for all $t \in [0,1]$ and all $a, b \in A$ with $a \leq b$. Order-concave and order-convex operators are studied in Amann [3, chapter V], along with their differential characterizations. Observe that order concavity is weaker than concavity because it involves only ordered pairs.

Subhomogeneity. The study of subhomogeneity for operators was pioneered by Krasnoselskii [22]. An operator $T : K \to K$ is called

1. *Subhomogeneous* if $T(\alpha x) \geq \alpha T(x)$ for all $x \in K$ and all $\alpha \in [0,1]$[4];
2. *Strictly subhomogeneous* if the inequality is strict when $0 \neq x \in K$ and $\alpha \in (0,1)$;
3. *Strongly subhomogeneous* if

$$T(\alpha x) \geq \varphi(x, \alpha) T(x) \qquad \forall 0 \neq x \in K, \forall \alpha \in (0,1), \tag{1}$$

with $\alpha < \varphi(x, \alpha) < 1$[5];

4. *Subhomogeneous of order* $p \in (0,1)$ if $T(\alpha x) \geq \alpha^p T(x)$ for all $x \in K$ and all $\alpha \in [0,1]$.

Note that $T(0) \geq 0$ holds for a subhomogeneous operator. Subhomogeneous operators of order p are strongly subhomogeneous with $\varphi(x, \alpha) = \alpha^p$ (they are, actually, the most convenient class of such operators). For brevity, throughout the paper operators in (4) will be called p-subhomogeneous.[6]

Norms and Units. A positive element $u \in V$ is an *order unit* for V if the interval $[-u, u]$ is absorbing—that is, $V = \bigcup_{\lambda > 0} \lambda[-u, u]$. If V is Archimedean, then it can be equipped with an *order unit norm* (see Krasnoselskii [22] and Amann [3]):

$$\|x\|_u = \inf\{\lambda > 0 : -\lambda u \leq x \leq \lambda u\} \qquad \forall x \in V.$$

The positive cone K is closed in the normed space $(V, \|\cdot\|_u)$, with nonempty interior consisting of the order units of V.

Links. Two elements $x, y \in K$ are *linked* (see Thompson [41]), written $x \sim y$, if there exist scalars $\alpha, \beta > 0$ such that

$$\alpha y \leq x \leq \beta y.$$

The binary relation $\sim$ is an equivalence relation that partitions the positive cone K in disjoint components, which form the quotient set $K/\sim$. We denote by $Q(x)$ the equivalence class with representative element $x \in K$—that is, $Q(x) = \{y \in K : x \sim y\}$. It will be always understood that the components Q differ from the trivial one $\{0\}$.

3. Lower Perimeter

Let A be a set in an ordered vector space V. The *lower perimeter* $\partial_\lozenge A$ of A is defined by

$$\partial_\lozenge A = \{x \in A : \exists y \in A \text{ s.t. } y > x \text{ and } tx + (1-t)y \notin A \text{ for every } t > 1\}.$$

In words, $\partial_\lozenge A$ consists of the dominated elements a of A such that the segments that join them with a dominant element b of A cannot be prolonged beyond a without exiting A. In contrast, an element $a \in A$ does not belong to $\partial_\lozenge A$ if it is either undominated (i.e., it is maximal) or $ta + (1-t)b \in A$ holds for some $t > 1$ whenever $a < b \in A$.

Proposition 1. *A dominated and minimal element of a convex set A belongs to $\partial_\lozenge A$.*

Proof. Let $x \in A$ be dominated, with $x < z \in A$, and minimal. Suppose by contradiction that $x \notin \partial_\lozenge A$. Then there exists $t > 1$ such that $tx + (1-t)z \in A$, which contradicts minimality because $tx + (1-t)z < x$. □

Of course, $\partial_\lozenge A$ may contain nonminimal elements, as the characterizations of lower perimeters that we are about to establish will show. We first characterize lower perimeters of intervals via the link equivalence relation $\sim$.

Proposition 2. *Let $I = [a, b] \subseteq V$, with $a < b$. An element $x \in I$ does not belong to $\partial_\lozenge I$ if and only if $x - a \sim b - a$.*

Proof. Let $x \in I \backslash \partial_\lozenge I$. If $x = b$, the result is obvious. Thus, suppose $a \leq x < b$. By definition, $(1-t)b + tx \geq a$ for some $t > 1$. Setting $t = 1 + \delta$, this is equivalent to $(1+\delta)x - \delta b \geq a$ for some $\delta > 0$, namely $(1+\delta)x \geq a + \delta b$. By subtracting $(1+\delta)a$ from both sides, we get

$$(1+\delta)(x-a) \geq \delta(b-a).$$

Hence,

$$b - a \geq x - a \geq \frac{\delta}{1+\delta}(b-a).$$

So $x - a$ and $b - a$ are linked. Conversely, suppose $a \leq x < b$ and $x - a \sim b - a$. Given that $x - a < b - a$, this means that $\lambda(x-a) \geq b - a$ and that $\lambda > 1$. Otherwise, $x - a \geq b - a$, which implies $x = b$. Because $\lambda > 1$, we can set $\delta = (\lambda - 1)^{-1} > 0$—that is, $1/\lambda = \delta/(1+\delta)$. Consequently,

$$x - a \geq \frac{1}{\lambda}(b-a) = \frac{\delta}{1+\delta}(b-a) \geq \frac{\delta}{1+\delta}(b'-a),$$

for every $b' \leq b$. So $(1+\delta)x - \delta b' \geq a$. By the substitution $t = 1 + \delta$, it becomes $tx + (1-t)b' \geq a$ for some $t > 1$. This suffices to conclude that $x \in I \backslash \partial_\lozenge I$. □

Next, we characterize lower perimeters of positive cones.

Proposition 3. *Let V be Archimedean, with order units. An element $x \in K$ does not belong to $\partial_\lozenge K$ if and only if $x \sim u$ for some order unit $u \in K$.*

Proof. Let $x \in K \backslash \partial_\lozenge K$. We have $\lambda u > x$ for some λ, because u is an order unit. In view of the preceding proof, $(1+\delta)x \geq \delta\lambda u$ for some $\delta > 0$. Hence,

$$\lambda u \geq x \geq \frac{\delta\lambda}{1+\delta}u,$$

and so $x \sim u$. Conversely, let $x \sim u$ and $b > x$. Then $\lambda u \geq b$ and $x \geq \mu u$, for some $\lambda, \mu > 0$, because $x \sim u$. Hence,

$$b > x \geq \frac{\mu}{\lambda}b.$$

It follows that $\mu/\lambda < 1$. Therefore, $\mu/\lambda = \delta/(1+\delta)$ for some $\delta > 0$, namely $(1+\delta)x - \delta b \geq 0$, which means $x \in K \backslash \partial_\lozenge K$. □

This proposition establishes a sharp topological characterization of the lower perimeter of K. Indeed, the space V can be equipped with the order unit norm $\|\cdot\|_u$, where u is an order unit of V. Hence, by Proposition 3 we have

$$K \backslash \partial_\lozenge K = \operatorname{int} K,$$

according to the topology induced by $\|\cdot\|_u$. So $\partial_\lozenge K$ is the boundary of K under this topology.

A dual notion of *upper perimeter* $\partial^{\diamond} A$ can be defined, for which dual results hold. For instance, in the dual version of Proposition 2, we have

$$x \in I \backslash \partial^{\diamond} I \Longleftrightarrow b - x \sim b - a. \tag{2}$$

In what follows, whenever needed, we take for granted such dual results for upper perimeters.

We close with an example of a lower perimeter that will be useful in the rest of the paper. We consider the space $\mathbb{R}^X$ of the real-valued functions $f : X \to \mathbb{R}$ defined on a set X, endowed with the pointwise order between functions. A piece of notation: if $f, g \in \mathbb{R}^X$, we write $f \ll g$ when $\inf_{x \in X} [g(x) - f(x)] > 0$.[7]

Proposition 4. *Let V be a vector subspace of $\mathbb{R}^X$. Consider an interval $I = [f, g] \subseteq V$, with $f < g$. Then*[8]

$$\partial_{\diamond} I = \left\{ h \in I : \inf_{x \in X} \frac{h(x) - f(x)}{g(x) - f(x)} = 0 \right\}. \tag{3}$$

In particular, if $f \ll g$ and $\sup_{x \in X} [g(x) - f(x)] < \infty$, then

$$\partial_{\diamond} I = \left\{ h \in I : \inf_{x \in X} \left[h(x) - f(x) \right] = 0 \right\}. \tag{4}$$

Proof. By Proposition 2, we have $h \in I \backslash \partial_{\diamond} I$ if and only if $h - f \geq \varepsilon (g - f)$ for some $\varepsilon > 0$. Note that $h(x) - f(x) = 0$ implies $g(x) - f(x) = 0$. Therefore, we have $h \in I \backslash \partial_{\diamond} I$ if and only if

$$\inf_{x \in X} \frac{h(x) - f(x)}{g(x) - f(x)} > 0,$$

and so (3) holds. The two conditions $f \ll g$ and $\sup_{x \in X} [g(x) - f(x)] < \infty$ mean that $M \geq g - f \geq \varepsilon > 0$. Hence, in this case,

$$\frac{1}{\varepsilon}(h - f) \geq \frac{h - f}{g - f} \geq \frac{1}{M}(h - f),$$

which shows the equivalence of (3) and (4). □

By similar methods one can show that the lower perimeter of the positive cone $B_+(X)$ of the space of bounded functions $B(X)$ is $\partial_{\diamond} B_+(X) = \{h \in B_+(X) : \inf_{x \in X} h(x) = 0\}$.

4. Existence and Uniqueness: Order Concavity

Throughout this section, V denotes a Dedekind σ-complete ordered vector space.

Lemma 1. *Let $T : A \to A$ be a monotone and order-concave self-map defined on an order-convex subset A of V. Assume that either A is Dedekind complete or T is order continuous. Suppose that*

1. *for each $b \in A \backslash \partial_{\diamond} A$, there is $b \geq a \in A$ such that $T(a) > a$;*
2. *$T(a) \neq a$ for all $a \in \partial_{\diamond} A$.*

Then T has a least fixed point in A if and only if it has a unique fixed point in A.

