Paul Downen (Lecture 1) [2018/07/03] ①

Today's Adgenda

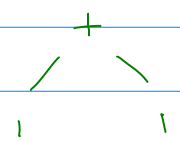
1. Syntax & scope ... (Downen)
2. Static & Dynamic Semantics ... (Hoffmann)

   ... of a programming language

(Lunch)

3. Type Safety
4. λ-calculus

Example $1+1=2$ symmantically equal, but not syntactically

Syntactically, LHS =  $\neq$ 2 = RHS

Let's start by defining a language with

numbers $n \in Num ::= 0 \mid 1 \mid 2 \mid \cdots$

booleans $b \in Bool ::= True \mid False$

variables $Var ::= x \mid y \mid z \mid \cdots$

... and expressions,

$$e \in Expr ::= x \mid Num[n] \mid Bool[b] \mid Plus(e_1, e_2)$$
$$\mid Less(e_1; e_2) \mid If(e_1; e_2; e_3) \mid Let(e_1; x.e_2)$$

Notation Note:

$x.e_2$ means "bind free occurrences of x in e_2"

i.e. e_2 might have a free occurrence of x in it, but $x.e_2$ binds the x.

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Static Scope

(i.e. α-equivalence)

1. names of local (bound) variables don't matter
2. substitution does not "capture" free variables.

EX: Let $x = 5$ in
  Let $y = x$ in
    Let $x = 10$ in y

(static binding = good)

Let $x = 5$
 in let $x = 10$ in x

(Dynamic binding)
(no good because names & variables matter)

Substitution

$e'[e/x]$ want e' but whenever free x occurs in e', replace it with e.

Definition of substitution

Base cases
- $x[e/x] = e$
- $y[e/x] = y$
- $\mathrm{Num}[n][e/x] = \mathrm{Num}[n]$
- $\mathrm{Bool}[b][e/x] = \mathrm{Bool}[b]$

Inductive cases
- $\mathrm{Plus}(e_1; e_2)[e/x] = \mathrm{Plus}(e_1[e/x]; e_2[e/x])$
- $\mathrm{Less}(e_1; e_2) = \mathrm{Less}(e_1[e/x]; e_2[e/x])$
- $\mathrm{If}(e_1; e_2; e_3)[e/x] = \mathrm{If}(e_1[e/x]; e_2[e/x]; e_3[e/x])$
- Let ↓

Substitution for Let

EX: $(\text{Let } x = 1 \text{ in } x)[5/x] = \text{let } x = 1 \text{ in } x$

Case: $\text{Let}(e_1; x.e_2)[e/x] = \text{Let}(e_1[e/x]; x.e_2)$

plug in for any free x's in e_1.   don't plug in here because x is bound.

Case: $\text{Let}(e_1; y.e_2)[e/x] = \text{Let}(e_1[e/x]; y.e_2[e/x])$

$x \neq y$ & $y \notin FV(e)$

But these don't cover all the cases (To be cont)

Bound and Free Variables

$BV(x) = \emptyset$

$BV(\text{Num}[n]) = \emptyset$

$BV(\text{Bool}[b]) = \emptyset$

$BV(\text{Plus}(e_1; e_2)) = BV(e_1) \cup BV(e_2)$

Similar for If and Less.

$BV(\text{Let}(e_1; x.e_2)) = BV(e_1) \cup BV(e_2) \cup \{x\}$.

$FV(x) = \{x\}$

$FV(\text{Num}[n]) = \emptyset = FV(\text{Bool}[b])$

$FV(\text{Plus}(e_1; e_2)) = FV(e_1) \cup FV(e_2) = FV(\text{Less}(e_1; e_2))$.

$FV(\text{Let}(e_1; x.e_2)) = FV(e_1) \cup (FV(e_2) \setminus \{x\})$

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α-equivalence

syntactically, 'let $x=1$ in x' is not equal to 'let $y=1$ in y'
But 'let $x=1$ in x' $=_\alpha$ 'let $y=1$ in y'

$$\text{let } x = e_1 \text{ in } e_2 =_\alpha \text{let } y = e_1 \text{ in } e_2[y/x]$$
$$(y \notin FV(e_2))$$

Properties of Substitution

1. For all e and e', if $BV(e') \cap FV(e) = \emptyset$ then $e'[e/x]$ is defined.

2. For all e, e', if $x \notin FV(e')$, then $e'[e/x] = e'$.

Properties of α-equivalence

1. For all e, e', $\exists e''. (e =_\alpha e''$ and $e''[e'/x]$ is defined.$)$
(so, with α-equivalence, substitution becomes a total fn.)

2. If $e =_\alpha e'$, then $e[e''/x] =_\alpha e'[e''/x]$.

Homework

1. Add more operations to this language (e.g. minus, equals)
2. Prove some of the properties listed above (by structural induction)