OPLSS-2018-Foundations-day5

Sunday, July 8, 2018 9:13 AM

Logical Relations + Termination

- STLC-"Reducibility"
- System F-"Reducibility Candiates"
- Closed, well-Typed, Terminating

Properties of expansion

Terminating it the property of…opps
Reducibility candidate,
Two typing rules for polymorphism type

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well-Typed e is well-typed when there is a Γ and τ such that $\Gamma \vdash e : \tau$ is derivable

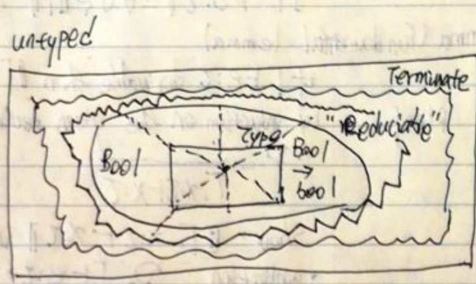
Theorem if e is closed and well-typed, then e is terminating

e is terminating when there is an e' such that $e \mapsto^* e'$ and $e' \not\mapsto$

Theorem if $\cdot \vdash e : \tau$ is derivable, then there is an e' s.t. $e \mapsto^* e' \not\mapsto$

proof: by induction on the derivation $\mathcal{D}$ of $\cdot \vdash e : \tau$

- $\mathcal{D} \neq \dfrac{}{\cdot \vdash x : \tau}$ don't happen



- $\mathcal{D} = \dfrac{}{\cdot \vdash \text{True} : \tau}$

  True $\mapsto^*$ True $\not\mapsto$

- $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1:}{\cdot \vdash e_1 : \text{Bool}} \quad \overset{\mathcal{D}_2:}{\cdot \vdash e_2 : \tau} \quad \overset{\mathcal{D}_3:}{\cdot \vdash e_3 : \tau}}{\cdot \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau}$

IH: for $i = 1, 2, 3$, there is some e_i' s.t. $e_i \mapsto^* e_i' \not\mapsto$

suppose $e_1' = b$: $\mapsto e_i$ ($j = 2$ if b true else $j = 3$)

$\mapsto e_j' \not\mapsto$

else $e_1' \neq b$ 1. if e_i' the e_2 else $e_3 \not\mapsto$ then $e_i' \neq b$

2. Type safety, can't happen.

- $\mathcal{D} = \dfrac{x{:}\tau \vdash e : \tau'}{\cdot \vdash \lambda x. e : \tau \to \tau'}$ $\quad \lambda x.e \mapsto^* \lambda x.e \not\mapsto$

- $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1:}{\cdot \vdash e_1 : \tau \to \tau'} \quad \overset{\mathcal{D}_2:}{\cdot \vdash e_2 : \tau}}{\cdot \vdash e_1\, e_2 : \tau'}$

IH: for $i = 1, 2$, $e_i \mapsto^* e_i' \not\mapsto$

Suppose $e_1' = \lambda x. e'$: $e_1 e_2 \mapsto^* (\lambda x. e') e_2 \mapsto e'[e_2/x]$

Define reducible: $\cdot \vdash e : \tau \Rightarrow e \in [\![\tau]\!] \subseteq$ Terminate

$[\![\text{bool}]\!] = \{\text{True} \mid \text{False}\}^*$ $\quad \mathbb{C}^* = \{e \mid \exists e' \in \mathbb{C}.\ e \mapsto^* e'\}$

(a typed based definition of termination) $(\lambda x. x)$ True : Bool (Semantically evaluates to boolean).

Properties of expansion: ① if $\mathbb{C}$ Termin. $\mathbb{C}^* \subseteq$ Termin

② $\mathbb{C} \subseteq \mathbb{C}^*$

③ $\mathbb{C}^{**} = \mathbb{C}^*$

$[\![\tau_1 \to \tau_2]\!] = \{ e \in \text{Termin} \mid \underline{\forall e' \in [\![\tau_1]\!] \;\; e\, e' \in [\![\tau_2]\!]} \}$

a function, you know, but you have to show these two things

$[\![\Gamma \vdash e : \tau]\!]$

$[\![\Gamma]\!] = \{ \sigma \in \text{Subst} \mid \forall (x:\tau) \in \Gamma.\;\; x[\sigma] \in [\![\tau]\!] \}$

