0.1 Introduction

We start our dissertation by studying deterministic optimal control. The system we aim to control is governed by ordinary differential equations.
Short description of what is done in this chapter.

0.2 Finite horizon

Let us consider a finite interval $I = [t, t_1] \subset \mathbb{R}$ as the operating time of the system. At each time $s \in I$ the system is described by $x(s) \in O \subseteq \mathbb{R}^n$ and controlled by $u(s) \in U \subseteq \mathbb{R}^n$ called control space. The system is described by:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)) & s \in I \\ x(t) = x \end{cases} \tag{1}$$

For a given $x \in O$ and suitable $f : \overline{Q} \times U \to \mathbb{R}^m$, where $Q_0 = [t, t_1) \times O$. That is we impose $f \in C(\overline{Q} \times U)$ and the existence of $K_\rho > 0$ for all $\rho > 0$:

$$|f(t, x, v) - f(t, y, v)| \leqslant K_\rho |x - y| \tag{2}$$

For all $t \in I$, $x, y \in O$ and $v \in U$ such that $|v| \leqslant \rho$. Under this conditions the system 1 has a unique solution. Controls $u(\cdot)$ are assumed to be in the set $L^\infty([t, t_1]; U)$. We will soon specify more about the set of controls.
We have described a control problem. The concept of optimality is related some value function, specified by payoffs (or costs) associated to the system's states. Let $L \in C(\overline{Q} \times U)$ be the *running cost* and $\Psi \in C(I \times O)$ the *terminal cost* defined as:

$$\Psi(t, x) = \begin{cases} g(t, x) & \text{if } (t, x) \in [t, t_1) \times O \\ \psi(x) & \text{if } (t, x) \in \{t_1\} \times O \end{cases} \tag{3}$$

We define the *payoff* J as:

$$J(t, x; u) = \int_t^\tau L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \tag{4}$$

Where τ is the exit time of $(s, x(s))$ from $\overline{Q}$, that is:

$$\tau = \begin{cases} \inf\{s \in [t, t_1) \,|\, x(s) \notin \overline{O}\} & \text{if } \exists\ s \in [t, t_1) : x(s) \notin \overline{O} \\ t_1 & \text{if } x(s) \in \overline{O}\ \ \forall\ s \in [t, t_1) \end{cases} \tag{5}$$

Then a control $u^*(\cdot)$ is *optimal* if:

$$J(t, x; u^*) \leqslant J(t, x; u) \qquad \forall\ u \in L^\infty(I; U) \tag{6}$$

Actually, we are being to generous with the control space. We have to impose a further condition on it, the *switching condition*. Let us assume that we have $u \in \mathcal{U}(t, x)$ and $u' \in \mathcal{U}(r, x(r))$ for $r \in [t, \tau]$. If we define:

$$\tilde{u}(s) = \begin{cases} u(s) & s \in [t, r) \\ u'(s) & s \in [r, t_1] \end{cases} \tag{7}$$

Then we impose:

$$\tilde{u}_s \in \mathcal{U}(s, \tilde{x}(s)) \qquad \forall\ s \in [t, \tilde{\tau}] \tag{8}$$

Where $\tilde{x}$ is the solution to the control problem 1 with control $\tilde{u}$ and initial condition x, $\tilde{u}_s$ is the restriction of $\tilde{u}$ to $[s, t_1]$ and $\tilde{\tau}$ is the exit time of $(s, \tilde{x}(s))$ from $\overline{Q}$. This condition assures that admissible controls can be replaced as the time evolves and the resulting control is still admissible.

0.3 Dynamic programming principle

One way of tackling certain optimal control problems is via *dynamic programming.* Let us define the *value function*:

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} J(t,x;u) \tag{9}$$

For all $(t,x) \in \overline{Q}$. We get rid of the instance in which $V(t,x) = -\infty$ assuming Q to be compact, or L and Ψ to be bounded below. We aim at retrieving the argument which attains the infimum of 9. In order to immerse this optimal control problem into a dynamic programming one we see the state of the system as the state of the variable and the control function as the decision function. The basic idea behind dynamic programming techniques is to subdivide a problem into smaller problems, what does this mean in our context? We will be able to find instantaneous the value function V via a partial differential equation (PDE) called Hamilton-Jacobi-Bellman equation.
We start by stating and proving the following proposition, which provides us with an equivalent definition of the value function.

