0.1 Introduction

We now study stochastic optimal control. The system we aim to control is governed by stochastic differential equations.
Short description of what is done in this chapter.

0.2 Markov diffusion processes

I now recall some definitions, give new ones and set the notation. Let $\Sigma \subseteq \mathbb{R}^n$ and $\mathcal{B}(\Sigma)$ the associated Borel σ-algebra. Let $(\Omega, \mathcal{F}, P)$ a general probability space. Given $x(s, \omega)$ a Σ-valued random process from $I_0 = [t_0, t_1)$ and $(\Omega, \mathcal{F})$, let us denote by:

$$P(C \,|\, x(s_1), \ldots, x(s_m)),\ C \in \mathcal{F}$$

The conditional probability of C given the sigma algebra $\bigvee_{i=1}^{m} \sigma(x(s_i))$.

Definition 0.2.1. *A stochastic process x satisfies the Markov property if there exists a function $p : I_0 \times \Sigma \times I_0 \times \mathcal{B}(\Sigma) \to \mathbb{R}$ such that:*

1. *For all t, s, B the function $x \mapsto p(t, x, s, B)$ is borel measurable on Σ*
2. *For all t, x, s the function $A \mapsto p(t, x, s, B)$ is a probability measure on $(\Omega, \mathcal{F})$*
3. *The Chapman-Kolmogorov equation holds for all $s, t, r \in I_0$ such that $t < r < s$:*

$$p(t, x, s, B) = \int_{\Sigma} p(r, y, s, B)\, p(t, x, r, dy) \tag{1}$$

And such that for all $r, s \in I_0$ where r, s and for all $B \in \mathcal{B}(\Sigma)$ then:

$$P(x(s) \in B \,|\, \mathcal{F}_r^x) = p(r, x(r), s, B) \tag{2}$$

Where $\mathcal{F}_r^x = \sigma\left(x(l) \,:\, l \in [t_0, r]\right)$.

Function p is called *Markov Transition Kernel.* We shall see a Markov transition kernel as the probability that the system starting from x at time t will be in B at time s. This heuristic interpretation clarifies the following notation:

$$E_{tx}\phi(x(s)) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) \tag{3}$$

For a real valued borel-measurable function ϕ. Given a Markov process x we can define a family of linear operators associated to it. Let $t < s$, hereafter all time indices will always be in I_0, and define:

$$T_{t,s}\phi(x) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) = E_{tx}\phi(x(s)) \tag{4}$$

Integrability assumptions on ϕ vary from case to case. For now, we can take ϕ to be bounded. Because of Chapman-Kolmogorov equation 1 the family $(T_{t,s})_{t,s \in I_0}$ satisfies the property:

$$T_{tr}\left[T_{rs}\phi\right] = T_{ts}\phi \tag{5}$$

For all $t < r < s$. This family of linear operators defines another operator, the *backward evolution operator.* Let $A : \{\Phi : I_0 \times \Sigma \to \mathbb{R}\} \to \mathbb{R}$:

$$A\Phi(t, x) = \lim_{h \to 0+} \frac{E_{tx}\Phi(t + h, x(t + h)) - \Phi(t, x)}{h} \tag{6}$$

provided that the limit exists. We define $\mathcal{D}(A)$ the space of functions such that limit 6 exists. The following holds.

Proposition 0.2.1. *Let A as before, then for all $\Phi \in \mathcal{A}$ the following hold:*

1. *$\Phi, \frac{\partial \Phi}{\partial t}$ and $A\Phi$ are continuous*
2. *For all $t, s \in \overline{I}_0$, $t < s$ then:*

$$E_{tx}|\Phi(s, x(s))| < +\infty, E_{tx} \int_t^s |A\Phi(r, x(r))|\, dr < +\infty$$

3. *Dynkin's formula holds for all $t < s$:*

$$E_{tx}\Phi(s, x(s)) - \Phi(t, x) = E_{tx} \int_t^s A\Phi(r, x(r))\, dr \tag{7}$$

Dynkin's formula can be proved in different instances, subject to the nature of the random process. We will see that it is a natural consequence of Ito formula for continuous state space processes.
If the random process x is autonomous (time-homogeneous) then the linear operator family is a semigroup. Recall that a Markov process is homogeneous if for all $t < s$ in I_0 then:

$$p(t, x, s, B) = p(0, x, s - t, B)$$

If so, by calling $T_s = T_{0s}$ property 5 is:

$$\begin{aligned}
T_{s+r}\phi(x) &= \int_\Sigma \phi(y)\, p(0, x, s + r, dy) && (8)\\
&= \int_\Sigma \phi(y) \int_\Sigma p(r, z, r + s, dy)\, p(0, x, r, dz) && (9)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(r, z, r + s, dy)\, p(0, x, r, dz) && (10)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(0, z, s, dy)\, p(0, x, r, dz) && (11)\\
&= \int_\Sigma T_s\phi(z)\, p(0, x, r, dz) && (12)\\
&= T_r \left[T_s\phi(x)\right]. && (13)
\end{aligned}$$

While the backward evolution operator analogous is called the *generator* and is defined as:

$$G\phi(x) = -\lim_{h\to 0^+} \frac{T_h\phi(x) - \phi(x)}{h} \tag{14}$$

With $D(G)$ as $\mathcal{D}(A)$ before. It is worth noting that, formally, the following equality holds:

$$A\Phi = \frac{\partial \Phi}{\partial t} - G\Phi(t, \cdot) \tag{15}$$

This relation links the two operators and the autonomous to the non-autonomous case. We now turn our attention to a subset of Markov processes: diffusion processes. A diffusion process is a Markov process whose paths are continuous. More formally.

