Cascade Schemes for Calculus Exercises

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April 23, 2025

Contents

Introduction

This document collects cascade schemes for exercises, techniques, and fundamental concepts in calculus, organized by chapter. Each scheme follows a hierarchical and synthetic model to facilitate quick consultation and memorization.

1 Integrals

In this section we will explore some types of exercises that can be asked in a typical calculus exam. Specifically we will explore two categories:

- Integral function exercises
- Improper integral exercises

1.1 Integral function exercises

Computing the derivative of a composition of the integral function

Key Question: How can we compute the derivative of a function that is the composition of the integral function with another function?

- **Remember the derivative formula for composite functions**
  - $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- **Consider Torricelli's Theorem**
  - $F(x) = \int_a^x f(t)dt$
  - $F'(x) = f(x)$
- **Example**
  - $G(x) = \int_a^{m(x)} f(t)\,dt = F(m(x))$
  - $G'(x) = F'(m(x)) \cdot m'(x) = f(m(x)) \cdot m'(x)$

Computing the derivative of an integral with variable limits

Key Question: How can we compute the derivative of a function defined by an integral where both limits depend on x, i.e., $G(x) = \int_{n(x)}^{m(x)} f(t)dt$?

- **1. Split the integral using a constant lower limit**
  - Use the property of definite integrals: $\int_a^b f(t)dt = \int_c^b f(t)dt - \int_c^a f(t)dt$.
  - Apply this to $G(x)$: Choose an arbitrary constant a (often 0 or 1) in the domain of f.
$$G(x) = \int_{n(x)}^{m(x)} f(t)dt = \int_a^{m(x)} f(t)dt - \int_a^{n(x)} f(t)dt$$
- **2. Define an auxiliary function using Torricelli's Theorem**
  - Let $F(x) = \int_a^x f(t)dt$.
  - By the Fundamental Theorem of Calculus (Part 1 / Torricelli's Theorem), $F'(x) = f(x)$.
  - Rewrite $G(x)$ using F: $G(x) = F(m(x)) - F(n(x))$.
- **3. Differentiate using the Chain Rule**

- Apply the chain rule to differentiate $F(m(x))$ and $F(n(x))$:

$$(F(h(x)))' = F'(h(x)) \cdot h'(x) = f(h(x)) \cdot h'(x)$$

- Therefore, the derivative of $G(x)$ is:

$$G'(x) = \frac{d}{dx}[F(m(x))] - \frac{d}{dx}[F(n(x))]$$

$$G'(x) = F'(m(x)) \cdot m'(x) - F'(n(x)) \cdot n'(x)$$

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

- **4. Example**
  - Compute the derivative of $G(x) = \int_x^{2x} \frac{\sin t}{t} dt$.
  - Here, $f(t) = \frac{\sin t}{t}$, $m(x) = 2x$, and $n(x) = x$.
  - Choose $a = 1$ (any constant works). Let $F(x) = \int_1^x \frac{\sin t}{t} dt$.
  - Then $G(x) = F(2x) - F(x)$.
  - The derivatives are $m'(x) = 2$ and $n'(x) = 1$.
  - Applying the formula:

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

$$G'(x) = f(2x) \cdot 2 - f(x) \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{2x} \cdot 2 - \frac{\sin x}{x} \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{x} - \frac{\sin x}{x} = \frac{\sin(2x) - \sin x}{x}$$

Computing limits involving integrals and indeterminate forms

Key Question: How to compute the limit:

$$\lim_{x \to 0} \frac{x - \int_0^x (e^{-t^2} + \sin^2 t) dt}{x(x^2 - \sin^2 x)}$$

- **Initial Check: Identify Indeterminate Form**
  - Substitute $x = 0$ into the numerator: $0 - \int_0^0 (e^{-t^2} + \sin^2 t) dt = 0 - 0 = 0$.
  - Substitute $x = 0$ into the denominator: $0 \cdot (0^2 - \sin^2 0) = 0 \cdot (0 - 0) = 0$.
  - The limit presents the indeterminate form $\left[\frac{0}{0}\right]$.
- **Apply L'Hôpital's Rule**
  - Differentiate the numerator with respect to x. Requires the Fundamental Theorem of Calculus (Torricelli-Barrow) for the integral part:

$$\frac{d}{dx}\left(x - \int_0^x (e^{-t^2} + \sin^2 t) dt\right) = 1 - (e^{-x^2} + \sin^2 x)$$

  - Differentiate the denominator with respect to x using the product rule:

$$\frac{d}{dx}(x(x^2 - \sin^2 x)) = (x^2 - \sin^2 x) + (2x^2 - x\sin(2x))$$

- The limit becomes (applying L'Hôpital's Rule):

$$L \stackrel{H}{=} \lim_{x\to 0} \frac{1-e^{-x^2}-\sin^2 x}{(x^2-\sin^2 x)+(2x^2-x\sin(2x))}$$

- **Evaluate the New Limit (Using Taylor Series Approximation)**
  - The limit is still in the form $\left[\frac{0}{0}\right]$. Further application of L'Hôpital's Rule is possible, but Taylor expansions are often more efficient.
  - Approximate numerator and denominator for $x \to 0$:
    - $1-e^{-x^2}-\sin^2 x = \frac{5}{6}x^4 + o(x^4)$
    - $(x^2-\sin^2 x)+(2x^2-x\sin(2x)) = \frac{5}{3}x^4 + o(x^4)$
  - Substitute the approximations back into the limit:

$$L = \lim_{x\to 0} \frac{\frac{5}{6}x^4 + o(x^4)}{\frac{5}{3}x^4 + o(x^4)} = \frac{5/6}{5/3} = \frac{5}{6}\cdot\frac{3}{5} = \frac{1}{2}$$

- **Result:** The limit exists and its value is $\frac{1}{2}$.

Computing Maclaurin Polynomials for Composite Functions with Integrals

Key Question: How to determine the Maclaurin polynomial for $F(x) = \sin\left(\int_0^x e^{-t^2}dt\right)$?

- **Define Auxiliary Function for the Integral**
  - Let $G(x) = \int_0^x e^{-t^2}dt$.
  - The original function becomes $F(x) = \sin(G(x))$.
- **Find the Maclaurin Series for the Auxiliary Function** $G(x)$
  - Calculate derivatives of $G(x)$ and evaluate at $x = 0$:
    - $G(0) = \int_0^0 e^{-t^2}dt = 0$
    - $G'(x) = e^{-x^2}$ (by Fundamental Theorem of Calculus) $\to G'(0) = e^0 = 1$
    - $G''(x) = -2xe^{-x^2} \to G''(0) = 0$
    - $G'''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \to G'''(0) = -2$
  - Construct the Maclaurin series for $G(x)$ using $G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!}x^n$:

$$G(x) = G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + \frac{G'''(0)}{3!}x^3 + o(x^3)$$

$$G(x) = 0 + 1\cdot x + \frac{0}{2}x^2 + \frac{-2}{6}x^3 + o(x^3) = x - \frac{1}{3}x^3 + o(x^3)$$

- **Use the Known Maclaurin Series for the Outer Function (**$\sin$**)**
  - Recall the Maclaurin series for $\sin(z)$:

$$\sin(z) = z - \frac{z^3}{6} + o(z^3)$$

- **Substitute the Series for** $G(x)$ **into the Series for** $\sin(z)$

- Replace z with $G(x) = x - \frac{1}{3}x^3 + o(x^3)$:

$$F(x) = \sin(G(x)) = \left(x - \frac{1}{3}x^3 + o(x^3)\right) - \frac{1}{6}\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 + o\left((G(x))^3\right)$$

- Since $G(x) \approx x$ for $x \to 0$, we have $o((G(x))^3) = o(x^3)$.
- Expand and keep terms up to x^3:

$$\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 = (x + o(x))^3 = x^3 + o(x^3)$$

- Substitute back:

$$F(x) = \left(x - \frac{1}{3}x^3\right) - \frac{1}{6}(x^3) + o(x^3)$$

$$F(x) = x - \left(\frac{1}{3} + \frac{1}{6}\right)x^3 + o(x^3) = x - \frac{2+1}{6}x^3 + o(x^3) = x - \frac{3}{6}x^3 + o(x^3)$$

$$F(x) = x - \frac{1}{2}x^3 + o(x^3)$$

- **Result: Maclaurin Polynomial**
  - The Maclaurin polynomial of order 3 for $F(x)$ is $P_3(x) = x - \frac{1}{2}x^3$.

Proving Existence and Uniqueness of Solutions Involving Integral Functions

Key Question: How to prove that the equation $f(x) = 1 - x$, where $f(x) = \int_0^x e^{-t^2} dt$, has a unique solution?

- **Reformulate the Problem**
  - Define a new function $g(x) = f(x) - (1 - x) = f(x) + x - 1$.
  - The original problem is equivalent to proving that $g(x) = 0$ has exactly one solution (a unique zero).
- **Prove Existence of a Zero (Intermediate Value Theorem)**
  - **Continuity:** $f(x)$ is continuous because it's an integral function of a continuous integrand (e^{-t^2}). The term $x-1$ is also continuous. Therefore, $g(x)$ is continuous on $\mathbb{R}$.
  - **Find points with opposite signs:**
    - Evaluate $g(0)$: $g(0) = f(0) + 0 - 1 = \int_0^0 e^{-t^2} dt - 1 = 0 - 1 = -1$. So, $g(0) < 0$.
    - Evaluate the limit as $x \to +\infty$:

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \left(\int_0^x e^{-t^2} dt + x - 1\right)$$

      The integral $\int_0^\infty e^{-t^2} dt$ converges to a finite value ($\frac{\sqrt{\pi}}{2}$). The term x goes to $+\infty$.

$$\lim_{x \to +\infty} g(x) = \left(\lim_{x \to +\infty} \int_0^x e^{-t^2} dt\right) + \left(\lim_{x \to +\infty} x\right) - 1 = \frac{\sqrt{\pi}}{2} + \infty - 1 = +\infty$$

    - Since the limit is $+\infty$, there must exist some value $\bar{x}$ such that for all $x > \bar{x}$, $g(x) > 0$. Let's pick one such value $\bar{x}$.

