0.1 Introduction

We start our dissertation by studying deterministic dynamic control. The system we aim to control is governed by ordinary differential equations.
Short description of what is done in this chapter.

0.2 Motivating example

0.3 Finite horizon

Let us consider a finite interval $I = [t, t_1] \subset \mathbb{R}$ as the operating time of the system. At each time $s \in I$ the system is described by $x(s) \in O \subseteq \mathbb{R}^n$ and controlled by $u(s) \in U \subseteq \mathbb{R}^n$ called control space. The system is described by:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)) & s \in I \\ x(t) = x \end{cases} \tag{1}$$

For a given $x \in O$ and suitable $f : \overline{Q} \times U \to \mathbb{R}^m$, where $Q_0 = [t, t_1) \times O$. That is we impose $f \in C(\overline{Q} \times U)$ and the existence of $K_\rho > 0$ for all $\rho > 0$:

$$|f(t, x, v) - f(t, y, v)| \leqslant K_\rho |x - y| \tag{2}$$

For all $t \in I$, $x, y \in O$ and $v \in U$ such that $|v| \leqslant \rho$. Under this conditions the system 1 has a unique solution. Controls $u(\cdot)$ are assumed to be in the set $L^\infty([t, t_1]; U)$. We will soon specify more about the set of controls.
We have described a control problem. The concept of optimality is related some value function, specified by payoffs (or costs) associated to the system's states. Let $L \in C(\overline{Q} \times U)$ be the *running cost* and $\Psi \in C(I \times O)$ the *terminal cost* defined as:

$$\Psi(t, x) = \begin{cases} g(t, x) & \text{if } (t, x) \in [t, t_1) \times O \\ \psi(x) & \text{if } (t, x) \in \{t_1\} \times O \end{cases} \tag{3}$$

We define the *payoff* J as:

$$J(t, x; u) = \int_t^\tau L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \tag{4}$$

Where τ is the exit time of $(s, x(s))$ from $\overline{Q}$, that is:

$$\tau = \begin{cases} \inf\{s \in [t, t_1) \,|\, x(s) \notin \overline{O}\} & \text{if } \exists\, s \in [t, t_1) : x(s) \notin \overline{O} \\ t_1 & \text{if } x(s) \in \overline{O} \;\; \forall\, s \in [t, t_1) \end{cases} \tag{5}$$

Then a control $u^*(\cdot)$ is *optimal* if:

$$J(t, x; u^*) \leqslant J(t, x; u) \quad \forall\, u \in L^\infty(I; U) \tag{6}$$

Actually, we are being to generous with the control space. We have to impose a further condition on it, the *switching condition*. Let us assume that we have $u \in \mathcal{U}(t, x)$ and $u' \in \mathcal{U}(r, x(r))$ for $r \in [t, \tau]$. If we define:

$$\tilde{u}(s) = \begin{cases} u(s) & s \in [t, r) \\ u'(s) & s \in [r, t_1] \end{cases} \tag{7}$$

Then we impose:

$$\tilde{u}_s \in \mathcal{U}(s, \tilde{x}(s)) \quad \forall\, s \in [t, \tilde{\tau}] \tag{8}$$

Where $\tilde{x}$ is the solution to the control problem 1 with control $\tilde{u}$ and initial condition x, $\tilde{u}_s$ is the restriction of $\tilde{u}$ to $[s, t_1]$ and $\tilde{\tau}$ is the exit time of $(s, \tilde{x}(s))$ from $\overline{Q}$. This condition assures that admissible controls can be replaced as the time evolves and the resulting control is still admissible.

0.4 Dynamic programming principle

One way of tackling certain optimal control problems is via *dynamic programming.* Let us define the *value function*:

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} J(t,x;u) \tag{9}$$

For all $(t,x) \in \overline{Q}$. We get rid of the instance in which $V(t,x) = -\infty$ assuming Q to be compact, or L and Ψ to be bounded below. We aim at retrieving the argument which attains the infimum of 9. In order to immerse this optimal control problem into a dynamic programming one we see the state of the system as the state of the variable and the control function as the decision function. The basic idea behind dynamic programming techniques is to subdivide a problem into smaller problems, what does this mean in our context? We will be able to find instantaneous the value function V via a partial differential equation (PDE) called Hamilton-Jacobi-Bellman equation.
We start by stating and proving the following proposition, which provides us with an equivalent definition of the value function.