Definition 0.2.2. *A diffusion process $x : \overline{I}_0 \times \Omega \to \Sigma$ is a almost surely continuous Markov process with Markov transition kernel p such that:*

- *For every $\epsilon > 0$:*

$$\lim_{h\to 0^+} \int_{|x-y|>\epsilon} p(t, x, t + h, dy) = 0 \tag{16}$$

- *There exist functions* $a_{ij}(t,x), f_{ij}(t,x)$ *for* $(t,x) \in \overline{Q}_0$ *and* $i,j = 1,\dots,n$ *such that for every* $\epsilon > 0$*:*

$$\lim_{h \to 0^+} \int_{|x-y| \leqslant \epsilon} (y_i - x_i)\, p(t,x,t+h,dy) = f_i(t,x) \tag{17}$$

And:

$$\lim_{h \to 0^+} \int_{|x-y| \leqslant \epsilon} (y_i - x_i)(y_j - x_j)\, p(t,x,t+h,dy) = a_{ij}(t,x). \tag{18}$$

These limits are intended uniformly.

Functions $f = (f_1,\dots,f_n)$ and $a = (a_{ij})_{ij}$ are respectively called local drift and local covariance coefficients.
How does the backward evolution operator, and the generator in the autonomous case, adapt to this situation? To answer this question we reduce our problem to a stochastic differential one by relying on the differential structure of a diffusion process. Give the local drift and covariance f, a of a diffusion process x we claim that it satisfies:

$$dx(s) = f(s,x(s))ds + \sqrt{a}(s,x(s))dw(s) \tag{19}$$

Clearly we have to impose further conditions of the stochastic differential equation's coefficients to ensure existence of a solution. In particular, we want those coefficients to be Lipshitz and sub-linearly growing with respect to the second variable. In equation 19 We define the square root of a as a function $\sqrt{a} = \sigma$ such that:

$$\sigma(t,x) \cdot \sigma'(t,x) = a(t,x) \tag{20}$$

We recall that under existence hypothesis for every $\Phi \in C^{1,2}(\overline{Q}_0)$ Ito's formula holds:

$$d\Phi(s,x(s)) = \Phi_s(s,x(s))ds + \sum_{i=1}^n \Phi_{x^i}(s,x(s))dx^i(s) + \frac{1}{2}\sum_{i,j=1}^n \Phi_{x^i x^j}(s,x(s))d[x,x]^{ij}(s) \tag{21}$$

where $[x,y]$ is the covariation of process x and y. Recall that this relation has always to be intended in integral form, that is:

$$\Phi(s,x(s)) = \Phi(t,x) + \int_t^s \Phi_s(r,x(r))\, dr + \sum_{i=1}^n \int_t^s \Phi_{x^i}(r,x(r))\, dx^i(r) + \frac{1}{2}\sum_{i,j=1}^n \int_t^s \Phi_{x^i x^j}(r,x(r))\, d[x,x]^{ij}(r). \tag{22}$$

Via this relation we can reconstruct Dynkin's formula in this setting. By defining the operator A as in 7 we have:

$$\Phi(s,x(s)) = \Phi(t,x) + \int_t^s \Phi_s(r,x(r))\,dr \tag{23}$$

$$+\sum_{i=1}^{n}\left[\int_t^s \Phi_{x^i}(r,x(r))f_i(r,x(r))\,dr + \sum_{j=1}^{n}\int_t^s \Phi_{x^i}(r,x(r))\sigma_{ij}(r,x(r))\,dw^j(r)\right] \tag{24}$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{l=1}^{n}\int_t^s \Phi_{x^ix^j}(r,x(r))\sigma_{il}(r,x(r))\sigma_{jl}(r,x(r))\,dr \tag{25}$$

$$= \Phi(t,x) + \int_t^s \Phi_s(r,x(r)) + D_x\Phi\cdot f(r,x(r)) + \frac{1}{2}D_x^2\Phi\cdot a(r,x(r))\,dr \tag{26}$$

$$+\int_t^s D_x\Phi\cdot\sigma(r,x(r))\,dw(r) \tag{27}$$

$$\tag{28}$$

but the last (stochastic) integral can be seen as a martingale. In particular, if we take Φ to have polynomial growth of some order m:

$$|\Phi(t,x)| \leqslant K(1+|x|^m)\ \ \forall\ (t,x)\in\overline{Q}_0 \tag{29}$$

then $D_x\Phi\cdot\sigma\in\mathbb{L}^2(I_0)$, where:

$$\mathbb{L}^2(I_0) = \left\{x : I\times\Omega\to\Sigma\,|\,E\int_I |x(s)|^2\,ds<\infty\right\}$$

and therefore its stochastic integral is a martingale (with respect to the canonical filtration associated to the Brownian motion w). Therefore, if we take the (conditional) expectation:

$$E_{tx}\Phi(s,x(s)) = \Phi(t,x) + E_{tx}\int_t^s \Phi_s(r,x(r)) + D_x\Phi\cdot f(r,x(r)) + \frac{1}{2}D_x^2\Phi\cdot a(r,x(r))\,dr. \tag{30}$$

