Computational Higher Type Theory III: Univalent Universes and Exact Equality

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Abstract

This is the third in a series of papers extending Martin-Löf's *meaning explanations* of dependent type theory to a Cartesian cubical realizability framework that accounts for higher-dimensional types. We extend this framework to include a cumulative hierarchy of univalent Kan universes of Kan types; exact equality and other pretypes lacking Kan structure; and a cumulative hierarchy of pretype universes. As in Parts I and II, the main result is a *canonicity theorem* stating that closed terms of boolean type evaluate to either true or false. This establishes the computational interpretation of Cartesian cubical higher type theory based on cubical programs equipped with a deterministic operational semantics.

1 Introduction

In Parts I and II of this series [Angiuli et al., 2016; Angiuli and Harper, 2016] we developed *mathematical meaning explanations* for higher-dimensional type theories with Cartesian cubical structure [Angiuli et al., 2017]. In Part III, we extend these meaning explanations to support an infinite hierarchy of Kan, univalent universes [Voevodsky, 2010].

Mathematical meaning explanations We define the judgments of computational higher type theory as dimension-indexed relations between programs equipped with a deterministic operational semantics. These relations are cubical analogues of Martin-Löf's *meaning explanations* [Martin-Löf, 1984] and of the original Nuprl type theory [Constable, et al., 1985], in which types are merely specifications of the computational behavior of programs. Because types are defined behaviorally, we trivially obtain the *canonicity* property at every type. (Difficulties instead lie in checking formation, introduction, and elimination rules. In contrast, the type theory of Cohen et al. [2018] is defined by such rules, and a separate argument by Huber [2016] establishes canonicity.)

Theorem 1 (Canonicity)**.** *If M is a closed term of type* bool*, then* $M \Downarrow \mathsf{true}$ *or* $M \Downarrow \mathsf{false}$.

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In a sense, our meaning explanations serve as *cubical logical relations*, or a *cubical realizability model*, justifying the rules presented in Section 6. However, those rules are intended only for reference; the rules included in the RedPRL proof assistant [Sterling et al., 2017] differ substantially (as described in Section 6). Moreover, as $M \in A\ [x_1, \dots, x_n]$ means that M is a (n-dimensional) program with behavior A, programs do not have unique types, nor are typing judgments decidable.

Cartesian cubes Our programs are parametrized by *dimension names* $x, y, \dots$ ranging over an abstract interval with end points 0 and 1. Programs with at most n free dimension names represent n-dimensional cubes: points ($n = 0$), lines ($n = 1$), squares ($n = 2$), and so forth. Substituting $\langle 0/x\rangle$ or $\langle 1/x\rangle$ yields the left or right face of a cube in dimension x; substituting $\langle y/x\rangle$ yields the x, y diagonal; and weakening by y yields a cube degenerate in the y direction.

The resulting notion of cubes is Cartesian [Licata and Brunerie, 2014; Awodey, 2016; Buchholtz and Morehouse, 2017]. In contrast, the Bezem et al. [2014] model of type theory has only faces and degeneracies, while the Cohen et al. [2018] type theory uses a de Morgan algebra of cubes with connections ($x \wedge y$, $x \vee y$) and reversals ($1 - x$) in addition to faces, diagonals, and degeneracies. The Cartesian notion of cube is appealing because it results in a *structural* dimension context (with exchange, weakening, and contraction) and requires no equational reasoning at the dimension level.

Kan operations *Kan types* are types equipped with coercion (coe) and homogeneous composition (hcom) operations. If A is a Kan type varying in x, the *coercion* $\mathsf{coe}^{r\rightsquigarrow r'}_{x.A}(M)$ sends an element M of $A\langle r/x\rangle$ to an element of $A\langle r'/x\rangle$, such that the coercion is equal to M when $r = r'$. For example, given a point M in the $\langle 0/x\rangle$ side of the type A, written $M \in A\langle 0/x\rangle\ [\cdot]$, we can coerce it to a point $\mathsf{coe}^{0\rightsquigarrow 1}_{x.A}(M)$ in $A\langle 1/x\rangle$, or coerce it to an x-line $\mathsf{coe}^{0\rightsquigarrow x}_{x.A}(M)$ between M and $\mathsf{coe}^{0\rightsquigarrow 1}_{x.A}(M)$.

$$M \xrightarrow{\mathsf{coe}^{0\rightsquigarrow x}_{x.A}(M)} \mathsf{coe}^{0\rightsquigarrow 1}_{x.A}(M)$$

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "y"'] & {} \\ {} & \end{tikzcd}
\qquad
\begin{tikzcd} \cdot \arrow[r, "M"] \arrow[d, "N_0"'] \arrow[dr, phantom, "\mathsf{hcom}^{0\rightsquigarrow y}_{A}(M;\cdots)"] & \cdot \arrow[d, "N_1"] \\ \cdot \arrow[r, dashed, "\mathsf{hcom}^{0\rightsquigarrow 1}_{A}(M;\cdots)"'] & \cdot \end{tikzcd}$$

If A is a Kan type, then *homogeneous composition* in A states that any open box in A has a composite; for example, $\mathsf{hcom}^{0\rightsquigarrow 1}_{A}(M; x = 0 \hookrightarrow y.N_0, x = 1 \hookrightarrow y.N_1)$ is the bottom line of the above square. The cap M is a line on the $\langle 0/y\rangle$ side of the box; $y.N_0$ (resp., $y.N_1$) is a line on the $x = 0$ (resp., $x = 1$) side of the box; and the composite is on the $\langle 1/y\rangle$ side of the box. Furthermore, the cap and tubes must be equal where they coincide (the $x = 0$ side of M with the $\langle 0/y\rangle$ side of N_0), every pair of tubes must be equal where they coincide (vacuous here, as $x = 0$ and $x = 1$ are disjoint) and the composite is equal to the tubes where they coincide (the $x = 0$ side of the composite with the $\langle 1/y\rangle$ side of N_0). Fillers are the special case in which we compose to a free dimension name y; here, $\mathsf{hcom}^{0\rightsquigarrow y}_{A}(M; x = 0 \hookrightarrow y.N_0, x = 1 \hookrightarrow y.N_1)$ is the entire square.

These Kan operations are variants of the uniform Kan conditions first proposed by Bezem et al. [2014]. Notably, Bezem et al. [2014] and Cohen et al. [2018] combine coercion and composition into a single heterogeneous composition operation and do not allow compositions from or to dimension names. Unlike both Cohen et al. [2018] and related work by Licata and Brunerie [2014], we allow tubes along diagonals ($x = z$), and require every non-trivial box to contain at least one opposing pair of tubes $x = 0$ and $x = 1$. The latter restriction (detailed in Definition 21) allows us to achieve canonicity for zero-dimensional elements of the circle and weak booleans.

Pretypes and exact equality As in the "two-level type theories" of Voevodsky [2013], Altenkirch et al. [2016], and Boulier and Tabareau [2017], we allow for *pretypes* that are not necessarily Kan. In particular, we have types $\mathsf{Eq}_A(M, N)$ of *exact equalities* that internalize (and reflect into) judgmental equalities $M \doteq N \in A\ [\Psi]$. Exact equality types are not, in general, Kan, as one cannot compose exact equalities with non-degenerate lines. However, unlike in prior two-level type theories, certain exact equality types *are* Kan (for example, when $A = \mathsf{nat}$; see Section 7 for a precise characterization). We write $A\ \mathsf{type_{pre}}\ [\Psi]$ when A is a pretype, and $A\ \mathsf{type_{Kan}}\ [\Psi]$ when A is a Kan type. Pretypes and Kan types are both closed under most type formers; for example, if $A\ \mathsf{type}_\kappa\ [\Psi]$ and $B\ \mathsf{type}_\kappa\ [\Psi]$ then $A \to B\ \mathsf{type}_\kappa\ [\Psi]$.

Universes and univalence We have two cumulative hierarchies of universes $\mathcal{U}_j^{\mathsf{pre}}$ and $\mathcal{U}_j^{\mathsf{Kan}}$ internalizing pretypes and Kan types respectively. The Kan universes $\mathcal{U}_j^{\mathsf{Kan}}$ are both Kan and univalent. (See `https://git.io/vFjUQ` for a RedPRL-checked proof of the univalence theorem.) Homogeneous compositions of Kan types are types whose elements are formal boxes of elements of the constituent types. Every equivalence E between A and B gives rise to the $\mathsf{V}_x(A, B, E)$ type whose x-faces are A and B; such types are a special case of "Glue types" [Cohen et al., 2018].

RedPRL RedPRL is an interactive proof assistant for computational higher type theory in the tradition of LCF and Nuprl; the RedPRL logic is principally organized around dependent refinement rules [Spiwack, 2011; Sterling and Harper, 2017], which are composed using a simple language of proof tactics. Unlike the inference rules presented in Section 6, RedPRL's rules are given in the form of a goal-oriented sequent calculus which is better-suited for both programming and automation.

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This paper directly continues work previously described in Angiuli et al. [2016], Angiuli and Harper [2016], and Angiuli et al. [2017], whose primary antecedents are two-dimensional type theory [Licata and Harper, 2012], the Bezem et al. [2014] cubical model of type theory, and the cubical type theories of Cohen et al. [2018] and Licata and Brunerie [2014].

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2 Programming language

The programming language itself has two sorts—dimensions and terms—and binders for both sorts. Terms are an ordinary untyped lambda calculus with constructors; dimensions are either dimension constants (0 or 1) or dimension names ($x, y, \dots$), the latter behaving like nominal constants [Pitts, 2015]. Dimensions may appear in terms: for example, loop_r is a term when r is a dimension. The operational semantics is defined on terms that are closed with respect to term variables but may contain free dimension names.

Dimension names represent generic elements of an abstract interval whose end points are notated 0 and 1. While one may sensibly substitute any dimension for a dimension name, terms are *not* to be understood solely in terms of their dimensionally-closed instances (namely, their end points). Rather, a term's dependence on dimension names is to be understood generically; geometrically, one might imagine additional unnamed points in the interior of the abstract interval.

2.1 Terms

$$
\begin{aligned}
M := {} & (a{:}A) \to B \mid (a{:}A) \times B \mid \mathsf{Path}_{x.A}(M, N) \mid \mathsf{Eq}_A(M, N) \mid \mathsf{void} \mid \mathsf{nat} \mid \mathsf{bool} \\
& \mid \mathsf{wbool} \mid \mathbb{S}^1 \mid \mathcal{U}^{\mathsf{pre}}_j \mid \mathcal{U}^{\mathsf{Kan}}_j \mid \mathsf{V}_r(A, B, E) \mid \mathsf{Vin}_r(M, N) \mid \mathsf{Vproj}_r(M, F) \\
& \mid \lambda a.M \mid \mathsf{app}(M, N) \mid \langle M, N \rangle \mid \mathsf{fst}(M) \mid \mathsf{snd}(M) \mid \langle x \rangle M \mid M@r \mid \star \\
& \mid \mathsf{z} \mid \mathsf{s}(M) \mid \mathsf{natrec}(M; N_1, n.a.N_2) \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}_{b.A}(M; N_1, N_2) \\
& \mid \mathsf{base} \mid \mathsf{loop}_r \mid \mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M; N_1, x.N_2) \\
& \mid \mathsf{coe}^{r \rightsquigarrow r'}_{x.A}(M) \mid \mathsf{hcom}^{r \rightsquigarrow r'}_A(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \\
& \mid \mathsf{com}^{r \rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \mid \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \\
& \mid \mathsf{ghcom}^{r \rightsquigarrow r'}_A(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \mid \mathsf{gcom}^{r \rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \\
& \mid \mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}) \mid \mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i})
\end{aligned}
$$

We use capital letters like M, N, and A to denote terms, r, r', r_i to denote dimensions, x to denote dimension names, ε to denote dimension constants (0 or 1), and $\overline{\varepsilon}$ to denote the opposite dimension constant of ε. We write $x.-$ for dimension binders, $a.-$ for term binders, and $\mathsf{FD}(M)$ for the set of dimension names free in M. Additionally, in $(a{:}A) \to B$ and $(a{:}A) \times B$, a is bound in B. Dimension substitution $M\langle r/x \rangle$ and term substitution $M[N/a]$ are defined in the usual way.

The final argument of most composition operators is a (possibly empty) list of triples $(r_i, r'_i, y.N_i)$ whose first two components are dimensions, and whose third is a term (in some cases, with a bound dimension). We write $\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}$ to abbreviate such lists or transformations on such lists, and ξ_i to abbreviate $r_i = r'_i$ when their identity is irrelevant.

Definition 2. We write $M\ \mathsf{tm}\ [\Psi]$ when M is a term with no free term variables, and $\mathsf{FD}(M) \subseteq \Psi$. (Similarly, we write $M\ \mathsf{val}\ [\Psi]$ when $M\ \mathsf{tm}\ [\Psi]$ and $M\ \mathsf{val}$.)

Definition 3. A total dimension substitution $\psi : \Psi' \to \Psi$ assigns to each dimension name in Ψ either 0, 1, or a dimension name in Ψ'. It follows that if $M\ \mathsf{tm}\ [\Psi]$ then $M\psi\ \mathsf{tm}\ [\Psi']$.

2.2 Operational semantics

The following describes a deterministic weak head reduction evaluation strategy for (term-)closed terms in the form of a transition system with two judgments:

1. $M\ \mathsf{val}$, stating that M is a *value*, or *canonical form*.

2. $M \longmapsto M'$, stating that M takes *one step of evaluation* to M'.

These judgments are defined so that if $M\ \mathsf{val}$, then $M \not\longmapsto$, but the converse need not be the case. As usual, we write $M \longmapsto^* M'$ to mean that M transitions to M' in zero or more steps. We say M evaluates to V, written $M \Downarrow V$, when $M \longmapsto^* V$ and $V\ \mathsf{val}$.

The $\longmapsto$ judgment satisfies two additional conditions. Determinacy implies that a term has at most one value; dimension preservation states that evaluation does not introduce new (free) dimension names.

Lemma 4 (Determinacy)**.** *If $M \longmapsto M_1$ and $M \longmapsto M_2$, then $M_1 = M_2$.*

Lemma 5 (Dimension preservation)**.** *If $M \longmapsto M'$, then $\mathsf{FD}(M') \subseteq \mathsf{FD}(M)$.*

Many rules below are annotated with $\Box$. Those rules define an additional pair of judgments $M\ \mathsf{val}_{\Box}$ and $M \longmapsto_{\Box} M'$ by replacing every occurrence of val (resp., $\longmapsto$) in those rules with $\mathsf{val}_{\Box}$ (resp., $\longmapsto_{\Box}$). These rules define the *cubically-stable values* (resp., *cubically-stable steps*), characterized by the following property:

Lemma 6 (Cubical stability)**.** *If $M\ \mathsf{tm}\ [\Psi]$, then for any $\psi : \Psi' \to \Psi$,*

1. *if $M\ \mathsf{val}_{\Box}$ then $M\psi\ \mathsf{val}$, and*

2. *if $M \longmapsto_{\Box} M'$ then $M\psi \longmapsto M'\psi$.*

Cubically-stable values and steps are significant because they are unaffected by the cubical apparatus. All standard operational semantics rules are cubically-stable.

Types

$$\dfrac{}{(a{:}A) \to B\ \mathsf{val}}\ \Box \qquad \dfrac{}{(a{:}A) \times B\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{Path}_{x.A}(M,N)\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{Eq}_{A}(M,N)\ \mathsf{val}}\ \Box$$

$$\dfrac{}{\mathsf{void}\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{nat}\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{bool}\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{wbool}\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathbb{S}^1\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathcal{U}^{\mathsf{pre}}_j\ \mathsf{val}}\ \Box$$

$$\dfrac{}{\mathcal{U}^{\mathsf{Kan}}_j\ \mathsf{val}}\ \Box \qquad \dfrac{}{\mathsf{V}_x(A,B,E)\ \mathsf{val}} \qquad \dfrac{}{\mathsf{V}_0(A,B,E) \longmapsto A}\ \Box \qquad \dfrac{}{\mathsf{V}_1(A,B,E) \longmapsto B}\ \Box$$

Kan operations

$$\frac{A \longmapsto A'}{\mathsf{hcom}^{r\rightsquigarrow r'}_{A}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{hcom}^{r\rightsquigarrow r'}_{A'}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i})} \qquad \frac{A \longmapsto A'}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.A}(M) \longmapsto \mathsf{coe}^{r\rightsquigarrow r'}_{x.A'}(M)}$$

$$\frac{}{\mathsf{com}^{r\rightsquigarrow r'}_{y.A}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{hcom}^{r\rightsquigarrow r'}_{A\langle r'/y\rangle}(\mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i)})}$$

$$\frac{r = r'}{\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto M} \qquad \frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i < j) \qquad r_j = r'_j}{\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \longmapsto N_j\langle r'/y\rangle}$$

$$\frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i)}{\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i})\ \mathsf{val}} \qquad \frac{}{\mathsf{ghcom}^{r\rightsquigarrow r'}_{A}(M;\cdot) \longmapsto M}$$

$$\frac{T_\varepsilon = \mathsf{hcom}^{r\rightsquigarrow z}_{A}(M; s'=\varepsilon \hookrightarrow y.N, s'=\overline{\varepsilon} \hookrightarrow y.\mathsf{ghcom}^{r\rightsquigarrow y}_{A}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}),\overrightarrow{\xi_i \hookrightarrow y.N_i})}{\begin{array}{c}\mathsf{ghcom}^{r\rightsquigarrow r'}_{A}(M; s=s' \hookrightarrow y.N,\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \\ \mathsf{hcom}^{r\rightsquigarrow r'}_{A}(M;\overrightarrow{s=\varepsilon \hookrightarrow z.T_\varepsilon}, s=s' \hookrightarrow y.N,\overrightarrow{\xi_i \hookrightarrow y.N_i})\end{array}}$$

$$\frac{}{\mathsf{gcom}^{r\rightsquigarrow r'}_{y.A}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{ghcom}^{r\rightsquigarrow r'}_{A\langle r'/y\rangle}(\mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i)})}$$

Dependent function types

$$\frac{M \longmapsto M'}{\mathsf{app}(M,N) \longmapsto \mathsf{app}(M',N)} \qquad \frac{}{\mathsf{app}(\lambda a.M,N) \longmapsto M[N/a]} \qquad \frac{}{\lambda a.M\ \mathsf{val}}$$

$$\frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{(a:A)\to B}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \lambda a.\mathsf{hcom}^{r\rightsquigarrow r'}_{B}(\mathsf{app}(M,a);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{app}(N_i,a)})}$$

$$\frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.(a:A)\to B}(M) \longmapsto \lambda a.\mathsf{coe}^{r\rightsquigarrow r'}_{x.B[\mathsf{coe}^{r'\rightsquigarrow x}_{x.A}(a)/a]}(\mathsf{app}(M,\mathsf{coe}^{r'\rightsquigarrow r}_{x.A}(a)))}$$

Dependent pair types

$$\frac{M \longmapsto M'}{\mathsf{fst}(M) \longmapsto \mathsf{fst}(M')} \qquad \frac{M \longmapsto M'}{\mathsf{snd}(M) \longmapsto \mathsf{snd}(M')} \qquad \frac{}{\langle M,N\rangle\ \mathsf{val}} \qquad \frac{}{\mathsf{fst}(\langle M,N\rangle) \longmapsto M}$$

$$\frac{}{\mathsf{snd}(\langle M,N\rangle) \longmapsto N}$$

$$\frac{F = \mathsf{hcom}^{r\rightsquigarrow z}_{A}(\mathsf{fst}(M);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)})}{\begin{array}{c}\mathsf{hcom}^{r\rightsquigarrow r'}_{(a:A)\times B}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \\ \langle \mathsf{hcom}^{r\rightsquigarrow r'}_{A}(\mathsf{fst}(M);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)}), \mathsf{com}^{r\rightsquigarrow r'}_{z.B[F/a]}(\mathsf{snd}(M);\overrightarrow{\xi_i \hookrightarrow y.\mathsf{snd}(N_i)})\rangle\end{array}}$$

$$\frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.(a:A)\times B}(M) \longmapsto \langle \mathsf{coe}^{r\rightsquigarrow r'}_{x.A}(\mathsf{fst}(M)), \mathsf{coe}^{r\rightsquigarrow r'}_{x.B[\mathsf{coe}^{r\rightsquigarrow x}_{x.A}(\mathsf{fst}(M))/a]}(\mathsf{snd}(M))\rangle}$$

Path types

$$\frac{M \longmapsto M'}{M@r \longmapsto M'@r} \qquad \frac{}{(\langle x\rangle M)@r \longmapsto M\langle r/x\rangle} \qquad \frac{}{\langle x\rangle M\ \mathsf{val}}$$

$$\frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{Path}_{x.A}(P_0,P_1)}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \langle x\rangle\mathsf{hcom}^{r\rightsquigarrow r'}_{A}(M@x;\overrightarrow{x=\varepsilon \hookrightarrow \_.P_\varepsilon},\overrightarrow{\xi_i \hookrightarrow y.N_i@x})}$$

$$\frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{y.\mathsf{Path}_{x.A}(P_0,P_1)}(M) \longmapsto \langle x\rangle\mathsf{com}^{r\rightsquigarrow r'}_{y.A}(M@x;\overrightarrow{x=\varepsilon \hookrightarrow y.P_\varepsilon})}$$

Equality types

$$\frac{}{\star\ \mathsf{val}} \qquad \frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{Eq}_A(E_0,E_1)}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \star}$$

Natural numbers

$$\frac{}{\mathsf{z}\ \mathsf{val}} \qquad \frac{}{\mathsf{s}(M)\ \mathsf{val}} \qquad \frac{M \longmapsto M'}{\mathsf{natrec}(M;Z,n.a.S) \longmapsto \mathsf{natrec}(M';Z,n.a.S)}$$

$$\frac{}{\mathsf{natrec}(\mathsf{z};Z,n.a.S) \longmapsto Z} \qquad \frac{}{\mathsf{natrec}(\mathsf{s}(M);Z,n.a.S) \longmapsto S[M/n][\mathsf{natrec}(M;Z,n.a.S)/a]}$$

$$\frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{nat}}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto M} \qquad \frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{nat}}(M) \longmapsto M}$$

Booleans

$$\frac{}{\mathsf{true}\ \mathsf{val}} \qquad \frac{}{\mathsf{false}\ \mathsf{val}} \qquad \frac{M \longmapsto M'}{\mathsf{if}_{b.A}(M;T,F) \longmapsto \mathsf{if}_{b.A}(M';T,F)} \qquad \frac{}{\mathsf{if}_{b.A}(\mathsf{true};T,F) \longmapsto T}$$

$$\frac{}{\mathsf{if}_{b.A}(\mathsf{false};T,F) \longmapsto F} \qquad \frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto M} \qquad \frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{bool}}(M) \longmapsto M}$$

Weak booleans

$$\frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{wbool}}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i})}$$

$$\frac{r \neq r' \qquad r_i \neq r_i'\ (\forall i) \qquad H = \mathsf{fcom}^{r\rightsquigarrow z}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i})}{\begin{array}{c}\mathsf{if}_{b.A}(\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i});T,F) \longmapsto \\ \mathsf{com}^{r\rightsquigarrow r'}_{z.A[H/b]}(\mathsf{if}_{b.A}(M;T,F);\overrightarrow{r_i = r_i' \hookrightarrow y.\mathsf{if}_{b.A}(N_i;T,F)})\end{array}} \qquad \frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{wbool}}(M) \longmapsto M}$$

Circle

$$\frac{}{\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathbb{S}^1}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i})} \qquad \frac{}{\mathsf{loop}_\varepsilon \longmapsto \mathsf{base}} \qquad \frac{}{\mathsf{base}\ \mathsf{val}}$$

$$\frac{}{\mathsf{loop}_x\ \mathsf{val}} \qquad \frac{M \longmapsto M'}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M;P,x.L) \longmapsto \mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M';P,x.L)}$$

$$\frac{}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{base};P,x.L) \longmapsto P} \qquad \frac{}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{loop}_w;P,x.L) \longmapsto L\langle w/x\rangle}$$

$$\frac{r \neq r' \qquad r_i \neq r_i'\ (\forall i) \qquad F = \mathsf{fcom}^{r\rightsquigarrow z}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i})}{\begin{array}{c}\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i});P,x.L) \longmapsto \\ \mathsf{com}^{r\rightsquigarrow r'}_{z.A[F/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M;P,x.L);\overrightarrow{r_i = r_i' \hookrightarrow y.\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(N_i;P,x.L)})\end{array}} \qquad \frac{}{\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathbb{S}^1}(M) \longmapsto M}$$

Univalence

$$\frac{}{\mathsf{Vin}_x(M,N)\ \mathsf{val}} \qquad \frac{}{\mathsf{Vin}_0(M,N) \longmapsto M} \qquad \frac{}{\mathsf{Vin}_1(M,N) \longmapsto N}$$

$$\frac{}{\mathsf{Vproj}_0(M,F) \longmapsto \mathsf{app}(F,M)} \qquad \frac{}{\mathsf{Vproj}_1(M,F) \longmapsto M} \qquad \frac{M \longmapsto M'}{\mathsf{Vproj}_x(M,F) \longmapsto \mathsf{Vproj}_x(M',F)}$$

$$\frac{}{\mathsf{Vproj}_x(\mathsf{Vin}_x(M,N),F) \longmapsto N}$$

$$\frac{\begin{array}{c}O = \mathsf{hcom}^{r\rightsquigarrow y}_A(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \\ \overrightarrow{T} = x = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E),O), x = 1 \hookrightarrow y.\mathsf{hcom}^{r\rightsquigarrow y}_B(M;\overrightarrow{\xi_i \hookrightarrow y.N_i})\end{array}}{\begin{array}{c}\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{V}_x(A,B,E)}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \\ \mathsf{Vin}_x(O\langle r'/y\rangle, \mathsf{hcom}^{r\rightsquigarrow r'}_B(\mathsf{Vproj}_x(M,\mathsf{fst}(E));\overrightarrow{\xi_i \hookrightarrow y.\mathsf{Vproj}_x(N_i,\mathsf{fst}(E))},\overrightarrow{T}))\end{array}}$$

$$\frac{}{\mathsf{coe}^{0\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M) \longmapsto \mathsf{Vin}_{r'}(M,\mathsf{coe}^{0\rightsquigarrow r'}_{x.B}(\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle),M)))}$$

$$\frac{\begin{array}{c}O = \mathsf{fst}(\mathsf{app}(\mathsf{snd}(E\langle r'/x\rangle),\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N))) \\ P = \mathsf{hcom}^{1\rightsquigarrow 0}_{B\langle r'/x\rangle}(\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N); r' = 0 \hookrightarrow y.\mathsf{snd}(O)@y, r' = 1 \hookrightarrow \_.\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N))\end{array}}{\mathsf{coe}^{1\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(N) \longmapsto \mathsf{Vin}_{r'}(\mathsf{fst}(O),P)}$$

$$\frac{\begin{array}{c}O_\varepsilon = \mathsf{Vproj}_w(\mathsf{coe}^{\varepsilon\rightsquigarrow w}_{x.\mathsf{V}_x(A,B,E)}(M),\mathsf{fst}(E\langle w/x\rangle)) \\ P = \mathsf{com}^{y\rightsquigarrow x}_{x.B}(\mathsf{Vproj}_y(M,\mathsf{fst}(E\langle y/x\rangle));\overrightarrow{y = \varepsilon \hookrightarrow w.O_\varepsilon}) \\ Q_\varepsilon[a] = \langle \mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a), \langle z\rangle \mathsf{com}^{\varepsilon\rightsquigarrow y}_{y.B\langle 0/x\rangle}(P\langle 0/x\rangle\langle \varepsilon/y\rangle;\overrightarrow{U})\rangle \\ \overrightarrow{U} = z = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle),\mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a)), z = 1 \hookrightarrow y.P\langle 0/x\rangle \\ R = \mathsf{app}(\mathsf{app}(\mathsf{snd}(\mathsf{app}(\mathsf{snd}(E\langle 0/x\rangle),P\langle 0/x\rangle)),Q_0[M\langle 0/y\rangle]),Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle])@y \\ \overrightarrow{T} = \overrightarrow{y = \varepsilon \hookrightarrow \_.O_\varepsilon\langle r'/w\rangle}, y = r' \hookrightarrow \_.\mathsf{Vproj}_{r'}(M,\mathsf{fst}(E\langle r'/x\rangle)), r' = 0 \hookrightarrow z.\mathsf{snd}(R)@z\end{array}}{\mathsf{coe}^{y\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M) \longmapsto \mathsf{Vin}_{r'}(\mathsf{fst}(R),\mathsf{hcom}^{1\rightsquigarrow 0}_{B\langle r'/x\rangle}(P\langle r'/x\rangle;\overrightarrow{T}))}$$

$$\frac{x \neq y \qquad \overrightarrow{T} = x = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E), \mathsf{coe}^{r \rightsquigarrow y}_{y.A}(M)), x = 1 \hookrightarrow y.\mathsf{coe}^{r \rightsquigarrow y}_{y.B}(M)}{\mathsf{coe}^{r \rightsquigarrow r'}_{y.\mathsf{V}_x(A,B,E)}(M) \longmapsto \mathsf{Vin}_x(\mathsf{coe}^{r \rightsquigarrow r'}_{y.A}(M), \mathsf{com}^{r \rightsquigarrow r'}_{y.B}(\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y \rangle)); \overrightarrow{T}))}$$

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$$\frac{}{\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathcal{U}^{\mathsf{Kan}}_j}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \longmapsto \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i})} \qquad \frac{}{\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathcal{U}^{\kappa}_j}(M) \longmapsto M}$$

$$\frac{r = r'}{\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow N_i}) \longmapsto M} \qquad \frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i < j) \qquad r_j = r'_j}{\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}) \longmapsto N_j}$$

$$\frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i)}{\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i})\ \mathsf{val}} \qquad \frac{r = r'}{\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow y.B_i}) \longmapsto M}$$

$$\frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i < j) \qquad r_j = r'_j}{\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}) \longmapsto \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(M)}$$

$$\frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i) \qquad M \longmapsto M'}{\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}) \longmapsto \mathsf{cap}^{r \leftsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i})}$$

$$\frac{r \neq r' \qquad r_i \neq r'_i\ (\forall i)}{\mathsf{cap}^{r \leftsquigarrow r'}(\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow N_i}); \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}) \longmapsto M}$$

$$\frac{\begin{array}{c} s \neq s' \qquad s_j \neq s'_j\ (\forall j) \qquad P_j = \mathsf{hcom}^{r \rightsquigarrow r'}_{B_j}(\mathsf{coe}^{s' \rightsquigarrow z}_{z.B_j}(M); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}^{s' \rightsquigarrow z}_{z.B_j}(N_i)}) \\ F[c] = \mathsf{hcom}^{s' \rightsquigarrow z}_{A}(\mathsf{cap}^{s \leftsquigarrow s'}(c; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j}); \overrightarrow{s_j = s'_j \hookrightarrow z'.\mathsf{coe}^{z' \rightsquigarrow s}_{z.B_j}(\mathsf{coe}^{s' \rightsquigarrow z'}_{z.B_j}(c))}) \\ O = \mathsf{hcom}^{r \rightsquigarrow r'}_{A}((F[M])\langle s/z \rangle; \overrightarrow{r_i = r'_i \hookrightarrow y.(F[N_i])\langle s/z \rangle}) \\ Q = \mathsf{hcom}^{s \rightsquigarrow s'}_{A}(O; \overrightarrow{r_i = r'_i \hookrightarrow z.F[N_i\langle r'/y \rangle]}, \overrightarrow{s_j = s'_j \hookrightarrow z.\mathsf{coe}^{z \rightsquigarrow s}_{z.B_j}(P_j)}, r = r' \hookrightarrow z.F[M]) \end{array}}{\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathsf{fcom}^{s \rightsquigarrow s'}(A; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j})}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \longmapsto \mathsf{box}^{s \rightsquigarrow s'}(Q; \overrightarrow{s_j = s'_j \hookrightarrow P_j\langle s'/z \rangle})}$$

$$\frac{\begin{array}{c} s \neq s' \qquad s_i \neq s'_i\ (\forall i) \qquad N_i = \mathsf{coe}^{s' \rightsquigarrow z}_{z.B_i}(\mathsf{coe}^{r \rightsquigarrow x}_{x.B_i\langle s'/z \rangle}(M)) \\ O = (\mathsf{hcom}^{s' \rightsquigarrow z}_{A}(\mathsf{cap}^{s \leftsquigarrow s'}(M; \overrightarrow{s_i = s'_i \hookrightarrow z.B_i}); \overrightarrow{s_i = s'_i \hookrightarrow z.\mathsf{coe}^{z \rightsquigarrow s}_{z.B_i}(N_i)}))\langle r/x \rangle \\ P = \mathsf{gcom}^{r \rightsquigarrow r'}_{x.A}(O\langle s\langle r/x \rangle/z \rangle; \overrightarrow{s_i = s'_i \hookrightarrow x.N_i\langle s/z \rangle|_{(x \# s_i, s'_i)}}, s = s' \hookrightarrow x.\mathsf{coe}^{r \rightsquigarrow x}_{x.A}(M)|_{(x \# s, s')}) \\ Q_k = \mathsf{gcom}^{s\langle r'/x \rangle \rightsquigarrow z}_{z.B_k\langle r'/x \rangle}(P; \overrightarrow{s_i = s'_i \hookrightarrow z.N_i\langle r'/x \rangle|_{(x \# s_i, s'_i)}}, r = r' \hookrightarrow z.N_k\langle r'/x \rangle) \end{array}}{\begin{array}{c} \mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathsf{fcom}^{s \rightsquigarrow s'}(A; \overrightarrow{s_i = s'_i \hookrightarrow z.B_i})}(M) \longmapsto \\ (\mathsf{box}^{s \rightsquigarrow s'}(\mathsf{hcom}^{s \rightsquigarrow s'}_{A}(P; \overrightarrow{s_i = s'_i \hookrightarrow z.\mathsf{coe}^{z \rightsquigarrow s}_{z.B_i}(Q_i)}, r = r' \hookrightarrow z.O); \overrightarrow{s_i = s'_i \hookrightarrow Q_i\langle s'/z \rangle}))\langle r'/x \rangle \end{array}}$$

3 Cubical type systems

In this paper, we define the judgments of higher type theory relative to a *cubical type system*, a family of relations over values in the previously-described programming language. In this section we describe how to construct a particular cubical type system that will validate the rules given in Section 6; this construction is based on similar constructions outlined by Allen [1987] and Harper [1992].

Definition 7. A *candidate cubical type system* is a relation $\tau(\Psi, A_0, B_0, \varphi)$ over $A_0\ \mathsf{val}\ [\Psi]$, $B_0\ \mathsf{val}\ [\Psi]$, and binary relations $\varphi(M_0, N_0)$ over $M_0\ \mathsf{val}\ [\Psi]$ and $N_0\ \mathsf{val}\ [\Psi]$.

For any relation R with value arguments, we define $R^{\Downarrow}$ as its evaluation lifting to terms. For example, $\tau^{\Downarrow}(\Psi, A, B, \varphi)$ when there exist A_0 and B_0 such that $A \Downarrow A_0$, $B \Downarrow B_0$, and $\tau(\Psi, A_0, B_0, \varphi)$.

Definition 8. A Ψ*-relation* is a family of binary relations $\alpha_\psi(M, N)$ indexed by substitutions $\psi : \Psi' \to \Psi$, relating $M\ \mathsf{tm}\ [\Psi']$ and $N\ \mathsf{tm}\ [\Psi']$. (We will write $\alpha(M, N)$ in place of $\alpha_{\mathsf{id}_\Psi}(M, N)$.) We are often interested in Ψ*-relations over values*, which relate only values. If a Ψ-relation depends only on the choice of Ψ' and not ψ, we instead call it *context-indexed* and write $\alpha_{\Psi'}(M, N)$.

We can precompose any Ψ-relation α by a dimension substitution $\psi : \Psi' \to \Psi$ to yield a Ψ'-relation $(\alpha\psi)_{\psi'}(M, N) = \alpha_{\psi\psi'}(M, N)$. Context-indexed relations are indeed families of binary relations indexed by contexts Ψ', because the choice of Ψ and ψ are irrelevant—every Ψ, Ψ' have at least one dimension substitution between them. We write $R(\Psi')$ for the context-indexed relation R regarded as a Ψ'-relation.

Definition 9. For any candidate cubical type system τ, the relation $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ over $A\ \mathsf{tm}\ [\Psi]$, $B\ \mathsf{tm}\ [\Psi]$, and a Ψ-relation over values α holds if for all $\psi : \Psi' \to \Psi$ we have $\tau^{\Downarrow}(\Psi', A\psi, B\psi, \alpha_\psi)$, and for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, we have $A\psi_1 \Downarrow A_1$, $B\psi_1 \Downarrow B_1$, $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, A\psi_1\psi_2, \varphi)$, $\tau^{\Downarrow}(\Psi_2, A\psi_1\psi_2, A_1\psi_2, \varphi)$, $\tau^{\Downarrow}(\Psi_2, B_1\psi_2, B\psi_1\psi_2, \varphi)$, $\tau^{\Downarrow}(\Psi_2, B\psi_1\psi_2, B_1\psi_2, \varphi)$, and $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, B_1\psi_2, \varphi)$.

Definition 10. For any Ψ-relation on values α, the relation $\mathsf{Tm}(\alpha)(M, N)$ over $M\ \mathsf{tm}\ [\Psi]$ and $N\ \mathsf{tm}\ [\Psi]$ holds if for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, we have $M\psi_1 \Downarrow M_1$, $N\psi_1 \Downarrow N_1$, $\alpha^{\Downarrow}_{\psi_1\psi_2}(M_1\psi_2, M\psi_1\psi_2)$, $\alpha^{\Downarrow}_{\psi_1\psi_2}(M\psi_1\psi_2, M_1\psi_2)$, $\alpha^{\Downarrow}_{\psi_1\psi_2}(N_1\psi_2, N\psi_1\psi_2)$, $\alpha^{\Downarrow}_{\psi_1\psi_2}(N\psi_1\psi_2, N_1\psi_2)$, and $\alpha^{\Downarrow}_{\psi_1\psi_2}(M_1\psi_2, N_1\psi_2)$.

Definition 11. A Ψ-relation on values α is *value-coherent*, or $\mathsf{Coh}(\alpha)$, when for all $\psi : \Psi' \to \Psi$, if $\alpha_\psi(M_0, N_0)$ then $\mathsf{Tm}(\alpha\psi)(M_0, N_0)$.

These relations are closed under dimension substitution by construction—for any $\psi : \Psi' \to \Psi$, if $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ then $\mathsf{PTy}(\tau)(\Psi', A\psi, B\psi, \alpha\psi)$, if $\mathsf{Tm}(\alpha)(M, N)$ then $\mathsf{Tm}(\alpha\psi)(M\psi, N\psi)$, and if $\mathsf{Coh}(\alpha)$ then $\mathsf{Coh}(\alpha\psi)$.

3.1 Fixed points

Ψ-relations (and context-indexed relations) over values form a complete lattice when ordered by inclusion. By the Knaster-Tarski fixed point theorem, any order-preserving operator $F(x)$ on a complete lattice has a least fixed point $\mu x.F(x)$ that is also its least pre-fixed point [Davey and Priestley, 2002, 2.35].

We define the canonical element equality relations of inductive types—$\mathbb{N}$ for natural numbers, $\mathbb{B}$ for weak booleans, and $\mathbb{C}$ for the circle—as context-indexed relations (written here as three-place relations) that are least fixed points of order-preserving operators:

$$
\begin{aligned}
\mathbb{N} &= \mu R.(\{(\Psi, \mathsf{z}, \mathsf{z})\} \cup \{(\Psi, \mathsf{s}(M), \mathsf{s}(M')) \mid \mathsf{Tm}(R(\Psi))(M, M')\}) \\
\mathbb{B} &= \mu R.(\{(\Psi, \mathsf{true}, \mathsf{true}), (\Psi, \mathsf{false}, \mathsf{false})\} \cup \textsc{FKan}(R)) \\
\mathbb{C} &= \mu R.(\{(\Psi, \mathsf{base}, \mathsf{base}), ((\Psi, x), \mathsf{loop}_x, \mathsf{loop}_x)\} \cup \textsc{FKan}(R))
\end{aligned}
$$

where

$$
\begin{aligned}
\textsc{FKan}(R) = \{(\Psi, &\mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}), \mathsf{fcom}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i})) \mid \\
&(r \neq r') \wedge (\forall i. r_i \neq r'_i) \wedge (\exists i, j.(r_i = r_j) \wedge (r'_i = 0) \wedge (r'_j = 1)) \wedge \mathsf{Tm}(R(\Psi))(M, M') \\
&\wedge (\forall i, j, \psi : \Psi' \to (\Psi, y).((r_i\psi = r'_i\psi) \wedge (r_j\psi = r'_j\psi)) \implies \mathsf{Tm}(R(\Psi'))(N_i\psi, N'_j\psi)) \\
&\wedge (\forall i, \psi : \Psi' \to \Psi.(r_i\psi = r'_i\psi) \implies \mathsf{Tm}(R(\Psi'))(N_i\langle r/y\rangle\psi, M\psi))\}
\end{aligned}
$$

The operators Tm and $\textsc{FKan}$ are order-preserving because they only use their argument relations in positive positions.

