Chapter 1

Preordered Sets and Posets

This chapter introduces the most basic constructs of order theory. In the decreasing order of generality, these are the notions of preorders, partial orders, and linear orders. In particular, we discuss some basic features of these notions of "order," and look at several examples of them. We also consider how algebraic analysis may interact with order theory at a general level by having a brief look at preordered groups, rings, and linear spaces. Combining these with the variety of the examples considered throughout this chapter should illustrate how widely applicable order theory is within mathematics. To put some of the applications we shall later consider in context, we also briefly discuss here why order theory is essential for individual and social decision theory, and hence for economics and related disciplines at large. However, the main objective of this chapter is to introduce the various primitives of order theory that shall be used in a variety of contexts later on. In particular, we discuss at length the extremal and extremum elements of a set relative to a preorder, height and width of a poset, grade functions, suprema, infima, chains and antichains. Even though we will investigate them in a more detailed manner later, we also introduce here some of the primary notions of completeness for posets, such as chain-completeness and conditional (Dedekind) completeness. In turn, the former property is utilized in establishing a major theorem of order theory, namely, the Bourbaki-Witt Fixed Point Theorem.

1 Binary Relations

Binary Relations. Let X be a nonempty set. A subset R of $X \times X$ is called a **binary relation on** X. If $(x, y) \in R$, then we think of R as associating the object x with y, and if $\{(x, y), (y, x)\} \cap R = \varnothing$, we understand that there is no connection between x and y as instigated by R. In what follows, we adopt the convention of writing $x\ R\ y$ instead of $(x, y) \in R$. Moreover, we simply write $x\ R\ y\ R\ z$ to mean $x\ R\ y$ and $y\ R\ z$, and so on.

Reflexivity, Completeness and Symmetry. Let R be a binary relation on a nonempty set X. We say that R is **reflexive** if $x\ R\ x$ for every $x \in X$, and that it is **complete** if either $x\ R\ y$ or $y\ R\ x$ holds for every $x, y \in X$. (Obviously, every complete binary relation is reflexive.) We say that R is **symmetric** if $x\ R\ y$ implies $y\ R\ x$ for every $x, y \in X$, and that it is **antisymmetric** if $x\ R\ y\ R\ x$ cannot hold for any distinct x and y in X.

To wit, where X stands for the collection of all people alive at present, "being as tall as" is a complete binary relation on X which is neither symmetric nor antisymmetric. On the other hand, "being a sibling of" is a symmetric binary relation on X which is neither complete nor antisymmetric. Finally, "having heard of" is a binary relation on X which is not complete, not symmetric and not antisymmetric.

(A)Symmetric Part of a Binary Relation. Any binary relation can be decomposed into two binary relations, one symmetric and one asymmetric. (Some authors refer to the symmetric and asymmetric parts of a binary relation as the *weak* and *strong* parts of that relation, respectively.)

Definition. Let R be a binary relation on a nonempty set X. The **asymmetric part** of R is defined as the binary relation P_R on X with $x \ P_R \ y$ iff $x \ R \ y$ but not $y \ R \ x$. The binary relation $I_R := R \backslash P_R$ on X is then called the **symmetric part** of R.

For instance, in the context of the examples above, the symmetric parts of "being as tall as," "being a sibling of," and "having heard of" are the binary relations "being of the same height," "being a sibling of," and "having heard of each other" on the collection of all people alive at present, respectively. The asymmetric parts of these relations are similarly deduced.

We note that, for any binary relation R on a nonempty set X, the binary relations P_R and I_R are disjoint, and we have $R = P_R \cup I_R$. Here I_R is a symmetric binary relation on X, which is reflexive if so is R. By contrast, P_R is neither reflexive nor symmetric.

Transitivity. A binary relation R on a nonempty set X is said to be **transitive** if

$$x \ R \ y \ R \ z \text{ implies } x \ R \ z \quad \text{for every } x, y, z \in X.$$

For instance, in the context of the examples above, the binary relations "being as tall as," and "being a sibling of," are transitive, and "having heard of" is not transitive, on the collection of all people alive at present.

Transitivity is, in effect, the starting point of order theory. In the abstract, one could even think of "order theory" as the systematic analysis of transitive binary relations. This perspective is put forth, for instance, by Ivan Rival in the first article of the journal *Order*: Order theory "... is inspired by the binary relation 'is contained in,' 'is part of,' 'is less than.' And in just one word, *order* is about *transitivity*." The upshot of this quotation will become abundantly clear as we go further into this chapter.

Note. For any elements x, y and z in X, and any transitive binary relation R on X,

$$x \ P_R \ y \ R \ z \text{ (or } x \ R \ y \ P_R \ z) \quad \text{implies} \quad x \ P_R \ z.$$

Note. If a binary relation R is transitive, so are P_R and I_R.

Set-Theoretic Formulations. There are succinct, set-theoretic ways of formulating the properties we have introduced above for binary relations. First, for any binary relation R on X, we define the **inverse** of R as the binary relation R^{-1} on X with $x \ R^{-1} \ y$ iff $y \ R \ x$. Furthermore, we denote the **diagonal** of $X \times X$ by $\triangle_X$, that is,

$$\triangle_X := \{(x, x) : x \in X\}.$$

Then, reflexivity of R means $\triangle_X \subseteq R$, completeness of R means $R \cup R^{-1} = X \times X$, symmetry of R means $R \subseteq R^{-1}$, and antisymmetry of R means $R \cap R^{-1} \subseteq \triangle_X$. In turn, we have $P_R = R \backslash R^{-1}$ and $I_R = R \cap R^{-1}$.

To formulate the transitivity property in a set-theoretic manner, we need another definition. For any two binary relations R and R' on X, we define the **composition** of R and R' as the binary relation $R \circ R'$ on X with $x\ R \circ R'\ y$ iff $x\ R'\ z\ R\ y$ for some $z \in X$. Then, transitivity of R means that $R \circ R \subseteq R$.

Transitive Closure. For any binary relation R on a nonempty set X, and we let $R^1 := R$ and define the m**th iterate** of R as $R^m := R \circ R^{m-1}$ for any integer $m > 1$. In turn, we define

$$\mathrm{tran}(R) := \bigcup_{m=1}^{\infty} R^m,$$

which is called the **transitive closure** of R. It is not difficult to show that this relation is transitive, and it is contained in every transitive relation on X. That is, $\mathrm{tran}(R)$ is the smallest transitive relation on X that contains R.

Exercises

1.1. Let P be a binary relation on a nonempty set X such that, for any $x, y, z \in X$,

(a) [*Asymmetry*] $x\ P\ y$ implies that $y\ P\ x$ is false; and

(b) [*Negative Transitivity*] $x\ P\ z$ implies either $x\ P\ y$ or $y\ P\ z$.

Define the binary relation R on X by $x\ R\ y$ iff $y\ P\ x$ is false. Show that R is complete and transitive, and we have $P_R = P$.

1.2. For any binary relation R on a nonempty set X, prove that $\mathrm{tran}(R)$ is the smallest transitive relation that contains R. Also show that, in general, $R^>$ and $\mathrm{tran}(R)^>$ are not nested. But when R is complete, we have $\mathrm{tran}(R)^> \subseteq R^>$.

1.3. Let R be a binary relation on a nonempty set X. We say that R is **quasitransitive** if $R^>$ is transitive, and **acyclic** if there do not exist a positive integer k and $x_1, ..., x_k \in X$ such that $x_1\ R^> \cdots R^>\ x_k\ R^>\ x_1$. Prove that transitivity implies quasitransitivity, and quasitransitivity implies acyclicity. Give examples to show that the converses of these facts are false.

1.4. Let R be a binary relation on a nonempty set X. Prove that R is acyclic iff for every nonempty finite subset S of X, there is an $x \in S$ such that $\omega\ R^>\ x$ does not hold for any $\omega \in S$.

1.5. Let R be an acyclic binary relation on a nonempty countable set X. Prove or disprove: There is a map $f : X \to \mathbb{R}$ such that $f(x) > f(y)$ for every x and y in X with $x\ R^>\ y$.

1.6. Given any two reflexive binary relations R and S on a nonempty set X, we say that R is S**-transitive** if $R \circ S \subseteq R$ and $S \circ R \subseteq R$. By the **transitive core** of R, we mean the largest subset S of R such that R is S-transitive, and denote this subrelation as $\mathsf{T}(R)$. (Some authors refer to $\mathsf{T}(R)$ as the **trace** of R.) Prove that $\mathsf{T}(R)$ exists, it is a preorder, and it satisfies: $x\ \mathsf{T}(R)\ y$ iff for every $z \in X$, $z\ R\ x$ implies $z\ R\ y$, and $y\ R\ z$ implies $x\ R\ z$.

1.7. (Nishimura) Take any $\varepsilon \geq 0$. Let X be a metric space and $u : X \to \mathbb{R}$ be a continuous function such that $\sup\{|u(x) - u(y)| : x, y \in X\} < 2\varepsilon$. Define the binary relation R on X by $x\ R\ y$ iff $u(x) \geq u(y) - \varepsilon$, and prove that $x\ \mathsf{T}(R)\ y$ iff $u(x) \geq u(y)$.

1.8. (Graham-Knuth-Motzkin) For any binary relation R on a nonempty set X, let us denote the relation $(X \times X)\backslash R$ as R^-, which is called the **complement** of R. Prove that $\mathrm{tran}(\mathrm{tran}(R)^-)^-$ is transitive for any binary relation R on X. (*Corollary.* At most 10 binary relations can be generated from R by taking complements and transitive closures – this is the best bound.)

2 Equivalence Relations

The binary relations that are reflexive, symmetric and transitive are encountered in mathematical analysis routinely. They are given a special name.

Definition. A binary relation $\sim$ on a nonempty set X is called an **equivalence relation** if it is reflexive, symmetric and transitive. For any $x \in X$, the **equivalence class** of x relative to $\sim$ is defined as the set

$$[x]_\sim := \{\omega \in X : x \sim \omega\}.$$

The collection of all equivalence classes relative to $\sim$ – denoted as $X/_\sim$ – is called the **quotient set** of X relative to $\sim$, that is, $X/_\sim := \{[x]_\sim : x \in X\}$.

An equivalence relation can be used to decompose a grand set of interest into subsets such that the members of the same subset are thought of as "identical" while the members of distinct subsets are viewed as "distinct." Indeed, recall that a **partition** of a nonempty set X is a collection $\mathcal{A}$ of nonempty subsets of X such that $\bigcup \mathcal{A} = X$ and $A \cap B = \varnothing$ for every distinct A and B in $\mathcal{A}$. Unsurprisingly, the collection of equivalence classes induced by any equivalence relation on a set is a partition of that set.

Proposition 2.1. For any equivalence relation $\sim$ on a nonempty set X, the quotient set $X/_\sim$ is a partition of X.

Example 2.1. For any nonempty set X, the binary relation $\triangle_X$ on X, which we call **diagonal relation**, is the smallest equivalence relation that can be defined on X (in the sense that $\triangle_X$ is contained in every equivalence relation on X). Clearly, $[x]_{\triangle_X} = \{x\}$ for every $x \in X$, which shows that $\triangle_X$ is none other than the "equality" relation on X. We have $X/_{\triangle_X} = \{\{x\} : x \in X\}$. At the other extreme is $X \times X$, which is the largest equivalence relation that can be defined on X. We have $[x]_{X\times X} = X$ for every x in X, and hence this relation induces the coarsest possible partition of X, that is, $X/_{X\times X} = \{X\}$.

Example 2.2. The symmetric part of any reflexive and transitive binary relation on a nonempty set is an equivalence relation on that set.

Exercises

2.1. Give an example of a symmetric and transitive binary relation that is not an equivalence relation.

2.2. Prove Proposition 2.1.

2.3 (*Converse of Proposition 2.1*) Let $\mathcal{A}$ be a partition of a nonempty set X, and define the binary relation $\sim$ on X by $x \sim y$ iff $\{x, y\} \subseteq A$ for some $A \in \mathcal{A}$. Show that $\sim$ is an equivalence relation on X.

2.4. (*Factorization of a Function*) Let X and Y be two nonempty sets and $f : X \to Y$ a function. Define the equivalence relation $\sim$ on X by $x \sim y$ iff $f(x) = f(y)$. Prove that there is a surjection $g : X \to X/_\sim$ and an injection $h : X/_\sim \to Y$ such that $f = h \circ g$. (Thus: Every function is a composition of a surjection and an injection.)

2.5. Two binary relations R and R' on a nonempty set X isa said to **commute** iff $R \circ R' = R' \circ R$. Prove: Where R and R' are equivalence relations, R and R' commute iff $R \circ R'$ is an equivalence relation.

3 Order Relations

3.1 Fundamentals of Order Theory

Preorders, Partial Orders and Linear Orders. A binary relation $\succsim$ on a nonempty set X is said to be a **preorder** on X if it is transitive and reflexive. It is said to be a **partial order** on X if it is an antisymmetric preorder on X. Finally, a partial order on X is called a **linear order** on X provided that it is complete.

Notation. For any preorder $\succsim$, we denote by $\succ$ the asymmetric part of $\succsim$, and by $\sim$ the symmetric part of $\succsim$. (In the notation of Section 1, therefore, $\succ := P_{\succsim}$ and $\sim := I_{\succsim}$.) The inverse of $\succsim$ is denoted as $\precsim$.

A preorder on a nonempty set X may have a large symmetric part (which is necessarily an equivalence relation on X). By contrast, the symmetric part of a partial order on X is the smallest reflexive relation on X, that is, it equals $\triangle_X$. This is the main difference between a preorder and a partial order.

Preordered Sets. The primary objects of study in order theory are the preordered sets. Put precisely, by a **preordered set**, we mean an ordered pair $(X, \succsim)$, where X is a nonempty set and $\succsim$ is a preorder on X. When the ambient preorder is not relevant, however, we may treat a preordered set simply as a set. That is, when we say something like "x is in a preordered set $(X, \succsim)$," what we mean is simply that "x is an element of X." This is abuse of terminology, but one that makes it easier to talk about preordered sets.

We say that $(X, \succsim)$ is a **finite** (**countable**) **preordered set**, if $(X, \succsim)$ is a preordered set and X is a finite (countable) set. Besides, two elements x and y of X are said to be **$\succsim$-comparable** (or that "x is $\succsim$-comparable to y") if either $x \succsim y$ or $y \succsim x$. By contrast, we say that x and y are **$\succsim$-incomparable** if neither $x \succsim y$ nor $y \succsim x$. The latter situation is denoted as $x \bowtie y$ (but note that some authors prefer the notation $x \,||\, y$ for this). Of course, every element x of X is $\succsim$-comparable to itself (by virtue of the reflexivity of $\succsim$). Also obvious is that if $\succsim$ is a complete preorder on X, then $\{(x, y) : x \bowtie y\} = \varnothing$.

Order-Boundedness. Let $(X, \succsim)$ be a preordered set. A subset S of X is said to be **$\succsim$-bounded from above** if there is an $x^* \in X$ such that $x^* \succsim x$ for every $x \in S$. It is said to be **$\succsim$-bounded from below** if there is an $x_* \in X$ such that $x \succsim x_*$ for every $x \in S$, and **$\succsim$-bounded** if it is $\succsim$-bounded from both above and below. If X is itself $\succsim$-bounded, we say that $(X, \succsim)$ is a **bounded preordered set**. (Preordered sets that are bounded from above, or below, are defined in the obvious way.)

> **Note.** The "dual" of a preordered set $(X, \succsim)$ is $(X, \precsim)$, which is the preordered set obtained by reversing the rankings declared by $\succsim$. This duality is exploited routinely in order theory.

Principal Ideals and Filters. Given a preordered set $(X, \succsim)$ and an element x of X, we denote by $x^{\downarrow}$ (resp., $x^{\uparrow}$) the set of all elements of X that are ranked lower (resp. higher) than x by $\succsim$. That is,

$$x^{\downarrow} := \{\omega \in X : x \succsim \omega\} \quad \text{and} \quad x^{\uparrow} := \{\omega \in X : \omega \succsim x\}.$$

We denote by ${}^{\downarrow}x$ (resp., ${}^{\uparrow}x$) the set of all elements of X that are ranked *strictly* lower (resp. higher) than x by $\succsim$, that is,

$${}^{\downarrow}x := \{\omega \in X : x \succ \omega\} \quad \text{and} \quad {}^{\uparrow}x := \{\omega \in X : \omega \succ x\}.$$

For any preordered set $(X, \succsim)$ and $x \in X$, a set of the form $x^{\downarrow}$ is often referred to as a **principal ideal**, and a set of the form $x^{\uparrow}$ as a **principal filter**, in $(X, \succsim)$. (Clearly, we have $x^{\downarrow} \cap x^{\uparrow} = [x]_{\sim}$.) On the other hand, a set of the form ${}^{\downarrow}x$ is sometimes referred to as an **initial segment**, and a set of the form ${}^{\uparrow}x$ as a **final segment**, in $(X, \succsim)$. (Obviously, we have ${}^{\downarrow}x \cup [x]_{\sim} = x^{\downarrow}$.)

> **Note.** Any principal ideal (initial segment) in a preordered set $(X, \succsim)$ is a principal filter (final segment) in $(X, \precsim)$, and conversely.

Posets and Losets. A preordered set $(X, \succcurlyeq)$ is called a **poset** (short for *partially ordered set*) if $\succcurlyeq$ is a partial order on X. Similarly, $(X, \succcurlyeq)$ is called a **loset** (short for *linearly ordered set*) if $\succcurlyeq$ is a linear order on X.

In the context of a poset $(X, \succcurlyeq)$, the principal ideal and filter associated with any given point overlap exactly at that point, that is, $x^{\downarrow} \cap x^{\uparrow} = \{x\}$. Similarly, $x^{\downarrow}$ and ${}^{\downarrow}x$ differ from each other exactly by $\{x\}$.

> **Note.** It appears that Felix Hausdorff was the first mathematician who has defined the notion of "poset" as this term is understood today. This he did in his 1914 opus *Grundzüge der Mengenlehre* under the title "teilweise geordnete menge."

Remark. Every poset is obviously a preordered set. Conversely, every preordered set $(X, \succsim)$ induces a poset by identifying "equivalent" elements. More precisely, we define the **poset induced by** $(X, \succsim)$ as $(X/_{\sim}, \succcurlyeq)$ where $[x]_{\sim} \succcurlyeq [y]_{\sim}$ iff $x \succsim y$, for every $x, y \in X$. (As the symmetric part of $\succsim$ is an equivalence relation, this poset is well-defined.) As we shall see, many concepts that are defined for posets can be extended to the more general context of preordered sets by applying them to the posets induced by preordered sets.

Chains and Antichains. Those subsets of a poset whose any two elements are comparable are of great importance for understanding the structure of that poset. Such sets, and their complements, are given special names.

Definition. Let $(X, \succsim)$ be a preordered set. A **$\succsim$-chain** in X is a nonempty subset S of X such that every x and y in S are $\succsim$-comparable. Dually, a subset S of X is an **$\succsim$-antichain** in X if every distinct x and y in S are $\succsim$-incomparable. In what follows, we denote the set of all $\succsim$-chains in X by $\mathbb{C}(X, \succsim)$, and the set of all $\succsim$-antichains in X by $\mathbb{A}(X, \succsim)$.

> **Note.** According to our definition, any singleton subset of a preordered set $(X, \succsim)$ is both a $\succsim$-chain and an $\succsim$-antichain. On the other hand, $\varnothing \in \mathbb{A}(X, \succsim) \backslash \mathbb{C}(X, \succsim)$.

We shall make use of the notions of "chain" and "antichain" extensively in the subsequent material. For now, we introduce another essential concept of the theory of ordered sets.

Extensions of Preorders. Let $(X, \succsim)$ be a preordered set. An **extension** of $\succsim$ is a preorder $\trianglerighteq$ on X such that $\succsim \subseteq \trianglerighteq$ and $\succ \subseteq \triangleright$, where $\succ$ and $\triangleright$ are the asymmetric parts of $\succsim$ and $\trianglerighteq$, respectively.

