

The transformation matrix from the physical coordinate  $(x, p_x, y, p_y)$  to the  $x$ - $y$  decoupled coordinate  $(X, P_X, Y, P_Y)$  is written as

$$\mathbf{R} = \begin{pmatrix} \mu\mathbf{I} & \mathbf{J}\mathbf{r}^T\mathbf{J} \\ \mathbf{r} & \mu\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mu & . & -\mathbf{R4} & \mathbf{R2} \\ . & \mu & \mathbf{R3} & -\mathbf{R1} \\ \mathbf{R1} & \mathbf{R2} & \mu & . \\ \mathbf{R3} & \mathbf{R4} & . & \mu \end{pmatrix} \quad (60)$$

with a submatrix

$$\mathbf{r} = \begin{pmatrix} \mathbf{R1} & \mathbf{R2} \\ \mathbf{R3} & \mathbf{R4} \end{pmatrix}, \quad (61)$$

where

$$\mu^2 + \det(\mathbf{r}) = 1, \quad (62)$$

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (63)$$

$$\mathbf{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (64)$$

The inverse of  $\mathbf{R}$  is obtained by reversing the sign of  $\mathbf{r}$ :

$$\mathbf{R}^{-1} = \begin{pmatrix} \mu\mathbf{I} & -\mathbf{J}\mathbf{r}^T\mathbf{J} \\ -\mathbf{r} & \mu\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mu & . & \mathbf{R4} & -\mathbf{R2} \\ . & \mu & -\mathbf{R3} & \mathbf{R1} \\ -\mathbf{R1} & -\mathbf{R2} & \mu & . \\ -\mathbf{R3} & -\mathbf{R4} & . & \mu \end{pmatrix} \quad (65)$$

The value of the function DETR is equal to  $\det(\mathbf{r})$  in this case.

Let  $\mathbf{T}$  stand for the physical transfer matrix from location 1 to location 2, then the transformation in the decoupled coordinate is diagonalized as

$$\mathbf{R}_2\mathbf{T}\mathbf{R}_1^{-1} = \begin{pmatrix} \mathbf{T}_X & 0 \\ 0 & \mathbf{T}_Y \end{pmatrix}. \quad (66)$$

The Twiss parameters are defined for the 2 by 2 matrices  $\mathbf{T}_X$  and  $\mathbf{T}_Y$ .

If  $\det(\mathbf{r}) \geq 1$ , the above condition for  $\mu$  is violated. In such a case, an alternative form of  $\mathbf{R}$  is used:

$$\mathbf{R} = \begin{pmatrix} \mathbf{J}\mathbf{r}^T\mathbf{J} & \mu\mathbf{I} \\ \mu\mathbf{I} & \mathbf{r} \end{pmatrix}, \quad (67)$$

where  $\mu^2 + \det(\mathbf{r}) = 1$ . The function DETR shows a number  $a - \det(\mathbf{r})$ , where  $a = 1.375$ . thus the alternative form is used when  $\det(\mathbf{r}) \geq 0.625$ .