Cascade Schemes for Calculus Exercises

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Contents

Introduction

This document collects cascade schemes for exercises, techniques, and fundamental concepts in calculus, organized by chapter. Each scheme follows a hierarchical and synthetic model to facilitate quick consultation and memorization.

1 Integrals

In this section we will explore some types of exercises that can be asked in a typical calculus exam. Specifically we will explore two categories:

- Integral function exercises
- Improper integral exercises

1.1 Integral function exercises

Computing the derivative of a composition of the integral function

Key Question: How can we compute the derivative of a function that is the composition of the integral function with another function?

- **Remember the derivative formula for composite functions**
  - $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- **Consider Torricelli's Theorem**
  - $F(x) = \int_a^x f(t)dt$
  - $F'(x) = f(x)$
- **Example**
  - $G(x) = \int_a^{m(x)} f(t)\,dt = F(m(x))$
  - $G'(x) = F'(m(x)) \cdot m'(x) = f(m(x)) \cdot m'(x)$

Computing the derivative of an integral with variable limits

Key Question: How can we compute the derivative of a function defined by an integral where both limits depend on x, i.e., $G(x) = \int_{n(x)}^{m(x)} f(t)dt$?

- **1. Split the integral using a constant lower limit**
  - Use the property of definite integrals: $\int_a^b f(t)dt = \int_c^b f(t)dt - \int_c^a f(t)dt$.
  - Apply this to $G(x)$: Choose an arbitrary constant a (often 0 or 1) in the domain of f.
  $$G(x) = \int_{n(x)}^{m(x)} f(t)dt = \int_a^{m(x)} f(t)dt - \int_a^{n(x)} f(t)dt$$
- **2. Define an auxiliary function using Torricelli's Theorem**
  - Let $F(x) = \int_a^x f(t)dt$.
  - By the Fundamental Theorem of Calculus (Part 1 / Torricelli's Theorem), $F'(x) = f(x)$.
  - Rewrite $G(x)$ using F: $G(x) = F(m(x)) - F(n(x))$.
- **3. Differentiate using the Chain Rule**

- Apply the chain rule to differentiate $F(m(x))$ and $F(n(x))$:

$$(F(h(x)))' = F'(h(x)) \cdot h'(x) = f(h(x)) \cdot h'(x)$$

- Therefore, the derivative of $G(x)$ is:

$$G'(x) = \frac{d}{dx}[F(m(x))] - \frac{d}{dx}[F(n(x))]$$

$$G'(x) = F'(m(x)) \cdot m'(x) - F'(n(x)) \cdot n'(x)$$

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

- **4. Example**
  - Compute the derivative of $G(x) = \int_x^{2x} \frac{\sin t}{t} dt$.
  - Here, $f(t) = \frac{\sin t}{t}$, $m(x) = 2x$, and $n(x) = x$.
  - Choose $a = 1$ (any constant works). Let $F(x) = \int_1^x \frac{\sin t}{t} dt$.
  - Then $G(x) = F(2x) - F(x)$.
  - The derivatives are $m'(x) = 2$ and $n'(x) = 1$.
  - Applying the formula:

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

$$G'(x) = f(2x) \cdot 2 - f(x) \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{2x} \cdot 2 - \frac{\sin x}{x} \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{x} - \frac{\sin x}{x} = \frac{\sin(2x) - \sin x}{x}$$

Computing limits involving integrals and indeterminate forms

Key Question: How to compute the limit:

$$\lim_{x \to 0} \frac{x - \int_0^x (e^{-t^2} + \sin^2 t) dt}{x(x^2 - \sin^2 x)}$$

- **Initial Check: Identify Indeterminate Form**
  - Substitute $x = 0$ into the numerator: $0 - \int_0^0 (e^{-t^2} + \sin^2 t) dt = 0 - 0 = 0$.
  - Substitute $x = 0$ into the denominator: $0 \cdot (0^2 - \sin^2 0) = 0 \cdot (0 - 0) = 0$.
  - The limit presents the indeterminate form $\left[\frac{0}{0}\right]$.
- **Apply L'Hôpital's Rule**
  - Differentiate the numerator with respect to x. Requires the Fundamental Theorem of Calculus (Torricelli-Barrow) for the integral part:

$$\frac{d}{dx}\left(x - \int_0^x (e^{-t^2} + \sin^2 t) dt\right) = 1 - (e^{-x^2} + \sin^2 x)$$

  - Differentiate the denominator with respect to x using the product rule:

$$\frac{d}{dx}(x(x^2 - \sin^2 x)) = (x^2 - \sin^2 x) + (2x^2 - x \sin(2x))$$

- The limit becomes (applying L'Hôpital's Rule):

$$L \overset{H}{=} \lim_{x\to 0} \frac{1 - e^{-x^2} - \sin^2 x}{(x^2 - \sin^2 x) + (2x^2 - x\sin(2x))}$$

- **Evaluate the New Limit (Using Taylor Series Approximation)**
  - The limit is still in the form $\left[\frac{0}{0}\right]$. Further application of L'Hôpital's Rule is possible, but Taylor expansions are often more efficient.
  - Approximate numerator and denominator for $x \to 0$:
    - $1 - e^{-x^2} - \sin^2 x = \frac{5}{6}x^4 + o(x^4)$
    - $(x^2 - \sin^2 x) + (2x^2 - x\sin(2x)) = \frac{5}{3}x^4 + o(x^4)$
  - Substitute the approximations back into the limit:

$$L = \lim_{x\to 0} \frac{\frac{5}{6}x^4 + o(x^4)}{\frac{5}{3}x^4 + o(x^4)} = \frac{5/6}{5/3} = \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$$

- **Result:** The limit exists and its value is $\frac{1}{2}$.

