Cascade Schemes for Calculus Exercises

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Introduction

This document collects cascade schemes for exercises, techniques, and fundamental concepts in calculus, organized by chapter. Each scheme follows a hierarchical and synthetic model to facilitate quick consultation and memorization.

1 Integrals

In this section we will explore some types of exercises that can be asked in a typical calculus exam. Specifically we will explore two categories:

- Integral function exercises
- Improper integral exercises

1.1 Integral function exercises

Computing the derivative of a composition of the integral function

Key Question: How can we compute the derivative of a function that is the composition of the integral function with another function?

- **Remember the derivative formula for composite functions**
  - $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- **Consider Torricelli's Theorem**
  - $F(x) = \int_a^x f(t)dt$
  - $F'(x) = f(x)$
- **Example**
  - $G(x) = \int_a^{m(x)} f(t)\, dt = F(m(x))$
  - $G'(x) = F'(m(x)) \cdot m'(x) = f(m(x)) \cdot m'(x)$

Computing the derivative of an integral with variable limits

Key Question: How can we compute the derivative of a function defined by an integral where both limits depend on x, i.e., $G(x) = \int_{n(x)}^{m(x)} f(t)dt$?

1. Split the integral using a constant lower limit

- Use the property of definite integrals: $\int_a^b f(t)dt = \int_c^b f(t)dt - \int_c^a f(t)dt$.
- Apply this to $G(x)$: Choose an arbitrary constant a (often 0 or 1) in the domain of f.

$$G(x) = \int_{n(x)}^{m(x)} f(t)dt = \int_a^{m(x)} f(t)dt - \int_a^{n(x)} f(t)dt$$

2. Define an auxiliary function using Torricelli's Theorem

- Let $F(x) = \int_a^x f(t)dt$.
- By the Fundamental Theorem of Calculus (Part 1 / Torricelli's Theorem), $F'(x) = f(x)$.
- Rewrite $G(x)$ using F: $G(x) = F(m(x)) - F(n(x))$.

3. Differentiate using the Chain Rule

- Apply the chain rule to differentiate $F(m(x))$ and $F(n(x))$:

$$(F(h(x)))' = F'(h(x)) \cdot h'(x) = f(h(x)) \cdot h'(x)$$

- Therefore, the derivative of $G(x)$ is:

$$G'(x) = \frac{d}{dx}[F(m(x))] - \frac{d}{dx}[F(n(x))]$$

$$G'(x) = F'(m(x)) \cdot m'(x) - F'(n(x)) \cdot n'(x)$$

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

4. Example

- Compute the derivative of $G(x) = \int_x^{2x} \frac{\sin t}{t} dt$.
- Here, $f(t) = \frac{\sin t}{t}$, $m(x) = 2x$, and $n(x) = x$.
- Choose $a = 1$ (any constant works). Let $F(x) = \int_1^x \frac{\sin t}{t} dt$.
- Then $G(x) = F(2x) - F(x)$.
- The derivatives are $m'(x) = 2$ and $n'(x) = 1$.
- Applying the formula:

$$G'(x) = f(m(x)) \cdot m'(x) - f(n(x)) \cdot n'(x)$$

$$G'(x) = f(2x) \cdot 2 - f(x) \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{2x} \cdot 2 - \frac{\sin x}{x} \cdot 1$$

$$G'(x) = \frac{\sin(2x)}{x} - \frac{\sin x}{x} = \frac{\sin(2x) - \sin x}{x}$$

Computing limits involving integrals and indeterminate forms

Key Question: How to compute the limit:

$$\lim_{x \to 0} \frac{x - \int_0^x (e^{-t^2} + \sin^2 t) dt}{x(x^2 - \sin^2 x)}$$

- **Initial Check: Identify Indeterminate Form**
  - Substitute $x = 0$ into the numerator: $0 - \int_0^0 (e^{-t^2} + \sin^2 t) dt = 0 - 0 = 0$.
  - Substitute $x = 0$ into the denominator: $0 \cdot (0^2 - \sin^2 0) = 0 \cdot (0 - 0) = 0$.
  - The limit presents the indeterminate form $\left[\frac{0}{0}\right]$.
- **Apply L'Hôpital's Rule**
  - Differentiate the numerator with respect to x. Requires the Fundamental Theorem of Calculus (Torricelli-Barrow) for the integral part:

$$\frac{d}{dx}\left(x - \int_0^x (e^{-t^2} + \sin^2 t) dt\right) = 1 - (e^{-x^2} + \sin^2 x)$$