Observe that $\partial_{\diamond} A$ might be empty. A dual result holds for order-convex operators by considering the upper perimeter $\partial^{\diamond} A$. It actually suffices to consider the conjugate map $\tilde{T} : -A \to -A$ defined by $\tilde{T}(x) = -T(-x)$.

Proof. Let ξ be the least fixed point in A. Suppose, per contra, that it is not unique. Let ζ be another fixed point. Clearly, $\xi < \zeta$ (so ξ is a dominated element of A). By (2), $\xi \notin \partial_{\diamond} A$. By (1), there exists $a \leq \xi$ such that $T(a) > a$. This implies that $a < \xi$. For each $n \in \mathbb{N}$, set

$$x_n = \xi - \frac{1}{n}(\zeta - \xi),$$

that is,

$$\xi = \frac{1}{1 + n^{-1}} x_n + \frac{n^{-1}}{1 + n^{-1}} \zeta.$$

Clearly, $x_n < \xi$ for each $n \in \mathbb{N}$, and so $x_n \neq T(x_n)$ for all $n \in \mathbb{N}$. Because $\xi \notin \partial_{\diamond} A$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. Hence, by the order concavity of T, for all $n \geq n_0$ we have

$$\frac{1}{1 + n^{-1}} x_n + \frac{n^{-1}}{1 + n^{-1}} \zeta = \xi = T(\xi) \geq \frac{1}{1 + n^{-1}} T(x_n) + \frac{n^{-1}}{1 + n^{-1}} \zeta.$$

Hence,

$$x_n > T(x_n) \qquad \forall n \geq n_0. \tag{5}$$

Because V is Dedekind σ-complete, $\sup_n x_n$ exists. Let us show that $\sup_n x_n = \xi$. Suppose not. Then there exists an element η such that

$$\xi - \frac{1}{n}(\zeta - \xi) \leq \eta < \xi \qquad \forall n \in \mathbb{N}.$$

Hence, $n(\xi - \eta) \leq \zeta - \xi$ for all $n \in \mathbb{N}$. Because a Dedekind σ-complete ordered vector space is Archimedean, we have the contradiction $\xi \leq \eta$. We conclude that $\sup_n x_n = \xi$. In turn, this implies $x_{\bar{n}} > a$ for some $\bar{n} \in \mathbb{N}$. Because A is order convex, $[a, x_{\bar{n}}] \subseteq A$. By (5), T maps the interval $[a, x_{\bar{n}}]$ into itself because $T(a) > a$. The set $[a, x_{\bar{n}}]$ is chain σ-complete. Consider two cases.

1. If A is Dedekind complete, then $[a, x_{\bar{n}}]$ is chain complete, so by the generalized Tarski's theorem there is a fixed point of T that belongs to $[a, x_{\bar{n}}]$.
2. If T is order continuous, the same is true by Kantorovich's theorem.

In both cases, because ξ is the least fixed point in A, we then have $\xi \in [a, x_{\bar{n}}]$, which contradicts $x_{\bar{n}} < \xi$. We conclude that $\zeta = \xi$. It is hence proved that if T has a least fixed point, then it is unique. The converse implication is immediate. □

In view of the generalized Tarski's theorem and of Kantorovich's theorem, we have the following existence and uniqueness result for fixed points.

Theorem 1. *Let $T: A \to A$ be a monotone and order-concave self-map defined on an order-convex and chain σ-complete subset A of V. Assume that either A is chain complete or T is order continuous. If $T(x) \neq x$ for all $x \in \partial_\diamond A$, then T has a unique fixed point.*

Proof. (1) Assume that A is chain complete. Then the existence of a least fixed point is guaranteed by the generalized Tarski's theorem. Because A is chain complete, it has a minimum element a. In view of Proposition 1, $a \in \partial_\diamond A$, and so $a < T(a)$. The hypotheses of Lemma 1 are then satisfied, so the fixed point is unique.

(2) Assume that T is order continuous. The existence of a least fixed point is then ensured by Kantorovich's theorem. Because A is chain σ-complete, it has a minimum element a. Because $a \in \partial_\diamond A$, we have $a < T(a)$. The hypotheses of Lemma 1 are then satisfied, so the fixed point is unique. □

In Riesz spaces, order-convex and chain-complete subsets are order intervals. Therefore, the previous theorem is usually applicable to self-maps defined on order intervals, as some examples will show later in the paper. The following result deals, instead, with self-maps defined on positive cones of chain σ-complete ordered vector spaces, a case not covered by Theorem 1.

Theorem 2. *Let $T: K \to K$ be a monotone and order concave self-map. Let $u \in K$ be an order unit of V. Then T has a unique fixed point in K provided that*

1. *$T(x) \neq x$ for all $x \in \partial_\diamond K$;*
2. *$T(\lambda u) \leq \lambda u$ for all sufficiently large $\lambda > 0$;*
3. *the intervals $[0, \lambda u]$ are chain complete or T is order continuous.*

Proof. By (2), there is $\lambda_0 > 0$ such that T is a monotone self-map on every interval $[0, \lambda u]$ if $\lambda \geq \lambda_0$. By (3), there is a least fixed point $\xi \in [0, \lambda_0 u]$. For the same reason, there is a least fixed point ζ in the interval $[0, \lambda u] \supseteq [0, \lambda_0 u]$. Hence, $\zeta \leq \xi$. So $\zeta \in [0, \lambda_0 u]$; therefore, $\zeta \geq \xi$ because ξ is the least fixed point in $[0, \lambda_0 u]$. We infer that ξ is the least fixed point for every interval $[0, \lambda u]$. If now η is any fixed point of $T: K \to K$, we have $\eta \in [0, \lambda u]$ for some λ because u is an order unit. Therefore, $\xi \leq \eta$, and so ξ is the least fixed point in K. By (1), $T(0) > 0$; hence, Lemma 1 guarantees the existence of the unique fixed point ξ in K. □

The lower perimeter plays a key role in the previous results. Indeed, the requirement that T has no fixed points on the lower perimeter cannot be weakened. For instance, consider the self-map $T(x_1, x_2) = (1/2, \sqrt{x_2})$ defined on the cone $\mathbb{R}^2_+$. It is monotone and concave, with $T(0,0) > (0,0)$, but it has the two fixed points $(1/2, 0)$ and $(1/2, 1)$. Clearly, $(1/2, 0) \in \partial_\diamond \mathbb{R}^2_+$.

Example 1. Define $T: [0,1]^n \to [0,1]^n$ by

$$T(x_1, x_2, .., x_n) = \left(\lambda_i \prod_{k=1}^{n} x_k^{\alpha_k^i} + \varepsilon_i \right)_{i=1}^{n}.$$

If $\varepsilon_i \geq 0$, $0 < \lambda_i + \varepsilon_i \leq 1$, and $\alpha_k^i > 0$ for each i and k, then T maps monotonically $[0,1]^n$ into itself. Assume now that

1. $\alpha_k^i > 1$ for every $i, k = 1, \ldots, n$;
2. $\lambda_i + \varepsilon_i < 1$ for some $i = 1, \ldots, n$.

From (1) it follows that T is order convex, although it is not convex.[9] The upper perimeter of $[0,1]^n$ is

$$\partial^\diamond [0,1]^n = \{x \in [0,1]^n : x_1 \vee \cdots \vee x_n = 1\}.$$

It is then easy to check that by (2) we have $T(x) < x$ for all $x \in \partial^\diamond [0,1]^n$. So no fixed point belongs to $\partial^\diamond [0,1]^n$. Theorem 1 guarantees the existence of a unique fixed point ξ in $[0,1]^n$, which can be computed recursively as $T^n(\bar{1}) \downarrow \xi$. ▲

5. Existence and Uniqueness: Subhomogeneity

5.1. Subhomogeneity and Order Concavity

In this section, we consider versions of the previous uniqueness results for subhomogeneous operators. The techniques that we use here are altogether different from those of the previous section and rely on a connection with the Thompson metric. Note that the fixed point problems based on subhomogeneous operators are often defined on cones, whereas those based on order concave operators are often defined on bounded intervals.

To best appreciate the scope of these results, it is important to understand first the relations between order concavity and subhomogeneity.

Proposition 5. *An order concave operator $T : K \to K$ is subhomogeneous.*

Proof. Clearly, $T(0) \geq 0$. If T is order concave, then

$$T(\alpha x) = T(\alpha x + (1-\alpha)0) \geq \alpha T(x) + (1-\alpha) T(0) \geq \alpha T(x),$$

for all $x \in K$ and $\alpha \in [0,1]$. So T is subhomogeneous. □

This result may seem to suggest, at a first glance, that the approach based on subhomogeneity is superior to the order concave one of Section 4 for operators that are defined on cones. The order-concavity approach, however, is more flexible because it extends, via a standard duality, to order convex operators. In contrast, subhomogeneous operators do not feature a similar duality.

The following notion is useful to verify the various kinds of subhomogeneity. Say that an operator $T : K \to K$ is *h-subconcave* (*at* 0), with $h \in K$, if

$$T(\alpha x) \geq \alpha T(x) + (1-\alpha) h \quad \forall x \in K, \forall \alpha \in [0,1].$$

An order concave operator T is h-subconcave for all $0 \leq h \leq T(0)$. Moreover, h-subconcavity implies subhomogeneity, which is, in turn, equivalent to 0-subconcavity. The next proposition shows that under some qualifications, the class of h-subconcave operators has stronger properties of subhomogeneity, provided $h > 0$.