Intuitively speaking, an extension of the preorder $\succsim$ on a nonempty set X is "more complete" than $\succsim$ in the sense that it compares more elements, but it certainly agrees with $\succsim$ whenever the latter applies. If $\trianglerighteq$ is a partial order, then it is an extension of $\succsim$ iff $\succsim \subseteq \trianglerighteq$. (Why?)

Examples. We now go through some examples of posets and illustrate the order-theoretic concepts introduced above. Many other examples will be encountered in the sequel.

Example 3.1.1. Let X be a nonempty set. Then, $(X, \triangle_X)$ is a poset, and for any x in X, we have $x^{\downarrow} = \{x\} = x^{\uparrow}$ and ${}^{\downarrow}x = \varnothing = {}^{\uparrow}x$ in the context of this poset. In fact, the diagonal relation $\triangle_X$ is the only partial order on X which is also an equivalence relation. Moreover, $(X, \triangle_X)$ is bounded iff $|X| = 1$, X is an $\triangle_X$-antichain, and every preorder on X is an extension of $\triangle_X$.

The binary relation $X \times X$ is, on the other hand, a complete preorder. This preorder is not a partial order unless $|X| = 1$. (For any x in X, we have $x^{\downarrow} = X = x^{\uparrow}$ and ${}^{\downarrow}x = \varnothing = {}^{\uparrow}x$ in the context of the poset $(X, X \times X)$.) Furthermore, $(X, X \times X)$ is bounded, and the only extension of $X \times X$ is itself.

Example 3.1.2. $(\mathbb{R}, \geq)$ is a loset, where $\geq$ is the usual linear order on $\mathbb{R}$. In the context of this loset, we have $x^{\downarrow} = (-\infty, x]$, ${}^{\downarrow}x = (-\infty, x)$, $x^{\uparrow} := [x, \infty)$ and ${}^{\uparrow}x = (x, \infty)$ for any real number x. Obviously, $(\mathbb{R}, \geq)$ is not bounded, and every nonempty subset of $\mathbb{R}$ is a $\geq$-chain. By intersecting the linear order $\geq$ with $S \times S$ for various subsets S of $\mathbb{R}$, we obtain other interesting losets of the form $(S, \geq \cap (S \times S))$. Among the important choices for S here are $\mathbb{N}$ and $\mathbb{Q}$, as well as the finite set $[n]$, where

$$[n] := \{1, ..., n\}, \quad n = 1, 2, ...,$$

a notation that we will use throughout the exposition.

Convention. For any preordered set $(X, \succsim)$ and a nonempty subset S of X, the preordered set $(S, \succsim \cap (S \times S))$ will be denoted simply as $(S, \succsim)$ thoughout this text. For instance, this is how $(\mathbb{N}, \geq)$ and $([n], \geq)$ should be interpreted.

Example 3.1.3. $(\mathbb{R}^n, \geq)$ is a poset for any positive integer n, where $\geq$ is defined coordinatewise, that is, $\mathbf{x} \geq \mathbf{y}$ iff $x_i \geq y_i$ for each $i \in [n]$. When we talk of $\mathbb{R}^n$, or any subset of it, without specifying an alternative preorder, we have this partial order in mind, which is complete iff $n = 1$. (Throughout this text, we use the same notation for the (partial) order of $\mathbb{R}^n$ and (linear) order of $\mathbb{R}$.) In the context of the poset $(\mathbb{R}^n, \geq)$, we have

$$\mathbf{x}^{\downarrow} = (-\infty, x_1] \times \cdots \times (-\infty, x_n]$$

and

$${}^{\downarrow}\mathbf{x} = \{\mathbf{y} \in \mathbf{x}^{\downarrow} : y_i < x_i \text{ for some } i \in [n]\},$$

and similarly for $\mathbf{x}^{\uparrow}$ and ${}^{\uparrow}\mathbf{x}$. The set $\{\mathbf{x} \in \mathbb{R}^n : x_1 = \cdots = x_n\}$ is an example of a $\geq$-chain in $\mathbb{R}^n$, and $\{\mathbf{x} \in \mathbb{R}^n : -x_1 = x_2 \text{ and } x_3 = \cdots = x_n = 0\}$ is an example of an $\geq$-antichain, provided that $n \geq 2$. For each $k \in [n]$, the binary relation $\trianglerighteq^k$ on $\mathbb{R}^n$ defined by $\mathbf{x} \trianglerighteq^k \mathbf{y}$ iff $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$, is also a preorder on $\mathbb{R}^n$; it is an extension of $\geq$ iff $k = n$.

Example 3.1.4. Let $n \geq 2$ be an integer. The **lexicographic order** on $\mathbb{R}^n$ is the binary relation $\geq_{\text{lex}}$ on $\mathbb{R}^n$ defined as $\mathbf{x} \geq_{\text{lex}} \mathbf{y}$ iff $x_1 > y_1$, or $x_1 = y_1$ and $x_2 > y_2$, or ..., or $x_i = y_i$ for each $i \in [n-1]$ and $x_n \geq y_n$. It is easily checked that $(\mathbb{R}^n, \geq_{\text{lex}})$ is a loset and that $\geq_{\text{lex}}$ is an extension of $\geq$.

Example 3.1.5. Let X be a nonempty set. Then, $(2^X, \supseteq)$ is a bounded poset, and for any subset S of X, we have $S^{\downarrow} = 2^S$ and ${}^{\downarrow}S = 2^S \backslash \{S\}$ in the context of this poset. The complete preorder $\trianglerighteq$ on 2^X, defined by $A \trianglerighteq B$ iff $|A| \geq |B|$, is an extension of $\supseteq$, provided that X is a finite set.

Example 3.1.6. Let X and Y be nonempty sets. We define

$$[Y^X] := \{f|_S : S \subseteq X \text{ and } f \in Y^X\}.$$

(Alternatively, $[Y^X] = \bigcup\{Y^S : S \subseteq X\}$.) Any one member of $[Y^X]$ is called a Y-valued **partial function** on X. Consider the binary relation $\sqsupseteq$ on $[Y^X]$ defined by $f|_S \sqsupseteq g|_T$ iff $S \supseteq T$ and $f|_T = g|_T$. It is readily checked that $([Y^X], \sqsupseteq)$ is a poset; such a poset is commonly referred to as a **partial function poset**.

The poset $([Y^X], \sqsupseteq)$ is bounded from below because $\varnothing \in [Y^X]$ and $h \sqsupseteq \varnothing$ for every $h \in [Y^X]$. It is not bounded from above, however, unless Y is a singleton. Besides, for any $h \in [Y^X]$, the principal ideal $h^{\downarrow}$ consists of all restrictions of h to the subsets of the domain of h, and the principal filter $h^{\uparrow}$ consists of all Y-valued extensions of h to the subsets of X that contain the domain of h. The partial order $\sqsupseteq'$ on $[Y^X]$, defined by $f|_S \sqsupseteq' g|_T$ iff $S \supseteq T$, is an extension of $\sqsupseteq$.

Example 3.1.7. Let $\mathbf{C}[0,1]$ stand for the linear space of continuous real maps on the interval $[0,1]$, and define the binary relation $\geq$ on $\mathbf{C}[0,1]$ as $f \geq g$ iff $f(t) \geq g(t)$ for every t in $[0,1]$. Then, $(\mathbf{C}[0,1], \geq)$ is a poset which is not bounded. (The unit ball of $\mathbf{C}[0,1]$ is an example of a $\geq$-bounded set in $\mathbf{C}[0,1]$.) For any f in $\mathbf{C}[0,1]$, the set $f^{\uparrow}$ consists of all continuous real functions on $[0,1]$ whose graphs are everywhere above that of f, and of course, ${}^{\uparrow}f = f^{\uparrow} \backslash \{f\}$. If, for any positive integer m, f_m is the function in $\mathbf{C}[0,1]$ that maps t to t^m, then $\{f_1, f_2, ...\}$ is a $\geq$-chain. The linear order $\trianglerighteq$ on $\mathbf{C}[0,1]$, defined by

$$f \trianglerighteq g \quad \text{iff} \quad \int_0^1 f(t)dt \geq \int_0^1 g(t)dt,$$

is an extension of $\geq$.

Products of Preordered Sets. There are many methods of obtaining a new preordered set by combining a given (finite) collection of preordered sets. The following is the most important of these methods.

Definition. The **product** of two preordered sets $(X, \succsim_X)$ and $(Y, \succsim_Y)$ is defined as the preordered set $(X \times Y, \succsim)$, where $(x, y) \succsim (z, w)$ iff $x \succsim_X z$ and $y \succsim_Y w$.

Obviously, this definition can be extended to the case of products of an arbitrary (finite) number of preordered sets by mathematical induction.

Example 3.1.8. For any positive integer $n \geq 2$, the poset $(\mathbb{R}^n, \geq)$ is the product of n copies of the loset $(\mathbb{R}, \geq)$. Equivalently, it is the product of $(\mathbb{R}^k, \geq)$ and $(\mathbb{R}^{n-k}, \geq)$ for any $k \in [n-1]$.

Exercises

3.1.1. Let $\mathbb{R}^\infty$ denote the collection of all real sequences, and define the binary relation $\succsim$ on $\mathbb{R}^\infty$ as $(x_m) \succsim (y_m)$ iff (y_m) is a subsequence of (x_m). Show that $(\mathbb{R}^\infty, \succsim)$ is a preordered set but not a poset.

3.1.2. (*Loewner Ordering*) Let Sym_n denote the linear space of all symmetric $n \times n$ matrices. Recall that a matrix $\mathbf{Q}$ in Sym_n is **positive semidefinite** if $\langle \mathbf{Qx}, \mathbf{x} \rangle \geq 0$ for every n-vector $\mathbf{x}$, where $\langle \cdot, \cdot \rangle$ stands for the usual inner product map on $\mathbb{R}^n \times \mathbb{R}^n$. Define the binary relation $\succcurlyeq$ on Sym_n by $\mathbf{A} \succcurlyeq \mathbf{B}$ iff $\mathbf{A} - \mathbf{B}$ is positive semidefinite. Show that $(\mathrm{Sym}_n, \succcurlyeq)$ is a poset.

3.1.3. (*Specialization Order*) Let X be a topological space, and define the binary relation $\succsim$ on X by $x \succsim y$ iff every open neighborhood of y is also an open neighborhood of x. Prove that $\succsim$ is a preorder on X, but it need not be a partial order. Also show that if X is a T_0-space (Section A.1.1), then $(X, \succsim)$ is a poset such that $x^{\downarrow} = \mathrm{cl}\{x\}$ for each $x \in X$.

3.1.4. Let $(X, \succcurlyeq)$ be a poset, and take any two elements x and y in X with $x \bowtie y$. Prove:

$$\mathrm{tran}(\succcurlyeq \cup \{(x, y)\}) = \ \succcurlyeq \cup \{(a, b) \in X \times X : a \succcurlyeq x \text{ and } y \succcurlyeq b\}.$$

3.1.5. (*Direct Sum of Posets*) Let $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ be two posets with $X \cap Y = \varnothing$. We define the binary relation $\succcurlyeq_{X \oplus Y}$ on $X \cup Y$ by

$$x \succcurlyeq_{X \oplus Y} y \quad \text{iff} \quad (x, y) \in \ \succcurlyeq_X \cup \succcurlyeq_Y \cup (X \times Y).$$

Show that $(X \cup Y, \succcurlyeq_{X \oplus Y})$ is a poset. (This poset is called the **direct sum** of $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$.)

3.1.6. (*Completion of a Preorder*) A complete preorder that extends a preorder $\succsim$ is said to be a **completion** of $\succsim$. Prove that every preorder on a nonempty finite set admits a completion.

3.1.7. (*Covering Relation*) The **covering relation** induced by a poset $(X, \succcurlyeq)$ is the binary relation $\rhd$ on X defined by $x \rhd y$ iff x $\succcurlyeq$-covers y, that is, $x \succ y$ and there is no z in X with $x \succ z \succ y$. Prove that a binary relation R on a nonempty set X is the covering relation induced by some poset $(X, \succcurlyeq)$ iff R is **totally non-transitive**, that is, for every integer $m \geq 3$ and elements $x_1, ..., x_m$ of X,

$$x_1 \ R \ \cdots \ R \ x_m \quad \text{implies} \quad x_1 \neq x_m \text{ and } x_1 \ R \ x_m \text{ is false.}$$

Hint. For the "if" part, consider the partial order $\mathrm{tran}(R \cup \triangle_X)$.

3.1.8. (*The Scott-Suppes Theorem*) Let $\mathcal{I}$ be the collection of all closed intervals of unit length, and define the binary relation $\trianglerighteq$ on $\mathcal{I}$ by $[a, b] \trianglerighteq [c, d]$ iff either $a \geq d$ or $[a, b] = [c, d]$. We say that a binary relation $\succcurlyeq$ on a nonempty set X is a **semiorder** on X if there is an injection $\varphi : X \to \mathcal{I}$ such that $x \succcurlyeq y$ iff $\varphi(x) \trianglerighteq \varphi(y)$.

Let $\succcurlyeq$ be a semiorder on X. Prove first that $(X, \succcurlyeq)$ is a poset. Next, assume that X is finite, and prove that there exists a function $f : X \to \mathbb{R}$ such that $x \succ y$ iff $f(x) > f(y) + 1$ for every x and y in X.

3.1.9. Let X be a nonempty finite set and $\succ$ a binary relation on X. We say that $\succ$ is an **interval order** if

$$x \succ y \text{ and } x' \succ y' \quad \text{imply} \quad \text{either } x \succ y' \text{ or } x' \succ y$$

for every x, y, x' and y' in X. Prove that $\succ$ is an interval order iff there exist two real functions f and g on X such that $x \succ y$ iff $f(x) > g(y)$ for every x and y in X.

3.2 Monotonic Sets

Monotonic Subsets of a Preordered Set. As we shall see later, monotonic subsets of a preordered set are useful aids toward understanding the general structure of that preordered set. We define such sets as follows:

Definition. Let $(X, \succsim)$ be a preordered set. A subset S of X is said to be **$\succsim$-increasing** (in X) if $x^{\uparrow} \subseteq S$ for every $x \in S$. Dually, we say that S is **$\succsim$-decreasing** (in X) if it is $\precsim$-increasing (that is, if $x^{\downarrow} \subseteq S$ for every $x \in S$).

Note. Relative to a preordered set $(X, \succsim)$, a subset S of X is $\succsim$-increasing iff $X \backslash S$ is $\succsim$-decreasing.

Given any preordered set $(X, \succsim)$, we denote the collections of all $\succsim$-increasing and $\succsim$-decreasing subsets of X by $2^X_\uparrow$ and $2^X_\downarrow$, respectively. Throughout this text, we shall consider $2^X_\uparrow$ as poset relative to the partial order $\subseteq$, that is, when we talk about $2^X_\uparrow$ as a poset, what we have in mind is $(2^X_\uparrow, \subseteq)$. Similarly, we order $2^X_\downarrow$ by the inclusion ordering $\supseteq$. That is, when we talk about $2^X_\downarrow$ as a poset, what we have in mind is $(2^X_\uparrow, \supseteq)$. (There is a formal sense in which $2^X_\uparrow$ and $2^X_\downarrow$ are the "same" posets, but we will talk about this matter later.)

Monotonic Closure. Given a preordered set $(X, \succsim)$, a subset S of X is $\succsim$-decreasing iff $x^\downarrow \subseteq S$ for every $x \in S$, and it is $\succsim$-increasing iff $x^\uparrow \subseteq S$ for every $x \in S$. (Thus, the collections $2^X_\uparrow$ and $2^X_\downarrow$ are never empty, for $\varnothing$ and X are, trivially, both $\succsim$-decreasing and $\succsim$-increasing.) This way of looking at things shows clearly that the union and intersection of any collection of $\succsim$-decreasing subsets of X are, again, $\succsim$-decreasing, and of course, the same goes for $\succsim$-increasing subsets of X as well. It also prompts the following:

Definition. Let $(X, \succsim)$ be a preordered set. We define $\varnothing^\downarrow := \varnothing$ and $\varnothing^\uparrow := \varnothing$. Moreover, for any nonempty subset S of X, we let

$$S^\downarrow := \bigcup\{x^\downarrow : x \in S\} \quad \text{and} \quad S^\uparrow := \bigcup\{x^\uparrow : x \in S\}.$$

Here, $S^\downarrow$ is called the **$\succsim$-decreasing closure** of S, and $S^\uparrow$ the **$\succsim$-increasing closure** of S.

> **Note.** The notation we adopt here is consistent with the one we use for principal ideals and filters in a preordered set $(X, \succsim)$: We have $x^\downarrow = \{x\}^\downarrow$ and $x^\uparrow = \{x\}^\uparrow$ for every x in X.

Where $(X, \succsim)$ is a preordered set, the $\succsim$-decreasing closure of a subset S of X is the smallest subset of X that contains S, that is, not only that it is a $\succsim$-decreasing subset of X that contains S, but it is also contained within any other such subset of X. It follows that a subset S of X is $\succsim$-decreasing iff it equals its $\succsim$-decreasing closure, and similarly for $\succsim$-increasing sets.

Convex Subsets of a Preordered Set. Let $(X, \succsim)$ be a preordered set. A subset S of X is said to be **$\succsim$-convex** (in X) if $x^\downarrow \cap y^\uparrow \subseteq S$ for every $x, y \in S$. Clearly, every $\succsim$-increasing, or $\succsim$-decreasing, subset of X is $\succsim$-convex, but the converse is false (as, for instance, every singleton subset of X is trivially $\succsim$-convex). Besides, X is trivially $\succsim$-convex and the intersection of any collection of $\succsim$-convex subsets of X is again $\succsim$-convex. This fact allows us to define the **$\succsim$-convex closure** of any subset S of X, which we denote by $\mathrm{co}_\succsim(S)$, as the intersection of all $\succsim$-convex subsets of X that contains S. Thus, $\mathrm{co}_\succsim(S)$ is the smallest $\succsim$-convex subset of X that contains S in the sense that it is contained in any other $\succsim$-convex subset of X that contains S. It is easily verified that $\mathrm{co}_\succsim(S) = S^\downarrow \cap S^\uparrow$.

Examples.

Example 3.2.1. Let X be a nonempty set. Then, every subset of X is both $\triangle_X$-increasing and $\triangle_X$-decreasing. Similarly, every subset of X is $\triangle_X$-convex. By contrast, the only $X \times X$-increasing (and hence $X \times X$-decreasing) subsets of X are $\varnothing$ and X.

Example 3.2.2. The $\geq$-decreasing subsets of $\mathbb{R}$ are $\varnothing$, $\mathbb{R}$ and any set of the form $x^\downarrow$ or ${}^\downarrow x$ for some real number x. Similarly, a subset of $\mathbb{R}$ is $\geq$-convex iff it is a closed interval. But there are losets whose (nonempty and proper) decreasing subsets cannot be written as principal ideals or initial segments. (Think of $(\mathbb{Q}, \geq)$!)

Example 3.2.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an increasing function. Then, for any real number α, both of the sets $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq \alpha\}$ and $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < \alpha\}$ are $\geq$-decreasing.

Example 3.2.4. Let X be a nonempty set, and $\mathcal{S}$ a collection of subsets of X. Then, in the context of $(2^X, \supseteq)$, we have $\mathcal{S}^{\downarrow} = \bigcup\{2^S : S \in \mathcal{S}\}$. Thus, $\mathcal{S}$ is $\supseteq$-decreasing iff every subset of a member of $\mathcal{S}$ is also a member of $\mathcal{S}$.

Example 3.2.5. Let $(X \times Y, \succsim)$ be the product of two preordered sets $(X, \succsim_X)$ and $(Y, \succsim_Y)$. Then, for any subsets S and T of X and Y, respectively, we have

$$(S \times T)^{\downarrow} = S^{\downarrow} \times T^{\downarrow}.$$

It follows that $S \times T$ is $\succsim$-decreasing iff S is $\succsim_X$-decreasing and T is $\succsim_Y$-decreasing. But, of course, not every $\succsim$-decreasing subset of $X \times Y$ arises this way. For instance, $\{(a, b) : a + b \leq 1\}$ is clearly a $\geq$-decreasing subset of $\mathbb{R}^2$ which cannot be written as the cartesian product of two $\geq$-decreasing subsets of $\mathbb{R}$.

