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Back to Fundamentals: Equilibrium in Abstract Economies†

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We propose a new abstract definition of equilibrium in the spirit of competitive equilibrium: a profile of alternatives and a public ordering (expressing prestige, price, or a social norm) such that each agent prefers his assigned alternative to all lower-ranked ones. The equilibrium operates in an abstract setting built upon a concept of convexity borrowed from convex geometry. We apply the concept to a variety of convex economies and relate it to Pareto optimality. The "magic" of linear equilibrium prices is put into perspective by establishing an analogy between linear functions in the standard convexity and "primitive orderings" in the abstract convexity. (*JEL* C90, D11)

In this paper, we return to the fundamental concept of competitive equilibrium and extend the notion to a more abstract setting. The extension is based on the idea that competitive equilibrium is a method of creating harmony in an interactive situation with a feasibility restriction and self-interested agents. It is built around a public ordering of the alternatives which either limits the choices available to the agents or systematically influences their preferences. In the standard economic setting, this ordering is given by prices that apply equally to all agents. These prices determine consumers' choice sets and producers' preferences. We propose an analogous solution concept adjusted to fit more abstract situations in which valuation using prices is replaced by valuation according to a public ordering.

The road to the construction of the equilibrium concept starts with a discussion of the notion of convexity. In the Euclidean setting, the algebraic notion of convexity is central to the standard analysis of competitive equilibrium. Since we primarily consider settings that lack an algebraic structure, we employ a more abstract form of convexity. Definitions of convexity involve the primitive phrase "*b* is between the elements $a^1, \dots, a^L$." In an Euclidean space, this means that b is an algebraic convex combination of $a^1, \dots, a^L$. However, the phrase also has a common use in daily conversation. For example, one can say that game theory is between mathematics and economics and that Canada is culturally between the United States, the United Kingdom, and France. To accommodate this, the first step will be to borrow a formal

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concept of convexity from the existing literature of convex geometry (see Edelman and Jamison 1985). This concept specifies, for each set $\mathcal{A}$, a set $K(\mathcal{A})$ of elements that are "between elements in $\mathcal{A}$" and defines a set $\mathcal{A}$ as *convex* if $K(\mathcal{A}) = \mathcal{A}$. We then present a new characterization result for convex geometries based upon a collection of "primitive orderings": a set $K(\mathcal{A})$ contains any element satisfying that for every primitive ordering there exists some element in $\mathcal{A}$ ranked beneath it. We say that such a set of primitive orderings *generates* the convexity. The primitive orderings play an analogous role to that of linear functions in the case of standard convexity.

The second step involves defining the economic object to be studied. A *convex economy* is a model that consists of: (i) a set of agents; (ii) a set of elements from which each agent chooses; (iii) the agents' preference relations over the set of elements; (iv) a feasibility constraint on choice profiles; and (v) a set of primitive orderings that generates a notion of convexity.

The next step is to present and analyze several related definitions of equilibrium. The first is that of *unrestricted equilibrium* (UE), which is in the spirit of Shapley and Scarf's (1974) concept of equilibrium for the housing economy model. A UE is defined as a profile of choices together with an arbitrary ordering on the set of elements that satisfies two conditions: (i) each choice assigned to an agent is preferred by him to all lower-ordered alternatives; and (ii) the profile is feasible. We refer to the equilibrium ordering as a *public ordering* whose interpretation will be discussed later. In our setting, it plays an analogous role to that of prices in the standard economic setting. It is shown that every Pareto optimal outcome is supported by some UE but there may be UE which are not Pareto optimal.

We then proceed to discuss restrictions on equilibrium public orderings by imposing connections between them and the underlying convexity notion. In particular, we introduce the concept of a *primitive equilibrium* (PE), which is a UE with the additional requirement that the public ordering must be one of the primitive orderings (which generate the convexity). This is analogous to the requirement that prices be linear in the standard setting. We apply these solution concepts to convex economies in which agents have convex preferences. In particular, we study the relationship between PE outcomes and Pareto optimality (the first and second fundamental welfare theorems).

Two directions in which the model can be extended are discussed. The first introduces into the model an initial profile interpreted as an initial distribution of rights and plays a role analogous to that of the initial endowment in the standard exchange economy. The second introduces into the model producer-like agents who influence the consumers' feasibility constraint and whose preferences are given by the public ordering.

The paper also contains ample examples of convex economies, each with an underlying economic story. These examples demonstrate the variety of economic (and noneconomic) models which fit into the framework and illustrate several natural notions besides prices which play in life the role of a public ordering in harmonizing society.

Finally, the reader is advised to read the paper bearing in mind that the goal of the paper is two-fold: to introduce a new solution concept similar in nature to that of competitive equilibrium but defined in a more abstract setting, and to highlight the structure and logic of the standard economic model and the competitive equilibrium notion.

I. Convex Geometry

The basic concepts of convex geometry are given in Edelman and Jamison (1985). Let $\mathcal{X}$ be a set (finite unless stated otherwise) whose members we call *elements*. Convexity is defined through an operator $K : 2^{\mathcal{X}} \rightarrow 2^{\mathcal{X}}$ with the interpretation that $K(\mathcal{A})$ is the set of elements that are "between elements in $\mathcal{A}$" (including the elements of $\mathcal{A}$ themselves). In the standard analysis, K is the convex hull operator. A set $\mathcal{A}$ is *convex* if $K(\mathcal{A}) = \mathcal{A}$.

Note that this concept of convexity allows us to say that "c is between a and b" (by stating that $c \in K(\{a, b\})$ but not that "c is one-quarter of the way from a to b" (expressed as $c = 0.75a + 0.25b$ in the standard algebraic convexity).

A *convex geometry* is an operator K that satisfies the following properties:

(A1) $\mathcal{A} \subseteq K(\mathcal{A})$ and $K(\emptyset) = \emptyset$ ("extensitivity").

(A2) $\mathcal{A} \subseteq \mathcal{B}$ implies $K(\mathcal{A}) \subseteq K(\mathcal{B})$ ("monotonicity").

(A3) $K(K(\mathcal{A})) = K(\mathcal{A})$ ("idempotence").

(A4) If $\mathcal{A}$ is convex, $a, b \notin \mathcal{A}$, and $a \in K(\mathcal{A} \cup b)$, then $b \notin K(\mathcal{A} \cup a)$ ("anti-exchange").

(A1) captures the degenerate sense in which each element in a set is between the set's elements. (A2) means that an element which is between some elements of a set is also between some elements of any larger set. One direction of (A3), $K(K(\mathcal{A})) \supseteq K(\mathcal{A})$, follows from (A1). The other direction, $K(K(\mathcal{A})) \subseteq K(\mathcal{A})$, means that any element which is between elements that are themselves between elements of $\mathcal{A}$ is also between elements of $\mathcal{A}$. (A4) states that if (i) $\mathcal{A}$ is convex; (ii) a and b are not in $\mathcal{A}$; and (iii) a is between b and elements of $\mathcal{A}$, then it is impossible for b to be between a and elements of $\mathcal{A}$. Of course, all four properties hold for the standard Euclidean case where K is the convex hull operator. A nonexample is the operator $K(\mathcal{A}) \equiv \mathcal{X}$ for all nonempty $\mathcal{A}$ (and $K(\emptyset) = \emptyset$) which satisfies (A1), (A2), and (A3), but not (A4) (since $\emptyset$ is convex and $K(\{a\}) = K(\{b\}) = \mathcal{X}$).

Crucial to our discussion of equilibrium is a new representation theorem of convex geometries. The theorem generalizes a property that holds for the standard convex geometry in Euclidean spaces: *a point is in the convex hull of a set if and only if, for every linear ordering, there is a weakly lower element in the set.* In other words, a point is outside the convex hull of a set if there is a linear ordering that places it below all members of the set. By a linear ordering (on an Euclidean space) we mean a binary relation represented by a nontrivial linear function.

To illustrate, in Figure 1, w is not in the convex hull of $\{x, y, z\}$ since there is a linear ordering (depicted by the arrow and the dashed line) that ranks w below x, y, and z. On the other hand, w is in the convex hull of $\mathcal{A} = \{x, y, z, v\}$ since every linear ordering ranks w above at least one of the elements in $\mathcal{A}$.

The representation theorem states that for any finite convex geometry there exists a set of orderings that play a role analogous to that of the linear orderings in the standard Euclidean setting. By an *ordering*, we mean a reflexive, complete,

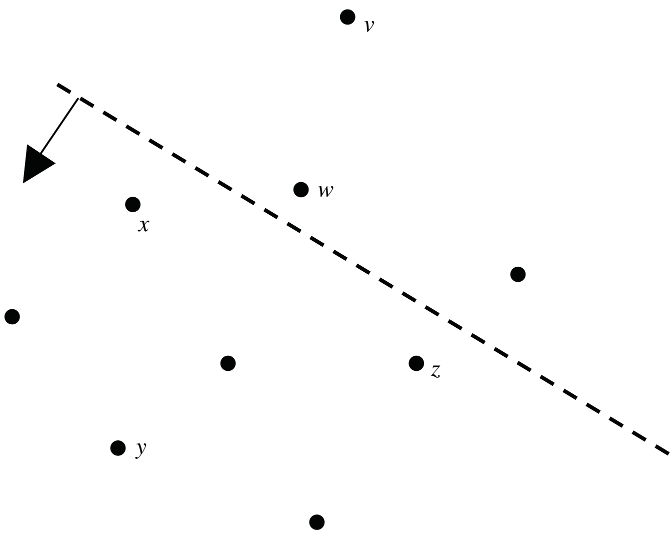

FIGURE 1

antisymmetric, and transitive binary relation. Note that in our terminology orderings are strict.