Proposition 6. *Let $T : K \to K$ be h-subconcave, with $h > 0$.*

1. *If $T([0,a]) \subseteq [0,a]$ for some $a \in K$, and $a \sim h$, then T is p-subhomogeneous on $[0,a]$ for some $p \in (0,1)$.*
2. *If $T([0,\lambda_0 u]) \subseteq [0, \lambda_0 u]$ for some $\lambda_0 > 0$, where u is an order unit with $u \sim h$, then T is strongly subhomogeneous on K.*
3. *If $T(K) \subseteq Q(h)$, then T is strongly subhomogeneous.*

Proof. (1) By hypothesis, $h \leq T(0) \leq a$ and $a \sim h$. Consequently, there is a scalar $\lambda \leq 1$ for which $h \geq \lambda a$. We can assume $\lambda < 1$ [the case $\lambda = 1$ is trivial because we would have $T(0) = a$—that is, $T([0,a]) = \{a\}$]. Therefore, for every $x \in [0,a]$,

$$\begin{aligned} T(\alpha x) &\geq \alpha T(x) + (1-\alpha) h \geq \alpha T(x) + (1-\alpha)\lambda a \geq \alpha T(x) + (1-\alpha)\lambda T(x) \\ &= [\alpha + (1-\alpha)\lambda]\, T(x). \end{aligned} \tag{6}$$

By the superdifferentiability at $\alpha = 1$ of the concave function $\alpha \to \alpha^{1-\lambda}$, defined on $\mathbb{R}_+$ with $\lambda \in (0,1)$, it follows the inequality

$$\alpha^{1-\lambda} \leq 1 + (1-\lambda)(\alpha - 1) = \alpha + (1-\alpha)\lambda. \tag{7}$$

In view of (6), we get $T(\alpha x) \geq \alpha^{1-\lambda} T(x)$ for all $x \in [0,a]$ and $\alpha \in [0,1]$. Hence, T is $(1-\lambda)$-subhomogeneous.

(2) Because T is h-subconcave, it is subhomogeneous. Take any $\mu \geq 1$, then $T(\mu \lambda_0 u) \leq \mu T(\lambda_0 u) \leq \mu \lambda_0 u$; namely $T([0, \lambda u]) \subseteq [0, \lambda u]$ holds for every $\lambda \geq \lambda_0$. Now take a vector $x \in K$. Because u is an order unit, we have

$x \in [0, \lambda u]$ and $T([0, \lambda u]) \subseteq [0, \lambda u]$ if λ is large enough. Point (1) implies that T is p-subhomogeneous on any bounded interval. Hence, it is strongly subhomogeneous on K.

(3) By hypothesis, $T(\alpha x) \geq \alpha T(x) + (1-\alpha) h$. Because $T(x)$ lies in $Q(h)$, we have $T(x) \leq \lambda_x h$ for some $\lambda_x > 0$. Therefore,

$$T(\alpha x) \geq \alpha T(x) + (1-\alpha) h \geq \left(\alpha + \lambda_x^{-1}(1-\alpha)\right) T(x).$$

Hence, $T(\alpha x) \geq \varphi(x, \alpha) T(x)$, with $\varphi(x, \alpha) > \alpha$, when $\alpha \in [0, 1]$, as desired. □

The next example shows that subhomogeneous operators might well be convex, a further stark illustration that order concavity and subhomogeneity are only partially related.

Example 2. The monotone and strictly convex functions $f_n : (0, \infty) \to (0, \infty)$ defined by $f_n(x) = (1 + x^n)^{1/n}$ are strongly subhomogeneous on $(0, \infty)$ and p-subhomogeneous on each bounded interval of $(0, \infty)$. Because $x f'_n(x)/f_n(x) = x^n/(1 + x^n) < 1$, these assertions will be a consequence of Corollary 3 in the appendix. However, the algebraic decomposition $f_n(\alpha x) = \varphi_n(x, \alpha) f_n(x)$, for $\alpha \in [0, 1]$ and

$$\varphi_n(x, \alpha) = \alpha \left(1 + \frac{\alpha^{-n} - 1}{1 + x^n}\right)^{1/n},$$

shows directly that each f_n is strongly subhomogeneous. ▲

A more interesting example is the Bellman operator $T : B(X) \to B(X)$, defined by

$$T(v)(x) = \sup_{y \in D(x)} u(x, y) + \beta v(y), \tag{8}$$

where $B(X)$ is endowed with the pointwise order. Here $D : X \rightrightarrows X$ is a nonempty valued correspondence on some set X, with graph $\mathrm{Gr}D = \{(x, y) \in X \times X : y \in D(x)\}$; the function $u : \mathrm{Gr}D \to \mathbb{R}$ is a bounded (short-run) objective function, and $\beta \in (0, 1)$ is a discount factor.[10]

Although the Bellman operator is manifestly convex, it is subhomogeneous when positive. Set $\kappa = \inf_{x \in X} \inf_{y \in D(x)} u(x, y)$.

Proposition 7. *If $\kappa \geq 0$, the Bellman operator $T : B_+(X) \to B_+(X)$ is subhomogeneous (strongly if $\kappa > 0$).*

Proof. Let $\kappa \geq 0$. Take $v \in B_+(X)$. Then

$$\begin{aligned} T(\alpha v)(x) &= \sup_{y \in D(x)} u(x, y) + \alpha \beta v(y) = \sup_{y \in D(x)} \alpha u(x, y) + \alpha \beta v(y) + (1-\alpha) u(x, y) \\ &\geq \alpha \sup_{y \in D(x)} [u(x, y) + \beta v(y)] + (1-\alpha) \kappa = \alpha T(v)(x) + (1-\alpha) \kappa. \end{aligned}$$

Thus, T is h-subconcave with $h = \kappa 1_X$, so it is subhomogeneous.

Now let $\kappa > 0$. Because u is bounded, there is $M > 0$ such that $u(x, y) \leq M$ for all $(x, y) \in \mathrm{Gr}D$. For any $0 \leq v \leq \lambda_0 1_X$, where $\lambda_0 = (1-\beta)^{-1} M$, we have $T(v)(x) \leq M + \beta(1-\beta)^{-1} M = \lambda_0$. So T maps $[0, \lambda_0 1_X]$ into itself. Because $\kappa > 0$ and $\kappa 1_X \sim 1_X$, 1_X is an order unit of $B_+(X)$. By Proposition 6(2), the operator T is strongly subhomogeneous on $B_+(X)$. □

5.2. Subhomogeneity and Thompson Distance

If $x, y \in K$ are linked, the relation

$$d(x, y) = \inf\{\lambda \geq 0 : e^{-\lambda} x \leq y \leq e^{\lambda} x\} \tag{9}$$

defines a distance, called *Thompson metric*, on each component Q of K provided that V is Archimedean (see Nussbaum [34] and Thompson [41]). Note that if vectors x and y are ordered, say $x \leq y$, then (9) reduces to $d(x, y) = \inf\{\lambda \geq 0 : y \leq e^{\lambda} x\}$.

The positive cone K in a normed ordered vector space is called *normal* if there exists a constant $\gamma > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \gamma \|y\|$.

Theorem 3 (Thompson). *Let V be a normed ordered vector space. If K is normal, then convergence in the Thompson metric implies convergence in norm. If, in addition, V is Banach, then each metric space (Q, d) is complete.*

There is a close connection between the subhomogeneity property and the Thompson distance d. Next, we establish some results along this line. Throughout this section, V denotes an Archimedean ordered vector space and $Q \in K/\sim$ a nontrivial component of K.

Proposition 8. *Let $T : Q \to K$ be monotone.*

1. *T is subhomogeneous if and only if it is not expansive; that is,*

$$d(T(x), T(y)) \leq d(x, y) \qquad \forall x, y \in Q.$$

2. *T is strongly subhomogeneous if and only if*

$$d(T(x), T(y)) < d(x, y) \qquad \forall x \neq y \in Q. \tag{10}$$

3. *T is p-subhomogeneous if and only if it is a p-contraction; that is,*

$$d(T(x), T(y)) \leq pd(x, y) \qquad \forall x, y \in Q. \tag{11}$$

Moreover, for a subhomogeneous T, we have $T(Q) \subseteq Q$ if and only if $T(x_0) \in Q$ for some element $x_0 \in Q$.

Proof. (3) Let T be p-subhomogeneous. By Definition (9) and the fact that the infimum is a minimum, being V Archimedean, we have

$$e^{-d(x,y)} x \leq y \leq e^{d(x,y)} x \qquad \forall x, y \in Q. \tag{12}$$

It follows that

$$e^{-pd(x,y)} T(x) \leq T(y) \leq e^{pd(x,y)} T(x).$$

In turn, this gives $d(T(x), T(y)) \leq pd(x, y)$. Conversely, assume that (11) holds. In particular, $d(T(x), T(\alpha x)) \leq pd(x, \alpha x)$ holds for $x \in Q$ and $\alpha \in (0, 1)$. Because $d(x, \alpha x) = -\log \alpha$, we get $d(T(x), T(\alpha x)) \leq -p \log \alpha$. Moreover,

$$T(x) \leq e^{d(T(x), T(\alpha x))} T(\alpha x), \tag{13}$$

which provides $T(x) \leq e^{-\log \alpha^p} T(\alpha x) = \alpha^{-p} T(\alpha x)$, as desired.

(2) Let T be strongly subhomogeneous, and take $x \neq y$. By the definition of d, we have the relation (12). By $\alpha < \varphi(x, \alpha)$, strong subhomogeneity and monotonicity, we have

$$T(y) \geq T(e^{-d(x,y)} x) \geq \varphi(x, e^{-d(x,y)}) T(x) > e^{-d(x,y)} T(x)$$

and

$$T(x) \geq T(e^{-d(x,y)} y) \geq \varphi(y, e^{-d(x,y)}) T(y) > e^{-d(x,y)} T(y).$$

Both inequalities are strict because $x \neq 0$ and $y \neq 0$. So

$$e^{-d(x,y)} T(x) < T(y) < e^{d(x,y)} T(x).$$

In view of (9), we get $d(T(x), T(y)) < d(x, y)$.

Conversely, suppose that (10) holds. Observe that the relation (13) can be interpreted as $T(\alpha x) \geq \varphi(x, \alpha) T(x)$, where $\varphi(x, \alpha) = e^{-d(T(x), T(\alpha x))}$. By contrast, $d(T(x), T(\alpha x)) < -\log \alpha$ if $0 < \alpha < 1$. This implies that $\varphi(x, \alpha) > \alpha$ whenever $\alpha \in (0, 1)$. This proves that T is strongly subhomogeneous.

