Set theory, relations, and functions (II)

Review: set theory

- Principle of Extensionality
- Special sets: singleton set, empty set
- Ways to define a set: list notation, predicate notation, recursive rules
- Relations of sets: identity, subset, powerset
- Operations on sets: union, intersection, difference, complement
- Venn diagram
- Applications in natural language semantics: (i) categories denoting sets of individuals: common nouns, predicative ADJ, VPs, IV; (ii) quantificational determiners denote relations of sets

1 Relations

1.1 Ordered pairs and Cartesian products

- The elements of a set are not ordered. To describe functions and relations we will need the notion of an *ordered pair*, written as $\langle a, b\rangle$, where a is the first element of the pair and b is the second.

  Compare: If $a \neq b$, then ...

  $\{a, b\} = \{b, a\}$, but $\langle a, b\rangle \neq \langle b, a\rangle$
  $\{a, a\} = \{a, a, a\}$, but $\langle a, a\rangle \neq \langle a, a, a\rangle$

- The *Cartesian product* of two sets A and B (written as $A \times B$) is the set of ordered pairs which take an element of A as the first member and an element of B as the second member.

  (1) $A \times B = \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$
  E.g. Let $A = \{1, 2\}$, $B = \{a, b\}$, then $A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle\}$,
  $B \times A = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle\}$

1.2 Relations, domain, and range

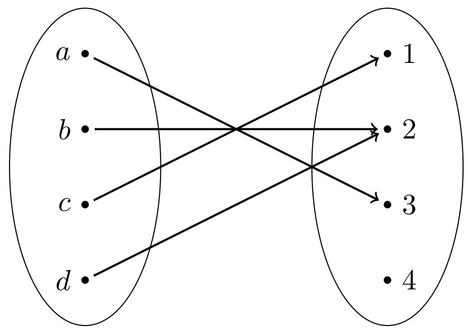
- A *relation* is a set of ordered pairs. For example:
  - Relations in math: =, >, ≠, ...
  - Relations in natural languages: the instructor of, the capital city of, ...

    (2) a. ⟦the capital city of⟧ = {⟨USA, Washington⟩, ⟨China, Beijing⟩, ⟨France, Paris⟩, ...}
    $= \{\langle x, y\rangle : y \text{ is the capital city of } x\}$
    b. ⟦invited⟧ = {⟨Andy, Billy⟩, ⟨Cindy, Danny⟩, ⟨Emily, Flori⟩, ...}
    $= \{\langle x, y\rangle : x \text{ invited } y\}$

- R is *a relation from A to B* iff R is a subset of the Cartesian product $A \times B$, written as $R \subseteq A \times B$.
  R is *a relation in A* iff R is a subset of the Cartesian product $A \times A$, written as $R \subseteq A \times A$.

  (3) a. $[\![$the capital city of$]\!] \subseteq \{x : x$ is a country $\} \times \{y : y$ is a city$\}$
  b. $[\![$the mother of$]\!] \subseteq \{x : x$ is a human $\} \times \{y : y$ is a human$\}$

- We can use a *mapping diagram* to illustrate a relation:



- A and B are the *domain* and *range* of R respectively, iff $A \times B$ is the **smallest** Cartesian product of which R is a subset. For example:

  (4) Let $R = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, b\rangle, \langle 3, b\rangle\}$, then: Dom$(R) = \{1, 2, 3\}$, Range$(R) = \{a, b\}$

  Discussion: Why is that the following definitions are problematic? Given counterexamples.

  (5) a. A and B are the domain and range of R iff $R = A \times B$.
  b. A and B are the domain and range of R iff $R \subseteq A \times B$.

1.3 Properties of relations

- Properties of relations: reflexivity, symmetry, transitivity
  Given a set A and a relation R in A, ...
  - R is *reflexive* iff for every x in A, $\langle x, x\rangle \in R$; otherwise R is *nonreflexive*.
    If there is no pair of the form $\langle x, x\rangle$ in R, then R is *irreflexive*.
  - R is *symmetric* iff for every xy in A, if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$; otherwise R is *nonsymmetric*.
    If there is no pair $\langle x, y\rangle$ in R such that the pair $\langle y, x\rangle$ is in R, then R is *asymmetric*.
  - R is *transitive* iff for every xyz in A, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$; otherwise R is *nontransitive*.
    If there are no pairs $\langle x, y\rangle$ and $\langle y, z\rangle$ in R such that the pair $\langle x, z\rangle$ is in R, then R is *intransitive*.