Similarly, candidate cubical type systems form a complete lattice, and we define a sequence of candidate cubical type systems as least fixed points of order-preserving operators, using the following auxiliary definitions for each type former:

$$
\begin{aligned}
\textsc{Fun}(\tau) = \{(\Psi, &(a{:}A) \to B, (a{:}A') \to B', \varphi) \mid \\
&\exists \alpha, \beta^{(-,-,-)}.\mathsf{PTy}(\tau)(\Psi, A, A', \alpha) \wedge \mathsf{Coh}(\alpha) \\
&\wedge (\forall \psi, M, M'.\mathsf{Tm}(\alpha\psi)(M, M') \implies \\
&\qquad \mathsf{PTy}(\tau)(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'}) \wedge \mathsf{Coh}(\beta^{\psi,M,M'})) \\
&\wedge (\varphi = \{(\lambda a.N, \lambda a.N') \mid \forall \psi, M, M'.\mathsf{Tm}(\alpha\psi)(M, M') \implies \\
&\qquad \mathsf{Tm}(\beta^{\psi,M,M'})(N\psi[M/a], N'\psi[M'/a])\})\} \\
\textsc{Pair}(\tau) = \{(\Psi, &(a{:}A) \times B, (a{:}A') \times B', \varphi) \mid \\
&\exists \alpha, \beta^{(-,-,-)}.\mathsf{PTy}(\tau)(\Psi, A, A', \alpha) \wedge \mathsf{Coh}(\alpha) \\
&\wedge (\forall \psi, M, M'.\mathsf{Tm}(\alpha\psi)(M, M') \implies \\
&\qquad \mathsf{PTy}(\tau)(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'}) \wedge \mathsf{Coh}(\beta^{\psi,M,M'})) \\
&\wedge (\varphi = \{(\langle M, N\rangle, \langle M', N'\rangle) \mid \mathsf{Tm}(\alpha)(M, M') \wedge \mathsf{Tm}(\beta^{\mathsf{id}_\Psi,M,M'})(N, N')\})\} \\
\textsc{Path}(\tau) = \{(\Psi, &\mathsf{Path}_{x.A}(P_0, P_1), \mathsf{Path}_{x.A'}(P'_0, P'_1), \varphi) \mid \\
&\exists \alpha.\mathsf{PTy}(\tau)((\Psi, x), A, A', \alpha) \wedge \mathsf{Coh}(\alpha) \wedge (\forall \varepsilon.\mathsf{Tm}(\alpha\langle\varepsilon/x\rangle)(P_\varepsilon, P'_\varepsilon)) \\
&\wedge (\varphi = \{(\langle x\rangle M, \langle x\rangle M') \mid \mathsf{Tm}(\alpha)(M, M') \wedge (\forall \varepsilon.\mathsf{Tm}(\alpha\langle\varepsilon/x\rangle)(M\langle\varepsilon/x\rangle, P_\varepsilon))\})\} \\
\textsc{Eq}(\tau) = \{(\Psi, &\mathsf{Eq}_A(M, N), \mathsf{Eq}_{A'}(M', N'), \varphi) \mid \\
&\exists \alpha.\mathsf{PTy}(\tau)(\Psi, A, A', \alpha) \wedge \mathsf{Coh}(\alpha) \wedge \mathsf{Tm}(\alpha)(M, M') \wedge \mathsf{Tm}(\alpha)(N, N') \\
&\wedge (\varphi = \{(\star, \star) \mid \mathsf{Tm}(\alpha)(M, N)\})\} \\
\mathrm{V}(\tau) = \{((\Psi, x), &\mathsf{V}_x(A, B, E), \mathsf{V}_x(A', B', E'), \varphi) \mid \\
&\exists \beta, \alpha^{(-)}, \eta^{(-)}.\mathsf{PTy}(\tau)((\Psi, x), B, B', \beta) \wedge \mathsf{Coh}(\beta)
\end{aligned}
$$

$$
\begin{aligned}
&\wedge (\forall\psi.(x\psi = 0) \implies \mathsf{PTy}(\tau)(\Psi', A\psi, A'\psi, \alpha^{\psi}) \wedge \mathsf{Coh}(\alpha^{\psi})\\
&\qquad \wedge \mathsf{PTy}(\tau)(\Psi', \mathsf{Equiv}(A\psi, B\psi), \mathsf{Equiv}(A\psi, B\psi), \eta^{\psi}) \wedge \mathsf{Tm}(\eta^{\psi})(E\psi, E'\psi))\\
&\wedge (\varphi = \{(\mathsf{Vin}_x(M,N), \mathsf{Vin}_x(M',N')) \mid \mathsf{Tm}(\beta)(N,N') \wedge (\forall\psi.(x\psi = 0) \implies\\
&\qquad \mathsf{Tm}(\alpha^{\psi})(M\psi, M'\psi) \wedge \mathsf{Tm}(\beta\psi)(\mathsf{app}(\mathsf{fst}(E\psi), M\psi), N\psi))\}\}
\end{aligned}
$$

$$
\begin{aligned}
\textsc{Fcom}(\tau) = \{&(\Psi, \mathsf{fcom}^{r\rightsquigarrow r'}(A; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}), \mathsf{fcom}^{r\rightsquigarrow r'}(A'; \overrightarrow{r_i = r'_i \hookrightarrow y.B'_i})) \mid\\
&\exists\alpha, \beta^{(-,-,-)}.(r \neq r') \wedge (\forall i.r_i \neq r'_i) \wedge (\exists i,j.(r_i = r_j) \wedge (r'_i = 0) \wedge (r'_j = 1))\\
&\wedge \mathsf{PTy}(\tau)(\Psi, A, A', \alpha) \wedge \mathsf{Coh}(\alpha)\\
&\wedge (\forall i,j,\psi : \Psi' \to (\Psi, y).((r_i\psi = r'_i\psi) \wedge (r_j\psi = r'_j\psi)) \implies\\
&\qquad \mathsf{PTy}(\tau)(\Psi', B_i\psi, B'_j\psi, \beta^{\psi,i,j}) \wedge \mathsf{Coh}(\beta^{\psi,i,j}))\\
&\wedge (\forall i,\psi.(r_i\psi = r'_i\psi) \implies \mathsf{PTy}(\tau)(\Psi', B_i\langle r/y\rangle\psi, A\psi, \_))\})\\
&\wedge (\varphi = \{(\mathsf{box}^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}), \mathsf{box}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow N'_i})) \mid \mathsf{Tm}(\alpha)(M, M')\\
&\qquad \wedge (\forall i,j,\psi.((r_i\psi = r'_i\psi) \wedge (r_j\psi = r'_j\psi)) \implies \mathsf{Tm}(\beta^{\psi,i,j}\langle r'\psi/y\rangle)(N_i\psi, N'_j\psi))\\
&\qquad \wedge (\forall i,\psi.(r_i\psi = r'_i\psi) \implies \mathsf{Tm}(\alpha\psi)(M\psi, \mathsf{coe}^{r'\psi\rightsquigarrow r\psi}_{y.B_i\psi}(N_i\psi)))\})\}\\
\textsc{Void} &= \{(\Psi, \mathsf{void}, \mathsf{void}, \{\})\}\\
\textsc{Nat} &= \{(\Psi, \mathsf{nat}, \mathsf{nat}, \mathbb{N}_\Psi)\}\\
\textsc{Bool} &= \{(\Psi, \mathsf{bool}, \mathsf{bool}, \{(\mathsf{true}, \mathsf{true}), (\mathsf{false}, \mathsf{false})\})\}\\
\text{WB} &= \{(\Psi, \mathsf{wbool}, \mathsf{wbool}, \mathbb{B}_\Psi)\}\\
\textsc{Circ} &= \{(\Psi, \mathbb{S}^1, \mathbb{S}^1, \mathbb{C}_\Psi)\}\\
\text{UP}\textsc{re}(\nu) &= \{(\Psi, \mathcal{U}^{\mathsf{pre}}_j, \mathcal{U}^{\mathsf{pre}}_j, \varphi) \mid \nu(\Psi, \mathcal{U}^{\mathsf{pre}}_j, \mathcal{U}^{\mathsf{pre}}_j, \varphi)\}\\
\text{UK}\textsc{an}(\nu) &= \{(\Psi, \mathcal{U}^{\mathsf{Kan}}_j, \mathcal{U}^{\mathsf{Kan}}_j, \varphi) \mid \nu(\Psi, \mathcal{U}^{\mathsf{Kan}}_j, \mathcal{U}^{\mathsf{Kan}}_j, \varphi)\}
\end{aligned}
$$

In the V case, and for the remainder of this paper, we use the abbreviations

$$
\begin{aligned}
\mathsf{isContr}(C) &:= C \times ((c{:}C) \to (c'{:}C) \to \mathsf{Path}_{\_.C}(c, c'))\\
\mathsf{Equiv}(A, B) &:= (f{:}A \to B) \times ((b{:}B) \to \mathsf{isContr}((a{:}A) \times \mathsf{Path}_{\_.B}(\mathsf{app}(f, a), b))).
\end{aligned}
$$

For candidate cubical type systems ν, σ, τ, define

$$
\begin{aligned}
P(\nu, \sigma, \tau) &= \textsc{Fun}(\tau) \cup \textsc{Pair}(\tau) \cup \textsc{Path}(\tau) \cup \textsc{Eq}(\tau) \cup \text{V}(\tau) \cup \textsc{Fcom}(\sigma)\\
&\quad \cup \textsc{Void} \cup \textsc{Nat} \cup \textsc{Bool} \cup \text{WB} \cup \textsc{Circ} \cup \text{UP}\textsc{re}(\nu) \cup \text{UK}\textsc{an}(\nu)\\
K(\nu, \sigma) &= \textsc{Fun}(\sigma) \cup \textsc{Pair}(\sigma) \cup \textsc{Path}(\sigma) \cup \text{V}(\sigma) \cup \textsc{Fcom}(\sigma)\\
&\quad \cup \textsc{Void} \cup \textsc{Nat} \cup \textsc{Bool} \cup \text{WB} \cup \textsc{Circ} \cup \text{UK}\textsc{an}(\nu)
\end{aligned}
$$

The operator P includes Eq and UPre while K does not; furthermore, in P only Fcom varies in σ. The operators P and K are order-preserving in all arguments because PTy and each type operator only use their argument in strictly positive positions.

Lemma 12. *In any complete lattice,*

1. *If $F(x)$ and $G(x)$ are order-preserving and $F(x) \subseteq G(x)$ for all x, then $\mu x.F(x) \subseteq \mu x.G(x)$.*

2. *If $F(x, y)$ and $G(x, y)$ are order-preserving and $F(x, y) \subseteq G(x, y)$ whenever $x \subseteq y$, then $\mu_f \subseteq \mu_g$ where $(\mu_f, \mu_g) = \mu(x, y).(F(x, y), G(x, y))$.*

Proof. For part (1), $\mu x.G(x)$ is a pre-fixed point of F because $F(\mu x.G(x)) \subseteq G(\mu x.G(x)) = \mu x.G(x)$. But $\mu x.F(x)$ is the least such, so $\mu x.F(x) \subseteq \mu x.G(x)$.

For part (2), let $\mu_\cap = \mu_f \cap \mu_g$. Note $(\mu_\cap, \mu_g)$ is a pre-fixed point of $(x, y) \mapsto (F(x, y), G(x, y))$ because, by assumption and (μ_f, μ_g) being a fixed point, $F(\mu_\cap, \mu_g) \subseteq F(\mu_f, \mu_g) = \mu_f$ and $F(\mu_\cap, \mu_g) \subseteq G(\mu_\cap, \mu_g) \subseteq G(\mu_f, \mu_g) = \mu_g$. This implies $(\mu_f, \mu_g) \subseteq (\mu_\cap, \mu_g)$ and thus $\mu_f \subseteq \mu_g$. □

Lemma 13. *Let $\mu^{\mathsf{pre}}(\nu, \sigma) = \mu\tau.P(\nu, \sigma, \tau)$ and let $\mu^{\mathsf{Kan}}(\nu) = \mu\sigma.K(\nu, \sigma)$. Then $\mu^{\mathsf{pre}}(\nu, \sigma)$ and $\mu^{\mathsf{Kan}}(\nu)$ are order-preserving and $\mu^{\mathsf{Kan}}(\nu) \subseteq \mu^{\mathsf{pre}}(\nu, \mu^{\mathsf{Kan}}(\nu))$ for all ν.*

Proof. Part (1) is immediate by part (1) of Lemma 12, because whenever $\nu \subseteq \nu'$ and $\sigma \subseteq \sigma'$, $P(\nu, \sigma, -) \subseteq P(\nu', \sigma', -)$ and $K(\nu, -) \subseteq K(\nu', -)$. For part (2), a theorem of Bekić [1984] on simultaneous fixed points implies $(\mu^{\mathsf{Kan}}(\nu), \mu^{\mathsf{pre}}(\nu, \mu^{\mathsf{Kan}}(\nu))) = \mu(\sigma, \tau).(K(\nu, \sigma), P(\nu, \sigma, \tau))$. Because each type operator is order-preserving, $K(\nu, \sigma) \subseteq P(\nu, \sigma, \tau)$ whenever $\sigma \subseteq \tau$. The result follows by part (2) of Lemma 12. □

We mutually define three sequences of candidate cubical type systems: ν_{i+1} containing i universes, $\tau^{\mathsf{pre}}_{i+1}$ containing the pretypes in a system with i universes, and $\tau^{\mathsf{Kan}}_{i+1}$ containing the Kan types in a system with i universes:

$$
\begin{aligned}
\nu_0 &= \emptyset \\
\nu_n &= \{(\Psi, \mathcal{U}^\kappa_j, \mathcal{U}^\kappa_j, \varphi) \mid (j < n) \land (\varphi = \{(A_0, B_0) \mid \tau^\kappa_j(\Psi, A_0, B_0, \_)\})\} \\
\tau^{\mathsf{pre}}_n &= \mu^{\mathsf{pre}}(\nu_n, \mu^{\mathsf{Kan}}(\nu_n)) \\
\tau^{\mathsf{Kan}}_n &= \mu^{\mathsf{Kan}}(\nu_n) \\
\nu_\omega &= \{(\Psi, \mathcal{U}^\kappa_j, \mathcal{U}^\kappa_j, \varphi) \mid \varphi = \{(A_0, B_0) \mid \tau^\kappa_j(\Psi, A_0, B_0, \_)\}\} \\
\tau^{\mathsf{pre}}_\omega &= \mu^{\mathsf{pre}}(\nu_\omega, \mu^{\mathsf{Kan}}(\nu_\omega)) \\
\tau^{\mathsf{Kan}}_\omega &= \mu^{\mathsf{Kan}}(\nu_\omega)
\end{aligned}
$$

Observe that $\nu_n \subseteq \nu_{n+i}$, $\nu_n \subseteq \nu_\omega$, $\tau^\kappa_n \subseteq \tau^\kappa_{n+i}$, $\tau^\kappa_n \subseteq \tau^\kappa_\omega$, $\tau^{\mathsf{Kan}}_n \subseteq \tau^{\mathsf{pre}}_n$, and $\tau^{\mathsf{Kan}}_\omega \subseteq \tau^{\mathsf{pre}}_\omega$.

3.2 Cubical type systems

In the remainder of this paper, we consider only candidate cubical type systems satisfying a number of additional conditions:

Definition 14. A *cubical type system* is a candidate cubical type system τ satisfying:

Functionality. If $\tau(\Psi, A_0, B_0, \varphi)$ and $\tau(\Psi, A_0, B_0, \varphi')$ then $\varphi = \varphi'$.

PER-valuation. If $\tau(\Psi, A_0, B_0, \varphi)$ then φ is symmetric and transitive.

Symmetry. If $\tau(\Psi, A_0, B_0, \varphi)$ then $\tau(\Psi, B_0, A_0, \varphi)$.

Transitivity. If $\tau(\Psi, A_0, B_0, \varphi)$ and $\tau(\Psi, B_0, C_0, \varphi)$ then $\tau(\Psi, A_0, C_0, \varphi)$.

Value-coherence. If $\tau(\Psi, A_0, B_0, \varphi)$ then $\mathsf{PTy}(\tau)(\Psi, A_0, B_0, \alpha)$ for some α.

If τ is a cubical type system, then $\mathsf{PTy}(\tau)$ is functional, symmetric, transitive, and Ψ-PER-valued in the above senses. If α is a Ψ-PER, then every $\alpha\psi$ is a Ψ'-PER, and $\mathsf{Tm}(\alpha)$ is a PER.

Lemma 15. *If ν, σ are cubical type systems, then $\mu^{\mathsf{Kan}}(\nu)$ and $\mu^{\mathsf{pre}}(\nu, \sigma)$ are cubical type systems.*

Proof. Because the operators Fun, Pair... are disjoint, we can check them individually in each case. We describe the proof for $\mu^{\mathsf{pre}}(\nu, \sigma)$; the proof for $\mu^{\mathsf{Kan}}(\nu)$ follows analogously.

1. *Functionality.*

   Define a candidate cubical type system $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \forall \varphi'. \mu^{\mathsf{pre}}(\nu, \sigma)(\Psi, A_0, B_0, \varphi') \implies (\varphi = \varphi')\}$. Let us show that Φ is a pre-fixed point of $P(\nu, \sigma, -)$ (that is, $P(\nu, \sigma, \Phi) \subseteq \Phi$). Because $\mu^{\mathsf{pre}}(\nu, \sigma)$ is the least pre-fixed point, it will follow that $\mu^{\mathsf{pre}}(\nu, \sigma) \subseteq \Phi$, and that $\mu^{\mathsf{pre}}(\nu, \sigma)$ is functional.

   Assume that $\text{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Thus $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$, and in particular, for all $\psi : \Psi' \to \Psi$, $\mu^{\mathsf{pre}}(\nu, \sigma)^{\Downarrow}(\Psi', A\psi, A'\psi, \varphi')$ implies $\alpha_\psi = \varphi'$, so α is unique in $\mu^{\mathsf{pre}}(\nu, \sigma)$ when it exists. Similarly, each $\beta^{(-,-,-)}$ is unique in $\mu^{\mathsf{pre}}(\nu, \sigma)$ when it exists. The relation φ is determined uniquely by α and $\beta^{(-,-,-)}$. Now let us show $\Phi(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$, that is, assume $\mu^{\mathsf{pre}}(\nu, \sigma)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi')$ and show $\varphi = \varphi'$. It follows that $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu, \sigma))(\Psi, A, A', \alpha')$ for some α', and similarly for some family β', but $\alpha = \alpha'$ and each $\beta = \beta'$. Because φ' is defined using the same α and $\beta^{(-,-,-)}$ as φ, we conclude $\varphi = \varphi'$. Other cases are similar; for Fcom, UPre, UKan we use that ν, σ are functional.

2. *PER-valuation.*

   Define $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \varphi \text{ is a PER}\}$, and show that Φ is a pre-fixed point of $P(\nu, \sigma, -)$. It follows that $\mu^{\mathsf{pre}}(\nu, \sigma)$ is PER-valued, by $\mu^{\mathsf{pre}}(\nu, \sigma) \subseteq \Phi$.

   Assume that $\text{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Then $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$, and in particular, for all $\psi : \Psi' \to \Psi$, $\Phi^{\Downarrow}(\Psi', A\psi, A'\psi, \alpha_\psi)$, so each α_ψ is a PER. Similarly, each $\beta^{\psi,M,M'}_{\psi'}$ is a PER. Now we must show $\Phi(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. The relation φ is a PER because $\mathsf{Tm}(\alpha\psi)$ and $\mathsf{Tm}(\beta^{\psi,M,M'})$ are PERs, because α_ψ and $\beta^{\psi,M,M'}_{\psi'}$ are PERs. Most cases proceed in this fashion. For Nat, WB, and Circ we show that $\mathbb{N}$, $\mathbb{B}$, and $\mathbb{C}$ are symmetric and transitive at each dimension (employing the same strategy as in parts (3–4)); for Fcom, UPre, UKan we use that σ, ν are PER-valued.

3. *Symmetry.*

   Define $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \mu^{\mathsf{pre}}(\nu, \sigma)(\Psi, B_0, A_0, \varphi)\}$. Let us show that Φ is a pre-fixed point of $P(\nu, \sigma, -)$. It will follow that $\mu^{\mathsf{pre}}(\nu, \sigma)$ is symmetric, by $\mu^{\mathsf{pre}}(\nu, \sigma) \subseteq \Phi$.

   Assume that $\text{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Then $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$ and $\mathsf{Coh}(\alpha)$, and thus $\mu^{\mathsf{pre}}(\nu, \sigma)^{\Downarrow}(\Psi', A'\psi, A\psi, \alpha_\psi)$, $A\psi_1 \Downarrow A_1$, $A'\psi_1 \Downarrow A'_1$, and $\mu^{\mathsf{pre}}(\nu, \sigma)^{\Downarrow}(\Psi_2, -, -, \varphi)$ relates $(A\psi_1\psi_2, A_1\psi_2)$, $(A_1\psi_2, A\psi_1\psi_2)$, $(A'\psi_1\psi_2, A'_1\psi_2)$, $(A'_1\psi_2, A'\psi_1\psi_2)$, and $(A'_1\psi_2, A_1\psi_2)$. Similar facts hold by virtue of $\mathsf{PTy}(\Phi)(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'})$ and $\mathsf{Coh}(\beta^{\psi,M,M'})$. We must show $\Phi(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$, that is, $\mu^{\mathsf{pre}}(\nu, \sigma)(\Psi, (a{:}A') \to B', (a{:}A) \to B, \varphi)$. This requires $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu, \sigma))(\Psi, A', A, \alpha)$ and $\mathsf{Coh}(\alpha)$, which follows from the above facts; and also $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu, \sigma))(\Psi, B'\psi[M/a], B\psi[M'/a], \beta^{\psi,M,M'})$ and $\mathsf{Coh}(\beta^{\psi,M,M'})$ whenever $\mathsf{Tm}(\alpha\psi)(M, M')$, which follows from the symmetry of $\mathsf{Tm}(\alpha\psi)$ (since each α_ψ is a PER, by (2)), and the above facts. Other cases are similar; for Fcom we use that σ is symmetric.

4. *Transitivity.*

   Define $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \forall C_0.\mu^{\mathsf{pre}}(\nu,\sigma)(\Psi, B_0, C_0, \varphi) \implies \mu^{\mathsf{pre}}(\nu,\sigma)(\Psi, A_0, C_0, \varphi)\}$. Let us show that Φ is a pre-fixed point of $P(\nu, \sigma, -)$. It will follow that $\mu^{\mathsf{pre}}(\nu,\sigma)$ is transitive, by $\mu^{\mathsf{pre}}(\nu,\sigma) \subseteq \Phi$.

   Assume that $\textsc{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Then $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$, and thus if $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi', A'\psi, C_0, \alpha_\psi)$ then $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi', A\psi, C_0, \alpha_\psi)$. Furthermore, $A\psi_1 \Downarrow A_1$, $A'\psi_1 \Downarrow A'_1$, and for any C_0, $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi_2, -, -, \varphi)$ relates $(A\psi_1\psi_2, C_0)$ if and only if $(A_1\psi_2, C_0)$; $(A'\psi_1\psi_2, C_0)$ if and only if $(A'_1\psi_2, C_0)$; and if $(A'_1\psi_2, C_0)$ then $(A_1\psi_2, C_0)$. Similar facts hold by virtue of $\mathsf{PTy}(\Phi)(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'})$.

   Now we must show $\Phi(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$, that is, if $\mu^{\mathsf{pre}}(\nu,\sigma)(\Psi, (a{:}A') \to B', C_0, \varphi)$ then $\mu^{\mathsf{pre}}(\nu,\sigma)(\Psi, (a{:}A) \to B, C_0, \varphi)$. By inspecting P, we see this is only possible if $C_0 = (a{:}A'') \to B''$, in which case $\mu^{\mathsf{pre}}(\nu,\sigma)(\Psi, (a{:}A') \to B', (a{:}A'') \to B'', \varphi)$. Thus we have $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, A', A'', \alpha')$ and $\mathsf{Coh}(\alpha')$, so $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi', A'\psi, A''\psi, \alpha'_\psi)$, and by hypothesis, $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi', A\psi, A''\psi, \alpha_\psi)$ and $\mathsf{Coh}(\alpha)$. We already know $A\psi_1 \Downarrow A_1$, $A''\psi_1 \Downarrow A''_1$, and that $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi_2, -, -, \varphi)$ relates $(A''\psi_1\psi_2, A''_1\psi_2)$ and vice versa. By $(A'_1\psi_2, A''_1\psi_2)$ and the above, we have $(A_1\psi_2, A''_1\psi_2)$. Finally, by $(A'\psi_1\psi_2, A'_1\psi_2)$ and transitivity we have $(A'_1\psi_2, A'_1\psi_2)$, hence by transitivity and symmetry $(A'_1\psi_2, A_1\psi_2)$, and again by transitivity $(A_1\psi_2, A_1\psi_2)$; as needed, $(A_1\psi_2, A_0\psi_2)$ and vice versa follow by transitivity. As before, $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, B\psi[M/a], B''\psi[M'/a], \beta^{\psi,M,M'})$ and $\mathsf{Coh}(\beta^{\psi,M,M'})$ when $\mathsf{Tm}(\alpha\psi)(M, M')$ follows by transitivity of $\mathsf{Tm}(\alpha\psi)$ (since each α_ψ is a PER, by (2)). Other cases are similar; for $\textsc{Fcom}$ we use that σ is transitive.

5. *Value-coherence.*

   Define $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, A_0, B_0, \alpha)\}$. Let us show that Φ is a pre-fixed point of $P(\nu, \sigma, -)$. The property $P(\nu, \sigma, \Phi) \subseteq \Phi$ holds trivially for base types $\textsc{Void}$, $\textsc{Nat}$... as well as universes $\textsc{UPre}$ and $\textsc{UKan}$; we check $\textsc{Fun}$ ($\textsc{Pair}$, $\textsc{Path}$, and $\textsc{Eq}$ are similar) and V ($\textsc{Fcom}$ is similar). It will follow that $\mu^{\mathsf{pre}}(\nu,\sigma)$ is value-coherent, by $\mu^{\mathsf{pre}}(\nu,\sigma) \subseteq \Phi$.

   Assume that $\textsc{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Then by $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$ and $\mathsf{Coh}(\alpha)$, we have $\Phi^{\Downarrow}(\Psi', A\psi, A'\psi, \alpha_\psi)$, $A\psi_1 \Downarrow A_1$, $A'\psi_1 \Downarrow A'_1$, $\Phi^{\Downarrow}(\Psi_2, A_1\psi_2, A\psi_1\psi_2, \varphi')$, and so forth. Note that for values A_0, B_0, if $\mathsf{PTy}(\tau)(\Psi, A_0, B_0, \alpha)$ then $\tau(\Psi, A_0, B_0, \alpha_{\mathsf{id}_\Psi})$ by definition. Therefore $\mu^{\mathsf{pre}}(\nu,\sigma)^{\Downarrow}(\Psi', A\psi, A'\psi, \alpha_\psi)$, and so forth. We get similar facts for each $\mathsf{Tm}(\alpha\psi)(M, M')$ by $\mathsf{PTy}(\Phi)(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'})$ and $\mathsf{Coh}(\beta^{\psi,M,M'})$. We must show $\Phi(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi')$, that is, $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, (a{:}A) \to B, (a{:}A') \to B', \gamma)$. We know $(a{:}A) \to B\ \mathsf{val}_{\square}$, and by the above, $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, A, A', \alpha)$, $\mathsf{Coh}(\alpha)$, and when $\mathsf{Tm}(\alpha\psi)(M, M')$, $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi', B\psi[M/a], B'\psi[M'/a], \beta^{\psi,M,M'})$ and $\mathsf{Coh}(\beta^{\psi,M,M'})$. The result holds because PTy, Tm, and Coh are closed under dimension substitution.

   The V case is mostly similar, but not all instances of $\mathsf{V}_x(A, B, E)$ have the same head constructor. Repeating the previous argument, by $\mathrm{V}(\Phi)(\Psi, \mathsf{V}_x(A, B, E), \mathsf{V}_x(A', B', E'))$ we have that $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, B, B', \beta)$ and for all ψ with $x\psi = 0$, $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi', A\psi, A'\psi, \alpha^\psi)$. However, in order to prove $\mathsf{PTy}(\mu^{\mathsf{pre}}(\nu,\sigma))(\Psi, \mathsf{V}_x(A, B, E), \mathsf{V}_x(A', B', E'))$, we must observe that when $x\psi = 0$, $\mathsf{V}_0(A\psi, B\psi, E\psi) \longmapsto A\psi$; when $x\psi = 1$, $\mathsf{V}_1(A\psi, B\psi, E\psi) \longmapsto B\psi$; and for every ψ_1, ψ_2 the appropriate relations hold in $\mu^{\mathsf{pre}}(\nu,\sigma)$. See Rule 46 for the full proof, and Lemma 57 for the corresponding proof for $\textsc{Fcom}$. $\square$

Theorem 16. *τ^κ_n and τ^κ_ω are cubical type systems.*

Proof.

System τ_n^κ. Use strong induction on n. Clearly ν_0 is a cubical type system; by Lemma 15 so are τ_0^{Kan} and thus τ_0^{pre}. Suppose τ_j^κ are cubical type systems for $j < n$. Then ν_n is a cubical type system: functionality, symmetry, transitivity, and value-coherence are immediate; PER-valuation follows from the previous τ_j^κ being cubical type systems. The induction step follows by Lemma 15.

System τ_ω^κ. Because each τ_n^κ is a cubical type system, so is ν_ω (as before), and so are τ_ω^κ. □

The cubical type systems employed by Angiuli and Harper [2016] are equivalent to candidate cubical type systems satisfying conditions (1–4): define $A_0 \approx^\Psi B_0$ to hold when $\tau(\Psi, A_0, B_0, \varphi)$, and $M_0 \approx^\Psi_{A_0} N_0$ when $\varphi(M_0, N_0)$. Condition (5) is needed in the construction of universes.

4 Mathematical meaning explanations

In this section, we finally define the judgments of higher type theory as relations parametrized by a choice of cubical type system τ. In these definitions we suppress dependency on τ, but we will write $\tau \models \mathcal{J}\ [\Psi]$ to make the choice of τ explicit.

The presuppositions of a judgment are facts that must be true before one can even sensibly state that judgment. For example, in Definition 18 below, we presuppose that A is a pretype when defining what it means to be equal elements of A; if we do not know A to be a pretype, $[\![A]\!]$ has no meaning. In every judgment $\mathcal{J}\ [\Psi]$ we will presuppose that the free dimensions of all terms are contained in Ψ.

4.1 Judgments

Definition 17. The judgment $A \doteq B\ \mathsf{type_{pre}}\ [\Psi]$ holds when $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ and $\mathsf{Coh}(\alpha)$. Whenever $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ the choice of α is unique and independent of B, so we notate it $[\![A]\!]$.

Definition 18. The judgment $M \doteq N \in A\ [\Psi]$ holds, presupposing $A \doteq A\ \mathsf{type_{pre}}\ [\Psi]$, when $\mathsf{Tm}([\![A]\!])(M, N)$.

If A and B have no free dimensions and $A \doteq B\ \mathsf{type_{pre}}\ [\Psi]$, then for any Ψ', $\tau^{\Downarrow}(\Psi', A, B, [\![A]\!])$ and $[\![A]\!]$ is context-indexed; if M, N, and A have no free dimensions and $M \doteq N \in A\ [\Psi]$, then $([\![A]\!](\Psi'))^{\Downarrow}(M, N)$ for all Ψ'. Therefore one can regard the ordinary meaning explanations as an instance of these meaning explanations, in which all dependency on dimensions trivializes.

We are primarily interested in *Kan types*, pretypes equipped with Kan operations that implement composition, inversion, etc., of cubes. These Kan operations are best specified using judgments augmented by *dimension context restrictions.* We extend the prior judgments to restricted ones:

Definition 19. For any Ψ and set of unoriented equations $\Xi = (r_1 = r_1', \dots, r_n = r_n')$ in Ψ (that is, $\mathsf{FD}(\overrightarrow{r_i}, \overrightarrow{r_i'}) \subseteq \Psi$), we say that $\psi : \Psi' \to \Psi$ *satisfies* Ξ if $r_i\psi = r_i'\psi$ for each $i \in [1, n]$.

Definition 20.

1. The judgment $A \doteq B\ \mathsf{type_{pre}}\ [\Psi \mid \Xi]$ holds, presupposing $\mathsf{FD}(\Xi) \subseteq \Psi$, when $A\psi \doteq B\psi\ \mathsf{type_{pre}}\ [\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.
2. The judgment $M \doteq N \in A\ [\Psi \mid \Xi]$ holds, presupposing $A\ \mathsf{type_{pre}}\ [\Psi \mid \Xi]$, when $M\psi \doteq N\psi \in A\psi\ [\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.

Definition 21. A list of equations $\overrightarrow{r_i = r_i'}$ is *valid* if either $r_i = r_i'$ for some i, or $r_i = r_j$, $r_i' = 0$, and $r_j' = 1$ for some i, j.

Definition 22. The judgment $A \doteq B\ \mathsf{type_{Kan}}\ [\Psi]$ holds, presupposing $A \doteq B\ \mathsf{type_{pre}}\ [\Psi]$, when the following Kan conditions hold for any $\psi : \Psi' \to \Psi$:

1. If
   (a) $\overrightarrow{r_i = r_i'}$ is valid,
   (b) $M \doteq M' \in A\psi\ [\Psi']$,
   (c) $N_i \doteq N_j' \in A\psi\ [\Psi', y \mid r_i = r_i', r_j = r_j']$ for any i, j, and

(d) $N_i\langle r/y\rangle \doteq M \in A\psi\ [\Psi' \mid r_i = r_i']$ for any i,

then

(a) $\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{B\psi}(M';\overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in A\psi\ [\Psi']$;

(b) if $r = r'$ then $\mathsf{hcom}^{r\rightsquigarrow r}_{A\psi}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in A\psi\ [\Psi']$; and

(c) if $r_i = r_i'$ then $\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\psi\ [\Psi']$.

2. If $\Psi' = (\Psi'', x)$ and $M \doteq M' \in A\psi\langle r/x\rangle\ [\Psi'']$, then

(a) $\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{x.B\psi}(M') \in A\psi\langle r'/x\rangle\ [\Psi'']$; and

(b) if $r = r'$ then $\mathsf{coe}^{r\rightsquigarrow r}_{x.A\psi}(M) \doteq M \in A\psi\langle r/x\rangle\ [\Psi'']$.

We extend the closed judgments to open terms by functionality, that is, an open pretype (resp., element of a pretype) is an open term that sends equal elements of the pretypes in the context to equal closed pretypes (resp., elements). The open judgments are defined simultaneously, stratified by the length of the context. (We assume the variables $a_1, \dots, a_n$ in a context are distinct.)

Definition 23. We say $(a_1 : A_1, \dots, a_n : A_n)\ \mathsf{ctx}\ [\Psi]$ when

$$\begin{gathered} A_1\ \mathsf{type_{pre}}\ [\Psi], \\ a_1 : A_1 \gg A_2\ \mathsf{type_{pre}}\ [\Psi], \dots \\ \text{and } a_1 : A_1, \dots, a_{n-1} : A_{n-1} \gg A_n\ \mathsf{type_{pre}}\ [\Psi]. \end{gathered}$$

Definition 24. We say $a_1 : A_1, \dots, a_n : A_n \gg B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$, presupposing $(a_1 : A_1, \dots, a_n : A_n)\ \mathsf{ctx}\ [\Psi]$, when for any $\psi : \Psi' \to \Psi$ and any

$$\begin{gathered} N_1 \doteq N_1' \in A_1\psi\ [\Psi'], \\ N_2 \doteq N_2' \in A_2\psi[N_1/a_1]\ [\Psi'], \dots \\ \text{and } N_n \doteq N_n' \in A_n\psi[N_1, \dots, N_{n-1}/a_1, \dots, a_n]\ [\Psi'], \end{gathered}$$

$B\psi[N_1, \dots, N_n/a_1, \dots, a_n] \doteq B'\psi[N_1', \dots, N_n'/a_1, \dots, a_n]\ \mathsf{type_{pre}}\ [\Psi']$.

Definition 25. We say $a_1 : A_1, \dots, a_n : A_n \gg M \doteq M' \in B\ [\Psi]$, presupposing $a_1 : A_1, \dots, a_n : A_n \gg B\ \mathsf{type_{pre}}\ [\Psi]$, when for any $\psi : \Psi' \to \Psi$ and any

$$\begin{gathered} N_1 \doteq N_1' \in A_1\psi\ [\Psi'], \\ N_2 \doteq N_2' \in A_2\psi[N_1/a_1]\ [\Psi'], \dots \\ \text{and } N_n \doteq N_n' \in A_n\psi[N_1, \dots, N_{n-1}/a_1, \dots, a_n]\ [\Psi'], \end{gathered}$$

$M\psi[N_1, \dots, N_n/a_1, \dots, a_n] \doteq M'\psi[N_1', \dots, N_n'/a_1, \dots, a_n] \in B\psi[N_1, \dots, N_n/a_1, \dots, a_n]\ [\Psi']$.

One should read $[\Psi]$ as extending across the entire judgment, as it specifies the starting dimension at which to consider not only B and M but Γ as well. The open judgments, like the closed judgments, are symmetric and transitive. In particular, if $\Gamma \gg B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$ then $\Gamma \gg B\ \mathsf{type_{pre}}\ [\Psi]$. As a result, the earlier hypotheses of each definition ensure that later hypotheses are sensible; for example, $(a_1 : A_1, \dots, a_n : A_n)\ \mathsf{ctx}\ [\Psi]$ and $N_1 \in A_1\psi\ [\Psi']$ ensure that $A_2\psi[N_1/a_1]\ \mathsf{type_{pre}}\ [\Psi']$.

Definition 26. We say $a_1 : A_1, \ldots, a_n : A_n \gg B \doteq B'$ $\mathsf{type_{Kan}}$ $[\Psi]$, presupposing $a_1 : A_1, \ldots, a_n : A_n \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$, when for any $\psi : \Psi' \to \Psi$ and any

$$N_1 \doteq N_1' \in A_1\psi\ [\Psi'],$$
$$N_2 \doteq N_2' \in A_2\psi[N_1/a_1]\ [\Psi'], \ldots$$
$$\text{and } N_n \doteq N_n' \in A_n\psi[N_1, \ldots, N_{n-1}/a_1, \ldots, a_n]\ [\Psi'],$$

we have $B\psi[N_1, \ldots, N_n/a_1, \ldots, a_n] \doteq B'\psi[N_1', \ldots, N_n'/a_1, \ldots, a_n]$ $\mathsf{type_{Kan}}$ $[\Psi']$.

Finally, the open judgments can also be augmented by context restrictions. In order to make sense of Definition 27, the presuppositions of the open judgments require them to be closed under dimension substitution, which we will prove in Lemma 28.

Definition 27.

1. The judgment Γ ctx $[\Psi \mid \Xi]$ holds, presupposing $\mathsf{FD}(\Xi) \subseteq \Psi$, when $\Gamma\psi$ ctx $[\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.

2. The judgment $\Gamma \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi \mid \Xi]$ holds, presupposing Γ ctx $[\Psi \mid \Xi]$, when $\Gamma\psi \gg B\psi \doteq B'\psi$ $\mathsf{type_{pre}}$ $[\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.

3. The judgment $\Gamma \gg M \doteq M' \in B$ $[\Psi \mid \Xi]$ holds, presupposing Γ ctx $[\Psi \mid \Xi]$ and $\Gamma \gg B$ $\mathsf{type_{pre}}$ $[\Psi \mid \Xi]$, when $\Gamma\psi \gg M\psi \doteq M'\psi \in B\psi$ $[\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.

4. The judgment $\Gamma \gg B \doteq B'$ $\mathsf{type_{Kan}}$ $[\Psi \mid \Xi]$ holds, presupposing Γ ctx $[\Psi \mid \Xi]$, when $\Gamma\psi \gg B\psi \doteq B'\psi$ $\mathsf{type_{Kan}}$ $[\Psi']$ for every $\psi : \Psi' \to \Psi$ satisfying Ξ.

4.2 Structural properties

Every judgment is closed under dimension substitution.

Lemma 28. *For any $\psi : \Psi' \to \Psi$,*

1. *if $A \doteq B$ $\mathsf{type_{pre}}$ $[\Psi]$ then $A\psi \doteq B\psi$ $\mathsf{type_{pre}}$ $[\Psi']$;*

2. *if $M \doteq N \in A$ $[\Psi]$ then $M\psi \doteq N\psi \in A\psi$ $[\Psi']$;*

3. *if $A \doteq B$ $\mathsf{type_{Kan}}$ $[\Psi]$ then $A\psi \doteq B\psi$ $\mathsf{type_{Kan}}$ $[\Psi']$;*

4. *if Γ ctx $[\Psi]$ then $\Gamma\psi$ ctx $[\Psi']$;*

5. *if $\Gamma \gg A \doteq B$ $\mathsf{type_{pre}}$ $[\Psi]$ then $\Gamma\psi \gg A\psi \doteq B\psi$ $\mathsf{type_{pre}}$ $[\Psi']$;*

6. *if $\Gamma \gg M \doteq N \in A$ $[\Psi]$ then $\Gamma\psi \gg M\psi \doteq N\psi \in A\psi$ $[\Psi']$; and*

7. *if $\Gamma \gg A \doteq B$ $\mathsf{type_{Kan}}$ $[\Psi]$ then $\Gamma\psi \gg A\psi \doteq B\psi$ $\mathsf{type_{Kan}}$ $[\Psi']$.*

Proof. For proposition (1), by $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ we have $\mathsf{PTy}(\tau)(\Psi', A\psi, B\psi, \alpha\psi)$. We must show for all $\psi' : \Psi'' \to \Psi'$ that $(\alpha\psi)_{\psi'}(M_0, N_0)$ implies $\mathsf{Tm}(\alpha\psi\psi')(M_0, N_0)$; this follows from value-coherence of α at $\psi\psi'$. Propositions (2) and (3) follow from $[\![A\psi]\!] = [\![A]\!]\psi$ and closure of Tm and the Kan conditions under dimension substitution.

Propositions (4), (5), and (6) are proven simultaneously by induction on the length of Γ. If $\Gamma = \cdot$, then (4) is trivial, and (5) and (6) follow because the closed judgments are closed under dimension substitution. The induction steps for all three use all three induction hypotheses. Proposition (7) follows similarly. $\square$

Lemma 29. *For any $\psi : \Psi' \to \Psi$, if $\mathcal{J}\ [\Psi \mid \Xi]$ then $\mathcal{J}\psi\ [\Psi' \mid \Xi\psi]$.*

Proof. We know that $\mathcal{J}\psi\ [\Psi']$ for any $\psi : \Psi' \to \Psi$ satisfying Ξ, and want to show that $\mathcal{J}\psi\psi'\ [\Psi'']$ for any $\psi : \Psi' \to \Psi$ and $\psi' : \Psi'' \to \Psi'$ satisfying $\Xi\psi$. It suffices to show that if ψ' satisfies $\Xi\psi$, then $\psi\psi'$ satisfies Ξ. But these both hold if and only if for each $(r_i = r_i') \in \Xi$, $r_i\psi\psi' = r_i'\psi\psi'$. $\square$

Remark 30. The context-restricted judgments can be thought of as merely a notational device, because it is possible to systematically translate $\mathcal{J}\ [\Psi \mid \Xi]$ into ordinary judgments by case analysis:

1. All ψ satisfy an empty set of equations, so $\mathcal{J}\ [\Psi \mid \cdot]$ if and only if $\mathcal{J}\psi\ [\Psi']$ for all ψ, which by Lemma 28 holds if and only if $\mathcal{J}\ [\Psi]$.
2. A ψ satisfies $(\Xi, r = r)$ if and only if it satisfies Ξ, so $\mathcal{J}\ [\Psi \mid \Xi, r = r]$ if and only if $\mathcal{J}\ [\Psi \mid \Xi]$.
3. No ψ satisfies $(\Xi, 0 = 1)$, so $\mathcal{J}\ [\Psi \mid \Xi, 0 = 1]$ always.
4. By Lemma 29, $\mathcal{J}\ [\Psi, x \mid \Xi, x = r]$ if and only if $\mathcal{J}\langle r/x\rangle\ [\Psi \mid \Xi\langle r/x\rangle, r = r]$, which holds if and only if $\mathcal{J}\langle r/x\rangle\ [\Psi \mid \Xi\langle r/x\rangle]$.

The open judgments satisfy the *structural rules* of type theory, like hypothesis and weakening.

Lemma 31 (Hypothesis)**.** *If $(\Gamma, a_i : A_i, \Gamma')\ \mathsf{ctx}\ [\Psi]$ then $\Gamma, a_i : A_i, \Gamma' \gg a_i \in A_i\ [\Psi]$.*

Proof. We must show for any $\psi : \Psi' \to \Psi$ and equal elements $N_1, N_1', \dots, N_n, N_n'$ of the pretypes in $(\Gamma\psi, a_i : A_i\psi, \Gamma'\psi)$, that $N_i \doteq N_i' \in A_i\psi\ [\Psi']$. But this is exactly our assumption about N_i, N_i'. $\square$

Lemma 32 (Weakening)**.**

1. *If $\Gamma, \Gamma' \gg B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$ and $\Gamma \gg A\ \mathsf{type_{pre}}\ [\Psi]$, then $\Gamma, a : A, \Gamma' \gg B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$.*
2. *If $\Gamma, \Gamma' \gg M \doteq M' \in B\ [\Psi]$ and $\Gamma \gg A\ \mathsf{type_{pre}}\ [\Psi]$, then $\Gamma, a : A, \Gamma' \gg M \doteq M' \in B\ [\Psi]$.*

Proof. For the first part, we must show for any $\psi : \Psi' \to \Psi$ and equal elements

$$\begin{gathered} N_1 \doteq N_1' \in A_1\psi\ [\Psi'], \\ N_2 \doteq N_2' \in A_2\psi[N_1/a_1]\ [\Psi'], \dots \\ N \doteq N' \in A\psi[N_1, \dots/a_1, \dots]\ [\Psi'], \dots \\ \text{and } N_n \doteq N_n' \in A_n\psi[N_1, \dots, N, \dots, N_{n-1}/a_1, \dots, a, \dots, a_n]\ [\Psi'], \end{gathered}$$

that the corresponding instances of B, B' are equal closed pretypes. By $\Gamma, \Gamma' \gg B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$ we know that $a \# \Gamma', B, B'$—since the contained pretypes become closed when substituting for $a_1, \dots, a_n$. It also gives us $B\psi[N_1, \dots/a_1, \dots] \doteq B'\psi[N_1', \dots/a_1, \dots]\ \mathsf{type_{pre}}\ [\Psi']$ which are the desired instances of B, B' because $a \# B, B'$. The second part follows similarly. $\square$

The definition of equal pretypes was chosen to ensure that equal pretypes have equal elements.

Lemma 33. *If $A \doteq B$ $\mathsf{type_{pre}}$ $[\Psi]$ and $M \doteq N \in A$ $[\Psi]$ then $M \doteq N \in B$ $[\Psi]$.*

Proof. If $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ then $\mathsf{PTy}(\tau)(\Psi, B, A, \alpha)$; the result follows by $[\![A]\!] = [\![B]\!]$. $\square$

Lemma 34. *If $\Gamma \gg A \doteq B$ $\mathsf{type_{pre}}$ $[\Psi]$ and $\Gamma \gg M \doteq N \in A$ $[\Psi]$ then $\Gamma \gg M \doteq N \in B$ $[\Psi]$.*

Proof. If $\Gamma = (a_1 : A_1, \dots, a_n : A_n)$ then $\Gamma \gg M \doteq N \in A$ $[\Psi]$ means that for any $\psi : \Psi' \to \Psi$ and equal elements $N_1, N_1', \dots, N_n, N_n'$ of the pretypes in $\Gamma\psi$, the corresponding instances of M and N are equal in $A\psi[N_1, \dots, N_n/a_1, \dots, a_n]$. But $\Gamma \gg A \doteq B$ $\mathsf{type_{pre}}$ $[\Psi]$ implies this pretype is equal to $B\psi[N_1, \dots, N_n/a_1, \dots, a_n]$, so the result follows by Lemma 33. $\square$

4.3 Basic lemmas

The definition of $\mathsf{PTy}(\tau)(\Psi, A, B, \alpha)$ can be simplified when τ is a cubical type system: it suffices to check for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$ that $A\psi_1 \Downarrow A_1$, $B\psi_1 \Downarrow B_1$, $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, A\psi_1\psi_2, \varphi)$, $\tau^{\Downarrow}(\Psi_2, B_1\psi_2, B\psi_1\psi_2, \varphi')$, and $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, B_1\psi_2, \varphi'')$. Then $\varphi = \varphi' = \varphi''$ and α exists uniquely. The proof uses the observation that the following permissive form of transitivity holds for any functional PER R: if $R(\Psi, A, B, \alpha)$ and $R(\Psi, B, C, \beta)$ then $R(\Psi, A, C, \alpha)$ and $\alpha = \beta$.