Proposition 0.3.1. *For any $(t,x) \in \overline{Q}$ and any $r \in I$ then:*

$$V(t,x) = \inf_{u \in \mathcal{U}(t,x)} \left\{ \int_t^{r \wedge \tau} L(s, x(s), u(s))\, ds + g(\tau, x(\tau))\chi_{\tau < r} + V(r, x(r))\chi_{r \leqslant \tau} \right\} \tag{10}$$

Proof. Value function less than rhs. If $r > \tau$ then $\tau < t_1$ and $\Psi(r \wedge \tau, x(r \wedge \tau)) = g(\tau, x(\tau))$ and then 10 follows directly by definition. If $r \leqslant \tau$, let $\delta > 0$ then there exists $u^1 \in \mathcal{U}(r, x(r))$ such that:

$$\int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \leqslant V(r, x(r)) + \delta$$

Where x^1 is the state function corresponding to u^1 with initial condition $(r, x(r))$ and τ^1 the first exit from $\overline{Q}$ of $(s, x^1(s))$. By defining $\tilde{u}$ as for the switching condition 7 we have $\tau^1 = \tilde{\tau}$, because $\tau \geqslant r$ and then $\tilde{u}$ is u^1. Then:

$$\begin{aligned}
V(t,x) &\leqslant V(t,x;\tilde{u}) \\
&= \int_t^{\tilde{\tau}} L(s, \tilde{x}(s), \tilde{u}(s))\, ds + \Psi(\tilde{\tau}, \tilde{x}(\tilde{\tau})) \\
&= \int_t^{r} L(s, x(s), u(s))\, ds + \int_r^{\tau^1} L(s, x^1(s), u^1(s))\, ds + \Psi(\tau^1, x^1(\tau^1)) \\
&\leqslant \int_t^{r} L(s, x(s), u(s))\, ds + V(r, x(r)) + \delta
\end{aligned}$$

Since δ is arbitrary the first inequality is proved.
Value function is bigger than rhs. Let $\delta > 0$ and $U \in \mathcal{U}(t,x)$ such that:

$$\int_r^{\tau} L(s, x(s), u(s))\, ds + \Psi(\tau, x(\tau)) \leqslant V(t,x) + \delta$$

Then:

$$
\begin{aligned}
V(t,x) &\geqslant \int_r^{\tau} L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + \int_{r\wedge\tau}^{\tau} L(s,x(s),u(s))\,ds + \Psi(\tau,x(\tau)) - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + J(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta \\
&= \int_t^{r\wedge\tau} L(s,x(s),u(s))\,ds + V(r,x(r))\chi_{r\leqslant\tau} + g(\tau,x(\tau))\chi_{\tau<r} - \delta
\end{aligned}
$$

As δ is arbitrary we proved the proposition. □

In the proof we used the concept of δ-*optimal control*, that is the control function $u \in \mathcal{U}(r,x(r))$ such that:

$$\int_r^{\tau^1} L(s,x^1(s),u^1(s))\,ds + \Psi(\tau^1,x^1(\tau^1)) \leqslant V(r,x(r)) + \delta.$$

This new representation allows us to find the so-called *dynamic programming equation*. We have to impose that the value function is continuously differentiable, although this is not always the case. If differentiability fails, the notion of viscosity solution is needed.
Let us first impose boundary conditions of the value function. Clearly if $t = t_1$ then:

$$V(t_1,x) = \psi(x) \;\; \forall\, x \in \overline{O} \tag{11}$$

If $(t,x) \in [t_0,t_1) \times \partial O$ then the value function is g:

$$V(t,x) = g(t,x) \tag{12}$$

Before stating the fundamental theorem which gives sufficient conditions for a solution to the optimal problem we follow a heuristic reasoning which will help our intuition. Under the hypothesis of continuous differentiability of the value function let us rewrite the dynamic programming principle as:

$$\inf_{u\in\mathcal{U}} \left\{ \frac{1}{h}\int_t^{(t+h)\wedge\tau} L(s,x(s),u(s))\,ds + \frac{1}{h}g(\tau,x(\tau))\chi_{\tau<t+h} + \frac{1}{h}\left[V(t+h,x(t+h))\chi_{\tau\geqslant t+h} - V(t,x)\right] \right\} = 0 \tag{13}$$