We say that the set of orderings $\{\geq_k\}$ *generates* K and that its members are *primitive orderings* for K if for all $\mathcal{A}$

$$K(\mathcal{A}) = \{x \mid \forall k, \quad \exists a_k \in \mathcal{A} \quad \text{s.t.} \quad x \geq_k a_k\}.$$

One interpretation of this representation is that agents have a set of criteria (orderings) in mind which they use to evaluate alternatives. A set is convex if for any element outside the set, one of the criteria ranks it as "inferior" to all elements in the set. To illustrate, in the case that $\mathcal{X}$ is a finite set in an Euclidean space with the standard convexity, a set of primitive orderings is given by the set of linear orderings that do not have any "ties" between elements of $\mathcal{X}$.

Note that in the standard setting a coupling property holds: if an ordering is primitive, then so is its inverse. We do not make such an assumption here; some of the convex geometries we consider satisfy this property while others do not.

CLAIM 1 (Representation Theorem for Convex Geometries):

(*a*) *Any set of orderings generates a convex geometry.*

(*b*) *For every* (*finite*) *convex geometry, there is a set of orderings that generates it.*

PROOF:

(a) Let $\{\geq_k\}$ be a set of orderings. Define the operator K by $K(\mathcal{A}) = \{x \mid \forall k, \quad \exists a_k \in \mathcal{A}$ s.t. $x \geq_k a_k\}$. Clearly, K satisfies (A1) and (A2). To see that (A3) holds, recall that by (A1), $K(K(\mathcal{A})) \supseteq K(\mathcal{A})$. To see the opposite inclusion: if $x \in K(K(\mathcal{A}))$ then for every k, there is $a_k \in K(\mathcal{A})$ such that $x \geq_k a_k$ and there is $b_k \in \mathcal{A}$ such that $a_k \geq_k b_k$. By the transitivity of $\geq_k$, $x \geq_k b_k$. Thus, $x \in K(\mathcal{A})$. Regarding A4, assume that $\mathcal{A}$ is convex, $a, b \notin \mathcal{A}$ and $a \in K(\mathcal{A} \cup b)$. Since $a \notin K(\mathcal{A})$ there is an ordering $\geq_k$ such that $x >_k a$ for all $x \in \mathcal{A}$. Since $a \in K(\mathcal{A} \cup b)$ it must be that for the same ordering $a >_k b$. Thus, $x >_k b$ for any $x \in \mathcal{A} \cup a$ and therefore $b \notin K(\mathcal{A} \cup a)$.

(b) Let K be a convex geometry. A chain of convex sets is maximal if it is not part of a strictly longer such chain. Theorem 2.2 of Edelman and Jamison (1985) states that every maximal chain of convex sets $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset \mathcal{X}$ has length $|\mathcal{X}| + 1$. For completeness, here is a proof: since $\emptyset$ and $\mathcal{X}$ are convex, it suffices to show that for any two convex sets $\mathcal{A} \subset \mathcal{B}$ where $|\mathcal{B} \backslash \mathcal{A}| > 1$, there is a convex set $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{C} \subset \mathcal{B}$. By (A1) and (A2), for every $x \in \mathcal{B} \backslash \mathcal{A}$ we have $\mathcal{A} = K(\mathcal{A}) \subset K(\mathcal{A} \cup x) \subseteq K(\mathcal{B}) = \mathcal{B}$ and by (A3), $K(\mathcal{A} \cup x)$ is convex. Take two elements $a, b \in \mathcal{B} \backslash \mathcal{A}$. If neither $K(\mathcal{A} \cup a)$ nor $K(\mathcal{A} \cup b)$ is a proper subset of $\mathcal{B}$, then $K(\mathcal{A} \cup a) = K(\mathcal{A} \cup b) = \mathcal{B}$, thus violating (A4).

For any maximal chain of convex sets $\emptyset = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \mathcal{C}_{|\mathcal{X}|} = \mathcal{X}$, define $c_l = \mathcal{C}_l \backslash \mathcal{C}_{l-1}$ and attach an ordering $c_1 > \ldots > c_{|\mathcal{X}|-1} > c_{|\mathcal{X}|}$. We now show that the set of the orderings $\{\geq_k\}$ attached to all maximal chains of convex sets generates K:

If $y \in K(\mathcal{A})$ and there is an ordering $\geq_k$ such that $a >_k y$ for all $a \in \mathcal{A}$, then there is a convex set $\mathcal{L}$ in the chain generating $\geq_k$ containing $\mathcal{A}$ and not y. However, by (A2) $y \in K(\mathcal{A}) \subseteq K(\mathcal{L}) = \mathcal{L}$, a contradiction. Thus, $y \in \{x \,|\, \forall k, \exists a_k \in \mathcal{A} \text{ s.t. } x \geq_k a_k\}$.

If $y \notin K(\mathcal{A})$, take a maximal chain of convex sets that extends $\emptyset \subseteq K(\mathcal{A}) \subset \mathcal{X}$. By the corresponding ordering $\geq_k$, it must be that $a >_k y$ for all $a \in K(\mathcal{A})$ and thus for all $a \in \mathcal{A}$. Thus, $y \notin \{x \,|\, \forall k, \exists a_k \in \mathcal{A} \text{ s.t. } x \geq_k a_k\}$. ∎

Note that the strictness of the primitive orderings is essential for Claim 1(a) as otherwise the induced K may violate the anti-exchange property. We will return to this issue in Section VD. Additionally, Claim 1(a) does not require $\mathcal{X}$ to be finite. It is beyond the scope of this paper, but one can extend Claim 1(b) for infinite $\mathcal{X}$ using Zorn's lemma.

Following the standard terminology, we say that a preference relation $\succsim$ is *convex* if for every $x^* \in \mathcal{X}$ the *strict upper contour set* $U(\succsim, x^*) = \{x \,|\, x \succ x^*\}$ is convex. The first part of the following lemma states that primitive orderings are convex in the geometry that they induce. This is analogous to the Euclidean property that *the strict upper contour sets of linear functions, namely open half-spaces, are convex*. The second part shows that, as in Euclidean spaces, *the weak upper contour sets of convex preferences are convex* as well. The third part states that any convex ordering added to the set of primitives will still generate the same convexity which implies that the set of primitive orderings generating a convexity is not unique.

LEMMA 1: *Let K be a convex geometry generated by* $\{\geq_k\}$. *Then:*

(i) *Any ordering* $\geq_{k^*}$ *is convex.*

(ii) *For any element* x^* *and convex preferences* $\succsim$, *the set* $\{x \mid x \succsim x^*\}$ *is convex.*

(iii) *Let* K^+ *be the convexity generated by adding an ordering* $\geq$. *Then,* $K^+ = K$ *if and only if* $\geq$ *is convex in K.*

PROOF:

(i) Let $\mathcal{U} = \{x \mid x >_{k^*} a\}$ and take $y \in K(\mathcal{U})$. Since $\{\geq_k\}$ generates K, there is an element $z_{k^*} \in \mathcal{U}$ such that $y \geq_{k^*} z_{k^*} >_{k^*} a$. Thus, $y \in \mathcal{U}$ and $\mathcal{U}$ is convex.

(ii) If x^* is $\succsim$-minimal, then $\{x \mid x \succsim x^*\} = \mathcal{X}$, which is a convex set. Otherwise $\{x \mid x \succsim x^*\} = \cap_{\{z \mid x^* \succ z\}} \{x \mid x \succ z\}$ and in general the intersection of a collection of convex sets $\{\mathcal{A}_i\}$ is convex. To see this, notice that by (A2) for all j, $K(\cap \mathcal{A}_i) \subseteq K(\mathcal{A}_j) = \mathcal{A}_j$ and therefore $K(\cap \mathcal{A}_i) \subseteq \cap \mathcal{A}_i$. Combined with (A1), we obtain $K(\cap \mathcal{A}_i) = \cap \mathcal{A}_i$.

(iii) Take $\geq$ convex and let $\mathcal{A}$ be a set. Obviously $K^+(\mathcal{A}) \subseteq K(\mathcal{A})$. Take $x \in K(\mathcal{A}) \backslash K^+(\mathcal{A})$. Then it must be that for every $a \in \mathcal{A}$, $a > x$. Additionally, $x \notin \{z \mid z > x\}$, a convex set containing $\mathcal{A}$ and thus $x \notin K(\mathcal{A})$, a contradiction.

In the other direction, take $\geq$ nonconvex and thus has a strict upper contour set that is not convex in K. By part (i) this set is convex in K^+ and thus K^+ differs from K. ∎

Here are some examples of convex geometries:

Example 1 (**The Degenerate Convexity**)**:** The convex geometry $K(\mathcal{A}) \equiv \mathcal{A}$ captures the degenerate case in which no element is between any combination of other elements.

