OPLSS-2018-Foundations-day2

Thursday, July 5, 2018 9:01 AM

7/5/2018 9:15 AM
Lecture 5 Finite Data Structures
Recap..
Products

- Conjunctive combination of data
- Tuples, records, structs, unit

Today: pairs, type, unit, prod

Statics
Dynamics
Products are useful
Get for free:
Sum Types

- Disjunctive combination of data
- Enums, option type, void,

7/5/2018 11:01 AM

Reading Paper: Why lazy functional programming matters, John Hughes..

Numbers
Recursive types

7/5/2018 2:02 PM
Lecture 7: PCF and Cost Semantics
PCF (Plotkin)
-small language with general recursion
Fixed Points
Evaluation dynamics: also big-step operational semantics, another way of defining the dynamics.

OPLSS2018-Foundations-day2

Friday, July 6, 2018 3:20 AM

7/5/2018

Lecture 5

1

Base language typ τ ::=

review....

Exp e ::=

introduction forms $\begin{cases} \text{triv} & \langle\,\rangle \\ \text{pair}(e_1; e_2) & \langle e_1; e_2\rangle \end{cases}$ elimination forms $\begin{cases} pr[l]\,e & e\cdot l \\ pr[r]\,e & e\cdot r \end{cases}$

Statics $\dfrac{}{\Gamma \vdash \langle\,\rangle\ \text{unit}}$ $\dfrac{\Gamma\vdash e_1:\tau_1 \quad \Gamma\vdash e_2:\tau_2}{\Gamma\vdash\langle e_1, e_2\rangle\ \tau_1\times\tau_2}$ $\dfrac{\Gamma\vdash e:\tau_1\times\tau_2}{\Gamma\vdash e\cdot l:\tau_1}$ $\dfrac{\Gamma\vdash e:\tau_1\times\tau_2}{\Gamma\vdash e r:\tau_2}$

Dynamics $\dfrac{}{\langle\,\rangle\ \text{val}}$ $\dfrac{e_1\ \text{val} \quad e_2\ \text{val}}{\langle e_1; e_2\rangle\ \text{val}}$ $\dfrac{e_1\mapsto e_1'}{\langle e_1, e_2\rangle \mapsto \langle e_1', e_2\rangle}$ $\dfrac{e_1\ \text{val} \quad e_2\mapsto e_2'}{\langle e_1, e_2\rangle \mapsto \boxed{\dfrac{\langle e_1, e_2'\rangle}{e_2 \longrightarrow e_2'}}}$

$\dfrac{e\mapsto e'}{e\cdot l\mapsto e' l}$ $\dfrac{e_1\ \text{val} \quad e_2\ \text{val}}{\langle e_1, e_2\rangle\cdot l\mapsto e_1}$ $\dfrac{e\mapsto e'}{e r\mapsto e'\cdot r}$ $\dfrac{e_1\ \text{val} \quad e_2\ \text{val}}{\langle e_1, e_2\rangle\cdot r\mapsto e_2}$

example. $((\lambda(x\cdot\text{unit})\ \langle x, x\rangle \mid \langle\,\rangle)\cdot l$

$\rightarrow \langle\langle\,\rangle, \langle\,\rangle\rangle\cdot l \mapsto \langle\,\rangle$

get for free 1) multiple fun, args:

$[\tau_1\times\tau_2] \longrightarrow \tau$

example: Python $\Rightarrow (\tau_1\times\tau_2)\rightarrow \neq (\tau_1\times\tau_2)\rightarrow\tau$

2) multiple return val

$\tau_1 \rightarrow \tau_1\times\tau_2$

Sum Type

Nullary and binary sums

Type τ ::= Exp e ::=

void	void	introduction forms	$in[l]$	$\{\tau_1,\tau_2\}(e)$	$l\cdot e$
$\text{Sum}(\tau_1;\tau_2)$	$\tau_1+\tau_2$		$in[r]$	$\{\tau_1;\tau_2\}(e)$	$r\cdot e$