Proposition 0.4.1. *For any $(t,x) \in \overline{Q}$ and any $r \in I$ then:*

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} \left\{ \int_t^{r \wedge \tau} L(s, x(s), u(s))\, ds + g(\tau, x(\tau))\chi_{\tau < r} + V(r, x(r))\chi_{r \leqslant \tau} \right\} \tag{10}$$

Proof. Value function less than rhs. If $r > \tau$ then $\tau < t_1$ and $\Psi(r \wedge \tau, x(r \wedge \tau)) = g(\tau, x(\tau))$ and then 10 follows directly by definition. If $r \leqslant \tau$, let $\delta > 0$ then there exists $u^1 \in \mathcal{U}(r, x(r))$ such that:

$$\int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \leqslant V(r, x(r)) + \delta$$

Where x^1 is the state function corresponding to u^1 with initial condition $(r, x(r))$ and τ^1 the first exit from $\overline{Q}$ of $(s, x^1(s))$. By defining $\tilde{u}$ as for the switching condition 7 we have $\tau^1 = \tilde{\tau}$, because $\tau \geqslant r$ and then $\tilde{u}$ is u^1. Then:

$$\begin{aligned}
V(t,x) &\leqslant V(t,x;\tilde{u}) \\
&= \int_t^{\tilde{\tau}} L(s, \tilde{x}(s), \tilde{u}(s))\, ds + \Psi(\tilde{\tau}, \tilde{x}(\tilde{\tau})) \\
&= \int_t^{r} L(s, x(s), u(s))\, ds + \int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \\
&\leqslant \int_t^{r} L(s, x(s), u(s))\, ds + V(r, x(r)) + \delta
\end{aligned}$$

Since δ is arbitrary the first inequality is proved.
Value function is bigger than rhs. Let $\delta > 0$ and $U \in \mathcal{U}(t,x)$ such that:

$$\int_r^{\tau} L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \leqslant V(t,x) + \delta$$

Then:

$$
\begin{aligned}
V(t,x) &\geqslant \int_r^\tau L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + \int_{r\wedge\tau}^{\tau} L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + J(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + V(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta
\end{aligned}
$$

As δ is arbitrary we proved the proposition. □

In the proof we used the concept of δ-*optimal control*, that is the control function $u \in \mathcal{U}(r,x(r))$ such that:

$$\int_r^{\tau^1} L(s,x^1(s),u^1(s))\,ds + \Psi(\tau^1,x^1(\tau^1)) \leqslant V(r,x(r)) + \delta.$$

This new representation allows us to find the so-called *dynamic programming equation*. We have to impose that the value function is continuously differentiable, although this is not always the case. If differentiability fails, the notion of viscosity solution is needed.
Let us first impose boundary conditions of the value function. Clearly if $t = t_1$ then:

$$V(t_1,x) = \psi(x) \ \forall\, x \in \overline{O} \tag{11}$$

If $(t,x) \in [t_0,t_1) \times \partial O$ then the value function is g:

$$V(t,x) = g(t,x) \tag{12}$$

Before stating the fundamental theorem which gives sufficient conditions for a solution to the optimal problem we follow a heuristic reasoning which will help our intuition. Under the hypothesis of continuous differentiability of the value function let us rewrite the dynamic programming principle as:

$$\inf_{u\in\mathcal{U}} \left\{ \frac{1}{h}\int_t^{(t+h)\wedge\tau} L(s,x(s),u(s))\,ds + \frac{1}{h}g(\tau,x(\tau))\chi_{\tau<t+h} + \frac{1}{h}\left[V(t+h,x(t+h))\chi_{\tau\geqslant t+h} - V(t,x)\right] \right\} = 0 \tag{13}$$

Then if we formally let $h \to 0$ we get:

$$\inf_{u\in\mathcal{U}} \{L(t,x(t),u(t)) + \partial_t V(t,x(t)) + D_x V(t,x(t)) \cdot f(t,x(t),u(t))\} = 0$$

Which can be rewritten as:

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_x V(t,x)) = 0 \tag{14}$$

Where for $(t,x,p) \in \overline{Q} \times \mathbb{R}^n$ the Hamiltonian is defined as:

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n} \{-p \cdot f(t,x,v) - L(t,x,v)\}. \tag{15}$$

Equation 14 turns out to be the main sufficient condition for the control to be optimal.
Maybe only differentiability is needed (also for if, for only if we already know).

Theorem 0.4.2 (Verification Theorem). *Let $W \in C^1(\overline{Q})$ satisfy 14 and the boundary conditions 11 and 12 then:*

$$W(t,x) \leqslant V(t,x) \;\; \forall \; (t,x) \in \overline{Q}$$

Moreover, there exists $u^* \in \mathcal{U}$ such that:

$$\begin{cases} L(s,x^*(s),u^*(s)) + f(s,x^*,u^*(s)) \cdot D_x W(s,x^*(s)) = -H(s,x^*(s),D_xW(s,x^*(s))) & a.s. \text{ for } s \in [t,\tau^*] \\ W(\tau^*,x^*(\tau^*)) = g(\tau^*,x^*(\tau^*)) & \text{if } \tau^* < t_1 \end{cases} \tag{16}$$

if and only if u^* is optimal and $W = V$.