Lemma 35. *If $\mathsf{PTy}(\tau)(\Psi, A, A, \alpha)$, then $A \Downarrow A_0$ and $\mathsf{PTy}(\tau)(\Psi, A, A_0, \alpha)$.*

Proof. Check for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$ that $\tau^{\Downarrow}(\Psi_2, A\psi_1\psi_2, A_0\psi_1\psi_2, \varphi)$ and $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, A_1'\psi_2, \varphi')$ where $A\psi_1 \Downarrow A_1$ and $A_0\psi_1 \Downarrow A_1'$. The former holds by $\mathsf{PTy}(\tau)(\Psi, A, A, \alpha)$ at the substitutions id_Ψ and $\psi_1\psi_2$. For the latter, $\mathsf{PTy}(\tau)(\Psi, A, A, \alpha)$ at $\psi_1, \mathsf{id}_{\Psi_1}$ proves that $\tau^{\Downarrow}(\Psi_1, A_1, A\psi_1, \_)$ and at id_Ψ, ψ_1 proves $\tau^{\Downarrow}(\Psi_1, A_0\psi_1, A\psi_1, \_)$. By transitivity, $\tau(\Psi_1, A_1, A_1', \_)$ so $\mathsf{PTy}(\tau)(\Psi_1, A_1, A_1', \_)$ and thus $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, A_1'\psi_2, \varphi')$ as required. $\square$

Lemma 36. *If A $\mathsf{type_{pre}}$ $[\Psi]$, then $A \Downarrow A_0$ and $A \doteq A_0$ $\mathsf{type_{pre}}$ $[\Psi]$.*

Proof. By Lemma 35 we have $\mathsf{PTy}(\tau)(\Psi, A, A_0, \alpha)$; value-coherence follows by A $\mathsf{type_{pre}}$ $[\Psi]$. $\square$

Lemma 37. *If $M \in A$ $[\Psi]$, $N \in A$ $[\Psi]$, and $[\![A]\!]^{\Downarrow}(M, N)$, then $M \doteq N \in A$ $[\Psi]$.*

Proof. We check for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$ that $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M\psi_1\psi_2, N\psi_1\psi_2)$; the other needed relations follow from $M \in A$ $[\Psi]$ and $N \in A$ $[\Psi]$. By A $\mathsf{type_{pre}}$ $[\Psi]$, $[\![A]\!]^{\Downarrow}(M, N)$ implies $\mathsf{Tm}([\![A]\!])(M_0, N_0)$ where $M \Downarrow M_0$ and $N \Downarrow N_0$, hence $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M_0\psi_1\psi_2, N_0\psi_1\psi_2)$. By $M \in A$ $[\Psi]$ we have $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M_0\psi_1\psi_2, M\psi_1\psi_2)$ and similarly for N, so the result follows by transitivity. $\square$

Lemma 38. *If $M \in A$ $[\Psi]$, then $M \Downarrow M_0$ and $M \doteq M_0 \in A$ $[\Psi]$.*

Proof. By $M \in A$ $[\Psi]$, $M \Downarrow M_0$ and $[\![A]\!](M_0, M_0)$. By A $\mathsf{type_{pre}}$ $[\Psi]$, $M_0 \in A$ $[\Psi]$, so the result follows by Lemma 37. $\square$

Lemma 39. *If A $\mathsf{type_{Kan}}$ $[\Psi]$, B $\mathsf{type_{Kan}}$ $[\Psi]$, and for all $\psi : \Psi' \to \Psi$, $A_\psi \doteq B_\psi$ $\mathsf{type_{Kan}}$ $[\Psi']$ where $A\psi \Downarrow A_\psi$ and $B\psi \Downarrow B_\psi$, then $A \doteq B$ $\mathsf{type_{Kan}}$ $[\Psi]$.*

Proof. By Lemma 36 we have $A\psi \doteq A_\psi$ $\mathsf{type_{pre}}$ $[\Psi']$ and $B\psi \doteq B_\psi$ $\mathsf{type_{pre}}$ $[\Psi']$ for all $\psi : \Psi' \to \Psi$; thus $A\psi \doteq B\psi$ $\mathsf{type_{pre}}$ $[\Psi']$ for all $\psi : \Psi' \to \Psi$, and it suffices to establish that if

1. $\overrightarrow{r_i = r_i'}$ is valid,

2. $M \doteq M' \in A\psi\ [\Psi']$,

3. $N_i \doteq N'_j \in A\psi\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and

4. $N_i\langle r/y\rangle \doteq M \in A\psi\ [\Psi' \mid r_i = r'_i]$ for any i,

then $\mathsf{hcom}^{r \rightsquigarrow r'}_{A\psi}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r \rightsquigarrow r'}_{B\psi}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}) \in A\psi\ [\Psi']$. We already know both terms are elements of this type (by Definition 22 and $A\psi \doteq B\psi\ \mathsf{type_{pre}}\ [\Psi']$), so by Lemma 37 it suffices to check that these terms are related by $[\![A\psi]\!]^{\Downarrow}$ or equivalently $[\![A_\psi]\!]^{\Downarrow}$. This is true because $\mathsf{hcom}_{A\psi} \longmapsto^* \mathsf{hcom}_{A_\psi}$, $\mathsf{hcom}_{B\psi} \longmapsto^* \mathsf{hcom}_{B_\psi}$, and by $A_\psi \doteq B_\psi\ \mathsf{type_{Kan}}\ [\Psi']$ and $A\psi \doteq A_\psi\ \mathsf{type_{pre}}\ [\Psi']$, $\mathsf{hcom}_{A_\psi} \doteq \mathsf{hcom}_{B_\psi} \in A_\psi\ [\Psi']$. The remaining hcom equations of Definition 22 follow by transitivity and $A_\psi\ \mathsf{type_{Kan}}\ [\Psi']$; the coe equations follow by a similar argument. □

In order to establish that a term is a pretype or element, one must frequently reason about the evaluation behavior of its aspects. When all aspects compute in lockstep, a *head expansion* lemma applies; otherwise one must appeal to its generalization, *coherent expansion*:

Lemma 40. *Assume we have A tm $[\Psi]$ and a family of terms $\{A^{\Psi'}_\psi\}_{\psi:\Psi'\to\Psi}$ such that for all $\psi : \Psi' \to \Psi$, $A^{\Psi'}_\psi \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi\ \mathsf{type_{pre}}\ [\Psi']$ and $A\psi \longmapsto^* A^{\Psi'}_\psi$. Then $A \doteq A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type_{pre}}\ [\Psi]$.*

Proof. We must show that for any $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, $A\psi_1 \Downarrow A_1$, $(A^{\Psi}_{\mathsf{id}_\Psi})\psi_1 \Downarrow A'_1$, and $\tau^{\Downarrow}(\Psi_2, -, -, \_)$ relates $A_1\psi_2$, $A\psi_1\psi_2$, $(A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2$, and $A'_1\psi_2$.

1. $A\psi_1 \Downarrow A_1$ and $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, A\psi_1\psi_2, \varphi)$.
   We know $A\psi_1 \longmapsto^* A^{\Psi_1}_{\psi_1}$ and $A^{\Psi_1}_{\psi_1}\ \mathsf{type_{pre}}\ [\Psi_1]$, so $\tau^{\Downarrow}(\Psi_2, A_1\psi_2, (A^{\Psi_1}_{\psi_1})\psi_2, \varphi)$ where $A^{\Psi_1}_{\psi_1} \Downarrow A_1$.
   By $A^{\Psi_1}_{\psi_1} \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\ \mathsf{type_{pre}}\ [\Psi_1]$ under ψ_2 and $(A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2 \doteq A^{\Psi_2}_{\psi_1\psi_2}\ \mathsf{type_{pre}}\ [\Psi_2]$, we have $(A^{\Psi_1}_{\psi_1})\psi_2 \doteq A^{\Psi_2}_{\psi_1\psi_2}\ \mathsf{type_{pre}}\ [\Psi_2]$ and thus $\tau^{\Downarrow}(\Psi_2, (A^{\Psi_1}_{\psi_1})\psi_2, A^{\Psi_2}_{\psi_1\psi_2}, \varphi)$. The result follows by transitivity and $A\psi_1\psi_2 \longmapsto^* A^{\Psi_2}_{\psi_1\psi_2}$.

2. $\tau^{\Downarrow}(\Psi_2, A\psi_1\psi_2, (A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2, \varphi')$.
   By $A^{\Psi_2}_{\psi_1\psi_2} \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2\ \mathsf{type_{pre}}\ [\Psi_2]$ we have $\tau^{\Downarrow}(\Psi_2, A^{\Psi_2}_{\psi_1\psi_2}, (A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2, \varphi')$; the result follows by $A\psi_1\psi_2 \longmapsto^* A^{\Psi_2}_{\psi_1\psi_2}$.

3. $(A^{\Psi}_{\mathsf{id}_\Psi})\psi_1 \Downarrow A'_1$ and $\tau^{\Downarrow}(\Psi_2, (A^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2, A'_1\psi_2, \varphi'')$.
   Follows from $A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type_{pre}}\ [\Psi]$. □

Lemma 41. *Assume we have M tm $[\Psi]$, $A\ \mathsf{type_{pre}}\ [\Psi]$, and a family of terms $\{M^{\Psi'}_\psi\}_{\psi:\Psi'\to\Psi}$ such that for all $\psi : \Psi' \to \Psi$, $M^{\Psi'}_\psi \doteq (M^{\Psi}_{\mathsf{id}_\Psi})\psi \in A\psi\ [\Psi']$ and $M\psi \longmapsto^* M^{\Psi'}_\psi$. Then $M \doteq M^{\Psi}_{\mathsf{id}_\Psi} \in A\ [\Psi]$.*

Proof. We must show that for any $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, $M\psi_1 \Downarrow M_1$, $(M^{\Psi}_{\mathsf{id}_\Psi})\psi_1 \Downarrow M'_1$, and $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}$ relates $M_1\psi_2$, $M\psi_1\psi_2$, $(M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2$, and $M'_1\psi_2$.

1. $M\psi_1 \Downarrow M_1$ and $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M_1\psi_2, M\psi_1\psi_2)$.
   We know $M\psi_1 \longmapsto^* M^{\Psi_1}_{\psi_1}$ and $M^{\Psi_1}_{\psi_1} \in A\psi_1\ [\Psi_1]$, so $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M_1\psi_2, (M^{\Psi_1}_{\psi_1})\psi_2)$ where $M^{\Psi_1}_{\psi_1} \Downarrow M_1$.
   By $M^{\Psi_1}_{\psi_1} \doteq (M^{\Psi}_{\mathsf{id}_\Psi})\psi_1 \in A\psi_1\ [\Psi_1]$ under ψ_2 and $(M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2 \doteq M^{\Psi_2}_{\psi_1\psi_2} \in A\psi_1\psi_2\ [\Psi_2]$, we have

$(M^{\Psi_1}_{\psi_1})\psi_2 \doteq M^{\Psi_2}_{\psi_1\psi_2} \in A\psi_1\psi_2\ [\Psi_2]$ and thus $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}((M^{\Psi_1}_{\psi_1})\psi_2, M^{\Psi_2}_{\psi_1\psi_2})$. The result follows by transitivity and $M\psi_1\psi_2 \longmapsto^* M^{\Psi_2}_{\psi_1\psi_2}$.

2. $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M\psi_1\psi_2, (M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2)$.

   By $M^{\Psi_2}_{\psi_1\psi_2} \doteq (M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2 \in A\psi_1\psi_2\ [\Psi_2]$ we have $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(M^{\Psi_2}_{\psi_1\psi_2}, (M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2)$; the result follows by $M\psi_1\psi_2 \longmapsto^* M^{\Psi_2}_{\psi_1\psi_2}$.

3. $(M^{\Psi}_{\mathsf{id}_\Psi})\psi_1 \Downarrow M'_1$ and $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}((M^{\Psi}_{\mathsf{id}_\Psi})\psi_1\psi_2, M'_1\psi_2)$.

   Follows from $M^{\Psi}_{\mathsf{id}_\Psi} \in A\ [\Psi]$. □

Lemma 42. *Assume we have A tm $[\Psi]$ and a family of terms $\{A^{\Psi'}_{\psi}\}_{\psi:\Psi'\to\Psi}$ such that for all $\psi : \Psi' \to \Psi$, $A^{\Psi'}_{\psi} \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$ and $A\psi \longmapsto^* A^{\Psi'}_{\psi}$. Then $A \doteq A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.*

Proof. By Lemma 40, $A \doteq A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$; it suffices to establish the conditions in Definition 22. First, assume $\psi : \Psi' \to \Psi$,

1. $\overrightarrow{r_i = r'_i}$ is valid,
2. $M \doteq M' \in A\psi\ [\Psi']$,
3. $N_i \doteq N'_j \in A\psi\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and
4. $N_i\langle r/y\rangle \doteq M \in A\psi\ [\Psi' \mid r_i = r'_i]$ for any i,

and show that $\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{(A^{\Psi}_{\mathsf{id}_\Psi})\psi}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}) \in A\psi\ [\Psi']$. We apply Lemma 41 to $\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i})$ and the family

$$\{\mathsf{hcom}^{r\psi'\rightsquigarrow r'\psi'}_{A^{\Psi''}_{\psi\psi'}}(M\psi'; \overrightarrow{r_i\psi' = r'_i\psi' \hookrightarrow y.N_i\psi'})\}^{\Psi''}_{\psi'}$$

at $A\psi$ $\mathsf{type}_{\mathsf{pre}}$ $[\Psi']$. We know $\mathsf{hcom}_{A\psi\psi'} \longmapsto^* \mathsf{hcom}_{A^{\Psi''}_{\psi\psi'}}$ by $A\psi\psi' \longmapsto^* A^{\Psi''}_{\psi\psi'}$, and $\mathsf{hcom}_{A^{\Psi''}_{\psi\psi'}} \doteq \mathsf{hcom}_{(A^{\Psi'}_{\psi})\psi'} \in A\psi\psi'\ [\Psi'']$ by $A^{\Psi''}_{\psi\psi'} \doteq (A^{\Psi'}_{\psi})\psi'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi'']$ and $A^{\Psi''}_{\psi\psi'} \doteq A\psi\psi'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi'']$ (both by transitivity through $(A^{\Psi}_{\mathsf{id}_\Psi})\psi\psi'$). We conclude that $\mathsf{hcom}_{A\psi} \doteq \mathsf{hcom}_{A^{\Psi'}_{\psi}} \in A\psi\ [\Psi']$, and the desired result follows by $A^{\Psi'}_{\psi} \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$. The remaining hcom equations of Definition 22 follow by transitivity and $A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.

Next, assuming $\psi : (\Psi', x) \to \Psi$ and $M \doteq M' \in A\psi\langle r/x\rangle\ [\Psi']$, show that $\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{x.(A^{\Psi}_{\mathsf{id}_\Psi})\psi}(M') \in A\psi\langle r'/x\rangle\ [\Psi']$. We apply Lemma 41 to $\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(M)$ and $\{\mathsf{coe}^{r\psi'\rightsquigarrow r'\psi'}_{x.A^{\Psi''}_{\psi\psi'}}(M\psi')\}^{\Psi''}_{\psi'}$ at $A\psi\langle r'/x\rangle$ $\mathsf{type}_{\mathsf{pre}}$ $[\Psi']$, using the same argument as before; we conclude that $\mathsf{coe}_{x.A\psi} \doteq \mathsf{coe}_{x.A^{\Psi'}_{\psi}} \in A\psi\langle r'/x\rangle\ [\Psi']$, and the desired result follows by $A^{\Psi'}_{\psi} \doteq (A^{\Psi}_{\mathsf{id}_\Psi})\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', x]$. The remaining coe equation of Definition 22 follows by transitivity and $A^{\Psi}_{\mathsf{id}_\Psi}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$. □

Lemma 43 (Head expansion)**.**

1. *If A' $\mathsf{type}_{\mathsf{pre}}$ $[\Psi]$ and $A \longmapsto^*_{\square} A'$, then $A \doteq A'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.*

2. *If $M' \in A\ [\Psi]$ and $M \longmapsto^*_{\square} M'$, then $M \doteq M' \in A\ [\Psi]$.*

3. *If $A'\ \mathsf{type_{Kan}}\ [\Psi]$ and $A \longmapsto^*_{\square} A'$, then $A \doteq A'\ \mathsf{type_{Kan}}\ [\Psi]$.*

Proof.

1. By Lemma 40 with $A^{\Psi'}_{\psi} = A'\psi$, because $A\psi \longmapsto^* A'\psi$ and $A'\psi\ \mathsf{type_{pre}}\ [\Psi']$ for all ψ.

2. By Lemma 41 with $M^{\Psi'}_{\psi} = M'\psi$, because $M\psi \longmapsto^* M'\psi$ and $M'\psi \in A\psi\ [\Psi']$ for all ψ.

3. By Lemma 42 with $A^{\Psi'}_{\psi} = A'\psi$, because $A\psi \longmapsto^* A'\psi$ and $A'\psi\ \mathsf{type_{Kan}}\ [\Psi']$ for all ψ. □

The hcom operation implements *homogeneous* composition, in the sense that A must be degenerate in the bound direction of the tubes. We can obtain *heterogeneous* composition, com, by combining hcom and coe.

Theorem 44. *If $A \doteq B\ \mathsf{type_{Kan}}\ [\Psi, y]$,*

1. *$\overrightarrow{r_i = r'_i}$ is valid,*

2. *$M \doteq M' \in A\langle r/y\rangle\ [\Psi]$,*

3. *$N_i \doteq N'_j \in A\ [\Psi, y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and*

4. *$N_i\langle r/y\rangle \doteq M \in A\langle r/y\rangle\ [\Psi \mid r_i = r'_i]$ for any i,*

then

1. *$\mathsf{com}^{r\rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq \mathsf{com}^{r\rightsquigarrow r'}_{y.B}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}) \in A\langle r'/y\rangle\ [\Psi]$;*

2. *if $r = r'$ then $\mathsf{com}^{r\rightsquigarrow r}_{y.A}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq M \in A\langle r/y\rangle\ [\Psi]$; and*

3. *if $r_i = r'_i$ then $\mathsf{com}^{r\rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\langle r'/y\rangle\ [\Psi]$.*

Proof. For all $\psi : \Psi' \to (\Psi, y)$ satisfying $r_i = r'_i$ and $r_j = r'_j$, we know $N_i\psi \doteq N'_j\psi \in A\psi\ [\Psi']$. By Definition 22, $(\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i))\psi \doteq (\mathsf{coe}^{y\rightsquigarrow r'}_{y.B}(N'_j))\psi \in A\langle r'/y\rangle\psi\ [\Psi']$, and therefore $\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i) \doteq \mathsf{coe}^{y\rightsquigarrow r'}_{y.B}(N'_j) \in A\ [\Psi, y \mid r_i = r'_i, r_j = r'_j]$. By a similar argument we conclude $(\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i))\langle r/y\rangle \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M) \in A\langle r'/y\rangle\ [\Psi \mid r_i = r'_i]$, and by Definition 22 directly, $\mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{y.B}(M') \in A\langle r'/y\rangle\ [\Psi]$. By Definition 22 we conclude

$$\begin{aligned}
&\mathsf{hcom}^{r\rightsquigarrow r'}_{A\langle r'/y\rangle}(\mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i)}) \\
&\doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{B\langle r'/y\rangle}(\mathsf{coe}^{r\rightsquigarrow r'}_{y.B}(M'); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.B}(N'_i)}) \in A\langle r'/y\rangle\ [\Psi].
\end{aligned}$$

Result (1) follows by Lemma 43 on each side.

Result (2) follows by Lemma 43 and, by Definition 22 twice,

$$\mathsf{hcom}^{r'\rightsquigarrow r'}_{A\langle r'/y\rangle}(\mathsf{coe}^{r'\rightsquigarrow r'}_{y.A}(M); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i)}) \doteq M \in A\langle r'/y\rangle\ [\Psi].$$

Result (3) follows by Lemma 43 and, by Definition 22 twice,

$$\mathsf{hcom}^{r\rightsquigarrow r'}_{A\langle r'/y\rangle}(\mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}^{y\rightsquigarrow r'}_{y.A}(N_i)}) \doteq N_i\langle r'/y\rangle \in A\langle r'/y\rangle\ [\Psi]. \qquad \square$$

5 Types

In Section 3 we defined two sequences of cubical type systems, and in Section 4 we defined the judgments of higher type theory relative to any cubical type system. In this section we will prove that $\tau_\omega^{\mathsf{pre}}$ validates certain rules, summarized in part in Section 6. For non-universe connectives, we in fact prove that the rules hold in every τ_n^κ and τ_ω^κ.

5.1 Dependent function types

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu, \sigma)$ for any cubical type systems ν, σ; in τ, whenever $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi]$, $a{:}A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$, and $\varphi = \{(\lambda a.N, \lambda a.N') \mid a{:}A \gg N \doteq N' \in B\ [\Psi]\}$, we have $\tau(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Notice that whenever $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi]$ and $a : A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$, we have $\mathsf{PTy}(\tau)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \_)$ because $(a{:}A) \to B$ $\mathsf{val}_{\square}$ and judgments are preserved by dimension substitution.

Lemma 45. *If $a : A \gg M \doteq M' \in B\ [\Psi]$ then $\mathsf{Tm}(\llbracket(a{:}A) \to B\rrbracket)(\lambda a.M, \lambda a.M')$.*

Proof. By $\lambda a.M$ $\mathsf{val}_{\square}$, it suffices to check that $\llbracket(a{:}A) \to B\rrbracket_\psi(\lambda a.M\psi, \lambda a.M'\psi)$ for any $\psi : \Psi' \to \Psi$; this holds because $a : A\psi \gg M\psi \doteq M'\psi \in B\psi\ [\Psi']$ and $\llbracket(a{:}A) \to B\rrbracket_\psi = \llbracket(a{:}A\psi) \to B\psi\rrbracket$. □

Rule 1 (Pretype formation)**.** *If $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi]$ and $a : A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$ then $(a{:}A) \to B \doteq (a{:}A') \to B'$ $\mathsf{type_{pre}}$ $[\Psi]$.*

Proof. We have $\mathsf{PTy}(\tau)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \alpha)$, and by Lemma 45, $\mathsf{Coh}(\alpha)$. □

Rule 2 (Introduction)**.** *If $a : A \gg M \doteq M' \in B\ [\Psi]$ then $\lambda a.M \doteq \lambda a.M' \in (a{:}A) \to B\ [\Psi]$.*

Proof. Immediate by Lemma 45 and Rule 1. □

Lemma 46. *If $M \in (a{:}A) \to B\ [\Psi]$ and $N \in A\ [\Psi]$ then $M \Downarrow \lambda a.O$ and $\mathsf{app}(M, N) \doteq O[N/a] \in B[N/a]\ [\Psi]$.*

Proof. For any $\psi : \Psi' \to \Psi$, we know that $M\psi \Downarrow \lambda a.O_\psi$ and $\llbracket(a{:}A) \to B\rrbracket_\psi(\lambda a.O_{\mathsf{id}_\Psi}\psi, \lambda a.O_\psi)$, and therefore $a : A\psi \gg O_{\mathsf{id}_\Psi}\psi \doteq O_\psi \in B\psi\ [\Psi']$. We apply coherent expansion to $\mathsf{app}(M, N)$, $B[N/a]$ $\mathsf{type_{pre}}$ $[\Psi]$, and $\{O_\psi[N\psi/a]\}_\psi^{\Psi'}$, by $\mathsf{app}(M\psi, N\psi) \longmapsto^* \mathsf{app}(\lambda a.O_\psi, N\psi) \longmapsto O_\psi[N\psi/a]$ and $O_\psi[N\psi/a] \doteq (O_{\mathsf{id}_\Psi}[N/a])\psi \in B\psi[N\psi/a]\ [\Psi']$. We conclude by Lemma 41 that $\mathsf{app}(M, N) \doteq O_{\mathsf{id}_\Psi}[N/a] \in B[N/a]\ [\Psi]$, as desired. □

Rule 3 (Elimination)**.** *If $M \doteq M' \in (a{:}A) \to B\ [\Psi]$ and $N \doteq N' \in A\ [\Psi]$ then $\mathsf{app}(M, N) \doteq \mathsf{app}(M', N') \in B[N/a]\ [\Psi]$.*

Proof. By Lemma 46 we know $M \Downarrow \lambda a.O$, $M' \Downarrow \lambda a.O'$, $\mathsf{app}(M, N) \doteq O[N/a] \in B[N/a]\ [\Psi]$, and $\mathsf{app}(M', N') \doteq O'[N'/a] \in B[N'/a]\ [\Psi]$. By Lemma 38, $M \doteq \lambda a.O \in (a{:}A) \to B\ [\Psi]$ and $M' \doteq \lambda a.O' \in (a{:}A) \to B\ [\Psi]$, and so by $\llbracket(a{:}A) \to B\rrbracket(\lambda a.O, \lambda a.O')$, $a : A \gg O \doteq O' \in B\ [\Psi]$. We conclude $O[N/a] \doteq O'[N'/a] \in B[N/a]\ [\Psi]$ and $B[N/a] \doteq B[N'/a]$ $\mathsf{type_{pre}}$ $[\Psi]$, and the result follows by symmetry, transitivity, and Lemma 33. □

Rule 4 (Eta)**.** *If $M \in (a{:}A) \to B\ [\Psi]$ then $M \doteq \lambda a.\mathsf{app}(M, a) \in (a{:}A) \to B\ [\Psi]$.*

Proof. By Lemma 38, $M \Downarrow \lambda a.O$ and $M \doteq \lambda a.O \in (a{:}A) \to B\ [\Psi]$; by transitivity and Rule 2 it suffices to show $a : A \gg O \doteq \mathsf{app}(M, a) \in B\ [\Psi]$, that is, for any $\psi : \Psi' \to \Psi$ and $N \doteq N' \in A\psi\ [\Psi']$, $O\psi[N/a] \doteq \mathsf{app}(M\psi, N') \in B\psi[N/a]\ [\Psi']$. By Lemma 46, $O_\psi[N'/a] \doteq \mathsf{app}(M\psi, N') \in B\psi[N'/a]\ [\Psi']$, where $M\psi \Downarrow \lambda a.O_\psi$. The result then follows by $B\psi[N/a] \doteq B\psi[N'/a]\ \mathsf{type_{pre}}\ [\Psi']$ and $a : A\psi \gg O_{\mathsf{id}_\Psi}\psi \doteq O_\psi \in B\psi\ [\Psi']$, the latter by $[\![(a{:}A) \to B]\!]_\psi(\lambda a.O\psi, \lambda a.O_\psi)$. $\square$

Rule 5 (Computation)**.** *If $a : A \gg M \in B\ [\Psi]$ and $N \in A\ [\Psi]$ then $\mathsf{app}(\lambda a.M, N) \doteq M[N/a] \in B[N/a]\ [\Psi]$.*

Proof. Immediate by $M[N/a] \in B[N/a]\ [\Psi]$, $\mathsf{app}(\lambda a.M, N) \longmapsto_{\square} M[N/a]$, and Lemma 43. $\square$

Rule 6 (Kan type formation)**.** *If $A \doteq A'\ \mathsf{type_{Kan}}\ [\Psi]$ and $a : A \gg B \doteq B'\ \mathsf{type_{Kan}}\ [\Psi]$ then $(a{:}A) \to B \doteq (a{:}A') \to B'\ \mathsf{type_{Kan}}\ [\Psi]$.*

Proof. By Rule 1, it suffices to check the five Kan conditions.

(hcom) First, suppose that $\psi : \Psi' \to \Psi$,

1. $\overrightarrow{\xi_i} = \overrightarrow{r_i = r_i'}$ is valid,
2. $M \doteq M' \in (a{:}A\psi) \to B\psi\ [\Psi']$,
3. $N_i \doteq N_j' \in (a{:}A\psi) \to B\psi\ [\Psi', y \mid r_i = r_i', r_j = r_j']$ for any i, j, and
4. $N_i\langle r/y\rangle \doteq M \in (a{:}A\psi) \to B\psi\ [\Psi' \mid r_i = r_i']$ for any i,

and show $\mathsf{hcom}^{r \rightsquigarrow r'}_{(a:A\psi)\to B\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r \rightsquigarrow r'}_{(a:A'\psi)\to B'\psi}(M'; \overrightarrow{\xi_i \hookrightarrow N_i'}) \in (a{:}A\psi) \to B\psi\ [\Psi']$. By Lemma 43 on both sides and Rule 2, it suffices to show

$$
\begin{aligned}
a : A\psi \gg\ & \mathsf{hcom}^{r \rightsquigarrow r'}_{B\psi}(\mathsf{app}(M, a); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{app}(N_i, a)}) \\
&\doteq \mathsf{hcom}^{r \rightsquigarrow r'}_{B'\psi}(\mathsf{app}(M', a); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{app}(N_i', a)}) \in B\psi\ [\Psi']
\end{aligned}
$$

or that for any $\psi' : \Psi'' \to \Psi'$ and $N \doteq N' \in A\psi\psi'\ [\Psi'']$,

$$
\begin{aligned}
& \mathsf{hcom}^{r\psi \rightsquigarrow r'\psi}_{B\psi\psi'[N/a]}(\mathsf{app}(M\psi', N); \overrightarrow{\xi_i\psi' \hookrightarrow y.\mathsf{app}(N_i\psi', N)}) \\
&\doteq \mathsf{hcom}^{r\psi \rightsquigarrow r'\psi}_{B'\psi\psi'[N'/a]}(\mathsf{app}(M'\psi', N'); \overrightarrow{\xi_i\psi' \hookrightarrow y.\mathsf{app}(N_i'\psi', N')}) \in B\psi\psi'[N/a]\ [\Psi''].
\end{aligned}
$$

By $a : A \gg B \doteq B'\ \mathsf{type_{Kan}}\ [\Psi]$ we know $B\psi\psi'[N/a] \doteq B'\psi\psi'[N'/a]\ \mathsf{type_{Kan}}\ [\Psi'']$, so the result follows by Definition 22 once we establish

1. $\overrightarrow{r_i\psi' = r_i'\psi'}$ is valid,
2. $\mathsf{app}(M\psi', N) \doteq \mathsf{app}(M'\psi', N') \in B\psi\psi'[N/a]\ [\Psi'']$,
3. $\mathsf{app}(N_i\psi', N) \doteq \mathsf{app}(N_j'\psi', N') \in B\psi\psi'[N/a]\ [\Psi'', y \mid r_i\psi' = r_i'\psi', r_j\psi' = r_j'\psi']$ for any i, j, and
4. $\mathsf{app}(N_i\langle r/y\rangle\psi', N) \doteq \mathsf{app}(M\psi', N') \in B\psi\psi'[N/a]\ [\Psi'' \mid r_i\psi' = r_i'\psi']$ for any i.

These follow from our hypotheses and a context-restricted variant of Rule 3, namely that if $M \doteq M' \in (a{:}A) \to B\ [\Psi \mid \Xi]$ and $N \doteq N' \in A\ [\Psi \mid \Xi]$ then $\mathsf{app}(M,N) \doteq \mathsf{app}(M',N') \in B[N/a]\ [\Psi \mid \Xi]$. (This statement is easily proven by expanding the definition of context-restricted judgments.)

Next, we must show that if $r = r'$ then $\mathsf{hcom}^{r \rightsquigarrow r'}_{(a:A\psi) \to B\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq M \in (a{:}A\psi) \to B\psi\ [\Psi']$. By Lemma 43 on the left and Rule 4 on the right, it suffices to show that

$$\lambda a.\mathsf{hcom}^{r \rightsquigarrow r'}_{B\psi}(\mathsf{app}(M,a); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{app}(N_i,a)}) \doteq \lambda a.\mathsf{app}(M,a) \in (a{:}A\psi) \to B\psi\ [\Psi'].$$

By Rule 2, we show that for any $\psi' : \Psi'' \to \Psi'$ and $N \doteq N' \in A\psi\psi'\ [\Psi'']$,

$$\mathsf{hcom}^{r\psi \rightsquigarrow r'\psi}_{B\psi\psi'[N/a]}(\mathsf{app}(M\psi',N); \overrightarrow{\xi_i\psi' \hookrightarrow y.\mathsf{app}(N_i\psi',N)}) \doteq \mathsf{app}(M\psi',N') \in B\psi\psi'[N/a]\ [\Psi'].$$

By $B\psi\psi'[N/a]\ \mathsf{type_{Kan}}\ [\Psi'']$ and $r = r'$ on the left, it suffices to show $\mathsf{app}(M\psi',N) \doteq \mathsf{app}(M\psi',N') \in B\psi\psi'[N/a]\ [\Psi'']$, which holds by Rule 3.

For the final hcom property, show that if $r_i = r_i'$ then $\mathsf{hcom}^{r \rightsquigarrow r'}_{(a:A\psi) \to B\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in (a{:}A\psi) \to B\psi\ [\Psi']$. As before, by Lemma 43 on the left, Rule 4 on the right, and Rule 2, show that for any $\psi' : \Psi'' \to \Psi'$ and $N \doteq N' \in A\psi\psi'\ [\Psi'']$,

$$\mathsf{hcom}^{r\psi \rightsquigarrow r'\psi}_{B\psi\psi'[N/a]}(\mathsf{app}(M\psi',N); \overrightarrow{\xi_i\psi' \hookrightarrow y.\mathsf{app}(N_i\psi',N)}) \doteq \mathsf{app}(N_i\langle r'/y\rangle\psi',N') \in B\psi\psi'[N/a]\ [\Psi'].$$

This follows by $B\psi\psi'[N/a]\ \mathsf{type_{Kan}}\ [\Psi'']$ and $r_i\psi' = r_i'\psi'$ on the left, and Rule 3.

(coe) Now, suppose that $\psi : (\Psi', x) \to \Psi$ and $M \doteq M' \in (a{:}A\psi\langle r/x\rangle) \to B\psi\langle r/x\rangle\ [\Psi']$, and show that $\mathsf{coe}^{r \rightsquigarrow r'}_{x.(a:A\psi) \to B\psi}(M) \doteq \mathsf{coe}^{r \rightsquigarrow r'}_{x.(a:A'\psi) \to B'\psi}(M') \in (a{:}A\psi\langle r'/x\rangle) \to B\psi\langle r'/x\rangle\ [\Psi']$. By Lemma 43 on both sides and Rule 2, we must show that for any $\psi' : \Psi'' \to \Psi'$ and $N \doteq N' \in A\psi\psi'\langle r'\psi'/x\rangle\ [\Psi'']$,

$$\begin{aligned}&\mathsf{coe}^{r\psi' \rightsquigarrow r'\psi'}_{x.B\psi\psi'[\mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A\psi\psi'}(N)/a]}(\mathsf{app}(M\psi', \mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A\psi\psi'}(N)))\\ \doteq\ &\mathsf{coe}^{r\psi' \rightsquigarrow r'\psi'}_{x.B'\psi\psi'[\mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A'\psi\psi'}(N')/a]}(\mathsf{app}(M'\psi', \mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A'\psi\psi'}(N'))) \in B\psi\psi'\langle r'\psi'/x\rangle[N/a]\ [\Psi''].\end{aligned}$$

By $A\psi\psi' \doteq A'\psi\psi'\ \mathsf{type_{Kan}}\ [\Psi'', x]$, we have $\mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A\psi\psi'}(N) \doteq \mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A'\psi\psi'}(N') \in A\psi\psi'\langle r'\psi'/x\rangle\ [\Psi'']$, and the corresponding instances of $B\psi\psi'$ and $B'\psi\psi'$ are equal as Kan types. By Rule 3 we have

$$\mathsf{app}(M\psi', \mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A\psi\psi'}(N)) \doteq \mathsf{app}(M'\psi', \mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A'\psi\psi'}(N')) \in B\psi\psi'\langle r\psi'/x\rangle[\mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A\psi\psi'}(N)/a]\ [\Psi'']$$

so the above coe are equal in $B\psi\psi'\langle r'\psi'/x\rangle[\mathsf{coe}^{r'\psi' \rightsquigarrow r'\psi'}_{x.A\psi\psi'}(N)/a]$. The result follows by Lemma 33 and $\mathsf{coe}^{r'\psi' \rightsquigarrow r'\psi'}_{x.A\psi\psi'}(N) \doteq N \in A\psi\psi'\langle r'\psi'/x\rangle\ [\Psi'']$.

Finally, show that if $r = r'$ then $\mathsf{coe}^{r \rightsquigarrow r'}_{x.(a:A\psi) \to B\psi}(M) \doteq M \in (a{:}A\psi\langle r'/x\rangle) \to B\psi\langle r'/x\rangle\ [\Psi']$. By Lemma 43 on the left, Rule 4 on the right, and Rule 2, it suffices to show that for any $\psi' : \Psi'' \to \Psi'$ and $N \doteq N' \in A\psi\psi'\langle r'\psi'/x\rangle\ [\Psi'']$,

$$\mathsf{coe}^{r\psi' \rightsquigarrow r'\psi'}_{x.B\psi\psi'[\mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A\psi\psi'}(N)/a]}(\mathsf{app}(M\psi', \mathsf{coe}^{r'\psi' \rightsquigarrow r\psi'}_{x.A\psi\psi'}(N))) \doteq \mathsf{app}(M\psi',N') \in B\psi\psi'\langle r'\psi'/x\rangle[N/a]\ [\Psi''].$$

By $r\psi' = r'\psi'$, $A\psi\psi'\ \mathsf{type_{Kan}}\ [\Psi'', x]$, Rule 3, and $B\psi\psi'[\mathsf{coe}^{r'\psi' \rightsquigarrow x}_{x.A\psi\psi'}(N)/a]\ \mathsf{type_{Kan}}\ [\Psi'', x]$, it suffices to show $\mathsf{app}(M\psi',N) \doteq \mathsf{app}(M\psi',N') \in B\psi\psi'\langle r'\psi'/x\rangle[N/a]\ [\Psi'']$, which again follows by Rule 3. □

5.2 Dependent pair types

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu,\sigma)$ for any cubical type systems ν,σ; in τ, whenever $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi]$, $a : A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$, and $\varphi = \{(\langle M,N\rangle,\langle M',N'\rangle) \mid M \doteq M' \in A\ [\Psi] \wedge N \doteq N' \in B[M/a]\ [\Psi]\}$, we have $\tau(\Psi,(a{:}A)\times B,(a{:}A')\times B',\varphi)$.

Rule 7 (Pretype formation)**.** *If* $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi]$ *and* $a : A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$ *then* $(a{:}A)\times B \doteq (a{:}A')\times B'$ $\mathsf{type_{pre}}$ $[\Psi]$*.*

Proof. We have $\mathsf{PTy}(\tau)(\Psi,(a{:}A)\times B,(a{:}A')\times B',\_)$ because $(a{:}A)\times B$ $\mathsf{val}_{\square}$ and judgments are preserved by dimension substitution. For $\mathsf{Coh}(\llbracket(a{:}A)\times B\rrbracket)$, assume $\llbracket(a{:}A)\times B\rrbracket_{\psi}(\langle M,N\rangle,\langle M',N'\rangle)$. Then $M \doteq M' \in A\psi\ [\Psi']$ and $N \doteq N' \in B\psi[M/a]\ [\Psi']$; again, $\langle M,N\rangle$ $\mathsf{val}_{\square}$ and these judgments are preserved by dimension substitution, so $\mathsf{Tm}(\llbracket(a{:}A)\times B\rrbracket\psi)(\langle M,N\rangle,\langle M',N'\rangle)$. □

Rule 8 (Introduction)**.** *If* $M \doteq M' \in A\ [\Psi]$ *and* $N \doteq N' \in B[M/a]\ [\Psi]$ *then* $\langle M,N\rangle \doteq \langle M',N'\rangle \in (a{:}A)\times B\ [\Psi]$*.*

Proof. Immediate by Rule 7. □

Rule 9 (Elimination)**.** *If* $P \doteq P' \in (a{:}A)\times B\ [\Psi]$ *then* $\mathsf{fst}(P) \doteq \mathsf{fst}(P') \in A\ [\Psi]$ *and* $\mathsf{snd}(P) \doteq \mathsf{snd}(P') \in B[\mathsf{fst}(P)/a]\ [\Psi]$*.*

Proof. For any $\psi : \Psi' \to \Psi$, $P\psi \Downarrow \langle M_\psi,N_\psi\rangle$, $M_\psi \in A\psi\ [\Psi']$, and $N_\psi \in B\psi[M_\psi/a]\ [\Psi']$. For part (1), apply coherent expansion to $\mathsf{fst}(P)$ with family $\{M_\psi\}_{\psi}^{\Psi'}$; then $(M_{\mathsf{id}_\Psi})\psi \doteq M_\psi \in A\psi\ [\Psi']$ by $P \in (a{:}A)\times B\ [\Psi]$ at id_Ψ,ψ. By Lemma 41, $\mathsf{fst}(P) \doteq M_{\mathsf{id}_\Psi} \in A\ [\Psi]$, and part (1) follows by $M_{\mathsf{id}_\Psi} \doteq M'_{\mathsf{id}_\Psi} \in A\ [\Psi]$ and a symmetric argument on the right side.

For part (2), apply coherent expansion to $\mathsf{snd}(P)$ with family $\{N_\psi\}_{\psi}^{\Psi'}$. We have $(N_{\mathsf{id}_\Psi})\psi \doteq N_\psi \in B\psi[(M_{\mathsf{id}_\Psi})\psi/a]\ [\Psi']$ by $P \in (a{:}A)\times B\ [\Psi]$ at id_Ψ,ψ, so by Lemma 41, $\mathsf{snd}(P) \doteq N_{\mathsf{id}_\Psi} \in B[M_{\mathsf{id}_\Psi}/a]\ [\Psi]$. Part (2) follows by $B[M_{\mathsf{id}_\Psi}/a] \doteq B[\mathsf{fst}(P)/a]$ $\mathsf{type_{pre}}$ $[\Psi]$ (by $a : A \gg B \doteq B'$ $\mathsf{type_{pre}}$ $[\Psi]$ and $M_{\mathsf{id}_\Psi} \doteq \mathsf{fst}(P) \in A\ [\Psi]$), $N_{\mathsf{id}_\Psi} \doteq N'_{\mathsf{id}_\Psi} \in B[M_{\mathsf{id}_\Psi}/a]\ [\Psi]$, and a symmetric argument on the right side. □

Rule 10 (Computation)**.** *If* $M \in A\ [\Psi]$ *then* $\mathsf{fst}(\langle M,N\rangle) \doteq M \in A\ [\Psi]$*. If* $N \in B\ [\Psi]$ *then* $\mathsf{snd}(\langle M,N\rangle) \doteq N \in B\ [\Psi]$*.*

Proof. Immediate by Lemma 43. □

Rule 11 (Eta)**.** *If* $P \in (a{:}A)\times B\ [\Psi]$ *then* $P \doteq \langle\mathsf{fst}(P),\mathsf{snd}(P)\rangle \in (a{:}A)\times B\ [\Psi]$*.*

Proof. By Lemma 38, $P \Downarrow \langle M,N\rangle$, $P \doteq \langle M,N\rangle \in (a{:}A)\times B\ [\Psi]$, $M \in A\ [\Psi]$, and $N \in B[M/a]\ [\Psi]$. By Rule 8 and Lemma 37 and transitivity, we show $\llbracket A\rrbracket^{\Downarrow}(M,\mathsf{fst}(P))$ and $\llbracket B[M/a]\rrbracket^{\Downarrow}(N,\mathsf{snd}(P))$. This is immediate by $\mathsf{fst}(P) \longmapsto^* \mathsf{fst}(\langle M,N\rangle) \longmapsto M$ and $\mathsf{snd}(P) \longmapsto^* \mathsf{snd}(\langle M,N\rangle) \longmapsto N$. □

Rule 12 (Kan type formation)**.** *If* $A \doteq A'$ $\mathsf{type_{Kan}}$ $[\Psi]$ *and* $a : A \gg B \doteq B'$ $\mathsf{type_{Kan}}$ $[\Psi]$ *then* $(a{:}A)\times B \doteq (a{:}A')\times B'$ $\mathsf{type_{Kan}}$ $[\Psi]$*.*

Proof. It suffices to check the five Kan conditions.

(hcom) First, suppose that $\psi : \Psi' \to \Psi$,

1. $\overrightarrow{r_i = r_i'}$ is valid,

2. $M \doteq M' \in (a{:}A\psi) \times B\psi\ [\Psi']$,

3. $N_i \doteq N'_j \in (a{:}A\psi) \times B\psi\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and

4. $N_i\langle r/y\rangle \doteq M \in (a{:}A\psi) \times B\psi\ [\Psi' \mid r_i = r'_i]$ for any i,

and show $\mathsf{hcom}^{r\rightsquigarrow r'}_{(a:A\psi)\times B\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{(a:A'\psi)\times B'\psi}(M'; \overrightarrow{\xi_i \hookrightarrow y.N'_i}) \in (a{:}A\psi) \times B\psi\ [\Psi']$. By Lemma 43 on both sides and Rule 8, it suffices to show (the binary version of)

$$\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(\mathsf{fst}(M); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)}) \in A\psi\ [\Psi']$$
$$\mathsf{com}^{r\rightsquigarrow r'}_{z.B\psi[F/a]}(\mathsf{snd}(M); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{snd}(N_i)}) \in B\psi[\mathsf{hcom}_{A\psi}/a]\ [\Psi']$$
$$\text{where } F = \mathsf{hcom}^{r\rightsquigarrow z}_{A\psi}(\mathsf{fst}(M); \overrightarrow{\xi_i \hookrightarrow y.\mathsf{fst}(N_i)}).$$

We have $\mathsf{hcom}_{A\psi} \in A\psi\ [\Psi']$ and $F \in A\psi\ [\Psi', z]$ by $A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$ and Rule 9. We show $\mathsf{com}_{z.B\psi[F/a]} \in B\psi[\mathsf{hcom}_{A\psi}/a]\ [\Psi']$ by Theorem 44, observing that $B\psi[F/a]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', z]$, $F\langle r'/z\rangle = \mathsf{hcom}_{A\psi}$,

1. $\mathsf{snd}(M) \in B\psi[F\langle r/z\rangle/a]\ [\Psi']$ by $F\langle r/z\rangle \doteq \mathsf{fst}(M) \in A\psi\ [\Psi']$ and Rule 9,

2. $N_i \doteq N_j \in B\psi[F\langle y/z\rangle/a]\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ by $F\langle y/z\rangle \doteq \mathsf{fst}(N_i) \in A\psi\ [\Psi', y \mid r_i = r'_i]$ and Rule 9, and

3. $\mathsf{snd}(N_i\langle r/y\rangle) \doteq \mathsf{snd}(M) \in B\psi[F\langle r/z\rangle/a]\ [\Psi' \mid r_i = r'_i]$ by $F\langle r/z\rangle \doteq \mathsf{fst}(M) \in A\psi\ [\Psi']$ and Rule 9.

Next, we must show that if $r = r'$ then $\mathsf{hcom}_{(a:A\psi)\times B\psi} \doteq M \in (a{:}A\psi) \times B\psi\ [\Psi']$. By Lemma 43, $\mathsf{hcom}_{(a:A\psi)\times B\psi} \doteq \langle \mathsf{hcom}_{A\psi}, \mathsf{com}_{z.B\psi[F/a]}\rangle \in (a{:}A\psi) \times B\psi\ [\Psi']$. By Definition 22 and Theorem 44, $\mathsf{hcom}_{A\psi} \doteq \mathsf{fst}(M) \in A\psi\ [\Psi']$, $\mathsf{com}_{z.B\psi[F/a]} \doteq \mathsf{snd}(M) \in B\psi[F\langle r/z\rangle/a]\ [\Psi']$, and $B\psi[F\langle r/z\rangle/a] \doteq B\psi[\mathsf{fst}(M)/a]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$. The result follows by Rule 11.

For the final hcom property, show that if $r_i = r'_i$ then $\mathsf{hcom}_{(a:A\psi)\times B\psi} \doteq N_i\langle r'/y\rangle \in (a{:}A\psi) \times B\psi\ [\Psi']$. The result follows by $\mathsf{hcom}_{A\psi} \doteq \mathsf{fst}(N_i\langle r'/y\rangle) \in A\psi\ [\Psi']$, $\mathsf{com}_{z.B\psi[F/a]} \doteq \mathsf{snd}(N_i\langle r'/y\rangle) \in B\psi[F\langle r'/z\rangle/a]\ [\Psi']$, and $B\psi[F\langle r'/z\rangle/a] \doteq B\psi[\mathsf{fst}(N_i\langle r'/y\rangle)/a]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$.

