0.1 Introduction

We now study stochastic optimal control. The system we aim to control is governed by stochastic differential equations.
Short description of what is done in this chapter.

0.2 Markov diffusion processes

I now recall some definitions, give new ones and set the notation. Let $\Sigma \subseteq \mathbb{R}^n$ and $\mathcal{B}(\Sigma)$ the associated Borel σ-algebra. Let $(\Omega, \mathcal{F}, P)$ a general probability space. Given $x(s, \omega)$ a Σ-valued random process from $I_0 = [t_0, t_1)$ and $(\Omega, \mathcal{F})$, let us denote by:

$$P(C \,|\, x(s_1), \ldots, x(s_m)),\ C \in \mathcal{F}$$

The conditional probability of C given the sigma algebra $\bigvee_{i=1}^{m} \sigma(x(s_i))$.

Definition 0.2.1. *A stochastic process x satisfies the Markov property if there exists a function $p : I_0 \times \Sigma \times I_0 \times \mathcal{B}(\Sigma) \to \mathbb{R}$ such that:*

1. *For all t, s, B the function $x \mapsto p(t, x, s, B)$ is borel measurable on Σ*
2. *For all t, x, s the function $A \mapsto p(t, x, s, B)$ is a probability measure on $(\Omega, \mathcal{F})$*
3. *The Chapman-Kolmogorov equation holds for all $s, t, r \in I_0$ such that $t < r < s$:*

$$p(t, x, s, B) = \int_{\Sigma} p(r, y, s, B)\, p(t, x, r, dy) \tag{1}$$

And such that for all $r, s \in I_0$ where r, s and for all $B \in \mathcal{B}(\Sigma)$ then:

$$P(x(s) \in B \,|\, \mathcal{F}_r^x) = p(r, x(r), s, B) \tag{2}$$

Where $\mathcal{F}_r^x = \sigma\left(x(l)\ :\ l \in [t_0, r]\right)$.

Function p is called *Markov Transition Kernel.* We shall see a Markov transition kernel as the probability that the system starting from x at time t will be in B at time s. This heuristic interpretation clarifies the following notation:

$$E_{tx}\phi(x(s)) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) \tag{3}$$

For a real valued borel-measurable function ϕ. Given a Markov process x we can define a family of linear operators associated to it. Let $t < s$, hereafter all time indices will always be in I_0, and define:

$$T_{t,s}\phi(x) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) = E_{tx}\phi(x(s)) \tag{4}$$

Integrability assumptions on ϕ vary from case to case. For now, we can take ϕ to be bounded. Because of Chapman-Kolmogorov equation **??** the family $(T_{t,s})_{t,s \in I_0}$ satisfies the property:

$$T_{tr}\left[T_{rs}\phi\right] = T_{ts}\phi \tag{5}$$

For all $t < r < s$. This family of linear operators defines another operator, the *backward evolution operator.* Let $A : \{\Phi : I_0 \times \Sigma \to \mathbb{R}\} \to \mathbb{R}$:

$$A\Phi(t, x) = \lim_{h \to 0+} \frac{E_{tx}\Phi(t + h, x(t + h)) - \Phi(t, x)}{h} \tag{6}$$

provided that the limit exists. We define $\mathcal{D}(A)$ the space of functions such that limit **??** exists. The following holds.

Proposition 0.2.1. *Let A as before, then for all $\Phi \in \mathcal{A}$ the following hold:*

1. *Φ, $\frac{\partial \Phi}{\partial t}$ and $A\Phi$ are continuous*
2. *For all $t, s \in \overline{I}_0$, $t < s$ then:*

$$E_{tx}|\Phi(s, x(s))| < +\infty, E_{tx} \int_t^s |A\Phi(r, x(r))| \, dr < +\infty$$

3. *Dynkin's formula holds for all $t < s$:*

$$E_{tx}\Phi(s, x(s)) - \Phi(t, x) = E_{tx} \int_t^s A\Phi(r, x(r)) \, dr \tag{7}$$

