OPTIMAL CONTROL VIA DYNAMIC PROGRAMMING

Viscosity Solutions

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Non differentiability

addplot Let us consider the calculus of variation problem:

$$\inf_{x \in Lip([0,1];[-1,1])} \int_t^{t_1} 1 + \frac{1}{4}(\dot{x}(s))^2 \, ds,$$

then the H-J equations are

$$\dot{x}^*(s) = 2p^*(s), \; \dot{p}^*(s) = 0,$$

which define the value function

$$V(t,x) = \begin{cases} 1-t & x \leq t \\ 1-t & x \geq t, \end{cases}$$

Vanishing Viscosity

Let

$$\begin{cases} u_t + H(u, Du) = 0, & \mathbb{R}^n \times (0, +\infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

We perturbate by second order derivative the equation

$$u_t^\epsilon + H(u^\epsilon, Du^\epsilon) - \epsilon \Delta u^\epsilon = 0,$$

which happens to have a solution[1]. Usually, Ascoli-Arzelà's hypotheses are satisfied[2] we take the limit $u \xleftarrow{j \to +\infty} u^{\epsilon_j}$ as a candidate solution. We lack information about its derivatives.

[1]Galerkin's approximations, Evans section 7.1.2

[2]Easy applications have a uniform Lipschitz bound. Barles-Perthame procedure has a wide range of applications.

Vanishing Viscosity

Then we take v smooth and (t_0, x_0) s.t. $u - v$ has a local maximum and there it nullifies. It implies

$$(u^\epsilon - v)(x_{\epsilon_j}, t_{\epsilon_j}) \geq (u^\epsilon - v)(x, t),$$

for (x, t) close to (x_0, t_0) and $(x_{\epsilon_j}, t_{\epsilon_j}) \xrightarrow{j \to +\infty} (x_0, t_0)$[3]. Since $u_{\epsilon_j} - v$ is maximized at $(x_{\epsilon_j}, t_{\epsilon_j})$

$$u_t^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = v(x_{\epsilon_j}, t_{\epsilon_j}), Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = Dv(x_{\epsilon_j}, t_{\epsilon_j}), -\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \geq -\Delta v(x_{\epsilon_j}, t_{\epsilon_j}).$$

Letting $j \to +\infty$ we get

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0. \tag{2}$$

[3]Because of local uniform convergence.

Viscosity Solution

Definition

A viscosity solution of 1 is a function u bounded and uniformly continuous on $\mathbb{R}^n \times [0, T]$ for all $T > 0$ such that for all $v \in C^{+\infty}(\mathbb{R}^n \times (0, +\infty))$:

$$v_t(x,t) + H(Dv(x,t), x) \leq 0$$

for all $(x,t) \in \arg\max\{u - v\}$ and:

$$v_t(x,t) + H(Dv(x,t), x) \geq 0$$

for all $(x,t) \in \arg\min\{u - v\}$. Furthermore, $u \equiv g$ for $t = 0$.

Abstract Dynamic Programming

Let Σ be a closed subset of a Banach space and $\mathcal{C}$ a collection of functions on Σ, closed under addition

$$\mathcal{T}_{tt}\phi = \phi \tag{3}$$

and

$$\begin{aligned} \mathcal{T}_{tr}\phi \leq \mathcal{T}_{ts}\psi \text{ if } \phi \leq \mathcal{T}_{rs}\psi \\ \mathcal{T}_{tr}\phi \geq \mathcal{T}_{ts}\psi \text{ if } \phi \geq \mathcal{T}_{rs}\psi. \end{aligned} \tag{4}$$

Provided that $\mathcal{T}_{rt} : \mathcal{C} \to \mathcal{C}$ implies the semigroup property and 3 is equivalent to monotonicity.

The semigroup property will mimic the dynamic programming principle.

Abstract Dynamic Programming

Let $\Sigma = \overline{O} \subset \mathbb{R}^n$, $\mathcal{C} = \mathcal{M}(\Sigma)$, and

$$\mathcal{T}_{t,r;u}\psi(x) = \int_t^{\tau \wedge r} L(s, x(s), u(s)),\, ds + g(\tau, x(\tau))\chi_{\tau<r} + \psi(x(r))\chi_{\tau \geq r},$$

and $\mathcal{T}_{tr}\psi = \inf_{u \in \mathcal{U}(t,x)} \mathcal{T}_{t,r;u}\psi$. Under the usual assumption on the running and terminal costs $\mathcal{T}_{tr}\psi \in \mathcal{C}$, then the programming principle reads

$$\mathcal{T}_{tt_1}\psi(x) = \mathcal{T}_{tr}\left(\mathcal{T}_{rt_1}\psi\right)(x).$$

Abstract Dynamic Programming

Let us define $V(t,x) = (\mathcal{T}_{tt_1}\psi)(x)$. Then

$$-\frac{1}{h}\left[\mathcal{T}_{tt+h}V(t+h,\cdot)(x) - V(t,x)\right] = 0.$$

We ask for $\{\mathcal{G}_t\}_{t\in[t_0,t_1]}$ functions on Σ such that:

$$\lim_{h\to 0}\frac{1}{h}\left[\mathcal{T}_{tt+h}V(t+h,\cdot)(x) - V(t,x)\right] = \frac{\partial}{\partial t}w(t,x) - (\mathcal{G}_t w(t,\cdot))(x), \tag{5}$$

for all $w \in \mathcal{D}$[4]. Then the dynamic programming equation reads

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0,\ (t,x) \in Q. \tag{6}$$

[4]Continuity assumptions are made on $\mathcal{D}$.