Then if we formally let $h \to 0$ we get:

$$\inf_{u\in\mathcal{U}} \left\{ L(t,x(t),u(t)) + \partial_t V(t,x(t)) + D_x V(t,x(t)) \cdot f(t,x(t),u(t)) \right\} = 0$$

Which can be rewritten as:

$$-\frac{\partial}{\partial t}V(t,x) + H(t,x,D_xV(t,x)) = 0 \tag{14}$$

Where for $(t,x,p) \in \overline{Q} \times \mathbb{R}^n$ the Hamiltonian is defined as:

$$H(t,x,p) = \sup_{v\in\mathbb{R}^n} \left\{ -p \cdot f(t,x,v) - L(t,x,v) \right\}. \tag{15}$$

Equation 14 turns out to be the main sufficient condition for the value function to be optimal. Maybe only differentiability is needed (also for if, for only if we already know).

Theorem 0.3.2 (Verification Theorem)**.** *Let $W \in C^1(\overline{Q})$ satisfy 14 and the boundary conditions 11 and 12 then:*

$$W(t,x) \leqslant V(t,x) \ \forall \ (t,x) \in \overline{Q}$$

Moreover, there exists $u^* \in \mathcal{U}$ such that:

$$\begin{cases} L(s,x^*(s),u^*(s)) + f(s,x^*,u^*(s)) \cdot D_x W(s,x^*(s)) = -H(s,x^*(s),D_x W(s,x^*(s))) & a.s. \ for \ s \in [t,\tau^*] \\ W(\tau^*,x^*(\tau^*)) = g(\tau^*,x^*(\tau^*)) & if \ \tau^* < t_1 \end{cases} \tag{16}$$

if and only if u^* is optimal and $W = V$.

Proof. Let $u \in \mathcal{U}$, then:

$$\begin{aligned} \Psi(\tau,x(\tau)) = W(\tau,x(\tau)) &= W(t,x(t)) + \int_t^\tau \frac{d}{ds} W(s,x(s))\, ds \\ &= W(t,x(t)) + \int_t^\tau \frac{\partial}{\partial t} W(s,x(s)) + \dot{x}(s) \cdot D_x W(s,x(s))\, ds \\ &= W(t,x(t)) + \int_t^\tau \frac{\partial}{\partial t} W(s,x(s)) + f(s,x(s),u(s)) \cdot D_x W(s,x(s))\, ds \\ &\overset{\circledast}{\geqslant} W(t,x(t)) - \int_t^\tau L(s,x(s),u(s))\, ds \end{aligned}$$

Then:

$$W(t,x(t)) \leqslant J(t,x;u)$$

And therefore by taking the infimum over $\mathcal{U}$ and recalling $x(t) = x$ we get:

$$W(t,x) \leqslant V(t,x)$$

If furthermore u^* satisfies 16 then the inequality $\overset{\circledast}{\geqslant}$ is an equality, and therefore:

$$W(t,x) = J(t,x;u^*)$$

Which implies that u^* is optimal and $W(t,x) = J(t,x;u^*) = V(t,x)$. The converse will be proved in a more general setting. In particular, only differentiability is needed.

□

Theorem 0.3.2 is an important tool in determining the explicit form of and optimal control. Indeed, condition 16 can be restated as:

$$u^*(s) \in \arg\min_{v \in U} \{ f(s,x^*(s),v) \cdot D_x W(s,x^*(s)) + L(s,x^*(s),v) \} \tag{17}$$

For almost all $s \in [t,t_1]$.
I have to prove verification theorem in more general case ($O \neq \mathbb{R}^n$).
We can express the optimality condition on u 17 in a differential inclusion form.

Corollary 0.3.3. *A control u^* is optimal if the corresponding state function x^* satisfies:*

$$x^* \in \{ f(t,x,v) \,|\, v \in v^*(t,x) \} \tag{18}$$

0.4 Pontryagin's principle and dynamic programming

In the previous section we tackled the optimal control problem via dynamic programming. As mentioned earlier this approach is of wide applicability and provides an implicit characterization of an optimal control. We now present another technique: the Pontryagin's principle. As before, it will give rise to necessary condition on a control function to be optimal, but they'll come from a completely different perspective. We now present Pontryagin's principle in its full generality, and then we will see how it is connected with the dynamic programming approach.