It is now coherent to define the operator $A : C_p^{1,2}(\overline{Q}_0)\to\mathbb{R}$ as:

$$A\Phi(r,x(r)) = \Phi_s(r,x(r)) + D_x\Phi\cdot f(r,x(r)) + \frac{1}{2}D_x^2\Phi\cdot a(r,x(r)) \tag{31}$$

where $C_p^{1,2}(I)$ is the family of functions g from I into $\mathbb{R}$ such that $g, g_s, g_{x_i}, g_{x_ix_j}$ are continuous and with polynomial growth.

Remark. *Be careful that the stochastic integral:*

$$\int_t^s D_x\Phi\cdot\sigma(r,x(r))\,dw(r)$$

is a martingale because x satisfies:

$$E_{tx}|x(r)|^m \leqslant C_m(1+|x|^m)\ \ \forall\ r\in I_0$$

as it is solution of the SDE 19.[1]

Consequently, the generator G of the time-homogeneous case is defined as:

$$G\Phi(x) = -\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\Phi_{x_ix_j}(x) - \sum_{i=1}^{n} f_i(x)\Phi_{x_i}(x) \tag{32}$$

[1]This is a standard result in SDE theory.

0.3 Markov control processes

So far we talked about Markov processes without specifying any kind of control. A control process in any stochastic process $u : \Omega \to U$, where U is the control space, that influences the evolution of the random process x. Formally, let $Q = I_0 \times O$ and u as before and define:

$$\begin{cases} dx(r) = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r)\, r \in I_0 \\ x(t) = x \end{cases} \tag{33}$$

where $U \subset \mathbb{R}^m$ closed, $f, \sigma \in C(\overline{Q}_0 \times U)$, $f(\cdot,\cdot,v), \sigma(\cdot,\cdot,v)$ belong to $C^1(\overline{Q}_0)$ for all $v \in U$, such that there exists $C > 0$ such that:

$$|f_t| + |f_x| \leqslant C,\ |\sigma_t| + |\sigma_x| \leqslant C \tag{34}$$
$$|f(t, x, v)| \leqslant C(1 + |x| + |v|) \tag{35}$$
$$|\sigma(t, x, v)| \leqslant C(1 + |x| + |v|) \tag{36}$$

We can relax the assumption by imposing Lipschitz condition on t and x for every fixed v. Furthermore, we assume u to be *admissible*, that is:

$$E\int_t^{t_1} |u(s)|^m\, ds < \infty \ \forall\, m \in \mathbb{N}. \tag{37}$$

It is implied by U being compact. Under these hypotheses, equation 33 has a unique (indistinguishable) solution. Where does optimality play its role? We define running and terminal costs L, Ψ, both continuous and satisfying:

$$|L(s, x, v)| \leqslant C(1 + |x|^k + |v|^k) \tag{38}$$
$$|\Psi(s, x)| \leqslant C(1 + |x|^k) \tag{39}$$

for suitable $C, k > 0$. We also define τ to be the exit time of $(s, x(s))$ from Q. We define:

$$J(t, x; u) = E_{tx}\left\{\int_t^{\tau} L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau))\right\} \tag{40}$$

for every initial condition $(t, x) \in Q$ and control u. We aim to minimize this criterion, that is:

$$\inf_{u \in \mathcal{U}} J(t, x; u).$$

This formulation is not mathematically formal enough, let us restate it. We begin by defining an infimum criterion with respect to a probability space, or more formally a *probability system*, and then we'll take the infimum over all probability systems.

Definition 0.3.1. *A reference probability system is a tuple $(\Omega, \{\mathcal{F}_s\}, P, \omega)$ such that:*

a) $\nu = (\Omega, \mathcal{F}_{t_1}, P)$ is a probability space

b) $\{\mathcal{F}_s\}$ is a filtration on Ω

c) w is an $\mathcal{F}$-adapted Brownian motion on $[t, t_1]$.

We denote with $\mathcal{A}_{t\nu}$ the collection of all $\mathcal{F}$ progressively measurable (that is $\mathcal{B}([t, s]) \times \mathcal{F}_s$-adapted), U valued processes u such that condition 37 holds on $[t, t_1]$.

We define:

$$V_\nu = \inf_{u \in \mathcal{A}_{t\nu}} J(t,x;u) \tag{41}$$

while we define:

$$V_{PM} = \inf_{\nu} V_\nu. \tag{42}$$

Equation 41 and respectively define *ν-optimality* and *optimality* for those control that satisfy them. We adapt the definition of operator A to this situation by defining for every element of the control space v the functional:

$$A^v\Phi = \Phi_t + \sum_{i=1}^n f_i(t,x,v)\Phi_{x_i} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t,x,v)\Phi_{x_ix_j}, \Phi \in C_p^{1,2}(\overline{Q}_0) \tag{43}$$

where $a = \sigma\sigma'$. As we did in the determinist case, we provide a heuristic derivation of the Hamilton-Jacobi-Bellman equation (the verification theorem), and then we'll formally prove it. Let us suppose that $O = \mathbb{R}^n$, then J is:

$$J(t,x;u) = \int_t^{t_1} L(s,x(s),u(s))\,ds + \Phi(t_1,x(t_1)). \tag{44}$$

By the dynamic programming principle for every $h < t_1 - t$:

$$V(t,x) = \inf_{u \in \mathcal{A}} E_{tx}\left\{\int_t^{t+h} L(s,x(s),u(s))\,ds + V(t+h,x(t+h))\right\}.$$