- Differentiate the denominator with respect to x using the product rule:

$$\frac{d}{dx}(x(x^2 - \sin^2 x)) = (x^2 - \sin^2 x) + (2x^2 - x\sin(2x))$$

- The limit becomes (applying L'Hôpital's Rule):

$$L \stackrel{H}{=} \lim_{x\to 0} \frac{1 - e^{-x^2} - \sin^2 x}{(x^2 - \sin^2 x) + (2x^2 - x\sin(2x))}$$

- **Evaluate the New Limit (Using Taylor Series Approximation)**
  - The limit is still in the form $\left[\frac{0}{0}\right]$. Further application of L'Hôpital's Rule is possible, but Taylor expansions are often more efficient.
  - Approximate numerator and denominator for $x \to 0$:
    - $1 - e^{-x^2} - \sin^2 x = \frac{5}{6}x^4 + o(x^4)$
    - $(x^2 - \sin^2 x) + (2x^2 - x\sin(2x)) = \frac{5}{3}x^4 + o(x^4)$

    (Note: These approximations are taken from the image provided).
  - Substitute the approximations back into the limit:

$$L = \lim_{x\to 0} \frac{\frac{5}{6}x^4 + o(x^4)}{\frac{5}{3}x^4 + o(x^4)} = \frac{5/6}{5/3} = \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}$$

- **Result:** The limit exists and its value is $\frac{1}{2}$.

Computing Maclaurin Polynomials for Composite Functions with Integrals

Key Question: How to determine the Maclaurin polynomial for $F(x) = \sin\left(\int_0^x e^{-t^2} dt\right)$?

- **Define Auxiliary Function for the Integral**
  - Let $G(x) = \int_0^x e^{-t^2} dt$.
  - The original function becomes $F(x) = \sin(G(x))$.
- **Find the Maclaurin Series for the Auxiliary Function** $G(x)$
  - Calculate derivatives of $G(x)$ and evaluate at $x = 0$:
    - $G(0) = \int_0^0 e^{-t^2} dt = 0$
    - $G'(x) = e^{-x^2}$ (by Fundamental Theorem of Calculus) $\to G'(0) = e^0 = 1$
    - $G''(x) = -2xe^{-x^2} \to G''(0) = 0$
    - $G'''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \to G'''(0) = -2$
  - Construct the Maclaurin series for $G(x)$ using $G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} x^n$:

$$G(x) = G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + \frac{G'''(0)}{3!}x^3 + o(x^3)$$

$$G(x) = 0 + 1 \cdot x + \frac{0}{2}x^2 + \frac{-2}{6}x^3 + o(x^3) = x - \frac{1}{3}x^3 + o(x^3)$$

- **Use the Known Maclaurin Series for the Outer Function** (sin)

- Recall the Maclaurin series for $\sin(z)$:

$$\sin(z) = z - \frac{z^3}{6} + o(z^3)$$

- **Substitute the Series for $G(x)$ into the Series for $\sin(z)$**
  - Replace z with $G(x) = x - \frac{1}{3}x^3 + o(x^3)$:

$$F(x) = \sin(G(x)) = \left(x - \frac{1}{3}x^3 + o(x^3)\right) - \frac{1}{6}\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 + o\left((G(x))^3\right)$$

  - Since $G(x) \approx x$ for $x \to 0$, we have $o((G(x))^3) = o(x^3)$.
  - Expand and keep terms up to x^3:

$$\left(x - \frac{1}{3}x^3 + o(x^3)\right)^3 = (x + o(x))^3 = x^3 + o(x^3)$$

  - Substitute back:

$$F(x) = \left(x - \frac{1}{3}x^3\right) - \frac{1}{6}(x^3) + o(x^3)$$

$$F(x) = x - \left(\frac{1}{3} + \frac{1}{6}\right)x^3 + o(x^3) = x - \frac{2+1}{6}x^3 + o(x^3) = x - \frac{3}{6}x^3 + o(x^3)$$

$$F(x) = x - \frac{1}{2}x^3 + o(x^3)$$

- **Result: Maclaurin Polynomial**
  - The Maclaurin polynomial of order 3 for $F(x)$ is $P_3(x) = x - \frac{1}{2}x^3$.