- **Apply IVT:** Since $g(x)$ is continuous on $[0, \bar{x}]$, $g(0) < 0$, and $g(\bar{x}) > 0$, the Intermediate Value Theorem guarantees that there exists at least one $x_0 \in (0, \bar{x})$ such that $g(x_0) = 0$.

- **Prove Uniqueness of the Zero (Monotonicity)**
  - **Calculate the derivative $g'(x)$:**

$$g'(x) = \frac{d}{dx}(f(x) + x - 1) = f'(x) + 1$$

  - Apply the Fundamental Theorem of Calculus to find $f'(x)$:

$$f'(x) = \frac{d}{dx}\left(\int_0^x e^{-t^2} dt\right) = e^{-x^2}$$

  - Substitute back into $g'(x)$:

$$g'(x) = e^{-x^2} + 1$$

  - **Analyze the sign of $g'(x)$:** Since $e^{-x^2} > 0$ for all real x, we have $g'(x) = e^{-x^2} + 1 > 0 + 1 = 1$.
  - Since $g'(x) > 0$ for all $x \in \mathbb{R}$, the function $g(x)$ is strictly increasing on its entire domain.

- **Conclusion**
  - We have shown that $g(x) = 0$ has at least one solution (by IVT).
  - We have shown that $g(x)$ is strictly increasing, which means it can cross the x-axis (equal zero) at most once.
  - Therefore, the equation $g(x) = 0$ has exactly one unique solution. This implies the original equation $f(x) = 1 - x$ also has a unique solution.

Finding Order and Principal Part for Functions with Integrals (Derivative Method)

Key Question: Determine the order and principal part of $G(x) = x - \int_0^x e^{-(t+x)^2} dt$ as $x \to 0$.

- **Strategy: Analyze the Derivative $G'(x)$**
  - It's often easier to find the Maclaurin expansion (order/principal part) of the derivative first, and then integrate.
  - **Simplify the integral (Optional Variable Change):** Let $z = t + x$, $dz = dt$. Limits change from $t \in [0, x]$ to $z \in [x, 2x]$.

$$G(x) = x - \int_x^{2x} e^{-z^2} dz$$

  - **Rewrite using a fixed lower limit:** Let $H(x) = \int_0^x e^{-t^2} dt$. Then $\int_x^{2x} e^{-z^2} dz = H(2x) - H(x)$.

$$G(x) = x - (H(2x) - H(x)) = x - H(2x) + H(x)$$

- **Compute the Derivative $G'(x)$**

- Apply differentiation rules, including the Fundamental Theorem of Calculus and Chain Rule:

$$G'(x) = \frac{d}{dx}(x) - \frac{d}{dx}(H(2x)) + \frac{d}{dx}(H(x))$$

$$G'(x) = 1 - (H'(2x) \cdot 2) + H'(x)$$

Since $H'(x) = e^{-x^2}$:

$$G'(x) = 1 - (e^{-(2x)^2} \cdot 2) + e^{-x^2} = 1 - 2e^{-4x^2} + e^{-x^2}$$

- **Find Maclaurin Expansion of $G'(x)$**
  - Use the known expansion $e^u = 1 + u + o(u)$ for $u \to 0$.
  - $e^{-4x^2} = 1 - 4x^2 + o(x^2)$
  - $e^{-x^2} = 1 - x^2 + o(x^2)$
  - Substitute into $G'(x)$:

$$G'(x) = 1 - 2(1 - 4x^2 + o(x^2)) + (1 - x^2 + o(x^2))$$

$$G'(x) = 1 - 2 + 8x^2 + o(x^2) + 1 - x^2 + o(x^2)$$

$$G'(x) = (1 - 2 + 1) + (8x^2 - x^2) + o(x^2) = 7x^2 + o(x^2)$$

  - The principal part of $G'(x)$ is $7x^2$, and its order is 2.
- **Determine Order and Principal Part of $G(x)$**
  - Since $G'(x) \sim 7x^2$ as $x \to 0$, and $G(0) = 0 - \int_0^0 \cdots = 0$, we can integrate the principal part of $G'(x)$ to find the principal part of $G(x)$:

$$G(x) \approx \int_0^x 7t^2 dt = 7\left[\frac{t^3}{3}\right]_0^x = \frac{7}{3}x^3$$

  - The principal part of $G(x)$ is $\frac{7}{3}x^3$.
  - The order of $G(x)$ is 3.
  - Thus, $G(x) = \frac{7}{3}x^3 + o(x^3)$ for $x \to 0$.
- **Verification (Using L'Hôpital's Rule on the Remainder)**
  - To confirm $G(x) = \frac{7}{3}x^3 + o(x^3)$, we must show $\lim_{x\to 0} \frac{G(x) - \frac{7}{3}x^3}{x^3} = 0$.
  - Apply L'Hôpital's Rule (since form is $\frac{0}{0}$):

$$\lim_{x\to 0} \frac{G'(x) - \frac{d}{dx}(\frac{7}{3}x^3)}{\frac{d}{dx}(x^3)} = \lim_{x\to 0} \frac{G'(x) - 7x^2}{3x^2}$$

  - Substitute the expansion of $G'(x)$:

$$\lim_{x\to 0} \frac{(7x^2 + o(x^2)) - 7x^2}{3x^2} = \lim_{x\to 0} \frac{o(x^2)}{3x^2} = 0$$

  - The limit is 0, confirming the order and principal part.

Qualitative Graph Sketching of Integral Functions near x

Key Question: How to draw a qualitative graph of $f(x) = \int_0^x \frac{e^t}{2t^2+1} dt$ in a neighborhood of $x = 0$?

- **Strategy: Analyze Local Behavior using Derivatives at x=0**
  - The behavior of a function near a point (like $x = 0$) is determined by its value and the values of its derivatives at that point. This is the foundation of Taylor series approximations.
- **Calculate** $f(0)$
  - Substitute $x = 0$ into the integral definition:

$$f(0) = \int_0^0 \frac{e^t}{2t^2+1} dt = 0$$

  - The function passes through the origin $(0, 0)$.
- **Calculate** $f'(x)$ **and** $f'(0)$
  - Apply the Fundamental Theorem of Calculus (Part 1):

$$f'(x) = \frac{d}{dx}\left(\int_0^x \frac{e^t}{2t^2+1} dt\right) = \frac{e^x}{2x^2+1}$$

  - Evaluate at $x = 0$:

$$f'(0) = \frac{e^0}{2(0)^2+1} = \frac{1}{1} = 1$$

  - The slope of the tangent line at the origin is 1. The function is increasing at $x = 0$.
- **Calculate** $f''(x)$ **and** $f''(0)$
  - Differentiate $f'(x)$ using the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$:

$$u = e^x \implies u' = e^x$$

$$v = 2x^2 + 1 \implies v' = 4x$$

$$f''(x) = \frac{(e^x)(2x^2+1) - (e^x)(4x)}{(2x^2+1)^2} = \frac{e^x(2x^2 - 4x + 1)}{(2x^2+1)^2}$$

  - Evaluate at $x = 0$:

$$f''(0) = \frac{e^0(2(0)^2 - 4(0) + 1)}{(2(0)^2+1)^2} = \frac{1(1)}{(1)^2} = 1$$

  - Since $f''(0) > 0$, the function is concave up at $x = 0$.
- **Sketching the Graph near x=0**
  - The graph passes through $(0, 0)$.
  - The tangent line at $(0, 0)$ has a slope of 1 (like the line $y = x$).
  - The graph is concave up at $(0, 0)$, meaning it lies above its tangent line near the point of tangency.
  - Combining these: Start at the origin, draw a curve that is initially tangent to $y = x$ and curves upwards (concave up).

Limit Computation using Integral Inequalities and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x \to 0^+} x \int_{x-x^2}^{x} \frac{dt}{\sin^3 t}$, if it exists.

- **Step 1: Analyze Integrand and Integration Interval**
  - The integration variable is t, and the interval is $[x - x^2, x]$.
  - As $x \to 0^+$, both x and $x - x^2 = x(1-x)$ approach 0^+. Thus, $t \to 0^+$.
  - For $t \in (0, \pi/2)$, $\sin t$ is positive and strictly increasing.
  - Consequently, $\sin^3 t$ is also positive and strictly increasing for $t \in (0, \pi/2)$.
  - Therefore, the integrand $g(t) = \frac{1}{\sin^3 t}$ is positive and strictly decreasing for t in the interval $(0, \pi/2)$.
- **Step 2: Establish Inequalities for the Integrand**
  - Since $t \in [x-x^2, x]$ and $g(t) = \frac{1}{\sin^3 t}$ is decreasing on this interval (for sufficiently small positive x), the minimum value of $g(t)$ occurs at $t = x$ and the maximum value occurs at $t = x - x^2$.
  - For $t \in [x - x^2, x]$, we have:

$$\frac{1}{\sin^3 x} \le \frac{1}{\sin^3 t} \le \frac{1}{\sin^3(x - x^2)}$$

- **Step 3: Integrate the Inequalities**
  - Integrate all parts of the inequality over the interval $[x-x^2, x]$. Since the bounds $\frac{1}{\sin^3 x}$ and $\frac{1}{\sin^3(x-x^2)}$ are constant with respect to t, we get:

$$\int_{x-x^2}^{x} \frac{1}{\sin^3 x} dt \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \int_{x-x^2}^{x} \frac{1}{\sin^3(x - x^2)} dt$$

  - The length of the integration interval is $x - (x - x^2) = x^2$.