Elimination forms $\begin{cases} \text{case}(e; x_1.e_1 ; x_2.e_2) \\ \text{abort}\{\tau\}(e) \end{cases}$ $\quad$ case e $\{ l.x_1 \hookrightarrow e_1$ $\mid r.x_2 \hookrightarrow e_2 \}$

case e $\{ \ \}$

Statics
$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \text{in}[l]\{\tau_1 ; \tau_2\}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \text{in}[r]\{\tau_1 ; \tau_2\}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \tau \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash \text{case}(e ; x_1.e_1 ; x_2.e_2) : \tau} \qquad \frac{\Gamma \vdash e : \text{void}}{\Gamma \vdash \text{case } e \{\} : \tau \quad \text{abort}\{\tau\}(e)}$$

true $:= l.\langle\rangle$ $\quad$ bool $:=$ unit $+$ unit $\quad$ false $:= r.\langle\rangle$

Dynamics
$$\frac{e \text{ val}}{l.e \text{ val}} \qquad \frac{e \text{ val}}{r.e \text{ val}} \qquad \frac{e \longmapsto e'}{l.e \longmapsto l.e'}$$

$$\frac{e \longmapsto e'}{\text{case}(e ; x_1.e_1 ; x_2.e_2) \longmapsto \text{case}(e' ; x_1.e_1 ; x_2.e_2)}$$

$$\frac{e \text{ val}}{\text{case}(l.e ; x_1.e_1 ; x_2.e_2) \longmapsto [e/x_1]e_1}$$

Examples:
- bool $=$ unit $+$ unit
- Enums: unit $+ \cdots +$ unit

option type: $\quad$ null $:= l.\langle\rangle$

opt $\tau :=$ $\tau +$ unit $\quad$ just $e := r.e$

ifnull e $\{$null $\hookrightarrow e_1 \mid$ just $(x_2) \hookrightarrow e_2\}$

Java: if x=null then $\{e_1\}$ else $\{e_2\}$. $l.f(x)$ $\quad := \text{case}(e ; e_1.e_1 ; x_2.e_2)$

Lecture 7:

Fixed Points: Let $F : A \to A$ is a function and $F(f) = f$ then we call f a fixed point of F

7/5/2018

Lecture 6

2

$e ::= \ldots \mid z \mid s(e) \mid rec(e;\, e_0;\, ^{x.y}e_1)$

$e ::= \ldots \mid z \mid s_e \mid rec\ e\ as\ \{ z \Rightarrow e_0 \mid S\underline{x}\ with\ y \Rightarrow e_1 \}$

add z $n = n$

add (S_m) $n = S(add\ m\ n)$

add $= \lambda m{:}Nat.\ \lambda n{:}Nat$

rec m as $z \Rightarrow n$

$S_{m'}$ with $r \Rightarrow S_r$

mult z $n = z$

mult (S_m) $n = n + (mult\ m\ n)$

mult $= \lambda m.Nat\ \lambda n: Nat.$ rec m as

$z \Rightarrow z$

$S\ m'$ with $r \Rightarrow$ add n r

pred $z = z$

pred $(S_n) = n$

pred $= \lambda n{:}Nat.$ rec n as

$z \Rightarrow z$

$S_{n'}$ with $\Rightarrow n'$

$\tau ::= \ldots \mid Nat \mid$

$$\frac{}{\Gamma \vdash z : Nat}$$

$$\frac{\Gamma \vdash e : Nat}{\Gamma \vdash S_e\ Nat}$$

$$\frac{\Gamma \vdash e{:}Nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x{:}Nat, y{:}\tau \vdash e_1 : \tau}{\Gamma \vdash rec\ e\ as\ \{z \Rightarrow e_0 \mid S_x\ with\ y \Rightarrow e_1\} : \tau}$$