The set of all orderings on $\mathcal{X}$ generates this convexity. Actually, any set of orderings for which each element is minimally ranked by at least one ordering generates this convexity.

Example 2 (**Box Convexity**)**:** The box convexity is defined based on a set of orderings $\{\geq_\ell\}$ as $K(\mathcal{A}) = \{x \mid \forall \ell, \exists a_\ell, b_\ell \in \mathcal{A} \text{ s.t. } b_\ell \geq_\ell x \geq_\ell a_\ell\}$.

An element belongs to $K(\mathcal{A})$ if according to each criterion it is sandwiched between some pair of elements in $\mathcal{A}$. This geometry is generated by the set of all orderings $\geq_\ell$ and their reversals.

An example that fits this type of convexity is the case where the elements are characterized by a vector of attribute values and an element is included in $K(\mathcal{A})$ if its value for each attribute is not an extreme with respect to $\mathcal{A}$.

Notice that this convexity might have the property that both c and d are in $K(\{a, b\})$, but both $c \notin K(\{a, d\})$ and $d \notin K(\{a, c\})$. This is unlike the standard convexity as if c and d are in $K(\{a, b\})$, then both are on the line segment connecting a and b, and thus either c is between a and d, or d is between a and c. Therefore, a box convexity is not "homeomorphic" (preserving convexity) to a subset of an Euclidean space with the standard convexity.

Example 3 (**Set Union Convexity**)**:** Let $\mathcal{Z}$ be a set of elements and $\mathcal{X}$ be the set of all menus (nonempty subsets of $\mathcal{Z}$). Define $K(\mathcal{A})$ as the set of all menus that are unions of menus in $\mathcal{A}$.

We now show that K is generated by the set of all extensions of the transitive strict relations $\{R_z\}_{z \in Z}$ defined by $aR_z b$ if $a \supset b$ and $z \notin a \backslash b$.

On the one hand, take a menu $a \in K(\mathcal{A}) \backslash \mathcal{A}$. Then a is a union of its strict subsets in $\mathcal{A}$. For any $z \in a$, there is a menu $c_z \in \mathcal{A}$ such that $z \in c_z \subset a$ and thus $aR_z c_z$. For any $z \notin a$, take $c_z \in \mathcal{A}$ such that $c_z \subset a$ and thus $aR_z c_z$ Therefore, a is not minimal for any extension of any R_z.

On the other hand, take a menu a that is not minimal in $\mathcal{A} \cup \{a\}$ for every extension of every R_z. Then, for every z there must be a menu b_z such that $aR_z b_z$. This implies that for every $z \in a$, $z \in b_z \subseteq a$. Thus, $a = \cup_{z \in a} b_z$ and $a \in K(\mathcal{A})$.

Comments on the Use of Abstract Geometry in Economics:

(i) Koshevoy (1999) pointed out a connection between the literature on convex geometry and that of choice theory for finite sets. He compared the properties of choice correspondences to those of the operator $ext(\mathcal{A})$, defined as the set of all $x \in \mathcal{A}$ such that $x \notin K(\mathcal{A} - x)$ (an element is *extreme* in $\mathcal{A}$ if it is not between other elements of $\mathcal{A}$). The operator *ext* satisfies two familiar properties in the choice theory literature:

Heritage: If $\mathcal{A}' \subseteq \mathcal{A}$, then $ext(\mathcal{A}') \supseteq ext(\mathcal{A}) \cap \mathcal{A}'$.
Outcast: If $ext(\mathcal{A}) \subseteq \mathcal{A}' \subseteq \mathcal{A}$, then $ext(\mathcal{A}') = ext(\mathcal{A})$.

Heritage is actually the α property in Sen (1970) and Outcast is Postulate 5* in Chernoff (1954). The representation $ext(\mathcal{A}) = \{x \in \mathcal{A} | x$ is the $\geq_k$-minimum in $\mathcal{A}$ for some $\geq_k\}$ can be derived from Claim 1. Claim 1(b) could also have been proved using Koshevoy's observations and Aizerman and Malishevskii's (1981) result: a choice correspondence C satisfies Heritage and Outcast if and only if there is a finite number of orderings over $\mathcal{X}$, such that $C(\mathcal{A})$ is the set of the unique maximums of these orderings in $\mathcal{A}$.

(ii) Baldwin and Klemperer (2013) studies the existence and properties of standard competitive equilibria in an Euclidean setting with indivisibilities using *tropical geometry*, a new concept related to algebraic geometry. The mathematical concepts they use and the economic issues involved are far from those studied in this paper.

II. The Convex Economy

We now turn to defining the abstract economic environment. Let $\mathcal{N} = \{1, .., n\}$ be a set of agents. Each agent chooses an *element* from an abstract set $\mathcal{X}$ endowed with a convex geometry K generated by a set of primitive orderings $\{\geq_k\}$. No further structure is imposed on $\mathcal{X}$. A profile assigns one element to each agent. Not all profiles are feasible. The feasibility constraint is given by a set $\mathcal{F} \subset \mathcal{X}^{\mathcal{N}}$. Unless stated otherwise, we assume that $\mathcal{F}$ is closed under all permutations; our definitions are not contingent upon this assumption. A tuple $< \mathcal{N}, \mathcal{X}, \mathcal{F}, \{\geq_k\} >$ is called a *convex environment*.

We have in mind examples such as the following:

The Exchange Environment.—$\mathcal{X} = R^L_+$ is the set of bundles in a world with L commodities. The set $\mathcal{F}$ is the set of all allocations of a total endowment ω among the agents.

The Housing Environment.—The set $\mathcal{X}$ contains n houses. A feasible allocation assigns a distinct house to each of the n agents.

The Sequential Production Environment.—$\mathcal{X}$ is a set of products. A correspondence T describes the possible transformations of products, that is, $T(x)$ is the set of products which x could be transformed into. One element of $\mathcal{X}$, denoted x^0, stands for the starting point of the sequential production process. A production sequence is a vector $(x^1, .., x^n)$ such that $x^m \in T(x^{m-1})$ for all m. The set $\mathcal{F}$ consists of all permutations of production sequences.

The Roommate Environment.—The alternatives are the agents and each agent must choose a partner. The feasibility constraint is that if i chooses j, then j must choose i. This feasibility constraint does not satisfy the permutation condition.

Each agent i possesses a preference relation $\succsim^i$ on $\mathcal{X}$ (an upper index always indicates the agent). We assume that each $\succsim^i$ is *convex*. We will refer to a tuple $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i\in\mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ as a *convex economy*.

An agent, unlike in a game setting, is interested only in the element he himself chooses, independent of other agents' choices. Given the representation theorem in Claim 1, the convexity of preferences has the following novel interpretation: all agents have a set of criteria $\{\geq_k\}$ in mind. An agent who ranks all elements in a set $\mathcal{Y}$ to be superior to x^* must also rank z to be superior to x^*, if for each criterion $\geq_k$ there is $y^k \in \mathcal{Y}$ such that $z >_k y^k$.

Note that we included a set of primitive orderings in the definition of a convex economy rather than just a notion of convexity. As mentioned earlier, there are many different sets of primitive orderings that generate the same convexity. The specification of a particular set of primitive orderings is intended to capture some prominent aspect of the economy (as linear orderings do in the case of the standard economy) and will play an important role throughout the paper.

Notice also that our concept of an economy lacks "initial endowments." We will discuss this point in Section VII.

III. Unrestricted Equilibrium

The constraints imposed by the set $\mathcal{F}$ will typically introduce conflicts between the agents. An equilibrium concept provides a method of resolving such conflicts in a way that produces some form of stability:

DEFINITION 1: *An unrestricted equilibrium* (UE) *is a pair* $< \left(x^i\right)_{i \in \mathcal{N}}, P >$ *where* $\left(x^i\right)_{i \in \mathcal{N}}$ *is a profile and* P *is an ordering on* $\mathcal{X}$, *such that* (i) *the profile is in* $\mathcal{F}$, *and* (ii) *for each* i, *the element* x^i *is* $\succsim^i$*-optimal in the set* $B\left(P, x^i\right) = \left\{z \,|\, x^i P z \text{ or } z = x^i\right\}$.

We refer to the ordering P as a *public ordering* and use the letter B to emphasize the analogy to the familiar term "budget set." The name "unrestricted equilibrium" emphasizes that the definition imposes no restrictions on P (such as convexity), apart from being a strict ordering. The requirement that P be strict is without loss of generality since any equilibrium profile supported by an ordering that contains indifferences will also be an equilibrium profile with arbitrary breaking of the indifferences.

Our main interpretation of this concept of equilibrium views P as a social ordering that reflects the elements' worth or prestige. The term aPb means that *a is more expensive than b* or that *a is more prestigious than b*. For a profile to be an equilibrium, there must exist a public ordering such that each agent is satisfied with his assigned element given his ability to replace it only with an element that is considered less expensive or less prestigious according to the public ordering. An equilibrium ordering stabilizes the equilibrium profile in the sense that each agent is satisfied with his assignment *given* the "worth" of his assigned element.