Proving Existence and Uniqueness of Solutions Involving Integral Functions

Key Question: How to prove that the equation $f(x) = 1 - x$, where $f(x) = \int_0^x e^{-t^2}dt$, has a unique solution?

- **Reformulate the Problem**
  - Define a new function $g(x) = f(x) - (1 - x) = f(x) + x - 1$.
  - The original problem is equivalent to proving that $g(x) = 0$ has exactly one solution (a unique zero).
- **Prove Existence of a Zero (Intermediate Value Theorem)**
  - **Continuity:** $f(x)$ is continuous because it's an integral function of a continuous integrand (e^{-t^2}). The term $x-1$ is also continuous. Therefore, $g(x)$ is continuous on $\mathbb{R}$.
  - **Find points with opposite signs:**
    - Evaluate $g(0)$: $g(0) = f(0) + 0 - 1 = \int_0^0 e^{-t^2}dt - 1 = 0 - 1 = -1$. So, $g(0) < 0$.
    - Evaluate the limit as $x \to +\infty$:

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \left(\int_0^x e^{-t^2}dt + x - 1\right)$$

The integral $\int_0^\infty e^{-t^2} dt$ converges to a finite value ($\frac{\sqrt{\pi}}{2}$). The term x goes to $+\infty$.

$$\lim_{x\to+\infty} g(x) = \left(\lim_{x\to+\infty} \int_0^x e^{-t^2} dt\right) + \left(\lim_{x\to+\infty} x\right) - 1 = \frac{\sqrt{\pi}}{2} + \infty - 1 = +\infty$$

- Since the limit is $+\infty$, there must exist some value $\bar{x}$ such that for all $x > \bar{x}$, $g(x) > 0$. Let's pick one such value $\bar{x}$.

- **Apply IVT:** Since $g(x)$ is continuous on $[0, \bar{x}]$, $g(0) < 0$, and $g(\bar{x}) > 0$, the Intermediate Value Theorem guarantees that there exists at least one $x_0 \in (0, \bar{x})$ such that $g(x_0) = 0$.

- **Prove Uniqueness of the Zero (Monotonicity)**
  - **Calculate the derivative $g'(x)$:**

$$g'(x) = \frac{d}{dx}(f(x) + x - 1) = f'(x) + 1$$

  - Apply the Fundamental Theorem of Calculus to find $f'(x)$:

$$f'(x) = \frac{d}{dx}\left(\int_0^x e^{-t^2} dt\right) = e^{-x^2}$$

  - Substitute back into $g'(x)$:

$$g'(x) = e^{-x^2} + 1$$

  - **Analyze the sign of $g'(x)$:** Since $e^{-x^2} > 0$ for all real x, we have $g'(x) = e^{-x^2} + 1 > 0 + 1 = 1$.
  - Since $g'(x) > 0$ for all $x \in \mathbb{R}$, the function $g(x)$ is strictly increasing on its entire domain.

- **Conclusion**
  - We have shown that $g(x) = 0$ has at least one solution (by IVT).
  - We have shown that $g(x)$ is strictly increasing, which means it can cross the x-axis (equal zero) at most once.
  - Therefore, the equation $g(x) = 0$ has exactly one unique solution. This implies the original equation $f(x) = 1 - x$ also has a unique solution.

Finding Order and Principal Part for Functions with Integrals (Derivative Method)

Key Question: Determine the order and principal part of $G(x) = x - \int_0^x e^{-(t+x)^2} dt$ as $x \to 0$.

- **Strategy: Analyze the Derivative $G'(x)$**
  - It's often easier to find the Maclaurin expansion (order/principal part) of the derivative first, and then integrate.
  - **Simplify the integral (Optional Variable Change):** Let $z = t+x$, $dz = dt$.