$\overline{z\ val}$ $\quad \overline{S\, e\ val}$

rec z as $z \Rightarrow e_0$, S_x with $y \Rightarrow e_1$ $\mapsto e_0$

rec S_e as $z \Rightarrow e_0$, S_x with $y \Rightarrow e_1$ $\mapsto e_1 \left[e/x, \begin{pmatrix} rec\ e\ as \\ z \Rightarrow e_0 \\ Sx\ with\ y \Rightarrow e_1 \end{pmatrix} \Big/ y \right]$

$E ::= \ldots \mid rec\ E\ as\ \{\ldots\}$

$$\frac{e \mapsto e'}{\begin{matrix} rec\ e\ as \\ z \Rightarrow e_0 \\ Sx\ with\ y \Rightarrow e_1 \end{matrix} \mapsto \begin{matrix} rec\ e'\ as \\ z \Rightarrow e_0 \\ Sx\ with\ y \Rightarrow e_1 \end{matrix}}$$

eager / lazy

$\tau ::= \ldots \mid \mu_\alpha.\tau \mid \alpha$ $\qquad$ $e ::= \ldots \mid \text{fold} \mid \text{unfold}(e)$

$\mu_\alpha.\tau = \tau[\mu_\alpha.\tau/\alpha]$

$$\mu_\alpha.\tau \;\underset{\text{unfold}}{\overset{\text{fold}}{\approx}}\; \tau[\mu_\alpha.\tau/\alpha]$$

$$\frac{\Gamma \vdash e : \tau[\mu_\alpha\tau/\alpha]}{\Gamma \vdash \text{fold}(e) : \mu_\alpha.\tau} \qquad \frac{\Gamma \vdash e : \mu_\alpha\tau}{\Gamma \vdash \text{unfold}(e) : \tau[\mu_\alpha\tau/\alpha]}$$

CBN $\quad \text{unfold}(\text{fold}(e)) \mapsto e$

$$\frac{e \mapsto e'}{\text{unfold}(e) \mapsto \text{unfold}(e')}\ (\beta) \qquad (\eta)\ \text{fold}(\text{unfold}(e)) =_\eta e : \mu_\alpha\tau$$

Nat $\approx$ unit + Nat $\qquad$ List $\tau \approx$ unit + $(\tau \times \text{List } \tau)$

Nat $= \mu_\alpha.\,\text{unit} + \alpha$ $\qquad$ List $\tau = \mu_\alpha.\,\text{unit} + (\tau \times \alpha)$

$z = \text{fold}(l.\langle\rangle)$ $\qquad$ nil $= \text{fold}(l.\langle\rangle)$

$Se = \text{fold}(r.e)$ $\qquad$ cons $e\ e' = \text{fold}(r.\langle e, e'\rangle)$

case

$\omega : \mu_\alpha.(\alpha \to \tau) \to \tau$ $\qquad$ $\mu_\alpha.(\alpha \to \tau) \to \tau \;\underset{\text{fold}}{\overset{\text{unfold}}{\approx}}\; \mu_\alpha.(\alpha \to \tau)$

$\omega = \lambda x : \mu_\alpha.(\alpha \to \tau).\ (\text{unfold}(x))\ x$ $\qquad$ $Yf \mapsto^*$

$\Omega = \omega \quad \omega$ $\qquad$ $Y : (\tau \to \tau) \to \tau$

$\Omega \mapsto \Omega$ $\qquad$ $Y = \lambda f : \tau \to \tau.\ (\lambda x : \mu_\alpha(\alpha \to \tau).\ f((\text{unfold } x)\, x))$

$(\text{fold}(\lambda x.\ \mu_\alpha(\alpha \to \tau)\ f(\text{unfold } x\ x$

$\Omega = \omega(\text{fold}\,\omega) \mapsto (\text{unfold}(\text{fold}(\omega)))\ (\text{fold}\ \omega)$

$\mapsto \omega\ (\text{fold}\ \omega)$

7/5/2018

Lecture 7

3

Example . factorial

1) Operational view $f(n) = \begin{cases} 1 & \text{if } n=0 \\ n f(n-1) & \text{if } n>0 \end{cases}$