Proof. Let $u \in \mathcal{U}$, then:

$$\begin{aligned} \Psi(\tau,x(\tau)) = W(\tau,x(\tau)) &= W(t,x(t)) + \int_t^\tau \frac{d}{ds} W(s,x(s))\,ds \\ &= W(t,x(t)) + \int_t^\tau \frac{\partial}{\partial t} W(s,x(s)) + \dot{x}(s) \cdot D_x W(s,x(s))\,ds \\ &= W(t,x(t)) + \int_t^\tau \frac{\partial}{\partial t} W(s,x(s)) + f(s,x(s),u(s)) \cdot D_x W(s,x(s))\,ds \\ &\overset{\circledast}{\geqslant} W(t,x(t)) - \int_t^\tau L(s,x(s),u(s))\,ds \end{aligned}$$

Then:

$$W(t,x(t)) \leqslant J(t,x;u)$$

And therefore by taking the infimum over $\mathcal{U}$ and recalling $x(t) = x$ we get:

$$W(t,x) \leqslant V(t,x)$$

If furthermore u^* satisfies 16 then the inequality $\overset{\circledast}{\geqslant}$ is an equality, and therefore:

$$W(t,x) = J(t,x;u^*)$$

Which implies that u^* is optimal and $W(t,x) = J(t,x;u^*) = V(t,x)$. The converse will be proved in a more general setting. In particular, only differentiability is needed.

□

Theorem 0.4.2 is an important tool in determining the explicit form of and optimal control. Indeed, condition 16 can be restated as:

$$u^*(s) \in \arg\min_{v \in U} \{f(s,x^*(s),v) \cdot D_x W(s,x^*(s)) + L(s,x^*(s),v)\} \tag{17}$$

For almost all $s \in [t,t_1]$.
I have to prove verification theorem in more general case ($O \neq \mathbb{R}^n$).
We can express the optimality condition on u 17 in a differential inclusion form.

Corollary 0.4.3. *A control u^* is optimal if the corresponding state function x^* satisfies:*

$$x^* \in \{f(t,x,v) \,|\, v \in v^*(t,x)\} \tag{18}$$

0.5 Pontryagin's principle

We will use the notion of differentiability of the value function. The classical notion of differentiability which we use is the following.

Definition 0.5.1. *V is differentiable in (t, x) if there exists $V_t(t, x), V_x(t, x) \in \mathbb{R}$ such that:*

$$\lim_{(h,k)\to(0,0)} \frac{1}{|h| + |k|} |V(t + h, x + k) - V(t, x) - V_t(t, x) \cdot h - V_x(t, x) \cdot k| = 0 \tag{19}$$

Differentiability is somewhat a strong hypothesis, but it allows us to prove the folllwing proposition.

Theorem 0.5.1. *Let V be differentiable in $(t, x) \in Q$ and u^* an optimal control such that $u^* \xrightarrow{s\to t} v$, then:*

$$V_t(t, x) + L(t, x, v) + f(t, x, v) \cdot D_x V(t, x) = 0 \tag{20}$$

Proof. Let $h > 0$ s.t. $t + h < \tau$, then by 0.4.1 we have:

$$V(t, x) = \int_t^{t+h} L(s, x(s), u^*(s))\, ds + V(t + h, x(t + h))$$

But because of differentiability we have:

$$\lim_{h\to 0} \frac{1}{|h|} |V(t + h, x(t + h)) - V(t, x(t))| = V_t(t, x) + f(t, x, v) \cdot D_x V(t, x)$$

Then we get:

$$L(t, x, v) = \lim_{h\to 0} \frac{1}{|h|} \int_t^{t+h} L(s, x(s), u(s))\, ds = \lim_{h\to 0} \frac{1}{|h|} |V(t + h, x(t + h)) - V(t, x(t))| \tag{21}$$

$$= V_t(t, x) + f(t, x, v) \cdot D_x V(t, x) \tag{22}$$

□

0.6 Existence

We now prove an existence theorem for optimal controls. We study the fixed time interval case with $O = \mathbb{R}^n$ and the function f linear in v. Furthermore, we impose convexity of L in v. Under these assumptions a classical variational argument proves the optimal control existence.

