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PCF & COST SEMANTICS

PCF (Plotkin) small language w/ general recursion

Fixed Points

Def: Let $F : A \to A$ be a fn and $F(f) = f$.
Then we call f a fixed point of F.

Example (factorial)

1. operational view

$$f(n) = \begin{cases} 1, & \text{if } n = 0 \\ n \cdot f(n-1), & \text{if } n > 0 \end{cases}$$

2. equational view

$$f = \lambda n.\ \text{if } (n=0)\ 1,\ \text{else } n \cdot f(n-1)$$

Find a solution f to this equation.
($\lambda n.\, n!$ is a solution).

3. fixed point view

$$F(f) = \lambda n.\ \text{if } (n=0)\ 1,\ \text{else } n \cdot f(n-1)$$

Find a fixed point of F.
($\lambda n.\, n!$ is the unique fixed point of F)

$\Longrightarrow$ PCF features an operation
"fixed point of F"

Types & Expressions (of PCF)

Typ $\tau ::= \text{nat}$ nat

$\text{par}(\tau_1, \tau_2)$ $\tau_1 \to \tau_2$

Exp $e ::= x$ abstract   concrete

abstract	concrete
x	
z	
$S(e)$	
$\text{ifz}(e; e_0; x.e_1)$	$\text{ifz } e \{ z \hookrightarrow e_0 \mid S(x) \hookrightarrow e_1 \}$
$\text{lam}[\tau](x.e)$	
$\text{app}(e_1; e_2)$	
$\text{fix}\{\tau\}(x.e)$	$\text{fix } x{:}\tau \text{ is } e$

Example $\text{fac} \triangleq \text{fix } f{:}\text{nat} \to \text{nat} \text{ is}$
$\lambda(x{:}\text{nat})\ \text{ifz } x \{ z \hookrightarrow S(z)$
$S(y) \hookrightarrow x \cdot f(y) \}$

Statics

$$\frac{\Gamma \vdash e : \text{Nat} \qquad \Gamma \vdash e_0 : \tau \qquad \Gamma, x{:}\text{nat} \vdash e_1 : \tau}{\Gamma \vdash \text{ifz}(e; e_0, x.e_1) : \tau}$$

$$\frac{\Gamma, x{:}\tau \vdash e : \tau}{\Gamma \vdash \text{fix}\{\tau\}(x.e) : \tau}$$

Dynamics

$$\frac{e \mapsto e'}{\text{ifz}(e; e_0, x.e_1) \mapsto \text{ifz}(e'; e_0; x.e_1)}$$

$$\frac{}{\text{fix}\{\tau\}(x.e) \mapsto [\text{fix}\{\tau\}(x.e)/x]\, e}$$

Theorem: 1) Progress: If $e : \tau$ then either e val or $e \mapsto e'$.
2) Preservation: If $e : \tau$ and $e \mapsto e'$ then $e' : \tau$.

Evaluation Dynamics (aka Big Step operational Semantics)

(another way of defining the dynamics)

$$\frac{}{z \text{ val}} \qquad \frac{v \text{ val}}{s(v) \text{ val}}$$

(Some things will be easier to prove with "structural dynamics" e.g. type safety. others may be easier with Big step.)

$$\frac{}{\lambda(x{:}\tau)e \text{ val}}$$

Judgment: $e \Downarrow v$ "expr e evaluates to v"

(there are no intermed. steps)

Rules

$$\frac{}{z \Downarrow z} \qquad \frac{e \Downarrow v}{s(e) \Downarrow s(v)}$$

$$\frac{}{\lambda(x{:}\tau).e \Downarrow \lambda(x{:}\tau).e} \qquad \frac{e \Downarrow z \qquad e_0 \Downarrow v_0}{ifz(e;\, e_0;\, x.e_1) \Downarrow v_0}$$

$$\frac{e \Downarrow s(v) \qquad [v/x]e_1 \Downarrow v_1}{ifz(e;\, e_0;\, x.e_1) \Downarrow v_1} \qquad \frac{[fix\{\tau\}(x.e)/x]e \Downarrow v}{fix\{\tau\}(x.e) \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda(x{:}\tau)e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1(e_2) \Downarrow v}$$

Example

$fix\{nat\}(\omega\,\omega) \Downarrow ?$

$s(\lambda(x{:}nat)x) \Downarrow ?$

This is why Eval. Dynamics is not good for proving type safety.

Theorem: $e \Downarrow v$ iff $e \longmapsto^* v$ and v val.

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Cost Dynamics

$e \Downarrow^n v$ expr e evaluates to val v with cost n.

Want: Theorem $e \Downarrow^n v$ iff $e \longmapsto^n v$ and v val.

(Recall: $$\frac{}{e \longmapsto^0 e} \qquad \frac{e \longmapsto e' \quad e' \longmapsto^n e''}{e \longmapsto^{n+1} e''}$$)

Rules (revisited w/ cost)

$$\frac{}{z \Downarrow^0 z} \qquad \frac{e \Downarrow^n v}{s(e) \Downarrow^n s(v)}$$ } is this good?

$$\frac{e \Downarrow^{n_1} z \quad e_0 \Downarrow^{n_2} v_0}{\text{ifz}(e; e_0, x.e_1) \Downarrow^n v_0} \quad (n = n_1 + n_2)$$

$$\frac{}{\lambda(x:\tau)e \Downarrow^0 \lambda(x:\tau)e}$$

$$\frac{e \Downarrow^{n_1} s(v) \quad [v/x]e_1 \Downarrow^{n_2} v_1}{\text{ifz}(e; e_0; x.e_1) \Downarrow^n v_1} \quad (n = n_1 + n_2 + 1)$$

$$\frac{[\text{fix}\{\tau\}(x.e)/x]e \Downarrow^n v}{\text{fix}\{\tau\}(x.e) \Downarrow^{n+1} v}$$

$$\frac{e_1 \Downarrow^{n_1} \lambda(x:\tau).e \quad e_2 \Downarrow^{n_2} v_2 \quad [v_2/x]e \Downarrow^{n_3} v}{e_1(e_2) \Downarrow^n v}$$