Quantum Phase Estimation

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1 Introduction to the problem

Quantum Phase Estimation (QPE) is an important subroutine in many quantum algorithms. Quantum Phase Estimation is an algorithm that estimates the phase θ of the Eigenvalue $e^{2\pi i\theta}$ of a unitary operator U acted on its Eigenstate $|\psi\rangle$ (Kasirajan, 2022). The magnitude of the Eigenvalue is one. The desired parameter is the phase θ in the eigenvalue.

$$U|\psi\rangle = e^{2\pi i\theta}\,|\psi\rangle\,, 0 \leq \theta < 1 \tag{1}$$

Since θ cannot be represented in binary in general but only in special occasions, this algorithm approximates θ to the closest the binary number (Wong, 2022).

2 Mathematical Requirements

2.1 Controlled Unitary Operators

Applying multiple U Operators to one qubit results in multipliyng the state by the Eigenvalue.

$$U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = e^{2\pi i 2^j\theta}|\psi\rangle \tag{2}$$

The Controlled Gate, namely CU, will apply a phase to the qubits if and only if the control bit is in the state $|1\rangle$. So applying CU to the control state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and the target qubits $|\psi\rangle$ will yield (Kasirajan, 2022):

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes e^{2\pi i\theta}|\psi\rangle) = \frac{1}{\sqrt{2}}(\big(|0\rangle + e^{2\pi i\theta}|1\rangle\big) \otimes |\psi\rangle) \tag{3}$$

2.2 Inverse Quantum Fourier Transform

The Quantum Fourier Transform (QFT) transforms from the binary state $|x\rangle$ to the Fourier Basis $|\widetilde{x}\rangle$ (Kasirajan, 2022). This takes $\Theta(n^2)$ time (Nielsen and Chuang, 2018).

$$\text{QFT}|x\rangle = \frac{1}{2^{\frac{n}{2}}}\left(|0\rangle + e^{\frac{2\pi i}{2}x}|1\rangle\right)\otimes\left(|0\rangle + e^{\frac{2\pi i}{2^2}x}|1\rangle\right)\otimes\ldots\otimes\left(|0\rangle + e^{\frac{2\pi i}{2^{n-1}}x}|1\rangle\right)\otimes\left(|0\rangle + e^{\frac{2\pi i}{2^n}x}|1\rangle\right) = |\widetilde{x}\rangle \tag{4}$$

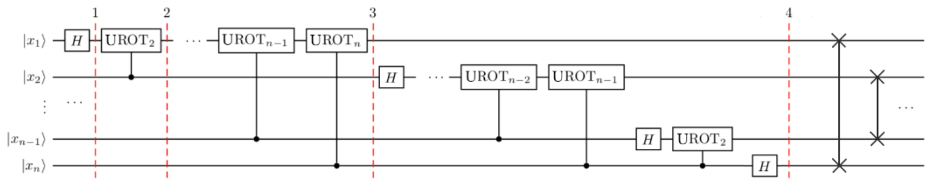

Figure 1: Implementation of Quantum Fourier Transform (IBM, 2022)

3 The QPE Algorithm

We will construct the general circuit now step by step:

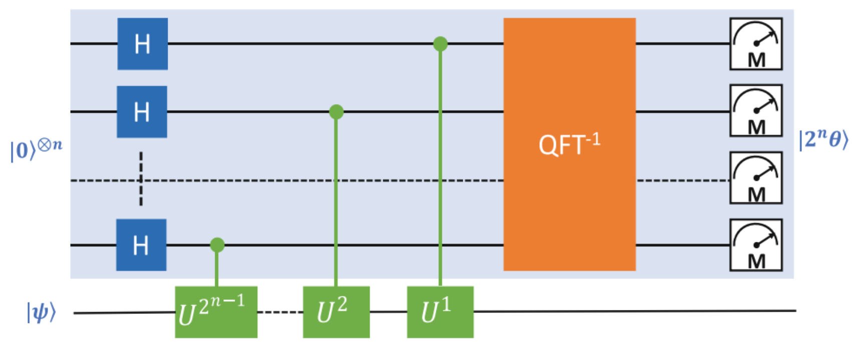

Figure 2: General Implementation of Quantum Phase Estimation (Kasirajan, 2022)

