1 Derivation of the Integrator Model

1.1 Assumptions

1. External accelerations can be directly controlled:

$$u = \begin{bmatrix} a_{x,b}, & a_{y,b}, & a_\psi \end{bmatrix}, \quad a_b = \begin{bmatrix} a_{x,b}, & a_{y,b} \end{bmatrix} \tag{1.1}$$

2. Orientation of the vehicle ψ equals the angle of the road θ:

$$\xi = \psi - \theta = 0 \tag{1.2}$$

which implies:

- $a_b = a_t$
- $\dot{\psi} = \dot{\theta} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = C(s)\dot{s}$
- $a_\psi = \ddot{\psi} = \ddot{\theta} = C'(s)\dot{s}^2 + C(s)\ddot{s}$

3. $C'(s) = C'$ is constant.

1.2 Further Simplification

1. Define artificial input variables:

$$\tilde{u} := \begin{bmatrix} u_t \\ u_n \end{bmatrix} = \begin{bmatrix} \dfrac{a_{x,tn} + 2\dot{n}C(s)\dot{s} + nC'(s)\dot{s}^2}{1 - nC(s)} \\ a_{y,tn} - C(s)\dot{s}^2(1 - nC(s)) \end{bmatrix} \tag{1.3}$$

1.3 Resulting Integrator Model

State:

$$x_{tn} = \begin{bmatrix} s, & n, & \dot{s}, & \dot{n} \end{bmatrix}$$

Input:

$$\tilde{u} = \begin{bmatrix} u_t, & u_n \end{bmatrix}$$

The integrator model is given by:

$$\dot{x}_{tn} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_{tn} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{u} \tag{1.4}$$

1.4 Constraints

The constraints are defined by:

$$g(x_{tn}, \tilde{u}) := \begin{bmatrix} (1 - nC(s))u_t - (2\dot{n}C(s)\dot{s} + nC'\dot{s}^2) \\ u_n + C(s)\dot{s}^2(1 - nC(s)) \end{bmatrix} \tag{1.5}$$

Define the constraint set $\mathcal{Z}$ as:

$$\mathcal{Z} = \left\{ \begin{bmatrix} x_{tn} \\ \tilde{u} \end{bmatrix} \middle| \begin{array}{l} C_{min} \leq C(s) \leq C_{max}, \\ n_{min} \leq n \leq n_{max}, \\ \begin{bmatrix} v_{xmin} \\ v_{ymin} \end{bmatrix} \leq \begin{bmatrix} \dot{s}(1 + nC(s)) \\ \dot{n} \end{bmatrix} \leq \begin{bmatrix} v_{ymax} \\ v_{xmax} \end{bmatrix}, \\ \dot{\psi}_{min} \leq C(s)\dot{s} \leq \dot{\psi}_{max}, \\ a_{\psi,b,min} \leq C'\dot{s}^2 + C(s)u_t \leq a_{\psi,b,max}, \\ a_{b,min} \leq g(x_{tn}, \tilde{u}) \leq a_{b,max}, \\ ||g(x_{tn}, \tilde{u})||^2 \leq \text{const} \end{array} \right\} \tag{1.6}$$

1.5 Constraint Approximation Problem

Given $\mathcal{Z}$, find an inner approximation $\underline{\mathcal{Z}}$ of $\mathcal{Z}$ such that $\underline{\mathcal{Z}}$ can be described with a set of constraints following the Disciplined Convex Programming (DCP) rules:

- affine == affine
- convex $\leq$ concave
- concave $\geq$ convex

1.6 $\forall$-Elimination

For a constraint not following the DCP rules, of the form:

$$c_{min} \leq f(x, y) \leq c_{max} \tag{1.7}$$

with $x \in \mathbb{R}$, $y \in \mathbb{R}^n$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}$, where $c_{min}, c_{max} \in \mathbb{R}$ are constants. Further, if f is affine in x, represented by:

$$f(x, y) = a(y)x + b(y) \tag{1.8}$$

with $a, b : \mathbb{R}^n \to \mathbb{R}$, bounds on $a(y)$ and $b(y)$ can be chosen:

$$a_{min} \leq a(y) \leq a_{max}, \quad b_{min} \leq b(y) \leq b_{max}$$

Thus, an inner approximation of the set Z defined by $c_{min} \leq f(x, y) \leq c_{max}$ can be given by:

$$\begin{aligned}&\underline{Z} = \\ &\{x \in \mathbb{R} \mid \forall y \in \mathbb{R}^n : a(y) \in [a_{min}, a_{max}] \wedge b(y) \in [b_{min}, b_{max}] \implies f(x, y) \in [c_{min}, c_{max}]\} \\ &\times \{y \in \mathbb{R}^n \mid a(y) \in [a_{min}, a_{max}] \wedge b(y) \in [b_{min}, b_{max}]\} \\ &=: X \times Y\end{aligned} \tag{1.9}$$