Limits change from $t \in [0, x]$ to $z \in [x, 2x]$.

$$G(x) = x - \int_x^{2x} e^{-z^2} dz$$

- **Rewrite using a fixed lower limit:** Let $H(x) = \int_0^x e^{-t^2} dt$. Then $\int_x^{2x} e^{-z^2} dz = H(2x) - H(x)$.

$$G(x) = x - (H(2x) - H(x)) = x - H(2x) + H(x)$$

- **Compute the Derivative $G'(x)$**
  - Apply differentiation rules, including the Fundamental Theorem of Calculus and Chain Rule:

$$G'(x) = \frac{d}{dx}(x) - \frac{d}{dx}(H(2x)) + \frac{d}{dx}(H(x))$$
$$G'(x) = 1 - (H'(2x) \cdot 2) + H'(x)$$

Since $H'(x) = e^{-x^2}$:

$$G'(x) = 1 - (e^{-(2x)^2} \cdot 2) + e^{-x^2} = 1 - 2e^{-4x^2} + e^{-x^2}$$

- **Find Maclaurin Expansion of $G'(x)$**
  - Use the known expansion $e^u = 1 + u + o(u)$ for $u \to 0$.
  - $e^{-4x^2} = 1 - 4x^2 + o(x^2)$
  - $e^{-x^2} = 1 - x^2 + o(x^2)$
  - Substitute into $G'(x)$:

$$G'(x) = 1 - 2(1 - 4x^2 + o(x^2)) + (1 - x^2 + o(x^2))$$
$$G'(x) = 1 - 2 + 8x^2 + o(x^2) + 1 - x^2 + o(x^2)$$
$$G'(x) = (1 - 2 + 1) + (8x^2 - x^2) + o(x^2) = 7x^2 + o(x^2)$$

  - The principal part of $G'(x)$ is $7x^2$, and its order is 2.
- **Determine Order and Principal Part of $G(x)$**
  - Since $G'(x) \sim 7x^2$ as $x \to 0$, and $G(0) = 0 - \int_0^0 \ldots = 0$, we can integrate the principal part of $G'(x)$ to find the principal part of $G(x)$:

$$G(x) \approx \int_0^x 7t^2 dt = 7 \left[\frac{t^3}{3} \right]_0^x = \frac{7}{3} x^3$$

  - The principal part of $G(x)$ is $\frac{7}{3}x^3$.
  - The order of $G(x)$ is 3.
  - Thus, $G(x) = \frac{7}{3}x^3 + o(x^3)$ for $x \to 0$.
- **Verification (Using L'Hôpital's Rule on the Remainder)**
  - To confirm $G(x) = \frac{7}{3}x^3 + o(x^3)$, we must show $\lim_{x \to 0} \frac{G(x) - \frac{7}{3}x^3}{x^3} = 0$.
  - Apply L'Hôpital's Rule (since form is $\frac{0}{0}$):

$$\lim_{x \to 0} \frac{G'(x) - \frac{d}{dx}(\frac{7}{3}x^3)}{\frac{d}{dx}(x^3)} = \lim_{x \to 0} \frac{G'(x) - 7x^2}{3x^2}$$

  - Substitute the expansion of $G'(x)$:

$$\lim_{x \to 0} \frac{(7x^2 + o(x^2)) - 7x^2}{3x^2} = \lim_{x \to 0} \frac{o(x^2)}{3x^2} = 0$$

  - The limit is 0, confirming the order and principal part.

Qualitative Graph Sketching of Integral Functions near x

Key Question: How to draw a qualitative graph of $f(x) = \int_0^x \frac{e^t}{2t^2+1} dt$ in a neighborhood of $x = 0$?

- **Strategy: Analyze Local Behavior using Derivatives at x=0**
  - The behavior of a function near a point (like $x = 0$) is determined by its value and the values of its derivatives at that point. This is the foundation of Taylor series approximations.
- **Calculate $f(0)$**
  - Substitute $x = 0$ into the integral definition:

$$f(0) = \int_0^0 \frac{e^t}{2t^2+1} dt = 0$$

  - The function passes through the origin $(0, 0)$.
- **Calculate $f'(x)$ and $f'(0)$**
  - Apply the Fundamental Theorem of Calculus (Part 1):

$$f'(x) = \frac{d}{dx}\left(\int_0^x \frac{e^t}{2t^2+1} dt\right) = \frac{e^x}{2x^2+1}$$

