# OPLSS-218-Foundations-day4

Saturday, July 7, 2018 9:08 AM

7/7/2018 9:12 AM
Linear Logic

- Logic & Programing Languages: propositions and type.
- It's all about behaviors

Curry-Howard Correspondences

- Proofs as programs
- Propositions as Types

| **Programming** | **Logic** |
|---|---|
| Functions | Implications ("if") |
| Product types | Conjunction ("and") |
| Sum types | Disjunction("or") |
| STLC | Intuitionistic logic |
| Simple type, ƛ-calculus | |
| polymorphism | Forall, 2nd order logic |
| Control-flow | Classical logic |
| Combination calculus | Hibert logic |
| Dependent types | 1st-order logic |

What is linear logic?
7/7/2018 11:11 AM
Beef…-_-->

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$$\frac{}{\Gamma, \tau \vdash \tau} \text{ Axiom}$$

$$\frac{\Gamma \vdash \tau \to \tau' \quad \Gamma \vdash \tau}{\Gamma \vdash \tau'} \to E$$

$$\frac{\Gamma, \tau \vdash \tau'}{\Gamma \vdash \tau \to \tau'} \to E$$

Linear Logic

$\tau_1, \tau_2, \ldots \tau_n \vdash \tau_1', \tau_2' \ldots \tau_m'$ → (propositions, things can be true

"if all $\tau_1 \ldots \tau_n$ are true then (at least) one of $\tau_1' \ldots \tau_n'$ is true"

Admissible Structured rules

$$\frac{\Gamma \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ (weakening left)} \qquad \frac{\Gamma, \tau, \tau \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ condition left} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau, \Delta} \text{ weakening right}$$

↓conjunction

$$\frac{\Gamma \vdash \tau, \tau, \Delta}{\Gamma \vdash \tau, \Delta} \text{ condition right}$$

Gentzen's Sequent Calculus

linear logicians reject above ⇑

Conjunction

binary propositions

$$\frac{\Gamma, \tau_1 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \& L_1 \qquad \frac{\Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \& L_2$$

| "zero" | "Bottom" $\perp$ | "Top" $\top$ | one "unit" $1$

$$\frac{\Gamma \vdash \tau_1, \Delta \quad \Gamma \vdash \tau_2}{\Gamma \vdash \tau_1 \& \tau_2, \Delta} \& R$$

$\tau ::= \ldots \tau_1 \& \tau_2$ | $\tau_1 \otimes \tau_2$ | $\tau_1 \oplus \tau_2$ | $\tau_1 ⅋ \tau_2$

"with" "times" "plus" "par"

$$\frac{}{\tau \vdash \tau} Ax \qquad \frac{\Gamma_1 \vdash \tau_1, \Delta_1 \quad \Gamma_2 \vdash \tau_2, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \tau_1 \otimes \tau_2, \Delta_1, \Delta_2} \otimes R \qquad \frac{\Gamma, \tau_1, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \otimes \tau_2 \vdash \Delta} \otimes L$$

Disjunction

$$\frac{\Gamma \vdash \tau_1, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta} \oplus R_1 \qquad \frac{\Gamma \vdash \tau_2, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta} \oplus R_2$$

$$\frac{\Gamma, \tau_1 \vdash \Delta \quad \Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \oplus \tau_2 \vdash \Delta} \oplus L \qquad \frac{}{\tau \vdash \tau} Ax \qquad \frac{\Gamma \vdash \tau, \Delta \quad \Gamma', \tau, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ cut}$$

$$\frac{\Gamma_1, \tau_1 \vdash \Delta_1 \quad \Gamma_2, \tau_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \tau_1 ⅋ \tau_2 \vdash \Delta_1, \Delta_2} ⅋ L \qquad \frac{\Gamma \vdash \tau_1, \tau_2, \Delta}{\Gamma \vdash \tau_1 ⅋ \tau_2, \Delta} ⅋ R$$

unit (1)

$$\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ 1L \qquad \frac{}{\cdot \vdash 1}\ 1R$$

T

$$\frac{}{\Gamma \vdash \top, \Delta}\ \top R \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0L \qquad \text{no } 0R \text{ rules} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot R$$

$$\frac{}{\bot \vdash \cdot}\ \bot L$$

$$\frac{\text{beef, bun} \vdash \text{burger} \quad \text{potato, oil} \vdash \text{fries}}{\text{beef, bun, potato, oil} \vdash \text{burger} \otimes \text{fries}}\ \otimes R$$