The general set up for the quantum circuit consists of 2 Quantum registers. The first register, which will be named counting register $|x\rangle$, consists of n qubits initialized as $|0\rangle$. The final state of this register will be the binary representation of the number $x = 2^n\theta$. Since θ is between 0 and 1, x doesn't exceed 2^n and you can store up to 2^n numbers in n bits. We measure the counting register to find x. Simple algebra allows us now to find $\theta = \frac{x}{2^n}$.
We call the second quantum register $|\psi\rangle$ b-register. The phase that U applies when acted on $|\psi\rangle$ is what we want to know.
The initialisation of the circuit is:

$$|\psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle \tag{5}$$

First we apply Hadamard Gates to the counting register, to create a equal superposition of $|0\rangle$ and $|1\rangle$.

$$|\psi_1\rangle = \frac{1}{2^{\frac{n}{2}}}\left(|0\rangle + |1\rangle\right)^{\otimes n}|\psi\rangle \tag{6}$$

The Quantum Gates now manipulate this superposition and make $|0\rangle$ or $|1\rangle$ more likely than the other. So when we measure the whole circuit often enough in the end, the propability distribution of every possible qubit configuration yield a binary number that will be x. Now we apply our unitary operator a specific number of times to all n qubits of the counting register. U is applied 2^j times to the j^{th} bit ($j = 0 \cdots n-1$) of the counting register as control bit and the target state $|\psi\rangle$ (Kasirajan, 2022).

$$|\psi_2\rangle = \frac{1}{2^{\frac{n}{2}}} \Big(|0\rangle + e^{\boldsymbol{2\pi i}\theta 2^{n-1}}|1\rangle\Big)\otimes\cdots\otimes\Big(|0\rangle + e^{\boldsymbol{2\pi i}\theta 2^{0}}|1\rangle\Big)\otimes|\psi\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{\boldsymbol{2\pi i}\theta k}|k\rangle_{10}\otimes|\psi\rangle \tag{7}$$

$|k\rangle_2$ denotes the binary representation of k (e.g.: for $n = 3, |k\rangle_{10} = |5\rangle, |k\rangle_2 = |101\rangle$).
It is remarkable that the counting register is a equal superposition but also algebraically representable as a number in the decimal system. Further we notice that we can write $|\psi_2\rangle$ as a tensorproduct where $|k\rangle_{10}$ and $|\psi\rangle$ are completely disctangled. So measuring one register doesn't affect the other. (Wong, 2022).
Since this last formula is in the form of the Fourier Transform it seems natural to apply the inverse Fourier Transform to convert back to desired state $|x\rangle$.

$$|\psi_3\rangle = \mathcal{QFT}_n^{-1}|\psi_2\rangle = \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}|x\rangle \otimes |\psi\rangle \tag{8}$$

We can see that the counting register $|x\rangle$ has a phase factor $e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}$ in front of it. This stems from the controlled unitary operators, since they apply a phase to the the b-register $|\psi\rangle$, if the "control" bit is in the state $|1\rangle$. But because $|\psi\rangle$ is an Eigenstate of U with the Eigenvalue $e^{i\psi}$ and one can put multiplicative factors anywhere in the tensorproduct (multilinearity, Tensorproduct of Hilbertspaces 2022) the phase actually is applied to the control qubit.

$$\mathrm{CU}|1+\rangle = \tfrac{1}{\sqrt{2}}(|0\psi\rangle + e^{i\psi}|1\Psi\rangle) \tag{9}$$

$$\tag{10}$$

$$= \tfrac{1}{\sqrt{2}}(|0\rangle + e^{i\psi}|1\rangle \otimes |\psi\rangle) \tag{11}$$

Measuring now means that we evaluate all qubits of the counting register many times to see what output state the counting register generates most often. This will be the $x = 2^n\theta$ we are looking for. Now we can compute $\theta = \frac{x}{2^n}$. Our accuracy is best when $e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)} = 1$ so $x - 2^n\theta$ has to be zero.
The final state would then be:

$$|\psi_4\rangle = |2^n\theta\rangle \otimes |\psi\rangle \tag{12}$$

4 Conclusion

We state the problem to approximately find the Eigenvalue of a unitary operator and solved it by builiding a general quantum algorithm to find the defining parameter θ for the Eigenvalue of U. This method is used to solve problems like the Traveling Salesman Problem and solving linear differential equations. Resolution is limited by the fact that we can only approximate the value of the phase. So the more qubits we add to the counting register, the more precise our phase estimation gets.

5 Sources

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