Dynkin's formula can be proved in different instances, subject to the nature of the random process. We will see that it is a natural consequence of Ito formula for continuous state space processes.
If the random process x is autonomous (time-homogeneous) then the linear operator family is a semigroup. Recall that a Markov process is homogeneous if for all $t < s$ in I_0 then:

$$p(t, x, s, B) = p(0, x, s - t, B)$$

If so, by calling $T_s = T_{0s}$ property **??** is:

$$\begin{aligned}
T_{s+r}\phi(x) &= \int_\Sigma \phi(y) \, p(0, x, s + r, dy) && (8)\\
&= \int_\Sigma \phi(y) \int_\Sigma p(r, z, r + s, dy) \, p(0, x, r, dz) && (9)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(r, z, r + s, dy) \, p(0, x, r, dz) && (10)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(0, z, s, dy) \, p(0, x, r, dz) && (11)\\
&= \int_\Sigma T_s\phi(z) \, p(0, x, r, dz) && (12)\\
&= T_r \left[T_s\phi(x)\right]. && (13)
\end{aligned}$$

While the backward evolution operator analogous is called the *generator* and is defined as:

$$G\phi(x) = - \lim_{h \to 0^+} \frac{T_h\phi(x) - \phi(x)}{h} \tag{14}$$

With $D(G)$ as $\mathcal{D}(A)$ before. It is worth noting that, formally, the following equality holds:

$$A\Phi = \frac{\partial \Phi}{\partial t} - G\Phi(t, \cdot) \tag{15}$$

This relation links the two operators and the autonomous to the non-autonomous case. We now turn our attention to a subset of Markov processes: diffusion processes. A diffusion process is a Markov process whose paths are continuous. More formally.

Definition 0.2.2. *A diffusion process $x : \overline{I}_0 \times \Omega \to \Sigma$ is a almost surely continuous Markov process with Markov transition kernel p such that:*

- *For every $\epsilon > 0$:*

$$\lim_{h \to 0^+} \int_{|x-y|>\epsilon} p(t, x, t + h, dy) = 0 \tag{16}$$

- *There exist functions $a_{ij}(t,x), f_{ij}(t,x)$ for $(t,x) \in \overline{Q}_0$ and $i,j = 1,\dots,n$ such that for every $\epsilon > 0$:*

$$\lim_{h \to 0^+} \int_{|x-y| \leqslant \epsilon} (y_i - x_i)\, p(t,x,t+h,dy) = f_i(t,x) \tag{17}$$

And:

$$\lim_{h \to 0^+} \int_{|x-y| \leqslant \epsilon} (y_i - x_i)(y_j - x_j)\, p(t,x,t+h,dy) = a_{ij}(t,x). \tag{18}$$

These limits are intended uniformly.

Functions $f = (f_1,\dots,f_n)$ and $a = (a_{ij})_{ij}$ are respectively called local drift and local covariance coefficients.
How does the backward evolution operator, and the generator in the autonomous case, adapt to this situation? To answer this question we reduce our problem to a stochastic differential one by relying on the differential structure of a diffusion process. Give the local drift and covariance f, a of a diffusion process x we claim that it satisfies:

$$dx(s) = f(s,x(s))ds + \sqrt{a}(s,x(s))dw(s) \tag{19}$$

Clearly we have to impose further conditions of the stochastic differential equation's coefficients to ensure existence of a solution. In particular, we want those coefficients to be Lipshitz and sub-linearly growing with respect to the second variable. In equation **??** We define the square root of a as a function $\sqrt{a} = \sigma$ such that:

$$\sigma(t,x) \cdot \sigma'(t,x) = a(t,x) \tag{20}$$

We recall that under existence hypothesis for every $\Phi \in C^{1,2}(\overline{Q}_0)$ Ito's formula holds:

