OPTIMAL CONTROL VIA DYNAMIC PROGRAMMING

Deterministic, Stochastic and Viscosity Solutions

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DETERMINISTIC

Introduction Optimal Control Problems

Notably, a dynamical system (physical, social, biological, etc.) can be described by its derivatives. Control theory assumes the derivative to be influenced by external factors, called controls.

It aims at finding the ”best” system behavior under a control. Optimality will be in the form of minimizing a cost function.

Control Problem Formulation

Consider a finite interval $I = [t, t_1] \subset \mathbb{R}$ as the operating time of a system, where at any time $s \in I$, the system is described by $x(s) \in O \subseteq \mathbb{R}^m$ and controlled by $u(s) \in U \subseteq \mathbb{R}^n$, known as control and the control space. The system is defined by:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), & s \in I \\ x(t) = x \end{cases}$$

With f satisfying the Lipschitz condition to ensure a unique solution. Controls $u(\cdot)$ are in $L^\infty([t, t_1]; U)$.

Optimality

Optimality is measured by a payoff J, incorporating continuous running cost L and terminal cost Ψ, defined as:

$$J(t,x;u) = \int_t^\tau L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)),$$

where τ is the exit time of $(s,x(s))$ from $Q = [t,t_1] \times O$. The terminal cost Ψ has the form:

$$\Psi(t,x) = \begin{cases} g(t,x) & \text{if } (t,x) \in [t,t_1) \times O \\ \psi(x) & \text{if } (t,x) \in \{t_1\} \times O \end{cases}$$

Optimal control theory aims at minimizing J across admissible controls.

Dynamic Programming Principle

We define the value function

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} J(t,x;u). \tag{1}$$

Dynamic Programming reduces the maximization problem over the whole time span to a maximization problem at each point in time.

Theorem
For any $(t,x) \in \overline{Q}$ and any $r \in I$ then:

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} \left\{ \int_t^{r \wedge \tau} L(s, x(s), u(s))\, ds + g(\tau, x(\tau))\chi_{\tau < r} + V(r, x(r))\chi_{r \leq \tau} \right\} \tag{2}$$

Dynamic Programming Equation

From the dynamic programming principle, we get a necessary condition for a value function to be optimal. Formally, we take $h > 0$ and rewrite 2 as:

$$\inf_{u\in\mathcal{U}}\{\frac{1}{h}\int_t^{(t+h)\wedge\tau} L(s,x(s),u(s))\,ds + \frac{1}{h}g(\tau,x(\tau))\chi_{\tau<t+h} \\ + \frac{1}{h}\left[V(t+h,x(t+h))\chi_{\tau\geq t+h} - V(t,x)\right]\} = 0.$$

Then letting $h \to 0^+$:

$$\inf_{u\in\mathcal{U}}\left\{L(t,x(t),u(t)) + \partial_t V(t,x(t)) + D_x V(t,x(t))\cdot f(t,x(t),u(t))\right\} = 0 \quad (3)$$

Dynamic Programming Equation

We can generally define:

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_xV(t,x)) = 0. \tag{4}$$

Where for $(t,x,p) \in \overline{Q} \times \mathbb{R}^n$ the Hamiltonian is defined as:

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n} \left\{-p \cdot f(t,x,v) - L(t,x,v)\right\}. \tag{5}$$

Equation 4 turns out to be the main sufficient condition for the value function to be optimal.

Verification Theorem

Theorem (Verification Theorem)
Let $W \in C^1(\overline{Q})$ satisfy 4 and the boundary conditions then:

$$W(t,x) \leq V(t,x) \ \forall \ (t,x) \in \overline{Q}$$

Moreover, there exists $u^* \in \mathcal{U}$ such that:

$$\begin{cases} L(s,x^*(s),u^*(s)) + f(s,x^*,u^*(s)) \cdot D_x W(s,x^*(s)) = -H(s,x^*(s),D_x W(s,x^*(s)) \\ W(\tau^*,x^*(\tau^*)) = g(\tau^*,x^*(\tau^*)) \end{cases} \tag{6}$$

if and only if u^* is optimal and $W = V$.

