Paul Downen (lecture 8) [2018/07/07]

~~$\lambda x : \tau . \langle x, x \rangle : \tau \to \tau \otimes \tau$~~ ← rule out arbitrary copying

Linear λ-calculus

(Linear logic for PL)

Types

$$\tau ::= \top \mid 0 \mid 1 \mid \tau_1 \to \tau_2 \mid !\tau \mid \tau_1 \,\&\, \tau_2 \mid \tau_1 \oplus \tau_2 \mid \tau_1 \otimes \tau_2$$

$$\Gamma ; \Delta \vdash e : \tau$$

nonlinear → Γ

linear → Δ

$$\frac{\Gamma ; \Delta \vdash e : \tau \qquad \Gamma ; \Delta', x : \tau \vdash e' : \tau'}{\Gamma \; \Delta \; \Delta' \vdash \text{let } x = e \text{ in } e' : \tau'} \ (\text{Cut})$$

$$\frac{}{\Gamma ; x : \tau \vdash x : \tau} \ Id^1 \qquad \frac{}{\Gamma , x : \tau ; \bullet \vdash x : \tau} \ Id^2$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1}{\Gamma ; \Delta \vdash l \cdot e : \tau_1 \oplus \tau_2} \qquad \frac{\Gamma ; \Delta \vdash e : \tau_2}{\Gamma ; \Delta \vdash r \cdot e : \tau_1 \oplus \tau_2}$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \oplus \tau_2 \qquad \Gamma ; \Delta', x : \tau_1 \vdash e_1 : \tau \qquad \Gamma ; \Delta', y : \tau_2 \vdash e_2 : \tau}{\Gamma ; \Delta , \Delta' \vdash \text{case } e \text{ of } l \cdot x \to e_1 \mid r \cdot x \to e_2 : \tau}$$

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$$\frac{\Gamma ; \Delta_1 \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta_2 \vdash e_2 : \tau_2}{\Gamma ; \Delta_1 , \Delta_2 \vdash \langle e_1 , e_2 \rangle : \tau_1 \otimes \tau_2} (\otimes I)$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \otimes \tau_2 \qquad \Gamma ; \Delta' , x : \tau_1 , y : \tau_2 \vdash e' : \tau'}{\Gamma ; \Delta , \Delta' \vdash \text{case } e \text{ of } \langle x , y \rangle \to e' : \tau'} (\otimes E)$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \& \tau_2}{\Gamma ; \Delta \vdash e . l : \tau_1} (\& E_1) \qquad \frac{\Gamma ; \Delta \vdash e : \tau_1 \& \tau_2}{\Gamma ; \Delta \vdash e . r : \tau_2} (\& E_2)$$

$$\frac{\Gamma ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash \{ l \to e_1 , r \to e_2 \} : \tau_1 \& \tau_2} (\& I)$$

(dual to plus)

$$\frac{\Gamma ; \Delta , x : \tau \vdash e : \tau'}{\Gamma ; \Delta \vdash \lambda x : \tau . e : \tau \multimap \tau'} (\multimap I)$$

$$\frac{\Gamma ; \Delta \vdash e : \tau \multimap \tau' \qquad \Gamma ; \Delta' \vdash e' : \tau}{\Gamma ; \Delta , \Delta' \vdash e \; e' : \tau'} (\multimap E)$$

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! "of course"

example: printer : !(paper $\otimes$ ink $\multimap$ page)

$$\frac{\Gamma ; \cdot \vdash e : \tau}{\Gamma ; \cdot \vdash \text{many } e : !\tau} \text{ (many-I)}$$

$$\frac{\Gamma ; \Delta \vdash e : !\tau \qquad \Gamma, x : \tau ; \Delta' \vdash e' : \tau'}{\Gamma ; \Delta, \Delta' \vdash \text{case } e \text{ of many } x \to e' : \tau'} \text{ (}\multimap\text{E)}$$

$$\frac{}{\Gamma ; \Delta \vdash \{\} : \top} \qquad \frac{}{\Gamma ; \cdot \vdash \langle\rangle : 1}$$

$$\frac{\Gamma ; \Delta \vdash e : 0}{\Gamma ; \Delta \vdash \text{case } e \text{ of } \{\} : \tau}$$

$$\frac{\Gamma ; \Delta \vdash e : 1 \qquad \Gamma ; \Delta' \vdash e' : \tau}{\Gamma ; \Delta, \Delta' \vdash \text{case } e \text{ of } \langle\rangle \Rightarrow e' : \tau}$$

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$\boxed{\text{Values}}$

$$v ::= \langle\ \rangle \mid \langle v, v\rangle \mid l \cdot v \mid r \cdot v \mid \lambda x{:}\tau.e \mid \{l \Rightarrow e,\ r \Rightarrow e\}$$
$$\mid \{\ \} \mid \text{many } v$$

Reduction (β) Rules

$\text{let } x = v \text{ in } e \longmapsto e[v/x]$

$(\lambda x.e)\, v \longmapsto e[v/x]$

$\{l \Rightarrow e_1, r \Rightarrow e_2\}.l \longmapsto e_1$

$\text{case } l \cdot v \text{ of } \{l \cdot x \Rightarrow e_1,\ r \cdot y \Rightarrow e_2\} \longmapsto e_1[v/x]$

$\text{case } \langle v_1\, v_2\rangle \text{ of } \langle x, y\rangle \longmapsto e[v_1/x, v_2/y]$

$\text{Case many } v \text{ of } x \Rightarrow e \longmapsto e[v/x]$

Lemma (Substitution)

- linear: If $\Gamma; \Delta \vdash e : \tau$ and $\Gamma; \Delta', x{:}\tau \vdash e' : \tau'$
  then $\Gamma; \Delta, \Delta' \vdash e'[e/x] : \tau'$.

- nonlinear: If $\Gamma; \circ \vdash e : \tau$ and $\Gamma, x{:}\tau; \Delta' \vdash e' : \tau'$
  then $\Gamma; \Delta' \vdash e'[e/x] : \tau'$.