0.1 Introduction

We start our dissertation by studying deterministic dynamic control. The system we aim to control is governed by ordinary differential equations.
Short description of what is done in this chapter.

0.2 Motivating example

0.3 Finite horizon

Let us consider a finite interval $I = [t, t_1] \subset \mathbb{R}$ as the operating time of the system. At each time $s \in I$ the system is described by $x(s) \in O \subseteq \mathbb{R}^n$ and controlled by $u(s) \in U \subseteq \mathbb{R}^n$ called control space. The system is described by:

$$\begin{cases} \dot{x}(s) = f(x(s), u(s)) & s \in I \\ x(t) = x \end{cases} \tag{1}$$

For a given $x \in O$ and suitable $f : \overline{Q} \times U \to \mathbb{R}^m$, where $Q_0 = [t, t_1) \times O$. That is we impose $f \in C(\overline{Q} \times U)$ and the existence of $K_\rho > 0$ for all $\rho > 0$:

$$|f(t, x, v) - f(t, y, v)| \leqslant K_\rho |x - y| \tag{2}$$

For all $t \in I$, $x, y \in O$ and $v \in U$ such that $|v| \leqslant \rho$. Under this conditions the system 1 has a unique solution. Controls $u(\cdot)$ are assumed to be in the set $L^\infty([t, t_1]; U)$. We will soon specify more about the set of controls.
We have described a control problem. The concept of optimality is related some value function, specified by payoffs (or costs) associated to the system's states. Let $L \in C(\overline{Q} \times U)$ be the *running cost* and $\Psi \in C(I \times O)$ the *terminal cost* defined as:

$$\Psi(t, x) = \begin{cases} g(t, x) & \text{if } (t, x) \in [t, t_1) \times O \\ \psi(x) & \text{if } (t, x) \in \{t_1\} \times O \end{cases} \tag{3}$$

We define the *payoff* J as:

$$J(t, x; u) = \int_t^\tau L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \tag{4}$$

Where τ is the exit time of $(s, x(s))$ from $\overline{Q}$, that is:

$$\tau = \begin{cases} \inf\{s \in [t, t_1) \mid x(s) \notin \overline{O}\} & \text{if } \exists\, s \in [t, t_1) : x(s) \notin \overline{O} \\ t_1 & \text{if } x(s) \in \overline{O} \;\; \forall\, s \in [t, t_1) \end{cases} \tag{5}$$

Then a control $u^*(\cdot)$ is *optimal* if:

$$J(t, x; u^*) \leqslant J(t, x; u) \qquad \forall\, u \in L^\infty(I; U) \tag{6}$$

Actually, we are being to generous with the control space. We have to impose a further condition on it, the *switching condition*. Let us assume that we have $u \in \mathcal{U}(t, x)$ and $u' \in \mathcal{U}(r, x(r))$ for $r \in [t, \tau]$. If we define:

$$\tilde{u}(s) = \begin{cases} u(s) & s \in [t, r) \\ u'(s) & s \in [r, t_1] \end{cases} \tag{7}$$

Then we impose:

$$\tilde{u}_s \in \mathcal{U}(s, \tilde{x}(s)) \quad \forall\, s \in [t, \tilde{\tau}] \tag{8}$$

Where $\tilde{x}$ is the solution to the control problem 1 with control $\tilde{u}$ and initial condition x, $\tilde{u}_s$ is the restriction of $\tilde{u}$ to $[s, t_1]$ and $\tilde{\tau}$ is the exit time of $(s, \tilde{x}(s))$ from $\overline{Q}$. This condition assures that admissible controls can be replaced as the time evolves and the resulting control is still admissible.

0.4 Dynamic programming principle

One way of tackling certain optimal control problems is via *dynamic programming.* Let us define the *value function*:

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} J(t,x;u) \tag{9}$$

For all $(t,x) \in \overline{Q}$. We get rid of the instance in which $V(t,x) = -\infty$ assuming Q to be compact, or L and Ψ to be bounded below. We aim at retrieving the argument which attains the infimum of 9. In order to immerse this optimal control problem into a dynamic programming one we see the state of the system as the state of the variable and the control function as the decision function. The basic idea behind dynamic programming techniques is to subdivide a problem into smaller problems, what does this mean in our context? We will be able to find instantaneous the value function V via a partial differential equation (PDE) called Hamilton-Jacobi-Bellman equation.
We start by stating and proving the following proposition, which provides us with an equivalent definition of the value function.

Proposition 0.4.1. *For any $(t,x) \in \overline{Q}$ and any $r \in I$ then:*

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} \left\{ \int_t^{r \wedge \tau} L(s, x(s), u(s))\, ds + g(\tau, x(\tau))\chi_{\tau < r} + V(r, x(r))\chi_{r \leqslant \tau} \right\} \tag{10}$$

Proof. Value function less than rhs. If $r > \tau$ then $\tau < t_1$ and $\Psi(r \wedge \tau, x(r \wedge \tau)) = g(\tau, x(\tau))$ and then 10 follows directly by definition. If $r \leqslant \tau$, let $\delta > 0$ then there exists $u^1 \in \mathcal{U}(r, x(r))$ such that:

$$\int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \leqslant V(r, x(r)) + \delta$$

Where x^1 is the state function corresponding to u^1 with initial condition $(r, x(r))$ and τ^1 the first exit from $\overline{Q}$ of $(s, x^1(s))$. By defining $\tilde{u}$ as for the switching condition 7 we have $\tau^1 = \tilde{\tau}$, because $\tau \geqslant r$ and then $\tilde{u}$ is u^1. Then:

$$\begin{aligned}
V(t,x) &\leqslant V(t,x;\tilde{u}) \\
&= \int_t^{\tilde{\tau}} L(s, \tilde{x}(s), \tilde{u}(s))\, ds + \Psi(\tilde{\tau}, \tilde{x}(\tilde{\tau})) \\
&= \int_t^{r} L(s, x(s), u(s))\, ds + \int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \\
&\leqslant \int_t^{r} L(s, x(s), u(s))\, ds + V(r, x(r)) + \delta
\end{aligned}$$