If we take the constant control $u \equiv v$ then by Dynkin's formula we get:

$$0 \leqslant E_{tx}V(t+h,x(t+h)) - V(t,x) + E_{tx}\int_t^{t+h} L(s,x(s),v)\,ds \tag{45}$$

$$= E_{tx}\int_t^{t+h} A^vV(s,x(s))\,ds + E_{tx}\int_t^{t+h} L(s,x(s),v)\,ds \tag{46}$$

dividing by h and taking the limit for $h \to 0^+$:

$$0 \leqslant A^vV(t,x) + L(t,x,v).$$

If we take u^* to be optimal, then equality holds:

$$A^{u^*}V(t,x) + L(t,x,u^*(t)) = 0.$$

We now present the verification theorem rigorously. Let us define the Hamiltonian for this problem. For every $(t,x) \in \overline{Q}_0$, $p \in \mathbb{R}^n$ and $A \in \mathcal{S}_+^n$ (set of symmetric, non-negative definite $n \times n$ matrices) we define:

$$\mathcal{H}(t,x,p,A) = \sup_{v \in U}\left\{-f(t,x,v)\cdot p - \frac{1}{2}tr\left[a(t,x,v)\cdot A\right] - L(t,x,v)\right\} \tag{47}$$

where for matrices $A, B \in \mathbb{R}^{n\times n}$:

$$tr\,(AB) = \sum_{i,j=1}^n A_{ij}B_{ji}, \tag{48}$$

which is equal to $\sum_{i,j=1}^n A_{ij}B_{ij}$ for symmetric matrices.
We can now state the verification theorem using the Hamiltonian defined in 47.

Theorem 0.3.1. *Let $W \in C^{1,2}(Q) \cap C_p(\overline{Q})$ such that:*

$$-\frac{\partial W}{\partial t}(t,x) + \mathcal{H}(t,x,D_x W, D_x^2 W) = 0, \forall\ (t,x) \in Q \tag{49}$$

$$W(t,x) = \Phi(t,x), \forall\ (t,x) \in \partial Q. \tag{50}$$

Then:

1. *for any system ν, initial condition $(t,x) \in Q$ and any $u \in \mathcal{A}_{t\nu}$ then:*

$$W(t,x) \leqslant J(t,x;u) \tag{51}$$

2. *If there exists $\nu^* = (\Omega^*, \{\mathcal{F}_s^*\}, P^*, w^*)$ and $u^* \in \mathcal{A}_{t\nu*}$ such that:*

$$u^*(s) \in \arg\min_{v \in U} \left\{ f(s,x^*(s),v) \cdot D_x W(s,x^*(s)) + \frac{1}{2} tr\left[a(s,x^*(s),v) \cdot D_x^2(s,x^*(s))\right] + L(s,x^*(s),v) \right\} \tag{52}$$

for almost all $(s,\omega) \in [t,\tau^*] \times \Omega^*$, then:

$$V_{PM}(t,x) = J(t,x;u^*). \tag{53}$$

Proof. We assume O to bounded and $W \in C^{1,2}(\overline{Q})$. Because of 49 for all $s \in [t,\tau]$:

$$0 \leqslant A^{u(s)} W(s,x(s)) + L(s,x(s),u(s)). \tag{54}$$

Because of Ito:

$$W(\tau,x(\tau)) - W(t,x) = \int_t^\tau A^{u(s)} W(s,x(s))\, ds + \int_t^\tau D_x\Phi(s,x(s)) \cdot \sigma(s,x(s),u(s))\, dw(s). \tag{55}$$

Because of estimates on SDE solution the last stochastic integral is a $\mathcal{F}_s$-martingale. Then if we take the expectation E_{tx} we get:

$$0 \leqslant E_{tx} \int_t^\tau A^{u(s)} W(s,x(s))\, ds + E_{tx} \int_t^\tau L(s,x(s),u(s))\, ds \tag{56}$$

$$= E_{tx}\left(W(\tau,x(\tau)) - W(t,x)\right) - E_{tx} \int_t^\tau D_x\Phi(s,x(s)) \cdot \sigma(s,x(s),u(s))\, dw(s) \tag{57}$$

$$+ E_{tx} \int_t^\tau L(s,x(s),u(s))\, ds \tag{58}$$

$$= -W(t,x) + E_{tx} \left\{ \int_t^\tau L(s,x(s),u(s))\, ds + W(\tau,x(\tau)) \right\} \tag{59}$$

$$= -W(t,x) + J(t,x;u). \tag{60}$$

If O is unbounded we define for every $\rho > 0$ such that $\rho^{-1} < t_1 - t_0$ the set:

$$O_\rho = O \cap \left\{ |x| < \rho \,|\, d(x,\partial O) > \frac{1}{\rho} \right\},\ Q_\rho = [t_0, t_1 - \rho^{-1}] \times O_\rho \tag{61}$$

and τ_ρ the exit time from Q_ρ. Then Q_ρ is bounded, and $W \in C^{1,2}(\overline{Q}_\rho)$, then:

$$W(t,x) \leqslant E_{tx} \left\{ \int_t^{\tau_\rho} L(s,x(s),u(s))\, ds + W(\tau_\rho, x(\tau_\rho)) \right\}. \tag{62}$$