$$d\Phi(s,x(s)) = \Phi_s(s,x(s))ds + \sum_{i=1}^n \Phi_{x^i}(s,x(s))dx^i(s) + \frac{1}{2}\sum_{i,j=1}^n \Phi_{x^i x^j}(s,x(s))d[x,x]^{ij}(s) \tag{21}$$

where $[x,y]$ is the covariation of process x and y. Recall that this relation has always to be intended in integral form, that is:

$$\Phi(s,x(s)) = \Phi(t,x) + \int_t^s \Phi_s(r,x(r))\, dr + \sum_{i=1}^n \int_t^s \Phi_{x^i}(r,x(r))\, dx^i(r) + \frac{1}{2}\sum_{i,j=1}^n \int_t^s \Phi_{x^i x^j}(r,x(r))\, d[x,x]^{ij}(r). \tag{22}$$

Via this relation we can reconstruct Dynkin's formula in this setting. By defining the operator A as in **??** we have:

$$\Phi(s,x(s)) = \Phi(t,x) + \int_t^s \Phi_s(r,x(r))\,dr \tag{23}$$

$$+ \sum_{i=1}^n \left[\int_t^s \Phi_{x^i}(r,x(r)) f_i(r,x(r))\,dr + \sum_{j=1}^n \int_t^s \Phi_{x^i}(r,x(r))\sigma_{ij}(r,x(r))\,dw^j(r) \right] \tag{24}$$

$$+ \frac{1}{2} \sum_{i,j=1}^n \sum_{l=1}^n \int_t^s \Phi_{x^i x^j}(r,x(r))\sigma_{il}(r,x(r))\sigma_{jl}(r,x(r))\,dr \tag{25}$$

$$= \Phi(t,x) + \int_t^s \Phi_s(r,x(r)) + D_x\Phi \cdot f(r,x(r)) + \frac{1}{2} D_x^2 \Phi \cdot a(r,x(r))\,dr \tag{26}$$

$$+ \int_t^s D_x\Phi \cdot \sigma(r,x(r))\,dw(r) \tag{27}$$

$$\tag{28}$$

but the last (stochastic) integral can be seen as a martingale. In particular, if we take Φ to have polynomial growth of some order m:

$$|\Phi(t,x)| \leqslant K(1+|x|^m) \ \forall \ (t,x) \in \overline{Q}_0 \tag{29}$$

then $D_x\Phi \cdot \sigma \in \mathbb{L}^2(I_0)$, where:

$$\mathbb{L}^2(I_0) = \left\{ x : I \times \Omega \to \Sigma \,|\, E \int_I |x(s)|^2\,ds < \infty \right\}$$

and therefore its stochastic integral is a martingale (with respect to the canonical filtration associated to the Brownian motion w). Therefore, if we take the (conditional) expectation:

$$E_{tx}\Phi(s,x(s)) = \Phi(t,x) + E_{tx} \int_t^s \Phi_s(r,x(r)) + D_x\Phi \cdot f(r,x(r)) + \frac{1}{2} D_x^2\Phi \cdot a(r,x(r))\,dr. \tag{30}$$

It is now coherent to define the operator $A : C_p^{1,2}(\overline{Q}_0) \to \mathbb{R}$ as:

$$A\Phi(r,x(r)) = \Phi_s(r,x(r)) + D_x\Phi \cdot f(r,x(r)) + \frac{1}{2} D_x^2\Phi \cdot a(r,x(r)) \tag{31}$$

where $C_p^{1,2}(I)$ is the family of functions g from I into $\mathbb{R}$ such that $g, g_s, g_{x_i}, g_{x_i x_j}$ are continuous and with polynomial growth.