  - Evaluate at $x = 0$:

$$f'(0) = \frac{e^0}{2(0)^2+1} = \frac{1}{1} = 1$$

  - The slope of the tangent line at the origin is 1. The function is increasing at $x = 0$.
- **Calculate $f''(x)$ and $f''(0)$**
  - Differentiate $f'(x)$ using the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$:

$$u = e^x u' = e^x$$

$$v = 2x^2 + 1v' = 4x$$

$$f''(x) = \frac{(e^x)(2x^2+1) - (e^x)(4x)}{(2x^2+1)^2} = \frac{e^x(2x^2-4x+1)}{(2x^2+1)^2}$$

  - Evaluate at $x = 0$:

$$f''(0) = \frac{e^0(2(0)^2-4(0)+1)}{(2(0)^2+1)^2} = \frac{1(1)}{(1)^2} = 1$$

  - Since $f''(0) > 0$, the function is concave up at $x = 0$.
- **Sketching the Graph near x=0**
  - The graph passes through $(0, 0)$.
  - The tangent line at $(0, 0)$ has a slope of 1 (like the line $y = x$).
  - The graph is concave up at $(0, 0)$, meaning it lies above its tangent line near the point of tangency.
  - Combining these: Start at the origin, draw a curve that is initially tangent to $y = x$ and curves upwards (concave up).

Limit Computation using Integral Inequalities and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x\to 0^+} x \int_{x-x^2}^{x} \frac{dt}{\sin^3 t}$, if it exists.

- **Step 1: Analyze Integrand and Integration Interval**
  - The integration variable is t, and the interval is $[x - x^2, x]$.
  - As $x \to 0^+$, both x and $x - x^2 = x(1-x)$ approach 0^+. Thus, $t \to 0^+$.
  - For $t \in (0, \pi/2)$, $\sin t$ is positive and strictly increasing.
  - Consequently, $\sin^3 t$ is also positive and strictly increasing for $t \in (0, \pi/2)$.
  - Therefore, the integrand $g(t) = \frac{1}{\sin^3 t}$ is positive and strictly decreasing for t in the interval $(0, \pi/2)$.
- **Step 2: Establish Inequalities for the Integrand**
  - Since $t \in [x-x^2, x]$ and $g(t) = \frac{1}{\sin^3 t}$ is decreasing on this interval (for sufficiently small positive x), the minimum value of $g(t)$ occurs at $t = x$ and the maximum value occurs at $t = x - x^2$.
  - For $t \in [x - x^2, x]$, we have:

$$\frac{1}{\sin^3 x} \le \frac{1}{\sin^3 t} \le \frac{1}{\sin^3(x - x^2)}$$

- **Step 3: Integrate the Inequalities**
  - Integrate all parts of the inequality over the interval $[x-x^2, x]$. Since the bounds $\frac{1}{\sin^3 x}$ and $\frac{1}{\sin^3(x-x^2)}$ are constant with respect to t, we get:

$$\int_{x-x^2}^{x} \frac{1}{\sin^3 x} dt \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \int_{x-x^2}^{x} \frac{1}{\sin^3(x - x^2)} dt$$

  - The length of the integration interval is $x - (x - x^2) = x^2$.

$$\frac{1}{\sin^3 x} \cdot x^2 \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{1}{\sin^3(x - x^2)} \cdot x^2$$

$$\frac{x^2}{\sin^3 x} \le \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{x^2}{\sin^3(x - x^2)}$$

- **Step 4: Incorporate the External Factor and Compute Limits**
  - Multiply the inequality by x (note $x > 0$ as $x \to 0^+$):

$$\frac{x^3}{\sin^3 x} \le x \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \le \frac{x^3}{\sin^3(x - x^2)}$$

  - Compute the limit of the lower bound as $x \to 0^+$:

$$\lim_{x\to 0^+} \frac{x^3}{\sin^3 x} = \lim_{x\to 0^+} \left(\frac{x}{\sin x}\right)^3 = (1)^3 = 1$$

    (using the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$)
  - Compute the limit of the upper bound as $x \to 0^+$:

$$\lim_{x\to 0^+} \frac{x^3}{\sin^3(x - x^2)} = \lim_{x\to 0^+} \left(\frac{x}{\sin(x - x^2)}\right)^3$$