Computing Maclaurin Polynomials for Composite Functions with Integrals

Key Question: How to determine the Maclaurin polynomial for $F(x) = \sin\left(\int_0^x e^{-t^2} dt\right)$?

- **Define Auxiliary Function for the Integral**
  - Let $G(x) = \int_0^x e^{-t^2} dt$.
  - The original function becomes $F(x) = \sin(G(x))$.
- **Find the Maclaurin Series for the Auxiliary Function** $G(x)$
  - Calculate derivatives of $G(x)$ and evaluate at $x = 0$:
    - $G(0) = \int_0^0 e^{-t^2} dt = 0$
    - $G'(x) = e^{-x^2}$ (by Fundamental Theorem of Calculus) $\to G'(0) = e^0 = 1$
    - $G''(x) = -2xe^{-x^2} \to G''(0) = 0$
    - $G'''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \to G'''(0) = -2$
  - Construct the Maclaurin series for $G(x)$ using $G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!}x^n$:

$$G(x) = G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + \frac{G'''(0)}{3!}x^3 + o(x^3)$$

$$G(x) = 0 + 1 \cdot x + \frac{0}{2}x^2 + \frac{-2}{6}x^3 + o(x^3) = x - \frac{1}{3}x^3 + o(x^3)$$

- **Use the Known Maclaurin Series for the Outer Function** ($\sin$)
  - Recall the Maclaurin series for $\sin(z)$:

$$\sin(z) = z - \frac{z^3}{6} + o(z^3)$$

- **Substitute the Series for $G(x)$ into the Series for $\sin(z)$**

- Replace z with $G(x) = x - \frac{1}{3}x^3 + o(x^3)$:

$$F(x) = \sin(G(x)) = \left(x - \frac{1}{3}x^3 + o(x^3)\right) - \frac{1}{6}\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 + o\left((G(x))^3\right)$$

- Since $G(x) \approx x$ for $x \to 0$, we have $o((G(x))^3) = o(x^3)$.
- Expand and keep terms up to x^3:

$$\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 = (x + o(x))^3 = x^3 + o(x^3)$$

- Substitute back:

$$F(x) = \left(x - \frac{1}{3}x^3\right) - \frac{1}{6}(x^3) + o(x^3)$$

$$F(x) = x - \left(\frac{1}{3} + \frac{1}{6}\right)x^3 + o(x^3) = x - \frac{2+1}{6}x^3 + o(x^3) = x - \frac{3}{6}x^3 + o(x^3)$$

$$F(x) = x - \frac{1}{2}x^3 + o(x^3)$$

- **Result: Maclaurin Polynomial**
  - The Maclaurin polynomial of order 3 for $F(x)$ is $P_3(x) = x - \frac{1}{2}x^3$.

Proving Existence and Uniqueness of Solutions Involving Integral Functions

Key Question: How to prove that the equation $f(x) = 1 - x$, where $f(x) = \int_0^x e^{-t^2} dt$, has a unique solution?

- **Reformulate the Problem**
  - Define a new function $g(x) = f(x) - (1 - x) = f(x) + x - 1$.
  - The original problem is equivalent to proving that $g(x) = 0$ has exactly one solution (a unique zero).
- **Prove Existence of a Zero (Intermediate Value Theorem)**
  - **Continuity:** $f(x)$ is continuous because it's an integral function of a continuous integrand (e^{-t^2}). The term $x-1$ is also continuous. Therefore, $g(x)$ is continuous on $\mathbb{R}$.
  - **Find points with opposite signs:**
    - Evaluate $g(0)$: $g(0) = f(0) + 0 - 1 = \int_0^0 e^{-t^2} dt - 1 = 0 - 1 = -1$. So, $g(0) < 0$.
    - Evaluate the limit as $x \to +\infty$:

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \left(\int_0^x e^{-t^2} dt + x - 1\right)$$

    The integral $\int_0^\infty e^{-t^2} dt$ converges to a finite value ($\frac{\sqrt{\pi}}{2}$). The term x goes to $+\infty$.

$$\lim_{x \to +\infty} g(x) = \left(\lim_{x \to +\infty} \int_0^x e^{-t^2} dt\right) + \left(\lim_{x \to +\infty} x\right) - 1 = \frac{\sqrt{\pi}}{2} + \infty - 1 = +\infty$$

    - Since the limit is $+\infty$, there must exist some value $\bar{x}$ such that for all $x > \bar{x}$, $g(x) > 0$. Let's pick one such value $\bar{x}$.

- **Apply IVT:** Since $g(x)$ is continuous on $[0, \bar{x}]$, $g(0) < 0$, and $g(\bar{x}) > 0$, the Intermediate Value Theorem guarantees that there exists at least one $x_0 \in (0, \bar{x})$ such that $g(x_0) = 0$.

- **Prove Uniqueness of the Zero (Monotonicity)**
  - **Calculate the derivative $g'(x)$:**
  $$g'(x) = \frac{d}{dx}(f(x) + x - 1) = f'(x) + 1$$
  - Apply the Fundamental Theorem of Calculus to find $f'(x)$:
  $$f'(x) = \frac{d}{dx}\left(\int_0^x e^{-t^2} dt\right) = e^{-x^2}$$
  - Substitute back into $g'(x)$:
  $$g'(x) = e^{-x^2} + 1$$
  - **Analyze the sign of $g'(x)$:** Since $e^{-x^2} > 0$ for all real x, we have $g'(x) = e^{-x^2} + 1 > 0 + 1 = 1$.
  - Since $g'(x) > 0$ for all $x \in \mathbb{R}$, the function $g(x)$ is strictly increasing on its entire domain.