An alternative interpretation views P as a socially agreed upon or imposed motive that systematically affects the agents' preference relations. The relation aPb means that "a is *less socially desirable* than b" (or "a is *less prestigious* than b"). An agent can choose any alternative in $\mathcal{X}$, but must rationalize his choice as furthering society's goals (or alternatively he can't bear to suffer a loss of prestige). Thus, the ordering P systematically affects the agents' preferences, such that given P and an assigned element x^i, the agent's lexicographical first priority is to "not move up the P ordering" while his second priority is to maximize his original preference. An equilibrium is a profile and a social ordering for which no agent wishes to deviate from his assigned element (given his personal preferences) and is able to justify it as furthering the social goal (or alternatively not decreasing his prestige).

Thus, according to the main interpretation, agents consider only the elements further down the public ordering to be feasible, while according to the alternative interpretation, all elements are feasible, but agents find only elements further down the P ordering (that is, higher in prestige) to be socially acceptable.

The existence of a UE does not require any further assumptions and follows from the following version of the second fundamental welfare theorem (SWT):

CLAIM 2 (SWT-UE): *Any Pareto optimal profile is an unrestricted equilibrium profile.*

PROOF:

Let $(a^i)_{i \in \mathcal{N}}$ be a Pareto optimal profile. Define the relation R by xRy if (i) there are i and j such that $x = a^i \succ^j a^j = y$ or (ii) $x \notin \{a^1, .., a^n\}$ and $y \in \{a^1, .., a^n\}$. The first condition guarantees that if j envies the element assigned to i he will not be able to replace his element with i's. The second condition guarantees that agents will find any unassigned element "unaffordable."

Since the profile is Pareto optimal and $\mathcal{F}$ is closed to permutations, R does not have cycles and thus can be extended to a complete ordering P. For this ordering, a^j is optimal in $B(P, a^j)$ for each agent j. ∎

Note that the above public ordering is not constructed with an eye to the convexity notion of the economy and may fail to be convex. Note also that any Pareto inefficient profile is also a UE profile if there are no cycles in the envy relation. In particular, any feasible profile that assigns the same element to all agents is supported by an *unrestricted* public ordering that makes it the cheapest good.

IV. Primitive Equilibrium

In standard economic models, where the set $\mathcal{X}$ is a subset of an Euclidean space, the following hold:

(i) The set of linear orderings generates the convex geometry.

(ii) The public ordering (price system) is a linear ordering.

Thus, a natural analogy is to require that the equilibrium public ordering is one of the primitive orderings generating the geometry. This brings us to the central definition of the paper of primitive equilibrium:

DEFINITION 2: *Let* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ *be a convex economy.*

A *primitive equilibrium* (*PE*) is a UE $< (x^i)_{i \in \mathcal{N}}, P >$ where P is one of the primitive orderings.

A *convex equilibrium* (*CE*) is a UE $< (x^i)_{i \in \mathcal{N}}, P >$ where P is a convex ordering.

By Lemma 1 any PE is a CE. The notion of CE depends only on the convexity induced by the set of primitive orderings and, unlike the notion of PE, does not depend on the particular set of primitive orderings. Convexity of the public ordering means that any set of the form $U(P, x^*) = \{x \mid xPx^*\}$ is convex. The set $U(P, x^*)$ is the set of elements that an agent who holds x^* finds unaffordable. The convexity of those sets is a reasonable requirement if the market is managed by a market maker who declares all the exchanges he is willing to make. The term yPx^* means that the market maker is not willing to exchange y for x^*. Accordingly, $U(P, x^*)$ is the set of all elements preferred by the market maker to x^*. The requirement that any set $U(P, x^*)$ be convex is equivalent to the assumption that the market maker's preferences are convex. In the standard consumer world convexity of the public ordering is an expression of nonlinear prices with quantity discounts.

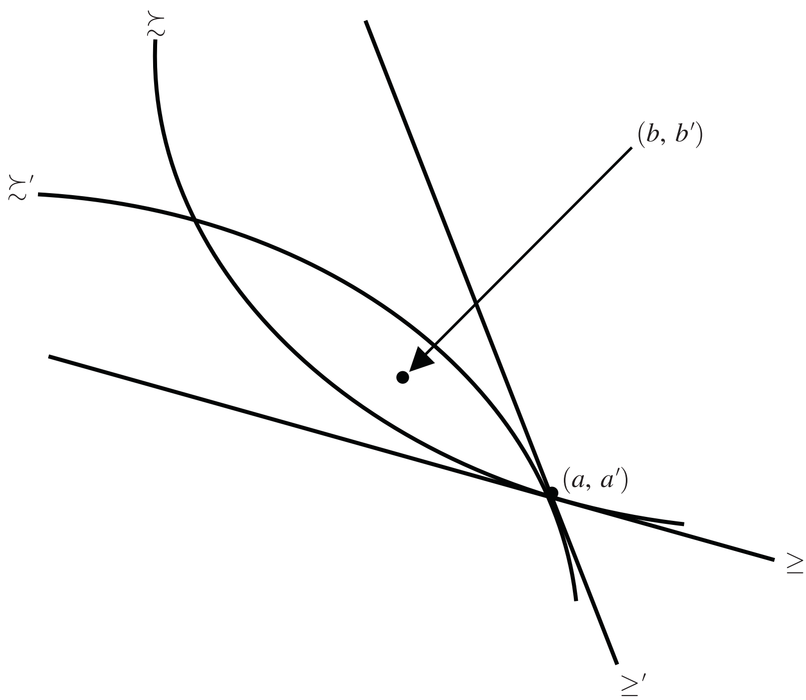

Figure 2

In general, the existence of a PE is not guaranteed. In fact, the following convex economy does not even have a CE.

Example 4: Consider the housing economy with four houses arranged on a line $a - b - c - d$, with the standard convexity. Two of the agents, 1 and 2, are "leftish" and hold the convex preferences $a \succ^i b \succ^i c \succ^i d$, while the other two, 3 and 4, are "rightish" and hold the convex preferences $a \prec^i b \prec^i c \prec^i d$. Claim 2 guarantees the existence of a UE (such as the profile (a, b, c, d) and the public ordering $aPdPbPc$). However, a CE does not exist since if it exists a or d would be minimal and there are two agents who top-rank this element. At least one of these agents is assigned a different element and thus strictly prefers an element that is lower ranked, violating the equilibrium condition.

What is special about the standard exchange economy that makes every Pareto optimal allocation a PE profile? It can be attributed to the following Richness property (illustrated in Figure 2 for the standard Edgeworth box):

We say that the environment $< \mathcal{N}, \mathcal{X}, \mathcal{F}, \{\geq_k\} >$ satisfies *Richness* if the following holds: let $\succsim$ and $\succsim'$ be two convex preferences over $\mathcal{X}$ and let a and a' be two elements in a profile $(a, a', a^3, .., a^n) \in \mathcal{F}$. Let $\geq$ and $\geq'$ be two different primitive orderings such that (i) a is $\succsim$-maximal in $B(\geq, a)$ but not in $B(\geq', a)$ and (ii) a' is $\succsim'$-maximal in $B(\geq', a')$ but not in $B(\geq, a')$. Then, there is a pair (b, b') such that:

(i) $(b, b', a^3, .., a^n) \in \mathcal{F}$,

(ii) (b, b') (weak) Pareto-dominates (a, a') (in the sense that (i) $(b, b') \neq (a, a')$; (ii) $b \succ a$ or $b = a$; and (iii) $b' \succ' a'$ or $b' = a'$).

The first part of the following claim states that the Richness condition is sufficient for any Pareto optimal profile to be a PE profile. However, the Richness condition is not necessary for the SWT to hold. An obvious example would be the case of an economy with degenerate convexity where the set of primitives consists of all orderings. For such an economy, regardless of whether the Richness condition holds, every Pareto optimal profile is a PE (Claim 2 showed that every Pareto optimal profile is a UE and the notions of PE and UE are equivalent for the degenerate convexity).

The second part of Claim 3 states that the Richness condition is necessary for the SWT under the following two conditions:

(i) Betweenness: $K(\{x\}) = \{x\}$ for each $x \in \mathcal{X}$. This property does not hold for monotonic convexities (for which we will show in Claim 7 that a SWT-CE always hold) but does hold for any case in which the convexity expresses a real betweenness.

(ii) Differentiability: for every convex preference relation $\succsim$ and for every non-top-ranked alternative x there is a unique primitive ordering $\geq$ such that x is $\succsim$-maximal in the set $B(\geq, x)$.

CLAIM 3 (SWT-PE): *Consider an environment* $< \mathcal{N}, \mathcal{X}, \mathcal{F}, \{\geq_k\} >$:

(*i*) *If Richness holds then for every extension of the environment to an economy* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ *any Pareto optimal profile is a PE profile.*

(*ii*) *If Betweenness and Differentiability hold and Richness fails, then there is an extension of the environment to an economy* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ *with a Pareto optimal profile that is not a PE profile.*

PROOF:

(i) Let $(x^i)_{i \in \mathcal{N}}$ be a Pareto optimal profile. By the convexity of preferences, the set $\mathcal{U}^i = \{z \mid z \succ^i x^i\}$ is convex and $x^i \notin \mathcal{U}^i$ for each agent i. Therefore, there is at least one primitive ordering that ranks x^i below all members of $\mathcal{U}^i$. Let $\mathcal{O}^i$ be the nonempty set of all such primitive orderings.