→ mandatory ~~two things.~~

~~beef, oil ⊢ t fs~~

$$\frac{\cancel{\text{tomato}}\text{, cream} \vdash \text{tomato sauce} \qquad \text{chicken-noodle} \vdash \text{ch-noodle soup}}{\text{tomato, cream} \vdash \text{tomato soup} \oplus \text{ch-noodle soup}}\ \oplus R$$

soup of the day

"or"? disjunction means the meaning of $\oplus$ and $\otimes$

Main = burgers $\otimes$ fries

Soup = tomato soup $\oplus$ ch noodle soup

$$\frac{\text{strawb, banana} \vdash \text{fruit salad} \qquad \text{straw, banana} \vdash \text{Smoothie}}{\text{strawb, banana} \vdash \text{fruit salad} \,\&\, \text{soomthie}}$$

dessert = fruit salad & Smoothie

lunch = main $\otimes$ soup $\otimes$ dessert

ingred = beef, bun, potato, oil, tomato, cream, strawb, banana

lunch of today (multiple ingredients)

$$\frac{\dfrac{\text{ingred} \vdash \text{lunch} \qquad \text{ingred} \vdash \text{lunch}}{\text{ingred, ingred} \vdash \text{lunch} \parr, \text{lunch}}\ \text{Mix}}{\text{ingred, ingred} \vdash \text{lunch} \parr \text{lunch}}\ \parr R$$

Mix

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \qquad \frac{}{\vdash \tau, \tau^\perp}\ \text{Id} \qquad \frac{\vdash \tau, \Delta \quad \vdash \tau^\perp, \Delta'}{\vdash \Delta, \Delta'}\ \text{cut}$$

Negation

$$(\tau_1 \,\&\, \tau_2)^\perp = \tau_1^\perp \oplus \tau_2^\perp \qquad (\tau_1 \oplus \tau_2)^\perp = \tau_1^\perp \,\&\, \tau_2^\perp$$

$$(\tau_1 \otimes \tau_2)^\perp = \tau_1^\perp \parr \tau_2^\perp \qquad (\tau_1 \parr \tau_2)^\perp = \tau_1^\perp \otimes \tau_2^\perp$$

$$\top^\perp = 0 \qquad 0^\perp = \top$$

$$\bot^\top = 1 \qquad 1^\perp = \bot$$

Theorem : $\tau = \tau^{\perp\perp} \qquad \neg\tau = \tau \rightarrow 0$

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Theorem (Duality) if $\Gamma \vdash \Delta$ is derivable then $\Delta^\perp \vdash \Gamma^\perp$ is, too.

$$\frac{\Gamma, \tau \vdash \Delta}{\Gamma \vdash \tau^\perp, \Delta}\ \text{invL} \qquad \frac{\Gamma \vdash \tau, \Delta}{\Gamma, \tau^\perp \vdash \Delta}\ \text{invR}$$

$$\tau_1 \multimap \tau_2 = \tau^\perp \parr \tau_2$$
$$\tau_1 \parr \tau_2 = \tau_1^\perp \multimap \tau_2$$

$$\frac{\dfrac{}{\tau \vdash \tau}\ \text{Id}}{\tau, \tau^\perp \vdash \cdot}\ \text{inv left} \qquad \frac{\dfrac{}{\tau \vdash \tau}\ \text{Id}}{\vdash \tau \multimap \tau}\ \text{invR} \qquad \frac{\dfrac{\dfrac{}{\tau^{\perp\perp} \vdash \tau}\ \text{Id}}{\cdot \vdash \tau^\perp, \tau}\ \text{In}}{\cdot \vdash \tau^\perp \parr \tau}\ \parr R$$

linear $\lambda$-calculus

$$\lambda x{:}\tau\ \langle x, x\rangle : \tau \to \tau \otimes \tau$$

$$\tau ::= \top \mid 0 \mid 1 \mid \tau_1 \multimap \tau_2 \mid !\tau \mid \tau_1 \& \tau_2 \mid \tau_1 \oplus \tau_2 \mid \tau_1 \otimes \tau_2$$

$$\frac{\Gamma;\Delta \vdash e : \tau \quad \Gamma;\Delta', x{:}\tau \vdash e' : \tau'}{\Gamma;\Delta,\Delta' \vdash \text{let } x = e \text{ in } e' : \tau'}$$

$\Gamma ; \Delta \vdash e : \tau$ (Type checking

$\Gamma$: non-linear, $\Delta$: linear

$$\frac{}{\Gamma; x{:}\tau \vdash x : \tau}\ \text{Id}^1 \qquad \frac{}{\Gamma, x{:}\tau; \cdot \vdash x : \tau}\ \text{Id}^*$$

$$\frac{\Gamma;\Delta \vdash e : \tau_1}{\Gamma;\Delta \vdash l.e : \tau_1 \oplus \tau_2} \qquad \frac{\Gamma;\Delta \vdash e : \tau_2}{\Gamma;\Delta \vdash r.e : \tau_1 \oplus \tau_2}$$