Remark. *Be careful that the stochastic integral:*

$$\int_t^s D_x\Phi \cdot \sigma(r,x(r))\,dw(r)$$

is a martingale because x satisfies:

$$E_{tx}|x(r)|^m \leqslant C_m(1+|x|^m) \ \forall \ r \in I_0$$

as it is solution of the SDE **??**.[1]

Consequently, the generator G of the time-homogeneous case is defined as:

$$G\Phi(x) = -\frac{1}{2} \sum_{i,j=1}^n a_{ij}(x)\Phi_{x_i x_j}(x) - \sum_{i=1}^n f_i(x)\Phi_{x_i}(x) \tag{32}$$

[1]This is a standard result in SDE theory.

0.3 Markov control processes

So far we talked about Markov processes without specifying any kind of control. A control process in any stochastic process $u : \Omega \to U$, where U is the control space, that influences the evolution of the random process x. Formally, let $Q = I_0 \times O$ and u as before and define:

$$\begin{cases} dx(r) = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r)\, r \in I_0 \\ x(t) = x \end{cases} \tag{33}$$

where $U \subset \mathbb{R}^m$ closed, $f, \sigma \in C(\overline{Q}_0 \times U)$, $f(\cdot, \cdot, v), \sigma(\cdot, \cdot, v)$ belong to $C^1(\overline{Q}_0)$ for all $v \in U$, such that there exists $C > 0$ such that:

$$|f_t| + |f_x| \leqslant C,\ |\sigma_t| + |\sigma_x| \leqslant C \tag{34}$$
$$|f(t, x, v)| \leqslant C(1 + |x| + |v|) \tag{35}$$
$$|\sigma(t, x, v)| \leqslant C(1 + |x| + |v|) \tag{36}$$

We can relax the assumption by imposing Lipschitz condition on t and x for every fixed v. Furthermore, we assume u to be *admissible*, that is:

$$E \int_t^{t_1} |u(s)|^m \, ds < \infty \ \forall\, m \in \mathbb{N}. \tag{37}$$

It is implied by U being compact. Under these hypotheses, equation **??** has a unique (indistinguishable) solution. Where does optimality play its role? We define running and terminal costs L, Ψ, both continuous and satisfying:

$$|L(s, x, v)| \leqslant C(1 + |x|^k + |v|^k) \tag{38}$$
$$|\Psi(s, x)| \leqslant C(1 + |x|^k) \tag{39}$$

for suitable $C, k > 0$. We also define τ to be the exit time of $(s, x(s))$ from Q. We define:

$$J(t, x; u) = E_{tx} \left\{ \int_t^{\tau} L(s, x(s), u(s)) \, ds + \Psi(\tau, x(\tau)) \right\} \tag{40}$$

for every initial condition $(t, x) \in Q$ and control u. We aim to minimize this criterion, that is:

$$\inf_{u \in \mathcal{U}} J(t, x; u).$$

This formulation is not mathematically formal enough, let us restate it. We begin by defining an infimum criterion with respect to a probability space, or more formally a *probability system*, and then we'll take the infimum over all probability systems.

Definition 0.3.1. *A reference probability system is a tuple* $(\Omega, \{\mathcal{F}_s\}, P, \omega)$ *such that:*

a) $\nu = (\Omega, \mathcal{F}_{t_1}, P)$ *is a probability space*

b) $\{\mathcal{F}_s\}$ *is a filtration on* Ω

c) w *is an* $\mathcal{F}$*-adapted Brownian motion on* $[t, t_1]$.

We denote with $\mathcal{A}_{t\nu}$ *the collection of all* $\mathcal{F}$ *progressively measurable (that is* $\mathcal{B}([t, s]) \times \mathcal{F}_s$*-adapted),* U *valued processes* u *such that condition* **??** *holds on* $[t, t_1]$.