$$\frac{1}{\sin^3 x} \cdot x^2 \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{1}{\sin^3(x - x^2)} \cdot x^2$$

$$\frac{x^2}{\sin^3 x} \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{x^2}{\sin^3(x - x^2)}$$

- **Step 4: Incorporate the External Factor and Compute Limits**
  - Multiply the inequality by x (note $x > 0$ as $x \to 0^+$):

$$\frac{x^3}{\sin^3 x} \le x \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{x^3}{\sin^3(x - x^2)}$$

  - Compute the limit of the lower bound as $x \to 0^+$:

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3 x} = \lim_{x \to 0^+} \left(\frac{x}{\sin x} \right)^3 = (1)^3 = 1$$

    (using the standard limit $\lim_{u \to 0} \frac{\sin u}{u} = 1$)
  - Compute the limit of the upper bound as $x \to 0^+$:

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3(x - x^2)} = \lim_{x \to 0^+} \left(\frac{x}{\sin(x - x^2)} \right)^3$$

    We use $\sin(x - x^2) \sim x - x^2$ as $x \to 0$.

$$\lim_{x \to 0^+} \left(\frac{x}{x - x^2} \right)^3 = \lim_{x \to 0^+} \left(\frac{x}{x(1 - x)} \right)^3 = \lim_{x \to 0^+} \left(\frac{1}{1 - x} \right)^3 = \left(\frac{1}{1 - 0} \right)^3 = 1$$

- **Step 5: Apply the Squeeze Theorem**
  - We have shown:
$$\lim_{x\to 0^+} \frac{x^3}{\sin^3 x} = 1$$
$$\lim_{x\to 0^+} \frac{x^3}{\sin^3(x-x^2)} = 1$$
$$\frac{x^3}{\sin^3 x} \le x\int_{x-x^2}^{x} \frac{1}{\sin^3 t}dt \le \frac{x^3}{\sin^3(x-x^2)}$$
  - By the Squeeze Theorem, the limit of the middle expression must also be 1.
  - Therefore, $L = \lim_{x\to 0^+} x\int_{x-x^2}^{x} \frac{dt}{\sin^3 t} = 1$.

Limit Computation using Mean Value Theorem and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x\to+\infty} x^3 \int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt$.

- **Step 1: Apply Mean Value Theorem for Integrals**
  - **Theorem Statement:** If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that $\int_a^b f(t)dt = f(c)(b-a)$.
  - **Application:**
    - Let $f(t) = \sin\left(\frac{1}{t^2}\right)$. This function is continuous on $[x^2, x^2+x]$ for large x (since $x^2 > 0$).
    - The interval length is $(x^2+x) - x^2 = x$.
    - By the MVT for Integrals, there exists $z \in [x^2, x^2+x]$ such that:
$$\int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt = \sin\left(\frac{1}{z^2}\right)\cdot x$$
  - **Rewrite the Limit:** Substitute this result back into the limit expression:
$$L = \lim_{x\to+\infty} x^3\left[x\sin\left(\frac{1}{z^2}\right)\right] = \lim_{x\to+\infty} x^4 \sin\left(\frac{1}{z^2}\right)$$
where z depends on x and satisfies $x^2 \le z \le x^2 + x$.
- **Step 2: Establish Bounds using $z \in [x^2, x^2 + x]$**
  - From $x^2 \le z \le x^2 + x$, we have:
$$\frac{1}{x^2+x} \le \frac{1}{z} \le \frac{1}{x^2}$$
Squaring (all terms are positive for large x):
$$\frac{1}{(x^2+x)^2} \le \frac{1}{z^2} \le \frac{1}{x^4}$$
  - As $x \to +\infty$, all terms in the inequality approach 0^+.
  - Since $\sin u$ is an increasing function for u near 0^+, we can apply sin to the inequalities (for sufficiently large x):
$$\sin\left(\frac{1}{(x^2+x)^2}\right) \le \sin\left(\frac{1}{z^2}\right) \le \sin\left(\frac{1}{x^4}\right)$$

- **Step 3: Apply the Squeeze Theorem**
  - Multiply the inequality by x^4 (which is positive):

$$x^4 \sin\left(\frac{1}{(x^2+x)^2}\right) \leq x^4 \sin\left(\frac{1}{z^2}\right) \leq x^4 \sin\left(\frac{1}{x^4}\right)$$

  - **Limit of the Upper Bound:**

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{x^4}\right)$$

Use the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$. Let $u = 1/x^4$. As $x \to +\infty$, $u \to 0^+$.

$$= \lim_{u\to 0^+} \frac{1}{u} \sin(u) = 1$$

  - **Limit of the Lower Bound:**
Let $v = 1/(x^2+x)^2$. As $x \to +\infty$, $v \to 0^+$. Use $\sin v \sim v$ for $v \to 0$.

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{(x^2+x)^2}\right)$$

$$= \lim_{x\to+\infty} x^4 \cdot \frac{1}{(x^2+x)^2} = \lim_{x\to+\infty} \frac{x^4}{(x^2(1+1/x))^2} = \lim_{x\to+\infty} \frac{x^4}{x^4(1+1/x)^2} = 1$$

  - **Conclusion:** Since the expression $x^4 \sin(1/z^2)$ is squeezed between two functions that both tend to 1 as $x \to +\infty$, by the Squeeze Theorem (Teorema dei Carabinieri):

$$L = \lim_{x\to+\infty} x^4 \sin\left(\frac{1}{z^2}\right) = 1$$

Integration using Weierstrass Substitution (Tangent Half-Angle)

Key Question: How to calculate the integral $I = \int \frac{\tan x}{\sin x + \tan x} dx$?

- **Step 1: Simplify the Integrand**
  - Rewrite $\tan x$ as $\sin x / \cos x$:

$$\frac{\tan x}{\sin x + \tan x} = \frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}}$$

  - Find a common denominator in the denominator:

$$\frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}} = \frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}}$$

  - Simplify (assuming $\sin x \neq 0, \cos x \neq 0$):

$$\frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}} = \frac{\sin x}{\sin x \cos x + \sin x} = \frac{\sin x}{\sin x(\cos x + 1)} = \frac{1}{\cos x + 1}$$

  - The integral becomes significantly simpler: $I = \int \frac{1}{\cos x + 1} dx$.

- **Step 2: Apply the Weierstrass Substitution ($t = \tan(x/2)$)**
  - **Motivation:** This substitution transforms rational functions of $\sin x$ and $\cos x$ into rational functions of t, which can often be integrated using standard techniques (like partial fractions, although not needed here).
  - **The Substitution Formulas:**
    - Let $\boxed{t = \tan\left(\frac{x}{2}\right)}$
    - Then $\cos x = \frac{1-t^2}{1+t^2}$
    - And $dx = \frac{2}{1+t^2} dt$
    - (Also $\sin x = \frac{2t}{1+t^2}$, though not required for this specific simplified integral).
- **Step 3: Substitute into the Simplified Integral**
  - Replace $\cos x$ and dx with their t-equivalents:

$$I = \int \underbrace{\frac{1}{\left(\frac{1-t^2}{1+t^2}\right) + 1}}_{\frac{1}{\cos x + 1}} \cdot \underbrace{\left(\frac{2}{1+t^2}\right) dt}_{dx}$$

  - Simplify the first part (the integrand in terms of x expressed in t):

$$\frac{1}{\frac{1-t^2+1+t^2}{1+t^2}} = \frac{1}{\frac{2}{1+t^2}} = \frac{1+t^2}{2}$$

  - Substitute this back into the integral expression:

$$I = \int \left(\frac{1+t^2}{2}\right) \cdot \left(\frac{2}{1+t^2}\right) dt$$

- **Step 4: Evaluate the Integral in t**
  - The integrand simplifies dramatically:

$$I = \int 1 \, dt$$

  - Compute the straightforward integral:

$$I = t + C$$

- **Step 5: Substitute Back to x**
  - Replace t with its definition in terms of x:

$$I = \tan\left(\frac{x}{2}\right) + C$$

- **Result:** The integral evaluates to $\int \frac{\tan x}{\sin x + \tan x} dx = \tan\left(\frac{x}{2}\right) + C$.

1.2 Improper Integral exercises

Evaluating Improper Integrals using the Definition

Key Question: How to compute the value of the improper integral $\int_0^1 \log x \, dx$?

- **Step 1: Identify the Point of Impropriety**
  - The function $f(x) = \log x$ is defined on $(0, 1]$.
  - The integral is improper because the integrand $\log x$ approaches $-\infty$ as x approaches the lower limit of integration, $x = 0$.
- **Step 2: Apply the Definition of Improper Integral**
  - Replace the problematic lower limit 0 with a variable ϵ and take the limit as ϵ approaches 0 from the right side (since the interval is $(0, 1]$):

$$\int_0^1 \log x \, dx \stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+} \int_\epsilon^1 \log x \, dx$$

- **Step 3: Find the Antiderivative**
  - Compute the indefinite integral $\int \log x \, dx$ using integration by parts:
    - Let $u = \log x \implies du = \frac{1}{x} dx$
    - Let $dv = dx \implies v = x$

$$\int \log x \, dx = uv - \int v \, du = x \log x - \int x \left(\frac{1}{x}\right) dx$$

$$= x \log x - \int 1 \, dx = x \log x - x + C = x(\log x - 1) + C$$

  - An antiderivative is $F(x) = x(\log x - 1)$.
- **Step 4: Evaluate the Definite Integral and Compute the Limit**
  - Use the antiderivative to evaluate the definite integral inside the limit (Fundamental Theorem of Calculus / Torricelli-Barrow):

$$\int_\epsilon^1 \log x \, dx = [x(\log x - 1)]_\epsilon^1 = (1(\log 1 - 1)) - (\epsilon(\log \epsilon - 1))$$

$$= (1(0 - 1)) - (\epsilon \log \epsilon - \epsilon) = -1 - \epsilon \log \epsilon + \epsilon$$

  - Now, compute the limit as $\epsilon \to 0^+$:

$$\lim_{\epsilon \to 0^+} (-1 - \epsilon \log \epsilon + \epsilon)$$

  - Use the standard limit $\lim_{\epsilon \to 0^+} \epsilon \log \epsilon = 0$.

$$= -1 - 0 + 0 = -1$$

- **Conclusion**
  - Since the limit exists and is finite, the improper integral converges.
  - The value of the integral is $\int_0^1 \log x \, dx = -1$.