- **Conclusion**
  - We have shown that $g(x) = 0$ has at least one solution (by IVT).
  - We have shown that $g(x)$ is strictly increasing, which means it can cross the x-axis (equal zero) at most once.
  - Therefore, the equation $g(x) = 0$ has exactly one unique solution. This implies the original equation $f(x) = 1 - x$ also has a unique solution.

Finding Order and Principal Part for Functions with Integrals (Derivative Method)

Key Question: Determine the order and principal part of $G(x) = x - \int_0^x e^{-(t+x)^2} dt$ as $x \to 0$.

- **Strategy: Analyze the Derivative $G'(x)$**
  - It's often easier to find the Maclaurin expansion (order/principal part) of the derivative first, and then integrate.
  - **Simplify the integral (Optional Variable Change):** Let $z = t + x$, $dz = dt$. Limits change from $t \in [0, x]$ to $z \in [x, 2x]$.
  $$G(x) = x - \int_x^{2x} e^{-z^2} dz$$
  - **Rewrite using a fixed lower limit:** Let $H(x) = \int_0^x e^{-t^2} dt$. Then $\int_x^{2x} e^{-z^2} dz = H(2x) - H(x)$.
  $$G(x) = x - (H(2x) - H(x)) = x - H(2x) + H(x)$$

- **Compute the Derivative $G'(x)$**

- Apply differentiation rules, including the Fundamental Theorem of Calculus and Chain Rule:

$$G'(x) = \frac{d}{dx}(x) - \frac{d}{dx}(H(2x)) + \frac{d}{dx}(H(x))$$

$$G'(x) = 1 - (H'(2x) \cdot 2) + H'(x)$$

Since $H'(x) = e^{-x^2}$:

$$G'(x) = 1 - (e^{-(2x)^2} \cdot 2) + e^{-x^2} = 1 - 2e^{-4x^2} + e^{-x^2}$$

- **Find Maclaurin Expansion of $G'(x)$**
  - Use the known expansion $e^u = 1 + u + o(u)$ for $u \to 0$.
  - $e^{-4x^2} = 1 - 4x^2 + o(x^2)$
  - $e^{-x^2} = 1 - x^2 + o(x^2)$
  - Substitute into $G'(x)$:

$$G'(x) = 1 - 2(1 - 4x^2 + o(x^2)) + (1 - x^2 + o(x^2))$$

$$G'(x) = 1 - 2 + 8x^2 + o(x^2) + 1 - x^2 + o(x^2)$$

$$G'(x) = (1 - 2 + 1) + (8x^2 - x^2) + o(x^2) = 7x^2 + o(x^2)$$

  - The principal part of $G'(x)$ is $7x^2$, and its order is 2.
- **Determine Order and Principal Part of $G(x)$**
  - Since $G'(x) \sim 7x^2$ as $x \to 0$, and $G(0) = 0 - \int_0^0 \cdots = 0$, we can integrate the principal part of $G'(x)$ to find the principal part of $G(x)$:

$$G(x) \approx \int_0^x 7t^2 dt = 7\left[\frac{t^3}{3}\right]_0^x = \frac{7}{3}x^3$$

  - The principal part of $G(x)$ is $\frac{7}{3}x^3$.
  - The order of $G(x)$ is 3.
  - Thus, $G(x) = \frac{7}{3}x^3 + o(x^3)$ for $x \to 0$.
- **Verification (Using L'Hôpital's Rule on the Remainder)**
  - To confirm $G(x) = \frac{7}{3}x^3 + o(x^3)$, we must show $\lim_{x\to 0} \frac{G(x) - \frac{7}{3}x^3}{x^3} = 0$.
  - Apply L'Hôpital's Rule (since form is $\frac{0}{0}$):

$$\lim_{x\to 0} \frac{G'(x) - \frac{d}{dx}(\frac{7}{3}x^3)}{\frac{d}{dx}(x^3)} = \lim_{x\to 0} \frac{G'(x) - 7x^2}{3x^2}$$

  - Substitute the expansion of $G'(x)$:

$$\lim_{x\to 0} \frac{(7x^2 + o(x^2)) - 7x^2}{3x^2} = \lim_{x\to 0} \frac{o(x^2)}{3x^2} = 0$$

  - The limit is 0, confirming the order and principal part.

Qualitative Graph Sketching of Integral Functions near x

Key Question: How to draw a qualitative graph of $f(x) = \int_0^x \frac{e^t}{2t^2+1} dt$ in a neighborhood of $x = 0$?

- **Strategy: Analyze Local Behavior using Derivatives at x=0**
  - The behavior of a function near a point (like $x = 0$) is determined by its value and the values of its derivatives at that point. This is the foundation of Taylor series approximations.
- **Calculate** $f(0)$
  - Substitute $x = 0$ into the integral definition:
$$f(0) = \int_0^0 \frac{e^t}{2t^2+1} dt = 0$$
  - The function passes through the origin $(0, 0)$.
- **Calculate** $f'(x)$ **and** $f'(0)$
  - Apply the Fundamental Theorem of Calculus (Part 1):
$$f'(x) = \frac{d}{dx}\left(\int_0^x \frac{e^t}{2t^2+1} dt\right) = \frac{e^x}{2x^2+1}$$
  - Evaluate at $x = 0$:
$$f'(0) = \frac{e^0}{2(0)^2+1} = \frac{1}{1} = 1$$
  - The slope of the tangent line at the origin is 1. The function is increasing at $x = 0$.
- **Calculate** $f''(x)$ **and** $f''(0)$
  - Differentiate $f'(x)$ using the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$:
$$u = e^x \implies u' = e^x$$
$$v = 2x^2+1 \implies v' = 4x$$
$$f''(x) = \frac{(e^x)(2x^2+1) - (e^x)(4x)}{(2x^2+1)^2} = \frac{e^x(2x^2-4x+1)}{(2x^2+1)^2}$$
  - Evaluate at $x = 0$:
$$f''(0) = \frac{e^0(2(0)^2-4(0)+1)}{(2(0)^2+1)^2} = \frac{1(1)}{(1)^2} = 1$$
  - Since $f''(0) > 0$, the function is concave up at $x = 0$.
- **Sketching the Graph near x=0**
  - The graph passes through $(0, 0)$.
  - The tangent line at $(0, 0)$ has a slope of 1 (like the line $y = x$).
  - The graph is concave up at $(0, 0)$, meaning it lies above its tangent line near the point of tangency.
  - Combining these: Start at the origin, draw a curve that is initially tangent to $y = x$ and curves upwards (concave up).