The intersection $\cap_i \mathcal{O}^i$ is not empty since otherwise there would be two agents i and j such that $\mathcal{O}^i$ and $\mathcal{O}^j$ are nonnested sets. Take $\geq^i \in \mathcal{O}^i \backslash \mathcal{O}^j$ and $\geq^j \in \mathcal{O}^j \backslash \mathcal{O}^i$. The element x^i is $\succsim^i$-maximal in $B(\geq^i, x^i)$ but not in $B(\geq^j, x^i)$, and likewise for agent j. Since $\mathcal{F}$ is closed under permutations, the Richness condition implies that there is a different pair of elements (y^i, y^j) such that the modified profile obtained by replacing the pair (x^i, x^j) with (y^i, y^j) is feasible and Pareto-dominating, contradicting the Pareto optimality of $(x^i)_{i \in \mathcal{N}}$.

Thus, there exists $\geq_k \in \cap_i \mathcal{O}^i$ and then $< (x^i)_{i\in\mathcal{N}}, \geq_k >$ is a PE.

(ii) Consider a pair of elements (a^1, a^2) from a feasible profile $(a^1, a^2, a^3, \ldots, a^n)$ together with a pair of convex preference relations $(\succsim^1, \succsim^2)$ and a pair of primitive orderings $(\geq^1, \geq^2)$ where the Richness condition fails, that is, (i) for $\{i, j\} = \{1, 2\}$, a^i is $\succsim^i$-optimal in $B(\geq^i, a^i)$ but not in $B(\geq^j, a^i)$, and (ii) there is no pair (b^1, b^2) such that (b^1, b^2) Pareto-dominates (a^1, a^2) and $(b^1, b^2, a^3, \ldots, a^n) \in \mathcal{F}$.

Endow agents 1 and 2 with the preferences $\succsim^1$ and $\succsim^2$. For each $i \geq 3$, let $\succsim^i$ be a strict convex preference relation so that a^i is top-ranked by $\succsim^i$. To verify the existence of such preferences recall the construction in the proof of Claim 1(b): take $\succsim^i$ to be the ordering attached to some maximal chain of convex sets which extends the chain $\emptyset \subseteq \{a^i\} \subseteq \mathcal{X}$ (Betweenness guarantees that $\{a^i\}$ is a convex set).

The profile $(a^i)_{i\in\mathcal{N}}$ is Pareto optimal: if a feasible Pareto-dominating profile $(b^i)_{i\in\mathcal{N}}$ exists, then for any $i \geq 3$, it must be that $b^i = a^i$ and then the pair (b^1, b^2) Pareto-dominates (a^1, a^2) and $(b^i) = (b^1, b^2, a^3, \ldots, a^n) \in \mathcal{F}$, contradicting the failure of the Richness condition.

Finally, there is no PE $< (a^i)_{i\in\mathcal{N}}, P >$. To see this, notice that for $i = 1, 2, a^i$ is not $\succsim^i$-top ranked in $\mathcal{X}$ because a^i is not $\succsim^i$-optimal even in $B(\geq^j, a^i)$. Differentiability then implies that for each $i = 1, 2$, P is the unique primitive ordering $\geq$ for which a^i is $\succsim^i$-maximal in the set $B(\geq, a^i)$, thus P is $\geq^i$. But $\geq^1$ and $\geq^2$ differ. ∎

In the case of a one-agent convex economy, the richness condition does not apply and every feasible element (and not just the agent's preference-maximal elements) is an outcome of some PE.

CLAIM 4: *In a one-agent convex economy, every feasible element x^* is a PE profile.*

PROOF:

Let x^* be a feasible element. By the convexity of the agent's preferences, the set $\{x \mid x \succ^1 x^*\}$ is convex. Since x^* is not a member of this convex set, there is a primitive ordering $\geq_k$ such that $x >_k x^*$ for every $x \succ^1 x^*$. Thus, x^* is $\succsim^1$-optimal in $\{x \mid x^* \geq_k x\}$ and $< (x^*), \geq_k >$ is a PE. ∎

With regard to the first fundamental welfare theorem (FWT), Claim 4 demonstrates that a PE profile need not be Pareto optimal. Furthermore, in multi-agent settings, if all agents have identical convex preferences equal to one of the primitive orderings, then every feasible profile combined with that primitive ordering is a PE.

The following claim identifies a necessary and sufficient condition for the FWT. A convex environment $< \mathcal{N}, \mathcal{X}, \mathcal{F}, \{\geq_k\} >$ satisfies *condition F* if there are no two feasible profiles (a^i) and (b^i) and a primitive ordering $\geq_k$ such that for all i either $b^i = a^i$ or $b^i >_k a^i$. In words, there are no two feasible profiles that Pareto-dominate one another according to some primitive ordering. Two prominent convex environments that satisfy condition F are: (i) the standard exchange economy with the standard convexity, and (ii) the housing economy with any convexity.

CLAIM 5 (First Fundamental Welfare Theorem): *Consider an environment* $< \mathcal{N}, \mathcal{X}, \mathcal{F}, \{\geq_k\} >$.

(i) *If it satisfies condition F, then for every extension of the environment to an economy* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$, *any PE profile* (a^i) *is (weak) Pareto optimal* $\big($*in the sense that there is no other feasible* (b^i) *such that for all i either* $b^i = a^i$ *or* $b^i \succ^i a^i\big)$.

(ii) *If it does not satisfy condition F, then there is an extension of the environment to an economy* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ *and a PE profile which is not Pareto optimal.*

PROOF:

(i) Consider a PE $< (a^i), \geq_k >$. If (a^i) is not Pareto optimal, then there is another feasible profile (b^i) such that for all i either $b^i = a^i$ or $b^i \succ^i a^i$. Then, for all i, either $b^i = a^i$ or $b^i >_k a^i$, a contradiction.

(ii) If the F-condition fails, then there are two distinct feasible profiles $(a^i)_{i \in \mathcal{N}}$ and $(b^i)_{i \in \mathcal{N}}$ and a primitive ordering $\geq$, such that for all i, $a^i \geq b^i$. Extend the environment to an economy by endowing each agent with the convex preference relation $\geq$. Then, b together with the primitive ordering $\geq$ is a PE which is not Pareto optimal. ∎

V. Economic Examples I

In this section, we analyze the various concepts of equilibria in the context of several simple convex economies. In particular, we examine the relationships between the equilibrium concepts and Pareto optimality.

A. *The "Give and Take" Economy*

Consider a society in which agents either give to or take from a voluntary public fund. The fund must be balanced. An agent's preferences reflect his attitude toward the trade-off between egalitarianism and selfishness. Each agent has in mind an ideal amount that he wishes to either contribute or withdraw. The problem is that agents' ideals may not match in the sense that the total of ideal contributions of those who wish to give may not be equal to the total of ideal withdrawals of those who wish to take.

In this economy, there are two natural orderings: one values more giving and less taking; the other is the opposite. The standard convexity is generated by these two orderings. An equilibrium maintains balance in the fund by assigning a contribution or withdrawal amount to each of the agents and provides a common social ordering. In equilibrium, no agent wishes to deviate from his assignment to a lower ordered point (interpreted as more socially acceptable). In a PE, the common social ordering is not arbitrary but rather is one of the two primitive orderings that generate the convexity.

The Economy.—Let $\mathcal{X} = [-1, 1]$; a positive number represents a contribution to the social fund and a negative number represents a withdrawal. We take the primitive orderings to be the increasing and decreasing orderings which generate the standard convexity. Assume that each agent i has convex preferences with a single peak denoted by $peak^i$. Let $\mathcal{F}$ be the set of all profiles that sum up to zero. The economy is interesting when $\sum peak^i \neq 0$. Without loss of generality, we assume that $\sum peak^i < 0$.

Only One of the Primitive Orderings Is an Equilibrium Public Ordering.—If there is a PE with the increasing ordering, then all agents must be at or to the left of their peak (otherwise, an agent would wish to move to his peak and could afford to), thus violating the feasibility constraint. On the other hand, with the decreasing ordering, any feasible profile that assigns to each agent an element at or to the right of his peak, is a PE.

The decreasing public ordering embodies the social norm that only allows an agent to consider giving more or taking less than his assigned element. This is a sound norm to govern a voluntary public fund in a society where the "average" tendency of agents, as reflected by their preferences, is to take rather than to give. Following the alternative interpretation of the equilibrium, agents are indoctrinated that giving more or taking less is more respected. The equilibrium maintains a balanced fund in such a society as agents do not take more than is assigned to them so as to avoid losing respect.

FWT.—The condition in Claim 5 holds and therefore any PE outcome is Pareto optimal. However, there can be a CE profile that is not Pareto optimal. Consider a case with an even number of agents, one-half of them "leftish" with negative peaks different than -1 who prefer -1 to 1 and one-half of them "rightish" with positive peaks different than 1 who prefer 1 to -1. The feasible Pareto inefficient profile in which each "leftish" agent is assigned -1 and each "rightish" agent is assigned 1 is supported by the convex public ordering P, where xPy if $|x| \leq |y|$. The social norm expressed by this ordering allows an agent to move from an assigned element only to a "more extreme" one.

