Computational Type Theory

Bob Harper
Carnegie Mellon University

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1 Recap

$\mathtt{Id}_A(M_1, M_2)$ identity / identification
$\mathtt{refl}_A(M)$
$J(a, b, c : C)(a : P)(Q) : [M_1, M_2, Q/a, b, c]C$ with $Q = \mathtt{Id}_A(M_1, M_2)$
if $\_\ \mathtt{Id}_A(M, N)$ then $M \equiv N : A$

1. adding extensionality
2. adding $ua(E) : \mathtt{Id}_{\mathcal{U}}(A, B)$ when $E : \mathtt{Equiv}(A, B)$
   Homotopy type theory
3. adding inductive types with identifications

2 Definition: Transport

$tr[a : F](P : \mathtt{Id}_A(M, N)) : F[M/a] \to F[N/a]$
$P \triangleq J(a, b, -, F(a) \to F(b))(a : \mathtt{id}_{F[a]})(P)$

2.1 Problem

Consider the "identity family" $a.a$
then $tr[a.a](P : \mathtt{Id}_{\mathcal{U}}(A, B)) : A \to B$
which produces an invertible function.
"coercion" function

$$\frac{M : A \quad A \equiv B}{M : B}$$

$$\frac{M \in A \quad A \doteq B}{M \in B}$$

$$\frac{P : \mathtt{Id}_{\mathcal{U}}(A, B) \quad M : A}{tr[M] : B}$$

Idea: identification as "proof relevant equality"
$tr[a.a](ua(E)) \equiv ???$ - STUCK!

But can use E which is an equivalence $\texttt{Equiv}(A, B)$ which at the very least is a function $f : A \to B$. So the idea ought to be to apply E! The interesting case is when the type is $\mathcal{U}$! Bob does not think this can be resolved. But we can do some ninja magic to make this work?

3 Adding axioms is suspicious

Axioms means elements of a type

1. Gentzen (vs Hilbert)
   entailment is prior to implication: $\frac{\Gamma \times A \vdash B}{\Gamma \vdash A \supset B}$ (logical consequence)
2. Eilenberg and Maclane
   maps are prior to functions: $\frac{f : C \times A \to B}{[f] : C \to B^A}$

Here: bring out the judgmental structure of identifications, then you can internalize them.
$\texttt{Id}_A$ as an inductive type vs. $\texttt{Path}_A$ internalizes paths.
Generic in A vs Dependent on A.

3.1 Paths

Two types: Between types and within types.
Where do we get paths between types?

1. Induced by paths between elements of index types.
   $a : A \vdash F$ $\texttt{type}$ - path with A will be preserved by F!
   $M \longleftrightarrow_P N$ induces $F[M] \longleftrightarrow_{F[P]} F[N]$
   where P resides in A and $F[P]$ resides in the multiverse of all types.
2. Univalence
   $\texttt{Bool} \times \texttt{Nat} \overset{\texttt{swap}=ua(\texttt{swapequiv})}{\longleftrightarrow} \texttt{Nat} \times \texttt{Bool}$
   Paths induce coercions!
   coe(swap): $\texttt{Bool} \times \texttt{Nat} \to \texttt{Nat} \times \texttt{Bool}$
   $\texttt{Bool} \times \texttt{Nat} \to C \overset{\texttt{swap} \to C}{\longleftrightarrow} \texttt{Nat} \times \texttt{Bool} \to C$
   coe($\texttt{swap} \to C$)

Paths are traced out by dimension variables that "range over unit intervals [0,1]"
$\texttt{swap}_x$ $\texttt{type}$ $[x]$
$\texttt{swap}_x\langle 0/x \rangle = \texttt{Bool} \times \texttt{Nat}$
$\texttt{swap}_x\langle 1/x \rangle = \texttt{Nat} \times \texttt{Bool}$
$\texttt{swap}_0 \mapsto \texttt{Bool} \times \texttt{Nat}$
$\texttt{swap}_1 \mapsto \texttt{Nat} \times \texttt{Bool}$
x is a "line of types"
What are the elements of $\texttt{swap}_x$? "heterogeneous Lines"
$\texttt{swapl}_x(N) \in \texttt{swap}_x$ $[x]$ if $N \in A_2 \times A_1$ $[x]$
$\texttt{swapl}_0(N) \mapsto \texttt{ap}(\texttt{swapfn}, N)$
$\texttt{swapl}_1(N) \mapsto N$

4 General Setup

1. Type lines induce coercions
2. must be able to <u>compose</u> lines
   - type lines must be reversible P, P^{-1}
   - must be able to concatenate $(P \to Q) \to R = P \to (Q \to R)$

Miraculous thing: <u>Kan condition</u>: a program for performing compositions that gives us all of above structure!

4.1 Cartesian Cubes

Have two notions:
A `type` $[x_1, \ldots, x_n]$
$M \in A \; [x_1, \ldots, x_n]$
Consider a square (Q) with top (T), bottom (B), left (L), and right (R)
$Q\langle 0/x \rangle \doteq L$
$Q\langle 1/x \rangle \doteq R$
$Q\langle 0/y \rangle \doteq T$
$Q\langle 1/y \rangle \doteq B$
Now the two notions are subject to coherence requirements.
One of the big issues is the diagonal, need to be able to plug in z for both x and y.

4.2 Example: swap

$coe^{0 \to 1}_{x:A_1 \to A_2}(M \in A_1\langle 0/x \rangle \to A_2\langle 0/x \rangle) \in A_1\langle 1/x \rangle \to A_2\langle 1/x \rangle$
$\mapsto \lambda a_1 \in A_1\langle 1/x \rangle$
$coe^{0 \to 1}_{x:A_2}(M(coe^{1 \to 0}_{x:A_1}(a_1)))$
Need these coercions:
$0 \to 1; 1 \to 0; 0 \to x; x \to 0; 1 \to x; x \to 1; x \to y$

$$coe^0 \to 0_y : B(M)[\doteq M] \overset{coe^0 \to x_y : B(M)}{\longleftrightarrow} coe^0 \to 0_y : B(M)$$

Evaluates to something derived from M itself!
(Secret to why $\mathtt{swapl}_x$ is as it is)

5 Concluding Remarks

See the paper!