The proof of (1) is similar. As to the final claim, let T be subhomogeneous and $x \in Q(x_0)$. Clearly, there are numbers $0 < \lambda < 1 < \mu$ such that $\lambda x \leq x_0 \leq \mu x$. We then have

$$\lambda T(x) \leq T(\lambda x) \leq T(x_0) \leq T(\mu x) \leq \mu T(x),$$

that is, $T(x) \sim T(x_0) \sim x_0$. □

A particularly elegant case is when V is the space of bounded functions $B(X)$ endowed with the supnorm, and the component Q contains a unit vector $u = 1_X$. This case, which is also related to the logarithmic transformation of an operator, will be treated in the appendix.

Next, we give a first consequence of the connection between subhomogeneity and the Thompson metric established in the last result. Note that, in the special case of p-subhomogeneous operators, the next result is well known and easily follows from the contraction property established in point (3) of the last result.

Proposition 9. *Let $T: K \to K$ be monotone and strongly subhomogeneous. If $\zeta \in Q$ is a fixed point of T, then $T(Q) \subseteq Q$, and given any initial condition $x_0 \in Q$, we have $T^n(x_0) \xrightarrow{\circ} \zeta$ as well as*

$$d(T^n(x_0), \zeta) \to 0.$$

The proof of this result will be given in the appendix. Recall that the Thompson convergence implies norm convergence when the space V is endowed with a norm and the positive cone is normal (Theorem 3).

Remark 1. (1) There is a large body of literature on subhomogeneity and uniqueness of fixed points, starting from Krasnoselskii [22]. What it is novel here is the observation that behind subhomogeneous operators, there is a metric. Proposition 9 is a first dividend of this dual nature of subhomogeneity. Note that this observation is independent of Thompson's theorem 3. Of course, when the cone is normal, his powerful theorem then ensures strong contractive properties for p-subhomogeneous operators.

(2) Thompson's theorem has already been applied in the earlier papers of Montrucchio [31] and Marinacci and Montrucchio [29]. The former paper applies a version of Thompson's theorem to establish results on the twice-differentiability of policy functions in infinite-horizon dynamic programming. The latter paper, instead, applies such a theorem to prove the unique existence of recursive utilities generated by aggregators. As mentioned in the Introduction, both papers use ad hoc arguments that, for instance, for Marinacci and Montrucchio [29] correspond to point (1) of Proposition 6 (the p-subhomogeneous case). In contrast, this paper provides a general analysis of the existence and uniqueness of fixed points, abstracting from specific applications (which will be considered later in the paper to illustrate our general results).

(3) Although for simplicity so far we considered a cone domain K, the different kinds of subhomogeneity that we introduced as well as the results that we established for them in this section hold on any star-shaped subset A of K—that is, $\alpha x \in A$ if $x \in A$ and $\alpha \in [0,1]$. Throughout this paper, we consider these more general domains whenever needed.

5.3. Existence and Uniqueness

We can now establish the subhomogeneous counterparts of the order concave existence and uniqueness results for fixed points of Section 4. Throughout this subsection, V denotes an Archimedean ordered vector space.

We begin with the counterpart of Theorem 1. Here d is the Thompson metric.

Theorem 4. *Let $T: [0,b] \to [0,b]$ be a monotone and strongly subhomogeneous self-map defined on a chain σ-complete interval of V. Assume that either $[0,b]$ is chain complete or T is order continuous. If $T(x) \neq x$ for all $x \in \partial_\lozenge [0,b]$, then T has a unique fixed point ζ. Moreover, for every initial point $x_0 \notin \partial_\lozenge [0,b]$, we have $T^n(x_0) \xrightarrow{\circ} \zeta$ as well as*

$$d(T^n(x_0), \zeta) \to 0. \tag{14}$$

Proof. By hypothesis, the fixed points must lie in $[0,b] \backslash \partial_\lozenge [0,b]$, which agrees with $Q(b)$, thanks to Proposition 2. By (10), the fixed point is unique whenever it exists. By contrast, it exists thanks to the theorems of Kantorovich and Tarski. The rest of the proof is now a consequence of Proposition 9. □

Note that the order interval $[0,b]$ is star shaped in V, a property that we need in the subhomogeneous case—so we cannot consider generic order intervals, as we did in the order-concave case. The global attracting properties established in Theorem 4 are remarkable because they make computable the fixed points via the iterations $T^n(x_0)$. In addition, when V is normed with a normal positive cone, then the convergence in (14) actually holds with respect to the norm (cf. Theorem 3).

Next, we state the subhomogeneous counterpart of Theorem 2.

Theorem 5. *Let $T: K \to K$ be a monotone and strongly subhomogeneous self-map. Let $u \in K$ be an order unit of V. Then T has a unique fixed point ζ in K provided that*

1. *$T(x) \neq x$ for each $x \in \partial_\lozenge K$;*
2. *$T(\lambda u) \leq \lambda u$ for some $\lambda > 0$;*
3. *$[0, \lambda u]$ is chain complete or T is order continuous and $[0, \lambda u]$ is chain σ-complete. Moreover, for every initial point $x_0 \notin intK$, we have $T^n(x_0) \xrightarrow{\circ} \zeta$ as well as*

$$d(T^n(x_0), \zeta) \to 0.$$

Proof. If (1) holds, then the fixed points will be located in $K \setminus \partial_\Diamond K$. By Proposition 3, $K \setminus \partial_\Diamond K = Q(u)$. Therefore, (2) of Proposition 8 implies the existence of at most a fixed point in $Q(u)$. Under (2), T maps monotonically $[0, \lambda u]$ into itself, and thus, (3) implies the existence of fixed points. Consequently, we have a unique fixed point $\zeta \in K$. Now we can invoke Proposition 9. □

6. Applications

6.1. Recursive Utilities and Bellman Equations

Although the study of recursive utilities dates back to Koopmans [21], the idea of using intertemporal aggregators to generate recursive utilities is due to Lucas and Stokey [26, 39]. Specifically, an *aggregator* is a function $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ that satisfies the following properties:

1. W is positive and monotone; and
2. the equation $W(c, \zeta) = \zeta$ has at least a positive solution for each $c \geq 0$.

A *recursive utility* $U : l_+^\infty \to \mathbb{R}$ generated by an aggregator W is a solution of the *Koopmans equation*

$$U(c_0, c_1, c_2, \ldots.) = W(c_0, U(c_1, c_2, \ldots.)). \tag{15}$$

More concisely,

$$U({}_0c) = W(c_0, U({}_1c)),$$

where ${}_0c = (c_0, c_1, c_2, \ldots)$ and ${}_1c = (c_1, c_2, \ldots)$ denotes the shift operator. For instance, standard time separable U are generated by the aggregators $W(c, \zeta) = u(c) + \beta\zeta$.

A key issue is whether an aggregator determines a unique recursive utility. The next proposition addresses this issue for a class of aggregators, introduced in Marinacci and Montrucchio [29] under the name of Thompson aggregators, that cannot be treated by the standard contraction methods employed by Lucas and Stokey [26].

To illustrate the techniques developed in this paper, we give just one unique existence result. For related results and discussions, we refer to our earlier work, Marinacci and Montrucchio [29], as well as to Hansen and Scheinkman [19], Balbus [6], Becker and Rincon-Zapatero [7], Borovička and Stachurski [10], and Guo and He [18].

Proposition 10. *Suppose the aggregator* $W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *satisfies the conditions*

a. $\zeta \longmapsto W(c, \zeta)$ *is concave at* 0 *for each* $c \in \mathbb{R}_+$[11];
b. $W(c, 0) > 0$ *for each* $c > 0$.

Then W *generates a unique recursive utility function* U *on* $\operatorname{int} l_+^\infty$. *Moreover, the orbit* $U_{n+1} = T(U_n)$ *generated by the Koopmans operator* (16) *converges uniformly to* U *from all initial* $U_0 \in B_+([\bar{\varepsilon}, \bar{L}])$, *where* $[\bar{\varepsilon}, \bar{L}]$ *is an interval in* l_+^∞.[12]

Proof. Under assumptions (a) and (b), the function $\zeta \longmapsto W(c, \zeta)/\zeta$ is strictly decreasing in $(0, \infty)$ for all $c > 0$.[13] This implies, in turn, that the equation $W(c, \zeta) = \zeta$ has a unique solution ζ_c for all $c > 0$ and, moreover, $W(c, \zeta) < \zeta$ if $\zeta > \zeta_c$.

Fix any two numbers $0 < \varepsilon < L$, and consider the interval $[\bar{\varepsilon}, \bar{L}] \subseteq \operatorname{int} l_+^\infty$, where we set $\bar{\varepsilon} = (\varepsilon, \ldots, \varepsilon, \ldots)$ and $\bar{L} = (L, \ldots, L, \ldots)$. Consider the space $B([\bar{\varepsilon}, \bar{L}])$ of the bounded functions on $[\bar{\varepsilon}, \bar{L}]$. Elements $U \in B_+([\bar{\varepsilon}, \bar{L}])$ are regarded as utility functions, and the recursive utilities will be fixed points of the operator $T : B_+([\bar{\varepsilon}, \bar{L}]) \to B_+([\bar{\varepsilon}, \bar{L}])$ defined by

$$T(U)({}_0c) = W(c_0, U({}_1c)). \tag{16}$$

By assumption (2), there is a positive scalar ζ_L such that $W(L, \zeta_L) = \zeta_L$. If we take any arbitrary $\lambda \geq \zeta_L$, the operator T maps the interval $[\mathbf{0}, \boldsymbol{\lambda}] \subseteq B_+([\bar{\varepsilon}, \bar{L}])$ into itself, where $\mathbf{0}$ and $\boldsymbol{\lambda}$ denote the constant utility functions $U(c) = 0$ and $U(c) = \lambda$, respectively. Actually, if $0 \leq U(c) \leq \lambda$, then by the monotonicity condition (1),

$$0 \leq T(U)(c) = W(c_0, U({}_1c)) \leq W(L, \lambda) \leq \lambda.$$

Clearly, T is monotone. Moreover, by (1),

$$\begin{aligned} T(\alpha U)({}_0c) = W(c_0, \alpha U({}_1c)) &\geq \alpha W(c_0, U({}_1c)) + (1 - \alpha) W(c_0, 0) \\ &\geq \alpha W(c_0, U({}_1c)) + (1 - \alpha) W(\varepsilon, 0). \end{aligned}$$

Hence, T is $\mathbf{h}$-subconcave by taking the constant function $\mathbf{h} = W(\varepsilon, 0) > 0$. It follows that $\mathbf{h}$ is an order unit of $B([\bar{\varepsilon}, \bar{L}])$, so Proposition 6(2) implies that $T : B_+([\bar{\varepsilon}, \bar{L}]) \to B_+([\bar{\varepsilon}, \bar{L}])$ is strongly subhomogeneous.