Exercise: Identify the properties of the following relations w.r.t. reflexivity, symmetry, and transitivity. (Make the strongest possible statement. For example, call a relation 'irreflexive' instead of 'nonreflexive', if satisfied.)

(6) a. $=$ reflective, symmetric, transitve
b. $\leqslant$
c. $\neq$
d. $\subseteq$

(7) a. is a sister of
b. is a mother of

- It is helpful in assimilating the notions of reflexivity, symmetry and transitivity to represent them in *relational diagrams*. If x is related to y (namely, $\langle x, y\rangle \in R$), an arrow connects the corresponding points.

2 Functions

- A relation is a function iff each element in the domain is paired with **just one** element in the range.

(8) $f(x) = x^2$ for $x \in \mathbb{N}$

In particular,

- a function f is a *from A (in)to B* (written as 'f: $A \to B$') iff $\text{Dom}(f) = A$ and $\text{Range}(f) \subseteq B$.
- a function f is a *from A onto B* iff $\text{Dom}(f) = A$ and $\text{Range}(f) = B$.

Exercise: For each set of ordered pairs, consider: what is its domain and range? Is it a function? If it is, draw a mapping diagram to illustrate this function.

(9) a. $\{\langle 2,3\rangle, \langle 5,4\rangle, \langle 0,3\rangle, \langle 4,1\rangle\}$
b. $\{\langle 3,-1\rangle, \langle 2,-2\rangle, \langle 0,2\rangle, \langle 2,1\rangle\}$

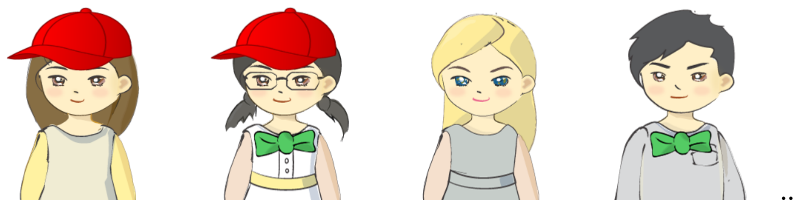
- We may specify functions with lists, tables, or words.

(10) a. $F = \{\langle a,b\rangle, \langle c,b\rangle, \langle d,e\rangle\}$
b. $F = \begin{bmatrix} a \to b \\ c \to b \\ d \to e \end{bmatrix}$
c. F is a function f with domain $\{a,b,c\}$ such that $f(a) = f(c) = b$ and $f(d) = e$.

Exercise: Define conjunctive *and*, disjunctive *or*, and negation *it is not the case that* as functions.

3 Characteristic function

- Recall that the following categories can be interpreted as sets of individuals: common nouns, predicative adjectives, intransitive verbs, VPs, ...



Annie Betty Cindy Danny

(11) a. $[\![\text{wears a hat}]\!] = \{\text{Andy}, \text{Betty}\}$

b. $[\![\text{wears a hat}]\!] = \{x : x \text{ wears a hat}\}$

Alternatively, the semantics of these expressions can also be modeled as functions from the set of individual entities to the set of truth values.

(12) a. $[\![\text{wears a hat}]\!] = \{\langle \text{Andy}, 1\rangle, \langle \text{Betty}, 1\rangle, \langle \text{Cindy}, 0\rangle, \langle \text{Danny}, 0\rangle\}$

b. $[\![\text{wears a hat}]\!] = \begin{bmatrix} \text{Andy} \to 1 \\ \text{Betty} \to 1 \\ \text{Cindy} \to 0 \\ \text{Danny} \to 0 \end{bmatrix}$

c. $[\![\text{wears a hat}]\!]$ = the function f from individual entities to truth values such that $f(x) = 1$ if x wears a hat, and $f(x) = 0$ otherwise.

Characteristic function: a function whose range is the set of possible truth values $\{1, 0\}$.

Exercise: provide the characteristic function for wears a bowtie.