3.3 Induced Weak Set-Ordering

There are various interesting ways in which we can order the nonempty subsets of a given set by using a preorder defined on that set. We now introduce an important example of such a method by using the notion of monotonic closure.

The Weak Set-Ordering. Let $(X, \succsim)$ be a preordered set. The **upper set-ordering** induced by $\succsim$ is the binary relation $\succsim^{\bullet}$ on 2^X such that $A \succsim^{\bullet} B$ iff $A^{\downarrow} \supseteq B^{\downarrow}$, which is an extension of the partial order $\supseteq$ on 2^X. It is readily checked that we can equivalently state the definition of $\succsim^{\bullet}$ as $A \succsim^{\bullet} B$ iff $A \cap y^{\uparrow} \neq \varnothing$ for every $y \in B$, thanks to the transitivity of $\succsim$. Similarly, the **lower set-ordering** induced by $\succsim$ is defined as the binary relation $\succsim_{\bullet}$ on 2^X such that $A \succsim_{\bullet} B$ iff $A^{\uparrow} \subseteq B^{\uparrow}$, or put equivalently, $A \succsim_{\bullet} B$ iff $B \cap x^{\downarrow} \neq \varnothing$ for every $x \in A$. ($\succsim_{\bullet}$ is an extension of $\subseteq$ on 2^X.) In turn, $\succsim^{\mathrm{w}} := \succsim_{\bullet} \cap \succsim^{\bullet}$ is called the **weak set-ordering** induced by $\succsim$. Put explicitly, we have $A \succsim^{\mathrm{w}} B$ iff for every $(x, y) \in A \times B$,

$$x \succsim w \text{ for some } w \in B \quad \text{and} \quad z \succsim y \text{ for some } z \in A.$$

Given any preordered set $(X, \succsim)$, the relations $\succsim_{\bullet}$, $\succsim^{\bullet}$ and $\succsim^{\mathrm{w}}$ are loyal to $\succsim$ in the sense that they agree with $\succsim$ on the ranking of singleton sets, that is, $\{x\} \succsim^{\mathrm{w}} \{y\}$ iff $x \succsim y$, for every $x, y \in X$. Furthermore, any one of these relations is a preorder.

Proposition 3.3.1. If $(X, \succsim)$ is a preordered set, then both $\succsim_{\bullet}$ and $\succsim^{\bullet}$ are preorders on 2^X. Thus: $(2^X, \succsim^{\mathrm{w}})$ is a preordered set.

The proof is a straightforward exercise, but note that the antisymmetry of $\succsim$ would not guarantee that of $\succsim^{\mathrm{w}}$, that is, the weak set-ordering induced by a partial order need not be a partial order. For instance, we have $[0, 1] \geq^{\mathrm{w}} \{0, 1\}$ and $\{0, 1\} \geq^{\mathrm{w}} [0, 1]$ simultaneously.

We will use the upper, lower and weak set-ordering in numerous times in our exposition. For now, we just offer only one order-theoretic example in which the upper set-ordering plays an essential role.

Example 3.3.1. (*Posets of Antichains*) Let $(X, \succsim)$ be a preordered set, and recall that $\mathbb{A}(X, \succsim)$ stands for the collection of all $\succsim$-antichains in X. We use the upper set-ordering induced by $\succsim$, that is, $\succsim^{\bullet}$, to turn $\mathbb{A}(X, \succsim)$ into a poset. By Proposition 3.3.1, $(\mathbb{A}(X, \succsim), \succsim^{\bullet})$ is a preordered set. To see that $\succsim^{\bullet}$ is antisymmetric on $\mathbb{A}(X, \succsim)$, take any two $\succsim$-antichains A and B in X such that $A \succsim^{\bullet} B \succsim^{\bullet} A$. Then, $A^{\downarrow} = B^{\downarrow}$, so, for any x in A, there is a $z \in B$ and $y \in A$ such that $y \succsim z \succsim x$. As A is an $\succsim$-antichain, we have $y = x$, and it thus follows that $x = z \in B$. This shows that $A \subseteq B$, but replacing the roles of A and B in this reasoning would also yield $B \subseteq A$, so we actually have $A = B$. Conclusion: $(\mathbb{A}(X, \succsim), \succsim^{\bullet})$ is a poset. This poset actually possesses quite an interesting structure; we shall return to it at various points in the following chapters.

> **Note.** One may also turn $\mathbb{C}(X, \succsim)$ into a preordered set by means of any one of the induced set-orderings induced by $\succsim$, but this does not yield a poset.

Exercises

3.3.1. Let $(X, \succsim)$ be a preordered set, and S a nonempty subset of X. We say that S is **$\succsim$-directed** if for every $x, y \in S$, there is a $z \in S$ with $z \succsim x$ and $z \succsim y$. Show that S is $\succsim$-directed iff so is $S^{\downarrow}$.

3.3.2. Is $(\mathbb{A}(X, \succsim), \succsim^{\mathrm{w}})$ a poset for any preordered set $(X, \succsim)$?

3.3.3. Let $(X, \succsim)$ be a preordered set.
a. Prove: For any $S \subseteq X$,

$$S^{\downarrow} = S^{\downarrow\downarrow}, \qquad S^{\uparrow} = S^{\uparrow\uparrow}, \qquad S \subseteq S^{\downarrow} \cap S^{\uparrow} \quad \text{and} \quad S^{\downarrow} = (S^{\downarrow} \cap S^{\uparrow})^{\downarrow}.$$

Give an example to show that equality need not hold in the third expression here.
b. In the context of $(\mathbb{R}^n, \geq)$, compute $\mathbf{0}^{\uparrow\downarrow} \cap \mathbf{0}^{\downarrow\uparrow}$, and also, give an example of a nonempty subset S of $\mathbb{R}^n$ such that $S = S^{\uparrow\downarrow}$.
c. Give an example of a poset $(X, \succcurlyeq)$ such that there is an $S \subseteq X$ with $S \subset S^{\downarrow} \subset S^{\downarrow\uparrow} \subset S^{\downarrow\uparrow\downarrow} \subset \cdots$.

3.4 Application: Preorders in Decision Theory

Preordered sets are, of course, ubiquitous in mathematics. They are also encountered quite frequently in various applied disciplines, including some of the social sciences. As we shall see throughout this text, in economics, in particular, order theory plays a foundational role, for preordered sets arise naturally in the context of individual decision making. Indeed, in individual choice theory, a **preference relation** $\succsim$ on a nonempty alternative set X is defined merely as a preorder on X. One thinks of X as the universal space of choice items in this context. In turn, $\succsim$ is assumed to contain all the information that concerns one's comparative preferences about the outcomes in X. If $x \succsim y$ holds, we understand that the individual with preference relation $\succsim$ views the alternative x at least as good as the alternative y. Thus, in the context of $(X, \succsim)$, the set of all alternatives in X that are at least as good as x for the involved individual corresponds to the principal ideal $x^{\uparrow}$, and $x^{\downarrow}$ is similarly interpreted.

Induced from $\succsim$ are the **strict preference relation** $\succ$ on X, which is the asymmetric part of $\succsim$, and the **indifference relation** $\sim$ on X, which is the symmetric part of $\succsim$. When

$x \succ y$ holds, we understand that the agent is strictly better off with the alternative x relative to y. (We may think of this as saying that the agent would be willing to pay a positive price to be able to move from y to x.) When $x \sim y$ is the case, we think of the agent being indifferent between the alternatives x and y. (The agent would then not pay a positive price to switch from x to y, and vice versa.) Finally, if neither $x \succsim y$ nor $y \succsim x$, a situation which is denoted as $x \bowtie y$ in this text, we understand that the agent is *indecisive* about comparing x to y.

Remark. Given a preference relation on a nonempty alternative set X, and any alternative x in X, the equivalence class $[x]_\sim$ is called the **indifference class** of x in the context of choice theory. This is a straightforward generalization of the familiar concept of "*the indifference curve that passes through* x." In particular, Proposition 2.1 says that no two distinct indifference sets can have a point in common. (In intermediate microeconomics, this is often paraphrased as: "Indifference curves do not cross.")

3.5 Hasse Diagrams

There is a simple method of visualizing a finite poset $(X, \succcurlyeq)$. To introduce this method, let us first agree to say that x is a **$\succcurlyeq$-cover** of y (or that "x $\succcurlyeq$-covers y") when x and y are two elements of X such that $x \succ y$ and $x \succ z \succ y$ does not hold for any z in X. In turn, we say that two elements x and y are **$\succcurlyeq$-adjacent** if either x $\succcurlyeq$-covers y, or y $\succcurlyeq$-covers x.

Now, consider representing each element of X by a point on the plane (with different elements being represented by different points) and connecting some of these points by line segments in such a way that

- if $x \succ y$, then the point corresponding to x appears higher than the one corresponding to y; and
- two points are connected by a (single) line segment iff x and y are $\succcurlyeq$-adjacent.

A graph drawn according to these rules is called the **Hasse diagram** of $(X, \succcurlyeq)$. Clearly, this graph determines the original poset $(X, \succcurlyeq)$ completely, because $x \succ y$ holds iff there is a string of connected line segments moving downward from x to y in the diagram. (Another, more formal, way of saying this is that every partial order $\succcurlyeq$ on a finite set X is the transitive closure of the union of $\triangle_X$ and the $\succcurlyeq$-covering relation.)

Of course, the geometric appearance of two graphs drawn according to these rules may be different, but this is of little relevance for order theory. One usually opts for the Hasse diagram that possesses a geometric sense of beauty. Here are a few illustrations.

Example 3.5.1. For $X := \{x, y\}$, the Hasse diagram of the poset $(2^X, \supseteq)$ can be drawn as in the first part of Figure 3.5.1. Similarly, for $X := \{x, y, z\}$, the Hasse diagram of the poset $(2^X, \supseteq)$ can be drawn as in the second part of Figure 3.5.1.

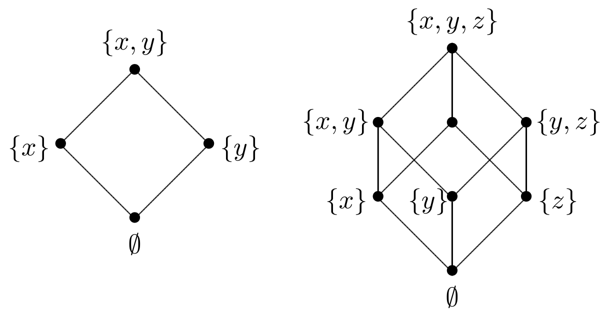

Figure 3.5.1

Example 3.5.2. Let X be the set of all divisors of 60, and consider the partial order $\succcurlyeq$ on X defined by $x \succcurlyeq y$ iff y is a divisor of x. Then, the Hasse diagram of the poset $(X, \succcurlyeq)$ can be drawn as in Figure 3.5.2.

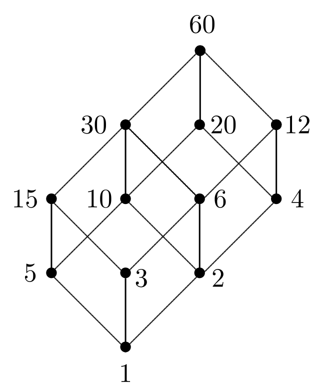

Figure 3.5.2

Example 3.5.3. For any positive integer n, and any n-element set X, which we enumerate as $\{x_1, ..., x_n\}$, let $\succcurlyeq$ be the partial order on X whose asymmetric part is defined as

$$x_{2i} \succ x_{2i-1} \text{ and } x_{2i} \succ x_{2i+1} \text{ for each } i \in [\tfrac{n-1}{2}]$$

if n is odd, and by

$$x_n \succ x_{n-1},\ x_{2i} \succ x_{2i-1} \text{ and } x_{2i} \succ x_{2i+1} \text{ for each } i \in [\tfrac{n-2}{2}]$$

if n is even. Then, the poset $(X, \succcurlyeq)$ is said to be an n-fence. In turn, when n is even, $(X, \succcurlyeq)$ is an n-fence, and $x_n \succ x_1$, we refer to $(X, \succcurlyeq)$ as an n**-crown**. Figure 3.5.3 depicts the Hasse diagrams of a 7-fence, an 8-fence and an 8-crown.

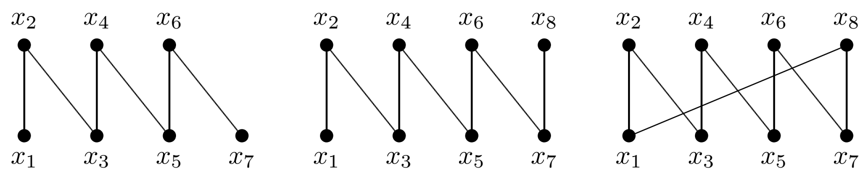

Figure 3.5.3

Example 3.5.4. The first part of Figure 3.5.4 depicts the Hasse diagram of a 6-crown. The second part of this figure illustrates the Hasse diagram of the poset of $\succcurlyeq$-antichains in this 6-crown. (Recall Example 3.3.1.)

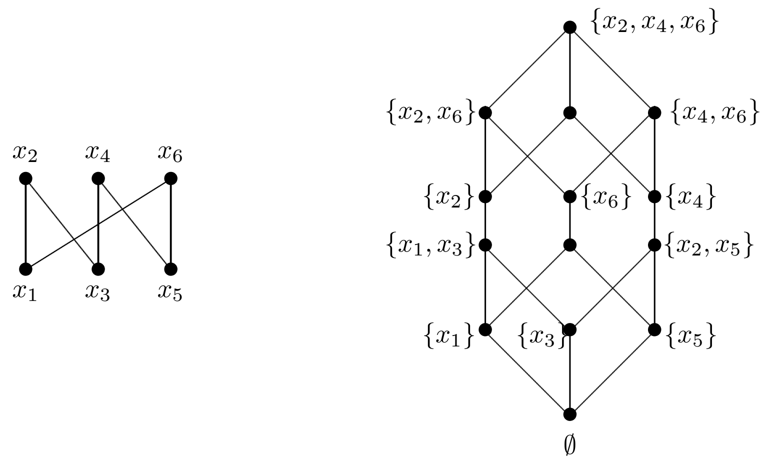

Figure 3.5.4

3.6 Atomic Posets

Let $(X, \succcurlyeq)$ be a poset such that there is a (necessarily unique) $x_* \in X$ with $x \succcurlyeq x_*$ for each $x \in X$. We say that an element a of X is an **$\succcurlyeq$-atom** of X if a $\succcurlyeq$-covers x_*. In turn, such a poset is said to be **atomic** if for every $x \in X \backslash \{x_*\}$ there is an $\succcurlyeq$-atom of X in $x^{\downarrow}$. Dually, provided that there is a (necessarily unique) $x^* \in X$ with $x^* \succcurlyeq x$ for each $x \in X$, an element x of X is a **$\succcurlyeq$-coatom** of X if x^* is a $\succcurlyeq$-cover of x. Such a poset is said to be **coatomic** if for every $x \in X \backslash \{x^*\}$ there is a $\succcurlyeq$-coatom of X in $x^{\uparrow}$.

Note. A bounded poset $(X, \succcurlyeq)$ is atomic iff $(X, \preccurlyeq)$ is coatomic.

It is plain that every bounded finite poset is both atomic and coatomic. On the other hand, $([0, 1], \geq)$ is, for instance, neither atomic nor coatomic. (Indeed, this loset does not possess any atoms or coatoms). Finally, $([0, 1] \cup \{2\}, \geq)$ is an example of a coatomic loset which is not atomic. Here are two further examples.

Example 3.6.1. Let X be any nonempty set. Then, every singleton subset of X is an $\supseteq$-atom of 2^X and hence $(2^X, \supseteq)$ is atomic. Similarly, any set of the form $X \backslash \{x\}$ (with $x \in X$) is a $\supseteq$-coatom of 2^X and hence $(2^X, \supseteq)$ is coatomic.

Example 3.6.2. The product of any two atomic (coatomic) posets is itself atomic (coatomic).

Exercises

3.6.1. Draw the Hasse diagrams of the posets of antichains of the 4-fence and 4-crown.

3.6.2. Consider the poset $(\mathbb{N}, \succcurlyeq)$ where $x \succcurlyeq y$ iff x is divisible by y. What are the $\succcurlyeq$-atoms of $\mathbb{N}$? Is $(\mathbb{N}, \succcurlyeq)$ atomic?

3.6.3. Consider the poset $(\mathrm{Sym}_n, \succcurlyeq)$ introduced in Exercise 3.1.2. Is this poset atomic? Coatomic?

3.6.4. Let X be a Hausdorff topological space, and let $(X, \succcurlyeq)$ be the poset introduced in Exercise 3.1.3. Is this poset atomic? Coatomic?

4 Preordered Algebraic Systems

Many preordered sets encountered in practice possess an additional mathematical structure that is consistent with the underlying order structure in some manner. From an applied perspective, the most important of such situations arises when the preordered set under consideration is endowed with an algebraic and/or topological structure. After a brief review of a few essentials of group theory, we consider the former situation in this section, focusing on preordered groups and linear spaces. (The case of preordered topological structures will be investigated later in the text.)

4.1 Groups: A Review

Basic Concepts. A **semigroup** is an ordered pair $(X, +)$ where X is a nonempty set and $+$ is an associative binary operation on X. If $(X, +)$ is a semigroup, therefore, we understand that $+$ is a map from $X \times X$ into X, but we write $x + y$ instead of $+(x, y)$ for any x and y in X. Associativity of X means that $(x + y) + z = x + (y + z)$ – writing $x + y + z$ is thus not ambiguous – for every $x, y, z \in X$. (In this text, we use the additive notation for the operation of a generic semigroup, even though that operation need not be commutative.)

A semigroup $(X, +)$ is said to be a **group** if (i) there is an element $\mathbf{0}$, called the **identity element** of X, such that $\mathbf{0} + x = x = x + \mathbf{0}$ for every $x \in X$; and (ii) for every $x \in X$ there is an element $-x$ of X, called the **inverse** of x, such that $-x + x = \mathbf{0} = x + -x$. If, in addition, $+$ is commutative, that is, $x + y = y + x$ for each $x, y \in X$, we say that $(X, +)$ is a **commutative** (Abelian) **group**. Finally, a **subgroup** (**subsemigroup**) of $(X, +)$ is a group (semigroup) $(Y, +')$, where $Y \subseteq X$ and $+'$ is the restriction of $+$ to Y. As $+$ and $+'$ agree on everywhere that $+'$ is defined, denoting it simply as $+$ does not cause ambiguity. As is common, we will follow this practice throughout the text.

Notation. Given a group $(X, +)$, and any two subsets A and B of X, we define

$$-A := \{-a : a \in A\} \quad \text{and} \quad A + B := \{a + b : (a, b) \in A \times B\}.$$

But, for any x in X, we write $A + x$ for $A + \{x\}$, and $x + B$ for $\{x\} + B$.

Notation. Given a group $(X, +)$, and $x \in X$, we set $0x := \mathbf{0}$, and for any integer k, we define $kx := x + \cdots + x$ (k times) if $k > 0$ and $kx := (-k)(-x)$ if $k < 0$.

As is common, we shall abuse terminology in what follows, and refer to a group $(X, +)$ simply as X. In fact, depending on the context, we may talk of X as a set, or as a group, interchangeably.

Notice that the defining properties of a group are precisely those of a linear space that pertain the vector addition. Thus, every linear space is a commutative group relative to its addition operation. In particular, in a group X, the identity element $\mathbf{0}$ is unique. (For, if $\mathbf{0}$ and $\mathbf{0}'$ are both identity elements in X, we have $\mathbf{0}' = \mathbf{0} + \mathbf{0}' = \mathbf{0}$.) It is also easily verified that the inverse of any x in X is unique. It follows that $-(x + y) = -y + -x$ for every $x, y \in X$. In what follows, we will denote an expression like $a + -x$ simply as $a - x$. Thus: $-(x + y) = -y - x$ for every $x, y \in X$.