We now take $\rho \to +\infty$ and get the thesis. We have convergence in probability for $\tau_\rho \xrightarrow{\rho\to+\infty} \tau$. We prove uniform integrability of the rhs and therefore get L^1 convergence. We have:

$$E_{tx}\int_t^{\tau_\rho} |L(s,x(s),u(s))|\,ds \leqslant E_{tx}\int_t^{t_1} |L(s,x(s),u(s))|\,ds \tag{63}$$
$$\leqslant E_{tx}\int_t^{t_1}\left(1+|x(s)|^k+|u(s)|^k\right)\,ds < +\infty \tag{64}$$

because u is admissible and estimates on SDE solutions. While we have:

$$E_{tx}|W(\tau_\rho,x(\tau_\rho))|^\alpha \leqslant KE_{tx}\left(1+|x(\tau_\rho)|^k\right)^\alpha \tag{65}$$
$$\leqslant 2^{\alpha-1}K\left(1+E_{tx}\|x\|^{\alpha k}\right) \leqslant C \tag{66}$$

for $\alpha > \frac{1}{k}$ and estimates on SDE solutions. Therefore we get:

$$\lim_{\rho\to+\infty} E_{tx}\left\{\int_t^{\tau_\rho} L(s,x(s),u(s))\,ds + W(\tau_\rho,x(\tau_\rho))\right\} = E_{tx}\left\{\int_t^{\tau} L(s,x(s),u(s))\,ds + W(\tau,x(\tau))\right\}. \tag{67}$$

Part $b)$ comes from equality in equation 54. □

0.4 Stochastic Maximum Principle

The stochastic analogous of Pontryagin's principle is the stochastic maximum principle. It relies on the notion of backward stochastic differential equation, whose solution will provide a necessary condition on the controlled system.

0.4.1 Backward Stochastic Differential Equation

A backward stochastic differential equation is a SDE where the initial date is replaced by a final distribution. We start by defining a formal concept of solution and provide a general result about existence and uniqueness.
We work with a filtered probability space $(\Omega,\mathcal{F},P,\{\mathcal{F}_s\})$ and a Brownian motion (w_s) adapted to the space filtration. We assume that the filtration is the natural one associated to w. A BSDE has the form:

$$\begin{cases} -dy_s = f(s,y_s,z_s)ds - z_s dw_s & \forall\ s\in[t,t_1] \\ y_{t_1} = \xi, \end{cases} \tag{68}$$

where f is real valued and ξ a suitable random variable. Further conditions on f and ξ will be imposed by the existence theorem. The above definitions has to be intended in integral form. To do so we have to specify some integrability conditions. Let us define the space:

$$\mathbb{S}^2(t,t_1) = \left\{(X_s)_{t\in[t,t_1]}\,|\,X_s\in\mathbb{R} \text{ is progressively measurable, } E\left[\sup_{s\in[t,t_1]}|X_s|^2\right] < +\infty\right\} \tag{69}$$

and:

$$\mathbb{H}^2(t,t_1)^n = \left\{(X_s)_{t\in[t,t_1]}\,|\,X_s\in\mathbb{R}^n \text{ is progressively measurable, } E\left[\int_t^{t_1}|Y_s|^2\,ds\right] < +\infty\right\}. \tag{70}$$

We can now define the solution concept of 68.

Definition 0.4.1. *A solution of* **??** *is a couple* $(y, z) \in \mathbb{S}^2(t, t_1) \times \mathbb{H}^2(t, t_1)^n$ *such that:*

$$y_s = \xi + \int_s^{t_1} f(r, y_r, z_r)\, dr - \int_s^{t_1} z_r\, dw_r$$

holds for all $s \in [t, t_1]$.

Some measurability and integrability conditions are necessary for equation 68 to make sense. Existence of a solution will be obtained through a classical fixed point method, which will rise from Lipschitz condition on f. We denote by m the Lebesgue measure on $[t, t_1]$.

Theorem 0.4.1. *Let* $f : \Omega \times [t, t_1] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that* $f(\cdot, \cdot, y, z)$ *is progressively measurable for all* $(y, z) \in \mathbb{R}^{n+1}$, $f(\cdot, \cdot, 0, 0) \in \mathbb{H}^2(t, t_1)^1$ *and there exists* $C > 0$ *such that:*

$$|f(s, y_1, z_1) - f(s, y_2, z_2)| \leqslant C(|y_1 - y_2| + |z_1 - z_2|), \ \forall\, y_1, y_2, z_1, z_2,\ m \otimes P\, a.s. \tag{71}$$

