<u>Corollary</u>: If $\bullet \vdash e : \text{bool}$ then there is a $b \in \{\text{True}, \text{False}\}$ such that $e \mapsto^* b$.

We defined $[\![\cdot]\!]$ so that

$$\bullet \vdash e : \tau \implies e \in [\![\tau]\!] \subseteq \text{Terminating}.$$

Next, Define $[\![\cdot]\!]$ for $\forall$.

$$[\![\forall\alpha.\tau]\!] = \{ e \in \text{Terminating} \mid \forall \tau' \in \text{Type}.\ e\,\tau \in [\![\tau[\tau'/\alpha]]\!] \}$$

(<u>for us</u> this is relevant)

BUT, This is not an inductive definition because τ' is not a subterm of $\forall\alpha.\tau$ so no IH applies to τ'.

We want a def of $[\![\cdot]\!]$ that satisfies the two lemmas – termination & expansion.

<u>Def</u>: A reducibility candidate is any set $\mathbb{C}$ of expressions such that

$$\mathbb{C}^* \subseteq \mathbb{C} \subseteq \text{Terminating}$$

Then we could rephrase the lemmas as follows:

Lemma: τ if $[\![\tau]\!]$ is a reducibility candidate.

Let θ = an "environment"

$[\![\alpha]\!]_\theta = \theta(\alpha)$ when α is a Var and θ subst. vals. for vars.

$[\![Bool]\!]_\theta = \{T, F\}^*$

$[\![\tau_1 \to \tau_2]\!]_\theta = \{e \in Terminating \mid \forall e' \in [\![\tau_1]\!]_\theta.\ e\,e' \in [\![\tau_2]\!]_\theta\}$

$[\![\forall\alpha.\tau]\!]_\theta = \{e \in Terminating \mid \forall \tau' \in Type.\ e\,\tau' \in [\![\tau]\!]_{\theta, [\![\tau']\!]/\alpha}\}$

Let θ : TypeVar $\longrightarrow$ CR

$\hookrightarrow$ reducibility candidates

Notation for extending environment:

$$(\theta, \mathbb{C}/\alpha)(\beta) \triangleq \begin{cases} \mathbb{C}, & \beta = \alpha \\ \theta(\beta), & \beta \neq \alpha \end{cases}$$

Finally, we define

$$[\![\forall\alpha.\tau]\!]_\Theta = \{ e \in \text{Terminating} \mid \forall \tau' \in \text{Type}.\ \forall \mathbb{C} \in CR.\ e\,\tau' \in [\![\tau]\!]_{\Theta, \mathbb{C}/\alpha} \}$$

But $\mathbb{C}$ may not have anything to do with the type τ'! So this seems like "false advertising"

Nonetheless, this works because of parametricity. Since e has to do

The same thing in all cases.

It cannot inspect the type α and do something depending on the type.

$$[\![\Theta, \Gamma]\!]_\Theta = \{ \sigma \in \text{Subst} \mid \forall (x:\tau) \in \Gamma.\ x[\sigma] \in [\![\tau]\!]_\Theta \ \wedge\ \forall \alpha \in \Theta.\ \alpha[\sigma] \in \text{Type} \}$$

$\Theta : \text{TypeVar} \to CR$

The meaning of the judgment $\Theta; \Gamma \vdash e : \tau$:

$$[\![\Theta; \Gamma \vdash e : \tau]\!] = \forall \Theta\ \forall \sigma \in [\![\Theta, \Gamma]\!]_\Theta\ e[\sigma] \in [\![\tau]\!]_\Theta$$

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Typing rule for using a $\forall$

$$(\#)\quad \mathcal{D} = \frac{\Theta ; \Gamma \vdash e : \forall \alpha . \tau \qquad \Theta \vdash \tau' : \star}{\Theta , \Gamma \vdash e\,\tau' : \tau[\tau'/\alpha]}$$