(coe) Now, suppose that $\psi : (\Psi', x) \to \Psi$ and $M \doteq M' \in ((a{:}A\psi) \times B\psi)\langle r/x\rangle\ [\Psi']$, and show $\mathsf{coe}^{r\rightsquigarrow r'}_{x.(a:A\psi)\times B\psi}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{x.(a:A'\psi)\times B'\psi}(M') \in ((a{:}A\psi) \times B\psi)\langle r'/x\rangle\ [\Psi']$. By Lemma 43 and Rule 8, it suffices to show (the binary version of)

$$\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(\mathsf{fst}(M)) \in A\psi\langle r'/x\rangle\ [\Psi']$$
$$\mathsf{coe}^{r\rightsquigarrow r'}_{x.B\psi[\mathsf{coe}^{r\rightsquigarrow x}_{x.A\psi}(\mathsf{fst}(M))/a]}(\mathsf{snd}(M)) \in B\psi\langle r'/x\rangle[\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(\mathsf{fst}(M))/a]\ [\Psi']$$

We know that $\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(\mathsf{fst}(M)) \in A\psi\langle r'/x\rangle\ [\Psi']$ and $B\psi[\mathsf{coe}^{r\rightsquigarrow x}_{x.A\psi}(\mathsf{fst}(M))/a]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', x]$ by $A\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', x]$, $a : A\psi \gg B\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', x]$, and Rule 9. We also know that $\mathsf{snd}(M) \in B\psi\langle r/x\rangle[\mathsf{fst}(M)/a]\ [\Psi']$ and $(\mathsf{coe}^{r\rightsquigarrow x}_{x.A\psi}(\mathsf{fst}(M)))\langle r/x\rangle \doteq \mathsf{fst}(M) \in A\langle r/x\rangle\ [\Psi']$, so $\mathsf{coe}_{x.B\psi[\dots/a]} \in B\psi\langle r'/x\rangle[\mathsf{coe}^{r\rightsquigarrow r'}_{x.A\psi}(\mathsf{fst}(M))/a]\ [\Psi']$ and the result follows.

Finally, show that if $r = r'$ then $\mathsf{coe}^{r\rightsquigarrow r}_{x.(a:A\psi)\times B\psi}(M) \doteq M \in ((a{:}A\psi)\times B\psi)\langle r/x\rangle\ [\Psi']$. By Lemma 43 and Rules 8 and 11, this follows from $\mathsf{coe}_{x.A\psi} \doteq \mathsf{fst}(M) \in A\psi\langle r/x\rangle\ [\Psi']$ and $\mathsf{coe}_{x.B\psi[\dots/a]} \doteq \mathsf{snd}(M) \in B\psi\langle r/x\rangle[\mathsf{fst}(M)/a]\ [\Psi']$. □

5.3 Path types

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu,\sigma)$ for any cubical type systems ν,σ; in τ, whenever $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi,x]$, $P_\varepsilon \doteq P'_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$ for $\varepsilon \in \{0,1\}$, and $\varphi = \{(\langle x\rangle M, \langle x\rangle M') \mid M \doteq M' \in A\ [\Psi,x] \wedge \forall\varepsilon.(M\langle\varepsilon/x\rangle \doteq P_\varepsilon \in A\langle\varepsilon/x\rangle\ [\Psi])\}$, we have $\tau(\Psi, \mathsf{Path}_{x.A}(P_0,P_1), \mathsf{Path}_{x.A'}(P'_0,P'_1), \varphi)$.

Rule 13 (Pretype formation)**.** *If* $A \doteq A'$ $\mathsf{type_{pre}}$ $[\Psi,x]$ *and* $P_\varepsilon \doteq P'_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$ *for* $\varepsilon \in \{0,1\}$*, then* $\mathsf{Path}_{x.A}(P_0,P_1) \doteq \mathsf{Path}_{x.A'}(P'_0,P'_1)$ $\mathsf{type_{pre}}$ $[\Psi]$*.*

Proof. We have $\mathsf{PTy}(\tau)(\Psi, \mathsf{Path}_{x.A}(P_0,P_1), \mathsf{Path}_{x.A'}(P'_0,P'_1), \_)$ because $\mathsf{Path}_{x.A}(P_0,P_1)$ $\mathsf{val}_{\square}$ and judgments are preserved by dimension substitution. To show $\mathsf{Coh}(\llbracket\mathsf{Path}_{x.A}(P_0,P_1)\rrbracket)$, suppose that $\llbracket\mathsf{Path}_{x.A}(P_0,P_1)\rrbracket(\langle x\rangle M, \langle x\rangle M')$. Then $M \doteq M' \in A$ $[\Psi,x]$ and $M\langle\varepsilon/x\rangle \doteq P_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$, so $M\psi \doteq M'\psi \in A\psi$ $[\Psi',x]$ and $M\psi\langle\varepsilon/x\rangle \doteq P_\varepsilon\psi \in A\psi\langle\varepsilon/x\rangle$ $[\Psi']$ for any $\psi : \Psi' \to \Psi$, so by $\langle x\rangle M$ $\mathsf{val}_{\square}$, $\mathsf{Tm}(\llbracket\mathsf{Path}_{x.A}(P_0,P_1)\rrbracket\psi)(\langle x\rangle M, \langle x\rangle M')$. □

Rule 14 (Introduction)**.** *If* $M \doteq M' \in A$ $[\Psi,x]$ *and* $M\langle\varepsilon/x\rangle \doteq P_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$ *for* $\varepsilon \in \{0,1\}$*, then* $\langle x\rangle M \doteq \langle x\rangle M' \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$*.*

Proof. Then $\llbracket\mathsf{Path}_{x.A}(P_0,P_1)\rrbracket(\langle x\rangle M, \langle x\rangle M')$, so the result follows by $\mathsf{Coh}(\llbracket\mathsf{Path}_{x.A}(P_0,P_1)\rrbracket)$. □

Rule 15 (Elimination)**.**

1. *If* $M \doteq M' \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$ *then* $M@r \doteq M'@r \in A\langle r/x\rangle$ $[\Psi]$*.*
2. *If* $M \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$ *then* $M@\varepsilon \doteq P_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$*.*

Proof. Apply coherent expansion to $M@r$ with family $\{M_\psi\langle r\psi/x\rangle \mid M\psi \Downarrow \langle x\rangle M_\psi\}^{\Psi'}_{\psi}$. By $M \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$ at id_Ψ,ψ we know $(M_{\mathsf{id}_\Psi})\psi \doteq M_\psi \in A\psi$ $[\Psi',x]$, so $(M_{\mathsf{id}_\Psi})\psi\langle r\psi/x\rangle \doteq M_\psi\langle r\psi/x\rangle \in A\langle r/x\rangle\psi$ $[\Psi']$. Thus by Lemma 41, $M@r \doteq M_{\mathsf{id}_\Psi}\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi]$; part (1) follows by the same argument on the right side and $M_{\mathsf{id}_\Psi} \doteq M'_{\mathsf{id}_\Psi} \in A$ $[\Psi,x]$. Part (2) follows from $M@\varepsilon \doteq M_{\mathsf{id}_\Psi}\langle\varepsilon/x\rangle \in A\langle\varepsilon/x\rangle$ $[\Psi]$ and $M_{\mathsf{id}_\Psi}\langle\varepsilon/x\rangle \doteq P_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$. □

Rule 16 (Computation)**.** *If* $M \in A$ $[\Psi,x]$ *then* $(\langle x\rangle M)@r \doteq M\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi]$*.*

Proof. Immediate by $(\langle x\rangle M)@r \longmapsto_{\square} M\langle r/x\rangle$, $M\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi]$, and Lemma 43. □

Rule 17 (Eta)**.** *If* $M \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$ *then* $M \doteq \langle x\rangle(M@x) \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$*.*

Proof. By Lemma 38, $M \Downarrow \langle x\rangle N$ and $M \doteq \langle x\rangle N \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$. By Rule 15, $M@x \doteq (\langle x\rangle N)@x \in A$ $[\Psi,x]$, so by Lemma 43 on the right, $M@x \doteq N \in A$ $[\Psi,x]$. By Rule 14, $\langle x\rangle(M@x) \doteq \langle x\rangle N \in \mathsf{Path}_{x.A}(P_0,P_1)$ $[\Psi]$, and the result follows by transitivity. □

Rule 18 (Kan type formation)**.** *If* $A \doteq A'$ $\mathsf{type_{Kan}}$ $[\Psi,x]$ *and* $P_\varepsilon \doteq P'_\varepsilon \in A\langle\varepsilon/x\rangle$ $[\Psi]$ *for* $\varepsilon \in \{0,1\}$*, then* $\mathsf{Path}_{x.A}(P_0,P_1) \doteq \mathsf{Path}_{x.A'}(P'_0,P'_1)$ $\mathsf{type_{Kan}}$ $[\Psi]$*.*

Proof. It suffices to check the five Kan conditions.

(hcom) First, suppose that $\psi : \Psi' \to \Psi$,

1. $\overrightarrow{\xi_i} = \overrightarrow{r_i = r'_i}$ is valid,
2. $M \doteq M' \in \mathsf{Path}_{x.A\psi}(P_0\psi, P_1\psi)$ $[\Psi']$,

3. $N_i \doteq N'_j \in \mathsf{Path}_{x.A\psi}(P_0\psi, P_1\psi)\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and

4. $N_i\langle r/y\rangle \doteq M \in \mathsf{Path}_{x.A\psi}(P_0\psi, P_1\psi)\ [\Psi' \mid r_i = r'_i]$ for any i,

and show the equality $\mathsf{hcom}^{r\rightsquigarrow r'}_{(\mathsf{Path}_{x.A}(P_0,P_1))\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{(\mathsf{Path}_{x.A'}(P'_0,P'_1))\psi}(M'; \overrightarrow{\xi_i \hookrightarrow N'_i}) \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\ [\Psi']$. By Lemma 43 and Rule 14 on both sides it suffices to show

$$\begin{aligned}&\mathsf{hcom}^{r\rightsquigarrow r'}_{A\psi}(M@x; \overrightarrow{x = \varepsilon \hookrightarrow \_.P_\varepsilon\psi}, \overrightarrow{\xi_i \hookrightarrow y.N_i@x})\\ \doteq\ &\mathsf{hcom}^{r\rightsquigarrow r'}_{A'\psi}(M'@x; \overrightarrow{x = \varepsilon \hookrightarrow \_.P'_\varepsilon\psi}, \overrightarrow{\xi_i \hookrightarrow y.N'_i@x}) \in A\psi\ [\Psi', x]\end{aligned}$$

and $(\mathsf{hcom}_{A\psi})\langle\varepsilon/x\rangle \doteq P_\varepsilon\psi \in A\psi\langle\varepsilon/x\rangle\ [\Psi']$. By our hypotheses and Rule 15,

1. $M@x \doteq M'@x \in A\psi\ [\Psi', x]$,

2. $P_\varepsilon\psi \doteq P'_\varepsilon\psi \in A\psi\ [\Psi', x \mid x = \varepsilon]$ and $P_\varepsilon\psi \doteq M@x \in A\psi\ [\Psi', x \mid x = \varepsilon]$,

3. $N_i@x \doteq N'_j@x \in A\psi\ [\Psi', x, y \mid r_i = r'_i, r_j = r'_j]$, $N_i@x \doteq P'_\varepsilon\psi \in A\psi\ [\Psi', x, y \mid r_i = r'_i, x = \varepsilon]$, and $N_i\langle r/y\rangle@x \doteq M@x \in A\psi\ [\Psi', x \mid r_i = r'_i]$,

and so by Definition 22, $\mathsf{hcom}_{A\psi} \doteq \mathsf{hcom}_{A'\psi} \in A\psi\ [\Psi', x]$ and $(\mathsf{hcom}_{A\psi})\langle\varepsilon/x\rangle \doteq P_\varepsilon\psi \in A\psi\ [\Psi]$.

Next, show if $r = r'$ then $\mathsf{hcom}^{r\rightsquigarrow r'}_{(\mathsf{Path}_{x.A}(P_0,P_1))\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq M \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\ [\Psi']$. By Rule 14 and Definition 22 the left side equals $\langle x\rangle(M@x)$, and Rule 17 completes this part.

Finally, if $r_i = r'_i$ then $\mathsf{hcom}^{r\rightsquigarrow r'}_{(\mathsf{Path}_{x.A}(P_0,P_1))\psi}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\ [\Psi']$. By Rule 14 and Definition 22 the left side equals $\langle x\rangle(N_i\langle r'/y\rangle@x)$, and Rule 17 completes this part.

(coe) Now, suppose that $\psi : (\Psi', y) \to \Psi$ and $M \doteq M' \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\langle r/y\rangle\ [\Psi']$, and show that $\mathsf{coe}^{r\rightsquigarrow r'}_{y.(\mathsf{Path}_{x.A}(P_0,P_1))\psi}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{y.(\mathsf{Path}_{x.A'}(P'_0,P'_1))\psi}(M') \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\langle r'/y\rangle\ [\Psi']$. By Lemma 43 on both sides and Rule 14, we show

$$\mathsf{com}^{r\rightsquigarrow r'}_{y.A\psi}(M@x; \overrightarrow{x = \varepsilon \hookrightarrow y.P_\varepsilon\psi}) \doteq \mathsf{com}^{r\rightsquigarrow r'}_{y.A'\psi}(M'@x; \overrightarrow{x = \varepsilon \hookrightarrow y.P'_\varepsilon\psi}) \in A\psi\langle r'/y\rangle\ [\Psi', x]$$

and $(\mathsf{com}_{y.A\psi})\langle\varepsilon/x\rangle \doteq P_\varepsilon\psi\langle r'/y\rangle \in A\psi\langle r'/y\rangle\langle\varepsilon/x\rangle\ [\Psi']$. By our hypotheses and Rule 15, $M@x \doteq M'@x \in A\psi\langle r/y\rangle\ [\Psi', x]$, $P_\varepsilon\psi \doteq P'_\varepsilon\psi \in A\psi\ [\Psi', x, y \mid x = \varepsilon]$, and $P_\varepsilon\psi\langle r/y\rangle \doteq M@x \in A\psi\langle r/y\rangle\ [\Psi', x \mid x = \varepsilon]$, so by Theorem 44, $\mathsf{com}_{y.A\psi} \doteq \mathsf{com}_{y.A'\psi} \in A\psi\langle r'/y\rangle\ [\Psi', x]$ and $(\mathsf{com}_{y.A\psi})\langle\varepsilon/x\rangle \doteq P_\varepsilon\psi\langle r'/y\rangle \in A\psi\langle r'/y\rangle\langle\varepsilon/x\rangle\ [\Psi']$.

Finally, show that if $r = r'$ then $\mathsf{coe}^{r\rightsquigarrow r'}_{y.(\mathsf{Path}_{x.A}(P_0,P_1))\psi}(M) \doteq M \in (\mathsf{Path}_{x.A}(P_0, P_1))\psi\langle r'/y\rangle\ [\Psi']$. By Rule 14 and Theorem 44 the left side equals $\langle x\rangle(M@x)$, and Rule 17 completes the proof. $\square$

5.4 Equality pretypes

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu, \sigma)$ for any cubical type systems ν, σ; in τ, whenever $A \doteq A'\ \mathsf{type_{pre}}\ [\Psi]$, $M \doteq M' \in A\ [\Psi]$, $N \doteq N' \in A\ [\Psi]$, and $\varphi = \{(\star, \star) \mid M \doteq N \in A\ [\Psi]\}$, $\tau(\Psi, \mathsf{Eq}_A(M, N), \mathsf{Eq}_{A'}(M', N'), \varphi)$.

Rule 19 (Pretype formation)**.** *If* $A \doteq A'\ \mathsf{type_{pre}}\ [\Psi]$*,* $M \doteq M' \in A\ [\Psi]$*, and* $N \doteq N' \in A\ [\Psi]$*, then* $\mathsf{Eq}_A(M, N) \doteq \mathsf{Eq}_{A'}(M', N')\ \mathsf{type_{pre}}\ [\Psi]$.

Proof. We have $\mathsf{PTy}(\tau)(\Psi, \mathsf{Eq}_A(M, N), \mathsf{Eq}_{A'}(M', N'), [\![\mathsf{Eq}_A(M, N)]\!])$ because $\mathsf{Eq}_A(M, N)\ \mathsf{val}_{\square}$ and judgments are preserved by dimension substitution. To show $\mathsf{Coh}([\![\mathsf{Eq}_A(M, N)]\!])$, suppose that $[\![\mathsf{Eq}_A(M, N)]\!]_\psi(\star, \star)$. Then $M \doteq N \in A\ [\Psi]$, so $M\psi \doteq N\psi \in A\psi\ [\Psi']$ for all $\psi : \Psi' \to \Psi$, so $\mathsf{Tm}([\![\mathsf{Eq}_A(M, N)]\!]\psi)(\star, \star)$ holds by this and $\star\ \mathsf{val}_{\square}$. $\square$

Rule 20 (Introduction). *If* $M \doteq N \in A$ $[\Psi]$ *then* $\star \in \mathsf{Eq}_A(M,N)$ $[\Psi]$.

Proof. Then $[\![\mathsf{Eq}_A(M,N)]\!](\star,\star)$, so the result follows by $\mathsf{Coh}([\![\mathsf{Eq}_A(M,N)]\!])$. $\square$

Rule 21 (Elimination). *If* $E \in \mathsf{Eq}_A(M,N)$ $[\Psi]$ *then* $M \doteq N \in A$ $[\Psi]$.

Proof. Then $[\![\mathsf{Eq}_A(M,N)]\!]^{\Downarrow}(E,E)$ so $E \Downarrow \star$ and $M \doteq N \in A$ $[\Psi]$. $\square$

Rule 22 (Eta). *If* $E \in \mathsf{Eq}_A(M,N)$ $[\Psi]$ *then* $E \doteq \star \in \mathsf{Eq}_A(M,N)$ $[\Psi]$.

Proof. Immediate by Lemma 38. $\square$

5.5 Void

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu,\sigma)$ for any cubical type systems ν,σ; we have $\tau(\Psi,\mathsf{void},\mathsf{void},\varphi)$ for φ the empty relation. By $\mathsf{void}\ \mathsf{val}_{\square}$, $\mathsf{PTy}(\tau)(\Psi,\mathsf{void},\mathsf{void},\alpha)$ where each $\alpha_{\Psi'}$ is empty.

Rule 23 (Pretype formation). $\mathsf{void}\ \mathsf{type_{pre}}$ $[\Psi]$.

Proof. We have already observed $\mathsf{PTy}(\tau)(\Psi,\mathsf{void},\mathsf{void},[\![\mathsf{void}]\!])$; $\mathsf{Coh}([\![\mathsf{void}]\!])$ trivially because each $[\![\mathsf{void}]\!]_{\Psi'}$ is empty. $\square$

Rule 24 (Elimination). *It is never the case that* $M \in \mathsf{void}$ $[\Psi]$.

Proof. If $\mathsf{Tm}([\![\mathsf{void}]\!])(M,M)$ then $[\![\mathsf{void}]\!]^{\Downarrow}_{\Psi}(M,M)$, but $[\![\mathsf{void}]\!]^{\Downarrow}_{\Psi}$ is empty. $\square$

If $\Gamma \gg M \in \mathsf{void}$ $[\Psi]$ then it must be impossible to produce elements of each pretype in Γ, in which case every (non-context-restricted) judgment holds under Γ. In Section 6, we say that if $M \in \mathsf{void}$ $[\Psi]$ then $\mathcal{J}$ $[\Psi]$.

Rule 25 (Kan type formation). $\mathsf{void}\ \mathsf{type_{Kan}}$ $[\Psi]$.

Proof. It suffices to check the five Kan conditions. In each condition, we suppose that $M \doteq M' \in \mathsf{void}$ $[\Psi']$, so by Rule 24 they vacuously hold. $\square$

5.6 Booleans

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu,\sigma)$ for any cubical type systems ν,σ; we have $\tau(\Psi,\mathsf{bool},\mathsf{bool},\varphi)$ for $\varphi = \{(\mathsf{true},\mathsf{true}),(\mathsf{false},\mathsf{false})\}$. By $\mathsf{bool}\ \mathsf{val}_{\square}$, $\mathsf{PTy}(\tau)(\Psi,\mathsf{bool},\mathsf{bool},\alpha)$ where each $\alpha_{\Psi'} = \varphi$.

Rule 26 (Pretype formation). $\mathsf{bool}\ \mathsf{type_{pre}}$ $[\Psi]$.

Proof. We have already observed $\mathsf{PTy}(\tau)(\Psi,\mathsf{bool},\mathsf{bool},[\![\mathsf{bool}]\!])$; for $\mathsf{Coh}([\![\mathsf{bool}]\!])$ we must show that $\mathsf{Tm}([\![\mathsf{bool}]\!])(\mathsf{true},\mathsf{true})$ and $\mathsf{Tm}([\![\mathsf{bool}]\!])(\mathsf{false},\mathsf{false})$. These hold by $\mathsf{true}\ \mathsf{val}_{\square}$, $[\![\mathsf{bool}]\!]_{\Psi'}(\mathsf{true},\mathsf{true})$, $\mathsf{false}\ \mathsf{val}_{\square}$, and $[\![\mathsf{bool}]\!]_{\Psi'}(\mathsf{false},\mathsf{false})$. $\square$

Rule 27 (Introduction). $\mathsf{true} \in \mathsf{bool}$ $[\Psi]$ *and* $\mathsf{false} \in \mathsf{bool}$ $[\Psi]$.

Proof. Immediate by $\mathsf{Coh}([\![\mathsf{bool}]\!])$. $\square$

Rule 28 (Computation). *If* $T \in B$ $[\Psi]$ *then* $\mathsf{if}_{b.A}(\mathsf{true};T,F) \doteq T \in B$ $[\Psi]$. *If* $F \in B$ $[\Psi]$ *then* $\mathsf{if}_{b.A}(\mathsf{false};T,F) \doteq F \in B$ $[\Psi]$.

Proof. Immediate by $\mathsf{if}_{b.A}(\mathsf{true};T,F) \longmapsto_{\square} T$, $\mathsf{if}_{b.A}(\mathsf{false};T,F) \longmapsto_{\square} F$, and Lemma 43. $\square$

Rule 29 (Elimination)**.** *If* $M \doteq M' \in \mathsf{bool}\ [\Psi]$*,* $b : \mathsf{bool} \gg C\ \mathsf{type_{pre}}\ [\Psi]$*,* $T \doteq T' \in C[\mathsf{true}/b]\ [\Psi]$*, and* $F \doteq F' \in C[\mathsf{false}/b]\ [\Psi]$*, then* $\mathsf{if}_{b.A}(M;T,F) \doteq \mathsf{if}_{b.A'}(M';T',F') \in C[M/b]\ [\Psi]$*.*

Proof. Apply coherent expansion to the left side with $\{\mathsf{if}_{b.A\psi}(M_\psi;T\psi,F\psi) \mid M\psi \Downarrow M_\psi\}^{\Psi'}_{\psi}$. We must show $\mathsf{if}_{b.A\psi}(M_\psi;T\psi,F\psi) \doteq \mathsf{if}_{b.A\psi}((M_{\mathsf{id}_\Psi})\psi;T\psi,F\psi) \in C\psi[M\psi/b]\ [\Psi']$. Either $M_\psi = \mathsf{true}$ or $M_\psi = \mathsf{false}$. In either case $M_{\mathsf{id}_\Psi} = M_\psi$ because $[\![\mathsf{bool}]\!]^{\Downarrow}_{\Psi'}((M_{\mathsf{id}_\Psi})\psi, M_\psi)$ and $M_{\mathsf{id}_\Psi} = \mathsf{true}$ or $M_{\mathsf{id}_\Psi} = \mathsf{false}$. Consider the case $M_\psi = \mathsf{true}$: we must show $\mathsf{if}_{b.A\psi}(\mathsf{true};T\psi,F\psi) \in C\psi[M\psi/b]\ [\Psi']$. By Lemma 38 we have $M\psi \doteq \mathsf{true} \in \mathsf{bool}\ [\Psi']$ so $C\psi[M\psi/b] \doteq C\psi[\mathsf{true}/b]\ \mathsf{type_{pre}}\ [\Psi']$. The result follows by Rule 28 (with $B = C\psi[\mathsf{true}/b]$). The $M_\psi = \mathsf{false}$ case is symmetric.

We conclude by Lemma 41 that $\mathsf{if}_{b.A}(M;T,F) \doteq \mathsf{if}_{b.A}(M_{\mathsf{id}_\Psi};T,F) \in C[M/b]\ [\Psi]$. By transitivity, Lemma 38, and the same argument on the right, it suffices to show $\mathsf{if}_{b.A}(M_{\mathsf{id}_\Psi};T,F) \doteq \mathsf{if}_{b.A'}(M'_{\mathsf{id}_\Psi};T',F') \in C[M_{\mathsf{id}_\Psi}/b]\ [\Psi]$. By $M \doteq M' \in \mathsf{bool}\ [\Psi]$, either $M_{\mathsf{id}_\Psi} = M'_{\mathsf{id}_\Psi} = \mathsf{true}$ or $M_{\mathsf{id}_\Psi} = M'_{\mathsf{id}_\Psi} = \mathsf{false}$, and in either case the result follows by Rule 28 on both sides. $\square$

Notice that Rule 29 places no restrictions on the motives $b.A$ and $b.A'$; these motives are only relevant in the elimination rule for wbool.

Lemma 47. *If* $M \in \mathsf{bool}\ [\Psi, y]$ *then* $M\langle r/y\rangle \doteq M\langle r'/y\rangle \in \mathsf{bool}\ [\Psi]$*.*

Proof. By $[\![\mathsf{bool}]\!]^{\Downarrow}_{(\Psi,y)}(M,M)$ we know $M \Downarrow \mathsf{true}$ or $M \Downarrow \mathsf{false}$, so by Lemma 38 either $M \doteq \mathsf{true} \in \mathsf{bool}\ [\Psi, y]$ or $M \doteq \mathsf{false} \in \mathsf{bool}\ [\Psi, y]$. In the former case, both $M\langle r/y\rangle \doteq \mathsf{true} \in \mathsf{bool}\ [\Psi]$ and $M\langle r'/y\rangle \doteq \mathsf{true} \in \mathsf{bool}\ [\Psi]$, and similarly in the latter case. $\square$

Rule 30 (Kan type formation)**.** $\mathsf{bool}\ \mathsf{type_{Kan}}\ [\Psi]$*.*

Proof. It suffices to check the five Kan conditions.

(hcom) Suppose that

1. $\overrightarrow{r_i = r'_i}$ is valid,
2. $M \doteq M' \in \mathsf{bool}\ [\Psi']$,
3. $N_i \doteq N'_j \in \mathsf{bool}\ [\Psi', y \mid r_i = r'_i, r_j = r'_j]$ for any i, j, and
4. $N_i\langle r/y\rangle \doteq M \in \mathsf{bool}\ [\Psi' \mid r_i = r'_i]$ for any i,

and show $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M';\overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}) \in \mathsf{bool}\ [\Psi']$. This is immediate by Lemma 43 on both sides, because $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \longmapsto_{\square} M$ and $M \doteq M' \in \mathsf{bool}\ [\Psi']$. Similarly, if $r = r'$ it is immediate that $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq M \in \mathsf{bool}\ [\Psi']$. Now suppose that $r_i = r'_i$, and show $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{bool}}(M;\overrightarrow{r_i = r'_i \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in \mathsf{bool}\ [\Psi']$. By Lemma 43 it suffices to show $M \doteq N_i\langle r'/y\rangle \in \mathsf{bool}\ [\Psi']$, which holds by $M \doteq N_i\langle r/y\rangle \in \mathsf{bool}\ [\Psi']$ and Lemma 47.

(coe) Suppose that $M \doteq M' \in \mathsf{bool}\ [\Psi']$, and show that $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{bool}}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{bool}}(M') \in \mathsf{bool}\ [\Psi']$. This is immediate by Lemma 43 on both sides, because $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{bool}}(M) \longmapsto_{\square} M$ and $M \doteq M' \in \mathsf{bool}\ [\Psi']$. Similarly, if $r = r'$ it is immediate that $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{bool}}(M) \doteq M \in \mathsf{bool}\ [\Psi']$. $\square$

5.7 Natural numbers

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu,\sigma)$ for any cubical type systems ν,σ; we have $\tau(\Psi,\mathsf{nat},\mathsf{nat},\mathbb{N}_\Psi)$, where $\mathbb{N}$ is the least context-indexed relation such that $\mathbb{N}_\Psi(\mathsf{z},\mathsf{z})$ and $\mathbb{N}_\Psi(\mathsf{s}(M),\mathsf{s}(M'))$ when $\mathsf{Tm}(\mathbb{N}(\Psi))(M,M')$. By $\mathsf{nat}\ \mathsf{val}_{\square}$, $\mathsf{PTy}(\tau)(\Psi,\mathsf{nat},\mathsf{nat},\mathbb{N}(\Psi))$.

Rule 31 (Pretype formation)**.** $\mathsf{nat}\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.

Proof. It suffices to show $\mathsf{Coh}(\llbracket\mathsf{nat}\rrbracket)$. We have $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket)(\mathsf{z},\mathsf{z})$ and $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket)(\mathsf{s}(M),\mathsf{s}(M'))$ when $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket)(M,M')$ by $\mathsf{z}\ \mathsf{val}_{\square}$, $\mathsf{s}(M)\ \mathsf{val}_{\square}$, and $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket\psi)(M\psi,M'\psi)$ for all $\psi:\Psi'\to\Psi$. $\square$

Rule 32 (Introduction)**.** $\mathsf{z}\in\mathsf{nat}\ [\Psi]$ *and if* $M\doteq M'\in\mathsf{nat}\ [\Psi]$ *then* $\mathsf{s}(M)\doteq\mathsf{s}(M')\in\mathsf{nat}\ [\Psi]$.

Proof. Immediate by $\mathsf{Coh}(\llbracket\mathsf{nat}\rrbracket)$. $\square$

Rule 33 (Elimination)**.** *If* $n:\mathsf{nat}\gg A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$, $M\doteq M'\in\mathsf{nat}\ [\Psi]$, $Z\doteq Z'\in A[\mathsf{z}/n]\ [\Psi]$, *and* $n{:}\mathsf{nat},a{:}A\gg S\doteq S'\in A[\mathsf{s}(n)/n]\ [\Psi]$, *then* $\mathsf{natrec}(M;Z,n.a.S)\doteq\mathsf{natrec}(M';Z',n.a.S')\in A[M/n]\ [\Psi]$.

Proof. We induct over the definition of $\llbracket\mathsf{nat}\rrbracket$. The equality relation of nat, $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket)$, is the lifting of the least pre-fixed point of an order-preserving operator N on context-indexed relations over values. Therefore, we prove (1) the elimination rule lifts from values to elements; (2) the elimination rule holds for values; and thus (3) the elimination rule holds for elements.

Define $\Phi_\Psi(M_0,M_0')$ to hold when $\llbracket\mathsf{nat}\rrbracket_\Psi(M_0,M_0')$ and for all $n:\mathsf{nat}\gg A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$, $Z\doteq Z'\in A[\mathsf{z}/n]\ [\Psi]$, and $n:\mathsf{nat},a:A\gg S\doteq S'\in A[\mathsf{s}(n)/n]\ [\Psi]$, we have $\mathsf{natrec}(M_0;Z,n.a.S)\doteq\mathsf{natrec}(M_0';Z',n.a.S')\in A[M_0/n]\ [\Psi]$.

1. If $\mathsf{Tm}(\Phi(\Psi))(M,M')$ then the elimination rule holds for M,M'.

   By definition, $\Phi\subseteq\llbracket\mathsf{nat}\rrbracket$, so because Tm is order-preserving, $\mathsf{Tm}(\llbracket\mathsf{nat}\rrbracket(\Psi))(M,M')$. Apply coherent expansion to $\mathsf{natrec}(M;Z,n.a.S)$ at $A[M/n]\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$ with $\{\mathsf{natrec}(M_\psi;Z\psi,n.a.S\psi)\mid M\psi\Downarrow M_\psi\}^{\Psi'}_{\psi}$. Then $\mathsf{natrec}(M_\psi;Z\psi,n.a.S\psi)\in A\psi[M_\psi/n]\ [\Psi']$ for all $\psi:\Psi'\to\Psi$ because $\Phi^{\Downarrow}_{\Psi}(M,M')$ by $\mathsf{Tm}(\Phi(\Psi))(M,M')$. We must show

   $$\mathsf{natrec}(M_\psi;Z\psi,n.a.S\psi)\doteq\mathsf{natrec}((M_{\mathsf{id}_\Psi})\psi;Z\psi,n.a.S\psi)\in A\psi[M_\psi/n]\ [\Psi']$$

   but by Lemma 37 and $(M_{\mathsf{id}_\Psi})\psi\doteq M_\psi\in\mathsf{nat}\ [\Psi']$ it suffices to show these natrec are related by $\llbracket A\psi[M_\psi/n]\rrbracket^{\Downarrow}$, which follows from $\Phi^{\Downarrow}_{\Psi'}((M_{\mathsf{id}_\Psi})\psi,M_\psi)$.

2. If $\llbracket\mathsf{nat}\rrbracket_\Psi(M_0,M_0')$ then $\Phi_\Psi(M_0,M_0')$.

   We prove that $N(\Phi)\subseteq\Phi$; then Φ is a pre-fixed point of N, and $\llbracket\mathsf{nat}\rrbracket\subseteq\Phi$ because $\llbracket\mathsf{nat}\rrbracket$ is the least pre-fixed point of N. Suppose $N(\Phi)_\Psi(M_0,M_0')$. There are two cases:

   (a) $M_0=M_0'=\mathsf{z}$.
   Show $\mathsf{natrec}(\mathsf{z};Z,n.a.S)\doteq\mathsf{natrec}(\mathsf{z};Z',n.a.S')\in A[\mathsf{z}/n]\ [\Psi]$, which is immediate by $Z\doteq Z'\in A[\mathsf{z}/n]\ [\Psi]$ and Lemma 43 on both sides.

   (b) $M_0=\mathsf{s}(M)$, $M_0'=\mathsf{s}(M')$, and $\mathsf{Tm}(\Phi(\Psi))(M,M')$.
   Show $\mathsf{natrec}(\mathsf{s}(M);Z,n.a.S)\doteq\mathsf{natrec}(\mathsf{s}(M');Z',n.a.S')\in A[\mathsf{s}(M)/n]\ [\Psi]$. By Lemma 43 on both sides, it suffices to show

   $$S[M/n][\mathsf{natrec}(M;Z,n.a.S)/a]\doteq S'[M'/n][\mathsf{natrec}(M';Z',n.a.S')/a]\in A[\mathsf{s}(M)/n]\ [\Psi].$$

We have $M \doteq M' \in \mathsf{nat}\ [\Psi]$ and $\mathsf{natrec}(M; Z, n.a.S) \doteq \mathsf{natrec}(M'; Z', n.a.S') \in A[M/n]\ [\Psi]$ by $\mathsf{Tm}(\Phi(\Psi))(M, M')$, so the result follows by $n : \mathsf{nat}, a : A \gg S \doteq S' \in A[\mathsf{s}(n)/n]\ [\Psi]$.

3. Assume $\mathsf{Tm}([\![\mathsf{nat}]\!](\Psi))(M, M')$; Tm is order-preserving and $[\![\mathsf{nat}]\!] \subseteq \Phi$, so $\mathsf{Tm}(\Phi(\Psi))(M, M')$. Thus the elimination rule holds for M, M', completing the proof. □

Rule 34 (Computation)**.**

1. *If* $Z \in A\ [\Psi]$ *then* $\mathsf{natrec}(\mathsf{z}; Z, n.a.S) \doteq Z \in A\ [\Psi]$.

2. *If* $n : \mathsf{nat} \gg A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$, $M \in \mathsf{nat}\ [\Psi]$, $Z \in A[\mathsf{z}/n]\ [\Psi]$, *and* $n : \mathsf{nat}, a : A \gg S \in A[\mathsf{s}(n)/n]\ [\Psi]$, *then* $\mathsf{natrec}(\mathsf{s}(M); Z, n.a.S) \doteq S[M/n][\mathsf{natrec}(M; Z, n.a.S)/a] \in A[\mathsf{s}(M)/n]\ [\Psi]$.

Proof. Part (1) is immediate by Lemma 43. For part (2), we have $\mathsf{natrec}(M; Z, n.a.S) \in A[M/n]\ [\Psi]$ and thus $S[M/n][\mathsf{natrec}(M; Z, n.a.S)/a] \in A[\mathsf{s}(M)/n]\ [\Psi]$ by Rule 33, so the result again follows by Lemma 43. □

Rule 35 (Kan type formation)**.** $\mathsf{nat}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.

Proof. Identical to Rule 30. □

5.8 Circle

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu, \sigma)$ for any cubical type systems ν, σ; we have $\tau(\Psi, \mathbb{S}^1, \mathbb{S}^1, \mathbb{C}_\Psi)$, where $\mathbb{C}$ is the least context-indexed relation such that:

1. $\mathbb{C}_\Psi(\mathsf{base}, \mathsf{base})$,

2. $\mathbb{C}_{(\Psi,x)}(\mathsf{loop}_x, \mathsf{loop}_x)$, and

3. $\mathbb{C}_\Psi(\mathsf{fcom}^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}), \mathsf{fcom}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}))$ whenever
   (a) $r \neq r'$; $r_i \neq r'_i$ for all i; $r_i = r_j$, $r'_i = 0$, and $r'_j = 1$ for some i, j;
   (b) $\mathsf{Tm}(\mathbb{C}(\Psi))(M, M')$;
   (c) $\mathsf{Tm}(\mathbb{C}(\Psi'))(N_i\psi, N'_j\psi)$ for all i, j and $\psi : \Psi' \to (\Psi, y)$ satisfying $r_i = r'_i, r_j = r'_j$; and
   (d) $\mathsf{Tm}(\mathbb{C}(\Psi'))(N_i\langle r/y\rangle\psi, M\psi)$ for all i, j and $\psi : \Psi' \to \Psi$ satisfying $r_i = r'_i$.

By $\mathbb{S}^1\ \mathsf{val}_{\square}$ it is immediate that $\mathsf{PTy}(\tau)(\Psi, \mathbb{S}^1, \mathbb{S}^1, \mathbb{C}(\Psi))$.

Lemma 48. *If*

1. $\overrightarrow{r_i = r'_i}$ *is valid,*

2. $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(M, M')$,

3. $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi'))(N_i\psi, N'_j\psi)$ *for all* i, j *and* $\psi : \Psi' \to (\Psi, y)$ *satisfying* $r_i = r'_i, r_j = r'_j$, *and*

4. $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi'))(N_i\langle r/y\rangle\psi, M\psi)$ *for all* i, j *and* $\psi : \Psi' \to \Psi$ *satisfying* $r_i = r'_i$,

then $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(\mathsf{fcom}^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}), \mathsf{fcom}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}))$.

Proof. Let us abbreviate the above fcom terms L and R respectively. Expanding the definition of Tm, for any $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$ we must show $L\psi_1 \Downarrow L_1$, $R\psi_1 \Downarrow R_1$, and $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}$ relates $L_1\psi_2$, $L\psi_1\psi_2$, $R_1\psi_2$, and $R\psi_1\psi_2$. We proceed by cases on the first step taken by $L\psi_1$ and $L\psi_1\psi_2$.

1. $r\psi_1 = r'\psi_1$.

   Then $L\psi_1 \longmapsto_{\square} M\psi_1$, $R\psi_1 \longmapsto_{\square} M'\psi_1$, and the result follows by $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(M, M')$.

2. $r\psi_1 \neq r'\psi_1$, $r_j\psi_1 = r'_j\psi_1$ (where $r_i\psi_1 \neq r'_i\psi_1$ for all $i < j$), and $r\psi_1\psi_2 = r'\psi_1\psi_2$.

   Then $L\psi_1 \longmapsto N_j\langle r'/y\rangle\psi_1$, $L\psi_1\psi_2 \longmapsto M\psi_1\psi_2$, $R\psi_1 \longmapsto N'_j\langle r'/y\rangle\psi_1$, and $R\psi_1\psi_2 \longmapsto M'\psi_1\psi_2$. Because ψ_1 satisfies $r_j = r'_j$, by (3) and (4) $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi_1, y))(N_j\psi_1, N'_j\psi_1)$ and $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi_1))(N_j\langle r/y\rangle\psi_1, M\psi_1)$. By the former at $\langle r'\psi_1/y\rangle, \psi_2$, $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}(N_j\langle r'/y\rangle\psi_1\psi_2, L_1\psi_2)$ and $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}(L_1\psi_2, R_1\psi_2)$. The latter at $\psi_2, \mathsf{id}_{\Psi_2}$ yields $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}(N_j\langle r/y\rangle\psi_1\psi_2, M\psi_1\psi_2)$; by transitivity and $r\psi_1\psi_2 = r'\psi_1\psi_2$ we have $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}(L_1\psi_2, L\psi_1\psi_2)$. Finally, by $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(M, M')$ we have $[\![\mathbb{S}^1]\!]^{\Downarrow}_{\Psi_2}(L\psi_1\psi_2, R\psi_1\psi_2)$.

3. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 = r'_i\psi_1$ (and this is the least such i), $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 = r'_j\psi_1\psi_2$ (and this is the least such $j \leq i$).

   Then $L\psi_1 \longmapsto N_i\langle r'/y\rangle\psi_1$, $L\psi_1\psi_2 \longmapsto N_j\langle r'/y\rangle\psi_1\psi_2$, $R\psi_1 \longmapsto N'_i\langle r'/y\rangle\psi_1$, and $R\psi_1\psi_2 \longmapsto N'_j\langle r'/y\rangle\psi_1\psi_2$. In this case, $\langle r'/y\rangle\psi_1\psi_2$ satisfies $r_i = r'_i, r_j = r'_j$, and the result follows because $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi_2))$ relates $N_i\langle r'/y\rangle\psi_1\psi_2$, $N_j\langle r'/y\rangle\psi_1\psi_2$, $N'_i\langle r'/y\rangle\psi_1\psi_2$, and $N'_j\langle r'/y\rangle\psi_1\psi_2$.

4. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r'_i\psi_1$ for all i, and $r\psi_1\psi_2 = r'\psi_1\psi_2$.

   Then $L\psi_1$ val, $L\psi_1\psi_2 \longmapsto M\psi_1\psi_2$, $R\psi_1$ val, and $R\psi_1\psi_2 \longmapsto M'\psi_1\psi_2$. In this case, $L_1\psi_2 = L\psi_1\psi_2$ and $R_1\psi_2 = R\psi_1\psi_2$, so the result follows by $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(M, M')$.

5. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r'_i\psi_1$ for all i, $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 = r'_j\psi_1\psi_2$ (the least such j).

   Then $L\psi_1$ val, $L\psi_1\psi_2 \longmapsto N_j\langle r'/y\rangle\psi_1\psi_2$, $R\psi_1$ val, and $R\psi_1\psi_2 \longmapsto N'_j\langle r'/y\rangle\psi_1\psi_2$. The result follows because $L_1\psi_2 = L\psi_1\psi_2$, $R_1\psi_2 = R\psi_1\psi_2$, and because $\langle r'/y\rangle\psi_1\psi_2$ satisfies $r_j = r'_j$, $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi_2))(N_j\langle r'/y\rangle\psi_1\psi_2, N'_j\langle r'/y\rangle\psi_1\psi_2)$.

6. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r'_i\psi_1$ for all i, and $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 \neq r'_j\psi_1\psi_2$ for all j.

   Then $L\psi_1$ val, $L\psi_1\psi_2$ val, $R\psi_1$ val, and $R\psi_1\psi_2$ val, so it suffices to show $[\![\mathbb{S}^1]\!]_{\Psi_2}(L\psi_1\psi_2, R\psi_1\psi_2)$. We know $\overrightarrow{r_i\psi_1\psi_2 = r'_i\psi_1\psi_2}$ is valid and $r_i\psi_1\psi_2 \neq r'_i\psi_1\psi_2$ for all i, so there must be some i, j for which $r_i\psi_1\psi_2 = r_j\psi_1\psi_2$, $r'_i\psi_1\psi_2 = 0$, and $r'_j\psi_1\psi_2 = 1$. The result follows immediately by the third clause of the definition of $[\![\mathbb{S}^1]\!]$. ◻

Rule 36 (Pretype formation)**.** $\mathbb{S}^1$ $\mathsf{type}_{\mathsf{pre}}$ $[\Psi]$.

Proof. It remains to show $\mathsf{Coh}([\![\mathbb{S}^1]\!])$. There are three cases:

1. $\mathsf{Tm}([\![\mathbb{S}^1]\!](\Psi))(\mathsf{base}, \mathsf{base})$.

   Immediate because base $\mathsf{val}_{\square}$.

2. $\mathsf{Tm}(\llbracket \mathbb{S}^1 \rrbracket(\Psi, x))(\mathsf{loop}_x, \mathsf{loop}_x)$.

   Show that if $\psi_1 : \Psi_1 \to (\Psi, x)$ and $\psi_2 : \Psi_2 \to \Psi_1$, $\mathsf{loop}_{x\psi_1} \Downarrow M_1$ and $\llbracket \mathbb{S}^1 \rrbracket^{\Downarrow}_{\Psi_2}(M_1\psi_2, \mathsf{loop}_{x\psi_1\psi_2})$. If $x\psi_1 = \varepsilon$ then $M_1 = \mathsf{base}$, $\mathsf{loop}_{x\psi_1\psi_2} \longmapsto \mathsf{base}$, and $\llbracket \mathbb{S}^1 \rrbracket_{\Psi_2}(\mathsf{base}, \mathsf{base})$. If $x\psi_1 = x'$ and $x'\psi_2 = \varepsilon$, then $M_1 = \mathsf{loop}_{x'}$, $\mathsf{loop}_{x'\psi_2} \longmapsto \mathsf{base}$, $\mathsf{loop}_{x\psi_1\psi_2} \longmapsto \mathsf{base}$, and $\llbracket \mathbb{S}^1 \rrbracket_{\Psi_2}(\mathsf{base}, \mathsf{base})$. Otherwise, $x\psi_1 = x'$ and $x'\psi_2 = x''$, so $M_1 = \mathsf{loop}_{x'}$ and $\llbracket \mathbb{S}^1 \rrbracket_{\Psi_2}(\mathsf{loop}_{x''}, \mathsf{loop}_{x''})$.