Theorem 0.6.1. *Let U compact and convex, $f_1, f_2 \in C^1(\overline{Q} \times U)$ such that $f(t, x, v) = f_1(t, x) + f_2(t, x)v$ and $\partial_x f_1, \partial_x f_2, f_2$ bounded. Let also $L \in C^1(\overline{Q} \times U)$ and $L(t, x, \cdot)$ be convex for all $(t, x) \in \overline{Q}$ and the temrinal cost $\phi \in C(\mathbb{R}^n)$.*
Then there exist an optimal control $u^*(\cdot)$.

Proof. Let u_n a minimizing sequence such that:

$$\lim_{n\to+\infty} J(t, x; u_n) = V(t, x) \tag{23}$$

Let $x_n(\cdot)$ be the solutions to 1 with $u = u_n$. If we show both sequence to converge respectively (weakly) to u^* and uniformly x^* (along subsequences) such that the latter is again the solution to 1 with $u = u^*$, then:

$$J(t, x; u_n) = \int_t^{t_1} L(s, x^*(s), u_n(s))\, ds + \int_t^{t_1} L(s, x_n(s), u_n(s)) - L(s, x^*(s), u_n(s))\, ds + \phi(x_n(t_1))$$

But then:

$$\liminf_{n\to+\infty} \int_t^{t_1} L(s, x_n(s), u_n(s)) - L(s, x^*(s), u_n(s))\, ds = 0$$

And $\psi(x_n(t_1)) \xrightarrow{n\to+\infty} \psi(x^*(t_1))$. But then:

$$\liminf_{n\to+\infty} J(t, x; u_n) = \liminf_{n\to+\infty} \int_t^{t_1} L(s, x^*(s), u_n(s))\, ds \geqslant \int_t^{t_1} L(s, x^*(s), u^*(s))\, ds \tag{24}$$

Because L is convex in u[1]. Therefore:

$$V(t, x) \leqslant J(t, x; u^*) \leqslant \liminf_{n\to+\infty} J(t, x; u_n) = V(t, x)$$

We need to prove convergence of x_n and u_n. Because U is compact and *convex* then $L^\infty([t, t_1]; U)$ is weakly sequentially compact. For what concerns x_n we use Ascoli-Arzela's theorem to show that is admits a uniformly convergent subsequence. Being uniformly limitated comes from:

$$|x_n(s)| \leqslant |x_n(s) - x| + |x| = |x| + \int_t^s \frac{d}{dr}|x_n(r)|\, dr \tag{25}$$

$$= |x| + \int_t^s |f_1(r, x_n(r)) + f_2(r, x_n(r)) \cdot u_n(r)|\, dr \tag{26}$$

$$\leqslant |x| + \int_t^s \|\partial_x f_1\|_\infty |x_n(r)| + \|f_2\|_\infty |u_n|\, dr \tag{27}$$

$$\leqslant C + K \left(\int_t^s |x_n(r)|\, dr \right) \tag{28}$$

Then by Gronwall's lemma x_n is uniformly limitated. Equicontinuity comes from the uniform buondedness of the derivative $\dot{x}_n(s)$. Therefore, we know that there exist the weak limit u^* and the uniform limit x^*. The latter is the solution of 1 with $u = u^*$. Indeed:

$$\begin{aligned} x_n(s) &= x + \int_t^s \frac{d}{dr} x_n(r)\, dr \\ &= x + \underbrace{\int_t^s f_1(r, x_n(r)) + f_2(r, x_n(r)) u^*(r)\, dr}_{A} + \underbrace{\int_t^s [f_2(r, x_n(r)) - f_2(r, x^*(r))]\, [u_n(r) - u^*(r)]\, dr}_{B} \\ &\quad + \underbrace{\int_t^s f_2(r, x^*(r))\, [u_n(r) - u^*(r)]\, dr}_{C} \end{aligned}$$

Letting $n \to +\infty$ we get B (by weak convergence and boundedness of f_2) and C (by weak convergence) going to 0 while we obtain:

$$A \xrightarrow{n\to+\infty} \int_t^s f_1(r, x^*(r)) + f_2(r, x^*(r)) u^*(r)\, dr$$

And therefore the thesis. □

Remark. *In the proof we asserted inequality 24 by convexity of the running cost. Indeed, by convexity and being C^1:*

$$L(s, x(s), u_n(s)) \geqslant L(s, x(s), u^*(s)) + [u_n(s) - u^*(s)]\, L_u(s, x(s), u^*(s))$$

Then by integrating and taking $\liminf_{n\to+\infty}$ *we get the inequality (using weak convergence of u_n).*

[1] Explained in remark 0.6