Integration using Weierstrass Substitution (Tangent Half-Angle)

Key Question: How to calculate the integral $I = \int \frac{\tan x}{\sin x + \tan x} dx$?

- **Step 1: Simplify the Integrand**
  - Rewrite $\tan x$ as $\sin x / \cos x$:

$$\frac{\tan x}{\sin x + \tan x} = \frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}}$$

  - Find a common denominator in the denominator:

$$= \frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}}$$

  - Simplify (assuming $\sin x \neq 0, \cos x \neq 0$):

$$= \frac{\sin x}{\sin x \cos x + \sin x} = \frac{\sin x}{\sin x(\cos x + 1)} = \frac{1}{\cos x + 1}$$

  - The integral becomes significantly simpler: $I = \int \frac{1}{\cos x + 1} dx$.
- **Step 2: Apply the Weierstrass Substitution ($t = \tan(x/2)$)**
  - **Motivation:** This substitution transforms rational functions of $\sin x$ and $\cos x$ into rational functions of t, which can often be integrated using standard techniques (like partial fractions, although not needed here).
  - **The Substitution Formulas:**
    - Let $\boxed{t = \tan\left(\frac{x}{2}\right)}$
    - Then $\cos x = \frac{1-t^2}{1+t^2}$
    - And $dx = \frac{2}{1+t^2} dt$
    - (Also $\sin x = \frac{2t}{1+t^2}$, though not required for this specific simplified integral).
- **Step 3: Substitute into the Simplified Integral**
  - Replace $\cos x$ and dx with their t-equivalents:

$$I = \int \underbrace{\frac{1}{\left(\frac{1-t^2}{1+t^2}\right) + 1}}_{\frac{1}{\cos x + 1}} \cdot \underbrace{\left(\frac{2}{1+t^2}\right) dt}_{dx}$$

  - Simplify the first part (the integrand in terms of x expressed in t):

$$\frac{1}{\frac{1-t^2+1+t^2}{1+t^2}} = \frac{1}{\frac{2}{1+t^2}} = \frac{1+t^2}{2}$$

  - Substitute this back into the integral expression:

$$I = \int \left(\frac{1+t^2}{2}\right) \cdot \left(\frac{2}{1+t^2}\right) dt$$

- **Step 4: Evaluate the Integral in t**

- The integrand simplifies dramatically:

$$I = \int 1\,dt$$

- Compute the straightforward integral:

$$I = t + C$$

- **Step 5: Substitute Back to x**
  - Replace t with its definition in terms of x:

$$I = \tan\left(\frac{x}{2}\right) + C$$

- **Result:** The integral evaluates to $\int \frac{\tan x}{\sin x + \tan x} dx = \tan\left(\frac{x}{2}\right) + C$.

Convergence of Improper Integrals with Oscillating Numerators

Key Question: How to discuss the convergence of the improper integral $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$?

- **Step 1: Identify Potential Issues**
  - **Infinite Interval:** The upper limit is $+\infty$, making it an improper integral of the first kind.
  - **Sign Changes:** The numerator $\cos x$ oscillates between -1 and 1, causing the integrand $f(x) = \frac{\cos x}{1+x^2}$ to change sign infinitely often.
  - **Problem with Direct Comparison:** Standard comparison tests (Direct Comparison, Limit Comparison) require the integrand to be non-negative (at least eventually), which is not the case here.
- **Step 2: Use Absolute Convergence**
  - **Strategy:** Test the convergence of the integral of the absolute value of the integrand:

$$I_{abs} = \int_0^\infty \left|\frac{\cos x}{1+x^2}\right| dx = \int_0^\infty \frac{|\cos x|}{1+x^2} dx$$

  - **Absolute Convergence Theorem:** If I_{abs} converges, then the original integral I also converges.
  - The new integrand $\frac{|\cos x|}{1+x^2}$ is now non-negative, allowing the use of comparison tests.
- **Step 3: Apply Direct Comparison Test to I_{abs}**
  - **Establish Inequality:** Use the property $|\cos x| \le 1$ for all x:

$$0 \le \frac{|\cos x|}{1+x^2} \le \frac{1}{1+x^2} \quad \text{for all } x \ge 0$$

  - **Analyze the Comparison Integral:** Consider the integral $J = \int_0^\infty \frac{1}{1+x^2} dx$.
  - **Convergence of J (Method 1: Direct Integration):**

$$J = \int_0^\infty \frac{1}{1+x^2} dx = [\arctan x]_0^\infty = \lim_{b\to\infty} (\arctan b) - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Since the result is finite, J converges.

- **Convergence of J (Method 2: Splitting & Asymptotic Comparison - Important General Technique):**
  - Split the integral: $J = \int_0^1 \frac{1}{1+x^2} dx + \int_1^\infty \frac{1}{1+x^2} dx$.
  - The integral $\int_0^1 \frac{1}{1+x^2} dx$ is proper and finite (integrand is continuous on $[0, 1]$).
  - For $\int_1^\infty \frac{1}{1+x^2} dx$, compare with $\int_1^\infty \frac{1}{x^2} dx$. Since $0 < \frac{1}{1+x^2} \leq \frac{1}{x^2}$ for $x \geq 1$ and $\int_1^\infty \frac{1}{x^2} dx$ converges (p-integral with $p = 2 > 1$), the integral $\int_1^\infty \frac{1}{1+x^2} dx$ converges by Direct Comparison.
  - Since both parts converge, J converges. (Note: Splitting is crucial when dealing with multiple improprieties or when asymptotic comparison is needed at different points).
- **Conclusion of Comparison Test:** Since $0 \leq \frac{|\cos x|}{1+x^2} \leq \frac{1}{1+x^2}$ and $\int_0^\infty \frac{1}{1+x^2} dx$ converges, by the Direct Comparison Test, $I_{abs} = \int_0^\infty \frac{|\cos x|}{1+x^2} dx$ must also converge.

- **Step 4: Final Conclusion**
  - We have shown that $I_{abs} = \int_0^\infty \left| \frac{\cos x}{1+x^2} \right| dx$ converges.
  - By the theorem on absolute convergence, if an integral converges absolutely, it also converges simply.
  - Therefore, the original integral $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$ converges.

Determining Improper Integral Convergence using Asymptotic Comparison

Key Question: How can we determine if an improper integral converges when the integrand is complex, especially near a point of discontinuity?

- **Problem Setup:** Consider the integral:

$$I = \int_0^1 f(x) dx = \int_0^1 \frac{\sqrt{x(x^2+1)} e^{2x}}{\tan \sqrt[3]{x}} dx$$

  The potential problem (discontinuity) is at $x = 0$.

- **Check Conditions for Comparison Test:**
  - The function $f(x)$ must be continuous on $(0, 1]$.
  - The function $f(x)$ must be non-negative on $(0, 1]$.
  - In this case, $f(x)$ is non-negative for $x \in (0, 1]$.
- **Apply Asymptotic Comparison Test (Limit Comparison Test):** We need to find a simpler function $g(x)$ such that $f(x) \sim g(x)$ as $x \to 0^+$. The convergence of $\int_0^1 f(x) dx$ will be the same as the convergence of $\int_0^1 g(x) dx$.
  - **Analyze numerator as $x \to 0^+$:**

$$\sqrt{x(x^2+1)} = \sqrt{x^3+x} \sim \sqrt{x} = x^{1/2}$$

$$e^{2x} \sim e^0 = 1$$

    So, Numerator $\sim x^{1/2}$.

- **Analyze denominator as $x \to 0^+$:**

$$\tan \sqrt{x} \sim \sqrt{x} = x^{1/2}$$

$$\tan \sqrt[3]{x} \sim x^{1/3} = x^{1/3}$$

- **Determine asymptotic behavior of $f(x)$:**

$$f(x) \sim \frac{\sqrt{x}}{\sqrt[3]{x}} = \frac{x^{1/2}}{x^{1/3}} = x^{1/2-1/3} = x^{1/6}$$

So, we use $g(x) = x^{1/6}$.

- **Analyze the Comparison Integral:** We examine the convergence of $\int_0^1 g(x)dx$:

$$\int_0^1 x^{1/6}dx = \int_0^1 \frac{1}{x^{-1/6}}dx$$

This is a generalized p-integral $\int_0^1 \frac{1}{x^p}dx$. It converges if and only if $p < 1$. In our case, $p = -1/6$. Since $p = -1/6 < 1$, the comparison integral $\int_0^1 x^{1/6}dx$ converges.

- **Conclusion:** Since $f(x) \sim x^{1/6}$ as $x \to 0^+$ and $\int_0^1 x^{1/6}dx$ converges, by the Limit Comparison Test, the original integral

$$\int_0^1 \frac{\sqrt{x(x^2+1)}e^{2x}}{\tan \sqrt[3]{x}}dx \quad \textbf{converges.}$$

Convergence of Improper Integrals with Parameters using Taylor Expansion

Key Question: How can we determine the values of a parameter α for which an improper integral converges, especially when the integrand's behavior near the problematic point requires Taylor series?

- **Problem Setup:** Consider the integral dependent on α:

$$I(\alpha) = \int_0^1 \frac{3}{[2(x - \log(1+x))]^{3-3\alpha}}dx$$

We need to find the values of α for which this integral converges. The potential problem point is $x = 0$.