3. $\mathsf{Tm}(\llbracket \mathbb{S}^1 \rrbracket(\Psi))(\mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}), \mathsf{fcom}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}))$ where...

   This is a special case of Lemma 48. (Note that $\overrightarrow{r_i = r_i'}$ is valid.) □

Rule 37 (Introduction)**.** $\mathsf{base} \in \mathbb{S}^1\ [\Psi]$, $\mathsf{loop}_\varepsilon \doteq \mathsf{base} \in \mathbb{S}^1\ [\Psi]$, *and* $\mathsf{loop}_r \in \mathbb{S}^1\ [\Psi]$.

Proof. The first is a consequence of $\mathsf{Coh}(\llbracket \mathbb{S}^1 \rrbracket)$; the second follows by $\mathsf{loop}_\varepsilon \longmapsto_{\square} \mathsf{base}$ and Lemma 43; the third is a consequence of $\mathsf{Coh}(\llbracket \mathbb{S}^1 \rrbracket)$ when $r = x$, and of Lemma 43 when $r = \varepsilon$. □

Rule 38 (Kan type formation)**.** $\mathbb{S}^1\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.

Proof. It suffices to check the five Kan conditions.

(hcom) First, suppose that

1. $\overrightarrow{r_i = r_i'}$ is valid,
2. $M \doteq M' \in \mathbb{S}^1\ [\Psi']$,
3. $N_i \doteq N_j' \in \mathbb{S}^1\ [\Psi', y \mid r_i = r_i', r_j = r_j']$ for any i, j, and
4. $N_i\langle r/y \rangle \doteq M \in \mathbb{S}^1\ [\Psi' \mid r_i = r_i']$ for any i,

and show $\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in \mathbb{S}^1\ [\Psi']$. This is immediate by Lemma 43 on both sides (because $\mathsf{hcom}_{\mathbb{S}^1} \longmapsto_{\square} \mathsf{fcom}$) and Lemma 48.

Next, show that if $r = r'$ then $\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in \mathbb{S}^1\ [\Psi']$. This is immediate by $\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \longmapsto_{\square} \mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \longmapsto_{\square} M$ and Lemma 43.

For the final hcom property, show that if $r_i = r_i'$ then $\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y \rangle \in \mathbb{S}^1\ [\Psi']$. We already know each side is an element of $\mathbb{S}^1$, so by Lemma 38 it suffices to show $\llbracket \mathbb{S}^1 \rrbracket^{\Downarrow}_{\Psi'}(\mathsf{hcom}^{r \rightsquigarrow r'}_{\mathbb{S}^1}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}), N_i\langle r'/y \rangle)$. If $r = r'$ then $\mathsf{hcom} \longmapsto^2 M$ and the result follows by $N_i\langle r/y \rangle \doteq M \in \mathbb{S}^1\ [\Psi' \mid r_i = r_i']$, because $\mathsf{id}_{\Psi'}$ satisfies $r_i = r_i'$. Otherwise, let $r_j = r_j'$ be the first true equation. Then $\mathsf{hcom} \longmapsto^2 N_j\langle r'/y \rangle$ and this follows by $N_i \doteq N_j \in \mathbb{S}^1\ [\Psi', y \mid r_i = r_i', r_j = r_j']$.

(coe) Now, suppose that $M \doteq M' \in \mathbb{S}^1\ [\Psi']$ and show $\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbb{S}^1}(M) \doteq \mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbb{S}^1}(M') \in \mathbb{S}^1\ [\Psi']$. This is immediate by $\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbb{S}^1}(M) \longmapsto_{\square} M$ and Lemma 43 on both sides. Similarly, if $r = r'$ then $\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathbb{S}^1}(M) \doteq M \in \mathbb{S}^1\ [\Psi']$ by Lemma 43 on the left. □

Rule 39 (Computation)**.** *If* $P \in B\ [\Psi]$ *then* $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{base}; P, x.L) \doteq P \in B\ [\Psi]$.

Proof. Immediate by $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{base}; P, x.L) \longmapsto_{\square} P$ and Lemma 43. □

Rule 40 (Computation)**.** *If* $L \in B\ [\Psi, x]$ *and* $L\langle \varepsilon/x \rangle \doteq P \in B\langle \varepsilon/x \rangle\ [\Psi]$ *for* $\varepsilon \in \{0, 1\}$, *then* $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{loop}_r; P, x.L) \doteq L\langle r/x \rangle \in B\langle r/x \rangle\ [\Psi]$.

Proof. If $r = \varepsilon$ then this is immediate by Lemma 43 and $L\langle\varepsilon/x\rangle \doteq P \in B\langle\varepsilon/x\rangle\ [\Psi]$. If $r = y$ then we apply coherent expansion to the left side with family $\{P\psi \mid y\psi = \varepsilon\}^{\Psi'}_{\psi} \cup \{L\psi\langle z/x\rangle \mid y\psi = z\}^{\Psi'}_{\psi}$. The id_Ψ element of this family is $L\langle y/x\rangle$; when $y\psi = \varepsilon$ we have $L\langle y/x\rangle\psi \doteq P\psi \in B\langle y/x\rangle\psi\ [\Psi']$ (by $\langle y/x\rangle\psi = \langle\varepsilon/x\rangle\psi$), and when $y\psi = z$ we have $L\langle y/x\rangle\psi \doteq L\psi\langle z/x\rangle \in B\langle y/x\rangle\psi\ [\Psi']$ (by $\psi\langle z/x\rangle = \langle y/x\rangle\psi$). Thus by Lemma 41, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{loop}_y; P, x.L) \doteq L\langle y/x\rangle \in B\langle y/x\rangle\ [\Psi]$. □

To establish the elimination rule we must induct over the definition of $[\![\mathbb{S}^1]\!]$. As $[\![\mathbb{S}^1]\!]$ was defined in Section 3 as the least pre-fixed point of an order-preserving operator C on context-indexed relations, we define our induction hypothesis as an auxiliary context-indexed PER on values $\Phi_\Psi(M_0, M_0')$ that holds when

1. $[\![\mathbb{S}^1]\!]_\Psi(M_0, M_0')$ and
2. whenever $c : \mathbb{S}^1 \gg A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$, $P \doteq P' \in A[\mathsf{base}/c]\ [\Psi]$, $L \doteq L' \in A[\mathsf{loop}_x/c]\ [\Psi, x]$, and $L\langle\varepsilon/x\rangle \doteq P \in A[\mathsf{base}/c]\ [\Psi]$ for $\varepsilon \in \{0,1\}$, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M_0; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M_0'; P', x.L') \in A[M_0/c]\ [\Psi]$. (In other words, the elimination rule holds for M_0 and M_0'.)

Lemma 49. *If* $\mathsf{Tm}(\Phi(\Psi))(M, M')$ *then whenever* $c{:}\mathbb{S}^1 \gg A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$, $P \doteq P' \in A[\mathsf{base}/c]\ [\Psi]$, $L \doteq L' \in A[\mathsf{loop}_x/c]\ [\Psi, x]$, *and* $L\langle\varepsilon/x\rangle \doteq P \in A[\mathsf{base}/c]\ [\Psi]$ *for* $\varepsilon \in \{0,1\}$, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M'; P', x.L') \in A[M/c]\ [\Psi]$.

Proof. First we apply coherent expansion to the left side with family $\{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M_\psi; P\psi, x.L\psi) \mid M\psi \Downarrow M_\psi\}^{\Psi'}_{\psi}$, by showing that

$$\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M_\psi; P\psi, x.L\psi) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}((M_{\mathsf{id}_\Psi})\psi; P\psi, x.L\psi) \in (A[M/c])\psi\ [\Psi'].$$

The left side is an element of this type by $\Phi_{\Psi'}(M_\psi, M_\psi)$ and $A\psi[M_\psi/c] \doteq A\psi[M\psi/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$ (by $M_\psi \doteq M\psi \in \mathbb{S}^1\ [\Psi']$). The right side is an element by $\Phi_\Psi(M_{\mathsf{id}_\Psi}, M_{\mathsf{id}_\Psi})$ and $A[M_{\mathsf{id}_\Psi}/c] \doteq A[M/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$. The equality follows from $(M_{\mathsf{id}_\Psi})\psi \Downarrow M_2$, $\Phi_{\Psi'}(M_\psi, M_2)$, and Lemma 38. Thus by Lemma 41, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M_{\mathsf{id}_\Psi}; P, x.L) \in A[M/c]\ [\Psi]$.

By the same argument on the right side, $A[M/c] \doteq A'[M'/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$ (by $M \doteq M' \in \mathbb{S}^1\ [\Psi]$), and transitivity, it suffices to show $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M_{\mathsf{id}_\Psi}; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M'_{\mathsf{id}_\Psi}; P', x.L') \in A[M/c]\ [\Psi]$; this is immediate by $\Phi_\Psi(M_{\mathsf{id}_\Psi}, M'_{\mathsf{id}_\Psi})$ and $A[M_{\mathsf{id}_\Psi}/c] \doteq A[M/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$. □

Lemma 50. *If* $C(\Phi)_\Psi(M_0, M_0')$ *then* $\Phi_\Psi(M_0, M_0')$.

Proof. We must show that $[\![\mathbb{S}^1]\!]_\Psi(M_0, M_0')$, and that if $c : \mathbb{S}^1 \gg A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$, $P \doteq P' \in A[\mathsf{base}/c]\ [\Psi]$, $L \doteq L' \in A[\mathsf{loop}_x/c]\ [\Psi, x]$, and $L\langle\varepsilon/x\rangle \doteq P \in A[\mathsf{base}/c]\ [\Psi]$ for $\varepsilon \in \{0,1\}$, then $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M_0; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M_0'; P', x.L') \in A[M_0/c]\ [\Psi]$. There are three cases to consider.

1. $C(\Phi)_\Psi(\mathsf{base}, \mathsf{base})$.

   Then $[\![\mathbb{S}^1]\!]_\Psi(\mathsf{base}, \mathsf{base})$ by definition, and the elimination rule holds by Rule 39 on both sides (with $B = A[\mathsf{base}/c]$) and $P \doteq P' \in A[\mathsf{base}/c]\ [\Psi]$.

2. $C(\Phi)_{(\Psi,y)}(\mathsf{loop}_y, \mathsf{loop}_y)$.

   Then $[\![\mathbb{S}^1]\!]_{(\Psi,y)}(\mathsf{loop}_y, \mathsf{loop}_y)$ by definition, and the elimination rule holds by Rule 40 on both sides (with $B = A[\mathsf{loop}_x/c]$ and $A[\mathsf{loop}_x/c]\langle\varepsilon/x\rangle \doteq A[\mathsf{base}/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$) and $L\langle y/x\rangle \doteq L'\langle y/x\rangle \in A[\mathsf{loop}_y/c]\ [\Psi]$.

3. $C(\Phi)_\Psi(\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{r_i=r_i'\hookrightarrow y.N_i}),\mathsf{fcom}^{r\rightsquigarrow r'}(M';\overrightarrow{r_i=r_i'\hookrightarrow y.N_i'}))$ where

   (a) $r\neq r'$; $r_i\neq r_i'$ for all i; $r_i=r_j$, $r_i'=0$, and $r_j'=1$ for some i,j;
   (b) $\mathsf{Tm}(\Phi(\Psi))(M,M')$;
   (c) $\mathsf{Tm}(\Phi(\Psi'))(N_i\psi,N_j'\psi)$ for all i,j and $\psi:\Psi'\to(\Psi,y)$ satisfying $r_i=r_i',r_j=r_j'$; and
   (d) $\mathsf{Tm}(\Phi(\Psi'))(N_i\langle r/y\rangle\psi,M\psi)$ for all i,j and $\psi:\Psi'\to\Psi$ satisfying $r_i=r_i'$.

By construction, $\Phi\subseteq[\![\mathbb{S}^1]\!]$, so $\mathsf{Tm}(\Phi)\subseteq\mathsf{Tm}([\![\mathbb{S}^1]\!])$ and $[\![\mathbb{S}^1]\!]_\Psi(\mathsf{fcom},\mathsf{fcom})$. By Lemma 49 and $\mathsf{Tm}(\Phi(\Psi))(M,M')$, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M)\doteq\mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M')\in A[M/c]\ [\Psi]$. For all ψ satisfying $r_i=r_i',r_j=r_j'$ we have $\mathsf{Tm}(\Phi(\Psi'))(N_i\psi,N_j'\psi)$, so by Lemma 49, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(N_i)\doteq\mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(N_j')\in A[N_i/c]\ [\Psi,y\mid r_i=r_i',r_j=r_j']$. Similarly, $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M)\doteq\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(N_i\langle r/y\rangle)\in A[M/c]\ [\Psi\mid r_i=r_i']$.

Apply coherent expansion to the term $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i\hookrightarrow y.N_i});P,x.L)$ at the type $A[\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i\hookrightarrow y.N_i})/c]\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$ with family:

$$\begin{cases}\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi;P\psi,x.L\psi) & r\psi=r'\psi\\ \mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(N_j\langle r'/y\rangle\psi;P\psi,x.L\psi) & r\psi\neq r'\psi,\text{ least } j \text{ s.t. } r_j\psi=r_j'\psi\\ \mathsf{com}^{r\psi\rightsquigarrow r'\psi}_{z.A\psi[F/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi;P\psi,x.L\psi);\overrightarrow{\xi_i\psi\hookrightarrow y.T_i}) & \text{otherwise}\\ \qquad F=\mathsf{fcom}^{r\psi\rightsquigarrow z}(M\psi;\overrightarrow{\xi_i\psi\hookrightarrow y.N_i\psi}) & \\ \qquad T_i=\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(N_i\psi;P\psi,x.L\psi) & \end{cases}$$

We must check three equations, noting that id_Ψ falls in the third category above. First:

$$\mathsf{com}^{r\psi\rightsquigarrow r'\psi}_{z.A\psi[F/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi);\overrightarrow{\xi_i\psi\hookrightarrow y.T_i})\doteq\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi)\in A\psi[\mathsf{fcom}\psi/c]\ [\Psi']$$

when $r\psi=r'\psi$. This follows from Theorem 44, $A\psi[\mathsf{fcom}\psi/c]=A\psi[F/c]\langle r'\psi/z\rangle$, and by Definition 22, $A\psi[F/c]\langle r'\psi/z\rangle\doteq A[M/c]\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$ and $A\psi[F/c]\doteq A[N_i\langle z/y\rangle/c]\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi',z\mid r_i\psi=r_i'\psi]$. Next, we must check

$$\mathsf{com}^{r\psi\rightsquigarrow r'\psi}_{z.A\psi[F/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi);\overrightarrow{\xi_i\psi\hookrightarrow y.T_i})\doteq\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(N_j\langle r'/y\rangle\psi)\in A\psi[\mathsf{fcom}\psi/c]\ [\Psi']$$

when $r\psi\neq r'\psi$, $r_j\psi=r_j'\psi$, and $r_i\psi\neq r_i'\psi$ for $i<j$; again this holds by Theorem 44. Finally, we must check

$$\mathsf{com}^{r\psi\rightsquigarrow r'\psi}_{z.A\psi[F/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A\psi}(M\psi);\overrightarrow{\xi_i\psi\hookrightarrow y.T_i})\in A\psi[\mathsf{fcom}\psi/c]\ [\Psi']$$

when $r\psi\neq r'\psi$ and $r_i\psi\neq r_i'\psi$ for all i; again this holds by Theorem 44. Therefore by Lemma 41,

$$\begin{aligned}&\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{fcom}^{r\rightsquigarrow r'}(M;\overrightarrow{\xi_i\hookrightarrow y.N_i});P,x.L)\\ \doteq\ &\mathsf{com}^{r\rightsquigarrow r'}_{z.A\psi[\mathsf{fcom}^{r\rightsquigarrow z}/c]}(\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M;P,x.L);\overrightarrow{\xi_i\hookrightarrow y.\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(N_i;P,x.L)})\in A[\mathsf{fcom}/c]\ [\Psi].\end{aligned}$$

By transitivity and a symmetric argument on the right side, it suffices to show that two coms are equal, which follows by Theorem 44. $\square$

Rule 41 (Elimination)**.** *If* $M \doteq M' \in \mathbb{S}^1$ $[\Psi]$*,* $c{:}\mathbb{S}^1 \gg A \doteq A'$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$*,* $P \doteq P' \in A[\mathsf{base}/c]$ $[\Psi]$*,* $L \doteq L' \in A[\mathsf{loop}_x/c]$ $[\Psi, x]$*, and* $L\langle\varepsilon/x\rangle \doteq P \in A[\mathsf{base}/c]$ $[\Psi]$ *for* $\varepsilon \in \{0,1\}$*, then* $\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M'; P', x.L') \in A[M/c]$ $[\Psi]$*.*

Proof. Lemma 50 states that Φ is a pre-fixed point of C; because $[\![\mathbb{S}^1]\!]$ is the least pre-fixed point of C, $[\![\mathbb{S}^1]\!] \subseteq \Phi$, and therefore $\mathsf{Tm}([\![\mathbb{S}^1]\!]) \subseteq \mathsf{Tm}(\Phi)$. We conclude that $\mathsf{Tm}(\Phi(\Psi))(M, M')$, and the result follows by Lemma 49. □

5.9 Weak booleans

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu, \sigma)$ for any cubical type systems ν, σ; we have $\tau(\Psi, \mathsf{wbool}, \mathsf{wbool}, \mathbb{B}_\Psi)$, where $\mathbb{B}$ is the least context-indexed relation such that:

1. $\mathbb{B}_\Psi(\mathsf{true}, \mathsf{true})$,
2. $\mathbb{B}_\Psi(\mathsf{false}, \mathsf{false})$, and
3. $\mathbb{B}_\Psi(\mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}), \mathsf{fcom}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}))$ whenever
   (a) $r \neq r'$; $r_i \neq r'_i$ for all i; $r_i = r_j$, $r'_i = 0$, and $r'_j = 1$ for some i, j;
   (b) $\mathsf{Tm}(\mathbb{B}(\Psi))(M, M')$;
   (c) $\mathsf{Tm}(\mathbb{B}(\Psi'))(N_i\psi, N'_j\psi)$ for all i, j and $\psi : \Psi' \to (\Psi, y)$ satisfying $r_i = r'_i, r_j = r'_j$; and
   (d) $\mathsf{Tm}(\mathbb{B}(\Psi'))(N_i\langle r/y\rangle\psi, M\psi)$ for all i, j and $\psi : \Psi' \to \Psi$ satisfying $r_i = r'_i$.

By wbool $\mathsf{val}_{\Box}$ it is immediate that $\mathsf{PTy}(\tau)(\Psi, \mathsf{wbool}, \mathsf{wbool}, \mathbb{B}(\Psi))$.

We have included wbool to demonstrate two Kan structures that one may equip to ordinary inductive types: trivial structure (as in bool) and free structure (as in wbool, mirroring $\mathbb{S}^1$). As the fcom structure of wbool is identical to that of $\mathbb{S}^1$, the proofs in this section are mostly identical to those in Section 5.8.

Lemma 51. *If*

1. $\overrightarrow{r_i = r'_i}$ *is valid,*
2. $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi))(M, M')$*,*
3. $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi'))(N_i\psi, N'_j\psi)$ *for all* i, j *and* $\psi : \Psi' \to (\Psi, y)$ *satisfying* $r_i = r'_i, r_j = r'_j$*, and*
4. $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi'))(N_i\langle r/y\rangle\psi, M\psi)$ *for all* i, j *and* $\psi : \Psi' \to \Psi$ *satisfying* $r_i = r'_i$*,*

then $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi))(\mathsf{fcom}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow y.N_i}), \mathsf{fcom}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow y.N'_i}))$.

Proof. Identical to Lemma 48. □

Rule 42 (Pretype formation)**.** wbool $\mathsf{type}_{\mathsf{pre}}$ $[\Psi]$*.*

Proof. Show $\mathsf{Coh}([\![\mathsf{wbool}]\!])$: $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi))(\mathsf{true}, \mathsf{true})$ and $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi))(\mathsf{false}, \mathsf{false})$ because true $\mathsf{val}_{\Box}$ and false $\mathsf{val}_{\Box}$, and $\mathsf{Tm}([\![\mathsf{wbool}]\!](\Psi))(\mathsf{fcom}, \mathsf{fcom})$ by Lemma 51. □

Rule 43 (Introduction)**.** *If* $M \doteq M' \in \mathsf{bool}$ $[\Psi]$ *then* $M \doteq M' \in \mathsf{wbool}$ $[\Psi]$*.*

Proof. Follows from $[\![\mathsf{bool}]\!] \subseteq [\![\mathsf{wbool}]\!]$ and the fact that Tm is order-preserving. $\square$

Rule 44 (Kan type formation)**.** $\mathsf{wbool}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.

Proof. Identical to Rule 38. $\square$

We already proved the computation rules in Rule 28. The elimination rule differs from that of bool, however: the motive $b.A$ must be Kan, because the eliminator must account and the proof must account for canonical fcom elements of wbool.

Rule 45 (Elimination)**.** *If* $M \doteq M' \in \mathsf{wbool}\ [\Psi]$*,* $b{:}\mathsf{wbool} \gg A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$*,* $T \doteq T' \in A[\mathsf{true}/b]\ [\Psi]$*, and* $F \doteq F' \in A[\mathsf{false}/b]\ [\Psi]$*, then* $\mathsf{if}_{b.A}(M; T, F) \doteq \mathsf{if}_{b.A'}(M'; T', F') \in A[M/b]\ [\Psi]$.

Proof. This proof is analogous to the proof of Rule 41. First, we define a context-indexed PER $\Phi_\Psi(M_0, M_0')$ that holds when $[\![\mathsf{wbool}]\!]_\Psi(M_0, M_0')$ and the elimination rule is true for M_0, M_0'. Next, we prove that if $\mathsf{Tm}(\Phi(\Psi))(M, M')$ then the elimination rule is true for M, M'. Finally, we prove that Φ is a pre-fixed point of the operator defining $\mathbb{B}$. (Here we must check that the elimination rule holds for true and false, which are immediate by Rule 28.) Therefore $\mathsf{Tm}([\![\mathsf{wbool}]\!]) \subseteq \mathsf{Tm}(\Phi)$, so the elimination rule applies to $M \doteq M' \in \mathsf{wbool}\ [\Psi]$. $\square$

5.10 Univalence

Recall the abbreviations:

$$\begin{aligned} \mathsf{isContr}(C) &:= C \times ((c{:}C) \to (c'{:}C) \to \mathsf{Path}_{\_.C}(c, c')) \\ \mathsf{Equiv}(A, B) &:= (f{:}A \to B) \times ((b{:}B) \to \mathsf{isContr}((a{:}A) \times \mathsf{Path}_{\_.B}(\mathsf{app}(f, a), b))) \end{aligned}$$

Let $\tau = \mu^{\mathsf{Kan}}(\nu)$ or $\mu^{\mathsf{pre}}(\nu, \sigma)$ for any cubical type systems ν, σ; in τ, when $A \doteq A'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi, x \mid x = 0]$, $B \doteq B'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi, x]$, $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi, x \mid x = 0]$, and $\varphi(\mathsf{Vin}_x(M, N), \mathsf{Vin}_x(M', N'))$ for

1. $N \doteq N' \in B\ [\Psi, x]$,
2. $M \doteq M' \in A\ [\Psi, x \mid x = 0]$, and
3. $\mathsf{app}(\mathsf{fst}(E), M) \doteq N \in B\ [\Psi, x \mid x = 0]$,

we have $\tau((\Psi, x), \mathsf{V}_x(A, B, E), \mathsf{V}_x(A', B', E'), \varphi)$.

Rule 46 (Pretype formation)**.**

1. *If* $A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$ *then* $\mathsf{V}_0(A, B, E) \doteq A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.
2. *If* $B\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$ *then* $\mathsf{V}_1(A, B, E) \doteq B\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.
3. *If* $A \doteq A'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi \mid r = 0]$*,* $B \doteq B'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$*, and* $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0]$*, then* $\mathsf{V}_r(A, B, E) \doteq \mathsf{V}_r(A', B', E')\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.

Proof. Parts (1–2) are immediate by Lemma 43. To show part (3), we must first establish that $\mathsf{PTy}(\tau)(\Psi, \mathsf{V}_r(A, B, E), \mathsf{V}_r(A', B', E'), \gamma)$, that is, abbreviating these terms L and R, for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, $L\psi_1 \Downarrow L_1$, $R\psi_1 \Downarrow R_1$, $\tau^{\Downarrow}(\Psi_2, L_1\psi_2, L\psi_1\psi_2, \_)$, $\tau^{\Downarrow}(\Psi_2, R_1\psi_2, R\psi_1\psi_2, \_)$, and $\tau^{\Downarrow}(\Psi_2, L_1\psi_2, R_1\psi_2, \_)$. We proceed by cases on the first step taken by $L\psi_1$ and $L\psi_1\psi_2$.

1. $r\psi_1 = 0$.

   Then $L\psi_1 \longmapsto_{\square} A\psi_1$, $R\psi_1 \longmapsto_{\square} A'\psi_1$, and the result follows by $A\psi_1 \doteq A'\psi_1\ \mathsf{type_{pre}}\ [\Psi_1]$.

2. $r\psi_1 = 1$.

   Then $L\psi_1 \longmapsto_{\square} B\psi_1$, $R\psi_1 \longmapsto_{\square} B'\psi_1$, and the result follows by $B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$.

3. $r\psi_1 = x$ and $r\psi_1\psi_2 = 0$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2 \longmapsto A\psi_1\psi_2$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2 \longmapsto A'\psi_1\psi_2$, and the result follows by $A\psi_1\psi_2 \doteq A'\psi_1\psi_2\ \mathsf{type_{pre}}\ [\Psi_2]$.

4. $r\psi_1 = x$ and $r\psi_1\psi_2 = 1$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2 \longmapsto B\psi_1\psi_2$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2 \longmapsto B'\psi_1\psi_2$, and the result follows by $B \doteq B'\ \mathsf{type_{pre}}\ [\Psi]$.

5. $r\psi_1 = x$ and $r\psi_1\psi_2 = x'$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2\ \mathsf{val}$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2\ \mathsf{val}$, and by $A\psi_1\psi_2 \doteq A'\psi_1\psi_2\ \mathsf{type_{pre}}\ [\Psi_2 \mid x' = 0]$, $B\psi_1\psi_2 \doteq B'\psi_1\psi_2\ \mathsf{type_{pre}}\ [\Psi_2]$, and $E\psi_1\psi_2 \doteq E'\psi_1\psi_2 \in \mathsf{Equiv}(A\psi_1\psi_2, B\psi_1\psi_2)\ [\Psi_2 \mid x' = 0]$, we have $\tau(\Psi_2, \mathsf{V}_{x'}(A\psi_1\psi_2, B\psi_1\psi_2, E\psi_1\psi_2), \mathsf{V}_{x'}(A'\psi_1\psi_2, B'\psi_1\psi_2, E'\psi_1\psi_2), \_)$.

To complete part (3), we must show $\mathsf{Coh}(\llbracket \mathsf{V}_r(A, B, E)\rrbracket)$, that is, for any $\psi : \Psi' \to \Psi$, if $\llbracket \mathsf{V}_{r\psi}(A\psi, B\psi, E\psi)\rrbracket(M_0, N_0)$ then $\mathsf{Tm}(\llbracket \mathsf{V}_{r\psi}(A\psi, B\psi, E\psi)\rrbracket)(M_0, N_0)$. If $r\psi = 0$ this follows by $\llbracket \mathsf{V}_0(A\psi, B\psi, E\psi)\rrbracket = \llbracket A\psi\rrbracket$ and $\mathsf{Coh}(\llbracket A\psi\rrbracket)$; if $r\psi = 1$ then this follows by $\llbracket \mathsf{V}_1(A\psi, B\psi, E\psi)\rrbracket = \llbracket B\psi\rrbracket$ and $\mathsf{Coh}(\llbracket B\psi\rrbracket)$. The remaining case is $\llbracket \mathsf{V}_x(A\psi, B\psi, E\psi)\rrbracket(\mathsf{Vin}_x(M, N), \mathsf{Vin}_x(M', N'))$, in which $N \doteq N' \in B\psi\ [\Psi']$, $M \doteq M' \in A\psi\ [\Psi' \mid x = 0]$, and $\mathsf{app}(\mathsf{fst}(E\psi), M) \doteq N \in B\psi\ [\Psi' \mid x = 0]$. Again we proceed by cases on the first step taken by the ψ_1 and $\psi_1\psi_2$ instances of the left side.

1. $x\psi_1 = 0$.

   Then $L\psi_1 \longmapsto_{\square} M\psi_1$, $R\psi_1 \longmapsto_{\square} M'\psi_1$, and the result follows by $\llbracket \mathsf{V}_0(A\psi\psi_1, \dots)\rrbracket = \llbracket A\psi\psi_1\rrbracket$ and $M\psi_1 \doteq M'\psi_1 \in A\psi\psi_1\ [\Psi_1]$.

2. $x\psi_1 = 1$.

   Then $L\psi_1 \longmapsto_{\square} N\psi_1$, $R\psi_1 \longmapsto_{\square} N'\psi_1$, and the result follows by $\llbracket \mathsf{V}_1(A\psi\psi_1, \dots)\rrbracket = \llbracket B\psi\psi_1\rrbracket$ and $N \doteq N' \in B\psi\ [\Psi']$.

3. $x\psi_1 = x'$ and $x\psi_1\psi_2 = 0$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2 \longmapsto M\psi_1\psi_2$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2 \longmapsto M'\psi_1\psi_2$, and the result follows by $\llbracket \mathsf{V}_0(A\psi\psi_1\psi_2, \dots)\rrbracket = \llbracket A\psi\psi_1\psi_2\rrbracket$ and $M\psi_1\psi_2 \doteq M'\psi_1\psi_2 \in A\psi\psi_1\psi_2\ [\Psi_2]$.

4. $x\psi_1 = x'$ and $x\psi_1\psi_2 = 1$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2 \longmapsto N\psi_1\psi_2$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2 \longmapsto N'\psi_1\psi_2$, and the result follows by $\llbracket \mathsf{V}_1(A\psi\psi_1\psi_2, \dots)\rrbracket = \llbracket B\psi\psi_1\psi_2\rrbracket$ and $N \doteq N' \in B\psi\ [\Psi']$.

5. $x\psi_1 = x'$ and $x\psi_1\psi_2 = x''$.

   Then $L\psi_1\ \mathsf{val}$, $L\psi_1\psi_2\ \mathsf{val}$, $R\psi_1\ \mathsf{val}$, $R\psi_1\psi_2\ \mathsf{val}$, and by $N\psi_1\psi_2 \doteq N'\psi_1\psi_2 \in B\psi\psi_1\psi_2\ [\Psi_2]$, $M\psi_1\psi_2 \doteq M'\psi_1\psi_2 \in A\psi\psi_1\psi_2\ [\Psi_2 \mid x'' = 0]$, and $\mathsf{app}(\mathsf{fst}(E\psi\psi_1\psi_2), M\psi_1\psi_2) \doteq N\psi_1\psi_2 \in B\psi\ [\Psi_2 \mid x'' = 0]$, $\llbracket \mathsf{V}_{x''}(A\psi\psi_1\psi_2, \dots)\rrbracket(\mathsf{Vin}_{x''}(M\psi_1\psi_2, N\psi_1\psi_2), \mathsf{Vin}_{x''}(M'\psi_1\psi_2, N'\psi_1\psi_2))$. $\square$

Rule 47 (Introduction)**.**

1. *If* $M \in A\ [\Psi]$ *then* $\mathsf{Vin}_0(M,N) \doteq M \in A\ [\Psi]$.

2. *If* $N \in B\ [\Psi]$ *then* $\mathsf{Vin}_1(M,N) \doteq N \in B\ [\Psi]$.

3. *If* $M \doteq M' \in A\ [\Psi \mid r=0]$, $N \doteq N' \in B\ [\Psi]$, $E \in \mathsf{Equiv}(A,B)\ [\Psi \mid r=0]$, *and* $\mathsf{app}(\mathsf{fst}(E),M) \doteq N \in B\ [\Psi \mid r=0]$, *then* $\mathsf{Vin}_r(M,N) \doteq \mathsf{Vin}_r(M',N') \in \mathsf{V}_r(A,B,E)\ [\Psi]$.

Proof. Parts (1–2) are immediate by $\mathsf{Vin}_0(M,N) \longmapsto_{\square} M$, $\mathsf{Vin}_1(M,N) \longmapsto_{\square} N$, and Lemma 43. For part (3), if $r=0$ (resp., $r=1$) the result follows by part (1) (resp., part (2)) and Rule 46. If $r=x$ then it follows by $\mathsf{Coh}(\llbracket \mathsf{V}_x(A,B,E) \rrbracket)$ and the definition of $\llbracket \mathsf{V}_x(A,B,E) \rrbracket$. $\square$

Rule 48 (Elimination)**.**

1. *If* $M \in A\ [\Psi]$ *and* $F \in A \to B\ [\Psi]$, *then* $\mathsf{Vproj}_0(M,F) \doteq \mathsf{app}(F,M) \in B\ [\Psi]$.

2. *If* $M \in B\ [\Psi]$ *then* $\mathsf{Vproj}_1(M,F) \doteq M \in B\ [\Psi]$.

3. *If* $M \doteq M' \in \mathsf{V}_r(A,B,E)\ [\Psi]$ *and* $F \doteq \mathsf{fst}(E) \in A \to B\ [\Psi \mid r=0]$, *then* $\mathsf{Vproj}_r(M,F) \doteq \mathsf{Vproj}_r(M',\mathsf{fst}(E)) \in B\ [\Psi]$.

Proof. Parts (1–2) are immediate by $\mathsf{Vproj}_0(M,F) \longmapsto_{\square} \mathsf{app}(F,M)$, $\mathsf{Vproj}_1(M,F) \longmapsto_{\square} M$, and Lemma 43. For part (3), if $r=0$ (resp., $r=1$) the result follows by part (1) (resp., part (2)), Rule 3, and Rule 46. If $r=x$ then we apply coherent expansion to the left side with family

$$\begin{cases} \mathsf{app}(F\psi, M\psi) & x\psi = 0 \\ M\psi & x\psi = 1 \\ N_\psi & x\psi = x',\ M\psi \Downarrow \mathsf{Vin}_{x'}(O_\psi, N_\psi) \end{cases}$$

where $O_\psi \in A\psi\ [\Psi' \mid x'=0]$, $N_\psi \in B\psi\ [\Psi']$, and $\mathsf{app}(\mathsf{fst}(E\psi),O_\psi) \doteq N_\psi \in B\psi\ [\Psi' \mid x'=0]$. First, show that if $x\psi=0$, $\mathsf{app}(F\psi,M\psi) \doteq (N_{\mathsf{id}_\Psi})\psi \in B\psi\ [\Psi']$. By Lemma 38, $M \doteq \mathsf{Vin}_x(O_{\mathsf{id}_\Psi}, N_{\mathsf{id}_\Psi}) \in \mathsf{V}_x(A,B,E)\ [\Psi]$, so by Rule 47, $M\psi \doteq (O_{\mathsf{id}_\Psi})\psi \in A\psi\ [\Psi']$. By assumption, $F\psi \doteq \mathsf{fst}(E\psi) \in A\psi \to B\psi\ [\Psi']$. This case is completed by Rule 3 and $\mathsf{app}(\mathsf{fst}(E\psi),(O_{\mathsf{id}_\Psi})\psi) \doteq (N_{\mathsf{id}_\Psi})\psi \in B\psi\ [\Psi']$. Next, show that if $x\psi=1$, $M\psi \doteq (N_{\mathsf{id}_\Psi})\psi \in B\psi\ [\Psi']$. This case is immediate by Rule 47 and $M \doteq \mathsf{Vin}_x(O_{\mathsf{id}_\Psi}, N_{\mathsf{id}_\Psi}) \in \mathsf{V}_x(A,B,E)\ [\Psi]$ under ψ. Finally, show that if $x\psi = x'$, $N_\psi \doteq (N_{\mathsf{id}_\Psi})\psi \in B\psi\ [\Psi']$. By $M \in \mathsf{V}_x(A,B,E)\ [\Psi]$ under id_Ψ,ψ we have $\llbracket \mathsf{V}_x(A,B,E) \rrbracket_\psi(\mathsf{Vin}_{x'}(O_\psi,N_\psi), \mathsf{Vin}_{x'}((O_{\mathsf{id}_\Psi})\psi,(N_{\mathsf{id}_\Psi})\psi))$, completing this case.

By Lemma 41 we conclude $\mathsf{Vproj}_x(M,F) \doteq N_{\mathsf{id}_\Psi} \in B\ [\Psi]$, and by a symmetric argument, $\mathsf{Vproj}_x(M',\mathsf{fst}(E)) \doteq N'_{\mathsf{id}_\Psi} \in B\ [\Psi]$. We complete the proof with transitivity and $N_{\mathsf{id}_\Psi} \doteq N'_{\mathsf{id}_\Psi} \in B\ [\Psi]$ by $\llbracket \mathsf{V}_x(A,B,E) \rrbracket(\mathsf{Vin}_x(O_{\mathsf{id}_\Psi},N_{\mathsf{id}_\Psi}), \mathsf{Vin}_x(O'_{\mathsf{id}_\Psi},N'_{\mathsf{id}_\Psi}))$. $\square$

Rule 49 (Computation)**.** *If* $M \in A\ [\Psi \mid r=0]$, $N \in B\ [\Psi]$, $F \in A \to B\ [\Psi \mid r=0]$, *and* $\mathsf{app}(F,M) \doteq N \in B\ [\Psi \mid r=0]$, *then* $\mathsf{Vproj}_r(\mathsf{Vin}_r(M,N),F) \doteq N \in B\ [\Psi]$.

Proof. If $r=0$ then by Lemma 43 it suffices to show $\mathsf{app}(F,\mathsf{Vin}_0(M,N)) \doteq N \in B\ [\Psi]$; by Rules 3 and 47 this holds by our hypothesis $\mathsf{app}(F,M) \doteq N \in B\ [\Psi]$. If $r=1$ the result is immediate by Lemma 43. If $r=x$ we apply coherent expansion to the left side with family

$$\begin{cases} \mathsf{app}(F\psi, \mathsf{Vin}_0(M\psi, N\psi)) & x\psi = 0 \\ N\psi & x\psi = 1 \text{ or } x\psi = x' \end{cases}$$

If $x\psi = 0$ then $\mathsf{app}(F\psi, \mathsf{Vin}_0(M\psi, N\psi)) \doteq N\psi \in B\psi\ [\Psi']$ by Rules 3 and 47 and $\mathsf{app}(F, M) \doteq N \in B\ [\Psi \mid x = 0]$. If $x\psi \neq 0$ then $N\psi \in B\psi\ [\Psi']$ and the result follows by Lemma 41. □

Rule 50 (Eta). *If* $N \in \mathsf{V}_r(A, B, E)\ [\Psi]$ *and* $M \doteq N \in A\ [\Psi \mid r = 0]$*, then* $\mathsf{Vin}_r(M, \mathsf{Vproj}_r(N, \mathsf{fst}(E))) \doteq N \in \mathsf{V}_r(A, B, E)\ [\Psi]$.

Proof. If $r = 0$ or $r = 1$ the result is immediate by Lemma 43 and Rule 46. If $r = x$ then by Lemma 38, $N \doteq \mathsf{Vin}_x(M', P') \in \mathsf{V}_x(A, B, E)\ [\Psi]$ where $M' \in A\ [\Psi \mid x = 0]$, $P' \in B\ [\Psi]$, and $\mathsf{app}(\mathsf{fst}(E), M') \doteq P' \in B\ [\Psi \mid x = 0]$. By Rule 47 it suffices to show that $M \doteq M' \in A\ [\Psi \mid x = 0]$, $\mathsf{Vproj}_x(N, \mathsf{fst}(E)) \doteq P' \in B\ [\Psi]$, and $\mathsf{app}(\mathsf{fst}(E), M') \doteq P' \in B\ [\Psi \mid x = 0]$ (which is immediate). To show $M \doteq M' \in A\ [\Psi \mid x = 0]$ it suffices to prove $N \doteq M' \in A\ [\Psi \mid x = 0]$, which follows from $N \doteq \mathsf{Vin}_x(M', P') \in \mathsf{V}_x(A, B, E)\ [\Psi]$ and Rules 46 and 47. To show $\mathsf{Vproj}_x(N, \mathsf{fst}(E)) \doteq P' \in B\ [\Psi]$, by Rule 48 it suffices to check $\mathsf{Vproj}_x(\mathsf{Vin}_x(M', P'), \mathsf{fst}(E)) \doteq P' \in B\ [\Psi]$, which holds by Rule 49. □

Lemma 52. *If* $A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid x = 0]$*,* $B \doteq B'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$*,* $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid x = 0]$*,*

1. $\overrightarrow{\xi_i} = \overrightarrow{r_i = r'_i}$ *is valid,*
2. $M \doteq M' \in \mathsf{V}_x(A, B, E)\ [\Psi]$*,*
3. $N_i \doteq N'_j \in \mathsf{V}_x(A, B, E)\ [\Psi, y \mid r_i = r'_i, r_j = r'_j]$ *for any* i, j*, and*
4. $N_i\langle r/y\rangle \doteq M \in \mathsf{V}_x(A, B, E)\ [\Psi \mid r_i = r'_i]$ *for any* i*,*

then

1. $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{V}_x(A,B,E)}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{V}_x(A',B',E')}(M'; \overrightarrow{\xi_i \hookrightarrow y.N'_i}) \in \mathsf{V}_x(A, B, E)\ [\Psi]$*;*
2. *if* $r = r'$ *then* $\mathsf{hcom}^{r\rightsquigarrow r}_{\mathsf{V}_x(A,B,E)}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq M \in \mathsf{V}_x(A, B, E)\ [\Psi]$*; and*
3. *if* $r_i = r'_i$ *then* $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{V}_x(A,B,E)}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in \mathsf{V}_x(A, B, E)\ [\Psi]$.

Proof. For part (1), apply coherent expansion to $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{V}_x(A,B,E)}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i})$ with family

$$\begin{cases}
\mathsf{hcom}^{r\psi\rightsquigarrow r'\psi}_{A\psi}(M\psi; \overrightarrow{\xi_i\psi \hookrightarrow y.N_i\psi}) & x\psi = 0 \\
\mathsf{hcom}^{r\psi\rightsquigarrow r'\psi}_{B\psi}(M\psi; \overrightarrow{\xi_i\psi \hookrightarrow y.N_i\psi}) & x\psi = 1 \\
(\mathsf{Vin}_x(O\langle r'/y\rangle, \mathsf{hcom}^{r\rightsquigarrow r'}_{B}(\mathsf{Vproj}_x(M, \mathsf{fst}(E)); \overrightarrow{T})))\psi & x\psi = x' \\
\quad O = \mathsf{hcom}^{r\rightsquigarrow y}_{A}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) & \\
\quad \overrightarrow{T} = \overrightarrow{\xi_i \hookrightarrow y.\mathsf{Vproj}_x(N_i, \mathsf{fst}(E))}, & \\
\qquad\quad x = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E), O), & \\
\qquad\quad x = 1 \hookrightarrow y.\mathsf{hcom}^{r\rightsquigarrow y}_{B}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) &
\end{cases}$$

Consider $\psi = \mathsf{id}_\Psi$. Using rules for dependent functions, dependent types, and univalence:

1. $O \in A\ [\Psi, y \mid x = 0]$ and $O\langle r/y\rangle \doteq M \in A\ [\Psi \mid x = 0]$ (by $\mathsf{V}_x(A, B, E) \doteq A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi \mid x = 0]$).
2. $\mathsf{Vproj}_x(M, \mathsf{fst}(E)) \in B\ [\Psi]$ where $\mathsf{Vproj}_x(M, \mathsf{fst}(E)) \doteq \mathsf{app}(\mathsf{fst}(E), M) \in B\ [\Psi \mid x = 0]$ and $\mathsf{Vproj}_x(M, \mathsf{fst}(E)) \doteq M \in B\ [\Psi \mid x = 1]$.

3. $\mathsf{Vproj}_x(N_i, \mathsf{fst}(E)) \doteq \mathsf{Vproj}_x(N_j, \mathsf{fst}(E)) \in B\ [\Psi, y \mid r_i = r_i', r_j = r_j']$ and $\mathsf{Vproj}_x(M, \mathsf{fst}(E)) \doteq \mathsf{Vproj}_x(N_i\langle r/y\rangle, \mathsf{fst}(E)) \in B\ [\Psi \mid r_i = r_i']$.

4. $\mathsf{app}(\mathsf{fst}(E), O) \in B\ [\Psi, y \mid x = 0]$, $\mathsf{app}(\mathsf{fst}(E), O) \doteq \mathsf{Vproj}_x(N_i, \mathsf{fst}(E)) \in B\ [\Psi, y \mid x = 0, r_i = r_i']$ (both equal $\mathsf{app}(\mathsf{fst}(E), N_i)$), and $\mathsf{app}(\mathsf{fst}(E), O\langle r/y\rangle) \doteq \mathsf{Vproj}_x(M, \mathsf{fst}(E)) \in B\ [\Psi \mid x = 0]$ (both equal $\mathsf{app}(\mathsf{fst}(E), M)$).

5. $\mathsf{hcom}_B^{r \rightsquigarrow y}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \in B\ [\Psi, y \mid x = 1]$ (by $\mathsf{V}_x(A, B, E) \doteq B\ \mathsf{type_{pre}}\ [\Psi \mid x = 1]$), $\mathsf{hcom}_B^{r \rightsquigarrow y}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{Vproj}_x(N_i, \mathsf{fst}(E)) \in B\ [\Psi, y \mid x = 1, r_i = r_i']$ (both equal N_i), and $\mathsf{hcom}_B^{r \rightsquigarrow r}(M; \overrightarrow{\xi_i \hookrightarrow y.N_i}) \doteq \mathsf{Vproj}_x(M, \mathsf{fst}(E)) \in B\ [\Psi \mid x = 1]$ (both equal M).