We define:

$$V_\nu = \inf_{u \in \mathcal{A}_\nu} J(t, x; u) \tag{41}$$

while we define:

$$V_{PM} = \inf_{\nu} V_\nu. \tag{42}$$

Equation **??** and respectively define ν-*optimality* and *optimality* for those control that satisfy them. We adapt the definition of operator A to this situation by defining for every element of the control space v the functional:

$$A^v \Phi = \Phi_t + \sum_{i=1}^{n} f_i(t, x, v)\Phi_{x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t, x, v)\Phi_{x_i x_j}, \Phi \in C_p^{1,2}(\overline{Q}_0) \tag{43}$$

where $a = \sigma\sigma'$. As we did in the determinist case, we provide a heuristic derivation of the Hamilton-Jacobi-Bellman equation (the verification theorem), and then we'll formally prove it. Let us suppose that $O = \mathbb{R}^n$, then J is:

$$J(t, x; u) = \int_t^{t_1} L(s, x(s), u(s))\, ds + \Phi(t_1, x(t_1)). \tag{44}$$

By the dynamic programming principle for every $h < t_1 - t$:

$$V(t, x) = \inf_{u \in \mathcal{A}} \left\{ E_{tx} \int_t^{t+h} L(s, x(s), u(s))\, ds + V(t + h, x(t + h)) \right\}.$$

If we take the constant control $u \equiv v$ then by Dynkin's formula we get:

$$0 \leqslant E_{tx} V(t + h, x(t + h)) - V(t, x) + E_{tx} \int_t^{t+h} L(s, x(s), v)\, ds \tag{45}$$

$$= E_{tx} \int_t^{t+h} A^v V(s, x(s))\, ds + E_{tx} \int_t^{t+h} L(s, x(s), v)\, ds \tag{46}$$

dividing by h and taking the limit for $h \to 0^+$:

$$0 \leqslant A^v V(t, x) + L(t, x, v).$$

If we take u^* to be optimal, then equality holds:

$$A^{u^*} V(t, x) + L(t, x, u^*(t)) = 0.$$

We now present the verification theorem rigorously. Let us define the Hamiltonian for this problem. For every $(t, x) \in \overline{Q}_0$, $p \in \mathbb{R}^n$ and $A \in \mathcal{S}_+^n$ (set of symmetric, non-negative definite $n \times n$ matrices) we define:

$$\mathcal{H}(t, x, p, A) = \sup_{v \in U} \left\{ -f(t, x, v) \cdot p - \frac{1}{2} tr\left[a(t, x, v) \cdot A\right] - L(t, x, v) \right\} \tag{47}$$

where for matrices $A, B \in \mathbb{R}^{n \times n}$:

$$tr\,(AB) = \sum_{i,j=1}^{n} A_{ij} B_{ij}. \tag{48}$$

We can now state the verification theorem using the Hamiltonian defined in **??**.

Theorem 0.3.1. *Let $W \in C^{1,2}(Q) \cap C_p(\overline{Q})$ such that:*

$$-\frac{\partial W}{\partial t} + \mathcal{H}(t, x, D_x W, D_x^2 W) = 0, \forall\ (t,x) \in Q \tag{49}$$

$$V(t,x) = \Psi(t,x), \forall\ (t,x) \in \partial Q. \tag{50}$$

Then:

1. *for any system ν, initial condition $(t,x) \in Q$ and any $u \in \mathcal{A}_{t\nu}$ then:*

$$W(t,x) \leqslant J(t,x;u) \tag{51}$$

2. *If there exists $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$ and $u^* \in \mathcal{A}_{t\nu^*}$ such that:*

$$u^*(s) \in \arg\min_{v \in U} \left\{ f(s, x^*(s), v) \cdot D_x W(s, x^*(s)) + \frac{1}{2} tr \left[a(s, x^*(s), v) \cdot D_x^2(s, x^*(s)) \right] + L(s, x^*(s), v) \right\} \tag{52}$$

for almost all $(s, \omega) \in [t, \tau^*] \times \Omega^*$, then:

$$V_{PM}(t,x) = J(t,x;u^*). \tag{53}$$

Proof. We assume O to bounded and $W \in C^{1,2}(\overline{Q})$. Because of **??** for all $s \in [t, \tau]$:

$$0 \leqslant A^{u(s)} W(s, x(s)) + L(s, x(s), u(s)). \tag{54}$$

Because of Ito:

$$W(\tau, x(\tau)) - W(t,x) = \int_t^\tau A^{u(s)} W(s, x(s))\, ds + \int_t^\tau D_x \Phi(s, x(s)) \cdot \sigma(s, x(s), u(s))\, dw(s). \tag{55}$$