Pontryagin's Principle

Porntryagin's principle gives another perspective on the problem. It asserts the existence of a *costate* variable that satisfies certain, conditions under optimality of the value function.

The *control state Hamiltonian* is defined as:

$$\tilde{H}(s,x,u,p) = -p \cdot f(s,x,u) - L(s,x,u),$$

with p representing the system's *costate*.

Pontryagin's Principle

Theorem

Let u^* be an optimal control and x^* its corresponding trajectory. Then there exists a function $p^* : [t, t_1] \to O$ such that:

$$\dot{x}^*(s) = D_p \tilde{H}(s, x^*(s), u^*(s), p^*(s)) \tag{7}$$

$$\dot{p}^*(s) = -D_x \tilde{H}(s, x^*(s), u^*(s), p^*(s)) \tag{8}$$

And also:

$$\tilde{H}(s, x^*(s), u^*(s), p^*(s)) = \sup_{v \in U} \tilde{H}(s, x^*(s), v, p^*(s)) \tag{9}$$

With:

$$p^*(t_1) = D\psi(x^*(t_1)) \tag{10}$$

Connection to Dynamic Programming

Pontryagin's Principle and dynamic programming, though seemingly different, are closely linked.

Theorem

Let u^* be an optimal right-continuous control and x^* its corresponding trajectory. Assume that the value function V is differentiable at $(s, x^*(s))$ for $s \in [t, t_1)$. If we define:

$$p(s) = D_x V(s, x^*(s)) \tag{11}$$

Then $p(s)$ satisfies 8, 9 and 10.

Existence Theorem for Optimal Controls

We now prove an existence theorem for optimal controls. We study the fixed time interval case with $O = \mathbb{R}^n$ and the function f linear in v. Furthermore, we impose convexity of L in v. Under these assumptions a classical variational argument proves the optimal control existence.

Theorem

Let U compact and convex, $f_1, f_2 \in C^1(\overline{Q} \times U)$ such that $f(t,x,v) = f_1(t,x) + f_2(t,x)v$[1] and $\partial_x f_1, \partial_x f_2, f_2$ bounded. Let also $L \in C^1(\overline{Q} \times U)$ and $L(t,x,\cdot)$ be convex for all $(t,x) \in \overline{Q}$ and the temrinal cost $\phi \in C(\mathbb{R}^n)$. Then there exist an optimal control $u^*(\cdot)$.

[1]We could generally ask for a convex f in v.

STOCHASTIC

Stochastic Optimal Control

What if the system evolves and is controlled stochastically? We enter the reign of stochastic optimal control.

We will study controlled diffusion processes, which can be represented by:

$$dx(r) = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r),\ r \in I_0 \tag{12}$$

where I_0 is a time interval and f, σ are drift and volatility coefficients. The control u is itself a random process.

Diffusion Processes

Generally, a diffusion process x is a Markov process with continuous paths. We also impose the existence of the following functions $a_{ij}(t,x), f_{ij}(t,x)$ and limits $\forall\, \epsilon > 0$:

$$\lim_{h\to 0^+} \int_{x-y>\epsilon} p(t,x,t+h,dy) = 0 \tag{13}$$

$$\lim_{h\to 0^+} \int_{x-y\leq\epsilon} (y_i - x_i)\, p(t,x,t+h,dy) = f_i(t,x) \tag{14}$$

$$\lim_{h\to 0^+} \int_{x-y\leq\epsilon} (y_i - x_i)(y_j - x_j)\, p(t,x,t+h,dy) = a_{ij}(t,x). \tag{15}$$

Under stricter conditions on f, a a diffusion process is described by 12.