In view of Theorem 5, to conclude that T has a unique fixed point in $B_+([\bar{\varepsilon},\bar{L}])$, it is sufficient to show that $\partial_\lozenge B_+([\bar{\varepsilon},\bar{L}])$ does not contain any recursive utility. Let $[U]_\infty = \inf_{c\in[\bar{\varepsilon},\bar{L}]} U(c)$. If U is recursive, then

$$[U]_\infty = \inf_{c\in[\bar{\varepsilon},\bar{L}]} U(c) = \inf_{c\in[\bar{\varepsilon},\bar{L}]} W(c_0, U({}_1c)) = \inf_{\varepsilon\le c_0\le L}\ \inf_{{}_1c\in[\bar{\varepsilon},\bar{L}]} W(c_0, U({}_1c))$$
$$\ge \inf_{\varepsilon\le c_0\le L} W(c_0, \inf_{{}_1c\in[\bar{\varepsilon},\bar{L}]} U({}_1c)) = \inf_{\varepsilon\le c_0\le L} W(c_0,[U]_\infty) = W(\varepsilon,[U]_\infty) \ge W(\varepsilon,0) > 0.$$

By Proposition 4, $U \notin \partial_\lozenge B_+([\bar{\varepsilon},\bar{L}])$. Thus, T has a unique fixed point in $B_+([\bar{\varepsilon},\bar{L}])$.

Because the two numbers $0<\varepsilon<L$ are arbitrary, we have established the existence and uniqueness of the recursive function defined on $\bigcup_{\varepsilon,L>0}[\bar{\varepsilon},\bar{L}] = \mathrm{int}\, l_+^\infty$. Finally, thanks to the normality of the cone $B_+([\bar{\varepsilon},\bar{L}])$, the convergence claimed in Theorem 5 implies the uniform convergence. □

Rather than solving directly the Koopmans equation (15), it is often useful to analyze an auxiliary "parametric" problem that leads to a family of fixed-point problems. For a given consumption stream $c\in l_+^\infty$, consider the operator $T_c : l_+^\infty \to l_+^\infty$ defined, at each time t, by

$$T_c(v)_t = W(c_t, v_{t+1}) \qquad \forall v\in l_+^\infty. \tag{17}$$

Clearly, the sequence $v_t = U({}_tc)$ is a fixed point of T_c if U solves (15). Conversely, the utility U may be recovered from the fixed point of T_c as the stream c varies. This approach, used in Marinacci and Montrucchio [29], is especially useful in the stochastic case. Using our techniques, we can prove the following uniqueness result that parallels Proposition 10 (for brevity, we omit the convergence properties).

Proposition 11. *Let W satisfy* (a) *and* (b). *The operator $T_c : l_+^\infty \to l_+^\infty$ has a unique fixed point provided* $\liminf_{t\to\infty} c_t > 0$.

Proof. We only sketch the proof. If $c_t \ge \eta > 0$ for all t, the proof follows the same lines of that of Proposition 10: the operator is strongly subhomogeneous, and Theorem 2 provides the desired result.

In the more general case, we have $c_t \ge \eta > 0$ for all $t \ge N$. Let $\bar{v}$ and $\underline{v}$ be the greatest and the least fixed points of T_c, respectively. We have ${}_N\bar{v} = {}_N\underline{v}$ because both are fixed points of $T_{{}_Nc}$, which has a unique fixed point thanks to the first part of the proof. By induction, we have

$$\bar{v}_{N-1} = W(c_{N-1},\bar{v}_N) = W(c_{N-1},\underline{v}_N) = \underline{v}_{N-1},$$

and so on. Therefore, $\bar{v} = \underline{v}$. □

We turn to the Bellman equations associated with recursive utilities.[14] Now the Bellman operator $T: B_+(X)\to B_+(X)$ is defined by

$$T(v)(x) = \sup_{y\in D(x)} W(u(x,y), v(y)), \tag{18}$$

where $u : \mathrm{Gr}D \to \mathbb{R}_+$ is positive and bounded. We get back to (8) when $W(c,\zeta) = c + \beta\zeta$. Under mild assumption—for example, the continuity of $W(c,\cdot)$—for any recursive utility function $U : l_+^\infty \to \mathbb{R}_+$ generated by W, the associated value function $v\in B_+(X)$ is a fixed point of the Bellman operator (see Bloise and Vailakis [9]). In general, the converse is no longer true—that is, its Bellman equation may admit multiple fixed points. Next, we formulate a uniqueness result that relies on some simple conditions. Recall that $\kappa = \inf_{x\in X}\inf_{y\in D(x)} u(x,y)$.

Proposition 12. *Let W satisfy* (a) *and* (b). *If either $W(0,0)>0$ or $\kappa>0$, then the Bellman operator has a unique fixed point in $B_+(X)$, which is globally convergent in the uniform topology.*

Proof. Because u is positive and bounded, there is $M>0$ such that $0\le u\le M$. By (a) and (b), there is a scalar ζ such that $W(M,\zeta)\le\zeta$. This implies that $T([0,\zeta 1_X])\subseteq[0,\zeta 1_X]$. As in Proposition 7, we can easily show that T is h-subconcave at 0. More specifically, $T(\alpha f)\ge \alpha T(f) + (1-\alpha)h$, where $h(x) = \inf_{y\in D(x)} W(u(x,y),0)$.

If $W(0,0)>0$, we have $h(x)\ge W(0,0)>0$ for all $x\in X$, whereas in the case $\kappa>0$, we have $h(x)\ge W(\kappa,0)>0$ for all $x\in X$. In both cases, we can invoke Proposition 6(2) and infer that T is strongly subhomogeneous. For the same reason, if v is a fixed point, then for each $x\in X$, either $v(x)\ge W(0,0)$ or $v(x)\ge W(\kappa,0)$. Hence, $v\notin\partial_\lozenge B_+(X)$. Theorem 5 then implies the existence of a unique solution of the Bellman equation. Finally, the normality of the positive cone $B_+(X)$ implies the claimed uniform convergence. □

As a further illustration of our methods, next we prove through them a deterministic version of a recent result of Bloise and Vailakis [9] that relies on an interiority condition based on feasible paths.

Proposition 13. *Suppose that*

1. *the set X is convex, the correspondence $D: X \rightrightarrows X$ is compact-valued and with convex graph, and the function $u: GrD \to \mathbb{R}$ is continuous and concave;*
2. *W is continuous, satisfies (a) and (b), and $W(\cdot, \zeta)$ is concave;*
3. *the Bellman operator $T: B_+(X) \to B_+(X)$ maps continuous functions into continuous functions.*

Then the Bellman operator has a unique fixed point in $B_+(X)$, which is a continuous function and is globally convergent in the uniform topology, provided that for each $x \in X$ either $D(x) = \{x\}$ or there is a feasible path $(x_t)_{t\geq 0}$ with $x_0 = x$ and

$$\liminf_{t\to\infty} u(x_t, x_{t+1}) > 0. \tag{19}$$

The *Feller property* (3) is, in our deterministic setting, usually obtained through a continuity assumption on the correspondence $D: X \rightrightarrows X$ (see Lucas and Stokey [26]). We refer to Bloise and Vailakis [9] for more comments about the interiority condition (19). It is, however, fairly mild, so this result guarantees the uniqueness of the fixed point of (18) for standard problems, avoiding cake-like models.[15]

Proof. Denote by $\bar{v}$ and $\underline{v}$ the greatest and the least fixed point of T in $B_+(X)$, respectively (their existence is ensured by Tarski's theorem). Because T is order continuous, that is, $f_n \uparrow f$ implies $T(f_n) \uparrow T(f)$, the least fixed point $\underline{v}$ is achieved by the sequence $f_n \uparrow \underline{v}$ with $f_{n+1} = T(f_n)$ and $f_0 \equiv 0$. Thanks to (3), $\underline{v}$ will be lower semicontinuous. Analogously, the greatest fixed point $\bar{v}$ is actually upper semicontinuous, but the argument to show it is slightly more involved, and we refer to Bloise and Vailakis [9, lemma 6] (observe that $f_n \downarrow f$ does not imply $T(f_n) \downarrow T(f)$, in general).