— Digression —

$$[\![\Theta]\!] = \{ \theta : \text{TypeVar} \to CR \}$$

Judgments (slightly elaborated)

$$[\![\Theta ; \Gamma \vdash e : \tau]\!] = \forall \theta \in [\![\Theta]\!] . \sigma \in [\![\Theta ; \Gamma]\!]_\theta \; e[\sigma] \in [\![\tau]\!]_\theta$$

$$[\![\Theta \vdash \tau : \star]\!] = \forall \theta \in [\![\Theta]\!]_\theta \; [\![\tau]\!]_\theta \in CR$$

Lemma: If $\Theta \vdash \tau : \star$ is derivable, then $[\![\Theta \vdash \tau : \star]\!]$ is true.

Back to proof of $(\#)$

IH: $[\![\Theta ; \Gamma \vdash e : \forall \alpha . \tau]\!]$ ok

$[\![\Theta \vdash \tau' : \star]\!]$ ok

Show $[\![\Theta ; \Gamma \vdash e\,\tau' : \tau[\tau'/\alpha]]\!]$

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Show $[\![\Theta;\Gamma \vdash e\,\tau' : \tau[\tau'/\alpha]]\!]$

Suppose $\theta \in [\![\Theta]\!]$ and $\sigma \in [\![\Theta;\Gamma]\!]_\theta$

Say $(e\,\tau')[\sigma] = e[\sigma]\,\tau'[\sigma] \in [\![\tau[\tau'/\alpha]]\!]_\theta$

Lemma: $[\![\tau[\tau'/\alpha]]\!]_\theta = [\![\tau]\!]_{\theta,\, [\![\tau']\!]_\theta/\alpha}$

Exercise: Prove this lemma!

By the lemma, $[\![\tau[\tau'/\alpha]]\!]_\theta = [\![\tau]\!]_{\theta,\, [\![\tau']\!]_\theta/\alpha}$

Lemma: $[\![\tau[\tau'/\alpha]]\!]_\theta = [\![\tau]\!]_{\theta,\, [\![\tau']\!]_\theta/\alpha}$

- $$\mathcal{D} = \frac{\Theta, \alpha ; \Gamma \vdash e : \tau}{\Theta ; \Gamma \vdash \Lambda\alpha. e : \forall\alpha.\tau}$$

IH $[\![\Theta, \alpha ; \Gamma \vdash e : \tau]\!]$ is true

Show $[\![\Theta ; \Gamma \vdash \Lambda\alpha. e : \forall\alpha.\tau]\!]$ is true

Suppose $\theta \in [\![\Theta]\!]$ and $\sigma \in [\![\Theta;\Gamma]\!]_\theta$

Show $(\Lambda\alpha.e)[\sigma] = \Lambda\alpha.e[\sigma] \in [\![\forall\alpha.\tau]\!]_\theta$

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(#)

$(\Lambda \alpha . e[\sigma])\ \tau' \longmapsto e[\sigma, \tau'/\alpha]$

If we extend $\sigma, \tau'/\alpha \in [\![\Theta, \Gamma]\!]_\Theta$

Note that (#) is in $[\![\tau]\!]_{\Theta, \mathcal{C}/\alpha}$

So by expansion,

$(\Lambda \alpha . e[\sigma])\tau' \in [\![\tau]\!]_{\Theta, \mathcal{C}/\alpha}$

"Free" Theorems

Example:

$\text{unit} = \forall \alpha . \alpha \to \alpha$

The only closed term of the type

$\bullet \vdash e : \forall \alpha . \alpha \to \alpha$ then

$e =_{\beta,\eta} \Lambda \alpha . \lambda x : \alpha . x$

i.e. e must be the identity.

Proof: From the F.L., we know

$e \in [\![\forall \alpha . \alpha \to \alpha]\!]_\varepsilon$ ε = empty

Then consider $\{x\}^* \in CR$.