$[\![\Gamma \vdash e : \tau]\!] = \forall \sigma \in [\![\Gamma]\!] \;\; e[\sigma] \in [\![\tau]\!]$

Lemma (Fundamental lemma)

if $\Gamma \vdash e : \tau$ derivable then $[\![\Gamma \vdash e:\tau]\!]$ true

proof: by induction on the given derivation of $\Gamma \vdash e:\tau$

- $\mathcal{D} = \dfrac{}{\Gamma, x:\tau \vdash x:\tau}$

Show: $[\![\Gamma, x:\tau \vdash x:\tau]\!]$ is true

- application $\mathcal{D} = \dfrac{\Gamma \vdash e_1 : \tau \to \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$

IV: $[\![\Gamma \vdash e_1 : \tau \to \tau']\!]$ is true $\;\; [\![\Gamma \vdash e_2 : \tau]\!]$ is true

Show: $[\![\Gamma \vdash e_1 e_2 . \tau']\!]$

Suppose $\sigma \in [\![\Gamma]\!]$

Show: $(e_1 e_2)[\sigma] \in [\![\tau']\!] = e_1(\sigma)\, e_2(\sigma)$

Show: $\underbrace{e_1(\sigma)}_{\in [\![\tau \to \tau']\!]} e_2(\sigma) \in [\![\tau']\!]$

- $\mathcal{D} = \dfrac{\Gamma, x:\tau \vdash e:\tau'}{\Gamma \vdash \lambda x e : \tau \to \tau'}$

IH: $[\![\Gamma, x.\tau \vdash e : \tau']\!]$ is true

show: $[\![\Gamma \vdash \lambda x.e : \tau \to \tau']\!]$

suppose: $\sigma \in [\![\Gamma]\!]$, show $(\lambda x.e)(\sigma) = \lambda x.(e[\sigma']) \in [\![\tau \to \tau']\!]$

suppose $e_2 \in [\![\tau]\!]$, show $(\lambda x.e\, e(\sigma))\, e_2 \in [\![\tau']\!] \mapsto e[\sigma, e_2/x] \in [\![\tau']\!] \;\; \sigma, e_2/x \in [\![\Gamma, x:\tau]\!]$

lemma: $[\![\tau]\!] \subseteq \text{Termin}$

Lemma (expansion) (two facts): $e \mapsto^* e'$ and $e' \in [\![\tau]\!]$ then $e \in [\![\tau]\!]$

i.e. $[\![\tau]\!]^* \subseteq [\![\tau]\!]$

Corollery: if $\cdot \vdash e:\tau$ terminable then $e \in [\![\tau]\!]$ and therefore e is terminating

Exercise: case of different type of boolens)

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case: if $\cdot \vdash e : bool$ then there is a $b = True, False$, such that $e \mapsto^* b$

$\mathbb{C}^* = \{e \mid \exists e' \in \mathbb{C}.\ e \mapsto^* e'\}$

$[\![bool]\!] = \{True, False\}^*$

$[\![\tau_1 \to \tau_2]\!] = \{e \in Termin \mid \forall e' \in [\![\tau_1]\!]\ e\,e' \in [\![\tau_2]\!]$

$[\![\forall \alpha . \tau]\!] = \{e \in Termin \mid \forall \tau' \in Type,\ e\,\tau' \in [\![\tau[\tau'/\alpha]]\!]\}$

$[\![\forall \alpha . \alpha]\!] = \{e \in Termin \mid \forall \tau .\ e\,\tau \in [\![\tau]\!]\}\quad \tau = \forall \alpha . \alpha$

Def:
Reducibility candidate: is any set of expressions $\mathbb{C}$, s.t. $\mathbb{C}^* \subseteq \mathbb{C} \subseteq Termin$.

Lemma: for every τ, $[\![\tau]\!]$ is reducibility candidate

$[\![\alpha]\!]_\theta = \theta(\alpha) \to$ (all the meanings included here?)