For any subset S of a group X, by the **subgroup of X generated** by S, we mean the smallest subgroup of X that contains S. We denote this subgroup by $\langle S \rangle$, but it is customary to write $\langle x_1, ..., x_k \rangle$ for $\langle \{x_1, ..., x_k\} \rangle$ for any $x_1, ..., x_k \in X$. By definition, $\langle S \rangle$ is the subgroup of X such that (i) $S \subseteq \langle S \rangle$; and (ii) $Y \subseteq Z$ for any subgroup Z of X with $S \subseteq Z$. It is easy to see that such a subgroup exists and it is unique. Indeed, $\langle S \rangle$ is the intersection of all subgroups of X that contains S, which is well-defined because the intersection of any collection of subgroups of X is a subgroup of X, and there is always a subgroup of X that contains S, namely, X itself. For instance, $\langle \varnothing \rangle = \{\mathbf{0}\}$ and $\langle x \rangle = \{kx : k \in \mathbb{Z}\}$ for any $x \in X$. More generally, $\langle S \rangle$ consists precisely of the elements of X of the form $a_1x_1 + \cdots + a_kx_k$ where $k \in \mathbb{N}$, $x_1, ..., x_k \in S$ and $a_1, ..., a_k \in \mathbb{Z}$.

Examples.

Example 4.1.1. Neither $\mathbb{N}$ nor $\mathbb{Z}_+$ are groups relative to the addition operation, but $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are commutative groups relative to this operation. Similarly, for any positive integer n, $\mathbb{Z}^n$, $\mathbb{Q}^n$ and $\mathbb{R}^n$ are commutative groups (relative to the usual (coordinatewise) addition operation). Also plain is that $\mathbb{Z}^n$ is a subgroup of $\mathbb{Q}^n$ and $\mathbb{Q}^n$ is a subgroup of $\mathbb{R}^n$. Similarly, $\mathbb{Q}_{++}$ and $\mathbb{R}_{++}$, as well as, $\mathbb{Q}\backslash\{0\}$ and $\mathbb{R}\backslash\{0\}$, are commutative groups relative to the multiplication operation.

Example 4.1.2. Let $n \geq 2$ be an integer. The collection of all invertible $n \times n$ real matrices is a group relative to matrix multiplication. This group is called the **general linear group** of degree n, and is denoted by $GL(n, \mathbb{R})$. This group is not commutative.

Example 4.1.3. For any nonempty set X, the set of all bijective self-maps on X is a group relative to the composition operation. This group is called the **symmetric group** on X, and is denoted by S_X, but it is customary to write S_n for $S_{[n]}$. Any subgroup of a symmetric group S_X is said to be a **group of permutations**.

Example 4.1.4. If X is a group such that $X = \langle x \rangle$ for some $x \in X$, we say that X is a **cyclic group**. For instance, $\mathbb{Z}$ is a cyclic group, and $k\mathbb{Z}$ is a cyclic subgroup of $\mathbb{Z}$ for every positive integer k. (Here $k\mathbb{Z} := \{km : m \in \mathbb{Z}\}$.) It is easily shown that every cyclic group is commutative, and that every subgroup of a cyclic group must itself be cyclic. (Any subgroup of $\mathbb{Z}$ is thus of the form $k\mathbb{Z}$ for some nonnegative integer k.)

Example 4.1.5. Let X be a group and Y a subgroup of X. For any x in X, the sets $x+Y$ and $Y+x$ need not be the same. (The former is said to be a **left coset** of Y in X, and the latter a **right coset** of Y in X.) However, if Y is a **normal subgroup** of X, that is, $x+y-x \in Y$ for every $(x,y) \in X \times Y$, then, and only then, we have $x+Y = Y+x$ for every $x \in X$.

We define $X/Y := \{x+Y : x \in X\}$. When Y is normal, there is a natural way of defining an addition operation $\boxplus$ on this collection: $(x+Y) \boxplus (y+Y) := (x+y)+Y$. (It is customary to denote $\boxplus$ simply as $+$, because the context always makes clear whether $+$ refers to the operation on X or that on X/Y.) It is easy to check that normality of Y ensures that this operation is well-defined – this happens only when Y is normal – and that X/Y is a group relative to it. This group is called the **factor group** of X by Y.

Example 4.1.6. Let X be a group. Then $Y := \{x \in X : x+y = y+x \text{ for each } y \in X\}$ is said to be the **center** of X. This is a commutative subgroup of X.

Group Isomorphisms. Recall that two linear spaces can be regarded as identical from the perspective of linear algebra if there exists a linear bijection from one onto the other. Analogously, to identify two groups, we need to find a bijection from one onto the other which preserves the structure of the operations of these groups.

Definition. Let X and Y be two groups. We say that a map $\Phi : X \to Y$ is a (group) **homomorphism** if

$$\Phi(x+y) = \Phi(x) + \Phi(y) \quad \text{for every } x, y \in X,$$

where we use the operation of X on the left-hand side of this equation, and that of Y on its right-hand side. (If X and Y are the same groups here, we refer to Φ as a (group) **automorphism**.) Such a map is said to be a **group isomorphism** if it is bijective. If there exists a group isomorphism from X onto Y, we say that X and Y are **isomorphic**.

Note. The inverse of a group isomorphism Φ from a group X onto a group Y is a group isomorphism from Y onto X.

Two isomorphic groups differ only in the labeling of their elements and operations. Naturally, the algebraic properties of any two such groups are identical.

Example 4.1.7. (Cayley's Theorem) *Every group is isomorphic to a group of permutations.* To see this, take any group X, and for each $x \in X$ define the self-map f_x on X by $f_x(\omega) := x + \omega$. It is routine to check that $Y := \{f_x : x \in X\}$ is a subgroup of S_X. Furthermore, the map $x \mapsto f_x$ is a group isomorphism from X onto Y.

Example 4.1.8. (The Fundamental Homomorphism Theorem) *Every homomorphic image of a group is isomorphic to a factor group of that group.* More precisely, where X and Y are two groups and $\Phi : X \to Y$ is a homomorphism, $\Phi(X)$ is a group, $\Phi^{-1}(\mathbf{0})$ is a normal subgroup of X, and the map $x + \Phi^{-1}(\mathbf{0}) \mapsto \Phi(x)$ is a group isomorphism from $X/\Phi^{-1}(\mathbf{0})$ onto $\Phi(X)$. (Here $\mathbf{0}$ is the identity element of Y.) We leave the proof as an exercise.

Subgroup Posets. Let X be a group and $\mathcal{G}(X)$ the collection of all ground sets of the subgroups of X. (That is, $Y \in \mathcal{G}(X)$ iff $(Y,+)$ is a subgroup of $(X,+)$.) Then, $(\mathcal{G}(X), \supseteq)$ is a poset.

We shall work on this poset quite a bit later in the text. For now, we provide only a simple example.

Let $X := \{\mathbf{0}, x, y, z\}$ and consider the binary relation $+$ on X where $\mathbf{0} + \omega = \omega = \omega + \mathbf{0}$ and $\omega+\omega = \mathbf{0}$ for each $\omega \in X$, while $x+y = z = y+x$, $x+z = y = z+x$ and $y+z = x = z+y$. Then, $(X, +)$ is a commutative group – it is called the **Klein 4-group**. The Hasse diagram of $(\mathcal{G}(X), \supseteq)$ in the context of this group is presented in Figure 4.1.1.

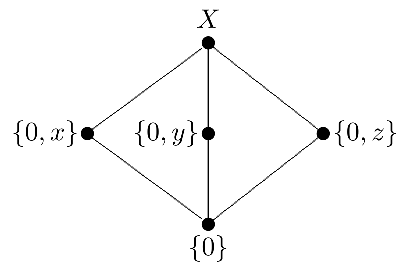

Figure 4.1.1

4.2 Preordered Groups

Translation Invariance. Let X be a group and $\succsim$ a preorder on X. We say that $\succsim$ is **translation-invariant** if

$$x \succsim y \quad \text{implies} \quad x + z \succsim y + z \text{ and } z + x \succsim z + y$$

for every $x, y, z \in X$. An immediate consequence of this property is that $x \succsim y$ implies $-y \succsim -x$. Moreover, for any $x_1, y_1, ..., x_k, y_k \in X$ with $x_i \succsim y_i$ for each i, we have $x_1+\cdots+x_k \succsim y_1 + \cdots + y_k$, which is easily proved by induction. In particular, $x \succsim y$ implies $kx \succsim ky$ for each $k \in \mathbb{N}$.

Translation invariance builds a natural compatibility between the group operation on X and the preorder $\succsim$. As we shall see, this allows for a surprisingly powerful way of making use of the algebraic structure of a group in the context of order-theoretic problems.

Ordered Groups. An ordered pair $(X, \succsim)$ is said to be a **preordered group** if X is a group and $\succsim$ is a translation-invariant preorder on X. If, in addition, $\succsim$ is a partial order, then $(X, \succsim)$ is called a **po-group** (short for *partially ordered group*). Finally, if $\succsim$ is a linear order here, we say that $(X, \succsim)$ is a **lo-group** (short for *linearly ordered set*). (For instance, under addition and the usual ordering, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are commutative lo-groups. The cartesian products of these groups yield examples of po-groups.)

Note. If $(X, \succsim)$ is a preordered group (or a po-group), so is $(X, \precsim)$.

Note. Let $(X, \succsim)$ be a preordered group, and define $\trianglerighteq$ on $X/_\sim$ by $[x]_\sim \trianglerighteq [y]_\sim$ iff $x \succsim y$. Then, it is readily checked that $\{x \in X : x \sim \mathbf{0}\}$ is a normal subgroup of X and $(X/_\sim, \trianglerighteq)$ is a po-group.

Positive Cone. Let $(X, \succsim)$ be a preordered group. For reasons that will come apparent in the next subsection, we will say that a subset C of X is a **cone** in X if C is a submonoid of X (that is, $\mathbf{0} \in C + C \subseteq C$) such that $-x + C + x \subseteq C$ for every $x \in X$. It is readily checked that $\mathbf{0}^\uparrow$ is a cone in X; we say that $\mathbf{0}^\uparrow$ is the **positive cone** of $(X, \succsim)$. As is largely common, we shall denote this cone as X_+ in what follows. (The **negative cone** of $(X, \succsim)$ is $-X_+$, that is, $\mathbf{0}^{\downarrow}$.) It should be stressed that X_+ contains all the information that $\succsim$ contains, in the sense that knowing X_+ determines $\succsim$ entirely:

$$x \succsim y \quad \text{iff} \quad x - y \in X_+ \quad \text{iff} \quad -y + x \in X_+$$

for every $x, y \in X$. Similarly, monotonic subsets of X are determined completely by X_+. In particular,

$$S^{\uparrow} = S + X_+ = X_+ + S \quad \text{and} \quad S^{\downarrow} = S + (-X_+) = -X_+ + S$$

for any subset S of X. (For instance, if $x \in S^{\uparrow}$, then $x \succsim y$ for some $y \in S$, so by translation invariance of $\succsim$, we have $-y + x \in X_+$, and hence $x = y + (-y + x) \in S + X_+$. Conversely, if $x = y + z$ for some $(y, z) \in S \times X_+$, then translation invariance of $\succsim$ and $z \succsim \mathbf{0}$ imply $x = y + z \succsim y + \mathbf{0} = y$, that is, $x \in S^{\uparrow}$. Thus: $S^{\uparrow} = S + X_+$.) These observations show that a subset S of X is $\succsim$-increasing iff $S + X_+ \subseteq S$, and $\succsim$-decreasing iff $S - X_+ \subseteq S$. In particular, X_+ is $\succsim$-increasing.

Cones in a given group X provide a natural way to turn X into a preordered group. Indeed, for any cone C in X, the binary relation $\succsim_C$ on X defined by $x \succsim_C y$ iff $x - y \in C$, is a translation-invariant preorder on X. (Exercise!) Furthermore, $(X, \succsim_C)$ is a po-group if $C \cap -C = \{\mathbf{0}\}$.

In fact, all preordered groups arise in this way. That is, for any preordered (ordered) group $(X, \succsim)$, there is a cone C in X such that $\succsim\, = \,\succsim_C$. Indeed, it is readily verified that we have, per force, $\succsim\, = \,\succsim_{X_+}$ in this case. (The positive cone of $(X, \succsim_{X_+})$ is X_+!)

> **Note.** Every group X can be viewed as a po-group by designating $\{\mathbf{0}\}$ as its positive cone; this po-group is $(X, \triangle_X)$.

Positive Cones of Subgroups. Let $(X, \succsim)$ be a preordered group and Y a subgroup of X. Then, $(Y, \succsim)$ is also a preordered group. (Recall that by $(Y, \succsim)$, we mean $(Y, \succsim \cap (Y \times Y))$ here.) It is readily verified that the positive cone of $(Y, \succsim)$ is the restriction of X_+ to Y, that is, $Y_+ = Y \cap X_+$. Morover, Y is generated by Y_+, provided that it is $\succsim$-directed (that is, for every $x, y \in Y$ there is a z in Y with $z \succsim x$ and $z \succsim y$).

Proposition 4.2.1. Let $(X, \succsim)$ be a preordered group and Y a $\succsim$-directed subgroup of X. Then, $Y = Y_+ - Y_+$.

Proof. If $x \in Y$, then $z \succsim x$ and $z \succsim \mathbf{0}$ for some $z \in Y$. Then, if we put $y := -x + z$, we see that both y and z belong to Y_+ and $x = z - y$. Thus: $Y \subseteq Y_+ - Y_+$. As the converse containment is obvious, we are done.

Exercises

4.2.1. Let $(X, \succsim)$ be a preordered group. Show that $X_+ \cap -X_+$ is a subgroup of X.

4.2.2. For any group X and any cone C in X, show that $(X, \succsim_C)$ is a preordered group.

4.2.3. Let $(X, \succsim)$ be a preordered group. For any nonempty subsets S and T of X, show that

$$-S^{\uparrow} = (-S)^{\downarrow}, \quad S^{\uparrow} + T^{\uparrow} \subseteq (S + T)^{\uparrow}, \quad \text{and} \quad x^{\uparrow} + T^{\uparrow} = (x + T)^{\uparrow}.$$

Give an example to show that the containment in the middle expression can be strict.

4.2.4. Let $(X, \succcurlyeq)$ be a po-group such that there is an $x \in X$ with $x \succcurlyeq y$ for every $y \in X$. Show that $|X| = 1$.

4.2.5. Let $(X, \succsim)$ be a preordered group and Y a subgroup of X. Prove that Y is $\succsim$-convex iff $x^{\downarrow} \cap X_{+} \subseteq Y$ for every $x \in Y$.

4.2.6. Let $(X, \succsim)$ be a preordered group and Y a subgroup of X. Prove that Y is $\succsim$-directed iff $Y = \langle Y_{+} \rangle$.

4.2.7. (*Factor Po-Groups*) Let $(X, \succsim)$ be a preordered group and Z a normal subgroup of X. We define the binary relation $\succsim_{X/Z}$ on X/Z by

$$x + Z \succsim_{X/Z} y + Z \quad \text{iff} \quad x \succsim y + z \text{ for some } z \in Z.$$

a. Show that $(X/Z, \succsim_{X/Z})$ is a preordered group whose positive cone is $\{x + Z : x \in X_{+}\}$.
b. Show that $(X/Z, \succsim_{X/Z})$ is a po-group iff Z is $\succsim$-convex.

4.3 Preordered Linear Spaces

Vector Preorders. For preorders on a linear space, it makes sense to ask for scale-invariance, in addition to translation invariance, to tie the order and algebraic structures at hand in a tighter way. Let X be a (real) linear space, and $\succsim$ a preorder on X. We say that $\succsim$ is **scale-invariant** if $x \succsim y$ iff $\lambda x \succsim \lambda y$ for every $x, y \in X$ and $\lambda > 0$. In turn, we refer to $\succsim$ as a **vector preorder** on X if $\succsim$ is both translation- and scale-invariant, that is,

$$x \succsim y \quad \text{iff} \quad \lambda x + z \succsim \lambda y + z$$

for every $x, y, z \in X$ and $\lambda > 0$.

Ordered Linear Spaces. An ordered pair $(X, \succsim)$ is said to be a **preordered linear space** if X is a linear space and $\succsim$ is a vector preorder on X. If, in addition, $\succsim$ is a partial order, then $(X, \succsim)$ is called a **partially ordered linear space**. In either of these cases, the set $\mathbf{0}^{\uparrow}$ is called the **positive cone** of $(X, \succsim)$, and is commonly denoted as X_{+}. (Here $\mathbf{0}$ stands for the origin of X.) This set is indeed a convex cone in X, that is, X_{+} is a nonempty subset of X such that $X_{+} + X_{+} \subseteq X_{+}$ and $\lambda X_{+} \subseteq X_{+}$ for every real number $\lambda \geq 0$. (This fact alone is responsible for the prevalence of convex analysis in the theory of preordered linear spaces.) Moreover, we have $X_{+} \cap -X_{+} = \{\mathbf{0}\}$ iff $\succsim$ is a partial order. It should be stressed that X_{+} contains all the information that $\succsim$ contains, in the sense that knowing X_{+} determines $\succsim$ entirely: $x \succsim y$ iff $x - y \in X_{+}$.

As for concrete examples, we note that the posets we considered in Examples 3.1.3 and 3.1.7 are partially ordered linear spaces. Indeed, $(\mathbb{R}^n, \geq)$ is a partially ordered linear space whose positive cone is the positive orthant of $\mathbb{R}^n$. (The notation $\mathbb{R}^n_{+}$ for the positive orthant of $\mathbb{R}^n$ is thus consistent with our notation for positive cones.) Similarly, the pointwise ordering $\geq$ of $\mathbf{C}[0,1]$ is the vector partial order on $\mathbf{C}[0,1]$ that induces the convex cone $\{f \in \mathbf{C}[0,1] : f(t) \geq 0 \text{ for each } t \in [0,1]\}$. Many other classical linear spaces, such as the ℓ^p spaces, $1 \leq p \leq \infty$, are also partially ordered linear spaces under their usual (coordinatewise) ordering.

As is the case with preordered groups, all order-theoretic statements in the context of a preordered linear space can be expressed in terms of the positive cone of that space. For instance, for any preordered linear space $(X, \succsim)$, it is plain that a vector preorder $\trianglerighteq$ on X is an extension of $\succsim$ iff the positive cone of $(X, \trianglerighteq)$ contains that of $(X, \succsim)$. Besides, for any preordered linear space $(X, \succsim)$, we have

$$x^{\downarrow} = x - X_{+} \quad \text{and} \quad x^{\uparrow} = x + X_{+}$$

and

$$^{\downarrow}x = x - (X_+ \cap (X\backslash -X_+)) \quad \text{and} \quad ^{\uparrow}x = x + (X_+ \cap (X\backslash -X_+))$$

for every x in X. (Thus: $(X, \succsim)$ is bounded iff $|X| = 1$.)

As a final observation, we note that it is usually easy to identify the chains and antichains in a preordered linear space $(X, \succsim)$. For instance, if y is in X and z in X_+, then $\{y+z, y+2z, ...\}$ is a $\succsim$-chain. More generally, for any x and y in X, the line $\{\lambda x + (1-\lambda)y : \lambda \in \mathbb{R}\}$ is a $\succsim$-chain if x and y are $\succsim$-comparable, and it is an $\succsim$-antichain if x and y are $\succsim$-incomparable.

Note. Preordered groups and linear spaces are but only two instances in which a given set X is endowed with a preorder that is in some way "compatible" with an algebraic structure on X. In particular, there are fairly well-developed theories for "preordered semigroups," "preordered rings", etc., but we shall not discuss such structures here. Blyth (2010) provides a very nice introduction to these topics.

Exercises

4.3.1. Let $(X, \succsim)$ be a preordered linear space. Show that $X_+ \cap -X_+$ is a subspace of X.

4.3.2. Let X be a linear space and $\succsim$ a binary relation on X. Prove that $(X, \succsim)$ is a preordered linear space iff there is a partial order $\succcurlyeq$ on X and a subspace Y of X such that $\{x \in X : x \succsim \mathbf{0}\} = Y + \{x \in X : x \succcurlyeq \mathbf{0}\}$.

