Review

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The Mathematical Theory of Optimal Processes. By Pontryagin, Boltyanskii, Gamkrelidze & Mishchenko. Wiley, New York, 1962. viii+360 pp. $11.95.

In the last decade, there has been a revival of mathematical research in certain aspects of the theory of ordinary differential equations and, to a lesser extent, in the calculus of variations as it relates to this theory. The renewed interest in both of these subjects seems to stem from a class of physical problems arising in the analysis of control systems, or more generally, dynamical systems. The scientific and engineering problems of the space age have spawned a large family of theoretical questions concerning the systems of ordinary differential equations which are the models of the physical systems of the new technology (e.g. control mechanisms, rocket trajectories, celestial mechanics, electrical circuits, etc.).

In particular, attention has been directed to systems of equations of the form,

$$dx_i/dt = f_i(x_1, \cdots, x_n, u_1, \cdots, u_r), \qquad i = 1, \cdots, n, \tag{1}$$

where the x_i are variables which characterize the "state" of the physical system at each instant of time t, and the u_j are "control" variables which can be specified as functions of t to determine a desired behavior of the system in some time interval $t_0 \leqq t \leqq t_1$. If the set of initial values, $x(t_0) = \{x_i(t_0)\}$, and a set of control functions $\{u_j(t)\}$ are given, the unique solution of (1) can be effected. What is usually sought is a particular set $\{u_j\}$ (the "optimal control") which will minimize a given integral functional,

$$J = \int_{t_0}^{t_1} f_0(x_1, \cdots, x_n, u_1, \cdots, u_r)dt, \tag{2}$$

and at the same time transfer the system from the given initial state to a prescribed final state, $x(t_1) = \{x_i(t_1)\}$. In this form, the problem is one in the classical calculus of variations. Very often, however, there are additional constraints on the control variables in the form of inequalities such as $|u_i(t)| \leqq 1$ or $\sum u_i^2 \leqq 1$. More generally, the point $(u_1, \cdots, u_r)$ is restricted to lie in some *closed* subset of r-dimensional euclidean space. This restriction makes the problem a "nonclassical" variational problem. For example, the class of admissible controls is not—indeed cannot be—limited to continuous functions as in the classical theory but is allowed to contain piecewise continuous functions. The state variables, however, are assumed to be continuous and piecewise differentiable functions of time. The f_i are assumed to be continuous and to have continuous partial derivatives. Under these hypotheses, necessary conditions for the existence of an optimal control can be formulated as the now well-known "maximum principle" of the authors. In this book, the authors present their new maximum principle in various settings, obtain a mathematical proof of it, show its relation to the classical calculus of variations, apply it to various familiar problems, and finally consider its use in solving statistical optimal control problems.

The volume under review is an English translation from the Russian. The translator and editor have done an excellent job. The reader will find very few traces of the awkward style which frequently occurs in translations from the Russian. Furthermore, the list of references has been edited somewhat to make it more complete and useful to readers in this country. Errors were corrected and are described in footnotes. One might say that the work has gained in the translation.

There are seven chapters. In Chapter 1, the maximum principle is stated as a theorem. To formulate the theorem, one must first extend system (1) by adjoining the equation, $dx_0/dt = f_0(x_1, \cdots, x_n, u_1, \cdots, u_r)$, where f_0 is the integrand in (2). The resulting system is written in vector form as

$$dx/dt = f(x, u). \tag{3}$$

The authors then introduce the auxiliary vector, $\psi = (\psi_0, \psi_1, \cdots, \psi_n)$, by means of the equations

$$d\psi_i/dt = -\sum_{j=0}^{n} (\partial f_j/\partial x_i)\psi_j, \qquad i = 0, 1, \cdots, n, \tag{4}$$

which are linear in ψ_j for a given *phase trajectory* $x(t)$ and control $u(t)$. A new function H is defined by

$$H(\psi, x, u) = \sum_{j=0}^{n} \psi_j f_j(x, u). \tag{5}$$

Its least upper bound is $M(\psi, x) = \sup_{u \in U} H(\psi, x, u)$ for fixed ψ, x and $u \in U$, where U is some set in the r-dimensional space of the vector u. Finally, let π be the line in the $(n+1)$-dimensional x space passing through the point $(0, x(t_1))$ and parallel to the x_0 axis. A necessary condition for a control $u(t)$ to be optimal is given by the following "maximum principle":

THEOREM. *Let $u(t)$, $t_0 \leqq t \leqq t_1$, be an admissible control such that the corresponding trajectory $x(t)$ passes through $x(t_0)$ at time t_0 and through a point on line π at time t_1. In order that $u(t)$ and $x(t)$ be optimal, it is necessary that there exist a nonzero continuous $\psi(t)$ such that:* (a) *for every t, $t_0 \leqq t \leqq t_1$, $H(\psi(t), x(t), u)$ attains its maximum at $u = u(t)$*; i.e. $H(\psi(t), x(t), u(t)) = M(\psi(t), x(t))$; *and* (b) $\psi_0(t_1) \leqq 0$ *and* $M(\psi(t_1), x(t_1)) = 0$.

Chapter 2 is devoted to a proof of the maximum principle. Chapter 3 is a study of linear time-optimal processes and contains proofs showing that the optimal controls are piecewise constant functions. In chapter 4, various applications of the maximum principle are discussed. In chapter 5, the relationship of the maximal principle to the classical calculus of variations is discussed within the framework of the problem of Lagrange. It is stressed that the two approaches are equivalent if the control region U is an open set, but that the classical theorems no longer hold if U is closed, whereas the maximum principle is still applicable. The interesting case in which the state variables also are restricted to lie in a fixed closed region B is considered in chapter 6. Necessary conditions analogous to the maximum principle are derived for this case.

The final chapter 7 deals with "statistical optimal control" problems. More precisely, the authors consider the problem of determining the optimal control which causes the trajectory $x(t)$ to intersect another trajectory $y(t)$ in minimum time, where $y(t)$ is known only with a given probability.

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Differential Equations. By C. W. Leininger. Harper and Row, New York, 1962. x+271 pp. $6.00.

In this introductory text numerical methods are ignored. No reference is given to a useful table to accompany the chapter on Laplace transforms. There are far too many ambiguous remarks, misleading discussions, poorly stated problems, and simply incorrect statements for the book to be of much use. The student should not find it difficult to produce counterexamples to many statements; e.g., Theorems 3.16, 3.20, 3.21, 3.29, and 8.6.

A careful scrutiny of the manuscript by the editors and publisher could have removed many of the errors.

R. G. BUSCHMAN, Oregon State University

Finite Mathematics with Business Applications. By Kemeny, Schleifer, Snell, Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1962. 494 pp. $7.95.

The major objective of this text is "to provide a sophisticated introduction for the non-mathematician to topics in modern mathematical analysis—what has come to be known as finite mathematics." Essentially a modified version of two books written by three of the above four authors, one for behavioral and social scientists and one for students in the physical sciences, this text treats finite mathematics "in the context of business and industrial administration."

Elements of mathematical logic, set theory, probability, Markov chains, vectors and matrices, game theory, and linear programming are discussed. Of particular note is the large number of excellent illustrations of the application of finite mathematical techniques to industrial and business problems. For example, probability is used to analyze some simple reliability component problems, the relationship of matrix methods to double classification bookkeeping is developed, a parts requirements listing problem is solved using matrix methods, difference equations are applied to some present value and interest rate problems that typically occur in business enterprises, the idea of critical path analysis is