SWT.—The Richness condition in Claim 3 holds and therefore any Pareto optimal profile is a PE profile. To see it directly, consider a Pareto optimal feasible profile. Given the assumption that $\sum peak^i < 0$, all agents are at or to the right of their peak. (All agents being to the left of their peak violates feasibility. If one agent is to the left of his peak and another is to the right of his peak, there is a Pareto improvement where each agent moves closer to his peak.) Such a profile is supported by the decreasing public ordering.

B. *The Consensus Economy*

Imagine a society of n agents where the set of possible political positions $\mathcal{X}$ consists of a finite or infinite set of points on a line. As above, endow $\mathcal{X}$ with the standard convexity and take the primitive orderings to be the "increasing" and "decreasing" orderings. Each agent i has a convex preference relation with a unique peak denoted by $peak^i$. Each agent chooses a position. The ruler wants to achieve a consensus and thus the feasibility constraint is the set of all constant profiles. We assume that he can indoctrinate the public only with primitive public norms.

It is easy to see that the only PE profiles for this economy will be of the form $(x^*, \ldots, x^*)$ where x^* is at or to the right of the right-most peak, supported by the decreasing ordering (or analogously in the other direction). An equilibrium at the right-most peak with the decreasing ordering means that the ruler brainwashes the agents to believe that it is sinful to express any view to the left of x^* and thus each agent can only consider moving even more to the right.

Note that positions which are in the middle of the peaks distribution cannot be supported by primitive orderings nor by any convex public ordering (since if x^* is between the right-most and left-most peaks then one of the two extreme peaks is affordable and preferred by the corresponding extreme agent).

This example formalizes a simple explanation of why regimes that require a consensus tend to be based on extreme positions as opposed to a middle-of-the-road policy.

Failure of the Welfare Theorems.—The Betweenness and Differentiability conditions hold while the F condition and Richness condition do not. Therefore, the welfare theorems do not hold. Regarding the FWT, every position which is to the right (or left) of all agents' peaks is a PE and Pareto dominated. As for the SWT, any position strictly between the left-most and right-most peaks is Pareto optimal and not a PE outcome.

C. *The Roommate Economy*

The standard roommate model fits into our framework with some modifications. Let $\mathcal{N}$ be a set of n (even number) agents and let $\mathcal{X} = \mathcal{N}$. The first modification is that each agent is allowed to choose only an agent other than himself. The feasibility constraint, $\mathcal{F} = \left\{ \left(x^i\right)_{i \in \mathcal{N}} \middle| \text{ if } x^i = j \text{ then } x^j = i \right\}$, imposes that agents choose each other. This constraint does not satisfy the permutation condition.

The commonly used solution concept for the roommate problem is stability. A profile is *stable* if there is no pair of agents such that each of them prefers the other over his assigned partner in the profile. Irving (1985) presented an algorithm for finding a stable assignment when it exists and the literature has investigated conditions for its existence (see Gudmundsson 2014 for a review).

Our notion of equilibrium is an alternative solution concept. The public ordering harmonizes the equilibrium profile in that no agent wishes to exchange his roommate for a lower-ranked agent. This public ordering might, for example, reflect a common scale of social status. As always, an antiprestige interpretation is available as well, where aPb means that b is more prestigious than a and in equilibrium, no agent finds any other agent to be both preferred and more prestigious than the one he is paired with.

The following two examples demonstrate that stability and equilibria are two separate solution concepts.

Example 5 (With PE and No Stable Matching): Consider $\mathcal{X} = \{1, 2, 3, 4\}$ with the standard convexity on a line $1 - 2 - 3 - 4$ and the following preferences:

Agent	1	2	3	4
1st preference	2	3	1	1
2nd preference	3	1	2	2
3rd preference	4	4	4	3

Each agent's preferences are convex on the set of the other agents (with the induced convexity for this set).

There are two PE profiles: the primitive ordering $1P2P3P4$ supports both $(3, 4, 1, 2)$ and $(4, 3, 2, 1)$. But, this economy does not have a stable profile: consider a profile. Let i be the agent matched with 4. He prefers any other agent to 4 and there is an agent j who ranks i first. Thus, the pair (i, j) blocks the profile from being stable.

Example 6 (With a Stable Matching Where There is No UE):

Agent	1	2	3	4
1st preference	3	4	2	1
2nd preference	4	3	1	2
3rd preference	2	1	4	3

The profile $(3, 4, 1, 2)$ is stable since 1 and 2 are matched with their top-ranked agent and agents 3 and 4 hate each other. This profile (and similarly $(4, 3, 2, 1)$) is not a UE profile because both 3 and 4 wish to exchange their partner for the one assigned to the other and one of them can "afford" it. The only other feasible profile, $(2, 1, 4, 3)$, is not a UE since in any equilibrium there must be at least one agent who is assigned his first best.

The literature offers two main sufficient conditions for the existence of a stable matching. The first is the "no odd cycles" condition which states that any list of agents $i_1, \ldots, i_m$ so that agent i_l prefers i_{l+1} to i_{l-1} (and agent i_m prefers i_1 to i_{m-1}) is of even length. This condition is satisfied in the previous example and therefore this condition does not guarantee the existence of UE. The other condition is α-reducibility, which states that for any subset of agents there are two agents, each of whom top-ranks the other among the subset.

The α-Reducibility Condition Is Sufficient for the Existence of a UE.—In $\mathcal{X}$, there are two agents who most prefer each other over all other agents. These agents are matched and removed. This process is then repeated among the remaining agents until all agents are removed. The public ordering of the agents is given by the order in which they were removed (earlier removed agents are higher). This algorithm constructs a UE because for any two unmatched agents, only the one who was removed earlier, i, can afford the one who is removed later, j. However, i does not prefer j to his match, because j was still in the pool when i was assigned to his top match from the pool.

Here are two examples of convex roommate economies which satisfy α-reducibility:

(i) The agents are arranged on a line with the standard convexity and each agent possesses a convex preference that top-ranks one of his two neighbors. This implies that an extreme agent ranks his unique neighbor first. Thus, in any set of agents there are always two neighbors such that the left one top-ranks his right neighbor and the right one top-ranks his left neighbor.

(ii) The agents live in some metric space and prefer closer agents to further ones. The α-reducibility condition holds because in any set of agents there is a distance-minimizing pair who most prefer each other.

D. *An Almost Standard Exchange Economy*

The Economy.—Let $\mathcal{X} = \Pi_{l=1}^{L}[0, z_l]$ be a set of bundles in an L-commodity world with total endowment $\mathbf{z}$. Feasibility is expressed by $\sum_{i=1}^{n} \mathbf{x}^i = \mathbf{z}$. All agents hold monotonic and convex preference relations. Agents have in mind a nonempty (finite or infinite) set of linear orderings $\Phi = \{\geq_k\}$ (*with indifferences*), each of which is characterized by a nonnegative vector $\boldsymbol{v}_k \neq 0$ such that $\boldsymbol{x} \geq_k \boldsymbol{y}$ if $\boldsymbol{v}_k \cdot \boldsymbol{x} \geq \boldsymbol{v}_k \cdot \boldsymbol{y}$. Define the operator K by $K(\mathcal{A}) = \{x \mid \forall k, \exists a_k \in \mathcal{A}, x \geq_k a_k\}$. This operator is defined as in our representation theorem but with the difference that the underlying orderings have indifferences. The operator satisfies (A1), (A2), and (A3), but satisfies only a slightly weaker version of (A4): if $\mathcal{A}$ is convex and $\boldsymbol{x}$ and $\boldsymbol{y}$ are not in $\mathcal{A}$ and if $\boldsymbol{x}$ is in the interior of $K(\mathcal{A} \cup \boldsymbol{y})$, then $\boldsymbol{y}$ is not in the interior of $K(\mathcal{A} \cup \mathbf{x})$.

When Φ is a singleton, all agents hold the unique convex preference relation represented by the utility function $\mathbf{v}_1 \cdot \mathbf{x}$. In the case that $\Phi = \{(1, 0), (0, 1)\}$, each indifference curve must be "right-angled."

FWT.—The economy satisfies the conditions of Claim 5 and the argument there holds and thus any PE profile is Pareto optimal.

SWT.—The Richness property used in Claim 3 holds and thus any Pareto optimal allocation is a PE profile. Note that this is slightly stronger than the statement of the textbook SWT that any Pareto optimal allocation is an equilibrium allocation supported by *some* linear ordering while the SWT here states that the equilibrium public ordering can be drawn from Φ.

VI. Examples with Monotonic Convexity

In this section we focus on a class of convex economies with a special type of convexity which has the flavor of monotonicity rather than of betweenness.

Let R be a partial ordering (reflexive, transitive, and antisymmetric, but not necessarily complete) of a set $\mathcal{X}$. Define the *monotonic convexity* generated by R as $K(\mathcal{A}) = \{x \mid \exists a \in \mathcal{A} \text{ s.t. } xRa\}$. For this type of convexity $a \in K(\mathcal{A})$ depends solely on the presence of certain other individual elements in $\mathcal{A}$ while in the previous examples it depended on the presence of certain combinations of elements in $\mathcal{A}$. By Theorem 3.2 in Edelman and Jamison (1985), a convex geometry K satisfies the additional *union property* $K(\mathcal{A} \cup \mathcal{B}) = K(\mathcal{A}) \cup K(\mathcal{B})$ if and only if it can be represented as a monotonic convexity.