0.4.1 Pontryagin's principle

Pontryagin gives us an elegant and unintuitive way of solving problems of the kind:

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)) & s \in [t, t_1] \\ x(t) = x \end{cases} \tag{19}$$

Where u is bounded measurable into $U \subset \mathbb{R}^m$ and $O = \mathbb{R}^n$. Having defined the functional $J(t, x; u)$ as usual we define the *control state Hamiltonian* as follows.

Definition 0.4.1. *The control state Hamiltonian of system 19 is:*

$$H(s, x, u, p) = -p \cdot f(s, x, u) - L(s, x, u) \tag{20}$$

For all $s \in [t, t_1]$, $x, p \in O$, $u \in U$.

The variable p is called *costate* of the system. Pontryagin's principle gives us information about the costate under an optimal trajectory, which in turn will characterize the optimal control.

Theorem 0.4.1. *Let u^* be an optimal control and x^* its corresponding trajectory. Then there exists a function $p^* : [t, t_1] \to O$ such that:*

$$\dot{x}^*(s) = D_p H(s, x^*(s), u^*(s), p^*(s)) \tag{21}$$

$$\dot{p}^*(s) = -D_x H(s, x^*(s), u^*(s), p^*(s)) \tag{22}$$

And also:

$$H(s, x^*(s), u^*(s), p^*(s)) = \sup_{v \in U} H(s, x^*(s), v, p^*(s)) \tag{23}$$

With:

$$p^*(t_1) = D\psi(x^*(t_1)) \tag{24}$$

Thanks to this result we can determine an optimal control via the costate. By solving equation 22 arisen in Theorem 0.4.1 we can obtain the explicit form of p^*, and then retrieve u^* from the maximization principle 23.

0.4.2 Dynamic programming interplay

As we just saw, Pontryagin's principle offers us an-even-though-quite-involved technique for finding an optimal control. As use of a control state Hamiltonian is crucial in this approach, it reminds us of the Hamiltonian defined in 15. The similarity is also fortified by the maximization principle 23. We shall prove that this similarity unveils the direct link between Pontryagin's principle and the dynamic programming equation.
We will use the notion of differentiability of the value function. The classical notion of differentiability which we use is the following.

Definition 0.4.2. *V is differentiable in (t,x) if there exists $V_t(t,x), V_x(t,x) \in \mathbb{R}$ such that:*

$$\lim_{(h,k)\to(0,0)} \frac{1}{|h|+|k|}|V(t+h,x+k) - V(t,x) - V_t(t,x)\cdot h - V_x(t,x)\cdot k| = 0 \tag{25}$$

Differentiability is somewhat a strong hypothesis, but it allows us to prove the following proposition. We must say that differentiability may easily fail in application, in such istances the notion of a weaker solution is needed, namely a viscosity solution.

Theorem 0.4.2. *Let V be differentiable in $(t,x) \in Q$ and u^* an optimal control such that $u^* \xrightarrow{s\to t} v$, then:*

$$V_t(t,x) + L(t,x,v) + f(t,x,v)\cdot D_xV(t,x) = 0 \tag{26}$$

Proof. Let $h > 0$ s.t. $t+h < \tau$, then by 0.3.1 we have:

$$V(t,x) = \int_t^{t+h} L(s,x(s),u^*(s))\,ds + V(t+h,x(t+h))$$

But because of differentiability we have:

$$\lim_{h\to 0}\frac{1}{|h|}|V(t+h,x(t+h)) - V(t,x(t))| = V_t(t,x) + f(t,x,v)\cdot D_xV(t,x)$$

Then we get:

$$\begin{aligned} L(t,x,v) &= \lim_{h\to 0}\frac{1}{|h|}\int_t^{t+h} L(s,x(s),u(s))\,ds = -\lim_{h\to 0}\frac{1}{|h|}|V(t+h,x(t+h)) - V(t,x(t))| &(27)\\ &= -V_t(t,x) - f(t,x,v)\cdot D_xV(t,x) &(28)\end{aligned}$$

□

Furthermore, we impose existence and continuity of all derivatives of f, L, g, ψ. The next theorem demonstrates that the costate in the Pontryagin Maximum Principle is in fact the gradient in x of the value function v, taken along an optimal trajectory.

Theorem 0.4.3. *Let u^* be an optimal right-continuous control and x^* its corresponding trajectory. Assume that the value function V is differentiable at $(s,x^*(s))$ for $s \in [t,t_1)$. If we define:*

$$p(s) = D_xV(s,x^*(s)) \tag{29}$$

Then $p(s)$ satisfies 22, 23 and 24.

