Paul Downen (Lecture 6) [2018/07/07]

Recall, Typing Rules for the λ-calc (STLC)

1

$$\frac{}{\Gamma, x:\tau \vdash x:\tau}$$

$$\frac{\Gamma \vdash e : \tau \to \tau' \qquad \Gamma \vdash e' : \tau}{\Gamma \vdash e e' : \tau'}$$

$$\frac{\Gamma, x:\tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : \tau \to \tau'}$$

If we erase the programs in the STLC we get a small logic

2

$$\frac{}{\Gamma, \tau \vdash \tau} \text{(Axiom)}$$

$$\frac{\Gamma \vdash \tau \to \tau' \qquad \Gamma \vdash \tau}{\Gamma \vdash \tau'} (\to E)$$

$$\frac{\Gamma\ \tau \vdash \tau'}{\Gamma \vdash \tau \to \tau'} (\to I)$$

The corresp. btwn 1 & 2 is the "Curry Howard" correspondence.

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Programming	Logic
function	implication ("if, then")
product	conjunction ("and")
sum	disjunction ("or")
STLC	Intuitionistic Logic — Gentzen's natural ded.
polymorphism	$\forall$ 2nd order logic (quantify over Props)
control-flow	classical logic
combinator calculus	Hilbert logic
dependent types	1st order logic (quantify over numbers, eg.)

Linear Logic

$$\tau_1, \ldots, \tau_n \vdash \tau_1', \ldots \tau_m'$$

The $\vdash$ relation here means: "if all τ_i 'true', then at least one of τ_i' 'true'"

(Admissible) Structural Rules

$$\frac{\Gamma \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ (weakening on left)} \qquad \frac{\Gamma \vdash \tau'}{\Gamma \vdash \tau, \tau'} \text{ (weakening right)}$$

$$\frac{\Gamma, \tau, \tau \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ (contraction left)} \qquad \frac{\Gamma \vdash \tau, \tau, \tau'}{\Gamma \vdash \tau, \tau'} \text{ (contract. right)}$$

(Gentzen's Sequent calculus)

The structural rules of weakening, contraction are sensible rules in some contexts/applications.

In Linear Logic, we deny them.

Linear Logic

Denote by Δ a comma separated list of τ's. (possibly empty)

$$\frac{\Gamma, \tau_1 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \ (\&\text{-}L_1)$$

$$\frac{\Gamma \vdash \tau_1, \Delta \qquad \Gamma \vdash \tau_2, \Delta}{\Gamma \vdash \tau_1 \& \tau_2, \Delta} \ (\& R)$$

$$\frac{\Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \ (\&\text{-}L_2)$$

$$\frac{}{\tau \vdash \tau} \ \left(\begin{array}{c}Ax \\ \text{linearity} \\ \text{axiom}\end{array}\right)$$

Think of Δ_1, Δ_2 as "resources on the right"

$$\frac{\Gamma_1 \vdash \tau_1, \Delta_1 \qquad \Gamma_2 \vdash \tau_2, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \tau_1 \otimes \tau_2, \Delta_1, \Delta_2} \ \otimes R$$

$$\frac{\Gamma, \tau_1, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \otimes \tau_2 \vdash \Delta} \ \otimes L$$

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$$\tau ::= \underset{\text{"with"}}{\tau_1 \& \tau_2} \mid \underset{\text{"times"}}{\tau_1 \otimes \tau_2} \mid \underset{\text{"plus"}}{\tau_1 \oplus \tau_2} \mid \underset{\text{"par"}}{\tau_1 \parr \tau_2} \mid \underset{\text{"top"}}{\top} \mid \underset{\text{"one"}}{1} \mid \underset{\text{"zero"}}{0} \mid \underset{\text{"bottom"}}{\bot}$$

$$\frac{\Gamma, \tau_1 \vdash \Delta \qquad \Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \oplus \tau_2 \vdash \Delta}\ \oplus L \qquad \frac{\Gamma \vdash \tau_1, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta}\ \oplus R_1$$

$$\frac{\Gamma \vdash \tau_2, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta}\ \oplus R_2$$

$$\frac{\Gamma \vdash \tau, \Delta \qquad \Gamma', \tau \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ (cut)$$

(This precludes contraction)

$$\frac{\Gamma_1, \tau_1 \vdash \Delta_1 \qquad \Gamma_2, \tau_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2\ \tau_1 \parr \tau_2 \vdash \Delta_1, \Delta_2}\ (\parr L) \qquad \frac{\Gamma \vdash \tau_1, \tau_2, \Delta}{\Gamma \vdash \tau_1 \parr \tau_2, \Delta}\ (\parr R)$$

(There is no T-L rule.)

$$\frac{}{\Gamma \vdash \top, \Delta}\ (\top\text{-}R)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ (1\text{-}L) \qquad \frac{}{\cdot \vdash 1}\ (1\text{-}R)$$

(multiplicative identity)

$$\frac{}{\Gamma, 0 \vdash \Delta}\ (0\text{-}L)$$

(no 0-R rule)

(additive identity)

$$\frac{}{\bot \vdash \cdot}\ (\bot\text{-}L) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ (\bot R)$$