Conditions on D

The space $\mathcal{D}$ is taken to be a vector space and such that for all functions $\omega \in \mathcal{D}$

$$\frac{\partial w}{\partial t} \text{ and } \mathcal{G}_t w(t, \cdot) \text{ are continuous on } Q, \tag{7}$$

and

$$w(t, \cdot) \in \mathcal{C}, \ \ \forall \ t \in [t_0, t_1]. \tag{8}$$

The elements of $\mathcal{D}$ are called test functions and $\mathcal{G}_t$ the infinitesimal generator of $\mathcal{T}_{tr}$. Explicit choices of C and D will vary from case to case, usually, they are chosen to satisfy certain integrability conditions on the functions.

Viscosity Solutions

Definition

Let $W \in C([t_0, t_1] \times \Sigma)$. W is a *viscosity subsolution* of 6 in Q if for every $w \in \mathcal{D}$:

$$-\frac{\partial}{\partial t} w\left(\bar{t}, \bar{x}\right) + \left(\mathcal{G}_{\bar{t}} w\left(\bar{t}, \cdot\right)\right)\left(\bar{x}\right) \leq 0, \tag{9}$$

at every $\left(\bar{t}, \bar{x}\right) \in \arg\max_{(t,x) \in Q} \{(W - w)(t, x)\}$, and $W\left(\bar{t}, \bar{x}\right) = w\left(\bar{t}, \bar{x}\right)$.
W is a *viscosity supersolution* of 6 in Q if for every $w \in \mathcal{D}$:

$$-\frac{\partial}{\partial t} w\left(\bar{t}, \bar{x}\right) + \left(\mathcal{G}_{\bar{t}} w\left(\bar{t}, \cdot\right)\right)\left(\bar{x}\right) \geq 0, \tag{10}$$

at every $\left(\bar{t}, \bar{x}\right) \in \arg\min_{(t,x) \in Q} \{(W - w)(t, x)\}$, and $W\left(\bar{t}, \bar{x}\right) = w\left(\bar{t}, \bar{x}\right)$.

W is a *viscosity solution* if it is a subsolution and a supersolution.

Strandard Approach

Historically the notion of viscosity solution was introduced for partial differential equations, that is when $\mathcal{G}_t$ is a partial differential operator.

The definition of viscosity solution for

$$-\frac{\partial}{\partial t}W(t,x)+F(t,x,D_xW(t,x),D_x^2W(t,x),W(t,x))=0, \qquad (11)$$

is the same we gave with $\mathcal{G}_t$, a part from the space of test functions:

$$w\in C^\infty(Q).$$

Theorem

Let all the previous assumptions and $W\in C_p(\overline{Q})\cap\mathcal{M}(\overline{Q})$ *and* $\mathcal{D}\subset C^{1,2}(Q)$. *Then the solution concepts coincide.*

Value Function as Viscosity solution

Recall

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0,\ (t,x) \in Q. \tag{6}$$

We have

Theorem

Let $\{\mathcal{T}_{tr}\}_{t_0 \le t \le r \le t_1}$ such that 3,3 and also there exists a vector space $\mathcal{D}$ and another family of operator $\{\mathcal{G}_t\}_{t \in [t_0,t_1]}$ such that 7 and 8 hold. Let

$$V(t,x) = (\mathcal{T}_{tt_1}\psi)(x).$$

If $V \in C(Q)$ then it is a viscosity solution of 6.

Value Function as Viscosity Solution

We now prove that it is a viscosity solution of the dynamic programming equation under two sets of assumptions.

Theorem

Let U be a bounded space of control and $f \in C(\overline{Q} \times U)$ such that $f(t,x,v) \leq K(1+|x|)$. Then for every $w \in C^1(Q) \cap \mathcal{M}(\overline{Q})$

$$\lim_{h \to 0} \frac{1}{h} \left[(\mathcal{T}_{tt+h} w(t+h,\cdot))(x) - w(t,x) \right] = \frac{\partial}{\partial t} w(t,x) - H(t,x,D_x w(t,x)), \tag{12}$$

for all $(t,x) \in \overline{Q}$.

It implies that under these assumptions, V is a viscosity solution (Theorem 4).