Proof. The trasversality condition 24 is straightforward from the definition of the value function, which implies also the maximality condition 23. We need to prove the "lagrangian multiplier condition" 22. Let us drop all * and rewrite this differential equation:

$$\frac{d}{dt}p_j(s) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_j}f_i(s,x(s),u(s))p_i(s) - \frac{\partial}{\partial x_j}L(s,x(s),u(s)) \tag{30}$$

This system admits solution $\overline{p}$ such that $\overline{p}(s) = D_xV(s,x(s))$; let us show that $\overline{p}(s) = p(s)$. Let u_s be the restriction of u to $[r,t_1)$, which is admissible by assumption. We have:

$$V(s,y) \leqslant J(s,y;u_s) \;\; \forall\, y \in \mathbb{R}^n$$

Then, because u is optimal, $y \mapsto J(s,y;u_s) - V(s,y)$ has its global minimum in $y = x(s)$, which implies by differentiability:

$$D_x V(s, x(s)) = D_x J(s, x(s); u_s) \tag{31}$$

Then we prove that $\overline{p}(s) = D_x J(s, x(s); u_s)$. We denote $x(r)$ the solution starting at $x(s)$ at time r. Because $L \in C^1$ then for all $i = 1, \dots, n$:

$$\frac{\partial}{\partial x_i} J(s, x(s), u) = \sum_{j=1}^{n} \int_s^{t_1} \left(L_{x_j}(r, x(r), u(r)) \frac{\partial x_j(r)}{\partial x_i} \right) dr + \psi_{x_j}(x(t_1)) \frac{\partial x_j(t_1)}{\partial x_i} \tag{32}$$

But then:

$$\overline{p}_i(s) = \sum_{j=1}^{n} \frac{\partial x_j(s)}{\partial x_i} \overline{p}_j(s) = \sum_{j=1}^{n} \frac{\partial x_j(t_1)}{\partial x_i} \overline{p}_j(t_1) - \int_s^{t_1} \frac{d}{dr} \left(\sum_{j=1}^{n} \frac{\partial x_j(r)}{\partial x_i} \overline{p}_j(r) \right) dr \tag{33}$$

$$= \sum_{j=1}^{n} \frac{\partial x_j(t_1)}{\partial x_i} \psi(x(t_1)) - \sum_{j=1}^{n} \int_s^{t_1} \frac{d}{dr} \left(\frac{\partial x_j(r)}{\partial x_i} \right) \overline{p}_j(r) + \frac{\partial x_j(r)}{\partial x_i} \frac{d}{dr} \left(\overline{p}_j(r) \right) dr \tag{34}$$

But under the integral:

$$\int_s^{t_1} \frac{d}{dr} \sum_{l=1}^{n} \frac{\partial x_j(r)}{\partial x_i} \left(\frac{\partial}{\partial x_j} f_l(r, x(r), u(r)) \overline{p}_j(r) - \sum_{l=1}^{n} \frac{\partial x_j(r)}{\partial x_i} \frac{\partial}{\partial x_j} f_l(r, x(r), u(r)) \overline{p}_j(r) - \frac{\partial}{\partial x_j} L(r, x(r), u(r)) \right) dr$$

Then we get:

$$\overline{p}_i(s) = \sum_{j=1}^{n} \left(\frac{\partial x_j(t_1)}{\partial x_i} \psi(x(t_1)) + \int_s^{t_1} \frac{\partial}{\partial x_j} L(r, x(r), u(r)) \frac{\partial x_j(r)}{\partial x_i} \, dr \right) \tag{35}$$

$$= \frac{\partial}{\partial x_i} J(s, x(s); u_r) \tag{36}$$

□

0.5 Existence

We now prove an existence theorem for optimal controls. We study the fixed time interval case with $O = \mathbb{R}^n$ and the function f linear in v. Furthermore, we impose convexity of L in v. Under these assumptions a classical variational argument proves the optimal control existence.