4.3.3. Let $(X, \succcurlyeq)$ be a preordered linear space and Y a subspace of X. Prove that $x^{\uparrow} \cap Y \neq \varnothing$ for each $x \in X$ iff $X = Y - X_+$.

4.3.4. Let $(X, \succsim)$ be a preordered linear space. Let X' stand for the **algebraic dual** of X, that is, X' is the linear space of all linear functionals on X. The **dual cone** of X_+ is defined as

$$X'_+ := \{f \in X' : f(x) \geq 0 \text{ for every } x \in X_+\}.$$

Define the binary relation $\trianglerighteq$ on X' by $f \trianglerighteq g$ iff $f - g \in X'_+$. Prove:

a. $(X', \trianglerighteq)$ is a preordered set.

b. If there is an $f \in X'$ with $f(x) > 0$ for every $x \in X_+\backslash\{\mathbf{0}\}$, then $(X, \succsim)$ is a poset.

c. If there is an $x \in X_+$ such that for every $y \in X$ there is a number $\lambda_y > 0$ with $x + \lambda y \in X_+$ for all λ in $[0, \lambda_y]$, then $(X', \trianglerighteq)$ is a poset.

5 Representation Through Complete Preorders

5.1 Representation of a Complete Binary Relation

There is an interesting way of representing a complete binary relation on a nonempty set X in terms of complete preorders on X.

Theorem 5.1.1. Let X be a nonempty finite set and R a complete binary relation on X. Then, there exists a nonempty set $\mathcal{P}$ of complete preorders on X such that

$$x \; R \; y \quad \text{iff} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| \geq |\{\succsim \in \mathcal{P} : y \succ x\}|$$

for every x and y in X.

Proof. If $|X| \leq 2$, the claim is trivial, so we assume $n := |X| > 2$ in what follows. Take any two elements x and y in X with $x \; R \; y$, and let $\{a_1, ..., a_{n-2}\}$ be an enumeration of $X\backslash\{x, y\}$. If $x \; P_R \; y$, we define $\succcurlyeq_{xy}$ as the linear order on X with

$$x \succ_{xy} y \succ_{xy} a_1 \succ_{xy} \cdots \succ_{xy} a_{n-2},$$

and $\trianglerighteq_{xy}$ as the linear order on X with

$$a_{n-2} \vartriangleright_{xy} \cdots \vartriangleright_{xy} a_1 \vartriangleright_{xy} x \vartriangleright_{xy} y.$$

(Here, of course, $\succ_{xy}$ is the asymmetric part of $\succcurlyeq_{xy}$, and similarly for $\vartriangleright_{xy}$.) Besides, if $x \; I_R \; y$, we leave $\succcurlyeq_{xy}$ exactly as above, and define $\trianglerighteq_{xy}$ as the linear order on X with

$$a_{n-2} \vartriangleright_{xy} \cdots \vartriangleright_{xy} a_1 \vartriangleright_{xy} y \vartriangleright_{xy} x.$$

Finally, we define $\mathcal{P}$ to stand for the collection of all $\succcurlyeq_{xy}$ and all $\trianglerighteq_{xy}$, where x and y are distinct elements of X. Then, by construction,

$$x \; P_R \; y \quad \text{implies} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| - |\{\succsim \in \mathcal{P} : y \succ x\}| = 2$$

and

$$x \; I_R \; y \quad \text{implies} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| - |\{\succsim \in \mathcal{P} : y \succ x\}| = 0,$$

and we are done.

Remark. There is a nice interpretation of Theorem 5.1.1 from the perspective of social choice theory. In fact, the main objective of McGarvey (1953), who has proved this result first, was to understand the structure of social preferences that arise from the majority voting of finitely many individuals with complete preference relations. From this viewpoint, Theorem 5.1.1 says that, when the outcome space X is finite, every complete binary relation R on X can be interpreted "as if" it arises from the majority voting of a group of agents with complete preference relations on X. (For a much deeper result along these lines, see Kalai (2006).)

This viewpoint also provides a useful synopsis of the proof of Theorem 5.1.1. When $x \; P_R \; y$, the element x is ranked strictly higher than y by both $\succcurlyeq_{xy}$ and $\trianglerighteq_{xy}$ (and hence both of these individuals "vote" for x over y). On the other hand, for any (a, b) in $X \times X$, distinct from (x, y), the complete preorders $\succcurlyeq_{ab}$ and $\trianglerighteq_{ab}$ disagree on the ranking of x and y (and hence the "votes" of the involved individuals cancel each other). The situation $x \; I_R \; y$ is similarly analyzed.

Remark. There is a literature (that goes under the garb of *representation of finite tournaments*) that attempts to find $\mathcal{P}$ in Theorem 5.1.1 with the smallest cardinality. For instance, a result in Stearns (1959) shows that we can choose $\mathcal{P}$ in Theorem 5.1.1 with $|\mathcal{P}| \leq |X| + 2$ when $|X|$ is even, and with $|\mathcal{P}| \leq |X| + 1$ when $|X|$ is odd.[1] Better bounds than this are known, but these sorts of theorems are mainly of combinatorial interest. From a purely order-theoretic perspective, it would be more interesting to see how one may extend Theorem 5.1.1 to a context in which X is no longer finite. This is easy when X is countable (Exercise 5.2.5), but the problem becomes quickly technical in the general case.

[1] In the proof above, we have included in $\mathcal{P}$ two preference relations for each pair of distinct elements in X. Therefore, in our construction, $|\mathcal{P}| = 2\binom{n}{2} = n(n-1)$.

5.2 Representation of a Preorder

There is also a useful way of representing a preorder on *any* nonempty set X in terms of complete preorders on X.

Proposition 5.2.1. Let X be a nonempty set and $\succsim$ a preorder on X. Then, there exists a nonempty set $\mathcal{P}$ of complete preorders on X such that

$$x \succsim y \quad \text{iff} \quad x \trianglerighteq y \text{ for each } \trianglerighteq \text{ in } \mathcal{P}$$

for every x and y in X.

Proof. For each ω in X, we define the complete preorder $\succsim_\omega$ on X by $x \succsim_\omega y$ iff $\mathbf{1}_{\omega^\uparrow}(x) \geq \mathbf{1}_{\omega^\uparrow}(y)$. (Here $\mathbf{1}_{\omega^\uparrow}$ is the $\{0,1\}$-valued function on X that takes value 1 on $\omega^\uparrow$, and 0 elsewhere.) It is an easy exercise to check that $x \succsim y$ iff $x \succsim_\omega y$ for each $\omega \in X$. Thus, letting $\mathcal{P} := \{\succsim_\omega : \omega \in X\}$ completes the proof.

Remark. From the perspective of individual choice theory (Section 3.3), Proposition 5.2.1 can be interpreted as saying that a preference relation $\succsim$ of an individual on a nonempty set X can always be considered "as if" this relation arises from the *unanimity* voting of a group of agents – these agents are usually thought of as the multiple "selves" (or "moods") of the individual – with complete preference relations on X. The situation $x \bowtie y$, that is, the case where the individual with preference relation $\succsim$ is indecisive about how to compare the alternatives x and y, is captured by the disagreement of the ranking of x and y by at least two "selves" of the individual.

Remark. In Proposition 5.2.1, no guarantee is given for the complete preorders in $\mathcal{P}$ to be extensions of $\succsim$. The representation certainly does not necessitate this to happen, and in fact, the construction we gave in the proof will fail to have this property in general. However, as we shall see later, it is actually possible to use only the extensions of $\succsim$ in this representation in a variety of situations.

Exercises

5.2.1. Let $\mathcal{P}$ be a nonempty collection of complete preorders on a nonempty set X. Show that $\bigcup \mathcal{P}$ is a complete and acyclic binary relation on X.

5.2.2. Let $\succsim$ be a reflexive binary relation on a nonempty set X. Show that $\succsim = \bigcup \mathcal{P}$ for some nonempty collection $\mathcal{P}$ of preorders on X.

5.2.3. Let $\succsim$ be a complete and acyclic binary relation on a nonempty finite set X. True or false: $\succsim = \bigcup \mathcal{P}$ for some nonempty collection $\mathcal{P}$ of complete preorders on X.

5.2.4. Let $\succsim$ be a binary relation on a nonempty finite set X. Prove: $\succsim$ is acyclic iff there is a complete preorder $\trianglerighteq$ such that $x \succ y$ implies $x \triangleright y$, for every $x, y \in X$.

5.2.5. Let R be a complete binary relation on a nonempty countable set X. Show that there is a nonempty collection $\mathcal{P}$ of complete preorders on X and a measure μ on $2^{\mathcal{P}}$ such that

$$x \; R \; y \quad \text{iff} \quad \mu(\{\succsim \in \mathcal{P} : x \succ y\}) \geq \mu(\{\succsim \in \mathcal{P} : y \succ x\})$$

for every x and y in X.

6 Extrema

Extremal and Extremum Elements. The *extremal* and *extremum* elements of a preordered set are essential to the analysis of that preordered set, and they arise routinely in applications. Put informally, an element of a preordered set is extremal if it is either never ranked strictly below another element in the set, or it is never ranked strictly above another element. By contrast, an extremum element is either ranked higher than every other element in the set, or it is ranked lower than every other element. Formally speaking:

Definition. Let X be a nonempty set, R a binary relation on X, and S a nonempty subset of X. An element x of S is called **R-maximal** in S if there is no $\omega \in S$ with $\omega\ P_R\ x$. It is said to be **R-minimal** in S if there is no $\omega \in S$ with $x\ P_R\ \omega$.

Such elements of X are often called **R-extremal**. In turn, the **R-extremum** elements of this set are defined as follows:

Definition. Let X be a nonempty set, R a binary relation on X, and S a nonempty subset of X. An element x of S is said to be an **R-maximum** in S if $x\ R\ \omega$ for every $\omega \in S$. It is said to be an **R-minimum** in S if $\omega\ R\ x$ for every $\omega \in S$.

Notation. Let X be a nonempty set, R a binary relation on X, and S a nonempty subset of X. We denote the set of all R-maximal and R-minimal elements in S by

$$\mathbf{MAX}(S,R) \quad \text{and} \quad \mathbf{MIN}(S,R),$$

respectively. Similarly, the set of all R-maximum and R-minimum elements in S are denoted by

$$\max(S,R) \quad \text{and} \quad \min(S,R),$$

respectively.

Note. For any preordered set $(X,\succsim)$ and nonempty subset S of X, we have $\mathbf{MAX}(S,\succsim) = \mathbf{MIN}(S,\precsim)$ and $\max(S,\succsim) = \min(S,\precsim)$. This duality allows one to deduce the properties minimal and minimum elements from those of maximal and maximum elements in general.

For any preordered set $(X,\succsim)$, a $\succsim$-maximum element of a subset S of X is $\succsim$-maximal in S, that is, we always have $\max(S,\succsim) \subseteq \mathbf{MAX}(S,\succsim)$. Easy examples show that this containment can hold strictly. However, this happens only when S does not have a $\succsim$-maximum element. That is, $\max(S,\succsim) = \mathbf{MAX}(S,\succsim)$ whenever $\max(S,\succsim) \neq \varnothing$. To see this, suppose $\max(S,\succsim) \neq \varnothing$, and pick any $\succsim$-maximal element x in S. Then, where y is a $\succsim$-maximum element in S, we have $y \succsim x$, but we cannot have $y \succ x$ in view of the $\succsim$-maximality of x in S. It follows that $x \sim y$, so, by transitivity of $\succsim$, we find that $x \succsim \omega$ for every $\omega \in X$, that is, x is a $\succsim$-maximum element in S.

A nonempty subset S in a preordered set $(X,\succsim)$ may have several maxima, but of course, we have $x \sim y$ for each $x, y \in \max(S,\succsim)$. In particular, if $(X,\succsim)$ is a poset, there can be at most one $\succsim$-maximum element in S. More generally, we have $|\max(S,\succsim)| \leq 1$, provided that $\succsim \cap (S \times S)$, which is a preorder on S, is antisymmetric.

Finally, we note that the extremal elements of a finite poset $(X,\succcurlyeq)$ is easily identified in the Hasse diagram of that poset: An element of X is $\succcurlyeq$-maximal ($\succcurlyeq$-minimal) iff the point that corresponds to that element is not connected by a line segment to any other point above (below) it. (For instance, in the first diagram of Figure 6.1 the highlighted points correspond to maximal elements of the involved poset, and in the second to minimal elements.)

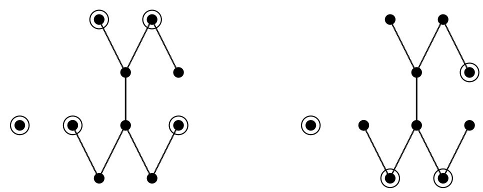

Figure 6.1

Examples.

Example 6.1. Let X be a set with $|X| \geq 2$. Then, no element of X is a $\triangle_X$-maximum in X, but every element of X is $\triangle_X$-maximal. Similarly, no element of X is a $\triangle_X$-minimum in X, but every element of X is $\triangle_X$-minimal.

Example 6.2. Let X be a nonempty set. Then, a nonempty collection $\mathcal{A}$ of subsets of X has a $\supseteq$-maximum if $\bigcup \mathcal{A} \in \mathcal{A}$. Similarly, this collection has a $\supseteq$-minimum if $\bigcap \mathcal{A} \in \mathcal{A}$. (In particular, the $\supseteq$-maximum of 2^X is X and the $\supseteq$-minimum of 2^X is $\varnothing$.) By contrast, an element of $\mathcal{A}$ is $\supseteq$-maximal iff it is not contained in any other element of $\mathcal{A}$. For instance, for any nonempty proper subset S of X, we have $\max(\{S, X\backslash S\}, \supseteq) = \varnothing$ while $\mathbf{MAX}(\{S, X\backslash S\}, \supseteq) = \{S, X\backslash S\}$.

Example 6.3. Let $S := \{(x, -x) : x \in \mathbb{R}\}$. Then, every element of S is both $\geq$-maximal and $\geq$-minimal in S, while no element of S is either $\geq$-maximum or $\geq$-minimum in S.

Example 6.4. Let $(\mathbb{Z}\backslash\{0\}, \succcurlyeq)$ be the preordered set where $x \succcurlyeq y$ iff y is a divisor of x. Then, both -1 and 1 are the $\succcurlyeq$-minimum elements of $\mathbb{Z}\backslash\{0\}$, while there are no $\succcurlyeq$-maximal elements in $\mathbb{Z}\backslash\{0\}$. On the other hand, $\min(\mathbb{N}, \succcurlyeq) = \{1\}$, and $\mathbf{MIN}(\{2, 3, ...\}, \succcurlyeq)$ consists of all prime numbers.

The following example shows that even when there is a unique maximal element in a poset, the presence of a maximum element is not guaranteed.

Example 6.5. Consider the partial order $\succcurlyeq$ on $\mathbb{N}$ defined by $i \succcurlyeq i$ for each $i \in \mathbb{N}$, and $i \succ j$ iff $i > j > 1$. (This partial order agrees with the usual ordering of positive integers, except that it renders 1 as non-comparable to any other number in $\mathbb{N}$.) Then, there is a unique $\succcurlyeq$-maximal element in $\mathbb{N}$, namely 1, while $\max(\mathbb{N}, \succcurlyeq) = \varnothing$.

These examples demonstrate that, in a preordered set, there may be a considerable wedge between maximal and maximum elements. Yet, if the preorder under consideration is complete, this discrepancy disappears. That is, for any preordered set $(X, \succsim)$ and a nonempty subset S of X, an element x is $\succsim$-maximal in S iff it is $\succsim$-maximum in S, *provided that* $\succsim$ *is complete*. (The same is true, of course, for minimal and minimum elements.)

Existence of Extremal Elements. Obviously, a poset need not have a maximal or a minimal element. (For instance, $\mathbb{R}$ has neither a $\geq$-maximal nor $\geq$-minimal element.) However, every *finite* preordered set is sure to possess extremal elements.

Proposition 6.1. Let $(X, \succsim)$ be a preordered set and S a nonempty finite subset of X. Then, $\mathbf{MAX}(S, \succsim) \neq \varnothing \neq \mathbf{MIN}(S, \succsim)$.

Proof. There is nothing to prove when $|S| = 1$. Then, take any positive integer k, and suppose our assertion holds for any k-element subset of X. Now take a subset S of X with $|S| = k+1$. Pick any x in S, and define $T := S \backslash \{x\}$. By the induction hypothesis, there exists a $\succsim$-maximal element, say, y, in S. Thus, if $x \succ y$ is false, y is a $\succsim$-maximal element in S. If, on the other hand, $x \succ y$ holds, then x is a $\succsim$-maximal element in S (by $\succsim$-maximality of y and transitivity of $\succ$). Thus, $\mathbf{MAX}(S, \succsim) \neq \varnothing$. Applying this finding to the preordered set $(X, \precsim)$, we find also that $\mathbf{MIN}(S, \succsim) \neq \varnothing$.

Evidently, finiteness of a poset is not enough for the existence of a maximum element (Example 6.2). But, if $\succsim$ is known to be complete, then an element is maximal in a set iff it is maximum in that set, and hence, Proposition 6.1 applies to yield:

Corollary 6.2. Let $(X, \succsim)$ be a preordered set and S a nonempty finite subset of X. If $\succsim \cap (S \times S)$ is complete, then $\max(S, \succsim) \neq \varnothing \neq \min(S, \succsim)$.

Later in our work, we will obtain substantial generalizations of these existence results.

Completion of a Preorder on a Finite Set. As a nice application of Proposition 6.1, we now show that it is always possible to extend any given partial order on a nonempty finite set to a linear order. More generally:

Corollary 6.3. Let $(X, \succsim)$ be a finite preordered set. Then, there is a complete preorder on X that extends $\succsim$.

Proof. Let $(X, \succcurlyeq)$ be a finite poset. By Proposition 6.1, there is a $\succcurlyeq$-minimal element of X, say, x_1. By the same token, there is a $\succcurlyeq$-minimal element of $X \backslash \{x_1\}$, say, x_2. Continuing this way, and putting $n := |X|$, we obtain an enumeration of X as $\{x_1, ..., x_n\}$ such that x_1 is $\succcurlyeq$-minimal in X and x_{i+1} is $\succcurlyeq$-minimal in $X \backslash \{x_1, ..., x_i\}$ for each $i \in [n-1]$. But then the linear order $\succcurlyeq'$ on X, defined by $x_n \succcurlyeq' \cdots \succcurlyeq' x_1$, extends $\succcurlyeq$. Conclusion: Every partial order on a finite set admits a linear extension.

For the general case, we apply what we have just found to the poset $(X/_\sim, \succcurlyeq)$ induced by $(X, \succsim)$ to find a linear order $\succcurlyeq'$ on $X/_\sim$ that extends $\succcurlyeq$. (Recall that we have $[x]_\sim \succcurlyeq [y]_\sim$ iff $x \succsim y$, for every $x, y \in X$.) We then define the preorder $\succsim'$ on X by $x \succsim' y$ iff $[x]_\sim \succcurlyeq' [y]_\sim$ for every $x, y \in X$. It is plain that $\succsim'$ is a complete preorder on X that extends $\succsim$.

Note. Finiteness requirement is in fact redundant in this result. By using the Axiom of Choice, we will prove later that every preorder admits a completion.

Exercises

6.1. Give an example of a poset $(X, \succcurlyeq)$ such that $|X| \geq 2$ and there is an $x \in X$ with $\mathbf{MAX}(X, \succcurlyeq) = \{x\} = \mathbf{MIN}(X, \succcurlyeq)$.

6.2. A poset $(X, \succcurlyeq)$ is said to be **well-founded** if every nonempty subset of X has a $\succcurlyeq$-minimal element, and **well-ordered** if every nonempty subset of X has a $\succcurlyeq$-minimum element. Give an example of a poset that is well-founded but not well-ordered.