Limit Computation using Integral Inequalities and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x \to 0^+} x \int_{x-x^2}^{x} \frac{dt}{\sin^3 t}$, if it exists.

- **Step 1: Analyze Integrand and Integration Interval**
  - The integration variable is t, and the interval is $[x - x^2, x]$.
  - As $x \to 0^+$, both x and $x - x^2 = x(1-x)$ approach 0^+. Thus, $t \to 0^+$.
  - For $t \in (0, \pi/2)$, $\sin t$ is positive and strictly increasing.
  - Consequently, $\sin^3 t$ is also positive and strictly increasing for $t \in (0, \pi/2)$.
  - Therefore, the integrand $g(t) = \frac{1}{\sin^3 t}$ is positive and strictly decreasing for t in the interval $(0, \pi/2)$.
- **Step 2: Establish Inequalities for the Integrand**
  - Since $t \in [x-x^2, x]$ and $g(t) = \frac{1}{\sin^3 t}$ is decreasing on this interval (for sufficiently small positive x), the minimum value of $g(t)$ occurs at $t = x$ and the maximum value occurs at $t = x - x^2$.
  - For $t \in [x - x^2, x]$, we have:

$$\frac{1}{\sin^3 x} \leq \frac{1}{\sin^3 t} \leq \frac{1}{\sin^3(x - x^2)}$$

- **Step 3: Integrate the Inequalities**
  - Integrate all parts of the inequality over the interval $[x-x^2, x]$. Since the bounds $\frac{1}{\sin^3 x}$ and $\frac{1}{\sin^3(x-x^2)}$ are constant with respect to t, we get:

$$\int_{x-x^2}^{x} \frac{1}{\sin^3 x} dt \leq \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \leq \int_{x-x^2}^{x} \frac{1}{\sin^3(x - x^2)} dt$$

  - The length of the integration interval is $x - (x - x^2) = x^2$.

$$\frac{1}{\sin^3 x} \cdot x^2 \leq \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \leq \frac{1}{\sin^3(x - x^2)} \cdot x^2$$

$$\frac{x^2}{\sin^3 x} \leq \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \leq \frac{x^2}{\sin^3(x - x^2)}$$

- **Step 4: Incorporate the External Factor and Compute Limits**
  - Multiply the inequality by x (note $x > 0$ as $x \to 0^+$):

$$\frac{x^3}{\sin^3 x} \leq x \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \leq \frac{x^3}{\sin^3(x - x^2)}$$

  - Compute the limit of the lower bound as $x \to 0^+$:

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3 x} = \lim_{x \to 0^+} \left(\frac{x}{\sin x} \right)^3 = (1)^3 = 1$$

    (using the standard limit $\lim_{u \to 0} \frac{\sin u}{u} = 1$)
  - Compute the limit of the upper bound as $x \to 0^+$:

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3(x - x^2)} = \lim_{x \to 0^+} \left(\frac{x}{\sin(x - x^2)} \right)^3$$

    We use $\sin(x - x^2) \sim x - x^2$ as $x \to 0$.

$$\lim_{x \to 0^+} \left(\frac{x}{x - x^2} \right)^3 = \lim_{x \to 0^+} \left(\frac{x}{x(1 - x)} \right)^3 = \lim_{x \to 0^+} \left(\frac{1}{1 - x} \right)^3 = \left(\frac{1}{1 - 0} \right)^3 = 1$$

- **Step 5: Apply the Squeeze Theorem**
  - We have shown:

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3 x} = 1$$

$$\lim_{x \to 0^+} \frac{x^3}{\sin^3(x - x^2)} = 1$$

$$\frac{x^3}{\sin^3 x} \le x \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{x^3}{\sin^3(x - x^2)}$$

  - By the Squeeze Theorem, the limit of the middle expression must also be 1.
  - Therefore, $L = \lim_{x \to 0^+} x \int_{x-x^2}^{x} \frac{dt}{\sin^3 t} = 1$.

Limit Computation using Mean Value Theorem and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x \to +\infty} x^3 \int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt$.