$f(2) = 2 f(1) = 2 \cdot 1 \cdot f(0) = 2 \cdot 1 \cdot 1 = 2$

2) As an equation

$f = \lambda n .$ if $n=0$ then 1 else $n . f(n-1)$ find a solution f to this equation

$succ_2 = \lambda n\, n+2$ is not a solution

$fac = \lambda n . n!$ is a solution

3) As a fixed point

$F(f) = \lambda n .$ if $n=0$ then 1 else $n . f(n-1)$. we are looking for a fixed point of F

fac is the unique fixed point

⇒ PCF features a operation "fixed point of F"

Types and expressions : Type $\tau ::=$ nat | nat

pair (τ_1, τ_2) | $\tau_1 \to \tau_1$

Exp $e ::= x$

x

Z

$S(e)$

ifz{e_0; $x.e_1$}(e) → ifz $(e; e_0; x e_1)$

lam {τ} $(x.e)$

app $(e_1; e_2)$

fix{τ}$(x.e)$ fix $x{:}\tau$ is e

Example: $fac \triangleq$ fix f: nat → nat is (→ defination)

$\lambda(x:nat)$ if$_z$ e { $z \hookrightarrow e_0$, $S(x) \hookrightarrow e_1$ }

$\lambda(x:nat)$ if$_z$ x { $Z \hookrightarrow S(Z)$,

$S(y) \hookrightarrow x . f(y)$ }

Statics

$$\frac{\Gamma \vdash e : nat \quad \Gamma \vdash e_0 : \tau \quad \Gamma, x : nat \vdash e_1 : \tau}{\Gamma \vdash \text{if}_z\,(e; e_0; x.e_1) : \tau}$$

$$\frac{\Gamma, x:\tau \vdash e : \tau}{\Gamma \vdash \text{fix}\{\tau\}(x.e)}$$

Dynamics

$$\frac{e \mapsto e'}{\text{if}_z(e; e_0 : e_1) \longmapsto \text{if}_z(e'; e_0; x.e_1)}$$

$$\frac{}{\text{fix}\{\tau\}(x.e) \longmapsto [\text{fix}\{\tau\}(x.e)/x]e}$$

Theorem:

progress: If $e:\tau$ then either e val or $e \mapsto e'$

preservation: if $e:\tau$ and $e \mapsto e'$ then $e':\tau$

Evaluation Dynamics

judgment: $e \Downarrow v$ expr. e evaluates to value v

Rules

$$\frac{}{z \Downarrow^0 z} \qquad \frac{e \Downarrow^n v}{s(e) \Downarrow^n s(v)} \qquad \frac{}{\lambda(x:\tau)e \Downarrow^0 \lambda(x:\tau)e} \qquad \frac{e \Downarrow^{n_1} z \quad e_0 \Downarrow^{n_2} v_0}{\text{if}z(e; e_0; x.e_1) \Downarrow^n v_0}$$

$n = n_1 + n_2 + 1$

$$\frac{e \Downarrow^{n_1} s(v) \quad [v/x]e_1 \Downarrow^{n_2} v_1}{\text{if}z(e; e_0; x.e_1) \Downarrow^n v_1} \qquad \frac{[\text{fix}\{\tau\}(x.e)/x]e \Downarrow^n v}{\text{fix}\{\tau\}(x.e) \Downarrow^{n+1} v} \qquad \frac{e_1 \Downarrow^{n_1} \lambda(x:\tau)e \quad e_2 \Downarrow^{n_2} v_2 \quad [v_2/x]e \Downarrow^{n_3} v}{e_1(e_2) \Downarrow^n v}$$

$n = n_1 + n_2 + 1$ $\qquad$ $n = n_1 + n_2 + n_3 + 1$

$\text{fix}\{\text{nat}\}(\omega.\omega) \Downarrow$

$s(\lambda(x:\text{nat})x) \Downarrow$

Theorem $\quad e \Downarrow^n v$ iff

$e \mapsto^{*n} v$ and v val

n might be 0

Cost Dynamics

$e \Downarrow^n v$ expr. e evaluates to value v with cost n