

Let  $\mathbf{V}$  denote the matrix to define the normal mode, i.e.,

$$\mathbf{U} = \mathbf{V}\mathbf{u}, \quad (68)$$

where  $\mathbf{U} = (X, P_x, Y, P_y, Z, P_z)$  and  $\mathbf{u} = (x, p_x, y, p_y, z, \delta \equiv p - 1)$  are the normal and physical coordinates, respectively. The matrix  $\mathbf{V}$  can be expressed as

$$\mathbf{V} = \mathbf{PBR}_6\mathbf{H}, \quad (69)$$

where

$$\mathbf{H} = \begin{pmatrix} \left(1 - \frac{\det \mathbf{H}_x}{1+a}\right) \mathbf{I} & \frac{\mathbf{H}_x \mathbf{J}_2 \mathbf{H}_y^T \mathbf{J}_2}{1+a} & -\mathbf{H}_x \\ \frac{\mathbf{H}_y \mathbf{J}_2 \mathbf{H}_x^T \mathbf{J}_2}{1+a} & \left(1 - \frac{\det \mathbf{H}_y}{1+a}\right) \mathbf{I} & -\mathbf{H}_y \\ -\mathbf{J}_2 \mathbf{H}_x^T \mathbf{J}_2 & -\mathbf{J}_2 \mathbf{H}_y^T \mathbf{J}_2 & a\mathbf{I} \end{pmatrix}, \quad (70)$$

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{I} \end{pmatrix} = \begin{pmatrix} b\mathbf{I} & \mathbf{J}_2 \mathbf{r}^T \mathbf{J}_2 & 0 \\ \mathbf{r} & b\mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{pmatrix}, \quad (71)$$

$$\mathbf{PB} = \begin{pmatrix} \mathbf{P}_x \mathbf{B}_x & 0 & 0 \\ 0 & \mathbf{P}_y \mathbf{B}_y & 0 \\ 0 & 0 & \mathbf{P}_z \mathbf{B}_z \end{pmatrix}, \quad (72)$$

with

$$a^2 + \det \mathbf{H}_x + \det \mathbf{H}_y = 1, \quad (73)$$

$$b^2 + \det \mathbf{R} = 1. \quad (74)$$

Symbols  $\mathbf{I}, \mathbf{J}_2, \mathbf{H}_{x,y}, \mathbf{r}, \mathbf{B}_{x,y,z}, \mathbf{P}_{x,y,z}$  above are 2 by 2 matrices:

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (75)$$

$$\mathbf{J}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (76)$$

$$\mathbf{r} \equiv \begin{pmatrix} \mathbf{R1} & \mathbf{R2} \\ \mathbf{R3} & \mathbf{R4} \end{pmatrix}, \quad (77)$$

$$\mathbf{B}_{x,y} \equiv \begin{pmatrix} \frac{1}{\sqrt{\beta_{x,y}}} & 0 \\ \frac{\alpha_{x,y}}{\sqrt{\beta_{x,y}}} & \sqrt{\beta_{x,y}} \end{pmatrix}, \quad (78)$$

$$\mathbf{P}_{x,y,z} \equiv \begin{pmatrix} \cos \psi_{x,y,z} & \sin \psi_{x,y,z} \\ -\sin \psi_{x,y,z} & \cos \psi_{x,y,z} \end{pmatrix}. \quad (79)$$

Matrices  $\mathbf{H}_{x,y}$  define dispersions as

$$\begin{pmatrix} \mathbf{ZX} & \mathbf{EX} \\ \mathbf{ZPX} & \mathbf{EPX} \\ \mathbf{ZY} & \mathbf{EY} \\ \mathbf{ZPY} & \mathbf{EPY} \end{pmatrix} \equiv \mathbf{R} \begin{pmatrix} \mathbf{H}_x \\ \mathbf{H}_y \end{pmatrix}. \quad (80)$$