For a monotonic convexity based on the relation R, a preference relation $\succsim$ is convex if and only if it is a completion of R. To verify this, suppose that $\succsim$ is convex and xRy. The set $\{z \mid z \succsim y\}$ is convex and contains y and therefore it contains x and thus $x \succsim y$. On the other hand, if $\succsim$ is an extension of R, any set $\{z \mid z \succsim y\}$ is convex since if wRz and $z \succsim y$, then $w \succsim z \succsim y$ since $\succsim$ extends R. Thus, requiring that agents' preferences are convex is the same as requiring that if aRb then all agents weakly prefer a to b. Additionally, it follows that the set of all orderings which complete R is the maximal set of primitives generating the convexity, but there are often smaller sets of primitives which are more natural.

Typically, the notion of UE is strictly more general than that of CE as it imposes no restrictions on the public orderings. The following claim shows that for any monotonic convex economy, any profile which can be supported with an unrestricted public ordering can also be supported with a convex public ordering and thus is also a PE profile if we take the maximal set of primitives.

CLAIM 6: *Consider a convex economy* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i \in \mathcal{N}}, \mathcal{F}, \{\geq_k\} >$ *with monotonic convexity. Then any UE profile is a CE profile.*

PROOF:

Let $< (x^i)_{i \in \mathcal{N}}, P >$ be a UE. Partition the set $\mathcal{X}$ into $\mathcal{A}$, the "assigned" elements and $\mathcal{U}$, the "unassigned" elements.

Define the binary relation E as yEz if there is an agent who is assigned z and strictly prefers y. Let $S = R \cup E$. To see that the relation S is acyclic, take a minimal cycle $z^1 S^1 z^2 S^2 z^3 S^3 \ldots z^m S^m z^1$ where each S^i is either R or E. The cycle cannot be only in R because R is acyclic. The cycle cannot be only in E since $z^{i-1} E z^i$ implies $z^{i-1} P z^i$ (because $< (x^i)_{i \in \mathcal{N}}, P >$ is an equilibrium) and an E cycle implies a P cycle. Additionally, a minimal cycle cannot contain both E and R. Suppose that such a minimal cycle exists, then there is a triple $z^{i-2} R z^{i-1} E z^i$. By $z^{i-1} E z^i$ then there is a j such that $z^i = x^j$ and $z^{i-1} \succ^j z^i$. As $\succ^j$ extends R, then $z^{i-2} \succsim^j z^{i-1} \succ^j z^i$ and therefore $z^{i-2} E z^i$ and the cycle can be shortened.

Finally, let P' be a transitive completion of S. It is convex because it extends R and $< (x^i)_{i \in \mathcal{N}}, P >$ is an equilibrium because it extends E. ∎

Claims 2 and 6 immediately imply a SWT:

CLAIM 7 (SWT-CE): *Consider a convex economy with monotonic convexity. Then, any Pareto optimal profile is a CE profile.*

E. *The Club Economy*

The Economy.—Let $\mathcal{X}$ be a set of clubs. Each agent chooses a single club. Feasibility is given by a vector of positive integers $(n_x)_{x \in \mathcal{X}}$ where n_x is the capacity of club x (nontriviality of $\mathcal{F}$ requires that the sum of the capacities be at least as large as the number of agents). Agents' preferences are convex with the convexity taken to be the monotonic convexity induced from some partial ordering R and the set of primitive orderings to contain all extensions of R. As mentioned earlier, this is the maximum set of primitive orderings and thus the notions of CE and PE coincide. Furthermore, in terms of equilibrium profiles, all three equilibrium notions coincide (Claim 6).

The housing model of Shapley and Scarf (1974) is a special case of the club economy when $n_x \equiv 1, |\mathcal{X}| = n$ and R is the empty relation (then, the preferences and the public ordering are unrestricted since all orderings are convex).

A public ordering in this example has an interpretation of social status. A member of a club cannot be admitted to a higher-status club but can switch to any lower-status club. We observe such status orderings in many walks of life, such as in employment, academia, and the caste system.

FWT.—The $\mathcal{F}$ condition holds if and only if the overall capacity of the clubs equals the number of agents. Thus, the first welfare theorem relies on the exact balance of supply and demand. This generalizes the Pareto optimality result for the standard housing economy. To see a failure of the FWT, consider the housing model of Shapley and Scarf (1974) where there are more houses than agents. It may be that all agents have the same preferences. Then an allocation which does not assign the universally top-ranked house is an equilibrium, but not Pareto optimal.

SWT.—The SWT follows from Claim 7 which is in line with the result of Piccione and Rubinstein (2007) for the case of single capacities.

F. *The Set Allocation Economy*

We now consider an economy with a set of indivisible goods in which (unlike in the housing economy) each agent can be allocated more than one good.

The Economy.—Let $\mathcal{Z}$ be a collection of items and $\mathcal{X}$ be the set of all of its subsets (menus). We will use lowercase letters for elements of $\mathcal{Z}$ and Greek symbols for menus. The set $\mathcal{F}$ contains all profiles that allocate each item to one agent. All agents have strict and convex preferences where the convexity is induced by taking R to be the inclusion relation, that is, $K(\mathcal{A}) = \{\Theta \mid \text{there exists } \Lambda \in \mathcal{A} \text{ s.t. } \Theta \supseteq \Lambda\}$. It is easy to verify that this convexity is generated by the set of all strict orderings $\{\geq_v\}$ where v is a positive-valued function on $\mathcal{Z}$ and $\geq_v$ is represented by the utility function $v(\Theta) = \sum_{z \in \Theta} v(z)$.

FWT.—All PE are Pareto optimal as the condition of Claim 5 holds. To see this, take a primitive ordering $\geq_v$. For any two feasible profiles, (Θ^i) and (Λ^i), $\sum_i v(\Theta^i) = \sum_i v(\Lambda^i) = v(\mathcal{Z})$. Therefore, if $v(\Theta^i) \geq v(\Lambda^i)$ for all i, then equality holds for all i and by the strictness of $\geq_v$, $\Theta^i = \Lambda^i$ for all i.

However, there can be CE that are not Pareto optimal. To see it, let $\mathcal{Z} = \{a, b, c, d\}$ and $n = 2$. Both agents have identical preferences that rank any cardinally larger set higher and therefore are convex. Additionally, $ac \succ^i ab$ and $bd \succ^i cd$. The profile (ab, cd) together with the convex public ordering that is identical to their preferences forms a CE which is Pareto-dominated by (ac, bd).

SWT.—Claim 7 applies and thus every Pareto optimal profile is a CE profile.

However, there can be Pareto optimal profiles that are not PE. To see it, let $\mathcal{Z} = \{a, b, c, d\}$. There are two agents, both of whom have preferences that rank any cardinally larger set higher and therefore are convex. Agent 1 ranks bd, ac above ab and ranks ab above any other two-element set. Agent 2 ranks ad, bc above cd and ranks cd above any other two-element set. The Pareto optimal profile (ab, cd) is not supported by a PE since a primitive public ordering $\geq_v$ must rank $ac \geq_v ab$ which implies that $v(c) > v(b)$. Similarly, we conclude that $v(c) > v(b) > v(d) > v(a) > v(c)$, a contradiction.

VII. Adding Initial Endowments

As mentioned earlier, our notion of a convex economy does not include initial endowments. This reflects our view of competitive equilibrium analysis as taking place in two stages. In the first stage, one defines an equilibrium concept, which involves a profile of elements and a public ordering which keeps the profile in harmony. In the second stage, one adds a method of selecting an equilibrium, perhaps given additional information such as a strength ordering of the agents, a welfare criterion, a profile of minimal satisfaction levels, or, as is standard in the competitive equilibrium literature, an initial endowment profile. These two stages do not have to be treated simultaneously as the standard competitive equilibrium analysis does. In this paper, we emphasize this by focusing only on the first stage, but in this section we demonstrate that the standard approach could be applied here as well.

Consider a convex economy with an additional initial profile. This profile can be thought of as a specification for each agent of one element which he has the right to choose independently of the elements chosen by the other agents. When the elements are objects, the initial profile can be thought of as a specification of initial ownership. We will refer to the model with an initial profile as the "extended convex economy." We define a *competitive equilibrium* (CompE) as a public ordering and a feasible profile such that each agent's assigned alternative is best for him given the "budget set" defined by his initial assignment and the public ordering.

Until now we could assume without loss of generality that the public ordering is strict. This is not the case with the CompE concept. Consider the housing model with two houses, two agents, and the initial profile (a, b). Assume that agent 1 prefers b and agent 2 prefers a. Then, the public ordering which assigns indifference to a and

b and the profile (b, a) constitutes a CompE. However, breaking the indifference will ruin the equilibrium since one of the two agents "will not be able to afford" the other element.

Any CompE profile in the extended convex economy is also an equilibrium in the model without initial endowments, so the first welfare theorems carry over to this setting.

The existence of a primitive CompE in the extended convex economy is not guaranteed even where equilibria exist in the convex economy. This is not surprising since the existence of standard competitive equilibrium is not guaranteed in an exchange economy with discrete quantities. An existence theorem would typically require augmenting the model with topological structure and applying a fixed point theorem. To illustrate, the next example shows the existence of CompE with a convex public ordering for the "give and take" economy (Section VA). The proof relies upon the fact that the space of alternatives is a continuous line segment.