6. By the above, $\mathsf{hcom}_B^{r \rightsquigarrow r'}(\mathsf{Vproj}_x(M, \mathsf{fst}(E)); \overrightarrow{T}) \in B\ [\Psi]$ and $\mathsf{hcom}_B \doteq \mathsf{app}(\mathsf{fst}(E), O\langle r'/y\rangle) \in B\ [\Psi \mid x = 0]$, so $\mathsf{Vin}_x(O\langle r'/y\rangle, \mathsf{hcom}_B^{r \rightsquigarrow r'}(\mathsf{Vproj}_x(M, \mathsf{fst}(E)); \overrightarrow{T})) \in \mathsf{V}_x(A, B, E)\ [\Psi]$.

When $x\psi = x'$, coherence is immediate. When $x\psi = 0$, $\mathsf{Vin}_0(O\langle r'\psi/y\rangle, \dots) \doteq \mathsf{hcom}_{A\psi} \in A\psi\ [\Psi']$ as required. When $x\psi = 1$, $\mathsf{Vin}_1(\dots, \mathsf{hcom}_{B\psi}^{r\psi \rightsquigarrow r'\psi}(\dots; \overrightarrow{T})) \doteq \mathsf{hcom}_{B\psi} \in B\psi\ [\Psi']$ as required. Therefore Lemma 41 applies, and part (1) follows by repeating this argument on the right side.

For part (2), show that $\mathsf{Vin}_x(O\langle r'/y\rangle, \mathsf{hcom}_B^{r \rightsquigarrow r'}(\mathsf{Vproj}_x(M, \mathsf{fst}(E)); \overrightarrow{T})) \doteq M \in \mathsf{V}_x(A, B, E)\ [\Psi]$ when $r = r'$. By the above, $\mathsf{Vin}_x(\dots) \doteq \mathsf{Vin}_x(M, \mathsf{Vproj}_x(M, \mathsf{fst}(E))) \in \mathsf{V}_x(A, B, E)\ [\Psi]$, so the result follows by Rule 50.

For part (3), show $\mathsf{Vin}_x(O\langle r'/y\rangle, \mathsf{hcom}_B^{r \rightsquigarrow r'}(\mathsf{Vproj}_x(M, \mathsf{fst}(E)); \overrightarrow{T})) \doteq N_i\langle r'/y\rangle \in \mathsf{V}_x(A, B, E)\ [\Psi]$ when $r_i = r_i'$. By the above, $\mathsf{Vin}_x(\dots) \doteq \mathsf{Vin}_x(N_i\langle r'/y\rangle, \mathsf{Vproj}_x(N_i\langle r'/y\rangle, \mathsf{fst}(E))) \in \mathsf{V}_x(A, B, E)\ [\Psi]$, so the result again follows by Rule 50. $\square$

Lemma 53. *If $A \doteq A'\ \mathsf{type_{Kan}}\ [\Psi, y \mid x = 0]$, $B \doteq B'\ \mathsf{type_{Kan}}\ [\Psi, y]$, $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi, y \mid x = 0]$, and $M \doteq M' \in (\mathsf{V}_x(A, B, E))\langle r/y\rangle\ [\Psi]$ for $x \neq y$, then $\mathsf{coe}_{y.\mathsf{V}_x(A,B,E)}^{r \rightsquigarrow r'}(M) \doteq \mathsf{coe}_{y.\mathsf{V}_x(A',B',E')}^{r \rightsquigarrow r'}(M') \in (\mathsf{V}_x(A, B, E))\langle r'/y\rangle\ [\Psi]$ and $\mathsf{coe}_{y.\mathsf{V}_x(A,B,E)}^{r \rightsquigarrow r}(M) \doteq M \in (\mathsf{V}_x(A, B, E))\langle r/y\rangle\ [\Psi]$.*

Proof. We apply coherent expansion to $\mathsf{coe}_{y.\mathsf{V}_x(A,B,E)}^{r \rightsquigarrow r'}(M)$ with family

$$\begin{cases} \mathsf{coe}_{y.A\psi}^{r\psi \rightsquigarrow r'\psi}(M\psi) & x\psi = 0 \\ \mathsf{coe}_{y.B\psi}^{r\psi \rightsquigarrow r'\psi}(M\psi) & x\psi = 1 \\ (\mathsf{Vin}_x(\mathsf{coe}_{y.A}^{r \rightsquigarrow r'}(M), \mathsf{com}_{y.B}^{r \rightsquigarrow r'}(\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)); \overrightarrow{T})))\psi & x\psi = x' \\ \quad \overrightarrow{T} = x = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E), \mathsf{coe}_{y.A}^{r \rightsquigarrow y}(M)), \\ \qquad\quad x = 1 \hookrightarrow y.\mathsf{coe}_{y.B}^{r \rightsquigarrow y}(M) \end{cases}$$

Consider $\psi = \mathsf{id}_\Psi$.

1. $\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)) \in B\langle r/y\rangle\ [\Psi]$ (by $M \in \mathsf{V}_x(A\langle r/y\rangle, \dots)\ [\Psi]$), $\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)) \doteq \mathsf{app}(\mathsf{fst}(E\langle r/y\rangle), M) \in B\langle r/y\rangle\ [\Psi \mid x = 0]$, and $\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)) \doteq M \in B\langle r/y\rangle\ [\Psi \mid x = 1]$.

2. $\mathsf{app}(\mathsf{fst}(E), \mathsf{coe}_{y.A}^{r \rightsquigarrow y}(M)) \in B\ [\Psi, y \mid x = 0]$ because $\mathsf{fst}(E) \in A \to B\ [\Psi, y \mid x = 0]$ and $\mathsf{coe}_{y.A}^{r \rightsquigarrow y}(M) \in A\ [\Psi, y \mid x = 0]$ (by $M \in A\langle r/y\rangle\ [\Psi \mid x = 0]$). Under $\langle r/y\rangle$ this $\doteq \mathsf{app}(\mathsf{fst}(E\langle r/y\rangle), M) \in B\langle r/y\rangle\ [\Psi \mid x = 0]$.

3. $\mathsf{coe}^{r\rightsquigarrow y}_{y.B}(M) \in B\ [\Psi, y \mid x = 1]$ (by $M \in B\langle r/y\rangle\ [\Psi \mid x = 1]$) and $\mathsf{coe}^{r\rightsquigarrow r}_{y.B}(M) \doteq M \in B\langle r/y\rangle\ [\Psi \mid x = 1]$.

4. Therefore $\mathsf{com}_{y.B} \in B\langle r'/y\rangle\ [\Psi]$, $\mathsf{com}_{y.B} \doteq \mathsf{app}(\mathsf{fst}(E\langle r'/y\rangle), \mathsf{coe}^{r\rightsquigarrow r'}_{y.A}(M)) \in B\langle r'/y\rangle\ [\Psi \mid x = 0]$, and $\mathsf{com}_{y.B} \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{y.B}(M) \in B\langle r'/y\rangle\ [\Psi \mid x = 1]$. It follows that $\mathsf{Vin}_x(\dots) \in \mathsf{V}_x(A\langle r'/y\rangle, B\langle r'/y\rangle, E\langle r'/y\rangle)\ [\Psi]$.

When $x\psi = x'$, coherence is immediate. When $x\psi = 0$, we have $\mathsf{Vin}_0(\mathsf{coe}^{r\psi\rightsquigarrow r'\psi}_{y.A\psi}(M\psi), \dots) \doteq \mathsf{coe}^{r\psi\rightsquigarrow r'\psi}_{y.A\psi}(M\psi) \in A\psi\langle r'\psi/y\rangle\ [\Psi']$. When $x\psi = 1$, $\mathsf{Vin}_1(\dots) \doteq \mathsf{coe}^{r\psi\rightsquigarrow r'\psi}_{y.B\psi}(M\psi) \in B\psi\langle r'\psi/y\rangle\ [\Psi']$. Therefore Lemma 41 applies, and the first part follows by the same argument on the right side.

For the second part, $\mathsf{coe}^{r\rightsquigarrow r}_{y.\mathsf{V}_x(A,B,E)}(M) \doteq \mathsf{Vin}_x(\mathsf{coe}^{r\rightsquigarrow r}_{y.A}(M), \mathsf{com}^{r\rightsquigarrow r}_{y.B}(\mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)); \overrightarrow{T})) \in (\mathsf{V}_x(A, B, E))\langle r/y\rangle\ [\Psi]$, which equals $\mathsf{Vin}_x(M, \mathsf{Vproj}_x(M, \mathsf{fst}(E\langle r/y\rangle)))$ and M by Rule 50. □

Lemma 54. *If $A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x \mid x = 0]$, $B \doteq B'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x]$, $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi, x \mid x = 0]$, and $M \doteq M' \in (\mathsf{V}_x(A, B, E))\langle 0/x\rangle\ [\Psi]$, then $\mathsf{coe}^{0\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M) \doteq \mathsf{coe}^{0\rightsquigarrow r'}_{x.\mathsf{V}_x(A',B',E')}(M') \in (\mathsf{V}_x(A, B, E))\langle r'/x\rangle\ [\Psi]$ and $\mathsf{coe}^{0\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M) \doteq M \in (\mathsf{V}_x(A, B, E))\langle 0/x\rangle\ [\Psi]$.*

Proof. By Lemma 43 on both sides, it suffices to show (the binary version of)

$$\mathsf{Vin}_{r'}(M, \mathsf{coe}^{0\rightsquigarrow r'}_{x.B}(\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), M))) \in (\mathsf{V}_x(A, B, E))\langle r'/x\rangle\ [\Psi].$$

By Rule 46, $M \in A\langle 0/x\rangle\ [\Psi]$, so $\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), M) \in B\langle 0/x\rangle\ [\Psi]$ and $\mathsf{coe}^{0\rightsquigarrow r'}_{x.B}(\dots) \in B\langle r'/x\rangle\ [\Psi]$. Then $M \in A\langle r'/x\rangle\ [\Psi \mid r' = 0]$ and $\mathsf{coe}^{0\rightsquigarrow r'}_{x.B}(\dots) \doteq \mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), M) \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$ so the first part follows by Rule 47. When $r' = 0$, $\mathsf{Vin}_0(M, \dots) \doteq M \in (\mathsf{V}_x(A, B, E))\langle 0/x\rangle\ [\Psi]$, completing the second part. □

Lemma 55. *If $A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x \mid x = 0]$, $B \doteq B'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x]$, $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi, x \mid x = 0]$, and $N \doteq N' \in (\mathsf{V}_x(A, B, E))\langle 1/x\rangle\ [\Psi]$, then $\mathsf{coe}^{1\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(N) \doteq \mathsf{coe}^{1\rightsquigarrow r'}_{x.\mathsf{V}_x(A',B',E')}(N') \in (\mathsf{V}_x(A, B, E))\langle r'/x\rangle\ [\Psi]$ and $\mathsf{coe}^{1\rightsquigarrow 1}_{x.\mathsf{V}_x(A,B,E)}(N) \doteq N \in (\mathsf{V}_x(A, B, E))\langle 1/x\rangle\ [\Psi]$.*

Proof. By Lemma 43 on both sides, it suffices to show (the binary version of) $\mathsf{Vin}_{r'}(\mathsf{fst}(O), P) \in (\mathsf{V}_x(A, B, E))\langle r'/x\rangle\ [\Psi]$ where

$$O = \mathsf{fst}(\mathsf{app}(\mathsf{snd}(E\langle r'/x\rangle), \mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N)))$$
$$P = \mathsf{hcom}^{1\rightsquigarrow 0}_{B\langle r'/x\rangle}(\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N); r' = 0 \hookrightarrow y.\mathsf{snd}(O)@y, r' = 1 \hookrightarrow \_.\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N)).$$

By Rule 46, $N \in B\langle 1/x\rangle\ [\Psi]$, so $\mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N) \in B\langle r'/x\rangle\ [\Psi]$ and

$$O \in (a{:}A\langle r'/x\rangle) \times \mathsf{Path}_{\_.B\langle r'/x\rangle}(\mathsf{app}(\mathsf{fst}(E\langle r'/x\rangle), a), \mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N))\ [\Psi].$$

Therefore $\mathsf{snd}(O)@y \in B\langle r'/x\rangle\ [\Psi, y \mid r' = 0]$ and $\mathsf{snd}(O)@1 \doteq \mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N) \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$, so by $B\langle r'/x\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$, $P \in B\langle r'/x\rangle\ [\Psi]$. We also have $\mathsf{fst}(O) \in A\langle r'/x\rangle\ [\Psi \mid r' = 0]$ and $\mathsf{app}(\mathsf{fst}(E\langle r'/x\rangle), \mathsf{fst}(O)) \doteq P \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$ (by $\mathsf{snd}(O)@0 \doteq \mathsf{app}(\mathsf{fst}(E\langle r'/x\rangle), \mathsf{fst}(O)) \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$) so the first part follows by Rule 47. When $r' = 1$, $\mathsf{Vin}_1(\mathsf{fst}(O), P) \doteq P \in (\mathsf{V}_x(A, B, E))\langle 1/x\rangle\ [\Psi]$, but $P \doteq \mathsf{coe}^{1\rightsquigarrow r'}_{x.B}(N) \doteq N$, completing the second part. □

Lemma 56. *If* $A \doteq A'$ $\mathsf{type_{Kan}}$ $[\Psi, x \mid x = 0]$, $B \doteq B'$ $\mathsf{type_{Kan}}$ $[\Psi, x]$, $E \doteq E' \in \mathsf{Equiv}(A, B)$ $[\Psi, x \mid x = 0]$, *and* $M \doteq M' \in (\mathsf{V}_x(A, B, E))\langle y/x\rangle$ $[\Psi]$, *then* $\mathsf{coe}^{y\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M) \doteq \mathsf{coe}^{y\rightsquigarrow r'}_{x.\mathsf{V}_x(A',B',E')}(M') \in (\mathsf{V}_x(A, B, E))\langle r'/x\rangle$ $[\Psi]$ *and* $\mathsf{coe}^{y\rightsquigarrow y}_{x.\mathsf{V}_x(A,B,E)}(M) \doteq M \in (\mathsf{V}_x(A, B, E))\langle y/x\rangle$ $[\Psi]$.

Proof. We apply coherent expansion to $\mathsf{coe}^{y\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M)$ with the family $\mathsf{coe}^{\varepsilon\rightsquigarrow r'\psi}_{x.\mathsf{V}_x(A\psi,B\psi,E\psi)}(M\psi)$ when $y\psi = \varepsilon$ and $(\mathsf{Vin}_{r'}(\mathsf{fst}(R), \mathsf{hcom}^{1\rightsquigarrow 0}_{B\langle r'/x\rangle}(P\langle r'/x\rangle; \overrightarrow{T})))\psi$ otherwise, where

$$O_\varepsilon = \mathsf{Vproj}_w(\mathsf{coe}^{\varepsilon\rightsquigarrow w}_{x.\mathsf{V}_x(A,B,E)}(M), \mathsf{fst}(E\langle w/x\rangle))$$
$$P = \mathsf{com}^{y\rightsquigarrow x}_{x.B}(\mathsf{Vproj}_y(M, \mathsf{fst}(E\langle y/x\rangle)); \overrightarrow{y = \varepsilon \hookrightarrow w.O_\varepsilon})$$
$$Q_\varepsilon[a] = \langle \mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a), \langle z\rangle \mathsf{com}^{\varepsilon\rightsquigarrow y}_{y.B\langle 0/x\rangle}(P\langle 0/x\rangle\langle \varepsilon/y\rangle; \overrightarrow{U})\rangle$$
$$\overrightarrow{U} = z = 0 \hookrightarrow y.\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), \mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a)), z = 1 \hookrightarrow y.P\langle 0/x\rangle$$
$$R = \mathsf{app}(\mathsf{app}(\mathsf{snd}(\mathsf{app}(\mathsf{snd}(E\langle 0/x\rangle), P\langle 0/x\rangle)), Q_0[M\langle 0/y\rangle]), Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle])@y$$
$$\overrightarrow{T} = \overrightarrow{y = \varepsilon \hookrightarrow \_.O_\varepsilon\langle r'/w\rangle}, y = r' \hookrightarrow \_.\mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)), r' = 0 \hookrightarrow z.\mathsf{snd}(R)@z.$$

Consider $\psi = \mathsf{id}_\Psi$.

1. $O_\varepsilon \in B\langle w/x\rangle$ $[\Psi, w \mid y = \varepsilon]$ by $\mathsf{coe}^{\varepsilon\rightsquigarrow w}_{x.\mathsf{V}_x(A,B,E)}(M) \in \mathsf{V}_w(A\langle w/x\rangle, \dots)$ $[\Psi, w \mid y = \varepsilon]$ (by $M \in \mathsf{V}_y(A\langle y/x\rangle, \dots)$ $[\Psi]$) and $\mathsf{fst}(E\langle w/x\rangle) \in A\langle w/x\rangle \to B\langle w/x\rangle$ $[\Psi, w \mid w = 0]$.

2. $P \in B$ $[\Psi, x]$ by $\mathsf{Vproj}_y(M, \mathsf{fst}(E\langle y/x\rangle)) \in B\langle y/x\rangle$ $[\Psi]$ and $O_\varepsilon\langle y/w\rangle \doteq \mathsf{Vproj}_y(M, \mathsf{fst}(E\langle y/x\rangle)) \in B\langle y/x\rangle$ $[\Psi \mid y = \varepsilon]$.

3. Let $C = (a' {:} A\langle 0/x\rangle) \times \mathsf{Path}_{\_.B\langle 0/x\rangle}(\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), a'), P\langle 0/x\rangle)$. Then $Q_\varepsilon[a] \in C$ $[\Psi]$ for any $a \in A\langle 0/x\rangle\langle \varepsilon/y\rangle$ $[\Psi]$ with $y \# a$ and $P\langle 0/x\rangle\langle \varepsilon/y\rangle \doteq \mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle\langle \varepsilon/y\rangle), a) \in B\langle 0/x\rangle\langle \varepsilon/y\rangle$ $[\Psi]$, because $\mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a) \in A\langle 0/x\rangle$ $[\Psi]$ and by

   (a) $P\langle 0/x\rangle\langle \varepsilon/y\rangle \in B\langle 0/x\rangle\langle \varepsilon/y\rangle$ $[\Psi]$,
   (b) $\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), \mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a)) \in B\langle 0/x\rangle$ $[\Psi]$,
   (c) $P\langle 0/x\rangle\langle \varepsilon/y\rangle \doteq \mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle\langle \varepsilon/y\rangle), \mathsf{coe}^{\varepsilon\rightsquigarrow \varepsilon}_{y.A\langle 0/x\rangle}(a)) \in B\langle 0/x\rangle\langle \varepsilon/y\rangle$ $[\Psi]$, and
   (d) $P\langle 0/x\rangle \in B\langle 0/x\rangle$ $[\Psi]$,

   we have $\langle z\rangle\mathsf{com} \in \mathsf{Path}_{\_.B\langle 0/x\rangle}(\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle), \mathsf{coe}^{\varepsilon\rightsquigarrow y}_{y.A\langle 0/x\rangle}(a)), P\langle 0/x\rangle)$ $[\Psi]$.

4. $Q_0[M\langle 0/y\rangle] \in C$ $[\Psi]$ because $M\langle 0/y\rangle \in A\langle 0/x\rangle\langle 0/y\rangle$ $[\Psi]$ and $P\langle 0/x\rangle\langle 0/y\rangle \doteq O_0\langle 0/w\rangle\langle 0/y\rangle \doteq \mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle\langle 0/y\rangle), M\langle 0/y\rangle)$.

5. $Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle] \in C$ $[\Psi]$ because $(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle \in A\langle 0/x\rangle\langle 1/y\rangle$ $[\Psi]$ (by $M\langle 1/y\rangle \in B\langle 1/x\rangle\langle 1/y\rangle$ $[\Psi]$) and $P\langle 0/x\rangle\langle 1/y\rangle \doteq O_1\langle 0/w\rangle\langle 1/y\rangle$ which in turn equals $\mathsf{app}(\mathsf{fst}(E\langle 0/x\rangle\langle 1/y\rangle), (\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle)$.

6. $R \in C$ $[\Psi]$ because $\mathsf{snd}(\mathsf{app}(\mathsf{snd}(E\langle 0/x\rangle), P\langle 0/x\rangle)) \in ((c{:}C) \to (c'{:}C) \to \mathsf{Path}_{\_.C}(c, c'))$ $[\Psi]$ and we further apply this to $Q_0[M\langle 0/y\rangle]$, $Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle]$, and y.

7. $\mathsf{hcom}^{1\rightsquigarrow 0}_{B\langle r'/x\rangle}(P\langle r'/x\rangle; \overrightarrow{T}) \in B\langle r'/x\rangle$ $[\Psi]$ because

(a) $O_\varepsilon\langle r'/w\rangle \in B\langle r'/x\rangle\ [\Psi \mid y = \varepsilon]$,
(b) $\mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)) \in B\langle r'/x\rangle\ [\Psi \mid y = r']$,
(c) $\mathsf{snd}(R)@z \in B\langle r'/x\rangle\ [\Psi, z \mid r' = 0]$ by $\mathsf{snd}(R)@z \in B\langle 0/x\rangle\ [\Psi, z]$,
(d) $P\langle r'/x\rangle \in B\langle r'/x\rangle\ [\Psi]$,
(e) $P\langle r'/x\rangle \doteq O_\varepsilon\langle r'/w\rangle \in B\langle r'/x\rangle\ [\Psi \mid y = \varepsilon]$,
(f) $P\langle r'/x\rangle \doteq \mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)) \in B\langle r'/x\rangle\ [\Psi \mid y = r']$,
(g) $P\langle r'/x\rangle \doteq \mathsf{snd}(R)@1 \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$ by $\mathsf{snd}(R)@1 \doteq P\langle 0/x\rangle \in B\langle 0/x\rangle\ [\Psi]$,
(h) $O_\varepsilon\langle r'/w\rangle \doteq \mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)) \in B\langle r'/x\rangle\ [\Psi \mid y = \varepsilon, y = r']$,
(i) $O_0\langle r'/w\rangle \doteq \mathsf{snd}(R)@z \in B\langle r'/x\rangle\ [\Psi, z \mid y = 0, r' = 0]$ by $\mathsf{snd}(R)@z \doteq \mathsf{snd}(Q_0[M\langle 0/y\rangle])@z \doteq (\langle z\rangle P\langle 0/x\rangle\langle 0/y\rangle)@z \doteq O_0\langle 0/w\rangle$,
(j) $O_1\langle r'/w\rangle \doteq \mathsf{snd}(R)@z \in B\langle r'/x\rangle\ [\Psi, z \mid y = 1, r' = 0]$ because we have $\mathsf{snd}(R)@z \doteq \mathsf{snd}(Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle])@z \doteq (\langle z\rangle P\langle 0/x\rangle\langle 1/y\rangle)@z \doteq O_1\langle 0/w\rangle$, and
(k) $\mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)) \doteq \mathsf{snd}(R)@z \in B\langle r'/x\rangle\ [\Psi, z \mid y = r', r' = 0]$ because $\mathsf{snd}(R)@z \doteq \mathsf{snd}(Q_0[M\langle 0/y\rangle])@z \doteq P\langle 0/x\rangle\langle 0/y\rangle \doteq \mathsf{Vproj}_y(M, \mathsf{fst}(E\langle y/x\rangle))$.

8. $\mathsf{Vin}_{r'}(\mathsf{fst}(R), \mathsf{hcom}_{B\langle r'/x\rangle}) \in \mathsf{V}_{r'}(A\langle r'/x\rangle, \dots)\ [\Psi]$ because $\mathsf{fst}(R) \in A\langle 0/x\rangle\ [\Psi \mid r' = 0]$, $\mathsf{hcom}_{B\langle r'/x\rangle} \in B\langle r'/x\rangle\ [\Psi]$, and $\mathsf{app}(\mathsf{fst}(E\langle r'/x\rangle), \mathsf{fst}(R)) \doteq \mathsf{hcom}_{B\langle r'/x\rangle} \in B\langle r'/x\rangle\ [\Psi \mid r' = 0]$ (by $\mathsf{hcom}_{B\langle r'/x\rangle} \doteq \mathsf{snd}(R)@0$).

When $y\psi = y'$, coherence is immediate. When $y\psi = \varepsilon$, we prove coherence by Rule 50, using $\mathsf{hcom}_{B\langle r'/x\rangle} \doteq \mathsf{Vproj}_{r'}(\mathsf{coe}^{\varepsilon\rightsquigarrow r'}_{x.\mathsf{V}_x(A,B,E)}(M), \mathsf{fst}(E\langle r'/x\rangle)) \in B\langle r'/x\rangle\ [\Psi \mid y = \varepsilon]$ (by $\doteq O_\varepsilon\langle r'/w\rangle$), $\mathsf{fst}(R) \doteq M \in A\langle r'/x\rangle\ [\Psi \mid y = 0, r' = 0]$ (by $\doteq \mathsf{fst}(Q_0[M\langle 0/y\rangle])$), and $\mathsf{fst}(R) \doteq \mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M) \in A\langle r'/x\rangle\ [\Psi \mid y = 1, r' = 0]$ (by $\doteq \mathsf{fst}(Q_1[(\mathsf{coe}^{1\rightsquigarrow 0}_{x.\mathsf{V}_x(A,B,E)}(M))\langle 1/y\rangle])$). Therefore Lemma 41 applies, and the first part follows by the same argument on the right side.

The second part follows by Rule 50, $\mathsf{hcom}_{B\langle r'/x\rangle} \doteq \mathsf{Vproj}_{r'}(M, \mathsf{fst}(E\langle r'/x\rangle)) \in B\langle r'/x\rangle\ [\Psi \mid y = r']$, and $\mathsf{fst}(R) \doteq M \in A\langle r'/x\rangle\ [\Psi \mid y = r', r' = 0]$ (as calculated previously). □

Rule 51 (Kan type formation)**.**

1. *If* $A\ \mathsf{type_{Kan}}\ [\Psi]$ *then* $\mathsf{V}_0(A, B, E) \doteq A\ \mathsf{type_{Kan}}\ [\Psi]$.
2. *If* $B\ \mathsf{type_{Kan}}\ [\Psi]$ *then* $\mathsf{V}_1(A, B, E) \doteq B\ \mathsf{type_{Kan}}\ [\Psi]$.
3. *If* $A \doteq A'\ \mathsf{type_{Kan}}\ [\Psi \mid r = 0]$*,* $B \doteq B'\ \mathsf{type_{Kan}}\ [\Psi]$*, and* $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0]$*, then* $\mathsf{V}_r(A, B, E) \doteq \mathsf{V}_r(A', B', E')\ \mathsf{type_{Kan}}\ [\Psi]$.

Proof. Parts (1–2) follow from Lemma 43. For part (3), we check the Kan conditions.

(hcom) For any $\psi : \Psi' \to \Psi$, consider a valid composition scenario in $\mathsf{V}_{r\psi}(A\psi, B\psi, E\psi)$. If $r\psi = 0$ (resp., 1) then the composition is in $A\psi$ (resp., $B\psi$) and the hcom Kan conditions follow from $A\psi \doteq A'\psi\ \mathsf{type_{Kan}}\ [\Psi']$ (resp., $B\psi \doteq B'\psi\ \mathsf{type_{Kan}}\ [\Psi']$). Otherwise, $r\psi = x$ and the hcom Kan conditions follow from Lemma 52 at $A\psi \doteq A'\psi\ \mathsf{type_{Kan}}\ [\Psi' \mid x = 0]$, $B\psi \doteq B'\psi\ \mathsf{type_{Kan}}\ [\Psi']$, and $E\psi \doteq E'\psi \in \mathsf{Equiv}(A\psi, B\psi)\ [\Psi' \mid x = 0]$.

(coe) Consider any $\psi : (\Psi', x) \to \Psi$ and $M \doteq M' \in (\mathsf{V}_r(A, B, E))\psi\langle s/x\rangle\ [\Psi']$. If $r\psi = 0$ (resp., 1) then the coe Kan conditions follow from $A\psi \doteq A'\psi\ \mathsf{type_{Kan}}\ [\Psi', x]$ (resp., $B\psi \doteq B'\psi\ \mathsf{type_{Kan}}\ [\Psi', x]$). If $r\psi = y \neq x$, then the coe Kan conditions follow from Lemma 53. Otherwise, $r\psi = x$; the result follows from Lemma 54 if $s = 0$, from Lemma 55 if $s = 1$, and from Lemma 56 if $s = y$. □

5.11 Composite types

Unlike the other type formers, fcoms are only pretypes when their constituents are Kan types. (For this reason, in Section 3 we only close τ_i^{pre} under fcoms of types from τ_i^{Kan}.) The results of this section hold in $\tau = \mu^{\mathsf{Kan}}(\nu)$ for any cubical type system ν, and therefore in each τ_i^{pre} as well. In this section, we will say that $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ and $A', \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}$ are *(equal) type compositions* $r \rightsquigarrow r'$ whenever:

1. $\overrightarrow{r_i = r_i'}$ is valid,
2. $A \doteq A'$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$,
3. $B_i \doteq B_j'$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, y \mid r_i = r_i', r_j = r_j']$ for any i, j, and
4. $B_i\langle r/y\rangle \doteq A$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi \mid r_i = r_i']$ for any i.

Lemma 57. *If $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ and $A', \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}$ are equal type compositions $r \rightsquigarrow r'$, then*

1. $\mathsf{PTy}(\tau)(\Psi, \mathsf{fcom}^{r\rightsquigarrow r'}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}), \mathsf{fcom}^{r\rightsquigarrow r'}(A'; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}), \_)$,
2. *if $r = r'$ then* $\mathsf{fcom}^{r\rightsquigarrow r}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq A$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$, *and*
3. *if $r_i = r_i'$ then* $\mathsf{fcom}^{r\rightsquigarrow r'}(A; \overrightarrow{r_i = r_i' \hookrightarrow B_i}) \doteq B_i\langle r'/y\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$.

Proof. Part (1) is precisely the statement of Lemma 48, applied to the context-indexed PER $\{(\Psi, A_0, B_0) \mid \tau(\Psi, A_0, B_0, \_)\}$ instead of $[\![\mathbb{S}^1]\!](\Psi)$; as the fcom structure of these PERs is defined identically, the same proof applies. Part (2) is immediate by Lemma 43. For part (3), if $r = r'$, the result follows by Lemma 43 and $B_i\langle r/y\rangle \doteq A$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi \mid r_i = r_i']$. Otherwise, there is some least j such that $r_j = r_j'$. Apply coherent expansion to the left side with family

$$\begin{cases} A\psi & r\psi = r'\psi \\ B_j\langle r'/y\rangle\psi & r\psi \neq r'\psi,\ r_j\psi = r_j'\psi, \text{ and } \forall k < j, r_k\psi \neq r_k'\psi. \end{cases}$$

If $r\psi = r'\psi$ then $B_j\langle r/y\rangle\psi \doteq A\psi$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi']$. If $r\psi \neq r'\psi$, there is some least k such that $r_k\psi = r_k'\psi$; then $B_j\langle r'/y\rangle\psi \doteq B_k\langle r'/y\rangle\psi$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi']$. By Lemma 42, $\mathsf{fcom} \doteq B_j\langle r'/y\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$, and part (3) follows by $B_i\langle r'/y\rangle \doteq B_j\langle r'/y\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi]$. □

Lemma 58. *If*

1. $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ *is a type composition* $r \rightsquigarrow r'$,
2. $M \doteq M' \in A$ $[\Psi]$,
3. $N_i \doteq N_j' \in B_i\langle r'/y\rangle$ $[\Psi \mid r_i = r_i', r_j = r_j']$ *for any i, j, and*
4. $\mathsf{coe}_{y.B_i}^{r'\rightsquigarrow r}(N_i) \doteq M \in A$ $[\Psi \mid r_i = r_i']$ *for any i,*

then $\mathsf{Tm}([\![\mathsf{fcom}^{r\rightsquigarrow r'}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i})]\!])(\mathsf{box}^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i}), \mathsf{box}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow N_i'}))$.

Proof. We focus on the unary case; the binary case follows similarly. For any $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$ we must show $\mathsf{box}\psi_1 \Downarrow X_1$ and $[\![\mathsf{fcom}^{r\rightsquigarrow r'}(A;\overrightarrow{r_i = r_i' \hookrightarrow y.B_i})]\!]^{\Downarrow}_{\psi_1\psi_2}(X_1\psi_2, \mathsf{box}\psi_1\psi_2)$. We proceed by cases on the first step taken by $\mathsf{box}\psi_1$ and $\mathsf{box}\psi_1\psi_2$.

1. $r\psi_1 = r'\psi_1$.

   Then $\mathsf{box}\psi_1 \longmapsto_{\square} M\psi_1$, $[\![\mathsf{fcom}]\!]_{\psi_1\psi_2} = [\![A]\!]_{\psi_1\psi_2}$ by Lemma 57, and $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(X_1\psi_2, M\psi_1\psi_2)$ by $M \in A\ [\Psi]$.

2. $r\psi_1 \neq r'\psi_1$, $r_j\psi_1 = r_j'\psi_1$ (where this is the least such j), and $r\psi_1\psi_2 = r'\psi_1\psi_2$.

   Then $\mathsf{box}\psi_1 \longmapsto N_j\psi_1$, $\mathsf{box}\psi_1\psi_2 \longmapsto M\psi_1\psi_2$, and $[\![\mathsf{fcom}]\!]_{\psi_1\psi_2} = [\![A]\!]_{\psi_1\psi_2}$ by Lemma 57. By $B_j\langle r'/y\rangle\psi_1\psi_2 \doteq A\psi_1\psi_2\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi_2]$ and $N_j\psi_1 \in B_j\langle r'/y\rangle\psi_1\ [\Psi_1]$ at $\mathsf{id}_{\Psi_1}, \psi_2$ we have $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(X_1\psi_2, N_j\psi_1\psi_2)$. We also have $[\![A]\!]^{\Downarrow}_{\psi_1\psi_2}(N_j\psi_1\psi_2, M\psi_1\psi_2)$ by $(\mathsf{coe}^{r'\rightsquigarrow r}_{y.B_j}(N_j))\psi_1\psi_2 \doteq M\psi_1\psi_2 \in A\psi_1\psi_2\ [\Psi_2]$ and $(\mathsf{coe}^{r'\rightsquigarrow r}_{y.B_j}(N_j))\psi_1\psi_2 \doteq N_j\psi_1\psi_2 \in A\psi_1\psi_2\ [\Psi_2]$; the result follows by transitivity.

3. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 = r_i'\psi_1$ (least such), $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 = r_j'\psi_1\psi_2$ (least such).

   Then $\mathsf{box}\psi_1 \longmapsto N_i\psi_1$, $\mathsf{box}\psi_1\psi_2 \longmapsto N_j\psi_1\psi_2$, and $[\![\mathsf{fcom}]\!]_{\psi_1\psi_2} = [\![B_i\langle r'/y\rangle]\!]_{\psi_1\psi_2}$ by Lemma 57. The result follows by $N_i\psi_1 \in B_i\langle r'/y\rangle\psi_1\ [\Psi_1]$ and $N_i\psi_1\psi_2 \doteq N_j\psi_1\psi_2 \in B_i\langle r'/y\rangle\psi_1\psi_2\ [\Psi_2]$.

4. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r_i'\psi_1$ for all i, and $r\psi_1\psi_2 = r'\psi_1\psi_2$.

   Then $\mathsf{box}\psi_1\ \mathsf{val}$, $\mathsf{box}\psi_1\psi_2 \longmapsto M\psi_1\psi_2$, $[\![\mathsf{fcom}]\!]_{\psi_1\psi_2} = [\![A]\!]_{\psi_1\psi_2}$ by Lemma 57, and the result follows by $M \in A\ [\Psi]$.

5. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r_i'\psi_1$ for all i, $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 = r_j'\psi_1\psi_2$ (the least such j).

   Then $\mathsf{box}\psi_1\ \mathsf{val}$, $\mathsf{box}\psi_1\psi_2 \longmapsto N_j\psi_1\psi_2$, $[\![\mathsf{fcom}]\!]_{\psi_1\psi_2} = [\![B_i\langle r'/y\rangle]\!]_{\psi_1\psi_2}$ by Lemma 57, and the result follows by $N_j\psi_1\psi_2 \in B_i\langle r'/y\rangle\psi_1\psi_2\ [\Psi_2]$.

6. $r\psi_1 \neq r'\psi_1$, $r_i\psi_1 \neq r_i'\psi_1$ for all i, and $r\psi_1\psi_2 \neq r'\psi_1\psi_2$, and $r_j\psi_1\psi_2 \neq r_j'\psi_1\psi_2$ for all j.

   Then $\mathsf{box}\psi_1\ \mathsf{val}$ and $\mathsf{box}\psi_1\psi_2\ \mathsf{val}$, and the result follows by the definition of $[\![\mathsf{fcom}]\!]$. □

Rule 52 (Pretype formation)**.** *If $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ and $A', \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}$ are equal type compositions $r \rightsquigarrow r'$, then*

1. $\mathsf{fcom}^{r\rightsquigarrow r'}(A;\overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{fcom}^{r\rightsquigarrow r'}(A';\overrightarrow{r_i = r_i' \hookrightarrow y.B_i'})\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$,
2. *if $r = r'$ then* $\mathsf{fcom}^{r\rightsquigarrow r}(A;\overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$, *and*
3. *if $r_i = r_i'$ then* $\mathsf{fcom}^{r\rightsquigarrow r'}(A;\overrightarrow{r_i = r_i' \hookrightarrow B_i}) \doteq B_i\langle r'/y\rangle\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$.

Proof. For part (1), by Lemma 57 it suffices to show $\mathsf{Coh}([\![\mathsf{fcom}]\!])$. Let $[\![\mathsf{fcom}]\!]_{\psi}(M_0, N_0)$ for any $\psi : \Psi' \to \Psi$. If $r\psi = r'\psi$ then $\mathsf{Tm}([\![\mathsf{fcom}]\!]\psi)(M_0, N_0)$ by $[\![\mathsf{fcom}]\!]\psi = [\![A]\!]\psi$ and $\mathsf{Coh}([\![A]\!])$. Similarly, if $r_i\psi = r_i'\psi$ for some i, then $\mathsf{Tm}([\![\mathsf{fcom}]\!]\psi)(M_0, N_0)$ by $[\![\mathsf{fcom}]\!]\psi = [\![B_i\langle r'/y\rangle\psi]\!]$ and $\mathsf{Coh}([\![B_i\langle r'/y\rangle\psi]\!])$. If $r\psi \neq r'\psi$ and $r_i\psi \neq r_i'\psi$, then M_0 and N_0 are boxes and the result follows by Lemma 58.

Parts (2–3) are immediate by Lemma 57. □

Rule 53 (Introduction)**.** *If*

1. $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ *is a type composition* $r \rightsquigarrow r'$,

2. $M \doteq M' \in A\ [\Psi]$,

3. $N_i \doteq N_j' \in B_i\langle r'/y\rangle\ [\Psi \mid r_i = r_i', r_j = r_j']$ *for any* i, j, *and*

4. $\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(N_i) \doteq M \in A\ [\Psi \mid r_i = r_i']$ *for any* i,

then

1. $\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i}) \doteq \mathsf{box}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow N_i'}) \in \mathsf{fcom}^{r \rightsquigarrow r'}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i})\ [\Psi]$*;*

2. *if* $r = r'$ *then* $\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i}) \doteq M \in A\ [\Psi]$*; and*

3. *if* $r_i = r_i'$ *then* $\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i}) \doteq N_i \in B_i\langle r'/y\rangle\ [\Psi]$.

Proof. Part (1) is immediate by Lemma 58 and Rule 52; part (2) is immediate by Lemma 43. For part (3), if $r = r'$, the result follows by Lemma 43. Otherwise, there is a least j such that $r_j = r_j'$, and we apply coherent expansion to the left side with family

$$\begin{cases} M\psi & r\psi = r'\psi \\ N_k\psi & r\psi \neq r'\psi,\ r_k\psi = r_k'\psi, \text{ and } \forall k' < k, r_{k'}\psi \neq r_{k'}'\psi. \end{cases}$$

If $r\psi = r'\psi$ then $M\psi \doteq N_j\psi \in B_i\langle r'/y\rangle\psi\ [\Psi']$ by $M\psi \doteq (\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(N_j))\psi \in A\psi\ [\Psi']$, $(\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(N_j))\psi \doteq N_j\psi \in B_i\langle r'/y\rangle\psi\ [\Psi']$, and $B_i\langle r'/y\rangle\psi \doteq A\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$. If $r\psi \neq r'\psi$ then $N_k\psi \doteq N_j\psi \in B_i\langle r'/y\rangle\psi\ [\Psi']$ by $N_k\psi \doteq N_j\psi \in B_j\langle r'/y\rangle\psi\ [\Psi']$ and $B_i\psi \doteq B_j\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', y]$. Thus by Lemma 42 we have $\mathsf{fcom} \doteq N_j \in B_i\langle r'/y\rangle\ [\Psi]$, and part (3) follows by $N_j \doteq N_i \in B_i\langle r'/y\rangle\ [\Psi]$. $\square$

Rule 54 (Elimination)**.** *If* $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ *and* $A', \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}$ *are equal type compositions* $r \rightsquigarrow r'$ *and* $M \doteq M' \in \mathsf{fcom}^{r \rightsquigarrow r'}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i})\ [\Psi]$, *then*

1. $\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{cap}^{r \leftsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}) \in A\ [\Psi]$*;*

2. *if* $r = r'$ *then* $\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq M \in A\ [\Psi]$*; and*

3. *if* $r_i = r_i'$ *then* $\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(M) \in A\ [\Psi]$.

Proof. Part (2) is immediate by Lemma 43 and Rule 52. For part (3), if $r = r'$ then the result follows by part (2), $B_i\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y]$, and $B_i\langle r/y\rangle \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$. Otherwise, $r \neq r'$ and there is a least j such that $r_j = r_j'$. Apply coherent expansion to the left side with family

$$\begin{cases} M\psi & r\psi = r'\psi \\ \mathsf{coe}^{r'\psi \rightsquigarrow r\psi}_{y.B_k\psi}(M\psi) & r\psi \neq r'\psi,\ r_k\psi = r_k'\psi, \text{ and } \forall i < k, r_i\psi \neq r_i'\psi \end{cases}$$

When $r\psi = r'\psi$, $(\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(M))\psi \doteq M\psi \in A\psi\ [\Psi']$ by $M \in B_j\langle r'/y\rangle\ [\Psi]$ (by Rule 52), $B_j\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y]$, and $B_j\langle r/y\rangle\psi \doteq A\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi']$. When $r\psi \neq r'\psi$ and $r_k\psi = r_k'\psi$ where k is the least such, we have $(\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(M))\psi \doteq \mathsf{coe}^{r'\psi \rightsquigarrow r\psi}_{y.B_k\psi}(M\psi) \in A\psi\ [\Psi']$ by $B_j\psi \doteq B_k\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi', y]$ and $B_j\langle r/y\rangle\psi \doteq$

$A\psi$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi']$. We conclude that $\mathsf{cap} \doteq \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(M) \in A\ [\Psi]$ by Lemma 41, and part (3) follows by $\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(M) \doteq \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(M) \in A\ [\Psi]$.

For part (1), if $r = r'$ or $r_i = r'_i$ then the result follows by the previous parts. If $r \neq r'$ and $r_i \neq r'_i$ for all i, then for any $\psi : \Psi' \to \Psi$, $M\psi \doteq \mathsf{box}^{r \rightsquigarrow r'}(O_\psi; \overrightarrow{\xi_i\psi \hookrightarrow N_{i,\psi}}) \in \mathsf{fcom}\psi\ [\Psi']$ by Lemma 38. Apply coherent expansion to the left side with family

$$\begin{cases} M\psi & r\psi = r'\psi \\ \mathsf{coe}^{r'\psi \rightsquigarrow r\psi}_{y.B_j\psi}(M\psi) & r\psi \neq r'\psi,\ r_j\psi = r'_j\psi, \text{ and } \forall i < j, r_i\psi \neq r'_i\psi \\ O_\psi & r\psi \neq r'\psi \text{ and } \forall i, r_i\psi \neq r'_i\psi \\ & \quad \text{where } M\psi \Downarrow \mathsf{box}^{r\psi \rightsquigarrow r'\psi}(O_\psi; \overrightarrow{\xi_i\psi \hookrightarrow N_{i,\psi}}). \end{cases}$$

When $r\psi = r'\psi$, $M\psi \doteq (O_{\mathsf{id}_\Psi})\psi \in A\psi\ [\Psi']$ because $M\psi \doteq \mathsf{box}\psi \in \mathsf{fcom}\psi\ [\Psi']$, $\mathsf{fcom}\psi \doteq A\psi$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi']$ (by Rule 52), and $\mathsf{box}\psi \doteq (O_{\mathsf{id}_\Psi})\psi \in \mathsf{fcom}\psi\ [\Psi']$ (by Rule 53). When $r\psi \neq r'\psi$ and $r_j\psi = r'_j\psi$ where j is the least such, $(O_{\mathsf{id}_\Psi})\psi \doteq \mathsf{coe}^{r'\psi \rightsquigarrow r\psi}_{y.B_j\psi}(M\psi) \in A\psi\ [\Psi']$ because $M\psi \doteq (N_{j,\mathsf{id}_\Psi})\psi \in B_j\langle r'/y\rangle\psi\ [\Psi']$ (by Rules 52 and 53) and $O_{\mathsf{id}_\Psi} \doteq \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_j}(N_{j,\mathsf{id}_\Psi}) \in A\ [\Psi \mid r_j = r'_j]$. When $r\psi \neq r'\psi$ and $r_i\psi \neq r'_i\psi$ for all i, $(O_{\mathsf{id}_\Psi})\psi \doteq O_\psi \in A\psi\ [\Psi]$ by $[\![\mathsf{fcom}]\!]((\mathsf{box}^{r \rightsquigarrow r'}(O_{\mathsf{id}_\Psi}; \overrightarrow{\xi_i \hookrightarrow N_{i,\mathsf{id}_\Psi}}))\psi, \mathsf{box}^{r\psi \rightsquigarrow r'\psi}(O_\psi; \overrightarrow{\xi_i\psi \hookrightarrow N_{i,\psi}}))$ (by $M \in \mathsf{fcom}\ [\Psi]$ at id_Ψ, ψ). Therefore $\mathsf{cap} \doteq O_{\mathsf{id}_\Psi} \in A\ [\Psi]$ by Lemma 41, and part (1) follows by a symmetric argument on the right side. □

Rule 55 (Computation)**.** *If*

1. $A, \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}$ *is a type composition* $r \rightsquigarrow r'$,
2. $M \doteq M' \in A\ [\Psi]$,
3. $N_i \doteq N'_j \in B_i\langle r'/y\rangle\ [\Psi \mid r_i = r'_i, r_j = r'_j]$ *for any* i, j, *and*
4. $\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(N_i) \doteq M \in A\ [\Psi \mid r_i = r'_i]$ *for any* i,

then $\mathsf{cap}^{r \leftsquigarrow r'}(\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}); \overrightarrow{r_i = r'_i \hookrightarrow y.B_i}) \doteq M \in A\ [\Psi]$.