Then for every $\xi \in L^2$ *the BSDE 68 has a unique solution.*

Proof. We use completeness of the space $(\mathbb{S}(t, t_1) \times \mathbb{H}^2(t, t_1)^n, \|\cdot\|_\beta)$ where:

$$\|(y, z)\|_\beta = \left(E\left[\int_t^{t_1} e^{\beta s} \left(|y_s|^2 + |z_s|^2 \right) ds \right] \right). \tag{72}$$

This result is proved in the appendix. We call the previous Banach space $(X, \|\cdot\|)$. We construct the map $\Phi : X \to X$ defined as $\Phi(u, v) = (y, z)$. The processes y and z are defined as follows. We define:

$$M_s = E\left[\xi + \int_0^{t_1} f(r, u_r, v_r)\, dr \mid \mathcal{F}_s \right]. \tag{73}$$

It is a square integrable martingale because:

$$E\left\{ E^2 \left[\xi + \int_t^{t_1} f(r, u_r, v_r)\, dr \mid \mathcal{F}_s \right] \right\} \leqslant C_0 E^2 \xi + C_1 E^2 \int_t^{t_1} f(r, 0, 0)\, dr \tag{74}$$

$$+ C_2 E^2 \int_t^{t_1} f(r, u_r, v_r) - f(r, 0, 0)\, dr \tag{75}$$

$$+ C_3 E\xi^2 E \int_t^{t_1} f(r, 0, 0)^2\, dr \tag{76}$$

$$+ C_4 E\xi^2 E \int_t^{t_1} (f(r, u_r, v_r) - f(r, 0, 0))^2\, dr, \tag{77}$$

which is finite because of the assumptions and given $(u, v) \in X$. It is a martingale. Therefore, by Ito's martingale representation theorem there exists a unique $z \in \mathbb{H}^2(t, t_1)^n$[2] and $m_t \in L2$ such that:

$$m_s = m_t + \int_t^s z_r\, dw_r. \tag{78}$$

We define (y, z) as z being the unique process of the martingale representation of m_s while y to be:

$$y_s = E\left[\xi + \int_s^{t_1} f(r, u_r, v_r)\, dr \mid \mathcal{F}_s \right] = m_s - \int_t^s f(r, u_r, v_r)\, dr. \tag{79}$$

We know that $z \in \mathbb{H}^2$. We show $y \in \mathbb{S}^2$:

$$E\left[\sup_{s \in [t, t_1]} |y_s|^2 \right] \leqslant C_0 E\left[\sup_{s \in [t, t_1]} |m_s|^2 \right] + C_1 E\left[\sup_{s \in [t, t_1]} \left| \int_t^s f(r, u_r, v_r)\, dr \right|^2 \right], \tag{80}$$

[2] Unique with respect to $\|\cdot\|_{\mathbb{H}}$.

but as before the second addend converges while for the first one:

$$E\left[\sup_{s\in[t,t_1]}|m_s|^2\right]\leqslant C_0Em_t^2+C_1E^{1/2}m_t^2E^{1/2}\sup_{s\in[t,t_1]}\left[\int_t^s z_r\,dw_r\right]^2+C_3E^{1/2}\sup_{s\in[t,t_1]}\left[\int_t^s z_r\,dw_r\right]^2,$$

which converges beacuse Doob's inequality implies:

$$E\left[\sup_{s\in[t,t_1]}\int_t^s z_r\,dw_r\right]^2\leqslant 4E\left[\int_t^{t_1} z_r^2\,dw_r\right]<+\infty.$$

If we prove Φ to be a contraction we'll have a unique (in $(X,\|\cdot\|)$) fixed point, therefore the thesis. Let $(U_1,V_1),(U_2,V_2)\in X$, $(X_1,Y_1),(X_2,Y_2)$ their images and $\overline{U},\overline{V},\overline{X},\overline{Y},\overline{f}_t$ the differences between subscript 1 and 2. By Ito's formula we have:

$$\begin{aligned}
\left|\overline{y_t}^2\right| &= -\int_t^{t_1}\frac{d}{dr}\left(e^{\beta r}\overline{y_r}^2\right)dr=-\int_t^{t_1}\beta e^{\beta r}\overline{y_r}^2+e^{\beta r}\frac{d}{dr}\overline{y_r}^2\,dr && (81)\\
&= -\int_t^{t_1}\beta e^{\beta r}\overline{y_r}^2\,dr-\int_t^{t_1}e^{\beta r}\left(2\overline{y_r}\right)d\overline{y_r}-\int_t^{t_1}e^{\beta r}\,d[\overline{y},\overline{y}]_r && (82)\\
&= -\int_t^{t_1}\beta e^{\beta r}\overline{y_r}^2\,dr+2\int_t^{t_1}e^{\beta r}\overline{y_r}\overline{f}_r\,dr-2\int_t^{t_1}e^{\beta r}\overline{y_r}\,dw_r-\int_t^{t_1}e^{\beta r}\overline{z}_r^2\,dr && (83)\\
&= -\int_t^{t_1}e^{\beta r}\left(\beta\overline{y}_r^2-2\overline{y}_r\overline{f}_r\right)dr-\int_t^{t_1}e^{\beta r}\overline{z}_r^2\,dr-2\int_t^{t_1}e^{\beta r}\overline{y}_r\overline{z}_r\,dw_r. && (84)
\end{aligned}$$