Hence, we must prove that $\bar{v}(x) = \underline{v}(x)$ for all $x \in X$. The case $D(x) = \{x\}$ is trivial because $\bar{v}(x) = \underline{v}(x)$. Fix, then, an initial vector x_0 for which the condition (19) holds. Because $\bar{v}$ is upper semicontinuous, the supremum in (8) is attained, so $\bar{v}(x) = \max_{y\in D(x)} W(u(x,y), \bar{v}(y))$. By iterating this equation from $x = x_0$, we obtain a sequence $\bar{x} = (\bar{x}_t)$ such that

$$\bar{v}(\bar{x}_t) = W(u(\bar{x}_t, \bar{x}_{t+1}), \bar{v}(\bar{x}_{t+1}))$$

and $\bar{x}_0 = x_0$. For the sake of simplicity, set $\bar{v}(\bar{x}_t) = \bar{v}_t$ and $\bar{c}_t = u(\bar{x}_t, \bar{x}_{t+1})$ so that

$$\bar{v}_t = W(\bar{c}_t, \bar{v}_{t+1}). \tag{20}$$

By hypothesis, there exists a feasible $x^* = (x_t^*)$, with $x_0^* = \bar{x}_0 = x_0$, that satisfies (19). Define the perturbed plan $\underline{x} = (1-\alpha)\bar{x} + \alpha x^*$ with $\alpha \in (0,1)$, which is feasible under the hypothesis of convexity. Because $\underline{v}$ is also a fixed point, we have

$$\underline{v}_t \geq W(\underline{c}_t, \underline{v}_{t+1}), \tag{21}$$

where, accordingly, we have set $\underline{v}_t = \underline{v}(\underline{x}_t)$ and $\underline{c}_t = u(\underline{x}_t, \underline{x}_{t+1})$. Observe that by the concavity of u, we have

$$\alpha > 0 \Rightarrow \liminf_{t\to\infty} \underline{c}_t > 0 \tag{22}$$

and that (21) is equivalent to $T_{\underline{c}}(\underline{v}) \leq \underline{v}$—using the auxiliary operator (17). Proposition 11 implies that $T_{\underline{c}}$ has a unique fixed point w^* in $[0, \underline{v}] \subseteq l_+^\infty$.

Now, by the concavity of u, we have $\underline{c}_t \geq (1-\alpha)\bar{c}_t + \alpha c_t^*$, where $c_t^* = u(x_t^*, x_{t+1}^*)$. So

$$W(\underline{c}_t, \bar{v}_{t+1}) \geq (1-\alpha) W(\bar{c}_t, \bar{v}_{t+1}) + \alpha W(c_t^*, \bar{v}_{t+1}) \geq (1-\alpha) W(\bar{c}_t, \bar{v}_{t+1}).$$

In view of (20), it follows that $(1-\alpha)\bar{v}_t \leq W(\underline{c}_t, \bar{v}_{t+1})$—namely $T_{\underline{c}}(\bar{v}) \geq (1-\alpha)\bar{v}$.

Let us first assume that $\underline{c}_t \geq \eta > 0$ for every $t \geq 0$ in place of (22). Then $T_{\underline{c}}$ turns out to be p-subhomogeneous for some $p \in (0,1)$—that is, $T_{\underline{c}}(\alpha w) \geq \alpha^p w$ if $\alpha \in [0,1]$. Set $\mu_0 = (1-\alpha)^{1/(1-p)} < 1$. We have

$$T_{\underline{c}}(\mu_0 \bar{v}) \geq \mu_0^p T_{\underline{c}}(\bar{v}) \geq \mu_0^p (1-\alpha)\bar{v} = \mu_0 \bar{v}.$$

Consequently, $\mu_0 \bar{v} \leq w^* \leq \underline{v}$. Because $\alpha \to 0^+$, we have that $\mu_0 \to 1$, so $\bar{v} \leq \underline{v}$. That is, $\bar{v}(x_0) = \underline{v}(x_0)$.

If now it holds condition (22), then $\underline{c}_t \geq \eta > 0$ for all $t \geq N$. In this case, we can use the preceding argument for the operator $T_{{}_N\underline{c}}$ and so conclude that $\bar{v}(x_N) = \underline{v}(x_N)$. Then

$$\bar{v}(\bar{x}_{N-1}) = W(u(\bar{x}_{N-1}, \bar{x}_N), \bar{v}(\bar{x}_N)) = W(u(\bar{x}_{N-1}, \bar{x}_N), \underline{v}(\bar{x}_N)) \leq \underline{v}(\bar{x}_{N-1}).$$

Therefore, $\bar{v}(\bar{x}_{N-1}) = \underline{v}(\bar{x}_{N-1})$, and recursively, we get again $\bar{v}(x_0) = \underline{v}(x_0)$. □

6.2. An Integral Equation

An interesting class of integral equations is

$$\varphi(x) = \int k(x, y, \varphi(y))\, \pi(\mathrm{d}y \mid x) \qquad \forall x \in X, \tag{23}$$

where $\varphi \in C(X)$, π is a transition function on a Polish space X, and $k : X \times X \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

This class of integral equations arises in economics, for instance, in finding the monetary equilibria of an overlapping-generations model studied by Stokey and Lucas [39, chapter 17]. Specifically, consider an infinitely lived economy with constant population. Each agent lives two periods, works when young, and consumes when old. An agent's separable utility is $U(l, c) = V(c) - H(l)$, where l is his or her labor supply when young and c is his or her consumption when old. Assume that $H : [0, L) \to \mathbb{R}_+$ is twice continuously differentiable, with $H'(l) > 0 = H'(0) > H''(l)$ for all $l \in (0, L)$ and $\lim_{l \to L} H'(l) = +\infty$.

The unique nonstorable consumption good is produced by a linear technology $y = xl$, where $x \in X = [a, b]$ is an exogenous random variable that follows a first-order Markov process $\pi(dy \mid x)$. Fiat money, with a fixed supply M, is the only store of value. In each period, the young agents produce quantities of the good and sell them to the old ones in exchange for fiat money.

In this economy, the technology shock x is the only state variable, so an equilibrium is a pair $(p(x), l(x))$, where $p : X \to \mathbb{R}_+$ is the price function at which the money is valued and $l : X \to [0, L]$ is the labor supply function such that $xl(x)p(x) = M$ and

$$l(x) \in \arg\max_{l \in [0, L]} -H(l) + \int V\left[\frac{xp(x)l}{p(y)}\right] \pi(\mathrm{d}y \mid x).$$

One can show (see Stokey and Lucas [39, chapter 17]) that the equilibrium problem is solved via the fixed point problem

$$\varphi(x) = \int k(y\zeta^{-1}(\varphi(y)))\, \pi(\mathrm{d}y \mid x), \tag{24}$$

where $\varphi(x) = \zeta(l(x))$, $\zeta(l) = lH'(l)$, and $k(y) = yV'(y)$. This is a special case of (23). Because the function ζ is invertible, owing to what is assumed on H, to each fixed point of $\bar{\varphi}(x)$ of (24), there corresponds the equilibrium $\bar{l}(x) = \zeta^{-1}(\bar{\varphi}(x))$ and $\bar{p}(x) = M(x\bar{l}(x))^{-1}$.

The integral Equation (23) may be handled with our techniques. To circumvent the unpleasant fact that the space $C(X)$ is not a σ-Dedekind complete lattice, we extend the operator T associated with Equation (23) to the operator $T^* : B_+(X) \to B_+(X)$ defined by

$$T^*(f)(x) = \int^* k(x, y, f(y))\, \pi(\mathrm{d}y \mid x) \qquad \forall x \in X,$$

where $\int^* \psi$ denotes the outer integral of a function $\psi \in B_+(X)$ (see, for instance, Bertsekas and Shreve [8, appendix A]).

Proposition 14. *The operator $T^* : B_+(X) \to B_+(X)$ has a unique fixed point $\varphi^* \in B_+(X)$, with $d((T^*)^n(\varphi_0), \varphi^*) \to 0$ for all $\varphi_0 \in B_+(X)$, provided that*

1. *$k(x, y, \cdot)$ is monotone on $\mathbb{R}_+$ and concave at 0 for all $x, y \in X$*[16];
2. $\inf_{x \in X} \int k(x, y, 0)\, \pi(\mathrm{d}y \mid x) > 0$;
3. *$\int k(x, y, \lambda)\, \pi(\mathrm{d}y \mid x) \le \lambda$ for all $x \in X$ and for all sufficiently large $\lambda > 0$.*

Proof. Set $\varepsilon = \inf_{x \in X} \int k(x, y, 0)\, \pi(dy \mid x)$. By (1), the operator T^* is monotone (see Bertsekas and Shreve [8, lemma A.3]). If now $f \in B_+(X)$, that is, $0 \le f \le \lambda$ for some λ satisfying (3), then $\varepsilon 1_X \le T(0) \le T^*(f) \le T(\lambda) \le \lambda 1_X$. Thus, T^* maps monotonically $B_+(X)$ into $Q(1_X)$. By contrast, T^* is $\varepsilon 1_X$-subconcave thanks to (1). Actually (see also Bertsekas and Shreve [8, lemma A.2]), we have

$$\begin{aligned}
T^*(\alpha f) &= \int^* k(x, y, \alpha f(y))\, \pi(\mathrm{d}y \mid x) \ge \int^* [\alpha k(x, y, f(y)) + (1 - \alpha) k(x, y, 0)]\, \pi(\mathrm{d}y \mid x) \\
&= \alpha \int^* k(x, y, f(y))\, \pi(\mathrm{d}y \mid x) + (1 - \alpha) \int k(x, y, 0)\, \pi(\mathrm{d}y \mid x) \\
&\ge \alpha T^*(f) + (1 - \alpha)\varepsilon 1_X.
\end{aligned}$$

By Proposition 6(3), it follows that T^* is strongly subhomogeneous.

Note further that by (2), $T^*(f) \neq f$ for each $f \in \partial_\Diamond K$. So Theorem 5 implies the existence and uniqueness of the fixed point. Moreover, it is attracting with respect to the Thompson metric (cf. Proposition 9). □

This result has the following implication for our original Equation (23).

Corollary 1. *Under the same conditions of Proposition 14 and the Feller property,*[17] *the operator T has a unique fixed point $\varphi^* \in C_b^+(X)$ and $T^n(\varphi_0) \to \varphi^*$ uniformly.*

Proof. By Proposition 14, $d((T^*)^n(\varphi_0), \varphi^*) \to 0$ if $\varphi_0 \in B_+(X)$. Now, by the Feller property, we have $(T^*)^n(\varphi_0) = T^n(\varphi_0)$ if $\varphi_0 \in C_b^+(X)$. By contrast, $d(T^n(\varphi_0), \varphi^*) \to 0$ implies that the convergence of the continuous function $T^n(\varphi_0)$ is uniform because the cone $B_+(X)$ is normal (see Theorem 3). Hence, φ^* is continuous as well. □

6.3. Complementary Problems and Variational Inequalities

Let V be a vector lattice. The complementary problem, associated with a map, $F: K \to V$, asks for a point $x^* \in K$ that satisfies the orthogonality condition[18]

$$F(x^*) \wedge x^* = 0.$$

For all $\lambda > 0$, we have

$$\begin{aligned} F(x) \wedge x = 0 &\Longleftrightarrow F(x) \wedge \lambda x = 0 \Longleftrightarrow [F(x) - \lambda x] \wedge 0 = -\lambda x \\ &\Longleftrightarrow [\lambda x - F(x)] \vee 0 = \lambda x \Longleftrightarrow \lambda^{-1}[\lambda x - F(x)]^+ = x. \end{aligned}$$

So the complementary problem amounts to finding the fixed points of the self-map $T_\lambda: K \to K$ defined by

$$T_\lambda(x) = \lambda^{-1}[\lambda x - F(x)]^+, \tag{25}$$

where $\lambda > 0$ is an arbitrarily fixed parameter. The next result, which relies on the results of Section 4 via the upper perimeter, provides conditions that ensure existence and uniqueness in a complementary problem through the fixed point of T_λ.

