Algebraic Effects and Handlers

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1 Recap

1.1 Terminology

Signature: $\Sigma = \{(\mathtt{op}_i, n_i)\}_{i \in I}$
Terms: $x_1, \ldots, x_n | t$
Equations: $x_1, \ldots, x_n | l = r$
Theory: $T = (\Sigma_T, \mathcal{E}_T)$
Interpretation: I
Model: M
Free Model: $\mathtt{Free}_T(X)$

2 General Arities and Parameters

Operation symbol $\mathtt{op}_i, n_i$
$[\![\mathtt{op}_1]\!]_I : |I| \times \cdots (n_i) \cdots \times |I| \to |I|$
$[\![\mathtt{op}_1]\!]_I : |I|^{n_i} \to |I|$
$[n] := \{0, 1, \ldots, n-1\}$
$X \times \cdots (n) \cdots \times X \cong X^{[n]}$
B^A set of all functions $A \to B$

2.1 Arity = Set

Signature (temporary): $\Sigma = \{(\mathtt{op}_i, A_i)\}_{i \in I}$, $\mathtt{op}_i$ "symbols", A_i sets.
$[\![\mathtt{op}_i]\!] : |I|^{A_i} \to |I|$
$\mathtt{op}_i(\ldots t_a \ldots)_{a \in A_i}$ NO! Use instead: $\mathtt{op}_i(\lambda a.t_a)$
$\mathtt{op}_i(\kappa)$

2.2 Example: Vector Space V

$v + w$ addition
0
$-v$
$s \cdot v$ $s \in \mathbb{R}$ scalar, $v \in V$
$\cdot : \mathbb{R} \times V \to V$
$\cdot : 1 \times V \times V \to V$

2.3 New Designations

Signature: $\Sigma := \{\mathtt{op}_i : P_i \Rightarrow A_i\}_{i \in I}$ Note: $\Rightarrow$ not a function.
$\mathtt{op}_i$ is a symbol, P_i set parameter, A_i set arity.
Trees: $\mathtt{Tree}_\Sigma(X)$:

- (return x) $\in \mathtt{Tree}_\Sigma(X)$ for $x \in X$
- $\mathtt{op}_i(p, \kappa) \in \mathtt{Tree}_Z(X)$ for $p \in P_i$, $\kappa : A_i \to \mathtt{Tree}_\Sigma(X)$

Equation: $X | l = r$ where $l, r \in \mathtt{Tree}_\Sigma(X)$
Theory: $T = (\Sigma_T, \mathcal{E}_T)$
Interpretation I :

- carrier set $|I|$
- for each $\mathtt{op}_i : P_i \Rightarrow A_i$, give $[\![\mathtt{op}_i]\!]_I : P_i \times |I|^{A_i} \to |I|$

Extend to interpretation of trees/terms:
For $t \in \mathtt{Tree}_\Sigma(X)$,

- $[\![t]\!]_I : |I|^X \to |I|$
- $[\![\mathtt{return}\ x]\!]_I(\gamma) = \gamma(x)$ for $x \in X$
- $[\![\mathtt{op}_i(p; \kappa)]\!]_I(\gamma) = [\![\mathtt{op}_i]\!]_I(p, \lambda a \in A_i.[\![\kappa(a)]\!]_I(\gamma))$

Model as before
Free Model: $\mathtt{Free}_T(X) = \mathtt{Tree}_\Sigma(X)/ \approx_T$
where $\approx_T$ is least congruence enforcing equations $\mathcal{E}_T$

3 Computational Effects as Algebraic Operations

Compuatations

- pure - terminates (i.e: `return v`)
- effectful (i.e.: `op(p,`κ`)` with `p` as parameter, κ the rest of the computation awaiting the result of `op`)

3.1 Example: `l++`

```
lookup(l,λx.update((l,x+1),λ_.return x))
print("Hello World!",λ_.return())
```

Signatures:
for `lookup`: $L \Rightarrow S$
for `update`: $L \times S \Rightarrow 1$
for `print`: `String` $\Rightarrow 1$

3.2 Example: State holding elements of a set S

Signatures:
`put`: $S \Rightarrow 1$
`get`: $1 \Rightarrow S$

```
get((), λx.get((), λy.κ(x, y))) = get((), λz.κ(z, z))
get((), λx.put(x, κ)) = κ()
put(s, get((), κ)) = put(s, λ_.κ(s))
put(s, λ_.put(t, κ)) = put(t, κ)
```

3.3 Example: Exceptions

`abort`: $1 \Rightarrow 0$
$[\![\texttt{abort}]\!]_M : 1 \times |M|^0 \to |M|$ - Non-reusable because 0 is empty.

3.4 Free Model

The free model $\texttt{Free}_T(V)$ is the set of compuatations with effects described by theory T and returning values from set V.
I.e.: $\texttt{Free}_T(V)$ with T as effects and V as return values.

3.4.1 Free Model of State

$\texttt{Free}_{\texttt{State}}(V) \cong S \to S \times V$
$\texttt{Free}_{\texttt{State}}(V) = \texttt{Tree}_{\texttt{State}}(V)/ \approx_{\texttt{State}}$

4 Generic Operations and Sequencing

4.1 New Notation

Generic Operation: $\bar{\texttt{op}}(p) = \texttt{op}(p, \lambda x.\texttt{return}\ x)$
Sequencing: $\texttt{do}\ x \leftarrow t_1\ \texttt{in}\ t_2(x)$
Equations:
$(\texttt{do}\ x \leftarrow \texttt{return}\ v\ \texttt{in}\ t_2(x)) = t_2(v)$
$(\texttt{do}\ x \leftarrow \texttt{op}(\texttt{op}(p, \kappa(y)))\ \texttt{in}\ t_2(x)) = \texttt{op}(p, \lambda y.\texttt{do}\ x \leftarrow \kappa(y)\ \texttt{in}\ t_2(x))$
$\texttt{op}(p, \lambda x.\kappa(x)) = \texttt{do}\ x \leftarrow \bar{\texttt{op}}(p)\ \texttt{in}\ \kappa(x)$
To the programmer:
$\bar{\texttt{op}} : P \to A$
Hence this notation: $\texttt{op} : P \Rightarrow A$