    We use $\sin(x - x^2) \sim x - x^2$ as $x \to 0$.

$$\lim_{x\to 0^+} \left(\frac{x}{x - x^2}\right)^3 = \lim_{x\to 0^+} \left(\frac{x}{x(1 - x)}\right)^3 = \lim_{x\to 0^+} \left(\frac{1}{1 - x}\right)^3 = \left(\frac{1}{1 - 0}\right)^3 = 1$$

- **Step 5: Apply the Squeeze Theorem**
  - We have shown:
$$\lim_{x\to 0^+} \frac{x^3}{\sin^3 x} = 1$$
$$\lim_{x\to 0^+} \frac{x^3}{\sin^3(x-x^2)} = 1$$
$$\frac{x^3}{\sin^3 x} \leq x \int_{x-x^2}^{x} \frac{1}{\sin^3 t} dt \leq \frac{x^3}{\sin^3(x-x^2)}$$
  - By the Squeeze Theorem, the limit of the middle expression must also be 1.
  - Therefore, $L = \lim_{x\to 0^+} x \int_{x-x^2}^{x} \frac{dt}{\sin^3 t} = 1$.

Limit Computation using Mean Value Theorem and Squeeze Theorem

Key Question: Calculate the limit $L = \lim_{x\to+\infty} x^3 \int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt$.

- **Step 1: Apply Mean Value Theorem for Integrals**
  - **Theorem Statement:** If f is continuous on $[a, b]$, there exists $c \in [a, b]$ such that $\int_a^b f(t)dt = f(c)(b-a)$.
  - **Application:**
    - Let $f(t) = \sin\left(\frac{1}{t^2}\right)$. This function is continuous on $[x^2, x^2+x]$ for large x (since $x^2 > 0$).
    - The interval length is $(x^2 + x) - x^2 = x$.
    - By the MVT for Integrals, there exists $z \in [x^2, x^2 + x]$ such that:
$$\int_{x^2}^{x^2+x} \sin\left(\frac{1}{t^2}\right) dt = \sin\left(\frac{1}{z^2}\right) \cdot x$$
  - **Rewrite the Limit:** Substitute this result back into the limit expression:
$$L = \lim_{x\to+\infty} x^3 \left[x \sin\left(\frac{1}{z^2}\right)\right] = \lim_{x\to+\infty} x^4 \sin\left(\frac{1}{z^2}\right)$$
where z depends on x and satisfies $x^2 \leq z \leq x^2 + x$.
- **Step 2: Establish Bounds using $z \in [x^2, x^2 + x]$**
  - From $x^2 \leq z \leq x^2 + x$, we have:
$$\frac{1}{x^2+x} \leq \frac{1}{z} \leq \frac{1}{x^2}$$
Squaring (all terms are positive for large x):
$$\frac{1}{(x^2+x)^2} \leq \frac{1}{z^2} \leq \frac{1}{x^4}$$
  - As $x \to +\infty$, all terms in the inequality approach 0^+.
  - Since $\sin u$ is an increasing function for u near 0^+, we can apply sin to the inequalities (for sufficiently large x):
$$\sin\left(\frac{1}{(x^2+x)^2}\right) \leq \sin\left(\frac{1}{z^2}\right) \leq \sin\left(\frac{1}{x^4}\right)$$

- **Step 3: Apply the Squeeze Theorem**
  - Multiply the inequality by x^4 (which is positive):

$$x^4 \sin\left(\frac{1}{(x^2+x)^2}\right) \le x^4 \sin\left(\frac{1}{z^2}\right) \le x^4 \sin\left(\frac{1}{x^4}\right)$$

  - **Limit of the Upper Bound:**

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{x^4}\right)$$

    Use the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$. Let $u = 1/x^4$. As $x \to +\infty$, $u \to 0^+$.

$$= \lim_{u\to 0^+} \frac{1}{u} \sin(u) = 1$$

  - **Limit of the Lower Bound:**
    Let $v = 1/(x^2+x)^2$. As $x \to +\infty$, $v \to 0^+$. Use $\sin v \sim v$ for $v \to 0$.