1.6.1 Calculate X

Assumptions:

$$c_{min} \leq b_{min} \text{ and } b_{max} \leq c_{max} \text{ (or } a(y) \neq 0 \text{ TODO)} \tag{1.10}$$

Definitions:

$$x_{min} := \max\left\{\min\left\{0, \frac{c_{min} - b_{min}}{a_{max}}\right\}, \min\left\{0, \frac{c_{max} - b_{max}}{a_{min}}\right\}\right\} \tag{1.11}$$

$$x_{max} := \min\left\{\max\left\{0, \frac{c_{max} - b_{max}}{a_{max}}\right\}, \max\left\{0, \frac{c_{min} - b_{min}}{a_{min}}\right\}\right\} \tag{1.12}$$

Claim:

$$X = [x_{min}, x_{max}] \tag{1.13}$$

Sub-Claim:

$$x_{min} < 0 < x_{max} \tag{1.14}$$

Proof of Claim (1.13)**:**

Let $x \in X$.

Case Distinction for x_{min}:

- **Case 1:** $x_{min} = \dfrac{c_{min} - b_{min}}{a_{max}}$

$$a_{max} x_{min} + b_{min} = c_{min} \leq a_{max} x + b_{min} \implies x_{min} \leq x \tag{1.15}$$

- **Case 2:** $x_{min} = \dfrac{c_{max} - b_{max}}{a_{min}}$

$$a_{min} x_{min} + b_{max} = c_{max} \geq a_{min} x + b_{max} \implies x_{min} \leq x \tag{1.16}$$

Case Distinction for x_{max}:

- **Case 1:** $x_{max} = \dfrac{c_{max} - b_{max}}{a_{max}}$

$$a_{max} x_{max} + b_{max} = c_{max} \geq a_{max} x + b_{max} \implies x_{max} \geq x \tag{1.17}$$

- **Case 2:** $x_{max} = \dfrac{c_{min} - b_{min}}{a_{min}}$

$$a_{min} x_{max} + b_{min} = c_{min} \leq a_{min} x + b_{min} \implies x_{max} \geq x \tag{1.18}$$

Therefore, we have:

$$x_{min} \leq x \leq x_{max} \tag{1.19}$$

Let $x \in [x_{min}, x_{max}]$, $y \in Y$.

Case Distinction for $a(y)$:

- **Case 1:** $a(y) > 0$

$$a(y)x + b(y) \leq a(y)x_{max} + b(y) \leq a_{max} \frac{c_{max} - b_{max}}{a_{max}} + b_{max} = c_{max} \tag{1.20}$$

$$a(y)x + b(y) \geq a(y)x_{min} + b(y) \geq a_{max} \frac{c_{min} - b_{min}}{a_{max}} + b_{min} = c_{min} \tag{1.21}$$

- **Case 2:** $a(y) < 0$

$$a(y)x + b(y) \leq a(y)x_{min} + b(y) \leq a_{min}\frac{c_{max} - b_{max}}{a_{min}} + b_{max} = c_{max} \tag{1.22}$$

$$a(y)x + b(y) \geq a(y)x_{max} + b(y) \geq a_{min}\frac{c_{min} - b_{min}}{a_{min}} + b_{min} = c_{min} \tag{1.23}$$

- **Case 3:** $a(y) = 0$

$$a(y)x + b(y) = b(y) \in [c_{min}, c_{max}] \tag{1.24}$$

Therefore,

$$\forall y \in Y : a(y)x + b(y) \in [c_{min}, c_{max}] \implies x \in X \tag{1.25}$$

1.6.2 Example

Given $v_{x min} \leq \dot{s}(1 + nC(s)) \leq v_{x min}$.
Set:

- $x = \dot{s}$
- $y = \begin{bmatrix} n \\ C(s) \end{bmatrix}$
- $a(y) = 1 + y_1 y_2$
- $b(y) = 0$

Bounds for $a(y), b(y)$:

- $a_{min} = 1 + \min\{C_{min}n_{min}, C_{min}n_{max}, C_{max}n_{min}, C_{max}n_{max}\}$
- $a_{max} = 1 + \max\{C_{min}n_{min}, C_{min}n_{max}, C_{max}n_{min}, C_{max}n_{max}\}$
- $b_{min} = b_{max} = 0$

One can now easily calculate $[x_{min}, x_{max}] = X$ and Y can be expressed as $[C_{min}, C_{max}] \times [n_{min}, n_{max}]$