- **Asymptotic Analysis near $x = 0$ using Taylor Expansion:** To understand the behavior of the integrand $f(x, \alpha) = \frac{3}{[2(x-\log(1+x))]^{3-3\alpha}}$ as $x \to 0^+$, we need the Taylor expansion of $\log(1+x)$ around $x = 0$.
  - **Recall Taylor expansion:**

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = x - \frac{x^2}{2} + o(x^2) \quad \text{as } x \to 0$$

  - **Analyze the term $x - \log(1+x)$:**

$$x - \log(1+x) = x - \left(x - \frac{x^2}{2} + o(x^2)\right) = \frac{x^2}{2} + o(x^2)$$

Therefore, $x - \log(1+x) \sim \frac{x^2}{2}$ as $x \to 0^+$.

- **Analyze the base of the power in the denominator:**

$$2(x - \log(1 + x)) \sim 2\left(\frac{x^2}{2}\right) = x^2 \quad \text{as } x \to 0^+$$

- **Determine the asymptotic behavior of the integrand $f(x, \alpha)$:**

$$f(x, \alpha) = \frac{3}{[2(x - \log(1 + x))]^{3-3\alpha}} \sim \frac{3}{[x^2]^{3-3\alpha}} = \frac{3}{x^{6-6\alpha}} \quad \text{as } x \to 0^+$$

- **Apply Asymptotic Comparison Test:** The convergence of $I(\alpha)$ is equivalent to the convergence of the comparison integral:

$$\int_0^1 \frac{3}{x^{6-6\alpha}} dx = 3 \int_0^1 \frac{1}{x^{6-6\alpha}} dx$$

- **Analyze the Comparison Integral (Generalized p-integral):** The integral $\int_0^1 \frac{1}{x^p} dx$ converges if and only if $p < 1$. In our case, the exponent is $p = 6 - 6\alpha$.
- **Determine the Condition for Convergence:** For convergence, we require $p < 1$:

$$6 - 6\alpha < 1$$

$$5 < 6\alpha$$
$$\alpha > \frac{5}{6}$$

- **Conclusion:** Based on the asymptotic comparison test and the condition for p-integral convergence, the original integral

$$\int_0^1 \frac{3}{[2(x - \log(1 + x))]^{3-3\alpha}} dx \quad \textbf{converges if and only if } \alpha > \frac{5}{6}.$$

2 Multivariate Calculus

In this section we will explore different types of exercises about multivariate calculus.

2.1 Limits exercises

In this subsection, we will tackle exercises about limits of multivariate functions and explore the different methods we can use for solving them or at least, prove that they exist or not.

Studying continuity of a two-variable function at a point

Key Question: How to determine if a piecewise function $f(x, y)$ is continuous at a point where its definition changes, typically $(0, 0)$?

Function Example:

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- **Check continuity away from the critical point:**
  - For $(x, y) \neq (0, 0)$, $f(x, y) = \frac{x^2y^2}{x^2+y^2}$. This is a ratio of polynomials (which are continuous everywhere).
  - The denominator $x^2 + y^2$ is zero only at $(0, 0)$.
  - Therefore, $f(x, y)$ is continuous for all $(x, y) \neq (0, 0)$ as it's a composition/ratio of continuous functions. The domain is $\mathbb{R}^2$.
- **Check continuity at the critical point** $(0, 0)$**:**
  - **Definition of continuity:** We need to verify if $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$.
  - We are given $f(0, 0) = 0$.
  - We need to compute $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$. If the limit exists and equals 0, the function is continuous at $(0, 0)$.
- **Method 1: Inequality / Squeeze Theorem**
  - Find bounds for the expression. Observe that $y^2 \leq x^2 + y^2$.
  - For $(x, y) \neq (0, 0)$, divide by $x^2 + y^2$: $\frac{y^2}{x^2+y^2} \leq 1$.
  - Since $x^2 \geq 0$, we can multiply by x^2: $x^2 \frac{y^2}{x^2+y^2} \leq x^2$.
  - The full expression is non-negative: $0 \leq \frac{x^2y^2}{x^2+y^2} \leq x^2$.
  - Take the limit as $(x, y) \to (0, 0)$:

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + y^2} \leq \lim_{(x,y)\to(0,0)} x^2$$

  - Since $\lim_{(x,y)\to(0,0)} 0 = 0$ and $\lim_{(x,y)\to(0,0)} x^2 = 0$.
  - By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$.
  - *Note:* If the expression involved terms that could be negative, absolute values would be necessary (e.g., $|f(x, y)| \leq g(x, y)$ where $g \to 0$).
- **Method 2: Polar Coordinates**
  - Substitute $x = \rho \cos \theta$ and $y = \rho \sin \theta$. The limit $(x, y) \to (0, 0)$ corresponds to $\rho \to 0^+$.

- Substitute into the expression:

$$\frac{(\rho\cos\theta)^2(\rho\sin\theta)^2}{(\rho\cos\theta)^2+(\rho\sin\theta)^2} = \frac{\rho^4\cos^2\theta\sin^2\theta}{\rho^2(\cos^2\theta+\sin^2\theta)} = \frac{\rho^4\cos^2\theta\sin^2\theta}{\rho^2} = \rho^2\cos^2\theta\sin^2\theta$$

- Compute the limit as $\rho \to 0^+$: $\lim_{\rho\to 0^+}(\rho^2\cos^2\theta\sin^2\theta)$.
- Use bounds for the trigonometric part: $0 \leq \cos^2\theta\sin^2\theta \leq 1$. The value depends on θ but is bounded.
- Multiply by ρ^2: $0 \leq \rho^2\cos^2\theta\sin^2\theta \leq \rho^2$.
- Apply the Squeeze Theorem for the limit in ρ: Since $\lim_{\rho\to 0^+} 0 = 0$ and $\lim_{\rho\to 0^+}\rho^2 = 0$.
- Therefore, $\lim_{\rho\to 0^+}(\rho^2\cos^2\theta\sin^2\theta) = 0$. The limit exists and is independent of θ.

- **Conclusion for $(0,0)$:** Both methods show $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Since this equals $f(0,0)$, the function is continuous at $(0,0)$.
- **Overall Conclusion:** Since $f(x,y)$ is continuous for $(x,y) \neq (0,0)$ and also continuous at $(0,0)$, the function is continuous on its entire domain $\mathbb{R}^2$.

Proving discontinuity using path restriction

Key Question: How can we show a function $f(x,y)$ is *not* continuous at a point, for example, $(0,0)$?
Function Example:

$$f(x,y) = \begin{cases} \arctan\left(\frac{x}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ \frac{\pi}{2} & \text{if } (x,y) = (0,0) \end{cases}$$

Domain: $\mathbb{R}^2$.

- **Check continuity away from the critical point:**
  - For $(x,y) \neq (0,0)$, $f(x,y) = \arctan\left(\frac{x}{x^2+y^2}\right)$.
  - This is a composition of continuous functions (arctan and a rational function whose denominator is non-zero away from the origin).
  - Thus, $f(x,y)$ is continuous for all $(x,y) \neq (0,0)$.
- **Check continuity at the critical point $(0,0)$:**
  - **Condition for continuity:** We need $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.
  - We are given $f(0,0) = \frac{\pi}{2}$.
  - We need to evaluate $\lim_{(x,y)\to(0,0)} \arctan\left(\frac{x}{x^2+y^2}\right)$.
- **Strategy: Show the limit does not exist by restricting to a path.**
  - If the limit exists, it must be the same regardless of the path taken towards $(0,0)$.
  - If we find even one path along which the limit doesn't exist, or two paths with different limits, then the overall limit does not exist.
  - *Insight:* "Sometimes a single curve is enough to show the limit doesn't exist."

- Let's analyze the argument of arctan: $g(x,y) = \frac{x}{x^2+y^2}$.
- **Try the path $y = x$:** Substitute $y = x$ into $g(x,y)$ for $x \neq 0$:

$$g(x,x) = \frac{x}{x^2 + x^2} = \frac{x}{2x^2} = \frac{1}{2x}$$

- Compute the limit along this path as $x \to 0$:

$$\lim_{x\to 0} g(x,x) = \lim_{x\to 0} \frac{1}{2x}$$

- This limit does not exist (it tends to $+\infty$ as $x \to 0^+$ and $-\infty$ as $x \to 0^-$).

- **Conclusion for the limit of $f(x,y)$:**
  - Since the limit of the argument $g(x,y) = \frac{x}{x^2+y^2}$ does not exist as $(x,y) \to (0,0)$ along the path $y = x$, the overall limit $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.
  - Consequently, the limit $\lim_{(x,y)\to(0,0)} \arctan(g(x,y))$ also does not exist. (As $g \to \pm\infty$, $\arctan(g) \to \pm\frac{\pi}{2}$, which are different values depending on the direction along the path $y = x$, confirming non-existence).
- **Conclusion for continuity at $(0,0)$:**
  - Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, the function cannot be continuous at $(0,0)$. The condition $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ is not met.
- **Overall Conclusion:** The function $f(x,y)$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ but is discontinuous at the origin $(0,0)$.

Limit Existence and Continuity in $\mathbb{R}^2$ vs. Restricted Domain

Key Questions:

1. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?
2. Is f continuous at $(0,0)$?
3. Is f continuous in $D = \{(x,y) \in \mathbb{R}^2 : |y| \leq x \leq 1\}$?

Function Example:

$$f(x,y) = \begin{cases} \frac{y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- **a) Existence of the limit at (0,0) in $\mathbb{R}^2$:**
  - **Strategy:** Check limits along different paths approaching $(0,0)$. If they differ, the limit does not exist.
  - **Path 1: $y = \sqrt{x}$ (approaching from $x > 0$):**

$$f(x,\sqrt{x}) = \frac{(\sqrt{x})^2}{x} = \frac{x}{x} = 1 \quad (\text{for } x > 0)$$

$$\lim_{x\to 0^+} f(x,\sqrt{x}) = 1$$

  - **Path 2: $y = x$ (approaching from $x \neq 0$):**

$$f(x,x) = \frac{x^2}{x} = x \quad (\text{for } x \neq 0)$$

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} x = 0$$

- **Conclusion (a):** Since the limits along different paths (1 and 0) are not equal, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (when considered in $\mathbb{R}^2$).