6.3. An $n \times n$ matrix is said to be **bistochastic** if every term of this matrix is nonnegative, the sum of the terms of each row is 1, and the sum of the terms of each column is 1. We define the **Lorenz order** $\geq_{\mathrm{L}}$ as the binary relation on $X := \{\mathbf{x} \in \mathbb{R}^n_+ : x_1 + \cdots + x_n = 1\}$ by $\mathbf{x} \geq_{\mathrm{L}} \mathbf{y}$ iff $\mathbf{x} = \mathbf{B}\mathbf{y}$ for some $n \times n$ bistochastic matrix $\mathbf{B}$. Prove that $\geq_{\mathrm{L}}$ is a preorder on X, and compute $\max(X, \geq_{\mathrm{L}})$ and $\min(X, \geq_{\mathrm{L}})$.

6.4. Let $(X, \succsim)$ be a finite preordered set, and let $\mathcal{P}$ stand for the collection of all completions of $\succsim$ (Exercise 3.1.6). Prove:

$$\mathbf{MAX}(X, \succsim) = \bigcup\{\max(X, \trianglerighteq) : \trianglerighteq \in \mathcal{P}\}.$$

6.5. (Cherepanov-Feddersen-Sandroni) Let X be a nonempty finite set, and f a map from $2^X \backslash \{\varnothing\}$ into X such that $f(A) \in A$. Consider the following property:

- For any distinct x and y in X, and any subsets A and B of X,

$$\{x, y\} \subseteq B \subseteq A \text{ and } f(\{x, y\}) = f(A) = x \quad \text{imply} \quad f(B) \neq y.$$

Prove: f satisfies this property iff there is a nonempty collection $\mathcal{P}$ of partial orders on X and a linear order $\succcurlyeq$ on X such that

$$\{f(S)\} = \max\left(\bigcup\{\max(S, \trianglerighteq) : \trianglerighteq \in \mathcal{P}\}, \succcurlyeq\right)$$

for every nonempty subset S of X.

6.6. (Manzini-Mariotti) Let X be a nonempty finite set, and f a map from $2^X \backslash \{\varnothing\}$ into X such that $f(A) \in A$. Consider the following property:

- $f(A) = f(A \cup B)$ for any subsets A and B of X with $f(A) = f(B)$.

Prove: f satisfies this property and the property we considered in the previous exercise iff there are two antisymmetric binary relations $\succcurlyeq_1$ and $\succcurlyeq_2$ on X such that $\{f(S)\} = \mathbf{MAX}(\mathbf{MAX}(X, \succcurlyeq_1), \succcurlyeq_2)$ for every nonempty subset S of X.

7 Parameters of Posets

There are two summary statistics of posets, namely, height and width, that often help describe the associated order-structure in economic terms. As we shall see, these parameters are particularly useful when working with finite posets.

7.1 Height

Order-Rank Function. Let $(X, \succcurlyeq)$ be a poset. The **$\succcurlyeq$-rank** function on X is the map $\mathfrak{r}_{\succcurlyeq} : X \to \mathbb{N} \cup \{\infty\}$ defined by

$$\mathfrak{r}_{\succcurlyeq}(x) := \sup\{|Y| : Y \text{ is a } \succcurlyeq\text{-chain in } X \text{ such that } x \in \max(Y, \succcurlyeq)\}.$$

Given a poset $(X, \succcurlyeq)$, therefore, the $\succcurlyeq$-rank of an element x of X is the size of a longest $\succcurlyeq$-chain in X whose $\succcurlyeq$-maximum is x. Clearly, when X is finite, two elements in X that have the same $\succcurlyeq$-rank cannot be $\succcurlyeq$-comparable. Indeed,

$$\mathfrak{r}_{\succcurlyeq}(x) \geq \mathfrak{r}_{\succcurlyeq}(y) + 1 \quad \text{whenever} \quad x \succ y$$

for every x and y in X. Thus: $\{\mathfrak{r}_{\succcurlyeq}^{-1}(i) : i \in [|X|]\} \backslash \{\varnothing\}$ is a collection of $\succcurlyeq$-antichains in X that partition X for any finite poset $(X, \succcurlyeq)$.

> **Note.** The previous observation can be used to give an even simpler proof for Corollary 6.3. If $(X, \succcurlyeq)$ is a finite poset, set $k := \mathrm{height}(X, \succcurlyeq) + 1$, and for each $i \in [k]$, take an arbitrary linear order $\trianglerighteq_i$ on $\mathfrak{r}_{\succcurlyeq}^{-1}(i)$. Next, define the binary relation $\trianglerighteq$ on X as $x \trianglerighteq y$ iff either $\mathfrak{r}_{\succcurlyeq}(x) > \mathfrak{r}_{\succcurlyeq}(y)$ or $\mathfrak{r}_{\succcurlyeq}(x) = \mathfrak{r}_{\succcurlyeq}(y)$ and $x \trianglerighteq_{\mathfrak{r}_{\succcurlyeq}(x)} y$. It is readily checked that $\trianglerighteq$ is a linear order on X that extends $\succcurlyeq$.

Height of a Poset. The **height** of a poset $(X, \succcurlyeq)$ – we denote this number by $\text{height}(X, \succcurlyeq)$ – is one less than the size of any longest $\succcurlyeq$-chain in X, that is,

$$\text{height}(X, \succcurlyeq) := \sup\{\mathfrak{r}_{\succcurlyeq}(x) : x \in X\} - 1.$$

For instance, the heights of the posets whose Hasse diagrams are given in Figure 7.1.1 are 3 and 4, respectively.

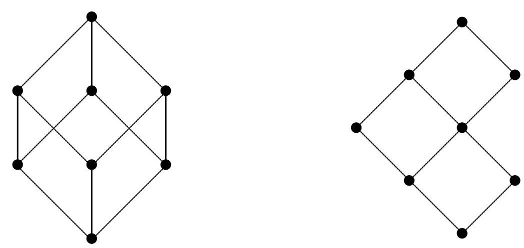

Figure 7.1.1

Example 7.1.1. $\text{height}(X, \triangle_X) = 0$ for any nonempty set X.

Example 7.1.2. $\text{height}(2^X, \supseteq) = |X|$ for any nonempty set X.

Example 7.1.3. $\text{height}(\mathbb{N}, \geq) = \infty$. On the other hand, if $(\mathbb{N} \times [k], \succcurlyeq)$ is the product of $(\mathbb{N}, \triangle_{\mathbb{N}})$ and $([k], \geq)$, we have $\text{height}(\mathbb{N} \times [k], \succcurlyeq) = k - 1$ for every $k \in \mathbb{N}$.

Posets of (Locally) Finite Height. Obviously, the size of every chain in a poset of finite height is finite. The following example shows that the converse of this false.

Example 7.1.4. Let $A_0 := \{0\}$, $A_1 := \{1, 2\}$, $A_2 := \{3, 4, 5\}$, and so on. In turn, define $X_k := \{-k\} \times A_k$, $k = 0, 1, ...$, and set $X := X_0 \cup X_1 \cup \cdots$. Then, $(X, \geq)$ is a poset of infinite height, and yet the cardinality of every $\geq$-chain in X is finite.

This example suggests the following generalization of the notion of "posets of finite height."

Definition. A poset $(X, \succcurlyeq)$ is said to be of **locally finite height** if for every x and y in X with $x \succ y$, we have $\text{height}(x^{\downarrow} \cap y^{\uparrow}, \succcurlyeq) < \infty$.

Put differently, in a poset $(X, \succcurlyeq)$ of locally finite height, for every x and y in X with $x \succ y$, there is a number $K_{x,y}$ such that the size of any chain between x and y is at most $K_{x,y}$. (For instance, the poset of Example 7.1.4 is of locally finite height.) We shall work with such posets frequently in what follows.

7.2 Graded Posets

Gradation of a Poset. Let $(X, \succcurlyeq)$ be a poset. For any two elements x and y of X such that x $\succcurlyeq$-covers y, we have $x \succ y$, and hence $\mathfrak{r}_{\succcurlyeq}(x) \geq \mathfrak{r}_{\succcurlyeq}(y) + 1$. But, in general, equality need not hold in this expression. For instance, relative to the poset whose Hasse diagram is given in the right part of Figure 7.2.1, we have $\mathfrak{r}_{\succcurlyeq}(x) = 3$ and $\mathfrak{r}_{\succcurlyeq}(y) = 1$. On the other hand, it is still possible to measure the "levels" of the elements in this poset by assigning an integer to each element such that the "level" of any element is exactly one more than that of any element that it covers. (Assign, for instance, 3 to x, 2 to both y and z, and 1 to w.) By contrast, no such assignment is possible in the context of the poset whose Hasse diagram appears in the left part of Figure 7.2.1.

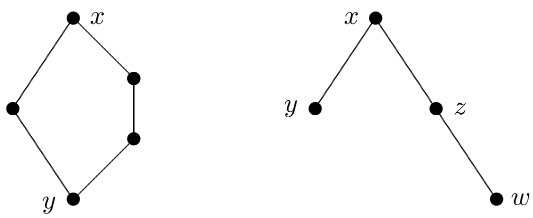

Figure 7.2.1

Definition. Let $(X, \succcurlyeq)$ be a poset. A map $g : X \to \mathbb{Z}$ is said to be a **grade function** for $(X, \succcurlyeq)$ if

$$g(x) > g(y) \quad \text{whenever} \quad x \succ y$$

and

$$g(x) = g(y) + 1 \quad \text{whenever} \quad x \succcurlyeq\text{-covers } y.$$

If there exists such a function, we say that $(X, \succcurlyeq)$ is a **graded poset**. If the $\succcurlyeq$-rank function on X is a grade function for $(X, \succcurlyeq)$, we say that $(X, \succcurlyeq)$ is **graded by rank**.

Thus, while the first poset in Figure 7.2.1 is not graded, the second poset there is graded, but it is not graded by rank. Here are some other examples.

Example 7.2.1. While $(\mathbb{N}, \geq)$ is graded by rank, $([0, 1], \geq)$ is not a graded poset.

Example 7.2.2. For any positive integer n, $(\mathbb{N}^n, \geq)$ is a graded poset. For instance, $(a_1, ..., a_n) \mapsto a_1 + \cdots + a_n$ is a grade function for $(\mathbb{N}^n, \geq)$. In fact, $(\mathbb{N}^n, \geq)$ is graded by rank.

Example 7.2.3. Consider the poset $(\mathbb{N}, \succcurlyeq)$ where $x \succcurlyeq y$ iff y is a divisor of x. This poset is graded. For instance, assigning to each positive integer x the number of prime factors of x (counted by multiplicity) yields a grade function for this poset. (Is this poset graded by rank?)

Example 7.2.4. Let X be a linear space and $\mathcal{X}$ the collection of all linear subspaces of X. Then, $(\mathcal{X}, \supseteq)$ is a graded poset. For instance, the map $g : \mathcal{X} \to \mathbb{N}$, defined by $g(Y) := \dim Y$, is a grade function for $(\mathcal{X}, \supseteq)$. In fact, this function is none other than the $\supseteq$-rank function on $\mathcal{X}$, so $(\mathcal{X}, \supseteq)$ is graded by rank.

The Jordan-Dedekind Chain Condition. In the context of finite posets, there is a useful characterization of being graded by rank in terms of the following property:

Definition. A poset $(X, \succcurlyeq)$ is said to satisfy the **Jordan-Dedekind chain condition** if for every x and y in X with $x \succ y$, all $\supseteq$-maximal $\succcurlyeq$-chains in X whose $\succcurlyeq$-minimum is y and $\succcurlyeq$-maximum is x have the same finite height.

Given any poset $(X, \succcurlyeq)$ and $x, y \in X$ with $x \succ y$, it is convenient to refer to a $\succcurlyeq$-chain S in X with $\min(S, \succcurlyeq) = \{x\}$ and $\max(S, \succcurlyeq) = \{y\}$ as a "$\succcurlyeq$-chain in X from y to x." Thus, if $(X, \succcurlyeq)$ satisfies the Jordan-Dedekind chain condition, we can say that the size of any two $\supseteq$-maximal $\succcurlyeq$-chains in X from y to x are finite and the same. In particular, when the Jordan-Dedekind chain condition is met, within the collection of $\succcurlyeq$-chains in X from y to x, the notions of being "$\supseteq$-maximal" and "longest" coincide. Thus, if $(X, \succcurlyeq)$ satisfies the Jordan-Dedekind chain condition, then it is of locally finite height.

It turns out that the Jordan-Dedekind chain condition is necessary and sufficient for the gradebility of a finite poset with a minimum element by means of its rank function. In fact, we have:

Proposition 7.2.1. Let $(X, \succcurlyeq)$ be a poset of locally finite height with a $\succcurlyeq$-minimum element. Then, $(X, \succcurlyeq)$ is graded by rank if, and only if, it satisfies the Jordan-Dedekind chain condition.

Proof. Suppose $(X, \succcurlyeq)$ is graded by rank. Take any $x, y \in X$ with $x \succ y$, and let S be a $\supseteq$-maximal $\succcurlyeq$-chain in X from y to x. As $(X, \succcurlyeq)$ is of locally finite height, there is a positive integer k such that we can enumerate S as $\{z_0, ..., z_k\}$ with $x = z_k \succ \cdots \succ z_0 = y$. As z_i $\succcurlyeq$-covers z_{i-1}, we have $\mathfrak{r}_{\succcurlyeq}(z_i) = \mathfrak{r}_{\succcurlyeq}(z_{i-1}) + 1$ for each $i \in [k]$. Thus, $\mathfrak{r}_{\succcurlyeq}(x) - \mathfrak{r}_{\succcurlyeq}(y) = k$, while, of course, $\text{height}(S, \succcurlyeq) = k$. It follows that the height of every $\supseteq$-maximal $\succcurlyeq$-chain in X from y to x equals $\mathfrak{r}_{\succcurlyeq}(x) - \mathfrak{r}_{\succcurlyeq}(y)$, thereby establishing that $(X, \succcurlyeq)$ satisfies the Jordan-Dedekind chain condition.

Conversely, suppose $(X, \succcurlyeq)$ satisfies the Jordan-Dedekind chain condition. Take any $x, y \in X$ such that x $\succcurlyeq$-covers y. Let x_* be the $\succcurlyeq$-minimum element of X. Obviously, there is a $\supseteq$-maximal $\succcurlyeq$-chain S in X from x_* to y such that $|S| = \mathfrak{r}_{\succcurlyeq}(y)$. But then $S \cup \{x\}$ is a $\supseteq$-maximal $\succcurlyeq$-chain in X from x_* to x (because x $\succcurlyeq$-covers y). As every such $\succcurlyeq$-chain must have the same size by the Jordan-Dedekind chain condition, and $\mathfrak{r}_{\succcurlyeq}(x)$ is the size of one such $\succcurlyeq$-chain, we must have $\mathfrak{r}_{\succcurlyeq}(x) = |S| + 1$, that is, $\mathfrak{r}_{\succcurlyeq}(x) = \mathfrak{r}_{\succcurlyeq}(y) + 1$.

7.3 Width

In a manner of speaking, the height of a poset tells us how "tall" we need to draw the Hasse diagram of that poset. By contrast, the next item in our agenda tells us how "wide" the associated Hasse diagram will have to be.

Width of a Poset. The **width** of a preordered set $(X, \succsim)$ – we denote this by $\text{width}(X, \succsim)$ – is the size of a longest $\succsim$-antichain in X, that is,

$$\text{width}(X, \succsim) := \sup\{|Y| : Y \text{ is an } \succsim\text{-antichain in } X\}.$$

For instance, the widths of the posets whose Hasse diagrams are given in Figure 7.1.1 are 3 and 2, respectively.

Example 7.3.1. $\text{width}(X, \triangle_X) = |X|$ and $\text{width}(X, X \times X) = 1$ for any nonempty set X.

Example 7.3.2. $\text{width}(X, \succcurlyeq) = 1$ for any loset $(X, \succcurlyeq)$.

Example 7.3.3. $\text{width}(\mathbb{R}^n, \geq) = \infty$, provided that $n > 1$.

Sperner's Theorem. The computation of the widths of special finite posets is a celebrated subfield of combinatorics. While this is not the place to delve into this topic, we next provide a brief illustration by computing the width of the poset of all subsets of an arbitrary finite set – this computation was performed first by Emanuel Sperner in 1928.

Sperner's Theorem. For any nonempty finite set X,

$$\text{width}(2^X, \supseteq) = \binom{|X|}{\lfloor |X| / 2 \rfloor}.$$

Clearly, for every positive integer k less than $|X|$, the collection of all k-element subsets of X is an $\supseteq$-antichain in 2^X, and the size of this collection is $\binom{|X|}{k}$. It is also easy to verify that the maximum of these numbers is $\binom{|X|}{\lfloor |X|/2 \rfloor}$, so there is certainly an $\supseteq$-antichain in 2^X with cardinality $\binom{|X|}{\lfloor |X|/2 \rfloor}$. Sperner's Theorem tells us that one cannot find a more crowded $\supseteq$-antichain in 2^X.

The modern proof of this celebrated result of combinatorics, which is due to David Lubell, is streamlined by means of the following inequality:

The Lubell-Yamamoto-Meshalkin Inequality. Let X be a nonempty finite set and $\mathcal{A}$ a collection of subsets of X such that no member of $\mathcal{A}$ contains another member of $\mathcal{A}$. If there are a_i many elements of $\mathcal{A}$ that contain i many elements, $i \in [|X|]$, then

$$\sum_{i \in [|X|]} \frac{a_i}{\binom{|X|}{i}} \leq 1.$$

Proof. Let us enumerate X as $\{x_1, ..., x_n\}$. For each member A of $\mathcal{A}$, let $\rho(A)$ be the set of all bijective maps f from $[n]$ onto X with $f([|A|]) = A$. (That is, any member of $\rho(A)$ is a permutation of the elements of X in which the first $|A|$ many elements come from A.) The key observation is that $\rho(A)$ and $\rho(B)$ are disjoint for any two distinct members of $\mathcal{A}$, because no member of $\mathcal{A}$ is contained in any other member of $\mathcal{A}$. Therefore, the number of elements in $\bigcup\{\rho(A) : A \in \mathcal{A}\}$ is

$$\sum_{A \in \mathcal{A}} |\rho(A)| = \sum_{A \in \mathcal{A}} |A|!(n - |A|)! = \sum_{i \in [n]} a_i i!(n-i)!.$$

As the total number of permutations of the elements of X is $n!$, therefore,

$$\sum_{i \in [n]} a_i i!(n-i)! \leq n!,$$

and dividing both sides of this inequality by $n!$ completes our proof.

We are now ready for:

Proof of Sperner's Theorem. Let X be a nonempty finite set, and set $n := |X|$. Let $\mathcal{A}$ be an $\supseteq$-antichain in 2^X. As $\binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{i}$, the Lubell-Yamamoto-Meshalkin Inequality gives us that

$$\sum_{i \in [n]} \frac{a_i}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{i \in [n]} \frac{a_i}{\binom{n}{i}} \leq 1,$$

where, for each i in $[n]$, a_i denotes the number of members of $\mathcal{A}$ with i elements. Hence

$$|\mathcal{A}| = \sum_{i \in [n]} a_i \leq \binom{n}{\lfloor n/2 \rfloor},$$

which proves that $\mathrm{width}(2^X, \supseteq) \leq \binom{n}{\lfloor n/2 \rfloor}$. As we have noted above, there is an $\supseteq$-antichain in 2^X with $\binom{n}{\lfloor n/2 \rfloor}$ elements, so our proof is complete.

7.4 Basic Properties of Finite Posets

Height-Width Inequality. The following basic result relates the size of a finite poset to its height and width.

Proposition 7.4.1. Let m and n be positive integers and $(X, \succcurlyeq)$ a finite poset with $|X| \geq mn + 1$. Then, either $\mathrm{height}(X, \succcurlyeq) \geq m$ or $\mathrm{width}(X, \succcurlyeq) \geq n + 1$.