Because of estimates on SDE solution the last stochastic integral is a $\mathcal{F}_s$-martingale. Then if we take the expectation E_{tx} we get:

$$0 \leqslant E_{tx} \int_t^\tau A^{u(s)} W(s, x(s))\, ds + E_{tx} \int_t^\tau L(s, x(s), u(s))\, ds \tag{56}$$

$$= E_{tx} \left(W(\tau, x(\tau)) - W(t,x) \right) - E_{tx} \int_t^\tau D_x \Phi(s, x(s)) \cdot \sigma(s, x(s), u(s))\, dw(s) \tag{57}$$

$$+ E_{tx} \int_t^\tau L(s, x(s), u(s))\, ds \tag{58}$$

$$= -W(t,x) + E_{tx} \left\{ \int_t^\tau L(s, x(s), u(s))\, ds + W(\tau, x(\tau)) \right\} \tag{59}$$

$$= -W(t,x) + J(t,x;u). \tag{60}$$

If O is unbounded we define for every $\rho > 0$ such that $\rho^{-1} < t_1 - t_0$ the set:

$$O_\rho = O \cap \left\{ |x| < \rho \,|\, d(x, \partial O) > \frac{1}{\rho} \right\}, \ Q_\rho = [t_0, t_1 - \rho^{-1}] \times O_\rho \tag{61}$$

and τ_ρ the exit time from Q_ρ. Then Q_ρ is bounded, and $W \in C^{1,2}(\overline{Q}_\rho)$, then:

$$W(t,x) \leqslant E_{tx} \left\{ \int_t^{\tau_\rho} L(s, x(s), u(s))\, ds + W(\tau_\rho, x(\tau_\rho)) \right\}. \tag{62}$$

We now take $\rho \to +\infty$ and get the thesis. We have convergence in probability for $\tau_\rho \xrightarrow{\rho\to+\infty} \tau$. We prove uniform integrability of the rhs and therefore get L^1 convergence. We have:

$$E_{tx}\int_t^{\tau_\rho} |L(s,x(s),u(s))|\,ds \leqslant E_{tx}\int_t^{t_1} |L(s,x(s),u(s))|\,ds \tag{63}$$
$$\leqslant E_{tx}\int_t^{t_1} \left(1+|x(s)|^k+|u(s)|^k\right)\,ds < +\infty \tag{64}$$

because u is admissible and estimates on SDE solutions. While we have:

$$E_{tx}|W(\tau_\rho,x(\tau_\rho))|^\alpha \leqslant KE\left(1+|x(\tau_\rho)|^k\right)^\alpha \tag{65}$$
$$\leqslant 2^{\alpha-1}K\left(1+E_{tx}\|x\|^{\alpha k}\right) \leqslant C \tag{66}$$

for $\alpha > \frac{1}{k}$ and estimates on SDE solutions. Therefore we get:

$$\lim_{\rho\to+\infty} E_{tx}\left\{\int_t^{\tau_\rho} L(s,x(s),u(s))\,ds + W(\tau_\rho,x(\tau_\rho))\right\} = E_{tx}\left\{\int_t^{\tau} L(s,x(s),u(s))\,ds + W(\tau,x(\tau))\right\}. \tag{67}$$

Part $b)$ comes from equality in equation **??**. $\square$

THEN EXAMPLE IF TIME ALLOWS.
THEN REREAD, REPEAT. I DONT THINK I'LL DO SLIDES. MAYBE THEY CAN BE A GOOD ASSET. LESS INFO, LESS THINGS TO KNOW PERFECTLY. BUT I HAVE TO KNOW EVERYTHING!!!