

The Hamiltonian  $H$  can be expanded up to the second order of the transverse coordinates as:

$$H = H_2 + \Delta H, \quad (18)$$

$$H_2 = - \left( p - \frac{(p_x - a_{x1})^2}{2p} - \frac{(p_y - a_{y1})^2}{2p} + a_{s2} \right) \left( 1 + \frac{x}{\rho} \right) + \frac{E}{p_0 v_0}, \quad (19)$$

where  $a_{x1}$  and  $a_{y1}$  are the transverse vector potentials up to the first order, and  $a_{s2}$  longitudinal one up to the second order of  $x$  and  $y$ , respectively. Note that  $a_{x,y} = 0$  where  $1/\rho \neq 0$ . Thus  $H_2$  represents up to quadrupole fields overlapped with a uniform solenoid. It is known that  $H_2$  has an *analytic solution*, if  $a_{s2}$  is time-independent. Unless  $H$  itself is solvable, SAD uses such an analytic solution of  $H_2$ , then apply the residual  $\Delta H$  by slicing.