- **b) Continuity at (0,0) in $\mathbb{R}^2$:**
  - For continuity at $(0,0)$, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ must exist and equal $f(0,0)$.
  - From part (a), the limit does not exist.
  - **Conclusion (b):** Therefore, f is not continuous at $(0,0)$ when considered as a function on $\mathbb{R}^2$.

- **c) Continuity in the domain $D = \{(x,y) \in \mathbb{R}^2 : |y| \le x \le 1\}$:**
  - **Visualize D:** This domain is a region bounded by the lines $y = x$, $y = -x$, and $x = 1$. It lies entirely in the $x \ge 0$ half-plane and includes the origin.
  - **Continuity for $x > 0$ in D:** In this region, $0 < x \le 1$. The function is $f(x,y) = y^2/x$. This is a ratio of polynomials with a non-zero denominator, hence continuous.
  - **Continuity at (0,0) *within D*:** We need to evaluate the limit as $(x,y) \to (0,0)$ such that (x,y) remains in D. This means we approach $(0,0)$ satisfying $|y| \le x$.
  - **Using Squeeze Theorem within D:** For $(x,y) \in D$ and $x \ne 0$:
    - Since $|y| \le x$, we have $y^2 \le x^2$ (as $x \ge 0$).
    - Since $x > 0$ in $D \setminus \{(0,0)\}$, we can divide by x: $\frac{y^2}{x} \le \frac{x^2}{x} = x$.
    - Also, $f(x,y) = \frac{y^2}{x} \ge 0$.
    - So, for $(x,y) \in D \setminus \{(0,0)\}$, we have $0 \le f(x,y) \le x$.
  - Take the limit as $(x,y) \to (0,0)$ *within D*. As $(x,y) \to (0,0)$ in D, we must have $x \to 0^+$.

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} f(x,y) \le \lim_{(x,y)\to(0,0)} x$$

$$0 \le \lim_{(x,y)\to(0,0)} f(x,y) \le 0$$

  - By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
  - This limit value (0) matches the function definition at the origin: $f(0,0) = 0$.
  - **Conclusion (c):** The function f is continuous at every point (x,y) with $x > 0$ in D. The limit exists at $(0,0)$ *when restricted to D* and equals $f(0,0)$. Therefore, f is continuous throughout the domain D.

Proving Limit Non-Existence & Pitfalls of Inequalities

Key Question: How to prove a limit does not exist at (0,0) and why might standard inequality methods fail? **Function Example:**

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Domain: $\mathbb{R}^2$.

- **Continuity away from (0,0):**
  - For $(x,y) \neq (0,0)$, $f(x,y)$ is a ratio of polynomials. The denominator $x^4+y^2 = 0$ only if $x = 0$ and $y = 0$.
  - Thus, $f(x,y)$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.
- **Investigating the limit at (0,0):**
  - **Conjecture:** The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
  - **Strategy: Path Restriction.** Find paths approaching (0,0) along which the function has different limits.
  - **Path 1:** $y = x^2$ **(Parabolic path):** Substitute $y = x^2$ into $f(x,y)$ for $x \neq 0$:

$$f(x,x^2) = \frac{x^2(x^2)}{x^4+(x^2)^2} = \frac{x^4}{x^4+x^4} = \frac{x^4}{2x^4} = \frac{1}{2}$$

$$\lim_{x\to 0} f(x,x^2) = \frac{1}{2}$$

  - **Path 2:** $y = 0$ **(Along the x-axis):** Substitute $y = 0$ into $f(x,y)$ for $x \neq 0$:

$$f(x,0) = \frac{x^2(0)}{x^4+0^2} = \frac{0}{x^4} = 0$$

$$\lim_{x\to 0} f(x,0) = 0$$

  - **Conclusion on Limit:** Since the limits along different paths (1/2 and 0) are not equal, the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
- **Continuity at (0,0):**
  - Since the limit does not exist, the function is not continuous at $(0,0)$. (The limit value would need to exist and be equal to $f(0,0) = 0$).
- **Observation: Why Squeeze Theorem / Inequalities Can Be Misleading Here**
  - **Attempt using AM-GM-like inequality:** We know $2ab \leq a^2+b^2$. Let $a = x^2$ and $b = y$. Then $2x^2|y| \leq (x^2)^2 + y^2 = x^4+y^2$.
  - This gives $|f(x,y)| = \frac{x^2|y|}{x^4+y^2} \leq \frac{\frac{1}{2}(x^4+y^2)}{x^4+y^2} = \frac{1}{2}$.
  - So, $0 \leq |f(x,y)| \leq \frac{1}{2}$. This shows the function is bounded near the origin.
  - **Why it fails:** This bound (1/2) does not tend to 0. Therefore, the Squeeze Theorem cannot be used to prove the limit is 0. It also doesn't prove the limit *isn't* 0, it's simply inconclusive for determining the limit value or existence based on squeezing towards 0.
  - **Another perspective:** Trying to bound $\frac{x^2}{x^4+y^2}$ by something that goes to zero fails because the x^4 term dominates the x^2 term along certain paths (like $y = 0$), but not along others (like $y = x^2$ where $x^4+y^2 = 2x^4$, making the fraction $x^2/(2x^4) = 1/(2x^2)$ which diverges). The different powers (x^2 vs x^4) prevent a simple bounding function that works for all paths and tends to zero.
  - **Key Takeaway:** Path restriction is often necessary when the degrees of terms in the numerator and denominator are 'unbalanced' in a way that depends on the relationship between x and y (e.g., $y \approx x^2$). Simple inequalities might hide this path-dependent behavior.

2.2 Exercises on continuity, derivability and differentiability

Analyzing Continuity Derivability and Differentiability

In this class of exercises we have to check continuity, derivability and differentiability of a function in a given point or on its domain.

Function: $f(x,y) = \begin{cases} \frac{|x|y^4}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

- **1. Check Continuity at (0,0)**
  - Compute the limit: $\lim_{(x,y)\to(0,0)} f(x,y)$.
  - Use Squeeze Theorem: $0 \leq \left|\frac{|x|y^4}{x^4+y^4}\right| = \frac{|x|y^4}{x^4+y^4}$.
  - Since $y^4 \leq x^4 + y^4$, we have $\frac{y^4}{x^4+y^4} \leq 1$.
  - Thus, $0 \leq \frac{|x|y^4}{x^4+y^4} \leq |x|$.
  - As $(x,y) \to (0,0)$, we know $|x| \to 0$.
  - By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
  - Since the limit equals the function value $f(0,0) = 0$, the function is **continuous** at (0,0).
- **2. Check Partial Derivatives at (0,0)**
  - Use the limit definition for partial derivatives:
  - $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$.
  - $\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$.
  - Both partial derivatives exist at (0,0). Thus, the function is **derivable** at (0,0) and the gradient is $\nabla f(0,0) = (0,0)$.
- **3. Check Directional Derivative at (0,0) for $w = (-1,-1)$**
  - Normalize the direction vector w: $||w|| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.
  - The unit vector is $v = \frac{w}{||w||} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.
  - Compute $D_v f(0,0)$ using the limit definition: $D_v f(0,0) = \lim_{t\to 0} \frac{f(tv)-f(0,0)}{t}$.
  - $$D_v f(0,0) = \lim_{t\to 0} \frac{f(-\frac{t}{\sqrt{2}}, -\frac{t}{\sqrt{2}})}{t} = \lim_{t\to 0} \frac{1}{t}\left[\frac{|-\frac{t}{\sqrt{2}}|(-\frac{t}{\sqrt{2}})^4}{(-\frac{t}{\sqrt{2}})^4 + (-\frac{t}{\sqrt{2}})^4}\right]$$
    $$= \lim_{t\to 0} \frac{1}{t}\left[\frac{\frac{|t|}{\sqrt{2}}\frac{t^4}{4}}{\frac{t^4}{4} + \frac{t^4}{4}}\right] = \lim_{t\to 0} \frac{1}{t}\left[\frac{\frac{|t|t^4}{4\sqrt{2}}}{\frac{2t^4}{4}}\right] = \lim_{t\to 0} \frac{|t|}{2\sqrt{2}t}$$
  - This limit **does not exist** because the limits from the left ($t \to 0^-$) and right ($t \to 0^+$) are different ($-\frac{1}{2\sqrt{2}}$ and $\frac{1}{2\sqrt{2}}$, respectively).
- **4. Check Differentiability at (0,0)**
  - **Method 1: Using Directional Derivative Result**
    - A necessary condition for differentiability at a point is the existence of all directional derivatives at that point.

- Since $D_v f(0,0)$ for $v = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ does not exist, f is **not differentiable** at (0,0).

- **Method 2: Using the Definition of Differentiability**
  - Check if the limit defining differentiability is zero:
  -

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - \nabla f(0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - 0 - (0,0)\cdot(h,k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\frac{|h|k^4}{h^4+k^4}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|k^4}{(h^4+k^4)\sqrt{h^2+k^2}}$$

  - Test the limit along the path $h = k$:
  - $\lim_{h\to 0} \frac{|h|h^4}{(h^4+h^4)\sqrt{h^2+h^2}} = \lim_{h\to 0} \frac{|h|h^4}{2h^4\sqrt{2h^2}} = \lim_{h\to 0} \frac{|h|h^4}{2h^4\sqrt{2}|h|} = \frac{1}{2\sqrt{2}}$.
  - Since the limit must be 0 for differentiability, and we found a path where the limit is $\frac{1}{2\sqrt{2}} \neq 0$, the function f is **not differentiable** at (0,0).