Proof. Let $k := \mathrm{height}(X, \succcurlyeq) + 1$, and recall that $\{\mathfrak{r}_{\succcurlyeq}^{-1}(i) : i \in [k]\}$ is a partition of X. Therefore,

$$\sum_{i \in [k]} \left|\mathfrak{r}_{\succcurlyeq}^{-1}(i)\right| = |X| \geq mn + 1,$$

and hence, either $k > m$ or $\left|\mathfrak{r}_{\succcurlyeq}^{-1}(i)\right| > n$ for some $i \in [k]$. As each nonempty $\mathfrak{r}_{\succcurlyeq}^{-1}(i)$ is an $\succcurlyeq$-antichain in X, our assertion is proved.

As simple as it is, this observation is quite informative. For instance, as an immediate consequence, we find:

Corollary 7.4.2. Either the height or the width of an infinite poset is infinite.

> **Note.** More is true: There is either an infinite chain or an infinite antichain in every infinite poset. Proving this, however, requires using the Axiom of Choice. (See Section 1 of Chapter 4.)

The Erdös-Szerekes Theorem. We conclude this section with another application of Proposition 7.4.1:

The Erdös-Szerekes Theorem. For any positive integer n, every $(n^2 + 1)$-vector of real numbers contains a monotonic $(n + 1)$-subvector.

Proof. Fix any positive integer n, and take any real numbers $x_1, ..., x_{n^2+1}$. Consider the poset $(X, \geq)$ where $X := \{(x_i, i) : i \in [n^2 + 1]\}$ and $(x_i, i) \geq (x_j, j)$ iff $x_i \geq x_j$ and $i \geq j$. Notice that a $\geq$-chain of X is an increasing subvector of $(x_1, ..., x_{n^2+1})$ and an $\geq$-antichain of X is a decreasing subvector of $(x_1, ..., x_{n^2+1})$. Thus, applying Proposition 7.4.1 to $(X, \geq)$ yields the result.

Exercises

7.1. (*A Generalization of Proposition 6.1*) We define the **height** of a preordered set $(X, \succsim)$ as the height of the poset $(X/_{\sim}, \succcurlyeq)$ induced by $(X, \succsim)$. Prove that every preordered set of finite height has a maximal and a minimal element.

7.2. Let $(X, \succcurlyeq)$ be a poset that is bounded from below, and define $\mathcal{X} := \{S \in 2^X_{\downarrow} : |S| < \infty\}$. Show that $(\mathcal{X}, \supseteq)$ is a graded poset.

7.3. Let $(X, \succcurlyeq)$ be a poset and $g : X \to \mathbb{Z}$ a map such that $g(x) > g(y)$ for every $x, y \in X$ with $x \succ y$. Prove: g is a grade function for $(X, \succcurlyeq)$ iff for every $x, y \in X$ with $x \succ y$, there is a $z \in g^{-1}(g(x) - 1)$ such that $x \succ z \succcurlyeq y$.

7.4. Let $(X, \succcurlyeq)$ be a bounded finite poset. Prove that $(X, \succcurlyeq)$ is graded iff all $\supseteq$-maximal $\succcurlyeq$-chains in X have the same height.

7.5. Prove: For any positive integers m and n, every $(mn + 1)$-vector of real numbers contains either an increasing $(m + 1)$-subvector or a decreasing $(n + 1)$-subvector.

7.6. Give an example of a poset of infinite width with no infinite antichains.

7.7. Let $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ be two posets with $X \cap Y = \varnothing$. Show that

$$\text{height}(X \cup Y, \succcurlyeq_{X \oplus Y}) = \text{height}(X, \succcurlyeq_X) + \text{height}(Y, \succcurlyeq_Y) + 1$$

and

$$\text{width}(X \cup Y, \succcurlyeq_{X \oplus Y}) = \max\{\text{width}(X, \succcurlyeq_X), \text{width}(Y, \succcurlyeq_Y)\}.$$

7.8. Let $(X \times Y, \succcurlyeq)$ be the product of two posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$. Prove that

$$\text{height}(X \times Y, \succcurlyeq) = \text{height}(X, \succcurlyeq_X) + \text{height}(Y, \succcurlyeq_Y).$$

Does a similar formula hold for the width of the product of two posets?

7.9. Let $(X \times Y, \succcurlyeq)$ be the product of two graded posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$. Show that $(X \times Y, \succcurlyeq)$ is graded.

7.10. (Freese-Hyndman-Nation) Let $(X, \succcurlyeq)$ be a finite poset and $n := |X|$. Prove that there exists a subset S of X such that $|S| \geq \lceil n^{1/3} \rceil$, and S is either a principal ideal, or a principal filter, in $(X, \succcurlyeq)$ or we have $x^{\uparrow} \cap y^{\uparrow} = \varnothing = x^{\downarrow} \cap y^{\downarrow}$ for every distinct x and y in S.

7.10. (Open Problem) Let n be a positive integer, and X a set with $n \geq 1$ elements. The number of distinct $\supseteq$-antichains in 2^X is called the nth **Dedekind number**, and is denoted by $D(n)$. At the time of this writing, only the first 8 Dedekind numbers are computed exactly.[2] The problem of finding a formula for $D(n)$ for any $n \geq 1$ is known as **Dedekind's Problem**.

8 Suprema and Infima

The notion of supremum of a set of real numbers – that is, the smallest of all numbers greater than the numbers in that set – plays an essential role in the order-theoretic construction of the real number system. This alone is enough of a motivation to explore this notion in the general context of posets. We will do this in this section. It will become clear later that much of order theory takes its cue from the notions of supremum and infimum.

8.1 Definitions and Examples

Upper and Lower Bounds for a Set. Let us begin with introducing the following bit of notation.

Notation. In what follows, for any preordered set $(X, \succsim)$, $x \in X$, and any subset S of X, we write $x \succsim S$ to mean $x \succsim \omega$ for every $\omega \in S$. The statement $S \succsim x$ is similarly interpreted.

Definition. Let $(X, \succsim)$ be a preordered set. For any subset S of X, an element x in X is said to be an **$\succsim$-upper bound** for S if $x \succsim S$. Dually, x is said to be a **$\succsim$-lower bound** for S if $S \succsim x$.

[2] These are 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788.

Supremum and Infimum of a Set. Among all upper bounds of a set S in a poset the one that is nearest to S (from the perspective of order theory) is the smallest of all those upper bounds. This smallest upper bound belongs to S iff it is the maximum element in S. On the other hand, when S lacks a maximum element, the smallest of all upper bounds for S, if it exists, is the best proxy of S from above. This prompts the following:

Definition. Let $(X, \succcurlyeq)$ be a poset. For any subset S of X, the **$\succcurlyeq$-supremum** of S (in X) is the $\succcurlyeq$-minimum element in the set of all $\succcurlyeq$-upper bounds for S, provided that such an element exists. Put more precisely, an element x of X is the $\succcurlyeq$-supremum of S (in X) if

- $x \succcurlyeq S$; and
- $y \succcurlyeq S$ implies $y \succcurlyeq x$ (for any $y \in X$).

(If there does not exist such an element x in X, we then say that "the $\succcurlyeq$-supremum of S does not exist.") The **$\succcurlyeq$-infimum** of S (in X) is defined dually (as the $\succcurlyeq$-maximum of all $\succcurlyeq$-lower bounds for S).

As there can be at most one maximum element in a poset, a supremum of a set (in a poset) is unique, provided that it exists. This is why we can talk about "the" supremum of a set. (The same goes also for "the" infimum of a set.)

Notation. For any poset $(X, \succcurlyeq)$ and a subset S of X, it is common to use the notation

$$\bigvee S \quad \text{and} \quad \bigwedge S$$

for the $\succcurlyeq$-supremum and $\succcurlyeq$-infimum of S (in X), respectively. While writing $\bigvee_X S$ and $\bigwedge_X S$ for these elements would be more informative, so long as one keeps in mind the ambient poset $(X, \succcurlyeq)$ – in particular, notes that $\bigvee S$ and $\bigwedge S$, when they exist, are elements of X, but not necessarily of S – using the simpler notation would not cause a problem.

Note. For any poset $(X, \succcurlyeq)$, and any subset S of X,

$$x \succcurlyeq S \text{ implies } x \succcurlyeq \bigvee S \quad \text{and} \quad S \succcurlyeq x \text{ implies } \bigwedge S \succcurlyeq x,$$

provided that $\bigvee S$ and $\bigwedge S$ exist. Furthermore, $\bigwedge S$ is the $\preccurlyeq$-supremum of S, provided that it exists.

Notation. For any poset $(X, \succcurlyeq)$ and elements x and y of X, the $\succcurlyeq$-supremum and $\succcurlyeq$-infimum of $\{x, y\}$ (in X) are commonly denoted as $x \vee y$ and $x \wedge y$, respectively. That is,

$$x \vee y := \bigvee \{x, y\} \quad \text{and} \quad x \wedge y := \bigwedge \{x, y\}.$$

Put precisely, if it exists, $x \vee y$ is the unique element of X such that $x \vee y \succcurlyeq \{x, y\}$ and $z \succcurlyeq x \vee y$ for every $z \in X$ with $z \succcurlyeq \{x, y\}$, and similarly for $x \wedge y$. It is often easy to identify such elements of a finite poset in the Hasse diagram of that poset. Figure 8.1.1 provides a simple illustration.

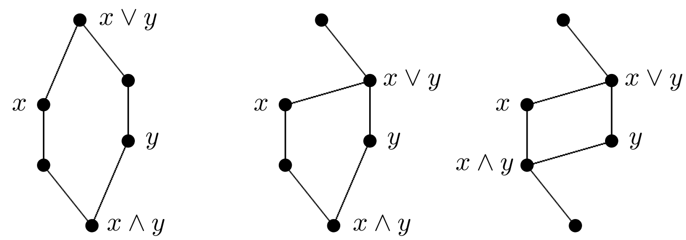

Figure 8.1.1

Examples.

Example 8.1.1. Let $(X, \succcurlyeq)$ be a poset. Then, for any $x, y \in X$,

$$x \vee y = x \quad \text{iff} \quad x \succcurlyeq y \quad \text{iff} \quad x \wedge y = y.$$

Example 8.1.2. Let X be a set with $|X| \geq 2$. Then, relative to the poset $(X, \triangle_X)$, neither $\bigvee X$ nor $\bigwedge X$ exist.

Example 8.1.3. Let $(X, \succcurlyeq)$ be a poset. Then, every element of X is an $\succcurlyeq$-upper bound for $\varnothing$. Consequently, $\bigvee \varnothing$ exists iff there is a $\succcurlyeq$-minimum element in X, and when there is a $\succcurlyeq$-minimum element in X, we have

$$\bigvee \varnothing = \text{ the } \succcurlyeq\text{-minimum element in } X.$$

(Yes?) Similarly,

$$\bigwedge \varnothing = \text{ the } \succcurlyeq\text{-maximum element in } X,$$

provided that a $\succcurlyeq$-maximum element in X exists. (If $\max(X, \succcurlyeq) = \varnothing$, then $\bigwedge \varnothing$ does not exist.)

Example 8.1.4. Let X be a nonempty set. Relative to the poset $(2^X, \supseteq)$, we have

$$\bigvee \mathcal{A} = \bigcup \mathcal{A} \quad \text{and} \quad \bigwedge \mathcal{A} = \bigcap \mathcal{A}$$

for any subset $\mathcal{A}$ of 2^X (provided that we adopt the convention that $\bigcup \varnothing = \varnothing$ and $\bigcap \varnothing = X$).

The formulas in Example 8.1.4 hold when the underlying poset is $(\mathcal{X}, \supseteq)$ for some nonempty collection $\mathcal{X}$ of sets that is closed under taking (arbitrary) unions and intersections. But they would have to be modified if $\mathcal{X}$ does not possess these closure properties. This points to a crucial difference between the maximum and supremum of a set S in a poset: The former depends only on the contents of S while for the latter elements outside S are also relevant (and similarly for the minimum and infimum of S.)

For instance, consider the posets $(2^{\{1,2,3\}}, \supseteq)$ and $(\{\{1\},\{2\},\{1,2,3\}\}, \supseteq)$. There is no $\supseteq$-maximum element of the set $\{\{1\},\{2\}\}$ regardless of which poset is under consideration. (The statement $\max(\{\{1\},\{2\}\}, \supseteq) = \varnothing$ is meaningful as is.) By contrast, we have $\{1\} \vee \{2\} = \{1,2\}$ relative to the first poset and $\{1\} \vee \{2\} = \{1,2,3\}$ relative to the second one. (The $\supseteq$-supremum of $\{\{1\},\{2\}\}$ cannot be determined unless we know the ground set of the underlying poset.)

Example 8.1.5. Consider the poset $(\mathbb{R}^n, \geq)$, and let S be a nonempty compact subset S of $\mathbb{R}^n$. Then,

$$\bigvee S = (\max S_1, ..., \max S_n)$$

where, for each i in $[n]$, we have $S_i := \{x_i \in \mathbb{R} : (x_i, \mathbf{x}_{-i}) \in S \text{ for some } \mathbf{x}_{-i} \in \mathbb{R}^{n-1}\}$, and similarly for $\bigwedge S$. (See Figure 8.1.2.)

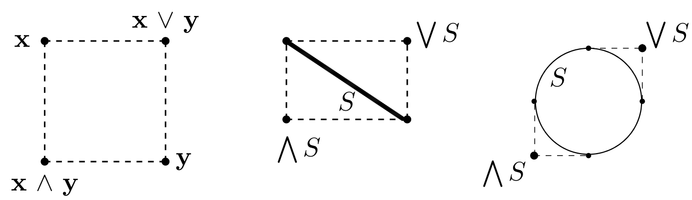

Figure 8.1.2

Of course, $\bigvee S$ need not exist if S is not compact, because S may not have an $\geq$-upper bound in that case. (For instance, neither $\bigvee \mathbb{R}^n_+$ nor $\bigvee \mathbb{R}$ exist.)

Example 8.1.6. For each positive integer m, define $f_m \in \mathbf{C}[0,1]$ by $f_m(t) := t^m$, and let $S := \{f_1, f_2, ...\}$. Then, relative to the poset $(\mathbf{C}[0,1], \geq)$, we have $\bigvee S = f_1$, but $\bigwedge S$ does not exist. (Why?) On the other hand, $\bigwedge\{f_1, ..., f_k\} = f_k$ for each positive integer k.

Example 8.1.7. Let X and Y be nonempty sets, and consider the partial function poset $([Y^X], \sqsupseteq)$ of Example 3.1.6. For subsets S and T of X and any $f, g \in Y^X$, we have $f|_S \wedge g|_T = f_{S \cap T}$ in the context of this poset. However, when $\min\{|X|, |Y|\} \geq 2$, the $\sqsupseteq$-supremum of two of the elements of $[Y^X]$ need not exist. For instance, for any distinct x and y in X, if we pick f and g in Y^X such that $f(x) \neq g(x) = g(y)$, then there is no $\sqsupseteq$-upper bound for $\{f|_{\{x\}}, g|_{\{x,y\}}\}$.

Exercises

8.1.1. Consider the poset $(\mathbb{N}, \succcurlyeq)$ where $x \succcurlyeq y$ iff y is a divisor of x. Take any two positive integers x and y, and show that, relative to this poset, $x \vee y$ is the lowest common multiple of x and y, and $x \wedge y$ is the greatest common divisor of x and y.

8.1.2. Let $(X, \succcurlyeq)$ be a poset of locally finite height. Take any elements a, x and y of X such that $a \bowtie x$, $a \bowtie y$, and x uniquely $\succcurlyeq$-covers y. Show that if $a \wedge y$ exists, then so does $a \wedge x$, and we have $a \wedge x = a \wedge y$.

8.1.3. Let $(X, \succcurlyeq)$ be a poset and S and T two subsets of X such that $\bigvee S$ and $\bigvee T$ exist. Prove that $\bigvee(S \cup T) = \bigvee S \vee \bigvee T$, provided that either side of this equation exists.

8.1.4. Let $(X, \succcurlyeq)$ be a poset and S a nonempty subset of X. Show that $\bigvee S = \bigvee S^{\downarrow}$, provided that either side of this equation exists.

8.1.5. Let $(X, \succcurlyeq)$ be a poset. Consider the binary relation $\succeq$ on X defined by $x \succeq y$ iff for every $\succcurlyeq$-directed (nonempty) subset D of X such that $\bigvee D$ exists,

$$\bigvee D \succcurlyeq x \quad \text{implies} \quad \omega \succcurlyeq y \text{ for some } \omega \in D.$$

Prove that $\succeq$ is antisymmetric and transitive, and give an example to show that it need not be reflexive. Also show that, if every nonempty subset of X has a $\succcurlyeq$-maximal element, then $\succeq$ is a partial order.

8.2 Conditionally Complete Posets

Conditional Completeness. Recall that every nonempty subset of real numbers that is bounded from above has a supremum. In fact, this property – often called the "completeness axiom" – singles $\mathbb{R}$ out among all ordered fields (up to ordered field isomorphism). It is thus natural to formulate this property in the general context of posets.

Definition. A poset $(X, \succcurlyeq)$ is said to be **conditionally complete** (or *Dedekind complete*) if $\bigvee S$ exists for every nonempty subset S of X such that $x \succcurlyeq S$ holds for at least one x in X.

In words, a poset $(X, \succcurlyeq)$ is conditionally complete iff every nonempty subset of X with an $\succcurlyeq$-upper bound in X has a $\succcurlyeq$-supremum in X.

Examples.

Example 8.2.1. For any nonempty set X, the poset $(X, \triangle_X)$ is conditionally complete.

Example 8.2.2. For any nonempty set X, the poset $(2^X, \supseteq)$ is trivially conditionally complete. (Indeed, every nonempty subset of 2^X has a $\supseteq$-supremum.)

Example 8.2.3. $\mathbb{Z}$ is conditionally complete, because any nonempty set of integers that is bounded from above must have a maximum element. On the other hand, $\mathbb{Q}$ is not conditionally complete. For instance, $\{r \in \mathbb{Q} : r^2 < 2\}$ is a subset of $\mathbb{Q}$ that is bounded from above, and yet, a bit of algebra shows that if q is the $\geq$-supremum of this set in $\mathbb{Q}$, we must have $q^2 = 2$, a property no rational number can satisfy.

Example 8.2.4. Let X be a nonempty set and $\mathbf{B}(X)$ the Banach space of all bounded real maps on X. (Recall Example 2.1.4 of Appendix.) Consider the partial order $\geq$ defined on $\mathbf{B}(X)$ pointwise: $f \geq g$ iff $f(x) \geq g(x)$ for each $x \in X$. We claim that $(\mathbf{B}(X), \geq)$ is conditionally complete. Indeed, if $\mathcal{F}$ is a nonempty set of bounded real maps on X such that $h \geq \mathcal{F}$ for some h in $\mathbf{B}(X)$, then, the map $f^* : X \to \mathbb{R}$ with

$$f^*(x) := \sup\{f(x) : f \in \mathcal{F}\},$$

is well-defined (because $\mathbb{R}$ is conditionally complete). Furthermore, f^* belongs to $\mathbf{B}(X)$, because $f^*(x) \leq h(x) \leq \|h\|_\infty$ for each x in X. It follows that f^* is the $\geq$-supremum of $\mathcal{F}$ in $\mathbf{B}(X)$, and our claim is proved.

Dual of Conditional Completeness. The dual of the conditional completeness property would require every nonempty set with a lower bound in a poset to have an infimum. As one would expect, conditional completeness already guarantees this dual property to hold.

Proposition 8.2.1. Let $(X, \succcurlyeq)$ be a conditionally complete poset. Then, $\bigwedge T$ exists for every nonempty subset T of X such that $T \succcurlyeq x$ holds for some $x \in X$.