$[\![\tau_1 \to \tau_2]\!]_\theta = \{e \in Termin \mid \forall e' \in [\![\tau_1]\!]_\theta.\ e\,e' \in [\![\tau_2]\!]_\theta\}$

$\forall \tau' \in Type.\ e\,\tau' \in [\![\tau]\!]_{\theta, [\![\tau']\!]/\alpha}\}$

θ: Type Var $\longrightarrow CR$ the set of all reducibility candidates is CR

$[\![\forall \alpha . \tau]\!]_\theta = \{e \in Termin \mid \forall \tau' \in Type.\ \forall \mathbb{C} \in CR.\ e\,\tau' \in [\![\tau]\!]_{\theta, \mathbb{C}/\alpha}\}$

i.e. $[\![\tau]\!] \in CR$

$(\theta, \mathbb{C}/\alpha)(\alpha) = \mathbb{C}$

$(\theta, \mathbb{C}/\alpha)(\beta) = \theta(\beta) \quad \alpha \neq \beta$

$[\![\Theta; \Gamma]\!]_\theta = \{\sigma \in Subst \mid \forall (x:\tau) \in \Gamma .\ \sigma(\alpha) \in [\![\tau]\!]_\theta) \wedge (\forall \alpha \in \Theta .\ \alpha[\sigma] \in Type\}$

θ: Type Variable $\longrightarrow CR$

$[\![\Theta ; \Gamma \vdash e : \tau]\!] = \forall \theta, \sigma \in [\![\Theta; \Gamma]\!]_\theta.\ e[\sigma] \in [\![\tau]\!]_\theta$

$\cdot\ \mathcal{D} = \dfrac{\Theta; \Gamma \vdash e : \forall \alpha \tau \quad \Theta \vdash \tau' : *}{\Theta, \Gamma \vdash e\,\tau' : \tau[\tau'/\alpha]}$

IH: $[\![\Theta; \Gamma \vdash e : \forall \alpha . \tau]\!]$ of $[\![\Theta \vdash \tau' : *]\!]$

show $[\![\Theta; \Gamma \vdash e\,\tau'; \tau' : \tau[\tau'/\alpha]]\!]$

suppose $\theta \in [\![\theta]\!]$ and $\partial \in [\![\Theta; \Gamma]\!]_\theta$

say $(e\,\tau')(\partial) = e(\partial)\ \tau'[\partial] \in [\![\tau(\tau'/\alpha)]\!]_\theta = [\![\tau]\!]_{\theta, \frac{\tau'(\theta)}{\alpha}}$

$[\![\Theta]\!] = \{\theta \in Type \longrightarrow CR\}$

the meaning of Θ

$[\![\Theta \vdash \tau : *]\!] = \forall \theta \in [\![\Theta]\!] .\ [\![\tau]\!]_\theta \in CR$

Lemma: if $\Theta \vdash \tau : *$ is derivable then $[\![\Theta \vdash \tau : *]\!]$ is true

Lemma: $[\![\tau[\tau'/\alpha]]\!]_\theta = [\![\tau]\!]_{\theta, [\![\tau']\!]_\theta/\alpha}$

$\cdot\ \mathcal{D} = \dfrac{\Theta, \alpha; \Gamma \vdash e : \tau}{\Theta; \Gamma \vdash \Lambda\alpha . e : \forall \alpha . \tau}$

IH: $[\![\Theta, \alpha; \Gamma \vdash e : \tau]\!]$ is true

Show: $[\![\Theta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau]\!]$ is true

suppose $\theta \in [\![\Theta]\!]$ and $\sigma \in [\![\Theta : \Gamma]\!]\theta$

Show $(\Lambda\alpha.e)[\sigma] = \Lambda\alpha.e[\sigma] \in [\![\forall\alpha.\tau]\!]_\theta$.

suppose τ' and $\mathcal{C} \in CR$

$(\Lambda\alpha.e(\sigma))\,\tau' \mapsto e[\sigma, \tau'/\alpha] \in [\![\tau]\!]_{\theta, \mathcal{C}/\alpha}$.

$\Uparrow \qquad \sigma, \tau'/\alpha \in [\![\theta; \Gamma]\!]_{\theta, \mathcal{C}/\alpha}$

$[\![\tau]\!]_{\theta, \mathcal{C}/\alpha}$

Unit type in System F: $\text{unit} = \forall\alpha.\alpha \to \alpha$

"Free theorem": $\cdot \vdash e : \forall\alpha.\alpha \to \alpha$ then $e =_{\beta\eta} \Lambda\alpha.\lambda x{:}\alpha.x$

tricky

Proof: from the foundamental λ lemma $e \in [\![\forall\alpha.\alpha \to \alpha]\!]_\cdot$

$\{x\}^* \in CR$