Proposition 15. *Let V be a Dedekind complete vector lattice. Consider the following assumptions:*

1. *$\lambda x - Fx$ is λ-weakly order-Lipschitz*[19];
2. *F is order concave on K;*
3. *There exists $b \in K$ such that $F(b) \geq 0$;*
4. *$T_\lambda(x) \neq x$ for all $x \in \partial^\Diamond[0, b]$.*

Under (1) *and* (3), *there exists a vector $\xi \in [0, b]$ such that $F(\xi) \wedge \xi = 0$. Under* (1)–(4), *such a vector ξ is unique.*

The linear case $F(x) = Lx + q$, where $L: V \to V$ is a linear operator and $q \in V$ has been studied by Borwein and Dempster [11]. The existence part of Proposition 15 may be interpreted as the nonlinear version of Borwein and Dempster [11, theorem 3.4] [note that (2) trivially holds in the linear case].

Proof. The operator $x \to x^+$ of V into V is monotone and convex. Therefore, by (1), it follows that $x \mapsto [\lambda x - F(x)]^+$ is monotone. Likewise, (1) and (2) imply that $x \mapsto [\lambda x - F(x)]^+$ is order convex.

Note that (3) means that $\lambda b - F(b) \leq \lambda b$. Hence, $T_\lambda(b) = \lambda^{-1}[\lambda b - F(b)] \vee 0 \leq b \vee 0 = b$, namely $T_\lambda(b) \leq b$. Consequently, the first claim follows from Tarski's theorem applied to the self-map $T_\lambda: [0, b] \to [0, b]$. Theorem 1 provides the uniqueness result. □

As is well known, the complementary problem is closely related with the solvability of variational inequalities. Specifically, let C be a nonempty closed and convex subset of an Hilbert lattice H, and let $F: C \to H$. The *variational problem*, associated with a pair (C, F), asks for an element $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \qquad \forall x \in C. \tag{26}$$

The variational property of the metric projection $\pi_C: H \to C$ entails the equivalence of (26) with the fixed point problem

$$x^* = \pi_C(x^* - \lambda F(x^*)),$$

where $\lambda > 0$ is a given parameter.

An order-theoretic approach to the variational problem (26) is studied by Li and Ok [24] and Nishimura and Ok [33]. The solvability of (26) in such an approach is based on the fact that π_C is order preserving if and only if C is a sublattice of H (see Nishimura and Ok [33, lemma 2.4]). Here, to further illustrate our approach, we establish a uniqueness result for the semiintervals $[-\infty, b] = \{x \in H : x \leq b\}$.

Proposition 16. *If $F : [-\infty, b] \to H$ is λ-weakly order Lipschitz and order convex, then there exists a unique vector $x^* \in [a, b]$ that solves the variational problem (26) provided that*

1. *$a < b$ and $F(a) \leq 0$;*
2. *$x \neq \pi_C(x - \lambda F(x))$ for all $x \in \partial_\diamond [a, b]$.*

Proof. Define $\Phi_\lambda : C \to C$ by $\Phi_\lambda(x) = \pi_C(x - \lambda F(x))$. Then

$$\Phi_\lambda(x) = [x - \lambda F(x)] \wedge b = b - [\lambda F(x) + b - x]^+. \tag{27}$$

The vector x^* solves (26) if and only if $x^* = \Phi_\lambda(x^*)$. Consider now the interval $[a, b]$. Clearly, $\Phi_\lambda(b) \leq b$. Furthermore, $a \leq a - \lambda F(a)$, which, in turn, implies $a = \pi_C(a) \leq \pi_C(a - \lambda F(a)) = \Phi_\lambda(a)$. Hence, $\Phi_\lambda([a, b]) \subseteq [a, b]$. By the by now usual arguments based on Theorem 1, we get the desired result because the self-map Φ_λ is monotone and, in view of (27), order concave. □

The concavity or convexity of the projections is a key condition to obtain unique solutions to the variational inequality.[20] Unfortunately, this condition can be rather demanding. For instance, it is easy to see that projections on intervals $[a, b]$ are neither convex nor concave. So we end with a positive result for cones. Here C° is the polar cone of a cone C.

Proposition 17. *Let $C \subseteq H$ be a closed and convex cone. The projection π_C is convex (resp., concave) if either C is a lattice and $C \supseteq K$ (resp., $C \supseteq -K$) or C° is a lattice and $C \subseteq K$ (resp., $C \subseteq -K$).*

Proof. Assume that C is a lattice with $C \supseteq K$. Let $x, y \in H$. The variational property of the metric projection for cones implies that $x - \pi_C(x) \in C^\circ$ and $y - \pi_C(y) \in C^\circ$. Therefore, $x + y - \pi_C(x) - \pi_C(y) \in C^\circ \subseteq -K$. Hence, $x + y \leq \pi_C(x) + \pi_C(y)$. Because C is a lattice, π_C is monotone. Consequently,

$$\pi_C(x + y) \leq \pi_C(\pi_C(x) + \pi_C(y)) = \pi_C(x) + \pi_C(y).$$

We conclude that π_C is subadditive. Because π_C is positively homogeneous (see, e.g., Deutsch [15, proposition 5.6]), π_C is convex on H. The concavity of π_C, provided the lattice $C \supseteq -K$, is proved similarly.

By the Moreau decomposition (see again Deutsch [15, proposition 5.6]), for each $x \in H$, we have $x = \pi_C(x) + \pi_{C^\circ}(x)$. For instance, if C° is a lattice such that $C \subseteq K$, then $C^\circ \supseteq -K$. By what has already been proved, π_{C° is concave. Hence, $\pi_C = I - \pi_{C^\circ}$ is convex. A similar argument holds when $C \subseteq -K$ holds. □

7. Related Literature

The starting point of our analysis was a special case of Theorem 1 proved in Baiocchi and Capelo [5, p. 224]. The results that we proved here are more general, partly because—by leveraging on the notion of lower perimeter that we introduced—they are able to best exploit the interplay between order and vector structures. Earlier results on unique fixed points of concave and monotone self-maps can also be found in Amann [2, 3]. They are, however, different from ours. For instance, also Amann [2] proves a version of Theorem 1 (see also Amann [3, theorem 24.4]). However, not relying on the results of Kantorovich and Tarski,[21] it uses order notions of monotonicity and concavity on a hybrid structure (ordered topological vector spaces with weak units) that are stronger than the standard ones. Our analysis takes, in contrast, advantage of the Tarski-type theorems that enable us to use standard notions of order concavity and monotonicity through the notion of lower perimeter.

The results on subhomogeneous operators offer often a powerful alternative to those related to the order concavity. The results presented here are inspired by Krasnoselskii's seminal work, although the key connection with the Thompson metric that we develop is new.

Similar topological results can be found in Amann [3]. More recently, the uniqueness part of the fixed point theorem established by Le Van et al. [23] is closely related to the strong subhomogeneity, although adapted to spaces of functions (their existence result rests on Kantorovich's theorem). We must also mention that several authors used related similar arguments to establish uniqueness results for equilibria of dynamic economies determined as fixed points (cf. Coleman [12] and Datta et al. [13]).

Finally, our analysis does not rely on any a priori given metric structure, so it is different from the recent fixed point literature that combines order and metric structures (see, e.g., Ran and Reurings [37], Gnana Bhaskar and Lakshmikantham [17], and Nieto and Rodríguez-Lopez [32]). It is, instead, closely related to the papers that—like Le Van et al. [23] and Bloise and Vailakis [9]—study the uniqueness of solutions of Bellman equations, as it was detailed in the paper.

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Appendix. Proof of Proposition 9 and Related Analysis.
Let V be the space of bounded functions $B(X)$ endowed with the supnorm, and the component Q is that containing a unit vector $u = 1_X$—that is, $Q(u) = \text{int}\, B_+(X) = B_+(X)\backslash\partial_\delta B_+(X)$.

Proposition 18. *The Thompson metric d on int$B_+(X)$ is*

$$d(f,g) = \sup_{x\in X} \left|\log f(x) - \log g(x)\right|.$$

Moreover, the map $\mathscr{L}:\text{int}B_+(X)\to B(X)$ defined by $\mathscr{L}(f)(x) = \log f(x)$ is an isometry of $(\text{int}B_+(X), d)$ onto $(B(X), \|\cdot\|)$.

Proof. For two functions $f, g \in \text{int}\, B_+(X)$, we have

$$e^{-\lambda} g \le f \le e^{\lambda} g \Longleftrightarrow \left|\log f - \log g\right| \le \lambda,$$

which provides the desired result. Moreover, observe that the transformation $\mathscr{L}: f \longmapsto \log f$ is a bijection with inverse $\mathscr{L}^{-1}: f \longmapsto e^f$. Therefore,

$$d(f,g) = \|\log f - \log g\| = \|\mathscr{L}(f) - \mathscr{L}(g)\|,$$

and so $\mathscr{L}$ is an isometry. □

The logarithmic transformation is frequently useful to solve the fixed problem $f = T(f)$ for operators $T:\text{int}B_+(X)\to \text{int}B_+(X)$.[22] If $\mathscr{L}:\text{int}B_+(X)\to B(X)$ denotes the log transformation $f \longmapsto \log f$, the conjugate operator is $\tilde{T} = \mathscr{L}\circ T\circ\mathscr{L}^{-1}: B(X)\to B(X)$. That is, $\tilde{T}(f) = \log T(e^f)$. Clearly, f^* is a fixed point of T if and only if $\mathscr{L}(f^*)$ is a fixed point of $\tilde{T}$.