Theorem 0.5.1. *Let U compact and convex, $f_1, f_2 \in C^1(\overline{Q} \times U)$ such that $f(t, x, v) = f_1(t, x) + f_2(t, x)v$ and $\partial_x f_1, \partial_x f_2, f_2$ bounded. Let also $L \in C^1(\overline{Q} \times U)$ and $L(t, x, \cdot)$ be convex for all $(t, x) \in \overline{Q}$ and the temrinal cost $\psi \in C(\mathbb{R}^n)$. Then there exist an optimal control $u^*(\cdot)$.*

Proof. Let u_n a minimizing sequence such that:

$$\lim_{n \to +\infty} J(t, x; u_n) = V(t, x) \tag{37}$$

Let $x_n(\cdot)$ be the solutions to 1 with $u = u_n$. If we show both sequence to converge respectively (weakly) to u^* and uniformly x^* (along subsequences) such that the latter is again the solution to 1 with $u = u^*$, then:

$$J(t, x; u_n) = \int_t^{t_1} L(s, x^*(s), u_n(s)) \, ds + \int_t^{t_1} L(s, x_n(s), u_n(s)) - L(s, x^*(s), u_n(s)) \, ds + \phi(x_n(t_1))$$

But then:

$$\liminf_{n\to+\infty}\int_t^{t_1} L(s,x_n(s),u_n(s))-L(s,x^*(s),u_n(s))\,ds=0$$

And $\psi(x_n(t_1))\xrightarrow{n\to+\infty}\psi(x^*(t_1))$. But then:

$$\liminf_{n\to+\infty} J(t,x;u_n)=\liminf_{n\to+\infty}\int_t^{t_1} L(s,x^*(s),u_n(s))\,ds \geqslant \int_t^{t_1} L(s,x^*(s),u^*(s))\,ds \tag{38}$$

Because L is convex in u[1]. Therefore:

$$V(t,x)\leqslant J(t,x;u^*)\leqslant \liminf_{n\to+\infty} J(t,x;u_n)=V(t,x)$$

We need to prove convergence of x_n and u_n. Because U is compact and *convex* then $L^\infty([t,t_1];U)$ is weakly sequentially compact. For what concerns x_n we use Ascoli-Arzela's theorem to show that is admits a uniformly convergent subsequence. Being uniformly limitated comes from:

$$\begin{aligned}
|x_n(s)| &\leqslant |x_n(s)-x|+|x| = |x|+\int_t^s \frac{d}{dr}|x_n(r)|\,dr && (39)\\
&= |x|+\int_t^s |f_1(r,x_n(r))+f_2(r,x_n(r))\cdot u_n(r)|\,dr && (40)\\
&\leqslant |x|+\int_t^s \|\partial_x f_1\|_\infty |x_n(r)|+\|f_2\|_\infty |u_n|\,dr && (41)\\
&\leqslant C+K\left(\int_t^s |x_n(r)|\,dr\right) && (42)
\end{aligned}$$

Then by Gronwall's lemma x_n is uniformly limitated. Equicontinuity comes from the uniform buondedness of the derivative $\dot{x}_n(s)$. Therefore, we know that there exist the weak limit u^* and the uniform limit x^*. The latter is the solution of 1 with $u=u^*$. Indeed:

$$\begin{aligned}
x_n(s) &= x+\int_t^s \frac{d}{dr}x_n(r)\,dr\\
&= x+\underbrace{\int_t^s f_1(r,x_n(r))+f_2(r,x_n(r))u^*(r)\,dr}_{A_n}+\underbrace{\int_t^s [f_2(r,x_n(r))-f_2(r,x^*(r))]\,[u_n(r)-u^*(r)]\,dr}_{B_n}\\
&\quad+\underbrace{\int_t^s f_2(r,x^*(r))\,[u_n(r)-u^*(r)]\,dr}_{C_n}
\end{aligned}$$

Letting $n\to+\infty$ we get B_n (by weak convergence and boundedness of f_2) and C_n (by weak convergence) going to 0 while we obtain:

$$A_n\xrightarrow{n\to+\infty}\int_t^s f_1(r,x^*(r))+f_2(r,x^*(r))u^*(r)\,dr$$

And therefore the thesis. $\square$

Remark. *In the proof we asserted inequality 38 by convexity of the running cost. Indeed, by convexity and being C^1:*

$$L(s,x(s),u_n(s))\geqslant L(s,x(s),u^*(s))+[u_n(s)-u^*(s)]\,L_u(s,x(s),u^*(s))$$

Then by integrating and taking $\liminf_{n\to+\infty}$ we get the inequality (using weak convergence of u_n).