Proof. By Rules 53 and 55, we know both sides have this type, so it suffices to show $[\![A]\!]^{\Downarrow}(\mathsf{cap}, M)$. If $r = r'$ then $\mathsf{cap} \longmapsto \mathsf{box} \longmapsto M$ and $[\![A]\!]^{\Downarrow}(M, M)$. If $r \neq r'$ and $r_i = r'_i$ where i is the least such, then $\mathsf{cap} \longmapsto \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(\mathsf{box})$, and $[\![A]\!]^{\Downarrow}(\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(\mathsf{box}), M)$ by $\mathsf{box} \doteq N_i \in B_i\langle r'/y\rangle\ [\Psi]$ and $\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(N_i) \doteq M \in A\ [\Psi]$. If $r \neq r'$ and $r_i \neq r'_i$ for all i, then $\mathsf{cap} \longmapsto M$ and $[\![A]\!]^{\Downarrow}(M, M)$. □

Rule 56 (Eta)**.** *If* $A, \overrightarrow{\xi_i \hookrightarrow y.B_i}$ *is a type composition* $r \rightsquigarrow r'$ *and* $M \in \mathsf{fcom}^{r \rightsquigarrow r'}(A; \overrightarrow{\xi_i \hookrightarrow y.B_i})\ [\Psi]$, *then* $\mathsf{box}^{r \rightsquigarrow r'}(\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow y.B_i}); \overrightarrow{\xi_i \hookrightarrow M}) \doteq M \in \mathsf{fcom}^{r \rightsquigarrow r'}(A; \overrightarrow{\xi_i \hookrightarrow y.B_i})\ [\Psi]$.

Proof. By $\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow y.B_i}) \in A\ [\Psi]$ (by Rule 54), $M \in B_i\langle r'/y\rangle\ [\Psi \mid r_i = r'_i]$ (by Rule 52), $\mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(M) \doteq \mathsf{cap} \in A\ [\Psi \mid r_i = r'_i]$ (by Rule 54), and Rule 53, we have $\mathsf{box} \in \mathsf{fcom}\ [\Psi]$. Thus, by Lemma 37, it suffices to show $[\![\mathsf{fcom}]\!]^{\Downarrow}(\mathsf{box}, M)$. If $r = r'$ then $\mathsf{box} \longmapsto \mathsf{cap} \longmapsto M$ and $[\![\mathsf{fcom}]\!]^{\Downarrow}(M, M)$. If $r \neq r'$ and $r_i = r'_i$ for the least such i, then $\mathsf{box} \longmapsto M$ and $[\![\mathsf{fcom}]\!]^{\Downarrow}(M, M)$. If $r \neq r'$ and $r_i \neq r'_i$ for all i, then $M \Downarrow \mathsf{box}^{r \rightsquigarrow r'}(O; \overrightarrow{\xi_i \hookrightarrow N_i})$ and $M \doteq \mathsf{box}^{r \rightsquigarrow r'}(O; \overrightarrow{\xi_i \hookrightarrow N_i}) \in \mathsf{fcom}\ [\Psi]$. The result follows by transitivity and Rule 53:

1. $\mathsf{cap}^{r\leftsquigarrow r'}(M;\overrightarrow{\xi_i \hookrightarrow y.B_i}) \doteq O \in A\ [\Psi]$ by Lemma 37 and $\mathsf{cap} \longmapsto^* O$,

2. $M \doteq N_i \in B_i\langle r'/y\rangle\ [\Psi \mid r_i = r_i']$ by $M \doteq \mathsf{box}^{r\rightsquigarrow r'}(O;\overrightarrow{\xi_i \hookrightarrow N_i}) \in \mathsf{fcom}\ [\Psi]$ and Rule 53, and

3. $\mathsf{coe}_{y.B_i}^{r'\rightsquigarrow r}(M) \doteq \mathsf{cap}^{r\leftsquigarrow r'}(M;\overrightarrow{\xi_i \hookrightarrow y.B_i}) \in A\ [\Psi \mid r_i = r_i']$ by Rule 54 as before. □

Our implementation of coercion for fcom requires Kan compositions whose lists of equations might be invalid (in the sense of Definition 21), although Kan types are only guaranteed to have compositions for valid lists of equations. However, we can implement such *generalized* homogeneous compositions ghcom using only ordinary homogeneous compositions hcom.

Theorem 59. *If $A \doteq B$ $\mathsf{type_{Kan}}$ $[\Psi]$,*

1. *$M \doteq M' \in A\ [\Psi]$,*

2. *$N_i \doteq N_j' \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j']$ for any i, j, and*

3. *$N_i\langle r/y\rangle \doteq M \in A\ [\Psi \mid r_i = r_i']$ for any i,*

then

1. *$\mathsf{ghcom}_A^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{ghcom}_B^{r\rightsquigarrow r'}(M';\overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in A\ [\Psi]$;*

2. *if $r = r'$ then $\mathsf{ghcom}_A^{r\rightsquigarrow r}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in A\ [\Psi]$; and*

3. *if $r_i = r_i'$ then $\mathsf{ghcom}_A^{r\rightsquigarrow r'}(M;\overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\ [\Psi]$.*

Proof. Use induction on the length of $\overrightarrow{r_i = r_i'}$. If there are zero tubes, for part (1) we must show $\mathsf{ghcom}_A^{r\rightsquigarrow r'}(M;\cdot) \doteq \mathsf{ghcom}_B^{r\rightsquigarrow r'}(M';\cdot) \in A\ [\Psi]$, which is immediate by Lemma 43 on each side. Part (2) is immediate by Lemma 43 on the left, and part (3) is impossible without tubes.

Now consider the case $\mathsf{ghcom}_A^{r\rightsquigarrow r'}(M; s = s' \hookrightarrow y.N, \overrightarrow{\xi_i \hookrightarrow y.N_i})$, where we know ghcoms with one fewer tube have the desired properties. By Lemma 43 we must show (the binary version of)

$$\mathsf{hcom}_A^{r\rightsquigarrow r'}(M;\overrightarrow{s = \varepsilon \hookrightarrow z.T_\varepsilon}, s = s' \hookrightarrow y.N, \overrightarrow{\xi_i \hookrightarrow y.N_i}) \in A\ [\Psi]$$
$$\text{where } T_\varepsilon = \mathsf{hcom}_A^{r\rightsquigarrow z}(M; s' = \varepsilon \hookrightarrow y.N, s' = \overline{\varepsilon} \hookrightarrow y.\mathsf{ghcom}_A^{r\rightsquigarrow y}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}), \overrightarrow{\xi_i \hookrightarrow y.N_i}).$$

First, show $T_\varepsilon \in A\ [\Psi, z \mid s = \varepsilon]$ by Definition 22, noting the composition is valid by $s' = \varepsilon, s' = \overline{\varepsilon}$,

1. $M \in A\ [\Psi \mid s = \varepsilon]$ by $M \in A\ [\Psi]$,

2. $N \in A\ [\Psi, y \mid s = \varepsilon, s' = \varepsilon]$ (by $N \in A\ [\Psi, y \mid s = s']$, because $s = s'$ whenever $s = \varepsilon, s' = \varepsilon$), $N \doteq N_i \in A\ [\Psi, y \mid s = \varepsilon, s' = \varepsilon, r_i = r_i']$ (by $N \doteq N_i \in A\ [\Psi, y \mid s = s', r_i = r_i']$), and $N\langle r/y\rangle \doteq M \in A\ [\Psi \mid s = \varepsilon, s' = \varepsilon]$ (by $N\langle r/y\rangle \doteq M \in A\ [\Psi \mid s = s']$), and

3. $\mathsf{ghcom}_A^{r\rightsquigarrow y}(M;\overrightarrow{\xi_i \hookrightarrow y.N_i}) \in A\ [\Psi, y \mid s = \varepsilon, s' = \overline{\varepsilon}]$ (by part (1) of the induction hypothesis), $\mathsf{ghcom}_A \doteq N_i \in A\ [\Psi, y \mid s = \varepsilon, s' = \overline{\varepsilon}, r_i = r_i']$ (by part (3) of the induction hypothesis), and $(\mathsf{ghcom}_A)\langle r/y\rangle \doteq M \in A\ [\Psi, y \mid s = \varepsilon, s' = \overline{\varepsilon}]$ (by part (2) of the induction hypothesis).

The remaining adjacency conditions are immediate. To check $\mathsf{hcom}_A \in A\ [\Psi]$ it suffices to observe that $T_\varepsilon \in A\ [\Psi, z \mid s = \varepsilon]$ (by the above); $T_\varepsilon \doteq N\langle z/y\rangle \in A\ [\Psi, z \mid s = \varepsilon, s = s']$ (by the $s' = \varepsilon$ tube in T_ε); $T_\varepsilon \doteq N_i\langle z/y\rangle \in A\ [\Psi, z \mid s = \varepsilon, r_i = r_i']$ (by the $r_i = r_i'$ tube in T_ε); $T_\varepsilon\langle r/z\rangle \doteq M \in A\ [\Psi \mid s = \varepsilon]$ (by $r = z\langle r/z\rangle$ in T_ε); and the $\overrightarrow{s = \varepsilon}$ tubes ensure the composition is valid. Part (1) follows by repeating this argument on the right side, and parts (2–3) follow from Definition 22. □

Theorem 60. *If $A \doteq B$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, y]$,*

1. *$M \doteq M' \in A\langle r/y\rangle\ [\Psi]$,*
2. *$N_i \doteq N_j' \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j']$ for any i, j, and*
3. *$N_i\langle r/y\rangle \doteq M \in A\langle r/y\rangle\ [\Psi \mid r_i = r_i']$ for any i,*

then

1. *$\mathsf{gcom}^{r\rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{gcom}^{r\rightsquigarrow r'}_{y.B}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in A\langle r'/y\rangle\ [\Psi]$;*
2. *if $r = r'$ then $\mathsf{gcom}^{r\rightsquigarrow r}_{y.A}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in A\langle r/y\rangle\ [\Psi]$; and*
3. *if $r_i = r_i'$ then $\mathsf{gcom}^{r\rightsquigarrow r'}_{y.A}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\langle r'/y\rangle\ [\Psi]$.*

Proof. The implementation of gcom by ghcom and coe mirrors exactly the implementation of com by hcom and coe; the proof is thus identical to that of Theorem 44, appealing to Theorem 59 instead of Definition 22. □

Lemma 61. *If $A, \overrightarrow{s_j = s_j' \hookrightarrow z.B_j}$ and $A', \overrightarrow{s_j = s_j' \hookrightarrow z.B_j'}$ are equal type compositions $s \rightsquigarrow s'$ and, letting $\mathsf{fcom} := \mathsf{fcom}^{s\rightsquigarrow s'}(A; \overrightarrow{s_j = s_j' \hookrightarrow z.B_j})$,*

1. *$\overrightarrow{r_i = r_i'}$ is valid,*
2. *$M \doteq M' \in \mathsf{fcom}\ [\Psi]$,*
3. *$N_i \doteq N_{i'}' \in \mathsf{fcom}\ [\Psi, y \mid r_i = r_i', r_{i'} = r_{i'}']$ for any i, i', and*
4. *$N_i\langle r/y\rangle \doteq M \in \mathsf{fcom}\ [\Psi \mid r_i = r_i']$ for any i,*

then

1. *$\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{fcom}}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{fcom}^{s\rightsquigarrow s'}(A'; \overrightarrow{s_j = s_j' \hookrightarrow z.B_j'})}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in \mathsf{fcom}\ [\Psi]$;*
2. *if $r = r'$ then $\mathsf{hcom}^{r\rightsquigarrow r}_{\mathsf{fcom}}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in \mathsf{fcom}\ [\Psi]$; and*
3. *if $r_i = r_i'$ then $\mathsf{hcom}^{r\rightsquigarrow r'}_{\mathsf{fcom}}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in \mathsf{fcom}\ [\Psi]$.*

Proof. If $s = s'$ or $s_j = s'_j$ for some j, the results are immediate by parts (2–3) of Lemma 57. Otherwise, $s \neq s'$ and $s_j \neq s'_j$ for all j; apply coherent expansion to $\mathsf{hcom}_{\mathsf{fcom}}$ with family

$$\begin{cases} \mathsf{hcom}_{A\psi}^{r\psi \rightsquigarrow r'\psi}(M\psi; \overrightarrow{r_i\psi = r'_i\psi \hookrightarrow y.N_i\psi}) & s\psi = s'\psi \\ \mathsf{hcom}_{B_j\langle s'/z\rangle\psi}^{r\psi \rightsquigarrow r'\psi}(M\psi; \overrightarrow{r_i\psi = r'_i\psi \hookrightarrow y.N_i\psi}) & s\psi \neq s'\psi, \text{ least } s_j\psi = s'_j\psi \\ (\mathsf{box}^{s \rightsquigarrow s'}(Q; \overrightarrow{s_j = s'_j \hookrightarrow P_j\langle s'/z\rangle}))\psi & s\psi \neq s'\psi, \forall j. s_j\psi \neq s'_j\psi \\ \quad P_j = \mathsf{hcom}_{B_j}^{r \rightsquigarrow r'}(\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(M); \overrightarrow{r_i = r'_i \hookrightarrow y.\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(N_i)}) \\ \quad F[c] = \mathsf{hcom}_A^{s' \rightsquigarrow z}(\mathsf{cap}^{s \leftsquigarrow s'}(c; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j}); \overrightarrow{T}) \\ \quad \overrightarrow{T} = \overrightarrow{s_j = s'_j \hookrightarrow z'.\mathsf{coe}_{z.B_j}^{z' \rightsquigarrow s}(\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z'}(c))} \\ \quad O = \mathsf{hcom}_A^{r \rightsquigarrow r'}((F[M])\langle s/z\rangle; \overrightarrow{r_i = r'_i \hookrightarrow y.(F[N_i])\langle s/z\rangle}) \\ \quad Q = \mathsf{hcom}_A^{s \rightsquigarrow s'}(O; \overrightarrow{r_i = r'_i \hookrightarrow z.F[N_i\langle r'/y\rangle]}, \overrightarrow{U}) \\ \quad \overrightarrow{U} = \overrightarrow{s_j = s'_j \hookrightarrow z.\mathsf{coe}_{z.B_j}^{z \rightsquigarrow s}(P_j)}, r = r' \hookrightarrow z.F[M] \end{cases}$$

Consider $\psi = \mathsf{id}_\Psi$.

1. $P_j \doteq P_{j'} \in B_j\ [\Psi, z \mid s_j = s'_j, s_{j'} = s'_{j'}]$ for all j, j', by
   (a) $B_j \doteq B_{j'}\ \mathsf{type_{Kan}}\ [\Psi, z \mid s_j = s'_j, s_{j'} = s'_{j'}]$,
   (b) $\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(M) \doteq \mathsf{coe}_{z.B_{j'}}^{s' \rightsquigarrow z}(M) \in B_j\ [\Psi, z \mid s_j = s'_j, s_{j'} = s'_{j'}]$ by $\mathsf{fcom} \doteq B_j\langle s'/z\rangle\ \mathsf{type_{Kan}}\ [\Psi \mid s_j = s'_j]$,
   (c) $\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(N_i) \doteq \mathsf{coe}_{z.B_{j'}}^{s' \rightsquigarrow z}(N_{i'}) \in B_j\ [\Psi, z, y \mid s_j = s'_j, s_{j'} = s'_{j'}, r_i = r'_i, r_{i'} = r'_{i'}]$ for all i, i', and
   (d) $\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(M) \doteq \mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z}(N_i\langle r/y\rangle) \in B_j\ [\Psi, z \mid s_j = s'_j, s_{j'} = s'_{j'}, r_i = r'_i]$ for all i by $M \doteq N_i\langle r/y\rangle \in B_j\langle s'/z\rangle\ [\Psi, z \mid s_j = s'_j, r_i = r'_i]$.

2. $F[c] \doteq F[c'] \in A\ [\Psi, z]$ for any $c \doteq c' \in \mathsf{fcom}\ [\Psi]$, by
   (a) $\mathsf{cap}^{s \leftsquigarrow s'}(c; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j}) \doteq \mathsf{cap}^{s \leftsquigarrow s'}(c'; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j}) \in A\ [\Psi]$,
   (b) $\mathsf{coe}_{z.B_j}^{z' \rightsquigarrow s}(\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z'}(c)) \doteq \mathsf{coe}_{z.B_{j'}}^{z' \rightsquigarrow s}(\mathsf{coe}_{z.B_{j'}}^{s' \rightsquigarrow z'}(c')) \in A\ [\Psi, z' \mid s_j = s'_j, s_{j'} = s'_{j'}]$ for all j, j' by $\mathsf{fcom} \doteq B_j\langle s'/z\rangle\ \mathsf{type_{Kan}}\ [\Psi \mid s_j = s'_j]$ and $B_j\langle s/z\rangle \doteq A\ \mathsf{type_{Kan}}\ [\Psi \mid s_j = s'_j]$, and
   (c) $(\mathsf{coe}_{z.B_j}^{z' \rightsquigarrow s}(\mathsf{coe}_{z.B_j}^{s' \rightsquigarrow z'}(c)))\langle s'/z'\rangle \doteq \mathsf{cap}^{s \leftsquigarrow s'}(c; \overrightarrow{s_j = s'_j \hookrightarrow z.B_j}) \in A\ [\Psi \mid s_j = s'_j]$ for all j because both sides $\doteq \mathsf{coe}_{z.B_j}^{s' \rightsquigarrow s}(c)$.

3. $O \in A\ [\Psi]$ by
   (a) $(F[M])\langle s/z\rangle \in A\ [\Psi]$ by $M \in \mathsf{fcom}\ [\Psi]$,
   (b) $(F[N_i])\langle s/z\rangle \doteq (F[N_{i'}])\langle s/z\rangle \in A\ [\Psi, y \mid r_i = r'_i, r_{i'} = r'_{i'}]$ for all i, i' by $N_i \doteq N_{i'} \in \mathsf{fcom}\ [\Psi, y \mid r_i = r'_i, r_{i'} = r'_{i'}]$, and
   (c) $(F[N_i\langle r/y\rangle])\langle s/z\rangle \doteq (F[M])\langle s/z\rangle \in A\ [\Psi \mid r_i = r'_i]$ for all i by $N_i\langle r/y\rangle \doteq M \in \mathsf{fcom}\ [\Psi \mid r_i = r'_i, r_{i'} = r'_{i'}]$.

4. $Q \in A\ [\Psi]$ by

(a) $F[N_i\langle r'/y\rangle] \doteq F[N_{i'}\langle r'/y\rangle] \in A\ [\Psi, z \mid r_i = r_i', r_{i'} = r_{i'}']$ for all i, i' by $N_i\langle r'/y\rangle \doteq N_{i'}\langle r'/y\rangle \in \mathsf{fcom}\ [\Psi \mid r_i = r_i', r_{i'} = r_{i'}']$,

(b) $\mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(P_j) \doteq \mathsf{coe}^{z\rightsquigarrow s}_{z.B_{j'}}(P_{j'}) \in A\ [\Psi, z \mid s_j = s_j', s_{j'} = s_{j'}']$ by $P_j \doteq P_{j'} \in B_j\ [\Psi, z \mid s_j = s_j', s_{j'} = s_{j'}']$,

(c) $F[M] \in A\ [\Psi, z \mid r = r']$ by $M \in \mathsf{fcom}\ [\Psi]$,

(d) $O \in A\ [\Psi]$,

(e) $(F[N_i\langle r'/y\rangle])\langle s/z\rangle \doteq O \in A\ [\Psi \mid r_i = r_i']$ for all i,

(f) $(\mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(P_j))\langle s/z\rangle \doteq O \in A\ [\Psi \mid s_j = s_j']$ for all j, because the left side $(\mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(P_j))\langle s/z\rangle \doteq \mathsf{hcom}^{r\rightsquigarrow r'}_{B_j\langle s/z\rangle}(\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(M); \overrightarrow{r_i = r_i' \hookrightarrow y.\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(N_i)}) \in A\ [\Psi \mid s_j = s_j']$, and this $\doteq O$ by $B_j\langle s/z\rangle \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid s_j = s_j']$, $\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(M) \doteq (F[M])\langle s/z\rangle \in A\ [\Psi \mid s_j = s_j']$ (because the right side $\doteq \mathsf{coe}^{s\rightsquigarrow s}_{z.B_j}(\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(M))$), and $\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(N_i) \doteq (F[N_i])\langle s/z\rangle \in A\ [\Psi \mid s_j = s_j', r_i = r_i']$ for all i (because the right side $\doteq \mathsf{coe}^{s\rightsquigarrow s}_{z.B_j}(\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(N_i))$),

(g) $(F[M])\langle s/z\rangle \doteq O \in A\ [\Psi \mid r = r']$,

(h) $F[N_i\langle r'/y\rangle] \doteq \mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(P_j) \in A\ [\Psi, z \mid r_i = r_i', s_j = s_j']$ for all i, j because both sides $\doteq \mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(\mathsf{coe}^{s'\rightsquigarrow z}_{z.B_j}(N_i\langle r'/y\rangle))$,

(i) $F[N_i\langle r'/y\rangle] \doteq F[M] \in A\ [\Psi, z \mid r_i = r_i', r = r']$ for all i by $N_i\langle r'/y\rangle \doteq M \in \mathsf{fcom}\ [\Psi \mid r_i = r_i']$, and

(j) $\mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(P_j) \doteq F[M] \in A\ [\Psi, z \mid s_j = s_j', r = r']$ for all j because both sides are $\doteq \mathsf{coe}^{z\rightsquigarrow s}_{z.B_j}(\mathsf{coe}^{s'\rightsquigarrow z}_{z.B_j}(M))$.

5. $\mathsf{box}^{s\rightsquigarrow s'}(Q; \overrightarrow{s_j = s_j' \hookrightarrow P_j\langle s'/z\rangle}) \in \mathsf{fcom}\ [\Psi]$ by $Q \in A\ [\Psi]$, $P_j\langle s'/z\rangle \doteq P_{j'}\langle s'/z\rangle \in B_j\langle s'/z\rangle\ [\Psi \mid s_j = s_j', s_{j'} = s_{j'}']$ for all j, j', and $\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_j}(P_j\langle s'/z\rangle) \doteq Q \in A\ [\Psi \mid s_j = s_j']$ for all j.

When $s\psi \neq s'\psi$ and $s_j\psi \neq s_j'\psi$ for all j, coherence is immediate. When $s\psi = s'\psi$, $\mathsf{box}\psi \doteq Q\psi \doteq O\psi \doteq \mathsf{hcom}^{r\psi\rightsquigarrow r'\psi}_{A\psi}(M\psi; \overrightarrow{\xi_i\psi \hookrightarrow y.N_i\psi}) \in A\psi\ [\Psi']$ by $(F[M])\langle s/z\rangle\psi \doteq M\psi \in A\psi\ [\Psi']$ and similarly for each tube. When $s\psi \neq s'\psi$ and $s_j\psi = s_j'\psi$ for the least such j, $\mathsf{box}\psi \doteq P_j\langle s'/z\rangle\psi \doteq (\mathsf{hcom}^{r\rightsquigarrow r'}_{B_j\langle s'/z\rangle}(\mathsf{coe}^{s'\rightsquigarrow s'}_{z.B_j}(M); \overrightarrow{r_i = r_i' \hookrightarrow y.\mathsf{coe}^{s'\rightsquigarrow s'}_{z.B_j}(N_i)}))\psi \doteq (\mathsf{hcom}^{r\rightsquigarrow r'}_{B_j\langle s'/z\rangle}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}))\psi$. By Lemma 41, $\mathsf{hcom}_{\mathsf{fcom}} \doteq \mathsf{box} \in \mathsf{fcom}\ [\Psi]$; part (1) follows by a symmetric argument on the right side.

For part (2), if $r = r'$ then $Q \doteq (F[M])\langle s'/z\rangle \doteq \mathsf{cap}^{s\leftsquigarrow s'}(M; \overrightarrow{s_j = s_j' \hookrightarrow z.B_j}) \in A\ [\Psi]$ and $P_j\langle s'/z\rangle \doteq \mathsf{coe}^{s'\rightsquigarrow s'}_{z.B_j}(M) \doteq M \in B_j\langle s'/z\rangle\ [\Psi \mid s_j = s_j']$ for all j, so $\mathsf{box}^{s\rightsquigarrow s'}(Q; \overrightarrow{s_j = s_j' \hookrightarrow P_j\langle s'/z\rangle}) \doteq M \in \mathsf{fcom}\ [\Psi]$ by Rule 56, and part (2) follows by transitivity.

For part (3), if $r_i = r_i'$ then $Q \doteq (F[N_i\langle r'/y\rangle])\langle s'/z\rangle \doteq \mathsf{cap}^{s\leftsquigarrow s'}(N_i\langle r'/y\rangle; \overrightarrow{s_j = s_j' \hookrightarrow z.B_j}) \in A\ [\Psi]$ and $P_j\langle s'/z\rangle \doteq \mathsf{coe}^{s'\rightsquigarrow s'}_{z.B_j}(N_i\langle r'/y\rangle) \doteq N_i\langle r'/y\rangle \in B_j\langle s'/z\rangle\ [\Psi \mid s_j = s_j']$ for all j, so $\mathsf{box} \doteq N_i\langle r'/y\rangle \in \mathsf{fcom}\ [\Psi]$ by Rule 56, and part (3) follows by transitivity. □

Lemma 62. *Let* $\mathsf{fcom} := \mathsf{fcom}^{s\rightsquigarrow s'}(A; \overrightarrow{s_i = s_i' \hookrightarrow z.B_i})$. *If*

1. $\overrightarrow{s_i = s_i'}$ *is valid in* (Ψ, x),

2. $A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x]$,

3. $B_i \doteq B'_j$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, x, z \mid s_i = s'_i, s_j = s'_j]$ *for any* i, j,

4. $B_i\langle s/z\rangle \doteq A$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, x \mid s_i = s'_i]$ *for any* i, *and*

5. $M \doteq M' \in \mathsf{fcom}\langle r/x\rangle$ $[\Psi]$,

then

1. $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{fcom}}(M) \doteq \mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{fcom}^{s\rightsquigarrow s'}(A';\overrightarrow{s_i=s'_i\hookrightarrow z.B'_i})}(M') \in \mathsf{fcom}\langle r'/x\rangle$ $[\Psi]$; *and*

2. *if* $r = r'$ *then* $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{fcom}}(M) \doteq M \in \mathsf{fcom}\langle r'/x\rangle$ $[\Psi]$.

Proof. If $s = s'$ or $s_i = s'_i$ for some i, the results are immediate by parts (2–3) of Lemma 57. Otherwise, $s \neq s'$ and $s_i \neq s'_i$ for all i; apply coherent expansion to $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{fcom}}(M)$ with family

$$\begin{cases}
\mathsf{coe}^{r\psi\rightsquigarrow r'\psi}_{x.A\psi}(M\psi) & s\psi = s'\psi \\
\mathsf{coe}^{r\psi\rightsquigarrow r'\psi}_{x.B_i\langle s'/z\rangle\psi}(M\psi) & s\psi \neq s'\psi, \text{ least } s_i\psi = s'_i\psi \\
(\mathsf{box}^{s\rightsquigarrow s'}(R;\overrightarrow{\xi_i \hookrightarrow Q_i\langle s'/z\rangle}))\langle r'/x\rangle\psi & s\psi \neq s'\psi, \forall i.s_i\psi \neq s'_i\psi \\
\quad N_i = \mathsf{coe}^{s'\rightsquigarrow z}_{z.B_i}(\mathsf{coe}^{r\rightsquigarrow x}_{x.B_i\langle s'/z\rangle}(M)) \\
\quad O = (\mathsf{hcom}^{s'\rightsquigarrow z}_{A}(\mathsf{cap}^{s\leftsquigarrow s'}(M;\overrightarrow{\xi_i \hookrightarrow z.B_i});\overrightarrow{\xi_i \hookrightarrow z.\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(N_i)}))\langle r/x\rangle \\
\quad P = \mathsf{gcom}^{r\rightsquigarrow r'}_{x.A}(O\langle s\langle r/x\rangle/z\rangle;\overrightarrow{\xi_i \hookrightarrow x.N_i\langle s/z\rangle}|_{(x\#\xi_i)}, T) \\
\quad T = s = s' \hookrightarrow x.\mathsf{coe}^{r\rightsquigarrow x}_{x.A}(M)|_{(x\#s,s')} \\
\quad Q_k = \mathsf{gcom}^{s\langle r'/x\rangle\rightsquigarrow z}_{z.B_k\langle r'/x\rangle}(P;\overrightarrow{\xi_i \hookrightarrow z.N_i\langle r'/x\rangle}|_{(x\#\xi_i)}, r = r' \hookrightarrow z.N_k\langle r'/x\rangle) \\
\quad R = \mathsf{hcom}^{s\rightsquigarrow s'}_{A}(P;\overrightarrow{\xi_i \hookrightarrow z.\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(Q_i)}, r = r' \hookrightarrow z.O)
\end{cases}$$

Consider $\psi = \mathsf{id}_\Psi$.

1. $N_i \doteq N_j \in B_i$ $[\Psi, x, z \mid \xi_i\langle r/x\rangle, \xi_j\langle r/x\rangle]$ for all i, j by $M \in B_i\langle s'/z\rangle\langle r/x\rangle$ $[\Psi \mid \xi_i\langle r/x\rangle]$ (by $M \in \mathsf{fcom}\langle r/x\rangle$ $[\Psi]$ and $\mathsf{fcom} \doteq B_i\langle s'/z\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, x \mid \xi_i]$) and $B_i \doteq B_j$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, x, z \mid \xi_i, \xi_j]$.

2. $O \in A\langle r/x\rangle$ $[\Psi, z]$ by

   (a) $(\mathsf{cap}^{s\leftsquigarrow s'}(M;\overrightarrow{\xi_i \hookrightarrow z.B_i}))\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi]$ by $M \in \mathsf{fcom}\langle r/x\rangle$ $[\Psi]$,

   (b) $\mathsf{coe}^{z\rightsquigarrow s\langle r/x\rangle}_{z.B_i\langle r/x\rangle}(N_i\langle r/x\rangle) \doteq \mathsf{coe}^{z\rightsquigarrow s\langle r/x\rangle}_{z.B_j\langle r/x\rangle}(N_j\langle r/x\rangle) \in A\langle r/x\rangle$ $[\Psi, z \mid \xi_i\langle r/x\rangle, \xi_j\langle r/x\rangle]$ for all i, j by $B_i\langle s/z\rangle\langle r/x\rangle \doteq A\langle r/x\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi \mid \xi_i\langle r/x\rangle]$, and

   (c) $(\mathsf{cap}^{s\leftsquigarrow s'}(M;\overrightarrow{\xi_i \hookrightarrow z.B_i}))\langle r/x\rangle \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(N_i))\langle s'/z\rangle\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi \mid \xi_i\langle r/x\rangle]$ for all i by $\mathsf{cap}\langle r/x\rangle \doteq (\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_i}(M))\langle r/x\rangle \in A\langle r/x\rangle$ $[\Psi \mid \xi_i\langle r/x\rangle]$ and $N_i\langle s'/z\rangle\langle r/x\rangle \doteq M \in B_i\langle s'/z\rangle\langle r/x\rangle$ $[\Psi \mid \xi_i\langle r/x\rangle]$.

3. $P \in A\langle r'/x\rangle$ $[\Psi]$ by

   (a) $O\langle s\langle r/x\rangle/z\rangle \in A\langle r/x\rangle$ $[\Psi]$,

   (b) $N_i\langle s/z\rangle \doteq N_j\langle s/z\rangle \in A$ $[\Psi, x \mid \xi_i, \xi_j]$ for all i, j such that $x \# \xi_i, \xi_j$ by $N_i\langle s/z\rangle \doteq N_j\langle s/z\rangle \in B_i\langle s/z\rangle$ $[\Psi, x \mid \xi_i\langle r/x\rangle, \xi_j\langle r/x\rangle]$ and $B_i\langle s/z\rangle \doteq A$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi, x \mid \xi_i]$,

   (c) $\mathsf{coe}^{r\rightsquigarrow x}_{x.A}(M) \in A$ $[\Psi, x \mid s = s']$ if $x \# s, s'$ by $\mathsf{fcom}\langle r/x\rangle \doteq A\langle r/x\rangle$ $\mathsf{type}_{\mathsf{Kan}}$ $[\Psi \mid s\langle r/x\rangle = s'\langle r/x\rangle]$,

(d) $O\langle s\langle r/x\rangle/z\rangle \doteq N_i\langle s/z\rangle\langle r/x\rangle \in A\langle r/x\rangle\ [\Psi \mid \xi_i]$ for all i such that $x \mathrel{\#} \xi_i$ by $O\langle s\langle r/x\rangle/z\rangle \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(N_i))\langle s/z\rangle\langle r/x\rangle \doteq N_i\langle s/z\rangle\langle r/x\rangle \in A\langle r/x\rangle\ [\Psi \mid \xi_i\langle r/x\rangle]$,

(e) $O\langle s\langle r/x\rangle/z\rangle \doteq (\mathsf{coe}^{r\rightsquigarrow x}_{x.A}(M))\langle r/x\rangle \in A\langle r/x\rangle\ [\Psi \mid s = s']$ if $x \mathrel{\#} s, s'$ by $O\langle s\langle r/x\rangle/z\rangle = O\langle s/z\rangle \doteq \mathsf{cap}\langle r/x\rangle \doteq M \in A\langle r/x\rangle\ [\Psi \mid s = s']$, and

(f) $N_i\langle s/z\rangle \doteq \mathsf{coe}^{r\rightsquigarrow x}_{x.A}(M) \in A\ [\Psi, x \mid \xi_i, s = s']$ for all i such that $x \mathrel{\#} \xi_i, s, s'$ by $N_i\langle s/z\rangle \doteq \mathsf{coe}^{r\rightsquigarrow x}_{x.B_i\langle s'/z\rangle}(M) \in B_i\langle s'/z\rangle\ [\Psi, x \mid \xi_i, s = s']$ and $B_i\langle s'/z\rangle \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x \mid \xi_i, s = s']$.

4. $Q_k \doteq Q_{k'} \in B_k\langle r'/x\rangle\ [\Psi, z \mid \xi_k\langle r'/x\rangle, \xi_{k'}\langle r'/x\rangle]$ for all k, k' by

(a) $P \in B_k\langle s/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_k\langle r'/x\rangle]$ by $A \doteq B_k\langle s/z\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x \mid \xi_k]$,

(b) $N_i\langle r'/x\rangle \doteq N_j\langle r'/x\rangle \in B_k\langle r'/x\rangle\ [\Psi, z \mid \xi_k\langle r'/x\rangle, \xi_i, \xi_j]$ for all i, j such that $x \mathrel{\#} \xi_i, \xi_j$ by $N_i \doteq N_j \in B_i\ [\Psi, x, z \mid \xi_i, \xi_j]$ and $B_i \doteq B_k\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x, z \mid \xi_i, \xi_k]$,

(c) $N_k\langle r'/x\rangle \doteq N_{k'}\langle r'/x\rangle \in B_k\langle r'/x\rangle\ [\Psi, z \mid \xi_k\langle r'/x\rangle, \xi_{k'}\langle r'/x\rangle]$,

(d) $P \doteq N_i\langle s/z\rangle\langle r'/x\rangle \in B_k\langle s/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_k\langle r'/x\rangle, \xi_i]$ for all i such that $x \mathrel{\#} \xi_i$ by $P \doteq N_i\langle s/z\rangle\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid \xi_i]$ and $A\langle r'/x\rangle \doteq B_k\langle s/z\rangle\langle r'/x\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid \xi_k\langle r'/x\rangle]$,

(e) $P \doteq N_k\langle s/z\rangle\langle r'/x\rangle \in B_k\langle s/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_k\langle r'/x\rangle, r = r']$ because $P \doteq O\langle s\langle r/x\rangle/z\rangle \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_k}(N_k))\langle s\langle r/x\rangle/z\rangle\langle r/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid \xi_k\langle r'/x\rangle, r = r']$, and

(f) $N_i\langle r'/x\rangle \doteq N_k\langle r'/x\rangle \in B_k\langle r'/x\rangle\ [\Psi, z \mid \xi_k\langle r'/x\rangle, \xi_i, r = r']$ for all i such that $x \mathrel{\#} \xi_i$.

5. $R\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi]$ by

(a) $P \in A\langle r'/x\rangle\ [\Psi]$,

(b) $\mathsf{coe}^{z\rightsquigarrow s\langle r'/x\rangle}_{z.B_i\langle r'/x\rangle}(Q_i) \doteq \mathsf{coe}^{z\rightsquigarrow s\langle r'/x\rangle}_{z.B_j\langle r'/x\rangle}(Q_j) \in A\langle r'/x\rangle\ [\Psi, z \mid \xi_i\langle r'/x\rangle, \xi_j\langle r'/x\rangle]$ for all i, j by $B_i \doteq B_j\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, z, x \mid \xi_i, \xi_j]$ and $B_i\langle s/z\rangle\langle r'/x\rangle \doteq A\langle r'/x\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid \xi_i\langle r'/x\rangle]$,

(c) $O \in A\langle r'/x\rangle\ [\Psi, z \mid r = r']$,

(d) $P \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(Q_i))\langle s/z\rangle\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle]$ for all i by $Q_i\langle s/z\rangle\langle r'/x\rangle \doteq P \in B_i\langle s/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle]$ and $B_i\langle s/z\rangle\langle r'/x\rangle \doteq A\langle r'/x\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid \xi_i\langle r'/x\rangle]$,

(e) $P \doteq O\langle s/z\rangle\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid r = r']$ by $O\langle s/z\rangle\langle r'/x\rangle = O\langle s\langle r'/x\rangle/z\rangle$, and

(f) $(\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(Q_i))\langle r'/x\rangle \doteq O\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi, z \mid \xi_i\langle r'/x\rangle, r = r']$ for all i by $O\langle r'/x\rangle = O \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(N_i))\langle r/x\rangle \in A\langle r'/x\rangle\ [\Psi, z \mid \xi_i\langle r'/x\rangle]$ and $Q_i\langle r'/x\rangle \doteq N_i\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi, z \mid \xi_i\langle r'/x\rangle, r = r']$.

6. $\mathsf{box}^{s\langle r'/x\rangle\rightsquigarrow s'\langle r'/x\rangle}(R\langle r'/x\rangle; \overrightarrow{\xi_i\langle r'/x\rangle \hookrightarrow Q_i\langle s'\langle r'/x\rangle/z\rangle}) \in \mathsf{fcom}\langle r'/x\rangle\ [\Psi]$ by

(a) $R\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi]$,

(b) $Q_i\langle s'\langle r'/x\rangle/z\rangle \doteq Q_j\langle s'\langle r'/x\rangle/z\rangle \in B_i\langle s'/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle, \xi_j\langle r'/x\rangle]$ for all i, j, and

(c) $(\mathsf{coe}^{s'\rightsquigarrow s}_{z.B_i}(Q_i\langle s'/z\rangle))\langle r'/x\rangle \doteq R\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle]$ for all i by $R\langle r'/x\rangle \doteq (\mathsf{coe}^{z\rightsquigarrow s}_{z.B_i}(Q_i))\langle s'/z\rangle\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle]$.

Consider $\psi : \Psi' \to \Psi$. When $s\psi \neq s'\psi$ and $s_i\psi \neq s_i'\psi$ for all i, coherence is immediate. When $s\psi = s'\psi$, then by $s \neq s'$, we must have $x \mathrel{\#} s, s'$ and thus $s\langle r'/x\rangle\psi = s'\langle r'/x\rangle\psi$ also. Thus $\mathsf{box}\langle r'/x\rangle\psi \doteq R\langle r'/x\rangle\psi \doteq P\langle r'/x\rangle\psi \doteq (\mathsf{coe}^{r\rightsquigarrow x}_{x.A}(M))\langle r'/x\rangle\psi \in A\langle r'/x\rangle\ [\Psi']$ as required. When $s\psi \neq s'\psi$ and $s_i\psi = s_i'\psi$ for the least such i, again $x \mathrel{\#} s_i, s_i'$ and $\mathsf{box}\langle r'/x\rangle\psi \doteq Q_i\langle s'/z\rangle\langle r'/x\rangle\psi \doteq$

$N_i\langle s'/z\rangle\langle r'/x\rangle\psi \doteq (\mathsf{coe}^{r\rightsquigarrow r'}_{x.B_i\langle s'/z\rangle}(M))\psi \in A\langle r'/x\rangle\ [\Psi']$. By Lemma 41, $\mathsf{coe}^{r\rightsquigarrow r'}_{x.\mathsf{fcom}}(M) \doteq \mathsf{box}\langle r'/x\rangle \in \mathsf{fcom}\langle r'/x\rangle\ [\Psi]$; part (1) follows by a symmetric argument on the right side.

For part (2), if $r = r'$ then $R\langle r'/x\rangle \doteq O\langle s'/z\rangle\langle r'/x\rangle \doteq (\mathsf{cap}^{s\leftsquigarrow s'}(M;\overrightarrow{\xi_i \hookrightarrow z.B_i}))\langle r'/x\rangle \in A\langle r'/x\rangle\ [\Psi]$ and $Q_i\langle s'/z\rangle\langle r'/x\rangle \doteq N_k\langle s'/z\rangle\langle r'/x\rangle \doteq M \in B_i\langle s'/z\rangle\langle r'/x\rangle\ [\Psi \mid \xi_i\langle r'/x\rangle]$ for all i, so $\mathsf{box}\langle r'/x\rangle \doteq M \in \mathsf{fcom}\langle r'/x\rangle\ [\Psi]$ by Rule 56, and part (2) follows by transitivity. □

Rule 57 (Kan type formation)**.** *If $A, \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}$ and $A', \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}$ are equal type compositions $r \rightsquigarrow r'$, then*

1. $\mathsf{fcom}^{r\rightsquigarrow r'}(A;\overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{fcom}^{r\rightsquigarrow r'}(A';\overrightarrow{r_i = r_i' \hookrightarrow y.B_i'})\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$,
2. *if* $r = r'$ *then* $\mathsf{fcom}^{r\rightsquigarrow r}(A;\overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$, *and*
3. *if* $r_i = r_i'$ *then* $\mathsf{fcom}^{r\rightsquigarrow r'}(A;\overrightarrow{r_i = r_i' \hookrightarrow B_i}) \doteq B_i\langle r'/y\rangle\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$.

Proof. We already showed parts (2–3) in Lemma 57. For part (1), the hcom conditions follow from Lemma 61 at $\mathsf{fcom}\psi$ for any $\psi : \Psi' \to \Psi$; the coe conditions follow from Lemma 62 at $x.\mathsf{fcom}\psi$ for any $\psi : (\Psi', x) \to \Psi$. □

5.12 Universes

Our type theory has two hierarchies of universes, $\mathcal{U}_j^{\mathsf{pre}}$ and $\mathcal{U}_j^{\mathsf{Kan}}$, constructed by two sequences τ_j^{pre} and τ_j^{Kan} of cubical type systems. To prove theorems about universe types in the cubical type system $\tau_\omega^{\mathsf{pre}}$, we must analyze these sequences as constructed in Section 3.

Lemma 63. *If τ, τ' are cubical type systems, $\tau \subseteq \tau'$, and $\tau \models \mathcal{J}$ for any judgment $\mathcal{J}$, then $\tau' \models \mathcal{J}$.*

Proof. The result follows by $\mathsf{PTy}(\tau) \subseteq \mathsf{PTy}(\tau')$ and the functionality of τ, τ'; the latter ensures that any (pre)type in τ has no other meanings in τ'. □

Lemma 64. *If τ is a cubical type system, $A\ \mathsf{tm}\ [\Psi]$, $B\ \mathsf{tm}\ [\Psi]$, and for all $\psi_1 : \Psi_1 \to \Psi$ and $\psi_2 : \Psi_2 \to \Psi_1$, we have $A\psi_1 \Downarrow A_1$, $A_1\psi_2 \Downarrow A_2$, $A\psi_1\psi_2 \Downarrow A_{12}$, $B\psi_1 \Downarrow B_1$, $B_1\psi_2 \Downarrow B_2$, $B\psi_1\psi_2 \Downarrow B_{12}$, $\tau \models (A_2 \doteq A_{12}\ \mathsf{type}_\kappa\ [\Psi_2])$, $\tau \models (B_2 \doteq B_{12}\ \mathsf{type}_\kappa\ [\Psi_2])$, and $\tau \models (A_2 \doteq B_2\ \mathsf{type}_\kappa\ [\Psi_2])$, then $\tau \models (A \doteq B\ \mathsf{type}_\kappa\ [\Psi])$.*

Proof. We apply coherent expansion to A and the family of terms $\{A_\psi^{\Psi'} \mid A\psi \Downarrow A_\psi^{\Psi'}\}_\psi^{\Psi'}$. By our hypotheses at $\psi, \mathsf{id}_{\Psi'}$ and $\mathsf{id}_\Psi, \mathsf{id}_\Psi$ we know $\tau \models (A_\psi^{\Psi'}\ \mathsf{type}_\kappa\ [\Psi'])$ and $\tau \models ((A_{\mathsf{id}_\Psi}^\Psi)\psi\ \mathsf{type}_\kappa\ [\Psi'])$; for any $\psi' : \Psi'' \to \Psi'$, our hypotheses at ψ, ψ' and $\mathsf{id}_\Psi, \psi\psi'$ show $\tau \models (A' \doteq A_{\psi\psi'}^{\Psi''}\ \mathsf{type}_\kappa\ [\Psi''])$ where $(A_\psi^{\Psi'})\psi' \Downarrow A'$, and $\tau \models (A'' \doteq A_{\psi\psi'}^{\Psi''}\ \mathsf{type}_\kappa\ [\Psi''])$ where $(A_{\mathsf{id}_\Psi}^\Psi)\psi\psi' \Downarrow A''$, hence $\tau \models (A' \doteq A''\ \mathsf{type}_\kappa\ [\Psi''])$.