- **Step 1: Apply Mean Value Theorem for Integrals**
  - **Theorem Statement:** If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that $\int_a^b f(t)dt = f(c)(b - a)$.
  - **Application:**
    - Let $f(t) = \sin\left(\frac{1}{t^2}\right)$. This function is continuous on $[x^2, x^2 + x]$ for large x (since $x^2 > 0$).
    - The interval length is $(x^2 + x) - x^2 = x$.
    - By the MVT for Integrals, there exists $z \in [x^2, x^2 + x]$ such that:

$$\int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt = \sin\left(\frac{1}{z^2}\right) \cdot x$$

  - **Rewrite the Limit:** Substitute this result back into the limit expression:

$$L = \lim_{x \to +\infty} x^3 \left[x \sin\left(\frac{1}{z^2}\right)\right] = \lim_{x \to +\infty} x^4 \sin\left(\frac{1}{z^2}\right)$$

    where z depends on x and satisfies $x^2 \le z \le x^2 + x$.
- **Step 2: Establish Bounds using $z \in [x^2, x^2 + x]$**
  - From $x^2 \le z \le x^2 + x$, we have:

$$\frac{1}{x^2 + x} \le \frac{1}{z} \le \frac{1}{x^2}$$

    Squaring (all terms are positive for large x):

$$\frac{1}{(x^2 + x)^2} \le \frac{1}{z^2} \le \frac{1}{x^4}$$

  - As $x \to +\infty$, all terms in the inequality approach 0^+.
  - Since $\sin u$ is an increasing function for u near 0^+, we can apply sin to the inequalities (for sufficiently large x):

$$\sin\left(\frac{1}{(x^2 + x)^2}\right) \le \sin\left(\frac{1}{z^2}\right) \le \sin\left(\frac{1}{x^4}\right)$$

- **Step 3: Apply the Squeeze Theorem**
  - Multiply the inequality by x^4 (which is positive):

$$x^4 \sin\left(\frac{1}{(x^2+x)^2}\right) \le x^4 \sin\left(\frac{1}{z^2}\right) \le x^4 \sin\left(\frac{1}{x^4}\right)$$

  - **Limit of the Upper Bound:**

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{x^4}\right)$$

Use the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$. Let $u = 1/x^4$. As $x \to +\infty$, $u \to 0^+$.

$$= \lim_{u\to 0^+} \frac{1}{u} \sin(u) = 1$$

  - **Limit of the Lower Bound:**
Let $v = 1/(x^2+x)^2$. As $x \to +\infty$, $v \to 0^+$. Use $\sin v \sim v$ for $v \to 0$.

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{(x^2+x)^2}\right)$$

$$= \lim_{x\to+\infty} x^4 \cdot \frac{1}{(x^2+x)^2} = \lim_{x\to+\infty} \frac{x^4}{(x^2(1+1/x))^2} = \lim_{x\to+\infty} \frac{x^4}{x^4(1+1/x)^2} = 1$$

  - **Conclusion:** Since the expression $x^4 \sin(1/z^2)$ is squeezed between two functions that both tend to 1 as $x \to +\infty$, by the Squeeze Theorem (Teorema dei Carabinieri):

$$L = \lim_{x\to+\infty} x^4 \sin\left(\frac{1}{z^2}\right) = 1$$

Integration using Weierstrass Substitution (Tangent Half-Angle)

Key Question: How to calculate the integral $I = \int \frac{\tan x}{\sin x + \tan x} dx$?

- **Step 1: Simplify the Integrand**
  - Rewrite $\tan x$ as $\sin x / \cos x$:

$$\frac{\tan x}{\sin x + \tan x} = \frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}}$$

  - Find a common denominator in the denominator:

$$\frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}} = \frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}}$$

  - Simplify (assuming $\sin x \neq 0, \cos x \neq 0$):

$$\frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}} = \frac{\sin x}{\sin x \cos x + \sin x} = \frac{\sin x}{\sin x(\cos x + 1)} = \frac{1}{\cos x + 1}$$

  - The integral becomes significantly simpler: $I = \int \frac{1}{\cos x + 1} dx$.

- **Step 2: Apply the Weierstrass Substitution ($t = \tan(x/2)$)**
    - **Motivation:** This substitution transforms rational functions of $\sin x$ and $\cos x$ into rational functions of t, which can often be integrated using standard techniques (like partial fractions, although not needed here).
    - **The Substitution Formulas:**
        - Let $\boxed{t = \tan\left(\frac{x}{2}\right)}$
        - Then $\cos x = \frac{1-t^2}{1+t^2}$
        - And $dx = \frac{2}{1+t^2} dt$
        - (Also $\sin x = \frac{2t}{1+t^2}$, though not required for this specific simplified integral).
- **Step 3: Substitute into the Simplified Integral**
    - Replace $\cos x$ and dx with their t-equivalents:

$$I = \int \underbrace{\frac{1}{\left(\frac{1-t^2}{1+t^2}\right) + 1}}_{\frac{1}{\cos x + 1}} \cdot \underbrace{\left(\frac{2}{1+t^2}\right) dt}_{dx}$$

    - Simplify the first part (the integrand in terms of x expressed in t):

$$\frac{1}{\frac{1-t^2+1+t^2}{1+t^2}} = \frac{1}{\frac{2}{1+t^2}} = \frac{1+t^2}{2}$$

    - Substitute this back into the integral expression:

$$I = \int \left(\frac{1+t^2}{2}\right) \cdot \left(\frac{2}{1+t^2}\right) dt$$

- **Step 4: Evaluate the Integral in t**
    - The integrand simplifies dramatically:

$$I = \int 1 \, dt$$

    - Compute the straightforward integral:

$$I = t + C$$

- **Step 5: Substitute Back to x**
    - Replace t with its definition in terms of x:

$$I = \tan\left(\frac{x}{2}\right) + C$$

- **Result:** The integral evaluates to $\int \frac{\tan x}{\sin x + \tan x} dx = \tan\left(\frac{x}{2}\right) + C$.

1.2 Improper Integral exercises

Evaluating Improper Integrals using the Definition

Key Question: How to compute the value of the improper integral $\int_0^1 \log x\,dx$?