[1]Explained in remark 0.5

An existence result can also be proved in the context of $O \neq \mathbb{R}^n$, where the cost function to be minimized has the form:

$$J(t,x;u) = \int_t^\tau L(s,x(s),u(s))\,ds + \Psi(\tau, x(\tau)) \tag{43}$$

with dynamical system described by $f(s,x,u) = u$. Therefore, the value function is:

$$V(t,x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^\tau L(s,x(s),\dot{x}(s))\,ds + \Psi(\tau, x(\tau)) \right\} \tag{44}$$

Under technical assumptions an optimal control exists, and the proof follows the same procedure as before: we take a minimizing sequence, show its (weak) convergence and its corresponding state evolution (uniform) convergence, then we show optimality, slightly modifying the proof because of convergence issues.

0.6 Infinite horizon

A particularly interesting version of the maximization problem arising from optimal control is with infinite horizon. Let us consider the usual Cauchy' problem:

$$\begin{cases} \dot{x}(s) = f(s,x(s),u(s)) & s \in [t,t_1] \\ x(t) = x \end{cases} \tag{45}$$

If we set $t_1 = +\infty$ we get an infinite horizon problem. Thus, the maximization problem becomes:

$$\inf_{u \in \mathcal{U}} \int_t^\tau L(s,x(s),u(s))\,ds + g(\tau, x(\tau))\chi_{\tau<+\infty} \tag{46}$$

Where τ is the exit time of $x(\cdot)$ from $\overline{O}$.[2]

0.7 Proof of Pontryagin's principle

We will show the maximum Pontryagin's principle in the simple context of no running cost. A cleaver reconstruction of the non-zero running cost problem as a zero running cost one will enlarge the thesis to this situation. The first concept we will need is the variation of a control.

Definition 0.7.1. *Given $u \in \mathcal{U}$. For $\epsilon, r > 0$ such that $0 < r - \epsilon < r$ and $a \in U$ we define the simple variation $u_\epsilon \in \mathcal{U}$ such that:*

$$u_\epsilon(t) = \begin{cases} a & s \in (r-\epsilon, r) \\ u(s) & s \notin (r-\epsilon, r) \end{cases} \tag{47}$$

By defining the matrix $A : [0,+\infty) \to \mathbb{R}^{n\times n} : s \mapsto D_x f(s,x(s),u(s))$ we state the following lemma.

Lemma 0.7.1. *Let x_ϵ be solution of:*

$$\begin{cases} \dot{x}_\epsilon(s) = f(s,x_\epsilon(s),u_\epsilon(s)) & s \in [t,t_1] \\ x_\epsilon(t) = x \end{cases} \tag{48}$$

Then the solution is:

$$x_\epsilon(s) = x(s) + \epsilon y(s) + o(\epsilon)\; \epsilon \to 0 \tag{49}$$

[2]More precisely, is the exit time of $(s, x(s))$ from $\overline{Q}$.

Where $y \equiv 0$ on $[t, r]$ and:

$$\begin{cases} \dot{y}(s) = A(s)y(s) \quad s \in [r, t_1] \\ y(r) = y^r \end{cases} \tag{50}$$

With $y^r = f(x(r), a) - f(x(r), u(r))$.

Proof. Let us divide the proof in three cases.

- $s \in [t, r - \epsilon]$: then $y(t) = 0$ and $u_\epsilon(t) = u(t)$, therefore:

$$x_\epsilon(t) = x(t) = x(t) + \epsilon y(t) + o(\epsilon)$$

- $s \in (r - \epsilon, r)$: then we have:

$$x_\epsilon(s) - x(s) = \int_{r-\epsilon}^{s} f(w, x_\epsilon(w), u_\epsilon(w)) - f(w, x(w), u(w))\, dw + o(\epsilon)$$

  Which is a little o of ϵ (because f is continuous).