Analysis a arctan function

Function: $f(x,y) = x \arctan\left(\frac{y}{x}\right)$

- **1. Determine the Domain**
  - The argument of arctan requires the denominator x to be non-zero.
  - Domain $D = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}$.
  - This domain represents the entire $\mathbb{R}^2$ plane excluding the y-axis.
  - The domain is open, not connected, and not bounded.
- **2. Determine where $f(x,y) \geq 0$**
  - We need $x \arctan(\frac{y}{x}) \geq 0$.
  - Case 1: $x > 0$. Requires $\arctan(\frac{y}{x}) \geq 0$, which implies $\frac{y}{x} \geq 0$. Since $x > 0$, this means $y \geq 0$.
  - Case 2: $x < 0$. Requires $\arctan(\frac{y}{x}) \leq 0$, which implies $\frac{y}{x} \leq 0$. Since $x < 0$, this means $y \geq 0$.
  - Combining both cases, $f(x,y) \geq 0$ if and only if $y \geq 0$ (and $x \neq 0$).
  - The set is $\{(x,y) \in \mathbb{R}^2 \mid x \neq 0, y \geq 0\}$. (The upper half-plane excluding the y-axis).
- **3. Continuous Extension to $\mathbb{R}^2$**
  - The function is continuous on its domain D as it's composed of continuous functions.
  - To extend it continuously to $\mathbb{R}^2$, we need to define it for $x = 0$ such that the limit matches the defined value.
  - Consider the limit as $(x,y) \to (0, y_0)$ for any $y_0 \in \mathbb{R}$.
  - $\lim_{(x,y)\to(0,y_0)} x \arctan\left(\frac{y}{x}\right)$.

- Since $x \to 0$ and $\arctan(\frac{y}{x})$ is bounded (between $-\pi/2$ and $\pi/2$), the limit is 0.
- We can define a continuous extension $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ as:

$$\tilde{f}(x,y) = \begin{cases} x\arctan(\frac{y}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- This function $\tilde{f}$ is continuous on all of $\mathbb{R}^2$.

- **4. Gradient and Directional Derivatives at (1,0)**
  - Compute partial derivatives for $x \neq 0$:
    - $\frac{\partial f}{\partial x} = \arctan\left(\frac{y}{x}\right) + x \cdot \frac{1}{1+(y/x)^2} \cdot \left(-\frac{y}{x^2}\right) = \arctan\left(\frac{y}{x}\right) - \frac{xy}{x^2+y^2}$.
    - $\frac{\partial f}{\partial y} = x \cdot \frac{1}{1+(y/x)^2} \cdot \left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}$.
  - Evaluate at the point $(1,0)$:
    - $\frac{\partial f}{\partial x}(1,0) = \arctan(0) - \frac{1\cdot 0}{1^2+0^2} = 0$.
    - $\frac{\partial f}{\partial y}(1,0) = \frac{1^2}{1^2+0^2} = 1$.
  - The gradient is $\nabla f(1,0) = (0,1)$.
  - Since the partial derivatives are continuous around $(1,0)$ (as $x = 1 \neq 0$), the function f is differentiable at $(1,0)$.
  - **Conclusion on Directional Derivatives:** Because f is differentiable at $(1,0)$, all directional derivatives $D_v f(1,0)$ exist and can be calculated using the formula $D_v f(1,0) = \nabla f(1,0) \cdot v$.

Gradient and Tangent Line to Contour Line

Function: $f(x,y) = \frac{\sqrt{xy}}{e^{2x}}$ (Note: Domain requires $xy \geq 0$)
Point: $P_0 = (-1,-1)$ (Point is in the domain as $(-1)(-1) = 1 \geq 0$)

- **1. Compute the Gradient** $\nabla f(-1,-1)$
  - Compute partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{(\frac{1}{2\sqrt{xy}} \cdot y)e^{2x} - \sqrt{xy}(2e^{2x})}{(e^{2x})^2} = \frac{e^{2x}(\frac{y}{2\sqrt{xy}} - 2\sqrt{xy})}{e^{4x}} = \frac{y - 4xy}{2\sqrt{xy}e^{2x}}$$

  - Evaluate $\frac{\partial f}{\partial x}$ at $(-1,-1)$:

$$\frac{\partial f}{\partial x}(-1,-1) = \frac{-1-4(-1)(-1)}{2\sqrt{(-1)(-1)}e^{2(-1)}} = \frac{-1-4}{2\sqrt{1}e^{-2}} = \frac{-5}{2e^{-2}} = -\frac{5}{2}e^2$$

  - Compute partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = \frac{1}{e^{2x}} \cdot \frac{\partial}{\partial y}(\sqrt{xy}) = \frac{1}{e^{2x}} \cdot \frac{1}{2\sqrt{xy}} \cdot x = \frac{x}{2\sqrt{xy}e^{2x}}$$

  - Evaluate $\frac{\partial f}{\partial y}$ at $(-1,-1)$:

$$\frac{\partial f}{\partial y}(-1,-1) = \frac{-1}{2\sqrt{(-1)(-1)}e^{2(-1)}} = \frac{-1}{2\sqrt{1}e^{-2}} = \frac{-1}{2e^{-2}} = -\frac{1}{2}e^2$$

- The Gradient at $(-1,-1)$ is:

$$\nabla f(-1,-1) = \left(-\frac{5}{2}e^2, -\frac{1}{2}e^2\right)$$

- **2. Find Tangent Line to contour line at** $(-1,-1)$
  - **Principle:** The gradient vector $\nabla f(x_0, y_0)$ is orthogonal to the contour line of f passing through (x_0, y_0).
  - The tangent line at (x_0, y_0) must be orthogonal to $\nabla f(x_0, y_0)$.
  - Our point is $P_0 = (-1,-1)$ and gradient is $\nabla f(-1,-1) = (-\frac{5}{2}e^2, -\frac{1}{2}e^2)$.
  - Find a direction vector $w = (w_1, w_2)$ for the tangent line such that $w \cdot \nabla f(-1,-1) = 0$.
$$w_1\left(-\frac{5}{2}e^2\right) + w_2\left(-\frac{1}{2}e^2\right) = 0$$
  - Divide by $-\frac{1}{2}e^2$: $5w_1 + w_2 = 0$.
  - Choose a simple direction vector satisfying this, e.g., $w_1 = 1$, $w_2 = -5$. So, $w = (1,-5)$.
  - The tangent line passes through $P_0 = (-1,-1)$ with direction vector $w = (1,-5)$.
  - **Parametric Equation:** $r(t) = P_0 + tw = (-1,-1) + t(1,-5)$

$$\begin{cases} x = -1 + t \\ y = -1 - 5t \end{cases}$$

  - **Cartesian Equation:** Eliminate t. From the first equation, $t = x + 1$.

$$y = -1 - 5(x+1) = -1 - 5x - 5$$

$$y = -5x - 6$$

  - The equation of the tangent line to the contour line at $(-1,-1)$ is $y = -5x - 6$.

3 Differential Equations

3.1 Exercises on separable differential equations

Solving a Separable Cauchy Problem

Problem: Solve the initial value problem:

$$\begin{cases} y' = \frac{y^2}{y^2+4}t \\ y(0) = 2 \end{cases}$$

- **1. Identify Equation Type and Separate Variables**
  - The equation is a first-order ODE and is separable.
  - Note: $y = 0$ is a constant solution to the DE ($0 = \frac{0}{0+4}t = 0$), but it does not satisfy the initial condition $y(0) = 2$.
  - Assuming $y \neq 0$, rewrite $y' = \frac{dy}{dt}$ and separate:

$$\frac{y^2+4}{y^2}dy = tdt$$

- **2. Integrate Both Sides**
  - Simplify the left side: $\int \left(1 + \frac{4}{y^2}\right) dy = \int (1 + 4y^{-2})dy$.
  - Integrate:

$$\int (1 + 4y^{-2})dy = y + 4\frac{y^{-1}}{-1} = y - \frac{4}{y}$$

$$\int tdt = \frac{t^2}{2}$$

  - Combine with the integration constant C:

$$y - \frac{4}{y} = \frac{t^2}{2} + C$$

- **3. Apply the Initial Condition $y(0) = 2$**
  - Substitute $t = 0$ and $y = 2$ into the implicit solution:

$$2 - \frac{4}{2} = \frac{0^2}{2} + C$$

$$2 - 2 = 0 + C \implies C = 0$$

- **4. Obtain the Implicit Solution**
  - Substitute $C = 0$ back into the equation:

$$y - \frac{4}{y} = \frac{t^2}{2}$$

- **5. Solve for the Explicit Solution $y(t)$**
  - Multiply by y (justified since $y(0) = 2$, so y is non-zero near $t = 0$):

$$y^2 - 4 = \frac{t^2}{2}y$$

- Rearrange into a quadratic equation for y:

$$y^2 - \left(\frac{t^2}{2}\right) y - 4 = 0$$

- Use the quadratic formula $y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ with $a = 1, b = -\frac{t^2}{2}, c = -4$:

$$y = \frac{\frac{t^2}{2} \pm \sqrt{\left(-\frac{t^2}{2}\right)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{\frac{t^2}{2} \pm \sqrt{\frac{t^4}{4} + 16}}{2} = \frac{\frac{t^2}{2} \pm \sqrt{\frac{t^4+64}{4}}}{2}$$

$$= \frac{\frac{t^2}{2} \pm \frac{\sqrt{t^4+64}}{2}}{2} = \frac{t^2 \pm \sqrt{t^4+64}}{4}$$

- Two potential solutions: $y_1(t) = \frac{t^2+\sqrt{t^4+64}}{4}$ and $y_2(t) = \frac{t^2-\sqrt{t^4+64}}{4}$.

