0.1 Introduction

We now study stochastic optimal control. The system we aim to control is governed by stochastic differential equations.
Short description of what is done in this chapter.

0.2 Markov diffusion process

I now recall some definitions, give new ones and set the notation. Let $\Sigma \subseteq \mathbb{R}^n$ and $\mathcal{B}(\Sigma)$ the associated Borel σ-algebra. Let $(\Omega, \mathcal{F}, P)$ a general probability space. Given $x(s, \omega)$ a Σ-valued random process from $I_0 = [t_0, t_1)$ and $(\Omega, \mathcal{F})$, let us denote by:

$$P(C \,|\, x(s_1), \ldots, x(s_m)),\ C \in \mathcal{F}$$

The conditional probability of C given the sigma algebra $\bigvee_{i=1}^{m} \sigma(x(s_i))$.

Definition 0.2.1. *A stochastic process x satisfies the Markov property if there exists a function $p : I_0 \times \Sigma \times I_0 \times \mathcal{B}(\Sigma) \to \mathbb{R}$ such that:*

1. *For all t, s, B the function $x \mapsto p(t, x, s, B)$ is borel measurable on Σ*
2. *For all t, x, s the function $A \mapsto p(t, x, s, B)$ is a probability measure on $(\Omega, \mathcal{F})$*
3. *The Chapman-Kolmogorov equation holds for all $s, t, r \in I_0$ such that $t < r < s$:*

$$p(t, x, s, B) = \int_{\Sigma} p(r, y, s, B)\, p(t, x, r, dy) \tag{1}$$

And such that for all $r, s \in I_0$ where r, s and for all $B \in \mathcal{B}(\Sigma)$ then:

$$P(x(s) \in B \,|\, \mathcal{F}_r^x) = p(r, x(r), s, B) \tag{2}$$

Where $\mathcal{F}_r^x = \sigma\,(x(l) \,:\, l \in [t_0, r])$.

Function p is called *Markov Transition Kernel.* We shall see a Markov transition kernel as the probability that the system starting from x at time t will be in B at time s. This heuristic interpretation clarifies the following notation:

$$E_{tx}\phi(x(s)) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) \tag{3}$$

For a real valued borel-measurable function ϕ. Given a Markov process x we can define a family of linear operators associated to it. Let $t < s$, hereafter all time indices will always be in I_0, and define:

$$T_{t,s}\phi(x) = \int_{\Sigma} \phi(y)\, p(t, x, s, dy) = E_{tx}\phi(x(s)) \tag{4}$$

Integrability assumptions on ϕ vary from case to case. For now, we can take ϕ to be bounded. Because of Chapman-Kolmogorov equation **??** the family $(T_{t,s})_{t,s \in I_0}$ satisfies the property:

$$T_{tr}\left[T_{rs}\phi\right] = T_{ts}\phi \tag{5}$$

For all $t < r < s$. This family of linear operators defines another operator, the *backward evolution operator.* Let $A : \{\Phi : I_0 \times \Sigma \to \mathbb{R}\} \to \mathbb{R}$:

$$A\Phi(t, x) = \lim_{h \to 0+} \frac{E_{tx}\Phi(t + h, x(t + h)) - \Phi(t, x)}{h} \tag{6}$$

provided that the limit exists. We define $\mathcal{D}(A)$ the space of functions such that limit **??** exists. The following holds.

Proposition 0.2.1. *Let A as before, then for all $\Phi \in \mathcal{A}$ the following hold:*

1. *Φ, $\frac{\partial \Phi}{\partial t}$ and $A\Phi$ are continuous*
2. *For all $t, s \in \overline{I}_0$, $t < s$ then:*

$$E_{tx}|\Phi(s, x(s))|, E_{tx}\int_t^s |A\Phi(r, x(r))|\, dr < +\infty$$

3. *Dynkin's formula holds for all $t < s$:*

$$E_{tx}\Phi(s, x(s)) - \Phi(t, x) = E_{tx}\int_t^s A\Phi(r, x(r))\, dr \tag{7}$$

Proof. I prove Dynkin's formula in the case of T_{ts} being a Feller semigroup. □

If the random process x is autonomous (time-homogeneous) then the linear operator family is a semigroup. Recall that a Markov process is homogeneous if for all $t < s$ in I_0 then:

$$p(t, x, s, B) = p(0, x, s - t, B)$$

If so, by calling $T_s = T_{0s}$ property **??** is:

$$\begin{aligned}
T_{s+r}\phi(x) &= \int_\Sigma \phi(y)\, p(0, x, s + r, dy) && (8)\\
&= \int_\Sigma \phi(y) \int_\Sigma p(r, z, r + s, dy)\, p(0, x, r, dz) && (9)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(r, z, r + s, dy)\, p(0, x, r, dz) && (10)\\
&= \int_\Sigma \int_\Sigma \phi(y) p(0, z, s, dy)\, p(0, x, r, dz) && (11)\\
&= \int_\Sigma T_s\phi(z)\, p(0, x, r, dz) && (12)\\
&= T_r\left[T_s\phi(x)\right]. && (13)
\end{aligned}$$

While the backward evolution operator analogous is called the *generator* and is defined as:

$$G\phi(x) = -\lim_{h\to 0^+} \frac{T_h\phi(x) - \phi(x)}{h} \tag{14}$$

With $D(G)$ as $\mathcal{D}(A)$ before. It is worth noting that, formally, the following equality holds:

$$A\Phi = \frac{\partial \Phi}{\partial t} - G\Phi(t, \cdot) \tag{15}$$

This relation links the two operators and the autonomous to the non-autonomous case. We now turn our attention to a subset of Markov processes: diffusion processes. A diffusion process is a Markov process whose paths are continuous. A diffusion process is completely determined by its infinitesimal mean and variance.

Definition 0.2.2. *The infinitesimal mean is:*

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