Backward Evolution Operator

The backward evolution operator A is defined for measurable functions Φ as:

$$A\Phi(t,x) = \lim_{h\to 0+} \frac{E_{tx}\Phi(t+h, x(t+h)) - \Phi(t,x)}{h} \tag{16}$$

We denote $\mathcal{D}(A)$ for the space of function such that the limit exists.
In various contexts, Dynkin's formula holds:

$$E_{tx}\Phi(s, x(s)) - \Phi(t,x) = E_{tx}\int_t^s A\Phi(r, x(r))\, dr. \tag{17}$$

In particular, it holds for diffusion processes with $\mathcal{D}(A) = C_p^{1,2}(\overline{Q})$.

Markov Control Processes

Moving to controlled dynamics, the controlled SDE becomes:

$$dx(r) = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r),\ r \in I_0$$

The objective is to minimize a cost criterion involving a running cost L and terminal cost Ψ.

$$J(t, x; u) = E_{tx}\left\{\int_t^\tau L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau))\right\}$$

The operator A is defined for $\Phi \in C_p^{1,2}(\overline{Q}_0)$ as:

$$A^v\Phi = \Phi_t + \sum_{i=1}^n f_i(t, x, v)\Phi_{x_i} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t, x, v)\Phi_{x_i x_j} \tag{18}$$

Markov Control Processes

The controlled SDE becomes:

$$dx(r) = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r), \; r \in I_0$$

The objective is to minimize a cost criterion involving a running cost L and terminal cost Ψ.

$$J(t, x; u) = E_{tx}\left\{\int_t^\tau L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau))\right\}$$

The operator A is defined for $\Phi \in C_p^{1,2}(\overline{Q}_0)$ as:

$$A^v\Phi = \Phi_t + \sum_{i=1}^{n} f_i(t, x, v)\Phi_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t, x, v)\Phi_{x_i x_j} \tag{19}$$

Optimality

Definition

A reference probability system is a tuple $(\Omega, \{\mathcal{F}_s\}, P, \omega)$ such that $\nu = (\Omega, \mathcal{F}_{t_1}, P)$ is a probability space, $\{\mathcal{F}_s\}$ is a filtration on Ω, w is an $\mathcal{F}$-adapted Brownian motion on $[t, t_1]$.
We denote with $\mathcal{A}_{t\nu}$ progressively measurable admissible controls u.
We define:

$$V_\nu = \inf_{u \in \mathcal{A}_{t\nu}} J(t, x; u) \tag{20}$$

$$V_{PM} = \inf_{\nu} V_\nu. \tag{21}$$

Equation 20 and respectively define ν-*optimality* and *optimality*

Verification Theorem

Analogously to the deterministic case we define the Hamiltonian $\mathcal{H}$ is defined as:

$$\mathcal{H}(t,x,p,A) = \sup_{v\in U}\left\{-f\cdot p - \frac{1}{2}\mathrm{tr}\left[a\cdot A\right] - L\right\}. \tag{22}$$

Theorem

Let $W \in C^{1,2}(Q)\cap C_p(\overline{Q})$ such that:

$$\begin{cases} -\frac{\partial W}{\partial t} + \mathcal{H}(t,x,D_xW,D_x^2W) = 0, & \forall\ (t,x)\in Q \\ V(t,x) = \Psi(t,x), & \forall\ (t,x)\in \partial Q. \end{cases} \tag{23}$$

Then, for any system ν, initial condition $(t,x)\in Q$ and any $u\in\mathcal{A}_{t\nu}$ we have $W(t,x)\le J(t,x;u)$. If there exists a system ν^* and a control u^* which realizes the Hamiltonian's minimum, then:

$$V_{PM}(t,x) = J(t,x;u^*). \tag{24}$$

Stochastic Maximum Principle and BSDE

The stochastic analogous of Pontryagin's principle is the stochastic maximum principle. It relies on the notion of the backward stochastic differential equation.

$$\begin{cases} -dy_s = f(s, y_s, z_s)ds - z_s dw_s \quad \forall\, s \in [t, t_1] \\ y_{t_1} = \xi, \text{ [2]} \end{cases} \tag{25}$$

A BSDE is a SDE where the initial date is replaced by a final distribution. We start by defining a formal concept of solution and provide a general result about existence and uniqueness.