$$\frac{\Gamma; \vdash e : \tau_1 \oplus \tau_2 \quad \Gamma;\Delta', x{:}\tau_1 \vdash e_1 : \tau \quad \Gamma;\Delta', y{:}\tau_2 \vdash e_2 : \tau}{\Gamma; \Delta\Delta' \vdash \text{case } e \text{ of } l.x \to e_1 : \tau \ \ r.y \to e_2}$$

$$\frac{\Gamma;\Delta_1 \vdash e_1 : \tau_1 \quad \Gamma;\Delta_2 \vdash e_2 : \tau_2}{\Gamma;\Delta_1,\Delta_2 \vdash \langle e_1, e_2\rangle : \tau_1 \otimes \tau_2}$$

$$\frac{\Gamma;\Delta \vdash e : \tau_1 \otimes \tau_2 \quad \Gamma;\Delta', x{:}\tau_1, y{:}\tau_2 \vdash e' : \tau'}{\Gamma;\Delta,\Delta' \vdash \text{case } e \text{ of } \langle x, y\rangle \to e' : \tau'}$$

$$\frac{\Gamma;\Delta \vdash e : \tau_1 \& \tau_2}{\Gamma;\Delta \vdash e.l : \tau_1} \qquad \frac{\Gamma;\Delta \vdash e : \tau_1 \& \tau_2}{\Gamma;\Delta \vdash e.r : \tau_2}$$

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$$\frac{\Gamma;\Delta\vdash e_1:\tau_1 \quad \Gamma;\Delta\vdash e_2:\tau_2}{\Gamma;\Delta\vdash \{l\Rightarrow e_1,\ r\Rightarrow e_2\}:\tau_1 \& \tau_2}$$

$$\frac{\Gamma;\Delta\vdash e:\tau_1\otimes\tau_2 \quad \Gamma;\Delta',x:\tau_1,y:\tau_2\vdash e':\tau'}{\Gamma;\Delta,\Delta'\vdash \text{case } e \text{ of } \langle x,y\rangle\to e':\tau'}$$

Linear Arrow

$$\frac{\Gamma;\Delta,x:\tau\vdash e:\tau'}{\Gamma;\Delta\vdash \lambda x:\tau.e:\tau\multimap\tau'}$$

$$\frac{\Gamma;\vdash e:\tau\multimap\tau' \quad \Gamma;\Delta'\vdash e':\tau}{\Gamma;\Delta,\Delta'\vdash e\ e':\tau'}$$

$$\frac{\Gamma;\cdot\vdash e:\tau}{\Gamma;\cdot\vdash \text{many } e:\tau}$$

$$\frac{\Gamma;\ e:!\tau \quad \Gamma,x:\tau;\ \vdash e:\tau'}{\Gamma;\ \text{case } e \text{ of many } x\Rightarrow e':\tau'}$$

$$\frac{}{\Gamma;\Delta\vdash\{\}:\tau}$$

$$\frac{\Gamma;\Delta\vdash e:0}{\Gamma;\Delta\vdash \text{case } e \text{ of}\{\}:\tau}$$

$$\frac{}{\Gamma;\cdot\vdash\langle\rangle:1}$$

$$\frac{\Gamma;\Delta\vdash e:1 \quad \Gamma;\Delta'\vdash e':\tau}{\Gamma;\Delta,\Delta'\vdash \text{case } e \text{ of}\langle\rangle\Rightarrow e':\tau}$$

Dynamics. reduction rules

$$V ::= \langle\rangle \mid \langle V,V\rangle \mid l.V \mid r.V \mid \lambda x:\tau.e \mid \{l\Rightarrow e,\ r\Rightarrow e\} \mid \{\}$$
$$\mid \text{many } V$$

$\text{let } x=v \text{ in } e \longmapsto e[v/x] \qquad (\lambda x.e)V \longmapsto e[V/x]$

$\{l\Rightarrow e_1, r\Rightarrow e_2\}.l \longmapsto e_1$

$\text{case } l.V \text{ of } \{l.x\Rightarrow e_1;\ r.y\Rightarrow e_2\} \longmapsto e_1[V/x]$

$\text{case } \langle V_1,V_2\rangle \text{ of } \langle x,y\rangle\Rightarrow e \longmapsto e[V_1/x, V_2/y]$

$\text{case many } V \text{ of } x\Rightarrow e \longmapsto e[V/x]$

Lemma (Substitution):

· linear. if $\Gamma;\Delta\vdash e:\tau$ and $\Gamma;\Delta',x:\tau\vdash e':\tau'$

then $\Gamma;\Delta,\Delta'\ e'[e/x]:\tau'$

non-linear

if $\Gamma;\cdot\vdash e:\tau$ and $\Gamma,x:\tau;\Delta'\vdash e':\tau'$

then

$\Gamma;\Delta'\vdash e'[e/x]:\tau'$

Type safety = resource safety