- $s \in [r, t_1]$: from before if $s = r$ then:

$$\begin{aligned} x_\epsilon(r) - x(r) &= \int_{r-\epsilon}^{r} f(w, x_\epsilon(w), u_\epsilon(w)) - f(w, x(w), u(w))\, dw + o(\epsilon) && (51) \\ &= \lim_{w \to s} [f(w, x_\epsilon(w), u_\epsilon(w)) - f(w, x(w), u(w))]\epsilon + o(\epsilon) && (52) \\ &= y^s \epsilon + o(\epsilon) && (53) \end{aligned}$$

  If $s > r$ then:

□

Let us now prove Pontryagin's principle with no running cost. The payoff functional is:

$$J(t, x; u) = \psi(x(t_1)) \tag{54}$$

And therefore the Hamiltonian is:

$$H(s, x, u, p) = -f(s, x, u) \cdot p \tag{55}$$

Theorem 0.7.2. *There exists a function $p^* : [t, t_1] \to \mathbb{R}^n$ such that:*

$$\dot{p}^*(s) = -D_x H(s, x^*(s), u^*(s), p^*(s))\; s \in [t, t_1] \tag{56}$$

together with the maximization:

$$H(s, x^*(s), u^*(s), p^*(s)) = \sup_{v \in U} H(s, x^*(s), v, p^*(s)) \tag{57}$$

and the trasversality condition:

$$p^*(t_1) = D\psi(x(t_1)) \tag{58}$$

Proof. Let us drop all the *. Let p be the unique solution of:

$$\begin{cases} \dot{p}(s) = -A'(s) \cdot p(s) & s \in [t, t_1] \\ p(t_1) = D\psi(x(t_1)) \end{cases} \tag{59}$$

It exists and is unique because the latter is a linear differential equation with integrable coefficient. We already satisfy the trasversality condition and the adjoint dynamics. We prove the maximization principle. Let $a \in U$. We define the variation u_ϵ for $\epsilon, r \in (t, t_1)$ as before. Since $\epsilon \mapsto J(t, x; u_\epsilon)$ for $\epsilon \in [0, 1]$ has a maximum in $\epsilon = 0$ we have:

$$\frac{d}{d\epsilon} J(t, x; u_\epsilon) \leqslant 0 \tag{60}$$

Computing the derivative, using 47:

$$\frac{d}{d\epsilon} J(t, x; u_\epsilon)\Big|_{\epsilon=0} = \frac{d}{d\epsilon} \psi(x_\epsilon(t_1))\Big|_{\epsilon=0} \tag{61}$$

$$= \frac{d}{d\epsilon} \psi(x(t_1) + \epsilon y(t_1) + o(\epsilon)) = D\psi(x(t_1)) \cdot y(t_1) \tag{62}$$

$$= p(t_1) \cdot y(t_1) = p(r) \cdot [f(r, x(r), a) - f(r, x(r), u(r))] \tag{63}$$

Where the last equality comes from:

$$\begin{aligned} \frac{d}{ds} (p(s) \cdot y(s)) &= \dot{p}(s) \cdot y(s) + p(s) \cdot \dot{y}(s) \\ &= -A'(s) \cdot p(s) \cdot y(s) + p(s) \cdot A(s) \cdot y(s) = 0 \end{aligned}$$

Therefore, by plugging into **??** we get:

$$0 \geqslant p(r) \cdot [f(r, x(r), a) - f(r, x(r), u(r))]$$

Which implies:

$$H(r, x(r), a, p(r)) = f(r, x(r), a) \cdot p(r) \leqslant f(r, x(r), u(r)) \cdot p(r) = H(r, x(r), u(r), p(r))$$

□

Given Pontryagin's principle for no running cost problems we can extend the result to the general case:

$$J(t, x; u) = \int_t^{t_1} L(s, x(s), u(s))\, ds + \psi(x(t_1)) \tag{64}$$

Where the Hamiltonian is:

$$H(s, x, u, p) = f(s, x, u) \cdot p + L(s, x, u) \tag{65}$$

Indeed, theorem 0.7.2 holds also under these conditions. We rewrite the problem as it has no running cost and then apply the theorem. Let us define x^{n+1} as:

$$x^{n+1}(s) = \int_t^s L(w, x(w), u(w))\, dw \tag{66}$$

Then by defining $\overline{f}, \overline{g}, \overline{x}, \overline{x}(s)$ as:

$$\overline{f}(s, x, u) = \begin{bmatrix} f(s, x, u) \\ L(s, x, u) \end{bmatrix}, \overline{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \overline{x}(s) = \begin{bmatrix} x(s) \\ x^{n+1}(s) \end{bmatrix}, \overline{g}(\overline{x}(t_1)) = g(x(t_1)) + x^{n+1}(t_1) \tag{67}$$

Thus, the problem has no running cost. We can apply the theorem and, noticing that $p^{n+1} \equiv 1$ we get the thesis.