We assume the focusing is uniform and $K = B'/(B\rho) > 0$, which can be always obtained by an appropriate rotation in the x-y plane. For a uniform longitudinal field $B_z = B/(B\rho)$ with the length of the section ℓ, the solution of H_2 for $1/\rho = 0$ is written as:

$$
\begin{aligned}
x &= x_0 + \Delta x\,, && (20)\\
p_x &= p_{x0} + pu_2 + B_z\left(v_2 - \frac{\Delta y}{2}\right)\,, && (21)\\
y &= y_0 + \Delta y\,, && (22)\\
p_y &= p_{y0} + pw_+ v_1 + B_z\left(-\frac{u_1}{w_+} + \frac{\Delta x}{2}\right)\,, && (23)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta x &= \frac{u_1}{w_1} + \frac{v_1 Bz}{pw_2}\,, && (24)\\
\Delta y &= \frac{u_2 B_z}{pw_1 w_+} + \frac{w_+ v_2}{w_2}\,, && (25)\\
u_1 &= aw_1(\cos\phi_1 - 1) + b\sin\phi_1\,, && (26)\\
u_2 &= -aw_1\sin\phi_1 + b(\cos\phi_1 - 1)\,, && (27)\\
v_1 &= cw_2(\cosh\phi_2 - 1) + d\sinh\phi_2\,, && (28)\\
v_2 &= cw_2\sinh\phi_2 + d(\cosh\phi_2 - 1)\,, && (29)
\end{aligned}
$$

with

$$
\begin{aligned}
a &= U \times \left(w_2 w_+ x_0 - \frac{B_z}{p^2} p_{ym}\right)\,, && (30)\\
b &= \frac{w_1 U}{p} \times (w_+ p_{xm} - B_z w_2 y_0)\,, && (31)\\
c &= \frac{U}{p} \times \left(\frac{w_1 B_z}{w_+} x_0 + p_{ym}\right)\,, && (32)\\
d &= U \times w_2\left(-\frac{B_z}{p^2 w_+} p_{xm} + w_1 y_0\right)\,, && (33)\\
p_{xm} &= p_{x0} + \frac{B_z}{2} y_0\,, && (34)\\
p_{ym} &= p_{y0} - \frac{B_z}{2} x_0\,. && (35)
\end{aligned}
$$

The parameters are:

$$
\begin{aligned}
\phi_1 &= w_1\ell\,, && (36)\\
\phi_2 &= w_2\ell\,, && (37)\\
w_1 &= \sqrt{\frac{(B_z/p)^2 + V}{2}}\,, && (38)\\
w_2 &= \frac{K}{pw_1}\,, && (39)\\
w_+ &= w_1 + w_2\,, && (40)\\
U &= \frac{1}{V}\,, && (41)\\
V &= \sqrt{(B_z/p)^4 + 4(K/p)^2} = w_1^2 + w_2^2\,. && (42)
\end{aligned}
$$

The subscript 0 above denotes the initial value.
The second order Hamiltonian H_2 can be rewritten to

$$H_{2u} = -p - iw_1 up_u - w_2 vp_v \tag{43}$$

in terms of a *complex* normal coordinate:

$$
\begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = \sqrt{U}
\begin{pmatrix}
\frac{w_1+w_2}{2} & -\frac{i}{p} & -\frac{\sqrt{w_1^2-w_2^2}}{2} & \frac{1}{p}\sqrt{\frac{w_1-w_2}{w_1+w_2}} \\
-\frac{ip}{4}(w_1+w_2)^2 & \frac{w_1+w_2}{2} & -\frac{p(w_1-w_2)}{4}\sqrt{w_1^2-w_2^2} & -\frac{w_1-w_2}{2}\sqrt{\frac{w_1-w_2}{w_1+w_2}} \\
-\frac{\sqrt{w_1^2-w_2^2}}{2} & \frac{1}{p}\sqrt{\frac{w_1-w_2}{w1+w2}} & \frac{w_1+w_2}{2} & \frac{1}{p} \\
-\frac{p(w_1-w_2)}{4}\sqrt{w_1^2-w_2^2} & -i\frac{w_1-w_2}{2}\sqrt{\frac{w_1-w_2}{w_1+w_2}} & -\frac{ip}{4}(w_1+w_2)^2 & \frac{w_1+w_2}{2}
\end{pmatrix}
\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}. \tag{44}
$$

Note that H_{2u} is real. Thus the transformation of the longitudinal coordinate is obtained as

$$
\begin{aligned}
z &= z_0 + \left(-iup_u\frac{\partial w_1}{\partial p} - vp_v\frac{\partial w_2}{\partial p} + \Delta v\right)\ell && (45)\\
&= z_0 + \frac{U}{p}\left(iw_1^3 u_0 p_{u0} + w_2^3 v_0 p_{v0} + \Delta v\right)\ell\,, && (46)
\end{aligned}
$$

using $up_u = u_0 p_{u0}$ and $vp_v = v_0 p_{v0}$.