We prove lim inf and lim sup.

lim sup) It follows from direct computations. It does not need neither boundedness of U or sub-linearity of f.

lim inf) Sub-linearity of f and Gronwall imply

$$|x(r) - x| \leq (1 + |x|)e^{K(t-r)-1}, \ \forall\, r \geq t.$$

Then for n sufficiently large we can find a $1/n^2$-optimal control $u^n(\cdot)$, which leads to

$$n\left[\left(\mathcal{T}_{tt+\frac{1}{n}} w\left(t+\frac{1}{n}\right)\right)(x) - w(t,x)\right] \geq \frac{\partial}{\partial t} w(t,x) + n\int_t^{t+1/n} L(s, x(s), u^n(s))\,ds$$

$$+ n\int_t^{t+1/n} f(t, x, u^n(s))\,ds \cdot D_x w(t,x) + e(n).$$

Value Function as Viscosity Solution

The previous result holds under quite stringent hypotheses. We can relax those assumptions by asking for the existence of an optimal control.

Theorem

If for each $(t,x) \in Q$ there exists a $u^* \in \mathcal{U}(t,x)$ be an optimal control, then a continuous value function is a viscosity solution of its dynamic programming equation.

Uniqueness of Solution

Let us consider

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_xV(t,x)) = 0,\ (t,x) \in Q.^{[5]} \tag{13}$$

Theorem

Let W and V viscosity subsolution and supersolution of 13 in Q, respectively. If Q is unbounded we assume W, V to be bounded and uniformly continuous on its closure. Then

$$\sup_{\overline{Q}}[W - V] = \sup_{\partial * Q}[W - V].$$

[5] H(t,x,p)-H(s,y,p')$\leq h(t-s+x-y) + h(t-s)p + Kx - yp + Kp - p',\ |H_p| \leq K,\ |H_t| + |H_x| \leq K'(1+|p|)$.

Continuity of Solution

We recall that

$$|f(t,x,v) - f(t,y,v)| \leq K_\rho |x-y|, \;\; \forall \; |v| \leq \rho. \tag{14}$$

Theorem

Let a bounded control space U, $Q = [t_0, t_1) \times \mathbb{R}^n$. Assume that f, L, ψ are bounded, f satisfies 14 and L, ψ uniformly continuous. Then the value function V is bounded and uniformly continuous.

Corollary

Under the previous assumptions, the value function is the unique viscosity solution of the dynamic programming equation with fixed terminal conditions

Pontryagin's Principle

Definition

Let $W \in C(\overline{Q})$ and $(t, x) \in Q$. The set of *superdifferentials* $D^{+}W(t, x)$ of W at (t, x) is the collection of all $(q, p) \in \mathbb{R} \times \mathbb{R}^n$ such that there exists some $w \in C^1(Q)$ for which:

$$(q, p) = \left(\frac{\partial}{\partial t} w(t, x), D_x w(t, x) \right), \tag{15}$$

and $(t, x) \in \arg\max \left\{ (W - w)(s, y) \,|\, (s, y) \in \overline{Q} \right\}$.

The set of *subdifferentials* $D^{-}W(t, x)$ of W at (t, x) as the collection of all $(q, p) \in \mathbb{R} \times \mathbb{R}^n$ such that there exists some $w \in C^1(Q)$ for which:

$$(q, p) = \left(\frac{\partial}{\partial t} w(t, x), D_x w(t, x) \right), \tag{16}$$

and $(t, x) \in \arg\min \left\{ (W - w)(s, y) \,|\, (s, y) \in \overline{Q} \right\}$.

Pontryagin's Principle

We recall the definition of the adjoint variable for a state variable x defined by the flow f, a control u, a terminal condition ψ, a Lagrangian L and a Hamiltonian H:

$$\dot{p}_j^*(s) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_j} f_i(s, x^*(s), u^*(s)) p_i(s) - \frac{\partial}{\partial x_j} L(s, x^*, u^*), \tag{17}$$

And also:

$$p(s) \cdot f(s, x^*(s), u^*(s)) + L(s, x^*(s), u^*(s)) = -H(s, x^*(s), u^*(s), p^*(s)), \tag{18}$$

With:

$$p^*(t_1) = D\psi(x^*(t_1)). \tag{19}$$

Pontryagin's Principle

Theorem

Let $u^*(\cdot)$ be an optimal control at (t, x) which is right continuous at each $[t, t_1)$, and $p^*(s)$ defined by 17, 18 and 19. Then for each $s \in [t, t_1)$

$$\Big(H(s, x^*(s), p^*(s)), p^*(s)\Big) \in D^+V(s, x^*(s)). \tag{20}$$

Proof Idea

By definition

$$V(r,y) \leq J(r,y;u^*),\ V(s,x^*(s)) \leq J(s,x^*(s);u^*).$$

We prove

$$\frac{\partial}{\partial r} J(s,x^*(s);u^*) = p^*(s),\ D_y J(s,x^*(s);u^*) = H(s,x^*(s),p^*(s)).$$

Both are proven by direct computation, via the FTC and

$$\frac{d}{dr}\left\{\sum_{j=1}^{n} \frac{\partial}{\partial x_i} x_j(r,x^*(r)) p_i(r)\right\} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_j} L(s,x^*(r),u^*(r)) \frac{\partial}{\partial x_i} x_j(r,x^*(r)).$$