- **Step 1: Identify the Point of Impropriety**
  - The function $f(x) = \log x$ is defined on $(0, 1]$.
  - The integral is improper because the integrand $\log x$ approaches $-\infty$ as x approaches the lower limit of integration, $x = 0$.
- **Step 2: Apply the Definition of Improper Integral**
  - Replace the problematic lower limit 0 with a variable ϵ and take the limit as ϵ approaches 0 from the right side (since the interval is $(0, 1]$):

$$\int_0^1 \log x\,dx \stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+} \int_\epsilon^1 \log x\,dx$$

- **Step 3: Find the Antiderivative**
  - Compute the indefinite integral $\int \log x\,dx$ using integration by parts:
    - Let $u = \log x \implies du = \frac{1}{x}dx$
    - Let $dv = dx \implies v = x$

$$\int \log x\,dx = uv - \int v\,du = x \log x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\log x - \int 1\,dx = x \log x - x + C = x(\log x - 1) + C$$

  - An antiderivative is $F(x) = x(\log x - 1)$.
- **Step 4: Evaluate the Definite Integral and Compute the Limit**
  - Use the antiderivative to evaluate the definite integral inside the limit (Fundamental Theorem of Calculus / Torricelli-Barrow):

$$\int_\epsilon^1 \log x\,dx = [x(\log x - 1)]_\epsilon^1 = (1(\log 1 - 1)) - (\epsilon(\log \epsilon - 1))$$

$$= (1(0 - 1)) - (\epsilon \log \epsilon - \epsilon) = -1 - \epsilon \log \epsilon + \epsilon$$

  - Now, compute the limit as $\epsilon \to 0^+$:

$$\lim_{\epsilon \to 0^+} (-1 - \epsilon \log \epsilon + \epsilon)$$

  - Use the standard limit $\lim_{\epsilon \to 0^+} \epsilon \log \epsilon = 0$.

$$= -1 - 0 + 0 = -1$$

- **Conclusion**
  - Since the limit exists and is finite, the improper integral converges.
  - The value of the integral is $\int_0^1 \log x\,dx = -1$.

Integration using Weierstrass Substitution (Tangent Half-Angle)

Key Question: How to calculate the integral $I = \int \frac{\tan x}{\sin x + \tan x} dx$?

- **Step 1: Simplify the Integrand**
  - Rewrite $\tan x$ as $\sin x / \cos x$:

$$\frac{\tan x}{\sin x + \tan x} = \frac{\frac{\sin x}{\cos x}}{\sin x + \frac{\sin x}{\cos x}}$$

  - Find a common denominator in the denominator:

$$= \frac{\frac{\sin x}{\cos x}}{\frac{\sin x \cos x + \sin x}{\cos x}}$$

  - Simplify (assuming $\sin x \neq 0, \cos x \neq 0$):

$$= \frac{\sin x}{\sin x \cos x + \sin x} = \frac{\sin x}{\sin x(\cos x + 1)} = \frac{1}{\cos x + 1}$$

  - The integral becomes significantly simpler: $I = \int \frac{1}{\cos x + 1} dx$.
- **Step 2: Apply the Weierstrass Substitution ($t = \tan(x/2)$)**
  - **Motivation:** This substitution transforms rational functions of $\sin x$ and $\cos x$ into rational functions of t, which can often be integrated using standard techniques (like partial fractions, although not needed here).
  - **The Substitution Formulas:**
    - Let $\boxed{t = \tan\left(\frac{x}{2}\right)}$
    - Then $\cos x = \frac{1-t^2}{1+t^2}$
    - And $dx = \frac{2}{1+t^2} dt$
    - (Also $\sin x = \frac{2t}{1+t^2}$, though not required for this specific simplified integral).
- **Step 3: Substitute into the Simplified Integral**
  - Replace $\cos x$ and dx with their t-equivalents:

$$I = \int \underbrace{\frac{1}{\left(\frac{1-t^2}{1+t^2}\right) + 1}}_{\frac{1}{\cos x + 1}} \cdot \underbrace{\left(\frac{2}{1+t^2}\right) dt}_{dx}$$

  - Simplify the first part (the integrand in terms of x expressed in t):

$$\frac{1}{\frac{1-t^2+1+t^2}{1+t^2}} = \frac{1}{\frac{2}{1+t^2}} = \frac{1+t^2}{2}$$

  - Substitute this back into the integral expression:

$$I = \int \left(\frac{1+t^2}{2}\right) \cdot \left(\frac{2}{1+t^2}\right) dt$$

- **Step 4: Evaluate the Integral in t**

- The integrand simplifies dramatically:

$$I = \int 1 \, dt$$

- Compute the straightforward integral:

$$I = t + C$$

- **Step 5: Substitute Back to x**
  - Replace t with its definition in terms of x:

$$I = \tan\left(\frac{x}{2}\right) + C$$

- **Result:** The integral evaluates to $\int \frac{\tan x}{\sin x + \tan x} dx = \tan\left(\frac{x}{2}\right) + C$.

Convergence of Improper Integrals with Oscillating Numerators

Key Question: How to discuss the convergence of the improper integral $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$?