The following corollaries are straightforward applications of Propositions 8 and 18. Point (1) of the next corollary is well known (see, e.g. Nussbaum [34, proposition 1.6]).

Corollary 2. *A monotone $T:\text{int}B_+(X)\to \text{int}B_+(X)$ is*
(1) *p-subhomogeneous if and only if $\tilde{T} = \mathscr{L}\circ T\circ\mathscr{L}^{-1}$ is a p-contraction on $(B(X), \|\cdot\|)$;*
(2) *strongly subhomogeneous if and only if $\left\|\tilde{T}f - \tilde{T}g\right\| < \|f-g\|$ for all $f \ne g \in B(X)$.*

When X is a singleton, that is, $B(X) = \mathbb{R}$ and $\text{int}\, B_+(X) = (0,\infty)$, the preceding result has the following useful consequence.

Corollary 3. *A monotone and differentiable function $f:(0,\infty)\to(0,\infty)$ is*
(1) *p-subhomogeneous if $f'(x)x/f(x) \le p < 1$ for all $x > 0$;*
(2) *strongly subhomogeneous if $f'(x)x/f(x) < 1$ for all $x > 0$.*

By setting $\tilde{f}(t) = \log f(e^t)$, the derivative is $\tilde{f}'(t) = f'(e^t)e^t/f(e^t) = f'(x)x/f(x)$. So $\left|\tilde{f}'(t)\right| \le p$ implies that $\tilde{f}$ is a contraction.

A remark before proving Proposition 9: Another condition that ensures a unique solution for Equation (23) is related to the previous logarithm transformation. Specifically, by replacing Condition (4) of Proposition 14 with the elasticity condition

$$\frac{D_3 k(x,y,t)\, t}{k(x,y,t)} \le p_L < 1 \qquad \forall x,y\in X, \forall t\in[0,L], \forall L > 0,$$

we have, by Corollary 3, that $k(x,y,\alpha t) \ge \alpha^{p_t} k(x,y,t)$ for all $x, y \in X$ and all $t > 0$. In turn, this implies that T^* is strongly subhomogeneous, so Proposition 14 follows.

Proof of Proposition 9. We begin with a claim.

Claim 1. *There exists $\alpha < \varphi(x,\alpha) < 1$ for $\alpha \in (0,1)$ and $x > 0$ such that (1) holds and $\varphi(x,\cdot)$ is continuous and monotone on $[0,1]$. Specifically, we can take*

$$\varphi(x,\alpha) = e^{-d(T(x),T(\alpha x))} = \max\{\beta > 0 : T(\alpha x) \ge \beta T(x)\}. \tag{28}$$

Proof of the Claim. The relation $\varphi(x,\alpha) = e^{-d(T(x),T(\alpha x))}$ has been already introduced in the proof of Proposition 8. The last equality is easily obtained. Also, the monotonicity property of $\varphi(x,\cdot)$ is easy. Let us show the continuity of $\varphi(x,\cdot)$. Let (α_n) be an increasing sequence in $(0,1)$ such that $\alpha_n \uparrow \alpha^*$. Then

$$T(\alpha_n x) = T\left(\frac{\alpha_n}{\alpha^*}\alpha^* x\right) \geq \frac{\alpha_n}{\alpha^*} T(\alpha^* x) \geq \frac{\alpha_n}{\alpha^*}\varphi(x,\alpha^*)\, T(x).$$

In view of (28), this implies that

$$\varphi(x,\alpha_n) \geq \frac{\alpha_n}{\alpha^*}\varphi(x,\alpha^*).$$

By taking limit, this leads to $\lim_n \varphi(x,\alpha_n) \geq \varphi(x,\alpha^*)$. Because $\varphi(x,\cdot)$ is increasing, we have also $\lim_n \varphi(x,\alpha_n) \leq \varphi(x,\alpha^*)$. The right limits are proved in the same way. □

We can now prove our proposition. The first statement is easy. Let now $x_0 \in Q(\zeta)$. This implies that there is some $t_0 \in (0,1)$ for which

$$t_0\zeta \leq x_0 \leq \frac{1}{t_0}\zeta. \tag{29}$$

Because T is strongly subhomogeneous, then there exists a continuous function φ for which $T(t\zeta) \geq \varphi(t)\, T(\zeta)$ holds for each $t \in [0,1]$ and $t < \varphi(t) < 1$ if $t \in (0,1)$. By (29), we get

$$\varphi(t_0)\,\zeta \leq T(x_0) \leq \frac{1}{\varphi(t_0)}\zeta,$$

and by iterating this procedure, we get the relation

$$t_n\zeta \leq T^n(x_0) \leq \frac{1}{t_n}\zeta,$$

where $t_n = \varphi(t_{n-1})$. Thanks to the continuity of φ, the increasing trajectory (t_n) must approach a fixed point of φ. Hence, $t_n \uparrow 1$. By definition of Thompson metric (9), it follows that

$$d(T^n(x_0),\zeta) \leq -\log t_n \to 0.$$

By contrast,

$$-(1-t_n)\,\zeta \leq T^n(x_0) - \zeta \leq \left(\frac{1}{t_n} - 1\right)\zeta,$$

where $(1-t_n) \downarrow 0$ and $1/t_n - 1 \downarrow 0$. Therefore, $T^n(x_0)$ is order convergent to ζ. □

Endnotes

[1] The minimum can be actually regarded as the supremum of the empty chain.

[2] That is, if $x \geq 0$ and $nx \leq y$ for every $n \in \mathbb{N}$, then $x = 0$. In fact, the countable chain $\cdots \geq nx \geq \cdots \geq x$ has a supremum $\sup nx$ in the order interval $[0,y]$. But $2^{-1}\sup(2n)x = \sup nx = \sup(2n)x$, so $0 \leq x \leq \sup nx = 0$.

[3] This result, as stated, can be found in Markowsky [30]. The existence of a least fixed point is due to Abian and Brown [1].

[4] Different authors use different terminologies. Subhomogeneous operators are called concave by Krasnoselskii [22] and sublinear by Amann [3]. For a class of subhomogeneous operators, Coleman [12] coined the term *pseudoconcave operators*.

[5] Under mild assumptions, φ can be chosen to be monotone and continuous in α (see the proof of Proposition 9).

[6] They are also called p-concave operators by some authors. Strongly subhomogeneous operators are also called φ-concave (see Liang et al. [25]).

[7] This notation is consistent with the familiar relation $x \ll y$ between vectors x and y of $\mathbb{R}^n$.

[8] In (3) we adopt the convention $0/0 = \infty$.

[9] The functions $\varphi(x_1, x_2, \ldots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $\alpha_i \geq 1$ are ultramodular (see Marinacci and Montrucchio [28]), so order convex.

[10] Though we consider the bounded case, most of the properties that we establish continue to hold in more general dynamic programming formulations. For instance (see Puterman and Brumelle [36]), we can consider self-maps $T : V \to V$ defined by $T(v) = \max_{L\in\mathscr{L}} u_L + Lv$, where $\mathscr{L}$ is a set of linear operators mapping V onto V such that the inverse operator of $I - L$ exists and is positive.

[11] That is, $W(c,\alpha\zeta) \geq \alpha W(c,\zeta) + (1-\alpha)\, W(c,0)$ for all $c,\zeta \geq 0$ and all $0 \leq \alpha \leq 1$.

[12] Here $\bar{\varepsilon} = (\varepsilon,\ldots,\varepsilon,\ldots)$ and $\bar{L} = (L,\ldots,L,\ldots)$, where $L > \varepsilon > 0$.

[13] See Marinacci and Montrucchio [29, lemma 1] for this and other related properties.

[14] This issue was not treated by Marinacci and Montrucchio [29].

[15] It actually suffices to postulate the existence of a sustainable state x^*, with strictly positive utility $u(x^*,x^*) > 0$, that may be reached from any state x for which $D(x) \neq \{x\}$.

[16] That is, $k(x,y,\alpha t) \geq \alpha k(x,y,t) + (1-\alpha)\, k(x,y,0)$ for all $t \geq 0$ and all $\alpha \in [0,1]$. In the appendix, after Corollary 3, we discuss an alternative condition.

[17] That is, T carries $C_b^+(X)$ into $C_b^+(X)$. Though the Feller property could be regarded as a primitive hypothesis, it may be also derived from other, more direct hypotheses (see, e.g., Bloise and Vailakis [9, assumption 5] and Marinacci and Montrucchio [29, proposition 3]).

[18] In $\mathbb{R}^n$ this complementary problem reduces to the familiar problem of finding a vector $x^* \geq 0$ for which $F(x^*) \geq 0$ and $x^* \cdot F(x^*) = 0$ hold. This finite-dimensional problem has two distinct extensions in infinite-dimensional settings: the topological complementary problem and the order complementary problem. For the latter, we refer readers to Fujimoto [16] and Borwein and Dempster [11].

[19] That is, $\lambda > 0$ is such that $F(x_2) - F(x_1) \leq \lambda(x_2 - x_1)$ for all $0 \leq x_1 \leq x_2$. See Nishimura and Ok [33], who study the close relation with the notion of Z-map in Riddell [38] and of λI map in Borwein and Dempster [11].

[20] The classical strict monotonicity condition $\langle x - y, F(x) - F(y) \rangle > 0$ for all $x, y \in C$ also guarantees a unique solution, but it is not related to order arguments.

[21] These authors are not mentioned in Amann [3]. Tarski's theorem is discussed in a later paper (Amann [4]), whose results are, however, not related to ours (it proves, inter alia, fixed point results a la Abian and Brown [1]).

[22] See, for instance, Stokey and Lucas [39, section 17.2] and Le Van et al. [23].

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