[2]Technical conditions on f and ξ are imposed

BSDE

Definition

A solution of 25 is a couple $(y, z) \in \mathbb{S}^2(t, t_1) \times \mathbb{H}^2(t, t_1)^n$ such that:

$$y_s = \xi + \int_s^{t_1} f(r, y_r, z_r)\, dr - \int_s^{t_1} z_r\, dw_r, \;\; \forall\, s \in [t, t_1].$$

Theorem

Let $f : \Omega \times [t, t_1] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $f(\cdot, \cdot, y, z)$ is progressively measurable for all $(y, z) \in \mathbb{R}^{n+1}$, $f(\cdot, \cdot, 0, 0) \in \mathbb{H}^2(t, t_1)^1$ and $\exists\, C > 0$ s.t.:

$$f(s, y_1, z_1) - f(s, y_2, z_2) \leq C(y_1 - y_2 + z_1 - z_2), \;\; \forall\, y_1, y_2, z_1, z_2,\; m \otimes P\, a.s. \tag{26}$$

Then for every $\xi \in L^2$ the BSDE 25 has a unique solution.

Stochastic Maximum Principle

As in the determinist case, under the assumption of optimality the value function V will define the solution of a differential problem: a BSDE. We adapt the Hamiltonian 22 to

$$\mathcal{G}(t,x,v,y,z) = -f(s,x,v)\cdot y - tr\left[\sigma'(s,x,v)\cdot z\right] - L(s,x,v). \tag{27}$$

Theorem

Let u^* be an optimal control and x^* the corresponding diffusion process, and the value function $V \in C^{1,2}(O) \cap C(\overline{O})$. Then V satisfies:

$$\mathcal{H}(s,x_s^*,u_s^*,D_xV(s,x_s^*),D_x^2V(s,x_s^*)) = \sup_{v\in U} \mathcal{H}(s,x_s^*,v,D_xV(s,x_s^*),D_x^2V(s,x_s^*)), \tag{28}$$

and the pair $(y_s,z_s) = \left(D_xV(s,x_s^*), D_x^2v(s,x_s^*)\sigma(s,x_s^*,u_s^*)\right)$ solves the BSDE:

$$-dy_s = D_x\mathcal{G}(s,x_s^*,u_s^*,y_s,z_s)ds - z_s dw_s,\; y_{t_1} = D_x\Psi(t_1,x_{t_1}). \tag{29}$$

VISCOSITY SOLUTION

Non differentiability

Let us consider the calculus of variation problem:

$$\inf_{x \in Lip([0,1];[-1,1])} \int_t^{t_1} 1 + \frac{1}{4}(\dot{x}(s))^2 \, ds,$$

then the H-J equations are

$$\dot{x}^*(s) = 2p^*(s), \; \dot{p}^*(s) = 0,$$

which define the value function

$$V(t,x) = \begin{cases} 1 - t & x \leq t \\ 1 - t & x \geq t, \end{cases}$$

Vanishing Viscosity

Let

$$\begin{cases} u_t + H(u, Du) = 0, & \mathbb{R}^n \times (0, +\infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{30}$$

We perturbate by second order derivative the equation

$$u_t^\epsilon + H(u^\epsilon, Du^\epsilon) - \epsilon \Delta u^\epsilon = 0,$$

which happens to have a solution[3]. Usually, Ascoli-Arzelà's hypotheses are satisfied[4] we take the limit $u \xleftarrow{j \to +\infty} u^{\epsilon_j}$ as a candidate solution. We lack information about its derivatives.