Proof. Let T be a nonempty subset of X with a $\succcurlyeq$-lower bound in X, and define $S := \{x \in X : T \succcurlyeq x\}$. Then S is a nonempty subset of X with an $\succcurlyeq$-upper bound, so, by conditional completeness, $\bigvee S$ exists in X. As $T \succcurlyeq x$ for every $x \in S$, we have $t \succcurlyeq S$ for every $t \in T$, and hence, $T \succcurlyeq \bigvee S$, that is, $\bigvee S$ is a $\succcurlyeq$-lower bound for T in X. Suppose $z \in X$ is another

$\succcurlyeq$-lower bound for T, that is, $T \succcurlyeq z$. Then, $z \in S$, and hence $\bigvee S \succcurlyeq z$. It follows that $\bigwedge T$ exists in X and equals $\bigvee S$.

The following is an immediate consequence of this observation.

Corollary 8.2.2. If $(X, \succcurlyeq)$ is a conditionally complete poset, then so is $(X, \preccurlyeq)$.

Example 8.2.5. $(\mathbf{C}[0,1], \geq)$ is not conditionally complete. For each integer $m \geq 2$, let f_m be the map that is equal to 1 on $[\frac{1}{2}, 1]$, to 0 on $[0, \frac{1}{2} - \frac{1}{m}]$, and to $m(x - \frac{1}{m}) + 1$ on $[\frac{1}{2} - \frac{1}{m}, \frac{1}{2}]$. Then, the zero function is a $\geq$-lower bound for $\{f_2, f_3, ...\}$ but there is no $\geq$-infimum of $\{f_2, f_3, ...\}$ in $\mathbf{C}[0,1]$. Applying Proposition 8.2.1 establishes our claim.

Products of Conditionally Complete Posets. The property of conditional completeness behaves well with respect to taking the product of two posets. Put precisely:

Proposition 8.2.3. The product $(X \times Y, \succcurlyeq)$ of two posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ is conditionally complete if, and only if, both $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ are conditionally complete.

Proof. For any subset S of $X \times Y$, we have $\bigvee S = (\bigvee A, \bigvee B)$, where A is the projection of S in X (that is, $A = \{x \in X : (x, y) \in S \text{ for some } y \in Y\}$) and B is the projection of S in Y. In particular, for any subsets A and B of X and Y, respectively, we have $\bigvee(A \times B) = (\bigvee A, \bigvee B)$. Our claim is an obvious consequence of this observation.

Example 8.2.6. Both $(\mathbb{Z}^n, \geq)$ and $(\mathbb{R}^n, \geq)$ are conditionally complete posets.

8.3 Chain-Complete Posets

Chain-Completeness. Most posets encountered in practice contains several subsets with no suprema and infima. Far more practical would be to ask for the existence of suprema and infima for certain "well-behaved" subsets of a poset. Indeed, the notion of conditional completeness is based on this premise where one regards "well-behaved" those sets that are order-bounded from above. Of course, as we alter what we mean by "well-behaved" here, alternative completeness conditions are obtained. For instance, if we wish to think of finite sets as well-behaved, then we are led to the notion of "lattice," which is the subject matter of the next chapter. Alternatively, it seems reasonable to consider those subsets of a poset on which the involved partial order acts as a linear order as having an appealing structure. Asking for the existence of suprema for such sets leads us to another useful notion of completeness for posets. (As we shall see in Chapter 6, this notion plays an indispensable role in order-theoretic fixed point theory.)

Definition. A poset $(X, \succcurlyeq)$ is said to be **chain-complete** if every $\succcurlyeq$-chain in X has a $\succcurlyeq$-supremum.

Remark. There is another, more encompassing way of looking at chain-completeness for a poset. Indeed, Markowsky (1976) has proved that a poset $(X, \succcurlyeq)$ is chain-complete iff every nonempty $\succcurlyeq$-directed subset D of X has a $\succcurlyeq$-supremum in X. (*Reminder.* Given a poset $(X, \succcurlyeq)$, we say that a nonempty subset D of X is $\succcurlyeq$**-directed** if for every x and y in D there is a $z \in D$ with $z \succcurlyeq \{x, y\}$ – recall Exercise 3.2.1.) We will be in a position to establish this result in Chapter 6.

Chain-Completeness vs. Conditional Completeness. The properties of conditional completeness and chain-completeness are independent, as we illustrate next.

Example 8.3.1. Every finite poset is chain-complete (Proposition 5.1). This observation shows that chain-completeness does not imply conditional completeness. For instance, where $\mathbf{e}^1$ and $\mathbf{e}^2$ are the unit vectors of $\mathbb{R}^2$, the poset $(\{\mathbf{e}^1, -\mathbf{e}^1, \mathbf{e}^2, -\mathbf{e}^2\}, \geq)$ is not conditionally complete, because $\{-\mathbf{e}^1, -\mathbf{e}^2\}$ is a $\geq$-bounded set in this poset without a $\geq$-supremum.

The converse implication is false as well. Indeed, $\mathbb{R}$ is conditionally complete (by its construction), but it is not chain-complete – for instance, $\sup \mathbb{N}$ does not exist in $\mathbb{R}$.

Examples. The rest of this section provide some illustrative examples of chain-complete posets. We will investigate the notion of chain-completeness in far greater detail later in the text.

Example 8.3.2. For any nonempty set X, the poset $(X, \triangle_X)$ is trivially chain-complete.

Example 8.3.3. Every partial function poset is chain-complete. In particular, $(2^X, \supseteq)$ is chain-complete for any nonempty set X.

Example 8.3.4. $(\mathbf{C}[0,1], \geq)$ is not chain-complete. For instance, where the map $f_m \in \mathbf{C}[0,1]$ is defined by $f_m(t) := t^{1/m}$ for each m, the set $\{f_1, f_2, ...\}$ is a $\geq$-chain in $\mathbf{C}[0,1]$ without a $\geq$-supremum.

The following example, which we will later generalize to cover infinite-dimensional spaces as well, points to a major source of chain-complete posets.

Example 8.3.5. $(X, \geq)$ *is chain-complete for every nonempty compact subset* X *of* $\mathbb{R}^n$. To see this, take any $\geq$-chain S in X, and note first that $\mathrm{cl}(S)$ is a $\geq$-chain in $\mathbb{R}^n$. Indeed, for any points $\mathbf{x}$ and $\mathbf{y}$ in $\mathrm{cl}(S)$, there exist two sequences $(\mathbf{x}^m)$ and $(\mathbf{y}^m)$ in S such that $\mathbf{x}^m \to \mathbf{x}$ and $\mathbf{y}^m \to \mathbf{y}$. Define $M := \{i \in \mathbb{N} : \mathbf{x}^i \geq \mathbf{y}^i\}$ and $N := \{i \in \mathbb{N} : \mathbf{y}^i \geq \mathbf{x}^i\}$. As $M \cup N = \mathbb{N}$, either M or N is an infinite set. In the former case, there exist a strictly increasing sequence (m_k) of integers such that $\mathbf{x}^{m_k} \to \mathbf{x}$, $\mathbf{y}^{m_k} \to \mathbf{y}$ and $\mathbf{x}^{m_k} \geq \mathbf{y}^{m_k}$ for each $k = 1, 2, ...$, and it follows that $\mathbf{x} \geq \mathbf{y}$. As we would similarly find $\mathbf{y} \geq \mathbf{x}$, in the latter case, we may conclude that $\mathbf{x}$ and $\mathbf{y}$ are $\geq$-comparable. Conclusion: $\mathrm{cl}(S)$ is a $\geq$-chain in $\mathbb{R}^n$.

Now, consider the real map f defined on X by $f(\mathbf{x}) := x_1 + \cdots + x_n$. As this map is continuous, and $\mathrm{cl}(S)$ is compact, being a closed subset of the compact set X, there is an $\mathbf{x}^*$ in $\mathrm{cl}(S)$ such that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for every $\mathbf{x}$ in $\mathrm{cl}(S)$. As X is a closed set, we have $\mathrm{cl}(S) \subseteq X$, and hence $\mathbf{x}^* \in X$. Besides, as f is a strictly increasing function, we cannot have $\mathbf{x} > \mathbf{x}^*$ for any $\mathbf{x} \in S$, so, as $\mathrm{cl}(S)$ is a $\geq$-chain in $\mathbb{R}^n$, we have $\mathbf{x}^* \geq S$. Now suppose $\mathbf{y}$ is another n-vector in X with $\mathbf{y} \geq S$. If $x_i^* > y_i$ for some $i \in [n]$, then, as $\mathbf{x}^*$ belongs to $\mathrm{cl}(S)$, there exists an $\mathbf{x} \in S$ with $x_i^* > x_i > y_i$, contradicting $\mathbf{y} \geq S$. Thus, $y_i \geq x_i^*$ for each $i \in [n]$, that is, $\mathbf{y} \geq \mathbf{x}^*$. Conclusion: $\mathbf{x}^* = \bigvee S$.

Our final example comes from within order theory.

Example 8.3.6. *The poset of all chains in a poset is chain-complete.* We can in fact prove something slightly stronger than this. Let $(X, \succcurlyeq)$ be a poset and let Y be either the empty set or a $\succcurlyeq$-chain in X. Let $\mathcal{C}_Y$ stand for the collection of all $\succcurlyeq$-chains in X that contain Y. We claim: $(\mathcal{C}_Y, \supseteq)$ is chain-complete.

To see this, pick an arbitrary $\supseteq$-chain $\mathcal{S}$ in $\mathcal{C}_Y$, and let $S := \bigcup \mathcal{S}$. We wish to show that S belongs to $\mathcal{C}_Y$. To see this, take any x and y in S. Then, $x \in A$ and $y \in B$ for some A and B in $\mathcal{S}$. But, as $\mathcal{S}$ is a $\supseteq$-chain, either $A \subseteq B$ or $B \subseteq A$, which means that either $\{x, y\} \subseteq B$ or $\{x, y\} \subseteq A$. As both A and B are $\succcurlyeq$-chains, therefore, x and y are $\succcurlyeq$-comparable. Conclusion: S is a $\succcurlyeq$-chain in X. As it is obvious that S contains Y, therefore, $S \in \mathcal{C}_Y$. Consequently, S is the $\supseteq$-supremum of $\mathcal{S}$ in $\mathcal{C}_Y$. Thus, in view of the arbitrariness of $\mathcal{S}$, we may conclude that $(\mathcal{C}_Y, \supseteq)$ is chain-complete.

Products of Chain-Complete Posets. Just like conditional completeness, and for the same reason, the chain-completeness property is preserved under taking the product of two (or more) posets. That is:

Proposition 8.3.1. The product $(X \times Y, \succcurlyeq)$ of two posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ is chain-complete if, and only if, both $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ are chain-complete.

Exercises

8.3.1. Let $\mathcal{M}$ be the collection of all subsets of $\mathbb{N}$ which are either finite or cofinite, that is, $\mathcal{M} := \{S \subseteq \mathbb{N} : \min\{|S|, |\mathbb{N}\backslash S|\} < \infty\}$. Show that the poset $(\mathcal{M}, \supseteq)$ is neither chain-complete nor conditionally complete.

8.3.2. Given a nonempty set X and a poset $(Y, \succcurlyeq)$, define the partial order $\trianglerighteq$ on Y^X by $f \trianglerighteq g$ iff $f(x) \succcurlyeq g(x)$ for each $x \in X$. Show that if $(Y, \succcurlyeq)$ is chain-complete (or conditionally complete), so is $(Y^X, \trianglerighteq)$.

8.3.3. Let $(X, \succcurlyeq)$ be a poset and $\mathcal{C}$ the collection of all $\succcurlyeq$-chains in X. Is $(\mathcal{C}, \supseteq)$ conditionally complete?

8.3.4. Let $(X, \succcurlyeq)$ be a poset, Y an $\succcurlyeq$-antichain in X and $\mathcal{A}$ the collection of all $\succcurlyeq$-antichains in X that contain Y. Is $(\mathcal{A}, \supseteq)$ chain-complete?

8.3.5. Is the direct sum of two conditionally complete posets (Exercise 3.9) conditionally complete? What is the situation for chain-completeness?

8.4 The Bourbaki-Witt Theorem

Even though we have barely scratched the surface of order theory, the tools that we have at our disposal so far is enough to derive a fundamental theorem. The "use" of this result, however, will become clear only in Chapter 5.[3]

The Bourbaki-Witt Theorem. Let $(X, \succcurlyeq)$ be a chain-complete poset. If f is a self-map on X such that

$$f(x) \succcurlyeq x \quad \text{for every } x \in X, \tag{1}$$

then $f(x) = x$ for some $x \in X$.

This is what one would call a "deep" theorem – its proof surely requires care. The basic idea here stems from the fact that if $(X, \succcurlyeq)$ were a chain-complete *loset*, we would be done right away, because, then, $\bigvee X \in X$, while $\bigvee X \succcurlyeq f(\bigvee X) \succcurlyeq \bigvee X$, and hence $\bigvee X$ is a fixed point of $(X, \succcurlyeq)$. The problem at hand is basically to come up with a chain-complete $\succcurlyeq$-chain in X, and then apply this argument to that $\succcurlyeq$-chain.

[3] The reader may wish to read this section after reading Sections 1 and 2 of Chapter 5. This would put the substance of the present material in clearer context.

Proof of the Bourbaki-Witt Theorem. Fix an element x_* of X. Let $\mathcal{A}$ denote the collection of all subsets A of X such that

- $x_* \in A$;
- $f(A) \subseteq A$; and
- $\bigvee S \in A$ for every $\succcurlyeq$-chain S in A.

$\mathcal{A}$ is nonempty, because it contains X. It is also easily checked that $\mathcal{A}$ is closed under taking intersections, that is, $\bigcap \mathcal{B} \in \mathcal{A}$ for any nonempty subset $\mathcal{B}$ of $\mathcal{A}$. We define $Y := \bigcap \mathcal{A}$ and note that $Y \in \mathcal{A}$. Clearly, Y is nonempty, because $x_* \in Y$. Furthermore, x_* is the $\succcurlyeq$-minimum element of Y. Indeed, $x_*^{\uparrow}$ belongs to $\mathcal{A}$ – check! – and hence, $Y \subseteq x_*^{\uparrow}$.

Now, for the moment, suppose that Y is a $\succcurlyeq$-chain in X. Then, as $Y \in \mathcal{A}$, we have $\bigvee Y \in Y$ and $f(Y) \subseteq Y$, and hence, $f(\bigvee Y) \in Y$. Therefore, by (1), we have $\bigvee Y \succcurlyeq f(\bigvee Y) \succcurlyeq \bigvee Y$, that is, $f(\bigvee Y) = \bigvee Y$. To complete our proof, therefore, it is enough to show that Y is a $\succcurlyeq$-chain.

For easy reference, let us agree to refer to an element x in Y as **nice** if

$$x \succ y \quad \text{implies} \quad x \succcurlyeq f(y)$$

for every $y \in Y$. We wish to prove the following claims about the nice elements of Y.

Claim 1. If $x \in Y$ is nice, then

$$\text{either } x \succcurlyeq y \text{ or } y \succcurlyeq f(x) \quad \text{for every } y \in Y. \tag{2}$$

Claim 2. Every element of Y is nice.

Proving these two claims will complete our proof, because, then, for any x and y in Y, we have either $x \succcurlyeq y$ or $y \succcurlyeq f(x)$, while in view of (1) and transitivity of $\succcurlyeq$, the latter statement entails $y \succcurlyeq x$. That is, Claims 1 and 2 jointly entail that Y is a $\succcurlyeq$-chain in X, as desired.

It remains to prove Claims 1 and 2.

Proof of Claim 1. Let x be a nice element of Y, and define

$$A := \{y \in Y : \text{ either } x \succcurlyeq y \text{ or } y \succcurlyeq f(x)\}.$$

As $A \subseteq Y = \bigcap \mathcal{A}$, if we can show that $A \in \mathcal{A}$, Claim 1 will be proved. To this end, first, notice that $x_* \in A$ because $Y \succcurlyeq x_*$ (and hence $x \succcurlyeq x_*$). Second, take any y in A, so we have either $x \succcurlyeq y$ or $y \succcurlyeq f(x)$. In the former case, if $x = y$, we obviously have $f(y) \succcurlyeq f(x)$, while if $x \succ y$, we have $x \succcurlyeq f(y)$ (because x is nice), so $f(y) \in A$ obtains either way. In the latter case, as $f(y) \succcurlyeq y$, transitivity of $\succcurlyeq$ ensures $f(y) \succcurlyeq f(x)$, so we find $f(y) \in A$ again. Conclusion: $f(A) \subseteq A$. Finally, let S be a $\succcurlyeq$-chain in A. We wish to show that $\bigvee S \in A$. Clearly, if $x \succcurlyeq S$, then $x \succcurlyeq \bigvee S$, so we are done. If, on the other hand, $x \succcurlyeq y$ is false for some $y \in S$, then, because $y \in A$, we have $\bigvee S \succcurlyeq y \succcurlyeq f(x)$, that is, $\bigvee S \succcurlyeq f(x)$, and we again find $\bigvee S \in A$.

Proof of Claim 2. Let B stand for the collection of all nice elements of Y. We wish to show that $B = Y$, and, again, it is enough to prove that $B \in \mathcal{A}$ for this. As x_* is the

$\succcurlyeq$-minimum element of Y, it is trivially nice. On the other hand, if x is a nice element of Y, and $f(x) \succ y$ for some $y \in Y$, then, by Claim 1, $x \succcurlyeq y$ must hold. If $x = y$, we obviously have $f(x) \succcurlyeq f(y)$, while if $x \succ y$, we have $x \succcurlyeq f(y)$ (because x is nice), and combining this with the fact that $f(x) \succcurlyeq x$ yields $f(x) \succcurlyeq f(y)$ again. Thus $f(x)$ is nice, so we may conclude: $f(B) \subseteq B$. Finally, let S be a $\succcurlyeq$-chain in B. We wish to show that, for any $y \in Y$,

$$\bigvee S \succ y \quad \text{implies} \quad \bigvee S \succcurlyeq f(y),$$

that is, $\bigvee S \in B$. To this end, fix any y in Y and assume $\bigvee S \succ y$. Then, $y \succcurlyeq S$ cannot hold, which means that $y \succcurlyeq x$ is false for some $x \in S$. We cannot have $x \bowtie y$ for such an x because, otherwise, Claim 1 entails $y \succcurlyeq f(x) \succcurlyeq x$, that is, $y \succcurlyeq x$, a contradiction. Therefore, we have $x \succ y$ for some $x \in S$. But then $x \succcurlyeq f(y)$ (since x is nice), so $\bigvee S \succcurlyeq x \succcurlyeq f(y)$, yielding $\bigvee S \succcurlyeq f(y)$ as desired. Conclusion: $\bigvee S \in B$ for every $\succcurlyeq$-chain S in B.

Note. The Bourbaki-Witt Theorem, the exact statement of which appears in Bourbaki (1949) and Witt (1951), is sometimes referred to as *Zermelo's Fixed Point Theorem* in the literature. This is mainly because the argument for this result is largely due to Ernst Zermelo, who used it in his derivation of the Well-Ordering Principle from the Axiom of Choice in the early 1900s.

The Bourbaki-Witt Theorem is a powerful result. As we shall see in Chapter 5, it is instrumental for the derivation of some of the most fundamental maximality principles of order theory. For now, however, we shall not pursue this matter any further.

Exercises

8.4.1. Define Y as in the proof of the Bourbaki-Witt Theorem, and prove that this set is well-ordered by $\succcurlyeq$, that is, there is a $\succcurlyeq$-minimum element of every nonempty subset S of Y.

8.4.2. (Lagler-Volkmann) Let $(X, \succcurlyeq)$ be a poset and denote the collection of all $\succcurlyeq$-chains in X by $\mathcal{C}$. Assume that there exists a function $F : \mathcal{C} \to X$ such that $F(S) \succcurlyeq S$ for every S in $\mathcal{C}$. Show that if a self-map f on X satisfies (1), then it has a fixed point.

Hint. Define $\Phi : \mathcal{C} \to 2^X$ by $\Phi(S) := S \cup \{f(F(S))\}$, and notice that $F(\text{Fix}(\Phi)) \subseteq \text{Fix}(f)$. Show that $\Phi(\mathcal{C}) \subseteq \mathcal{C}$, and apply the Bourbaki-Witt Theorem with respect to $(\mathcal{C}, \supseteq)$ and Φ.