$$\lim_{x\to+\infty} x^4 \sin\left(\frac{1}{(x^2+x)^2}\right)$$

$$= \lim_{x\to+\infty} x^4 \cdot \frac{1}{(x^2+x)^2} = \lim_{x\to+\infty} \frac{x^4}{(x^2(1+1/x))^2} = \lim_{x\to+\infty} \frac{x^4}{x^4(1+1/x)^2} = 1$$

  - **Conclusion:** Since the expression $x^4 \sin(1/z^2)$ is squeezed between two functions that both tend to 1 as $x \to +\infty$, by the Squeeze Theorem (Teorema dei Carabinieri):

$$L = \lim_{x\to+\infty} x^4 \sin\left(\frac{1}{z^2}\right) = 1$$

1.2 Improper Integral exercises

1.2.1 Exercises using definition

1.2.2 Exercises using the comparison criterion

1.2.3 Exercises using the asymptotic comparison criterion

2 Multivariate Differential Calculus

Sample Cascade Scheme: Analyzing a Function of Two Variables

Analyzing a Function $f(x$

Key Question: How do you analyze a function of two variables?

- **Domain of definition**
- **Continuity and differentiability**
- **Compute gradients**
- **Find critical points**
  - Solve $\nabla f = 0$
- **Classify critical points**
  - Hessian matrix
  - Maximum, minimum, saddle points
- **Constrained extrema (if present)**
  - Lagrange multipliers method

3 Differential Equations

Sample Cascade Scheme: Solving a First-Order Linear Differential Equation

First-Order Linear Differential Equation

Key Question: How do you solve $y' + p(x)y = q(x)$?

- **Find the integrating factor**
  - $\mu(x) = e^{\int p(x)dx}$
- **Multiply both sides by** $\mu(x)$
- **Rewrite the left side as the derivative of a product**
- **Integrate both sides**
- **Isolate the general solution**

Appendix: Taylor/Maclaurin Polynomials and Standard Limits

Taylor and Maclaurin Polynomials

General Taylor and Maclaurin Formulas

Taylor Polynomial of order n for $f(x)$ at $x = a$:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Polynomial: Taylor polynomial at $a = 0$:

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Remainder (Lagrange form):

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \quad for some \xi between a and x$$

Maclaurin Series for Major Functions

Exponential:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Cosine:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Tangent (first terms):

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + o(x^7)$$

Arctangent:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Natural Logarithm:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1$$

Binomial Series: For $|x| < 1$, $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Inverse:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Examples: Maclaurin Polynomials (Order 3 or 4)

- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
- $\tan x = x + \frac{x^3}{3} + o(x^3)$
- $\arctan x = x - \frac{x^3}{3} + o(x^3)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + o(x^3)$

Standard Limits

Standard Limits as $x \to 0$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\arcsin x}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ (a > 0)$
- $\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \ (\alpha \in \mathbb{R})$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{x}{\ln(1+x)} = 1$
- $\lim_{x\to 0} \frac{\sin(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\arctan(ax)}{x} = a$
- $\lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^{ax} - 1}{x} = a$

- $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n \ (n \in \mathbb{N})$

Standard Limits as $x \to \infty$

- $\lim_{x\to\infty} \frac{\ln x}{x} = 0$
- $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$ for any $a > 0$
- $\lim_{x\to\infty} \frac{e^{ax}}{x^b} = \{ 0 \ a < 0$
  $\infty a > 0$
- $\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a$
- $\lim_{x\to\infty} x^{1/x} = 1$
- $\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$ for $a > 0$
- $\lim_{x\to\infty} \frac{x^a}{\ln x} = \infty$ for $a > 0$
- $\lim_{x\to\infty} \frac{x^a}{x^b} = \{ 0 \ a < b$
  $1 a = b$
  $\infty a > b$
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$
- $\lim_{x\to\infty} \frac{a^x}{b^x} = 0$ if $0 < a < b$

Other Useful Limits

- $\lim_{x\to 0} (1+x)^{1/x} = e$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$
- $\lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = 1$
- $\lim_{x\to 0} \frac{\arctan x}{x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1$

These formulas and limits are fundamental tools for solving calculus problems, especially for evaluating limits, approximating functions, and analyzing local behavior.