[3]Galerkin's approximations, Evans section 7.1.2

[4]Easy applications have a uniform Lipschitz bound. Barles-Perthame procedure has a wide range of applications.

Vanishing Viscosity

Then we take v smooth and (t_0, x_0) s.t. $u - v$ has a local maximum and there it nullifies. It implies

$$(u^\epsilon - v)(x_{\epsilon_j}, t_{\epsilon_j}) \geq (u^\epsilon - v)(x, t),$$

for (x, t) close to (x_0, t_0) and $(x_{\epsilon_j}, t_{\epsilon_j}) \xrightarrow{j \to +\infty} (x_0, t_0)$[5]. Since $u_{\epsilon_j} - v$ is maximized at $(x_{\epsilon_j}, t_{\epsilon_j})$

$$u_t^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = v(x_{\epsilon_j}, t_{\epsilon_j}), Du^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) = Dv(x_{\epsilon_j}, t_{\epsilon_j}), -\Delta u^{\epsilon_j}(x_{\epsilon_j}, t_{\epsilon_j}) \geq -\Delta v(x_{\epsilon_j}, t_{\epsilon_j}).$$

Letting $j \to +\infty$ we get

$$v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0. \tag{31}$$

[5]Because of local uniform convergence.

Viscosity Solution

Definition

A viscosity solution of 30 is a function u bounded and uniformly continuous on $\mathbb{R}^n \times [0, T]$ for all $T > 0$ such that for all $v \in C^{+\infty}(\mathbb{R}^n \times (0, +\infty))$:

$$v_t(x, t) + H(Dv(x, t), x) \leq 0$$

for all $(x, t) \in \arg\max\{u - v\}$ and:

$$v_t(x, t) + H(Dv(x, t), x) \geq 0$$

for all $(x, t) \in \arg\min\{u - v\}$. Furthermore, $u \equiv g$ for $t = 0$.

Abstract Dynamic Programming

Let Σ be a closed subset of a Banach space and $\mathcal{C}$ a collection of functions on Σ, closed under addition

$$\mathcal{T}_{tt}\phi = \phi,\ \mathcal{T}_{tr}\phi \leq \mathcal{T}_{ts}\psi \text{ if } \phi \leq \mathcal{T}_{rs}\psi,\ \mathcal{T}_{tr}\phi \geq \mathcal{T}_{ts}\psi \text{ if } \phi \geq \mathcal{T}_{rs}\psi. \tag{32}$$

Provided that $\mathcal{T}_{rt} : \mathcal{C} \to \mathcal{C}$ implies the semigroup property and 32 is equivalent to monotonicity. Let $\Sigma = \overline{O} \subset \mathbb{R}^n$, $\mathcal{C} = \mathcal{M}(\Sigma)$, and

$$\mathcal{T}_{t,r;u}\psi(x) = \int_t^{\tau\wedge r} L(s, x(s), u(s)),\ ds + g(\tau, x(\tau))\chi_{\tau<r} + \psi(x(r))\chi_{\tau\geq r},$$

and $\mathcal{T}_{tr}\psi = \inf_{u\in\mathcal{U}(t,x)} \mathcal{T}_{t,r;u}\psi$. Under the usual assumption on the running and terminal costs $\mathcal{T}_{tr}\psi \in \mathcal{C}$, then the programming principle reads

$$\mathcal{T}_{tt_1}\psi(x) = \mathcal{T}_{tr}\left(\mathcal{T}_{rt_1}\psi\right)(x).$$

Let us define $V(t,x) = (\mathcal{T}_{tt_1}\psi)(x)$. Then

$$-\frac{1}{h}\left[\mathcal{T}_{tt+h}V(t+h,\cdot)(x) - V(t,x)\right] = 0.$$

We ask for $\{\mathcal{G}_t\}_{t\in[t_0,t_1]}$ functions on Σ such that:

$$\lim_{h\to 0}\frac{1}{h}\left[\mathcal{T}_{tt+h}V(t+h,\cdot)(x) - V(t,x)\right] = \frac{\partial}{\partial t}w(t,x) - (\mathcal{G}_t w(t,\cdot))(x), \quad (33)$$

for all $w \in \mathcal{D}$[6]. Then the dynamic programming equation reads

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0,\ (t,x) \in Q. \quad (34)$$

[6]Continuity assumptions are made on $\mathcal{D}$.