If $\kappa = \mathsf{pre}$ then by Lemma 36, $\tau \models ((A_{\mathsf{id}_\Psi}^\Psi)\psi \doteq A_0'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi'])$ where $(A_{\mathsf{id}_\Psi}^\Psi)\psi \Downarrow A_0'$; thus we have $\tau \models (A_\psi^{\Psi'} \doteq (A_{\mathsf{id}_\Psi}^\Psi)\psi\ \mathsf{type}_{\mathsf{pre}}\ [\Psi'])$ by transitivity, and by Lemma 40, $\tau \models (A \doteq A_0\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$ where $A \Downarrow A_0$. If $\kappa = \mathsf{Kan}$ then by Lemma 39, $\tau \models (A_\psi^{\Psi'} \doteq (A_{\mathsf{id}_\Psi}^\Psi)\psi\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi'])$, and by Lemma 42, $\tau \models (A \doteq A_0\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$ where $A \Downarrow A_0$. In either case, we repeat the argument for B to obtain $\tau \models (B \doteq B_0\ \mathsf{type}_\kappa\ [\Psi])$ where $B \Downarrow B_0$, and the result follows by symmetry and transitivity. □

Rule 58 (Pretype formation)**.** *If $i < j$ or $j = \omega$ then $\tau_j^{\mathsf{pre}} \models (\mathcal{U}_i^\kappa\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$ and $\tau_j^{\mathsf{Kan}} \models (\mathcal{U}_i^{\mathsf{Kan}}\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$.*

Proof. In each case we have $\mathsf{PTy}(\tau_j^{\kappa'})(\Psi, \mathcal{U}_i^\kappa, \mathcal{U}_i^\kappa, \_)$ by $\mathcal{U}_i^\kappa$ $\mathsf{val}_{\square}$ and the definition of $\tau_j^{\kappa'}$. For $\mathsf{Coh}(\llbracket \mathcal{U}_i^\kappa \rrbracket)$, show that if $\llbracket \mathcal{U}_i^\kappa \rrbracket_{\Psi'}(A_0, B_0)$ then $\mathsf{Tm}(\llbracket \mathcal{U}_i^\kappa \rrbracket(\Psi'))(A_0, B_0)$. But $\mathsf{Tm}(\llbracket \mathcal{U}_i^\kappa \rrbracket(\Psi'))(A, B)$ if and only if $\mathsf{PTy}(\tau_i^\kappa)(\Psi', A, B, \_)$, so this is immediate by value-coherence of τ_i^κ. ☐

Rule 59 (Cumulativity)**.** *If $\tau_\omega^{\mathsf{pre}} \models (A \doteq B \in \mathcal{U}_i^\kappa\ [\Psi])$ and $i \le j$ then $\tau_\omega^{\mathsf{pre}} \models (A \doteq B \in \mathcal{U}_j^\kappa\ [\Psi])$.*

Proof. In Section 3 we observed that $\tau_i^\kappa \subseteq \tau_j^\kappa$ whenever $i \le j$; thus $\llbracket \mathcal{U}_i^\kappa \rrbracket \subseteq \llbracket \mathcal{U}_j^\kappa \rrbracket$, and the result follows because Tm is order-preserving. ☐

Lemma 65.

1. *If $\tau_\omega^{\mathsf{pre}} \models (A \doteq B \in \mathcal{U}_i^{\mathsf{Kan}}\ [\Psi])$ then $\tau_i^{\mathsf{Kan}} \models (A \doteq B\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$.*
2. *If $\tau_\omega^{\mathsf{pre}} \models (A \doteq B \in \mathcal{U}_i^{\mathsf{pre}}\ [\Psi])$ then $\tau_i^{\mathsf{pre}} \models (A \doteq B\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$.*

Proof. We prove part (1) by strong induction on i. For each i, define $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \tau_i^{\mathsf{Kan}} \models (A_0 \doteq B_0\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])\}$, and show $K(\nu_i, \Phi) \subseteq \Phi$. We will conclude $\tau_i^{\mathsf{Kan}} \subseteq \Phi$ and so $\tau_i^{\mathsf{Kan}} \models (A_0 \doteq B_0\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$ whenever $\llbracket \mathcal{U}_i^{\mathsf{Kan}} \rrbracket(A_0, B_0)$; part (1) will follow by Lemma 64.

To establish $K(\nu_i, \Phi) \subseteq \Phi$, we check each type former independently. Consider the case $\textsc{Fun}(\Phi)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \varphi)$. Then $\mathsf{PTy}(\Phi)(\Psi, A, A', \alpha)$, which by Lemma 64 implies $\tau_i^{\mathsf{Kan}} \models (A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$; similarly, $\tau_i^{\mathsf{Kan}} \models (a : A \gg B \doteq B'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$. By Rule 6, we conclude $\tau_i^{\mathsf{Kan}} \models ((a{:}A) \to B \doteq (a{:}A') \to B'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$. The same argument applies for every type former except for $\textsc{UKan}$, where we must show $\tau_i^{\mathsf{Kan}} \models (\mathcal{U}_j^{\mathsf{Kan}}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$ for every $j < i$. The coe conditions are trivial by $\mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathcal{U}_j^{\mathsf{Kan}}}(M) \longmapsto_{\square} M$; the hcom conditions hold by $\mathsf{hcom}_{\mathcal{U}_j^{\mathsf{Kan}}} \longmapsto_{\square} \mathsf{fcom}$, $\tau_i^{\mathsf{Kan}} \models (A \doteq B \in \mathcal{U}_j^{\mathsf{Kan}}\ [\Psi])$ implies $\tau_i^{\mathsf{Kan}} \models (A \doteq B\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$ (by induction), and Rule 57.

We prove part (2) directly for all i, by establishing $P(\nu_i, \tau_i^{\mathsf{Kan}}, \Phi) \subseteq \Phi$ for $\Phi = \{(\Psi, A_0, B_0, \varphi) \mid \tau_i^{\mathsf{pre}} \models (A_0 \doteq B_0\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])\}$ and appealing to Lemma 64. Most type formers follow the same pattern as above; we only discuss $\textsc{Fcom}$, $\textsc{UPre}$, and $\textsc{UKan}$. For $\textsc{Fcom}$, we appeal to part (1) and Rule 52, observing that $\mathsf{PTy}(\tau_i^{\mathsf{Kan}})(\Psi, A, B, \_)$ if and only if $\mathsf{Tm}(\llbracket \mathcal{U}_i^{\mathsf{Kan}} \rrbracket(\Psi))(A, B)$. For $\textsc{UPre}$ and $\textsc{UKan}$, $\tau_i^{\mathsf{pre}} \models (\mathcal{U}_j^\kappa\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$ for all $j < i$ is immediate by Rule 58. ☐

Rule 60 (Elimination)**.** *If $\tau_\omega^{\mathsf{pre}} \models (A \doteq B \in \mathcal{U}_i^\kappa\ [\Psi])$ then $\tau_\omega^{\mathsf{pre}} \models (A \doteq B\ \mathsf{type}_\kappa\ [\Psi])$.*

Proof. Immediate by $\tau_i^\kappa \subseteq \tau_\omega^{\mathsf{pre}}$ and Lemmas 63 and 65. ☐

Rule 61 (Introduction)**.** *In $\tau_\omega^{\mathsf{pre}}$,*

1. *If $A \doteq A' \in \mathcal{U}_j^\kappa\ [\Psi]$ and $a : A \gg B \doteq B' \in \mathcal{U}_j^\kappa\ [\Psi]$ then $(a{:}A) \to B \doteq (a{:}A') \to B' \in \mathcal{U}_j^\kappa\ [\Psi]$.*
2. *If $A \doteq A' \in \mathcal{U}_j^\kappa\ [\Psi]$ and $a : A \gg B \doteq B' \in \mathcal{U}_j^\kappa\ [\Psi]$ then $(a{:}A) \times B \doteq (a{:}A') \times B' \in \mathcal{U}_j^\kappa\ [\Psi]$.*
3. *If $A \doteq A' \in \mathcal{U}_j^\kappa\ [\Psi, x]$ and $P_\varepsilon \doteq P'_\varepsilon \in A\langle \varepsilon/x \rangle\ [\Psi]$ for $\varepsilon \in \{0,1\}$ then $\mathsf{Path}_{x.A}(P_0, P_1) \doteq \mathsf{Path}_{x.A'}(P'_0, P'_1) \in \mathcal{U}_j^\kappa\ [\Psi]$.*
4. *If $A \doteq A' \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]$, $M \doteq M' \in A\ [\Psi]$, and $N \doteq N' \in A\ [\Psi]$ then $\mathsf{Eq}_A(M, N) \doteq \mathsf{Eq}_{A'}(M', N') \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]$.*
5. *$\mathsf{void} \in \mathcal{U}_j^\kappa\ [\Psi]$.*
6. *$\mathsf{nat} \in \mathcal{U}_j^\kappa\ [\Psi]$.*

7. $\mathsf{bool} \in \mathcal{U}_j^\kappa\ [\Psi]$.

8. $\mathsf{wbool} \in \mathcal{U}_j^\kappa\ [\Psi]$.

9. $\mathbb{S}^1 \in \mathcal{U}_j^\kappa\ [\Psi]$.

10. *If* $A \doteq A' \in \mathcal{U}_j^\kappa\ [\Psi \mid r = 0]$, $B \doteq B' \in \mathcal{U}_j^\kappa\ [\Psi]$, *and* $E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0]$, *then* $\mathsf{V}_r(A, B, E) \doteq \mathsf{V}_r(A', B', E') \in \mathcal{U}_j^\kappa\ [\Psi]$.

11. *If* $i < j$ *then* $\mathcal{U}_i^\kappa \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]$.

12. *If* $i < j$ *then* $\mathcal{U}_i^{\mathsf{Kan}} \in \mathcal{U}_j^{\mathsf{Kan}}\ [\Psi]$.

Proof. Note that Rule 60 is needed to make sense of these rules; for example, in part (1), by Rule 60 and $\tau_\omega^{\mathsf{pre}} \models (A \in \mathcal{U}_j^\kappa\ [\Psi])$ we conclude $\tau_\omega^{\mathsf{pre}} \models (A\ \mathsf{type}_\kappa\ [\Psi])$, which is a presupposition of $\tau_\omega^{\mathsf{pre}} \models (a : A \gg B \doteq B' \in \mathcal{U}_j^\kappa\ [\Psi])$.

For part (1), by $\tau_\omega^{\mathsf{pre}} \models (A \doteq A' \in \mathcal{U}_j^\kappa\ [\Psi])$ and Lemma 65, $\tau_j^\kappa \models (A \doteq A'\ \mathsf{type}_\kappa\ [\Psi])$; similarly, by $\tau_\omega^{\mathsf{pre}} \models (a : A \gg B \doteq B' \in \mathcal{U}_j^\kappa\ [\Psi])$ and Lemmas 63 and 65, $\tau_j^\kappa \models (a : A \gg B \doteq B'\ \mathsf{type}_\kappa\ [\Psi])$. By Rule 1, we conclude that $\tau_j^\kappa \models ((a{:}A) \to B \doteq (a{:}A') \to B'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi])$, and in particular, $\mathsf{PTy}(\tau_j^\kappa)(\Psi, (a{:}A) \to B, (a{:}A') \to B', \_)$. Therefore $\mathsf{Tm}(\llbracket \mathcal{U}_j^\kappa \rrbracket)((a{:}A) \to B, (a{:}A') \to B')$ as needed. Parts (2–12) follow the same pattern. $\square$

Rule 62 (Kan type formation)**.** $\tau_\omega^{\mathsf{pre}} \models (\mathcal{U}_i^{\mathsf{Kan}}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi])$.

Proof. By Rule 61, $\tau_\omega^{\mathsf{pre}} \models (\mathcal{U}_i^{\mathsf{Kan}} \in \mathcal{U}_{i+1}^{\mathsf{Kan}}\ [\Psi])$; the result follows by Rule 60. $\square$

Rule 63 (Subsumption)**.** *If* $\tau_\omega^{\mathsf{pre}} \models (A \doteq A' \in \mathcal{U}_i^{\mathsf{Kan}}\ [\Psi])$ *then* $\tau_\omega^{\mathsf{pre}} \models (A \doteq A' \in \mathcal{U}_i^{\mathsf{pre}}\ [\Psi])$.

Proof. By $\tau_i^{\mathsf{Kan}} \subseteq \tau_i^{\mathsf{pre}}$ we have $\llbracket \mathcal{U}_i^{\mathsf{Kan}} \rrbracket \subseteq \llbracket \mathcal{U}_i^{\mathsf{pre}} \rrbracket$ and thus $\mathsf{Tm}(\llbracket \mathcal{U}_i^{\mathsf{Kan}} \rrbracket) \subseteq \mathsf{Tm}(\llbracket \mathcal{U}_i^{\mathsf{pre}} \rrbracket)$. $\square$

6 Rules

In this section we collect the rules proven in Sections 4 and 5 (relative to $\tau_\omega^{\mathsf{pre}}$) for easy reference. Note, however, that these rules do not constitute our higher type theory, which was defined in Sections 3 and 4 and whose properties were verified in Section 5. One can settle on a different collection of rules depending on the need. For example, the RedPRL proof assistant [Sterling et al., 2017] based on this paper uses a sequent calculus rather than natural deduction, judgments without any presuppositions, and a unified context for dimensions and terms.

For the sake of concision and clarity, we state the following rules in *local form*, extending them to *global form* by *uniformity*, also called *naturality*. (This format was suggested by Martin-Löf [1984], itself inspired by Gentzen's original concept of natural deduction.) While the rules in Section 5 are stated only for closed terms, the corresponding generalizations to open-term sequents follow by the definition of the open judgments, the fact that the introduction and elimination rules respect equality (proven in Section 5), and the fact that all substitutions commute with term formers.

In the rules below, Ψ and Ξ are unordered sets, and the equations in Ξ are also unordered. $\mathcal{J}$ stands for any type equality or element equality judgment, and κ for either pre or Kan. The $\longmapsto_{\square}$ judgment is the *cubically-stable stepping* relation defined in Section 2.

Structural rules

$$\frac{A\ \mathsf{type}_\kappa\ [\Psi]}{a : A \gg a \in A\ [\Psi]} \qquad \frac{\mathcal{J}\ [\Psi] \qquad A\ \mathsf{type}_\kappa\ [\Psi]}{a : A \gg \mathcal{J}\ [\Psi]} \qquad \frac{\mathcal{J}\ [\Psi] \qquad \psi : \Psi' \to \Psi}{\mathcal{J}\psi\ [\Psi']} \qquad \frac{A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]}{A \doteq A'\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]}$$

$$\frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi]}{A' \doteq A\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi] \qquad A' \doteq A''\ \mathsf{type}_\kappa\ [\Psi]}{A \doteq A''\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{M' \doteq M \in A\ [\Psi]}{M \doteq M' \in A\ [\Psi]}$$

$$\frac{M \doteq M' \in A\ [\Psi] \qquad M' \doteq M'' \in A\ [\Psi]}{M \doteq M'' \in A\ [\Psi]} \qquad \frac{M \doteq M' \in A\ [\Psi] \qquad A \doteq A'\ \mathsf{type}_\kappa\ [\Psi]}{M \doteq M' \in A'\ [\Psi]}$$

$$\frac{a : A \gg B \doteq B'\ \mathsf{type}_\kappa\ [\Psi] \qquad N \doteq N' \in A\ [\Psi]}{B[N/a] \doteq B'[N'/a]\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{a : A \gg M \doteq M' \in B\ [\Psi] \qquad N \doteq N' \in A\ [\Psi]}{M[N/a] \doteq M'[N'/a] \in B[N/a]\ [\Psi]}$$

Restriction rules

$$\frac{\mathcal{J}\ [\Psi]}{\mathcal{J}\ [\Psi \mid \cdot]} \qquad \frac{\mathcal{J}\ [\Psi \mid \Xi]}{\mathcal{J}\ [\Psi \mid \Xi, \varepsilon = \varepsilon]} \qquad \frac{}{\mathcal{J}\ [\Psi \mid \Xi, \varepsilon = \overline{\varepsilon}]} \qquad \frac{\mathcal{J}\langle r/x\rangle\ [\Psi \mid \Xi\langle r/x\rangle]}{\mathcal{J}\ [\Psi, x \mid \Xi, x = r]}$$

Computation rules

$$\frac{A' \doteq B\ \mathsf{type}_\kappa\ [\Psi] \qquad A \longmapsto_{\square} A'}{A \doteq B\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{M' \doteq N \in A\ [\Psi] \qquad M \longmapsto_{\square} M'}{M \doteq N \in A\ [\Psi]}$$

Kan conditions

$$
\frac{r_i = r_j \qquad r_i' = 0 \qquad r_j' = 1}{\overrightarrow{r_i = r_i'}\ \mathsf{valid}}
\qquad\qquad
\frac{r_i = r_i'}{\overrightarrow{r_i = r_i'}\ \mathsf{valid}}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'}\ \mathsf{valid} \\
 & A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \\
 & M \doteq M' \in A\ [\Psi] \\
(\forall i,j) & N_i \doteq N_j' \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & N_i\langle r/y\rangle \doteq M \in A\ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{hcom}_A^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{hcom}_{A'}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in A\ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'}\ \mathsf{valid} \\
 & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \\
 & M \in A\ [\Psi] \\
(\forall i,j) & N_i \doteq N_j \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & N_i\langle r/y\rangle \doteq M \in A\ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{hcom}_A^{r\rightsquigarrow r}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in A\ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & r_i = r_i' \\
 & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \\
 & M \in A\ [\Psi] \\
(\forall i,j) & N_i \doteq N_j \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & N_i\langle r/y\rangle \doteq M \in A\ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{hcom}_A^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\ [\Psi]}
$$

$$
\frac{A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x] \qquad M \doteq M' \in A\langle r/x\rangle\ [\Psi]}{\mathsf{coe}_{x.A}^{r\rightsquigarrow r'}(M) \doteq \mathsf{coe}_{x.A'}^{r\rightsquigarrow r'}(M') \in A\langle r'/x\rangle\ [\Psi]}
\qquad
\frac{A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x] \qquad M \in A\langle r/x\rangle\ [\Psi]}{\mathsf{coe}_{x.A}^{r\rightsquigarrow r}(M) \doteq M \in A\langle r/x\rangle\ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'}\ \mathsf{valid} \\
 & A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y] \\
 & M \doteq M' \in A\langle r/y\rangle\ [\Psi] \\
(\forall i,j) & N_i \doteq N_j' \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & N_i\langle r/y\rangle \doteq M \in A\langle r/y\rangle\ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{com}_{y.A}^{r\rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq \mathsf{com}_{y.A'}^{r\rightsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i'}) \in A\langle r'/y\rangle\ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'}\ \mathsf{valid} \\
 & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y] \\
 & M \in A\langle r/y\rangle\ [\Psi] \\
(\forall i,j) & N_i \doteq N_j \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & N_i\langle r/y\rangle \doteq M \in A\langle r/y\rangle\ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{com}_{y.A}^{r\rightsquigarrow r}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}) \doteq M \in A\langle r/y\rangle\ [\Psi]}
$$

$$\frac{\begin{array}{ll} & r_i = r_i' \\ & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y] \\ & M \in A\langle r/y\rangle\ [\Psi] \\ (\forall i, j) & N_i \doteq N_j \in A\ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\ (\forall i) & N_i\langle r/y\rangle \doteq M \in A\langle r/y\rangle\ [\Psi \mid r_i = r_i'] \end{array}}{\mathsf{com}_{y.A}^{r \rightsquigarrow r'}(M; \overline{r_i = r_i' \hookrightarrow y.N_i}) \doteq N_i\langle r'/y\rangle \in A\langle r'/y\rangle\ [\Psi]}$$

Dependent function types

$$\frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi] \qquad a : A \gg B \doteq B'\ \mathsf{type}_\kappa\ [\Psi]}{(a{:}A) \to B \doteq (a{:}A') \to B'\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{a : A \gg M \doteq M' \in B\ [\Psi]}{\lambda a.M \doteq \lambda a.M' \in (a{:}A) \to B\ [\Psi]}$$

$$\frac{M \doteq M' \in (a{:}A) \to B\ [\Psi] \qquad N \doteq N' \in A\ [\Psi]}{\mathsf{app}(M, N) \doteq \mathsf{app}(M', N') \in B[N/a]\ [\Psi]} \qquad \frac{a : A \gg M \in B\ [\Psi] \qquad N \in A\ [\Psi]}{\mathsf{app}(\lambda a.M, N) \doteq M[N/a] \in B[N/a]\ [\Psi]}$$

$$\frac{M \in (a{:}A) \to B\ [\Psi]}{M \doteq \lambda a.\mathsf{app}(M, a) \in (a{:}A) \to B\ [\Psi]}$$

Dependent pair types

$$\frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi] \qquad a : A \gg B \doteq B'\ \mathsf{type}_\kappa\ [\Psi]}{(a{:}A) \times B \doteq (a{:}A') \times B'\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{M \doteq M' \in A\ [\Psi] \qquad N \doteq N' \in B[M/a]\ [\Psi]}{\langle M, N\rangle \doteq \langle M', N'\rangle \in (a{:}A) \times B\ [\Psi]}$$

$$\frac{P \doteq P' \in (a{:}A) \times B\ [\Psi]}{\mathsf{fst}(P) \doteq \mathsf{fst}(P') \in A\ [\Psi]} \qquad \frac{P \doteq P' \in (a{:}A) \times B\ [\Psi]}{\mathsf{snd}(P) \doteq \mathsf{snd}(P') \in B[\mathsf{fst}(P)/a]\ [\Psi]} \qquad \frac{M \in A\ [\Psi]}{\mathsf{fst}(\langle M, N\rangle) \doteq M \in A\ [\Psi]}$$

$$\frac{N \in B\ [\Psi]}{\mathsf{snd}(\langle M, N\rangle) \doteq N \in B\ [\Psi]} \qquad \frac{P \in (a{:}A) \times B\ [\Psi]}{P \doteq \langle \mathsf{fst}(P), \mathsf{snd}(P)\rangle \in (a{:}A) \times B\ [\Psi]}$$

Path types

$$\frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi, x] \qquad (\forall \varepsilon)\ P_\varepsilon \doteq P_\varepsilon' \in A\langle \varepsilon/x\rangle\ [\Psi]}{\mathsf{Path}_{x.A}(P_0, P_1) \doteq \mathsf{Path}_{x.A'}(P_0', P_1')\ \mathsf{type}_\kappa\ [\Psi]}$$

$$\frac{M \doteq M' \in A\ [\Psi, x] \qquad (\forall \varepsilon)\ M\langle \varepsilon/x\rangle \doteq P_\varepsilon \in A\langle \varepsilon/x\rangle\ [\Psi]}{\langle x\rangle M \doteq \langle x\rangle M' \in \mathsf{Path}_{x.A}(P_0, P_1)\ [\Psi]} \qquad \frac{M \doteq M' \in \mathsf{Path}_{x.A}(P_0, P_1)\ [\Psi]}{M@r \doteq M'@r \in A\langle r/x\rangle\ [\Psi]}$$

$$\frac{M \in \mathsf{Path}_{x.A}(P_0, P_1)\ [\Psi]}{M@\varepsilon \doteq P_\varepsilon \in A\langle \varepsilon/x\rangle\ [\Psi]} \qquad \frac{M \in A\ [\Psi, x]}{(\langle x\rangle M)@r \doteq M\langle r/x\rangle \in A\langle r/x\rangle\ [\Psi]}$$

$$\frac{M \in \mathsf{Path}_{x.A}(P_0, P_1)\ [\Psi]}{M \doteq \langle x\rangle(M@x) \in \mathsf{Path}_{x.A}(P_0, P_1)\ [\Psi]}$$

Equality pretypes

$$\frac{A \doteq A' \ \mathsf{type_{pre}}\ [\Psi] \qquad M \doteq M' \in A\ [\Psi] \qquad N \doteq N' \in A\ [\Psi]}{\mathsf{Eq}_A(M,N) \doteq \mathsf{Eq}_{A'}(M',N')\ \mathsf{type_{pre}}\ [\Psi]} \qquad \frac{M \doteq N \in A\ [\Psi]}{\star \in \mathsf{Eq}_A(M,N)\ [\Psi]}$$

$$\frac{E \in \mathsf{Eq}_A(M,N)\ [\Psi]}{M \doteq N \in A\ [\Psi]} \qquad \frac{E \in \mathsf{Eq}_A(M,N)\ [\Psi]}{E \doteq \star \in \mathsf{Eq}_A(M,N)\ [\Psi]}$$

Void

$$\frac{}{\mathsf{void}\ \mathsf{type_{Kan}}\ [\Psi]} \qquad \frac{M \in \mathsf{void}\ [\Psi]}{\mathcal{J}\ [\Psi]}$$

Natural numbers

$$\frac{}{\mathsf{nat}\ \mathsf{type_{Kan}}\ [\Psi]} \qquad \frac{}{\mathsf{z} \in \mathsf{nat}\ [\Psi]} \qquad \frac{M \doteq M' \in \mathsf{nat}\ [\Psi]}{\mathsf{s}(M) \doteq \mathsf{s}(M') \in \mathsf{nat}\ [\Psi]}$$

$$\frac{\begin{array}{c} n : \mathsf{nat} \gg A\ \mathsf{type}_\kappa\ [\Psi] \\ M \doteq M' \in \mathsf{nat}\ [\Psi] \qquad Z \doteq Z' \in A[\mathsf{z}/n]\ [\Psi] \qquad n : \mathsf{nat}, a : A \gg S \doteq S' \in A[\mathsf{s}(n)/n]\ [\Psi] \end{array}}{\mathsf{natrec}(M; Z, n.a.S) \doteq \mathsf{natrec}(M'; Z', n.a.S') \in A[M/n]\ [\Psi]}$$

$$\frac{Z \in A\ [\Psi]}{\mathsf{natrec}(\mathsf{z}; Z, n.a.S) \doteq Z \in A\ [\Psi]}$$

$$\frac{n : \mathsf{nat} \gg A\ \mathsf{type}_\kappa\ [\Psi] \qquad M \in \mathsf{nat}\ [\Psi] \qquad Z \in A[\mathsf{z}/n]\ [\Psi] \qquad n : \mathsf{nat}, a : A \gg S \in A[\mathsf{s}(n)/n]\ [\Psi]}{\mathsf{natrec}(\mathsf{s}(M); Z, n.a.S) \doteq S[M/n][\mathsf{natrec}(M; Z, n.a.S)/a] \in A[\mathsf{s}(M)/n]\ [\Psi]}$$

Booleans

$$\frac{}{\mathsf{bool}\ \mathsf{type_{Kan}}\ [\Psi]} \qquad \frac{}{\mathsf{true} \in \mathsf{bool}\ [\Psi]} \qquad \frac{}{\mathsf{false} \in \mathsf{bool}\ [\Psi]}$$

$$\frac{\begin{array}{c} b : \mathsf{bool} \gg C\ \mathsf{type_{pre}}\ [\Psi] \\ M \doteq M' \in \mathsf{bool}\ [\Psi] \qquad T \doteq T' \in C[\mathsf{true}/b]\ [\Psi] \qquad F \doteq F' \in C[\mathsf{false}/b]\ [\Psi] \end{array}}{\mathsf{if}_{b.A}(M; T, F) \doteq \mathsf{if}_{b.A'}(M'; T', F') \in C[M/b]\ [\Psi]}$$

$$\frac{T \in B\ [\Psi]}{\mathsf{if}_{b.A}(\mathsf{true}; T, F) \doteq T \in B\ [\Psi]} \qquad \frac{F \in B\ [\Psi]}{\mathsf{if}_{b.A}(\mathsf{false}; T, F) \doteq F \in B\ [\Psi]}$$

Weak Booleans

$$\frac{}{\mathsf{wbool}\ \mathsf{type_{Kan}}\ [\Psi]} \qquad \frac{M \doteq M' \in \mathsf{bool}\ [\Psi]}{M \doteq M' \in \mathsf{wbool}\ [\Psi]}$$

$$\frac{\begin{array}{c} b : \mathsf{wbool} \gg A \doteq A'\ \mathsf{type_{Kan}}\ [\Psi] \\ M \doteq M' \in \mathsf{wbool}\ [\Psi] \qquad T \doteq T' \in A[\mathsf{true}/b]\ [\Psi] \qquad F \doteq F' \in A[\mathsf{false}/b]\ [\Psi] \end{array}}{\mathsf{if}_{b.A}(M; T, F) \doteq \mathsf{if}_{b.A'}(M'; T', F') \in A[M/b]\ [\Psi]}$$

Circle

$$\frac{}{\mathbb{S}^1\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]} \qquad \frac{}{\mathsf{base} \in \mathbb{S}^1\ [\Psi]} \qquad \frac{}{\mathsf{loop}_r \in \mathbb{S}^1\ [\Psi]} \qquad \frac{}{\mathsf{loop}_\varepsilon \doteq \mathsf{base} \in \mathbb{S}^1\ [\Psi]}$$

$$\frac{\begin{array}{c} c:\mathbb{S}^1 \gg A \doteq A'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \qquad M \doteq M' \in \mathbb{S}^1\ [\Psi] \\ P \doteq P' \in A[\mathsf{base}/c]\ [\Psi] \qquad L \doteq L' \in A[\mathsf{loop}_x/c]\ [\Psi, x] \qquad (\forall \varepsilon)\ L\langle \varepsilon/x \rangle \doteq P \in A[\mathsf{base}/c]\ [\Psi] \end{array}}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(M; P, x.L) \doteq \mathbb{S}^1\text{-}\mathsf{elim}_{c.A'}(M'; P', x.L') \in A[M/c]\ [\Psi]}$$

$$\frac{P \in B\ [\Psi]}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{base}; P, x.L) \doteq P \in B\ [\Psi]} \qquad \frac{L \in B\ [\Psi, x] \qquad (\forall \varepsilon)\ L\langle \varepsilon/x \rangle \doteq P \in B\langle \varepsilon/x \rangle\ [\Psi]}{\mathbb{S}^1\text{-}\mathsf{elim}_{c.A}(\mathsf{loop}_r; P, x.L) \doteq L\langle r/x \rangle \in B\langle r/x \rangle\ [\Psi]}$$

Univalence

$$\mathsf{isContr}(C) := C \times ((c{:}C) \to (c'{:}C) \to \mathsf{Path}_{\_.C}(c, c'))$$

$$\mathsf{Equiv}(A, B) := (f{:}A \to B) \times ((b{:}B) \to \mathsf{isContr}((a{:}A) \times \mathsf{Path}_{\_.B}(\mathsf{app}(f, a), b)))$$

$$\frac{A \doteq A'\ \mathsf{type}_\kappa\ [\Psi \mid r = 0] \qquad B \doteq B'\ \mathsf{type}_\kappa\ [\Psi] \qquad E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0]}{\mathsf{V}_r(A, B, E) \doteq \mathsf{V}_r(A', B', E')\ \mathsf{type}_\kappa\ [\Psi]}$$

$$\frac{A\ \mathsf{type}_\kappa\ [\Psi]}{\mathsf{V}_0(A, B, E) \doteq A\ \mathsf{type}_\kappa\ [\Psi]} \qquad \frac{B\ \mathsf{type}_\kappa\ [\Psi]}{\mathsf{V}_1(A, B, E) \doteq B\ \mathsf{type}_\kappa\ [\Psi]}$$

$$\frac{\begin{array}{c} M \doteq M' \in A\ [\Psi \mid r = 0] \\ N \doteq N' \in B\ [\Psi] \qquad E \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0] \qquad \mathsf{app}(\mathsf{fst}(E), M) \doteq N \in B\ [\Psi \mid r = 0] \end{array}}{\mathsf{Vin}_r(M, N) \doteq \mathsf{Vin}_r(M', N') \in \mathsf{V}_r(A, B, E)\ [\Psi]}$$

$$\frac{M \in A\ [\Psi]}{\mathsf{Vin}_0(M, N) \doteq M \in A\ [\Psi]} \qquad \frac{N \in B\ [\Psi]}{\mathsf{Vin}_1(M, N) \doteq N \in B\ [\Psi]}$$

$$\frac{M \doteq M' \in \mathsf{V}_r(A, B, E)\ [\Psi] \qquad F \doteq \mathsf{fst}(E) \in A \to B\ [\Psi \mid r = 0]}{\mathsf{Vproj}_r(M, F) \doteq \mathsf{Vproj}_r(M', \mathsf{fst}(E)) \in B\ [\Psi]}$$

$$\frac{M \in A\ [\Psi] \qquad F \in A \to B\ [\Psi]}{\mathsf{Vproj}_0(M, F) \doteq \mathsf{app}(F, M) \in B\ [\Psi]} \qquad \frac{M \in B\ [\Psi]}{\mathsf{Vproj}_1(M, F) \doteq M \in B\ [\Psi]}$$

$$\frac{M \in A\ [\Psi \mid r = 0] \qquad N \in B\ [\Psi] \qquad F \in A \to B\ [\Psi \mid r = 0] \qquad \mathsf{app}(F, M) \doteq N \in B\ [\Psi \mid r = 0]}{\mathsf{Vproj}_r(\mathsf{Vin}_r(M, N), F) \doteq N \in B\ [\Psi]}$$

$$\frac{N \in \mathsf{V}_r(A, B, E)\ [\Psi] \qquad M \doteq N \in A\ [\Psi \mid r = 0]}{\mathsf{Vin}_r(M, \mathsf{Vproj}_r(N, \mathsf{fst}(E))) \doteq N \in \mathsf{V}_r(A, B, E)\ [\Psi]}$$

Universes

$$\frac{}{\mathcal{U}_j^{\mathsf{pre}}\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]} \qquad \frac{}{\mathcal{U}_j^{\mathsf{Kan}}\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]} \qquad \frac{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi]}{A \doteq A'\ \mathsf{type}_{\kappa}\ [\Psi]} \qquad \frac{A \doteq A' \in \mathcal{U}_i^{\kappa}\ [\Psi] \qquad i \leq j}{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{A \doteq A' \in \mathcal{U}_j^{\mathsf{Kan}}\ [\Psi]}{A \doteq A' \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]} \qquad \frac{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi] \qquad a : A \gg B \doteq B' \in \mathcal{U}_j^{\kappa}\ [\Psi]}{(a{:}A) \to B \doteq (a{:}A') \to B' \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi] \qquad a : A \gg B \doteq B' \in \mathcal{U}_j^{\kappa}\ [\Psi]}{(a{:}A) \times B \doteq (a{:}A') \times B' \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi, x] \qquad (\forall \varepsilon)\ P_\varepsilon \doteq P'_\varepsilon \in A\langle \varepsilon/x \rangle\ [\Psi]}{\mathsf{Path}_{x.A}(P_0, P_1) \doteq \mathsf{Path}_{x.A'}(P'_0, P'_1) \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{A \doteq A' \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi] \qquad M \doteq M' \in A\ [\Psi] \qquad N \doteq N' \in A\ [\Psi]}{\mathsf{Eq}_A(M, N) \doteq \mathsf{Eq}_{A'}(M', N') \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]} \qquad \frac{}{\mathsf{void} \in \mathcal{U}_j^{\kappa}\ [\Psi]} \qquad \frac{}{\mathsf{nat} \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{}{\mathsf{bool} \in \mathcal{U}_j^{\kappa}\ [\Psi]} \qquad \frac{}{\mathsf{wbool} \in \mathcal{U}_j^{\kappa}\ [\Psi]} \qquad \frac{}{\mathbb{S}^1 \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{A \doteq A' \in \mathcal{U}_j^{\kappa}\ [\Psi \mid r = 0] \qquad B \doteq B' \in \mathcal{U}_j^{\kappa}\ [\Psi] \qquad E \doteq E' \in \mathsf{Equiv}(A, B)\ [\Psi \mid r = 0]}{\mathsf{V}_r(A, B, E) \doteq \mathsf{V}_r(A', B', E') \in \mathcal{U}_j^{\kappa}\ [\Psi]}$$

$$\frac{i < j}{\mathcal{U}_i^{\kappa} \in \mathcal{U}_j^{\mathsf{pre}}\ [\Psi]} \qquad \frac{i < j}{\mathcal{U}_i^{\mathsf{Kan}} \in \mathcal{U}_j^{\mathsf{Kan}}\ [\Psi]}$$

$$\frac{\begin{array}{ll} & \overrightarrow{r_i = r'_i}\ \mathsf{valid} \\ & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \\ & M \doteq M' \in A\ [\Psi] \\ (\forall i, j) & B_i \doteq B_j\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y \mid r_i = r'_i, r_j = r'_j] \\ (\forall i, j) & N_i \doteq N'_j \in B_i\langle r'/y \rangle\ [\Psi \mid r_i = r'_i, r_j = r'_j] \\ (\forall i) & B_i\langle r/y \rangle \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid r_i = r'_i] \\ (\forall i) & \mathsf{coe}_{y.B_i}^{r' \rightsquigarrow r}(N_i) \doteq M \in A\ [\Psi \mid r_i = r'_i] \end{array}}{\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}) \doteq \mathsf{box}^{r \rightsquigarrow r'}(M'; \overrightarrow{r_i = r'_i \hookrightarrow N'_i}) \in \mathsf{hcom}_{\mathcal{U}_j^{\mathsf{Kan}}}^{r \rightsquigarrow r'}(A; \overrightarrow{r_i = r'_i \hookrightarrow y.B_i})\ [\Psi]}$$

$$\frac{M \in A\ [\Psi]}{\mathsf{box}^{r \rightsquigarrow r}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}) \doteq M \in A\ [\Psi]} \qquad \frac{\begin{array}{ll} & r_i = r'_i \\ & A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi] \\ & M \in A\ [\Psi] \\ (\forall i, j) & B_i \doteq B_j\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, y \mid r_i = r'_i, r_j = r'_j] \\ (\forall i, j) & N_i \doteq N_j \in B_i\langle r'/y \rangle\ [\Psi \mid r_i = r'_i, r_j = r'_j] \\ (\forall i) & B_i\langle r/y \rangle \doteq A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi \mid r_i = r'_i] \\ (\forall i) & \mathsf{coe}_{y.B_i}^{r' \rightsquigarrow r}(N_i) \doteq M \in A\ [\Psi \mid r_i = r'_i] \end{array}}{\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i}) \doteq N_i \in B_i\langle r'/y \rangle\ [\Psi]}$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'} \ \mathsf{valid} \\
 & A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi] \\
(\forall i, j) & B_i \doteq B_j' \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & B_i\langle r/y\rangle \doteq A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi \mid r_i = r_i'] \\
 & M \doteq M' \in \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathcal{U}^{\mathsf{Kan}}_j}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \ [\Psi]
\end{array}}{\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{cap}^{r \leftsquigarrow r'}(M'; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i'}) \in A \ [\Psi]}
$$

$$
\frac{M \in A \ [\Psi]}{\mathsf{cap}^{r \leftsquigarrow r}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq M \in A \ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & r_i = r_i' \\
 & A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi] \\
(\forall i, j) & B_i \doteq B_j' \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & B_i\langle r/y\rangle \doteq A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi \mid r_i = r_i'] \\
 & M \doteq M' \in \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathcal{U}^{\mathsf{Kan}}_j}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \ [\Psi]
\end{array}}{\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(M) \in A \ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'} \ \mathsf{valid} \\
 & A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi] \\
 & M \doteq M' \in A \ [\Psi] \\
(\forall i, j) & B_i \doteq B_j \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i, j) & N_i \doteq N_j' \in B_i\langle r'/y\rangle \ [\Psi \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & B_i\langle r/y\rangle \doteq A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi \mid r_i = r_i'] \\
(\forall i) & \mathsf{coe}^{r' \rightsquigarrow r}_{y.B_i}(N_i) \doteq M \in A \ [\Psi \mid r_i = r_i']
\end{array}}{\mathsf{cap}^{r \leftsquigarrow r'}(\mathsf{box}^{r \rightsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i}); \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \doteq M \in A \ [\Psi]}
$$

$$
\frac{\begin{array}{ll}
 & \overrightarrow{r_i = r_i'} \ \mathsf{valid} \\
 & A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi] \\
(\forall i, j) & B_i \doteq B_j \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi, y \mid r_i = r_i', r_j = r_j'] \\
(\forall i) & B_i\langle r/y\rangle \doteq A \ \mathsf{type}_{\mathsf{Kan}} \ [\Psi \mid r_i = r_i'] \\
 & M \in \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathcal{U}^{\mathsf{Kan}}_j}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \ [\Psi]
\end{array}}{\mathsf{box}^{r \rightsquigarrow r'}(\mathsf{cap}^{r \leftsquigarrow r'}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}); \overrightarrow{r_i = r_i' \hookrightarrow M}) \doteq M \in \mathsf{hcom}^{r \rightsquigarrow r'}_{\mathcal{U}^{\mathsf{Kan}}_j}(A; \overrightarrow{r_i = r_i' \hookrightarrow y.B_i}) \ [\Psi]}
$$

7 Future work

Formal Cartesian cubical type theory With Guillaume Brunerie, Thierry Coquand, and Dan Licata, we have developed a formal Cartesian cubical type theory with univalent universes, accompanied by a constructive cubical set model, most of which has been formalized in Agda in the style of Orton and Pitts [2016]. This forthcoming work explores the the Kan operations described in this paper—in particular, with the addition of $x = z$ diagonal constraints—in a proof-theoretic and model-theoretic setting, rather than the computational setting emphasized in this paper.

Cubical (higher) inductive types Evan Cavallo is currently extending this work to account for a general class of inductive types with higher-dimensional recursive constructors. In the cubical setting, such types are generated by dimension-parametrized constructors with prescribed boundaries. (For example, $\mathbb{S}^1$ is generated by base and loop_x, whose x-faces are base.)

Discrete, hcom, and coe types In this paper we divide types into pretypes and Kan types, but finer distinctions are possible. Some types support hcom but not necessarily coe, or vice versa. Exact equality types always have hcom structure because $\star$ is a suitable composite for every box, but not coe in general. Types with hcom or coe structure are not themselves closed under all type formers, but depend on each other; for example,

1. $(a{:}A) \to B\ \mathsf{type}_{\mathsf{hcom}}\ [\Psi]$ when $A\ \mathsf{type}_{\mathsf{pre}}\ [\Psi]$ and $a : A \gg B\ \mathsf{type}_{\mathsf{hcom}}\ [\Psi]$,
2. $(a{:}A) \times B\ \mathsf{type}_{\mathsf{hcom}}\ [\Psi]$ when $A\ \mathsf{type}_{\mathsf{hcom}}\ [\Psi]$ and $a : A \gg B\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi]$,
3. $(a{:}A) \to B\ \mathsf{type}_{\mathsf{coe}}\ [\Psi]$ when $A\ \mathsf{type}_{\mathsf{coe}}\ [\Psi]$ and $a : A \gg B\ \mathsf{type}_{\mathsf{coe}}\ [\Psi]$, and
4. $\mathsf{Path}_{x.A}(M, N)\ \mathsf{type}_{\mathsf{coe}}\ [\Psi]$ when $A\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi, x]$, $M \in A\langle 0/x\rangle\ [\Psi]$, and $N \in A\langle 1/x\rangle\ [\Psi]$.

Discrete Kan types, such as nat and bool, are not only Kan but also strict sets, in the sense that all paths are exactly equal to reflexivity. To be precise, we say $A \doteq B\ \mathsf{type}_{\mathsf{disc}}\ [\Psi]$ if for any $\psi_1 : \Psi_1 \to \Psi$, $\psi_2, \psi_2' : \Psi_2 \to \Psi_1$, we have $A\psi_1\psi_2 \doteq B\psi_1\psi_2'\ \mathsf{type}_{\mathsf{Kan}}\ [\Psi_2]$, and for any $M \in A\psi_1\ [\Psi_1]$, we have $M\psi_2 \doteq M\psi_2' \in A\psi_1\psi_2\ [\Psi_2]$. Discrete Kan types are closed under most type formers, including exact equality. Exact equality types do not in general admit coercion, because $\mathsf{coe}^{0\rightsquigarrow 1}_{x.\mathsf{Eq}_A(P\langle 0/x\rangle, P)}(\star)$ turns any line P into an exact equality $\mathsf{Eq}_A(P\langle 0/x\rangle, P\langle 1/x\rangle)$ between its end points. However, if $A\ \mathsf{type}_{\mathsf{disc}}\ [\Psi]$ then $a : A, a' : A \gg \mathsf{Eq}_A(a, a')\ \mathsf{type}_{\mathsf{disc}}\ [\Psi]$, because paths in A *are* exact equalities.

Further improvements in RedPRL Implementing and using this type theory in RedPRL has already led to several minor improvements not described in this paper:

1. We have added *line types* to RedPRL, $(x{:}\mathsf{dim}) \to A$, path types whose end points are not fixed. Elements of line types are simply terms with an abstracted dimension, which has proven cleaner in practice than the iterated sigma type $(a{:}A) \times (a'{:}A) \times \mathsf{Path}_{\_.A}(a, a')$.
2. We are experimenting with alternative implementations of the Kan operations for fcom and V types in RedPRL, some inspired by the work in the forthcoming formal Cartesian cubical type theory mentioned above.
3. The RedPRL proof theory includes discrete Kan, hcom, and coe types as described above, in addition to the Kan types and pretypes described in this paper.
4. The definitions of the $M \longmapsto_{\square} M'$ and $M\ \mathsf{val}_{\square}$ judgments have been extended to account for computations that are stable by virtue of taking place under dimension binders.

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