We show that $e^{\beta r}\overline{y}_r\overline{z}_r\in\mathbb{H}^2(t,t_1)^1$, which will guarantee the integral to vanish under expectation. We have:

$$E\left[\left(\int_t^{t_1}e^{2\beta r}\overline{y}_r^2\overline{z}_r^2\,dr\right)^{1/2}\right]\leqslant\frac{e^{\beta t_1}}{2}E\left[\sup_{s\in[t,t_1]\overline{y}_s^2+\int_t^{t_1}\overline{z}_r^2}dr\right]<+\infty.$$

By taking the expectation on 81:

$$\begin{aligned}
E\overline{y}_t^2+E\left[\int_t^{t_1}e^{\beta r}\left(\beta\overline{y}_r^2+\overline{z}_r^2\right)dr\right] &= 2E\left[\int_t^{t_1}e^{\beta r}\overline{y}_r\overline{f}_r\,dr\right] && (85)\\
&\leqslant 2CE\left[\int_t^{t_1}e^{\beta r}\overline{y}_r\left(\overline{u}_r+\overline{v}_r\right)dr\right] && (86)\\
&\leqslant 2CE\left[\int_t^{t_1}e^{\beta r}\overline{y}_r^2\,dr\right]+CE\left[\int_t^{t_1}e^{\beta r}\left(\overline{u}_r^2+\overline{v}_r^2\right)dr\right], && (87)\\
& && (88)
\end{aligned}$$

which implies:

$$E\left[\int_t^{t_1}e^{\beta r}\left(\overline{y}_r^2+\overline{z}_r^2\right)dr\right]\leqslant E\left[\int_t^{t_1}e^{\beta r}\left(\overline{u}_r^2+\overline{v}_r^2\right)dr\right]$$

□

The previous proof used the concept of local martingale and the Burkholder-Davis-Gundy inequality, they are detailed in the appendix. In particular the fact that if $X\in\mathbb{H}^2$ then $\int X\,dw$ is a martingale uses these concepts.
A notable case is the one with a generator f linear in y and z. That is, there exists a,b bounded progressively measurable processes valued in $\mathbb{R}$ and $\mathbb{R}^n$ and $c\in\mathbb{H}(t,t_1)^1$ which define:

$$-dy_s=\left(a_sy_s+z_sb_s+c_s\right)ds-z_sdw_s,\ y_{t_1}=\xi. \tag{89}$$

Proposition 0.4.2. *The unique solution* (y, z) *of 89 is:*

$$\Gamma_s y_s = E\left[\Gamma_{t_1}\xi + \int_s^{t_1} \Gamma_r c_r \, dr \,|\, \mathcal{F}_s\right], \tag{90}$$

and z_s *is defined via the Martingale representation of 90. The process* Γ *is defined by:*

$$d\Gamma_s = \Gamma_s \left(a_s ds + b_s dw_s\right), \, \Gamma_t = 1.$$

0.4.2 Stochastic Maximum Principle

The stochastic counterpart of Pontryagin's principle is the stochastic maximum principle. We consider the controlled diffusion process x on $\mathbb{R}^n$ defined by:

$$dx_s = f(r, x(r), u(r))dr + \sigma(r, x(r), u(r))dw(r) \tag{91}$$

and let the functional to be maximized:

$$J(t, x; u) = E_{tx}\left\{\int_t^{t_1} L(s, x(s), u(s)) \, ds + \Phi(t_1, x(t_1))\right\}, \tag{92}$$

where the running and terminal cost satisfy the usual assumptions. We aim at maximizing functional J over all admissible systems, that is:

$$\inf_{\nu} \inf_{u \in \mathcal{A}_{t\nu}} J(t, x; u).$$

We consider again the value function related to this problem, that is a function V, with suitable smoothness conditions, such that:

$$-\frac{\partial V}{\partial t} + \mathcal{H}(t, x, D_x V, D_x^2 V) = 0, \tag{93}$$

where:

$$\mathcal{H}(s, x, v, p, A) = -f(s, x, v) \cdot p - \frac{1}{2} tr\left[\sigma\sigma'(s, x, v) \cdot A\right] - L(s, x, v). \tag{94}$$

As in the determinist case, under the assumption of optimality the value function V will define the solution of a differential equation, more precisely a BSDE. The backward stochastic differential equation will be an analogous of the deterministic adjoint equation:

$$\begin{cases} \dot{p}(s) = D_x H(s, x(s), u(s), p(s)) \\ p(t_1) = D\psi(x(t_1)), \end{cases}$$

as stated in **??**. In particular, we will need the functional:

$$\mathcal{G}(t, x, v, y, z) = -f(s, x, v) \cdot y - tr\left[\sigma'(s, x, v) \cdot z\right] - L(s, x, v). \tag{95}$$

This functional will define the BSDE in the next theorem.