- **Step 1: Identify Potential Issues**
  - **Infinite Interval:** The upper limit is $+\infty$, making it an improper integral of the first kind.
  - **Sign Changes:** The numerator $\cos x$ oscillates between -1 and 1, causing the integrand $f(x) = \frac{\cos x}{1+x^2}$ to change sign infinitely often.
  - **Problem with Direct Comparison:** Standard comparison tests (Direct Comparison, Limit Comparison) require the integrand to be non-negative (at least eventually), which is not the case here.
- **Step 2: Use Absolute Convergence**
  - **Strategy:** Test the convergence of the integral of the absolute value of the integrand:

$$I_{abs} = \int_0^\infty \left| \frac{\cos x}{1+x^2} \right| dx = \int_0^\infty \frac{|\cos x|}{1+x^2} dx$$

  - **Absolute Convergence Theorem:** If I_{abs} converges, then the original integral I also converges.
  - The new integrand $\frac{|\cos x|}{1+x^2}$ is now non-negative, allowing the use of comparison tests.
- **Step 3: Apply Direct Comparison Test to I_{abs}**
  - **Establish Inequality:** Use the property $|\cos x| \leq 1$ for all x:

$$0 \leq \frac{|\cos x|}{1+x^2} \leq \frac{1}{1+x^2} \quad \text{for all } x \geq 0$$

  - **Analyze the Comparison Integral:** Consider the integral $J = \int_0^\infty \frac{1}{1+x^2} dx$.
  - **Convergence of J (Method 1: Direct Integration):**

$$J = \int_0^\infty \frac{1}{1+x^2} dx = [\arctan x]_0^\infty = \lim_{b\to\infty} (\arctan b) - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Since the result is finite, J converges.

- **Convergence of J (Method 2: Splitting & Asymptotic Comparison - Important General Technique):**
  - Split the integral: $J = \int_0^1 \frac{1}{1+x^2} dx + \int_1^\infty \frac{1}{1+x^2} dx$.
  - The integral $\int_0^1 \frac{1}{1+x^2} dx$ is proper and finite (integrand is continuous on $[0, 1]$).
  - For $\int_1^\infty \frac{1}{1+x^2} dx$, compare with $\int_1^\infty \frac{1}{x^2} dx$. Since $0 < \frac{1}{1+x^2} \leq \frac{1}{x^2}$ for $x \geq 1$ and $\int_1^\infty \frac{1}{x^2} dx$ converges (p-integral with $p = 2 > 1$), the integral $\int_1^\infty \frac{1}{1+x^2} dx$ converges by Direct Comparison.
  - Since both parts converge, J converges. (Note: Splitting is crucial when dealing with multiple improprieties or when asymptotic comparison is needed at different points).
- **Conclusion of Comparison Test:** Since $0 \leq \frac{|\cos x|}{1+x^2} \leq \frac{1}{1+x^2}$ and $\int_0^\infty \frac{1}{1+x^2} dx$ converges, by the Direct Comparison Test, $I_{abs} = \int_0^\infty \frac{|\cos x|}{1+x^2} dx$ must also converge.

- **Step 4: Final Conclusion**
  - We have shown that $I_{abs} = \int_0^\infty \left| \frac{\cos x}{1+x^2} \right| dx$ converges.
  - By the theorem on absolute convergence, if an integral converges absolutely, it also converges simply.
  - Therefore, the original integral $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$ converges.

Determining Improper Integral Convergence using Asymptotic Comparison

Key Question: How can we determine if an improper integral converges when the integrand is complex, especially near a point of discontinuity?

- **Problem Setup:** Consider the integral:

$$I = \int_0^1 f(x) dx = \int_0^1 \frac{\sqrt{x(x^2+1)}e^{2x}}{\tan \sqrt[3]{x}} dx$$

  The potential problem (discontinuity) is at $x = 0$.

- **Check Conditions for Comparison Test:**
  - The function $f(x)$ must be continuous on $(0, 1]$.
  - The function $f(x)$ must be non-negative on $(0, 1]$.
  - In this case, $f(x)$ is non-negative for $x \in (0, 1]$.
- **Apply Asymptotic Comparison Test (Limit Comparison Test):** We need to find a simpler function $g(x)$ such that $f(x) \sim g(x)$ as $x \to 0^+$. The convergence of $\int_0^1 f(x) dx$ will be the same as the convergence of $\int_0^1 g(x) dx$.
  - **Analyze numerator as $x \to 0^+$:**

$$\sqrt{x(x^2+1)} = \sqrt{x^3+x} \sim \sqrt{x} = x^{1/2}$$

$$e^{2x} \sim e^0 = 1$$

    So, Numerator $\sim x^{1/2}$.

- **Analyze denominator as $x \to 0^+$:**

$$\tan\sqrt{x} \sim \sqrt{x} = x^{1/2}$$

$$\tan\sqrt[3]{x} \sim x^{1/3} = x^{1/3}$$

- **Determine asymptotic behavior of $f(x)$:**

$$f(x) \sim \frac{\sqrt{x}}{\sqrt[3]{x}} = \frac{x^{1/2}}{x^{1/3}} = x^{1/2-1/3} = x^{1/6}$$

So, we use $g(x) = x^{1/6}$.

- **Analyze the Comparison Integral:** We examine the convergence of $\int_0^1 g(x)dx$:

$$\int_0^1 x^{1/6} dx = \int_0^1 \frac{1}{x^{-1/6}} dx$$

This is a generalized p-integral $\int_0^1 \frac{1}{x^p} dx$. It converges if and only if $p < 1$. In our case, $p = -1/6$. Since $p = -1/6 < 1$, the comparison integral $\int_0^1 x^{1/6} dx$ converges.

- **Conclusion:** Since $f(x) \sim x^{1/6}$ as $x \to 0^+$ and $\int_0^1 x^{1/6} dx$ converges, by the Limit Comparison Test, the original integral

$$\int_0^1 \frac{\sqrt{x(x^2+1)}e^{2x}}{\tan\sqrt[3]{x}} dx \quad \textbf{converges.}$$

Convergence of Improper Integrals with Parameters using Taylor Expansion

Key Question: How can we determine the values of a parameter α for which an improper integral converges, especially when the integrand's behavior near the problematic point requires Taylor series?