Viscosity Solutions

Definition

Let $W \in C([t_0, t_1] \times \Sigma)$. W is a *viscosity subsolution* of 34 in Q if for every $w \in \mathcal{D}$:

$$-\frac{\partial}{\partial t} w\left(\bar{t}, \bar{x}\right) + \left(\mathcal{G}_{\bar{t}} w\left(\bar{t}, \cdot\right)\right)\left(\bar{x}\right) \leq 0, \tag{35}$$

at every $\left(\bar{t}, \bar{x}\right) \in \arg\max_{(t,x)\in Q} \left\{(W - w)(t, x)\right\}$, and $W\left(\bar{t}, \bar{x}\right) = w\left(\bar{t}, \bar{x}\right)$.
W is a *viscosity supersolution* of 34 in Q if for every $w \in \mathcal{D}$:

$$-\frac{\partial}{\partial t} w\left(\bar{t}, \bar{x}\right) + \left(\mathcal{G}_{\bar{t}} w\left(\bar{t}, \cdot\right)\right)\left(\bar{x}\right) \geq 0, \tag{36}$$

at every $\left(\bar{t}, \bar{x}\right) \in \arg\min_{(t,x)\in Q} \left\{(W - w)(t, x)\right\}$, and $W\left(\bar{t}, \bar{x}\right) = w\left(\bar{t}, \bar{x}\right)$.

W is a *viscosity solution* if it is a subsolution and a supersolution.

Strandard Approach

Historically the notion of viscosity solution was introduced for partial differential equations, that is when $\mathcal{G}_t$ is a partial differential operator.

The definition of viscosity solution for

$$-\frac{\partial}{\partial t}W(t,x)+F(t,x,D_xW(t,x),D_x^2W(t,x),W(t,x))=0, \tag{37}$$

is the same we gave with $\mathcal{G}_t$, a part from the space of test functions:

$$w \in C^\infty(Q).$$

Theorem

Let all the previous assumptions and $W \in C_p(\overline{Q}) \cap \mathcal{M}(\overline{Q})$ and $\mathcal{D} \subset C^{1,2}(Q)$. Then the solution concepts coincide.

Value Function as Viscosity solution

Recall

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0,\ (t,x) \in Q. \tag{34}$$

We have

Theorem

Let $\{\mathcal{T}_{tr}\}_{t_0 \leq t \leq r \leq t_1}$ such that 32 and also there exists a vector space $\mathcal{D}$ and another family of operator $\{\mathcal{G}_t\}_{t \in [t_0,t_1]}$ such that $w_t, \mathcal{G}_t w$ are continuous and $w \in \mathcal{C}$. Let

$$V(t,x) = \left(\mathcal{T}_{tt_1}\psi\right)(x).$$

If $V \in C(Q)$ then it is a viscosity solution of 34.

Value Function as Viscosity Solution

We now prove that it is a viscosity solution of the dynamic programming equation under two sets of assumptions.

Theorem

Let U be a bounded space of control and $f \in C(\overline{Q} \times U)$ such that $f(t, x, v) \leq K(1 + |x|)$. Then for every $w \in C^1(Q) \cap \mathcal{M}(\overline{Q})$

$$\lim_{h \to 0} \frac{1}{h} \left[(\mathcal{T}_{tt+h} w(t+h, \cdot))(x) - w(t, x) \right] = \frac{\partial}{\partial t} w(t, x) - H(t, x, D_x w(t, x)), \tag{38}$$

for all $(t, x) \in \overline{Q}$.