Theorem 0.4.3. *Let* u^* *be an optimal control and* x^* *the corresponding diffusion process, and the value function* $V \in C^{1,2}(O) \cap C(\overline{O})$. *Then* V *satisfies:*

$$\mathcal{H}(s, x_s^*, u_s^*, D_x V(s, x_s^*), D_x^2 V(s, x_s^*)) = \sup_{v \in U} \mathcal{H}(s, x_s^*, v, D_x V(s, x_s^*), D_x^2 V(s, x_s^*)), \tag{96}$$

and the pair $(y_s, z_s) = (D_x V(s, x_s^*), D_x^2 v(s, x_s^*)\sigma(s, x_s^*, u_s^*))$ *solves the BSDE:*

$$-dy_s = D_x \mathcal{G}(s, x_s^*, u_s^*, y_s, z_s)ds - z_s dw_s, \tag{97}$$

with final condition:

$$y_{t_1} = D_x\Psi(t_1, x_{t_1}). \tag{98}$$

Proof. We drop the *. We consider the optimal system over which the control is defined. Since u is optimal we have:

$$\begin{aligned} V(s,x(s)) &= E_{sx(s)}\left[\int_s^{t_1} L(r,x(r),u(r))\,dr + \Psi(t_1,x(t_1))\right] && (99)\\ &= E_{tx}\left[\int_t^{t_1} L(r,x(r),u(r))\,dr + \Psi(t_1,x(t_1))\right] - \int_t^s L(r,x(r),u(r))\,dr. && (100)\end{aligned}$$

We then apply Ito's formula and get:

$$\begin{aligned} &\int_t^{t_1} \partial_s V(r,x(r)) + D_xV(r,x(r))f(r,x(r),u(r)) + \frac{1}{2}tr\left[\sigma\sigma'(r,x(r),u(r))D_x x^2 V(r,x(r))\right]dr && (101)\\ &\qquad + \int_t^{t_1} D_xV(r,x(r))\sigma(r,x(r),u(r))\,dw_r + V(t,x) && (102)\\ &= \int_t^{t_1} -f(r,x(r),u(r))\,dr + \int_t^{t_1} \alpha_r\,dw_r + V(t,x) && (103)\end{aligned}$$

where we used the integral representation of a martingale, which implies:

$$-\frac{\partial V}{\partial t} + \mathcal{H}(s,x(s),u(s),D_xV(s,x(s)),D_x^2V(s,x(s))) = 0. \tag{104}$$

We then prove the BSDE using the continuity of triple spacial derivatives of the value functions and the fact that 104 at s has maximum in $x(s)$, which implies:

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t}(s,x) - \mathcal{H}(s,x,u(s),D_xV(s,x),D_x^2V(s,x))\right)|_{x=x(s)} = 0.$$

We compute the derivative and get:

$$\begin{aligned} &\frac{\partial^2 V}{\partial x\partial t}(s,x(s)) + D_xf(s,x,u(s))|_{x=x(s)}D_xV(s,x(s)) + f(s,x(s),u(s))D_x^2V(s,x(s))\\ &+\frac{1}{2}tr\left[D_x(\sigma\sigma')(s,x,u(s))|_{x=x(s)}\cdot D_x^2V(s,x(s))\right] + \frac{1}{2}tr\left[\sigma\sigma'(s,x(s),u(s))\cdot D_x^3V(s,x(s))\right] + D_xL(s,x(s),u(s))\\ &= \frac{\partial^2 V}{\partial x\partial t}(s,x(s)) + f(s,x(s),u(s))D_x^2V(s,x(s)) + \frac{1}{2}tr\left[\sigma\sigma'(s,x(s),u(s))\cdot D_x^3V(s,x(s))\right]\\ &+f(s,x(s),u(s))D_xV(s,x(s)) + tr\left[D_x\sigma'(s,x(s),u(s))\cdot D_x^2V(s,x(s))\sigma(s,x(s),u(s))\right] + D_xL(s,x(s),u(s)),\end{aligned}$$

which implies:

$$\begin{aligned} &\frac{\partial^2 V}{\partial x\partial t}(s,x(s)) + f(s,x(s),u(s))D_x^2V(s,x(s)) + \frac{1}{2}tr\left[\sigma\sigma'(s,x(s),u(s))\cdot D_x^3V(s,x(s))\right]\\ &\qquad = -D_x\left[\mathcal{G}(s,x(s),u(s),D_xV(s,x(s)),D_x^2V(s,x(s))\sigma(s,x(s),u(s)))\right].\end{aligned} \tag{105}$$

Equation 105 implies that y_s is the unique solution to the BSDE 97:

$$\begin{aligned} -dy_s &= -d[D_xv(s,x(s))] = -\frac{\partial^2 V}{\partial x\partial t}(s,x(s))ds - f(s,x(s),u(s))D_x^2V(s,x(s))ds\\ &- D_x^2V(s,x(s))\sigma(s,x(s),u(s))dw_s + \frac{1}{2}tr\left[\sigma\sigma'(s,x(s),u(s))\cdot D_x^3V(s,x(s))\right]ds\\ &= D_x\left[\mathcal{G}(s,x(s),u(s),D_xV(s,x(s)),D_x^2V(s,x(s))\sigma(s,x(s),u(s)))\right]ds - D_x^2V(s,x(s))\sigma(s,x(s),u(s))dw_s,\end{aligned}$$

and:

$$V(t_1, x(t_1)) = E_{t_1 x(t_1)} \left[\int_{t_1}^{t_1} L(r, x(r), u(r))\, dr + \Psi(t_1, x(t_1)) \right] = \Psi(t_1, x(t_1)),$$

which implies:

$$D_x V(t_1, x(t_1)) = D_x \Psi(t_1, x(t_1)).$$

$\square$