- **Problem Setup:** Consider the integral dependent on α:

$$I(\alpha) = \int_0^1 \frac{3}{[2(x - \log(1+x))]^{3-3\alpha}} dx$$

We need to find the values of α for which this integral converges. The potential problem point is $x = 0$.

- **Asymptotic Analysis near $x = 0$ using Taylor Expansion:** To understand the behavior of the integrand $f(x, \alpha) = \frac{3}{[2(x-\log(1+x))]^{3-3\alpha}}$ as $x \to 0^+$, we need the Taylor expansion of $\log(1+x)$ around $x = 0$.

  - **Recall Taylor expansion:**

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = x - \frac{x^2}{2} + o(x^2) \quad \text{as } x \to 0$$

  - **Analyze the term $x - \log(1+x)$:**

$$x - \log(1+x) = x - \left(x - \frac{x^2}{2} + o(x^2)\right) = \frac{x^2}{2} + o(x^2)$$

Therefore, $x - \log(1+x) \sim \frac{x^2}{2}$ as $x \to 0^+$.

- **Analyze the base of the power in the denominator:**

$$2(x - \log(1 + x)) \sim 2\left(\frac{x^2}{2}\right) = x^2 \quad \text{as } x \to 0^+$$

- **Determine the asymptotic behavior of the integrand $f(x, \alpha)$:**

$$f(x, \alpha) = \frac{3}{[2(x - \log(1 + x))]^{3-3\alpha}} \sim \frac{3}{[x^2]^{3-3\alpha}} = \frac{3}{x^{6-6\alpha}} \quad \text{as } x \to 0^+$$

- **Apply Asymptotic Comparison Test:** The convergence of $I(\alpha)$ is equivalent to the convergence of the comparison integral:

$$\int_0^1 \frac{3}{x^{6-6\alpha}} dx = 3 \int_0^1 \frac{1}{x^{6-6\alpha}} dx$$

- **Analyze the Comparison Integral (Generalized p-integral):** The integral $\int_0^1 \frac{1}{x^p} dx$ converges if and only if $p < 1$. In our case, the exponent is $p = 6 - 6\alpha$.
- **Determine the Condition for Convergence:** For convergence, we require $p < 1$:

$$6 - 6\alpha < 1$$

$$5 < 6\alpha$$
$$\alpha > \frac{5}{6}$$

- **Conclusion:** Based on the asymptotic comparison test and the condition for p-integral convergence, the original integral

$$\int_0^1 \frac{3}{[2(x - \log(1 + x))]^{3-3\alpha}} dx \quad \textbf{converges if and only if } \alpha > \frac{5}{6}.$$

2 Differential Equations

Sample Cascade Scheme: Solving a First-Order Linear Differential Equation

First-Order Linear Differential Equation

Key Question: How do you solve $y' + p(x)y = q(x)$?

- **Find the integrating factor**
    - $\mu(x) = e^{\int p(x)dx}$
- **Multiply both sides by** $\mu(x)$
- **Rewrite the left side as the derivative of a product**
- **Integrate both sides**
- **Isolate the general solution**

3 Appendix: Taylor/Maclaurin Polynomials and Standard Limits

3.1 Taylor and Maclaurin Polynomials

General Taylor and Maclaurin Formulas

Taylor Polynomial of order n for $f(x)$ at $x = a$:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Polynomial: Taylor polynomial at $a = 0$:

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Remainder (Peano form):

$$R_n(x) = o((x-a)^n) \quad \text{as } x \to a$$

Maclaurin Series for Major Functions

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Tangent (first terms):

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + o(x^7)$$

Arctangent:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Natural Logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1$$

Binomial Series: For $|x| < 1$, $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Inverse:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Examples: Maclaurin Polynomials (Order 3 or 4)

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
- $\tan x = x + \frac{x^3}{3} + o(x^3)$
- $\arctan x = x - \frac{x^3}{3} + o(x^3)$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
- $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + o(x^3)$

3.2 Standard Limits

Standard Limits as $x \to 0$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ (a > 0)$
- $\lim_{x\to 0} \frac{(1 + x)^\alpha - 1}{x} = \alpha \ (\alpha \in \mathbb{R})$
- $\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$
- $\lim_{x\to 0} \frac{x}{\ln(1 + x)} = 1$
- $\lim_{x\to 0} \frac{\sin(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\arctan(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\ln(1 + \sin x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a$

- $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n \ (n \in \mathbb{N})$

Standard Limits as $x \to \infty$

- $\lim_{x \to \infty} \frac{\ln x}{x} = 0$
- $\lim_{x \to \infty} \frac{x^a}{e^x} = 0$ for any $a > 0$
- $\lim_{x \to \infty} \frac{e^{ax}}{x^b} = \begin{cases} 0 & a < 0 \\ \infty & a > 0 \end{cases}$
- $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$
- $\lim_{x \to \infty} x^{1/x} = 1$
- $\lim_{x \to \infty} \frac{\ln x}{x^a} = 0$ for $a > 0$
- $\lim_{x \to \infty} \frac{x^a}{\ln x} = \infty$ for $a > 0$
- $\lim_{x \to \infty} \frac{x^a}{x^b} = \begin{cases} 0 & a < b \\ 1 & a = b \\ \infty & a > b \end{cases}$
- $\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$
- $\lim_{x \to \infty} \frac{a^x}{b^x} = 0$ if $0 < a < b$

Other Useful Limits

- $\lim_{x \to 0} (1+x)^{1/x} = e$
- $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$
- $\lim_{x \to 0} \frac{\ln(1+\sin x)}{x} = 1$
- $\lim_{x \to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$

These formulas and limits are fundamental tools for solving calculus problems, especially for evaluating limits, approximating functions, and analyzing local behavior.