

Let  $V$  denote the matrix to define the normal mode, i.e.,

$$\boldsymbol{U} = \boldsymbol{V}\boldsymbol{u} \,,$$

where  $\boldsymbol{U} = (X, P_x, Y, P_y, Z, P_z)$  and  $\boldsymbol{u} = (x, p_x, y, p_y, z, \delta \equiv p - 1)$  are the normal and physical coordinates, respectively, expressed as

$$\boldsymbol{V} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}_6\boldsymbol{H} \,,$$

where

$$\boldsymbol{H} = \begin{pmatrix} \left(1 - \frac{\det \boldsymbol{H}_x}{1 + a}\right)\boldsymbol{I} & \frac{\boldsymbol{H}_x\boldsymbol{J}_2\boldsymbol{H}_y^T\boldsymbol{J}_2}{1 + a} & -\boldsymbol{H}_x \\ \frac{\boldsymbol{H}_y\boldsymbol{J}_2\boldsymbol{H}_x^T\boldsymbol{J}_2}{1 + a} & \left(1 - \frac{\det \boldsymbol{H}_y}{1 + a}\right)\boldsymbol{I} & -\boldsymbol{H}_y \\ -\boldsymbol{J}_2\boldsymbol{H}_x^T\boldsymbol{J}_2 & -\boldsymbol{J}_2\boldsymbol{H}_y^T\boldsymbol{J}_2 & a\boldsymbol{I} \end{pmatrix} \,,$$

$$\boldsymbol{R}_6 = \begin{pmatrix} \boldsymbol{R} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} b\boldsymbol{I} & \boldsymbol{J}_2\boldsymbol{r}^T\boldsymbol{J}_2 & 0 \\ \boldsymbol{r} & b\boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} \end{pmatrix} \,,$$

$$\boldsymbol{P}\boldsymbol{B} = \begin{pmatrix} \boldsymbol{P}_x\boldsymbol{B}_x & 0 & 0 \\ 0 & \boldsymbol{P}_y\boldsymbol{B}_y & 0 \\ 0 & 0 & \boldsymbol{P}_z\boldsymbol{B}_z \end{pmatrix} \,,$$

with

$$\begin{aligned} a^2 + \det \boldsymbol{H}_x + \det \boldsymbol{H}_y &= 1 \,, \\ b^2 + \det \boldsymbol{R} &= 1 \, . \end{aligned}$$

Symbols  $\boldsymbol{I}, \boldsymbol{J}_2, \boldsymbol{H}_{x,y}, \boldsymbol{r}, \boldsymbol{B}_{x,y,z}, \boldsymbol{P}_{x,y,z}$  above are 2 by 2 matrices:

$$\begin{aligned} \boldsymbol{I} &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \\ \boldsymbol{J}_2 &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \\ \boldsymbol{r} &\equiv \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 \\ \boldsymbol{R}_3 & \boldsymbol{R}_4 \end{pmatrix} \,, \\ \boldsymbol{B}_{x,y} &\equiv \begin{pmatrix} \frac{1}{\sqrt{\beta_{x,y}}} & 0 \\ \frac{\alpha_{x,y}}{\sqrt{\beta_{x,y}}} & \sqrt{\beta_{x,y}} \end{pmatrix} \,, \\ \boldsymbol{P}_{x,y,z} &\equiv \begin{pmatrix} \cos \psi_{x,y,z} & \sin \psi_{x,y,z} \\ -\sin \psi_{x,y,z} & \cos \psi_{x,y,z} \end{pmatrix} \, . \end{aligned}$$

Matrices  $\boldsymbol{H}_{x,y}$  define dispersions as

$$\begin{pmatrix} \boldsymbol{Z}\boldsymbol{X} & \boldsymbol{E}\boldsymbol{X} \\ \boldsymbol{Z}\boldsymbol{P}\boldsymbol{X} & \boldsymbol{E}\boldsymbol{P}\boldsymbol{X} \\ \boldsymbol{Z}\boldsymbol{Y} & \boldsymbol{E}\boldsymbol{Y} \\ \boldsymbol{Z}\boldsymbol{P}\boldsymbol{Y} & \boldsymbol{E}\boldsymbol{P}\boldsymbol{Y} \end{pmatrix} \equiv \boldsymbol{R} \begin{pmatrix} \boldsymbol{H}_x \\ \boldsymbol{H}_y \end{pmatrix} \, .$$