Since δ is arbitrary the first inequality is proved.
Value function is bigger than rhs. Let $\delta > 0$ and $U \in \mathcal{U}(t,x)$ such that:

$$\int_r^{\tau} L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \leqslant V(t,x) + \delta$$

Then:

$$
\begin{aligned}
V(t,x) &\geqslant \int_r^\tau L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + \int_{r\wedge\tau}^{\tau} L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + J(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + V(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta
\end{aligned}
$$

As δ is arbitrary we proved the proposition. □

In the proof we used the concept of δ-*optimal control*, that is the control function $u \in \mathcal{U}(r,x(r))$ such that:

$$\int_r^{\tau^1} L(s,x^1(s),u^1(s))\,ds + \Psi(\tau^1,x^1(\tau^1)) \leqslant V(r,x(r)) + \delta.$$

This new representation allows us to find the so-called *dynamic programming equation.* We have to impose that the value function is continuously differentiable, although this is not always the case. If differentiability fails, the notion of viscosity solution is needed.
We restrict our study to $O = \mathbb{R}^n$, as the other cases need viscosity solutions. In this situation the boundary condition is simply:

$$V(t_1,x) = \psi(x) \ \ \forall\, x \in \mathbb{R}^n \tag{11}$$

Before stating the fundamental theorem which gives sufficient conditions for a solution to the optimal problem we follow a heuristic reasoning which will help our intuition. Under the hypothesis of continuous differentiability of the value function let us rewrite the dynamic programming principle as:

$$\inf_{u\in\mathcal{U}}\left\{\frac{1}{h}\int_t^{(t+h)\wedge\tau} L(s,x(s),u(s))\,ds + \frac{1}{h}g(\tau,x(\tau))\chi_{\tau<t+h} + \frac{1}{h}\left[V(t+h,x(t+h))\chi_{\tau\geqslant t+h} - V(t,x)\right]\right\} = 0 \tag{12}$$

Then if we formally let $h \to 0$ we get:

$$\inf_{u\in\mathcal{U}}\{L(t,x(t),u(t)) + \partial_t V(t,x(t)) + D_x V(t,x(t))\cdot f(t,x(t),u(t))\} = 0$$

Which can be rewritten as:

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_xV(t,x)) = 0 \tag{13}$$

Where for $(t,x,p) \in \overline{Q}\times\mathbb{R}^n$ the Hamiltonian is defined as:

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n}\{-p\cdot f(t,x,v) - L(t,x,v)\}\,. \tag{14}$$

Equation 13 turns out to be the main sufficient condition for the control to be optimal.

Theorem 0.4.2 (Verification Theorem). *Let $W \in C^1(\overline{Q})$ satisfy 13 and 11 then:*

$$W(t,x) \leqslant V(t,x)$$

Moreover, there exists $u^* \in \mathcal{U}$ such that:

$$L(s,x^*(s),u^*(s)) + f(s,x^*,u^*(s))\cdot D_xW(s,x^*(s)) = -H(s,x^*(s),D_xW(s,x^*(s))) \tag{15}$$

For almost all $s \in [t,t_1]$ if and only if u^* is optimal and $W = V$.

Proof. Let $u \in \mathcal{U}$, then:

$$\begin{aligned}
\psi(x(t_1)) &= W(t_1, x(t_1)) = W(t, x(t)) + \int_t^{t_1} \frac{d}{ds} W(s, x(s))\, ds \\
&= W(t, x(t)) + \int_t^{t_1} \frac{\partial}{\partial t} W(s, x(s)) + \dot{x}(s) \cdot D_x W(s, x(s))\, ds \\
&= W(t, x(t)) + \int_t^{t_1} \frac{\partial}{\partial t} W(s, x(s)) + f(s, x(s), u(s)) \cdot D_x W(s, x(s))\, ds \\
&\overset{\circledast}{\geqslant} W(t, x(t)) - \int_t^{t_1} L(s, x(s), u(s))\, ds
\end{aligned}$$

Then:

$$W(t, x(t)) \leqslant J(t, x; u)$$

And therefore by taking the infimum over $\mathcal{U}$ and recalling $x(t) = x$ we get:

$$W(t, x) \leqslant V(t, x)$$

If furthermore u^* satisfies 15 then the inequality $\overset{\circledast}{\geqslant}$ is an equality, and therefore:

$$W(t, x) = J(t, x; u^*)$$

Which implies that u^* is optimal and $W(t, x) = J(t, x; u^*) = V(t, x)$. Conversely, if u^* is optimal:

$$\begin{aligned}
-\int_t^{t_1} L(s, x(s), u^*(s))\, ds &\leqslant \int_t^{t_1} \frac{\partial}{\partial t} W(s, x(s)) + f(s, x(s), u^*(s)) \cdot D_x W(s, x(s))\, ds \\
&= \int_t^{t_1} \frac{\partial}{\partial t} W(s, x(s)) + f(s, x(s), u(s)) \cdot D_x W(s, x(s))\, ds =
\end{aligned}$$

□

Theorem 0.4.2 is an important tool in determining the explicit form of and optimal control. Indeed, condition 15 can be restated as:

$$u^*(s) \in \underset{v \in U}{\arg\min} \{ f(s, x^*(s), v) \cdot D_x W(s, x^*(s)) + L(s, x^*(s), v) \} \tag{16}$$

For almost all $s \in [t, t_1]$.