It implies that under these assumptions, V is a viscosity solution (Theorem 14).

Value Function as Viscosity Solution

The previous result holds under quite stringent hypotheses. We can relax those assumptions by asking for the existence of an optimal control.

Theorem

If for each $(t, x) \in Q$ there exists a $u^* \in \mathcal{U}(t, x)$ be an optimal control, then a continuous value function is a viscosity solution of its dynamic programming equation.

Uniqueness of Solution

Let us consider

$$-\frac{\partial}{\partial t}V(t,x)+H(t,x,D_xV(t,x))=0,\ (t,x)\in Q.^{[7]} \tag{39}$$

Theorem

Let W and V viscosity subsolution and supersolution of 39 in Q, respectively. If Q is unbounded we assume W, V to be bounded and uniformly continuous on its closure. Then

$$\sup_{\overline{Q}}[W-V]=\sup_{\partial * Q}[W-V].$$

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[7] H(t,x,p)-H(s,y,p')$\leq h(t-s+x-y)+h(t-s)p+Kx-yp+Kp-p',\ |H_p|\leq K,\ |H_t|+|H_x|\leq K'(1+|p|)$.

Continuity of Solution

We recall that

$$|f(t,x,v) - f(t,y,v)| \leq K_\rho |x-y|, \ \ \forall \, |v| \leq \rho. \tag{40}$$

Theorem

Let a bounded control space U, $Q = [t_0, t_1) \times \mathbb{R}^n$. Assume that f, L, ψ are bounded, f satisfies 40 and L, ψ uniformly continuous. Then the value function V is bounded and uniformly continuous.

Corollary

Under the previous assumptions, the value function is the unique viscosity solution of the dynamic programming equation with fixed terminal conditions

Pontryagin's Principle

Definition

Let $W \in C(\overline{Q})$ and $(t, x) \in Q$. The set of *superdifferentials* $D^+W(t, x)$ of W at (t, x) is the collection of all $(q, p) \in \mathbb{R} \times \mathbb{R}^n$ such that there exists some $w \in C^1(Q)$ for which:

$$(q, p) = \left(\frac{\partial}{\partial t} w(t, x), D_x w(t, x) \right), \tag{41}$$

and $(t, x) \in \arg\max \left\{ (W - w)(s, y) \,|\, (s, y) \in \overline{Q} \right\}$.

The set of *subdifferentials* $D^-W(t, x)$ of W at (t, x) as the collection of all $(q, p) \in \mathbb{R} \times \mathbb{R}^n$ such that there exists some $w \in C^1(Q)$ for which:

$$(q, p) = \left(\frac{\partial}{\partial t} w(t, x), D_x w(t, x) \right), \tag{42}$$

and $(t, x) \in \arg\min \left\{ (W - w)(s, y) \,|\, (s, y) \in \overline{Q} \right\}$.

Pontryagin's Principle

We recall the definition of the adjoint variable for a state variable x defined by the flow f, a control u, a terminal condition ψ, a Lagrangian L and a Hamiltonian H:

$$\dot{p}_j^*(s) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_j} f_i(s, x^*(s), u^*(s)) p_i(s) - \frac{\partial}{\partial x_j} L(s, x^*, u^*), \tag{43}$$

And also:

$$p(s) \cdot f(s, x^*(s), u^*(s)) + L(s, x^*(s), u^*(s)) = -H(s, x^*(s), u^*(s), p^*(s)), \tag{44}$$

With:

$$p^*(t_1) = D\psi(x^*(t_1)). \tag{45}$$

Pontryagin's Principle

Theorem

Let $u^*(\cdot)$ be an optimal control at (t,x) which is right continuous at each $[t,t_1)$, and $p^*(s)$ defined by 43, 44 and 45. Then for each $s \in [t,t_1)$

$$\Big(H(s,x^*(s),p^*(s)),p^*(s)\Big) \in D^+V(s,x^*(s)). \tag{46}$$