Algebraic Effects and Handlers

Andrej Bauer
University of Ljubljana

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1 Schedule of Lectures

1. Algebraic Theories
2. Programming Language
3. Reasoning
4. Applications

2 Algebraic Theories (Universal Algebra)

2.1 Example

A group $(G, u, \cdot, ^{-1})$ is s.t.

- $u \cdot x = x = x \cdot u$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- $x \cdot x^{-1} = u = x^{-1} \cdot x$

Alternative Representation: $(G, u, \cdot)$ and monoid s.t. $\forall x \exists y. x \cdot y = u = y \cdot x$

2.2 Definition: Signature

A signature $\Sigma = \{(\mathtt{op}_i, n_i)\}_{i \in I}$ where $\mathtt{op}_i$ are operation symbols and $n_i \in \mathbb{N}$ is the arity of $\mathtt{op}_i$

2.3 Definition: Term

A term in context $x_1, \ldots, x_k$ is

- one of the variables x_i or
- $\mathtt{op}_i(t_1, \ldots, t_{n_i})$ where $t_1, \ldots, t_{n_i}$ are terms in context $x_1, \ldots, x_k$

2.4 Definition: Algebraic (Equational) Theory

T is $(\Sigma_T, \mathcal{E}_T)$ where Σ_T is a signature and $\mathcal{E}_T$ is a set of equations.
An equation is $x_1, \ldots, x_k | l = r$ where l and r are Σ_T-terms in $x_1, \ldots, x_k$

2.5 Example

Group signature $\Sigma_{Group} = \{(u, 0), (m, 2), (i, 1)\}$
$x | m(u(), x) = x$
$x | m(x, i(x)) = u()$

2.6 Examples

- ring
- empty theory
- field $+, -, \times, ^{-1}, 0, 1$
- Pointed Set:
  signature: $(\cdot, 0)$
  no equations
- semilattice:
  signature: $\{(\perp, 0), (\vee, 2)\}$
  $\perp \vee x = x$
  $x \vee (y \vee z) = (x \vee y) \vee z$
  $x \vee y = y \vee x$
  $x \vee x = x$

3 Interpretation and Models

Suppose T theory

3.1 Definition: Interpretation

An interpretation I of Σ_T is given by

- a carrier set $|I|$
- for each $(\mathtt{op}_i, n_i) \in \Sigma_T$ a map
  $[\![\mathtt{op}_i]\!]_I : |I| \times \cdots \times |I| \to |I|$ with the function taking n_i params $(|I|^{n_i})$

Each term $x_1, \ldots, x_k | t$ is interpreted as a map
$[\![x_i, \ldots, x_k | t]\!] : |I|^k \to |I|$ as follows:

- $[\![x_i, \ldots, x_k | x_i]\!] : |I|^k \to |I|$ i^{th} projection
- $[\![x_1, \ldots, x_k | \mathtt{op}_i(t_1, \ldots, t_{n_i})]\!]_I$ is
  $|I|^k \stackrel{[\![t_1]\!]_I, \ldots, [\![t_{n_i}]\!]_I}{\rightarrow} |I|^{n_i} \stackrel{[\![\mathtt{op}_i]\!]_I}{\rightarrow} |I|$

3.2 Definition: T-Model

A T-Model is an interpretation M of theory T s.t. for every
$x_1, \ldots, x_k | l = r$ in $\mathcal{E}_T$
the maps
$[\![x_1, \ldots, x_k | l]\!]_M : |M|^k \to |M|$ and
$[\![x_1, \ldots, x_k | r]\!]_M : |M|^k \to |M|$ are equal.

3.3 Example

A model M of the theory of a pointed set:

- a carrier set —M—
- a map $[\![\cdot]\!]_M : |M|^0 \to |M|$ (i.e.: $1 \to |M|$

Isomorphically: (S, s) where S is a set and $s \in S$

3.4 Example

Every theory T has the trivial model M:

- $|M| = 1$
- $[\![\mathtt{op}_i]\!]_M : 1^{n_i} \to 1$

3.5 Example

If M and L are T-Models:
$|M \times L| = |M| \times |L|$
$[\![\mathtt{op}_i]\!]_{M \times L} : (|M| \times |L|)^{n_i} \to |M| \times |L|$
$[\![\mathtt{op}_i]\!]_{m \times l}(a_1, \ldots, a_{n_i}) = ([\![\mathtt{op}_i]\!]_M(\pi_1 a_1, \ldots, \pi_1 a_{n_i}), [\![\mathtt{op}_i]\!]_L(\pi_2 a_1, \ldots, \pi_2 a_{n_i}))$

4 Free Models

Given theory T and set X, say that a T-model M with a map $\eta : X \to |M|$ is freely generated by X where $X \xrightarrow{\eta} |M|$
because $\forall f \exists! \bar{f}$ s.t. $X \xrightarrow{f} |L| \implies |M| \xrightarrow{\bar{f}} |L|$
where $\bar{f} : M \to L$ is a T-homomorphism

4.1 Definition: T-Homomorphism

A T-homomorphism $f : M \to L$
For every $\mathtt{op}_i$ in T:
$f([\![\mathtt{op}_i]\!]_M(a_1, \ldots, a_{n_i})) = [\![\mathtt{op}_i]\!]_L(f(a_i), \ldots, f(a_{n_i}))$

4.2 Terminology

$(M, \eta : X \to |M|)$ is a free model over X, free model given by X

4.3 Example

Define $P_{<\omega}(X) := \{S \subseteq X | S \text{finite}\}$
Claim: $(P_{<\omega}(X), 0, \cup)$ is the free semilatice generated by X
with $\eta : X \to P_{<\omega}(X)$, $\eta\ x = \{x\}$

4.4 Free Model Construction ($\texttt{Free}_T(X)$)

1. $\texttt{Tree}_T(X)$ set of well-founded trees defined inductively

   - for every $x \in X$ there is tree $\texttt{return}\ x \in \texttt{Tree}_T(X)$
   - if $(\texttt{op}_i, n_i) \in \Sigma_T$ and $t_1, \ldots, t_{n_1} \in \texttt{Tree}_T(X)$
     then $\texttt{op}_i(t_1, \ldots, t_{n_1}) \in \texttt{Tree}_T(X)$

2. Define $\approx_T$ on $\texttt{Tree}_T(X)$ to be least equiv. rel. s.t.

   - it's a congruence w.r.t. tree formation, and
   - it validates the equations of T