Example 7: Consider the "give and take" convex economy with n agents and initial profile $\{e^i\}_{i \in \mathcal{N}}$. Assume that each agent i has single-peaked preferences on the interval $\mathcal{X} = [-1, 1]$ with a peak at $peak^i$ where $\sum peak^i < 0$. Such an economy has a CompE with an ordering P_z that decreases from -1 to some point z and from there on all elements are indifferent to z. Any such ordering is convex. Given the total indifference ordering P_{-1}, each agent would choose his ideal and the sum of their chosen actions is equal to $\sum peak^i < 0$. Given the strictly decreasing ordering P_{+1} each agent chooses an alternative which is at or to the right of his initial action and thus the sum of the chosen elements is nonnegative. By continuity, there is a z between -1 and $+1$ for which the sum of chosen elements is zero.

A nontopological reason for the nonexistence of competitive equilibrium is demonstrated by the following example:

Example 8: Consider a three-agent economy where $\mathcal{X} = \{a, b\}$ and $\mathcal{F}$ is the set of all permutations of the initial profile (a, a, b). Assume that agents 1 and 2 strictly prefer b to a and agent 3 strictly prefers a to b. A CompE public ordering cannot rank a weakly above b because both agents 1 and 2 must then be assigned b, which is not feasible. A CompE public ordering cannot rank a strictly below b since then all agents would choose a, which is also not feasible. Thus, the concept of CompE does not allow for the natural exchange between (for example) 2 and 3 leading to the Pareto-dominating profile (a, b, a). This is due to the failure of the equilibrium to create different budget sets for the two identical agents 1 and 2.

Thus, the existence of CompE is not guaranteed when the initial profile assigns identical elements to different agents. The picture changes when all elements in the initial profile are distinct. A prime example of such an existence result is Gale's beautiful proof of existence of CompE for the housing model (cited by Shapley and Scarf 1974). The proof can be extended to show the existence of an unrestricted CompE for our model with an initial profile when every agent has a distinct endowment. To see this, consider a public ordering that orders the elements in the initial profile according to Gale's top-trading cycle argument and places all initially unassigned elements above them. This public ordering is not necessarily primitive or even convex. In fact, in a four agent housing economy example on the line (Example 4), there is no UE with a convex public ordering (and thus no CompE with a convex public ordering). The following claim shows however that competitive equilibrium with a convex public ordering always exists for a family of extended economies with monotonic convexity.

CLAIM 8: *Let* $< \mathcal{N}, \mathcal{X}, \{\succsim^i\}_{i\in\mathcal{N}}, \mathcal{F}, \{\geq_k\}, \{e^i\}_{i\in\mathcal{N}} >$ *be an extended convex economy with monotonic convexity where all endowments are distinct. Then, there is a CompE with a convex public ordering.*

PROOF:

Let R be the strict binary relation generating the monotonic convexity and let $\mathcal{Y}$ be the set of alternatives in the initial profile. To construct an equilibrium, start with the set $\mathcal{X}$ and eliminate elements sequentially as follows: in each stage, if there is a remaining element not in $\mathcal{Y}$ which is not R-inferior to some remaining element which is in $\mathcal{Y}$, then eliminate a R-maximal such element. Otherwise, find a top-trading cycle among the remaining elements and remove that cycle (by the convexity of the agents' preferences such a cycle exists among the R-maximal elements). This process terminates as $\mathcal{X}$ is finite. Define the public ordering P by the removal order (earlier removed elements are higher). The ordering P is convex because it is an extension of R (if aRb then b could not be removed before a and thus aPb). An agent is assigned an element according to the top-trading cycle in which his initial element is removed. The assignment and P consist of a CompE: for any agent, any element which is weakly P-lower than his initial endowment was available when his top-trading cycle was formed. ∎

VIII. Production

We now demonstrate for a version of the housing economy a possible expansion to a world with production. Continuing the analogy with the standard setting, while consumers' preferences are taken to be exogenous, producers' preferences are determined by the public ordering. The producers are agents who maximize the public ordering and their actions determine the set of feasible profiles for the consumers. An alternative model would have "profit-maximizing" producers who operate also as agents in the model described in the previous section where their profits form their initial endowments.

Let $\mathcal{X}$ be a set of house types. In the economy there are n consumers and n producers. Each consumer possesses a preference relation on $\mathcal{X}$ and chooses a single type of house. Each producer is equipped with a nonempty set of types, his "production set," and produces a single house of one of those types. The feasibility constraint of the economy is such that the demand for each type of house (the number of consumers who wish to purchase a house of this type) is equal to the supply (the number of producers who construct a house of this type).

An equilibrium is given by a public ordering on $\mathcal{X}$ (possibly with indifferences) and feasible production and consumption profiles, such that (i) no consumer wishes to switch from his assigned type to a type that is (weakly) lower-ranked, and (ii) no producer can produce a strictly higher-ranked type.

Note that ties in the equilibrium public ordering cannot be arbitrarily broken, unlike in the model without producers, since producers' preferences depend on the public ordering. To illustrate, consider the case of $n = 2$ and $\mathcal{X} = \{a, b\}$ where both producers can produce both types, one consumer prefers type a and the other type b. If the public ordering is indifferent between the two types, then each of the producers can produce a different type, which is consumed by the consumer who favors that type and this is an equilibrium. Breaking the ties between the two types will not make the consumers change their choices but will induce one of the producers to shift to the higher-ranked type, upsetting the equilibrium.

We say that a feasible profile of actions is *Pareto optimal* if there is no other feasible profile (of consumption and production) such that all consumers who are assigned a different element are strictly better off. This efficiency concept refers to the welfare of the consumers only. In the following we show that the fundamental welfare theorems hold for the housing economy with production.

FWT.—Consider an equilibrium. Any feasible profile which Pareto-dominates the consumers' profile reassigns to each consumer an element that is at least as highly ranked as the one assigned originally and for at least one consumer is strictly higher-ranked. Therefore, at least one producer is reassigned to produce a type that is strictly higher-ranked than the one he is assigned to produce in the original profile, thus violating the producers' equilibrium condition.

SWT.—Consider a Pareto optimal profile of choices for the consumers and producers. Define two relations:

(i) $aE_c b$ if there is a consumer assigned b who strictly prefers a and

(ii) $aE_P b$ if there is a producer who produces a and could produce b.

Define $E = E_c \cup E_p$. We will show that an E cycle may only involve E_P. Consider a minimal E-cycle $a_1 E_1 a_2 E_2 \ldots . E_{K-1} a_K E_K a_1$, where each E_k is either E_c or E_p with at least one of the relations being E_c. No element appears in the cycle more than once. We now show that such a cycle leads to a Pareto improvement. If $a_k E_P a_{k+1}$, then one producer assigned to produce a_k and who is able to produce a_{k+1} is reassigned to produce a_{k+1}. If $a_k E_C a_{k+1}$, then a consumer assigned a_{k+1} and who desires a_k is reassigned to consume a_k. Each commodity, x, either does not appear in the cycle (and its demand and supply remain equal) or there is a unique k for which $a_k = x$. If $a_{k-1} E_P x E_P a_{k+1}$ (or, $a_{k-1} E_C x E_C a_{k+1}$) then the number of consumers and producers of x remains the same. If $a_{k-1} E_P x E_C a_{k+1}$ (or $a_{k-1} E_C x E_P a_{k+1}$) then one consumer and one producer will both shift to (or away from) x. Thus, the shift maintains feasibility. Finally, this cycle reassignment Pareto improves the consumers since at least one of the relations is E_C.

Define the public ordering P to be a minimal completion of E. Note that P could involve indifferences only due to E_P cycles. This public ordering supports the Pareto optimal profile: if a consumer prefers another house type to the one assigned to him then P has it strictly higher-ranked (because $aE_c b$ implies aPb strictly). For a producer, any type he can produce is weakly lower-ranked by P than the one he is assigned to produce (because $aE_P b$ implies aPb weakly).

IX. Final Comments

The paper examined economic environments in which agents make individualistic decisions but not every profile of choices is feasible. Equilibrium is sustained by a public ordering that regulates agents' choice sets to resolve the conflicts between their individualistic desires and global feasibility. The public ordering provides a decentralization method that play a role analogous to that of prices in the standard economic setting.

We are led by two motivations. The first is to propose an abstract extension of competitive equilibrium which is suited to nonclassical environments. We suggest that rankings commonly used in everyday life (such as prestige, seniority, and social values) can be thought of as playing a similar role to that competitive prices play in the standard economic setting. Our approach suggests that situations which require some sort of coordination between agents and which involve conflicts of interest might be resolved by the introduction of a social ranking that systematically limits the ability of agents to deviate from their assigned action.

The second motivation is more didactic. Abstracting the concept of a competitive equilibrium illuminates the role of the convexity of individuals' preferences, the choice of linear prices, and in particular the close relationship between them. In addition, it becomes possible to emphasize some hidden assumptions underlying the two basic welfare theorems.

Although it may be a cliché to say so, we believe that this paper could be a first stage of a deeper investigation into the abstract definition of equilibrium presented here in economic and noneconomic contexts.

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