- **6. Select the Correct Branch using $y(0) = 2$**
  - Check $y_1(0)$: $y_1(0) = \frac{0^2+\sqrt{0^4+64}}{4} = \frac{\sqrt{64}}{4} = \frac{8}{4} = 2$. (Matches)
  - Check $y_2(0)$: $y_2(0) = \frac{0^2-\sqrt{0^4+64}}{4} = \frac{-\sqrt{64}}{4} = \frac{-8}{4} = -2$. (Does not match)
- **7. Final Explicit Solution**
  - The solution to the Cauchy problem is:

$$y(t) = \frac{t^2 + \sqrt{t^4+64}}{4}$$

Qualitative Sketch of Cauchy Problem Solution

Problem: Sketch the solution near $t = 0$ for:

$$\begin{cases} y' - (t+2)\left(1 + \frac{1}{y}\right) = 0 \\ y(0) = 1 \end{cases}$$

- **1. Attempt Explicit Solution (Separation of Variables)**
  - Rewrite DE: $y' = (t+2)\left(\frac{y+1}{y}\right)$.
  - Note: $y = -1$ is a constant solution to the DE, but $y(0) = 1 \neq -1$, so it's not our solution.
  - Separate (for $y \neq -1$): $\frac{y}{y+1} dy = (t+2) dt$.
  - Integrate: $\int \left(1 - \frac{1}{y+1}\right) dy = \int (t+2) dt$.

$$y - \ln|y+1| = \frac{t^2}{2} + 2t + C$$

  - Apply $y(0) = 1$: $1 - \ln|1+1| = 0 + 0 + C \implies C = 1 - \ln(2)$.
  - Implicit Solution: $y - \ln|y+1| = \frac{t^2}{2} + 2t + 1 - \ln(2)$.

- **Conclusion:** Finding an explicit $y(t)$ from this implicit form is difficult. We proceed with qualitative analysis.

- **2. Qualitative Analysis at $t = 0$**
  - **Value at $t = 0$:** Given $y(0) = 1$. The solution passes through the point $(0, 1)$.
  - **Slope at $t = 0$ ($y'(0)$):** Use the DE $y' = (t + 2)(1 + 1/y)$.

$$y'(0) = (0 + 2)\left(1 + \frac{1}{y(0)}\right) = (2)\left(1 + \frac{1}{1}\right) = 2(2) = 4$$

    Since $y'(0) = 4 > 0$, the solution is **increasing** at $t = 0$.
  - **Concavity at $t = 0$ ($y''(0)$):** Differentiate the expression for y' with respect to t.

$$y'(t) = (t + 2)\left(1 + \frac{1}{y(t)}\right)$$

$$y''(t) = \frac{d}{dt}\left[(t + 2)\left(1 + \frac{1}{y(t)}\right)\right]$$

    Using the product rule:

$$y''(t) = (1)\left(1 + \frac{1}{y(t)}\right) + (t + 2)\frac{d}{dt}\left(1 + \frac{1}{y(t)}\right)$$

$$y''(t) = 1 + \frac{1}{y(t)} + (t + 2)\left(-\frac{1}{y(t)^2} \cdot y'(t)\right)$$

  - Evaluate $y''(0)$ using $t = 0$, $y(0) = 1$, and $y'(0) = 4$:

$$y''(0) = 1 + \frac{1}{1} + (0 + 2)\left(-\frac{1}{1^2} \cdot 4\right)$$

$$y''(0) = 1 + 1 + (2)(-1 \cdot 4) = 2 - 8 = -6$$

    Since $y''(0) = -6 < 0$, the solution is **concave down** at $t = 0$.

- **3. Sketching Conclusion**
  - The graph of the solution $y(t)$ near $t = 0$:
    - Passes through the point $(0, 1)$.
    - Is increasing at this point (slope is 4).
    - Is concave down at this point (curves downwards).

Solving a Linear First-Order Cauchy Problem

Problem: Solve the initial value problem:

$$\begin{cases} y' = 4y + e^{3t} \\ y(0) = 0 \end{cases}$$

This is a linear first-order ODE. Standard form: $y' - 4y = e^{3t}$. Here, $a(t) = -4$ and $g(t) = e^{3t}$. Initial condition is at $t_0 = 0$.

- **Method 1: Using the General Formula (Integrating Factor)**

- The general solution formula for $y' + a(t)y = g(t)$ with $y(t_0) = y_0$ is:

$$y(t) = y_0 e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} g(s) e^{A(s)} ds$$

  where $A(t)$ is an antiderivative of $a(t)$.
- For our problem $y' - 4y = e^{3t}$: $a(t) = -4$, $g(t) = e^{3t}$, $t_0 = 0$, $y_0 = 0$.
- Choose the antiderivative $A(t) = \int_{t_0}^{t} a(\tau) d\tau = \int_0^t (-4) d\tau = -4t$. (This ensures $A(t_0) = A(0) = 0$, simplifying the formula slightly, though any antiderivative works).
- Apply the formula:

$$y(t) = 0 \cdot e^{-(-4t)} + e^{-(-4t)} \int_0^t e^{3s} e^{-4s} ds$$

$$y(t) = e^{4t} \int_0^t e^{-s} ds$$

$$y(t) = e^{4t} \left[-e^{-s} \right]_0^t$$

$$y(t) = e^{4t}(-e^{-t} - (-e^{0})) = e^{4t}(-e^{-t} + 1)$$

- Simplify:

$$y(t) = e^{4t} - e^{3t}$$

- **Method 2: Homogeneous Solution + Particular Solution (Variation of Constants)**
  - **Step 2a: Solve the Homogeneous Equation** $y' - 4y = 0$.
    - Let the homogeneous solution be $z(t)$. $z' = 4z$.
    - Separate variables: $\frac{dz}{z} = 4dt$.
    - Integrate: $\ln|z| = 4t + C_1$.
    - Exponentiate: $|z| = e^{4t + C_1} = e^{C_1} e^{4t}$.
    - General homogeneous solution: $z(t) = K e^{4t}$ (where $K = \pm e^{C_1}$ or $K = 0$).
  - **Step 2b: Find a Particular Solution using Variation of Constants**
    - Assume a particular solution $y_p(t)$ of the form $y_p(t) = K(t) e^{4t}$.
    - Find the derivative: $y_p'(t) = K'(t) e^{4t} + K(t)(4e^{4t})$.
    - Substitute y_p and y_p' into the full non-homogeneous equation $y' - 4y = e^{3t}$:

$$(K'(t) e^{4t} + 4K(t) e^{4t}) - 4(K(t) e^{4t}) = e^{3t}$$

    - Simplify: $K'(t) e^{4t} = e^{3t}$.
    - Solve for $K'(t)$: $K'(t) = e^{3t} e^{-4t} = e^{-t}$.
    - Integrate to find $K(t)$ (we only need one instance, constant=0):

$$K(t) = \int e^{-t} dt = -e^{-t}$$

    - The particular solution is: $y_p(t) = K(t) e^{4t} = (-e^{-t}) e^{4t} = -e^{3t}$.
  - **Step 2c: Combine for General Solution**
    - The general solution is $y(t) = z(t) + y_p(t) = K e^{4t} - e^{3t}$.
  - **Step 2d: Apply Initial Condition** $y(0) = 0$.

- $0 = Ke^{4(0)} - e^{3(0)} = Ke^0 - e^0 = K - 1$.
- Therefore, $K = 1$.

- **Step 2e: Final Solution**
  - Substitute $K = 1$ into the general solution: $y(t) = (1)e^{4t} - e^{3t}$.
  - $y(t) = e^{4t} - e^{3t}$

- **Conclusion:** Both methods yield the same solution $y(t) = e^{4t} - e^{3t}$.

4 Appendix: Taylor/Maclaurin Polynomials and Standard Limits

4.1 Taylor and Maclaurin Polynomials

General Taylor and Maclaurin Formulas

Taylor Polynomial of order n for $f(x)$ at $x = a$:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Polynomial: Taylor polynomial at $a = 0$:

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Remainder (Peano form):

$$R_n(x) = o((x-a)^n) \quad \text{as } x \to a$$

Maclaurin Series for Major Functions

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Tangent (first terms):

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + o(x^7)$$

Arctangent:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Natural Logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1$$

Binomial Series: For $|x| < 1$, $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Inverse:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Examples: Maclaurin Polynomials (Order 3 or 4)

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
- $\tan x = x + \frac{x^3}{3} + o(x^3)$
- $\arctan x = x - \frac{x^3}{3} + o(x^3)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + o(x^3)$

4.2 Standard Limits

Standard Limits as $x \to 0$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ (a > 0)$
- $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \ (\alpha \in \mathbb{R})$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{x}{\ln(1+x)} = 1$
- $\lim_{x\to 0} \frac{\sin(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\arctan(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a$

- $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n \ (n \in \mathbb{N})$

Standard Limits as $x \to \infty$

- $\lim_{x\to\infty} \frac{\ln x}{x} = 0$
- $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$ for any $a > 0$
- $\lim_{x\to\infty} \frac{e^{ax}}{x^b} = \begin{cases} 0 & a < 0 \\ \infty & a > 0 \end{cases}$
- $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$
- $\lim_{x\to\infty} x^{1/x} = 1$
- $\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$ for $a > 0$
- $\lim_{x\to\infty} \frac{x^a}{\ln x} = \infty$ for $a > 0$
- $\lim_{x\to\infty} \frac{x^a}{x^b} = \begin{cases} 0 & a < b \\ 1 & a = b \\ \infty & a > b \end{cases}$
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$
- $\lim_{x\to\infty} \frac{a^x}{b^x} = 0$ if $0 < a < b$

Other Useful Limits

- $\lim_{x\to 0} (1+x)^{1/x} = e$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$
- $\lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$

These formulas and limits are fundamental tools for solving calculus problems, especially for evaluating limits, approximating functions, and analyzing local behavior.