Outline

1. Algebraic Theories
2. Prog. Language
3. Reasoning (connecting 1 & 2)
4. Applications

math → programming

1 Algebraic Theories

Example: A group: $\langle G, u, \cdot, {}^{-1} \rangle$ and equations

- assoc. of $\cdot$
- $x \cdot u = u = u \cdot x$
- $x \cdot x^{-1} = u = x^{-1} \cdot x$

General Form: operations on a set + equations

$(u, \cdot, {}^{-1})$ ("laws")

nullary, binary, unary

Alternatively: $(G, u, \cdot)$ a monoid (w/ assoc.) $\cdot$ and unit u.

plus axiom: $\underbrace{\forall x \, \exists y}$ $x \cdot y = u = y \cdot x$

(not just an equation w/ $\forall$)

It has $\forall \exists \ldots$

(complicates the theory)

Def: signature $\Sigma = \{ (op_i, n_i) : i \in I \}$

where op_i is an operation symbol

and each n_i = arity of op_i.

Def: A term in context $x_1, \ldots, x_k$ is

- one of the variables or
- an operation applied to some terms:

$op_i(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are terms in context $x_1, \ldots, x_k$.

(Read as an inductive def. These are all/only ways to construct terms.)

Def: An algebraic (or equational) theory T is $(\Sigma_T, \mathcal{E}_T)$ where Σ_T is a signature & $\mathcal{E}_T$ is a set of equations, where an equation is $x_1, \ldots, x_k \mid l = r$

l, r are Σ_T-terms in $x_1, \ldots, x_k$.

Ex (Group) signature $\Sigma_{Group} = \{ (u, 0), (m, 2), (i, 1) \}$

$$\mathcal{E}_{Group} = \{ x \mid m(u(), x) = x, \quad x \mid m(x, i(x)) = u(), \ldots \}$$

Others:
- ring
- empty

Ex: Thy of a pointed set $\Sigma = \{(\circ, 0)\}$

Ex: Semilattice: $\Sigma_{SL} = \{(1,0), (\vee, 2)\}$

$$\mathcal{E}_{SL} = \{ x \mid \bot \vee x = x,\ xyz \mid x \vee (y \vee z) = (x \vee y) \vee z,\ x \mid x \vee x = x \}$$

Ex: Field $+, -, \times, {}^{-1}, 0, 1, \cdot$

problem: 0^{-1} undefined

2 Interpretations & Models

Suppose T is a theory

An interpretation I of Σ_T is given by

- a carrier set $|I|$
- for each $(op_i\ n_i) \in \Sigma_T$ a map

$$[\![op_i]\!]_I : \underbrace{|I| \times \cdots \times |I|}_{n_i} \longrightarrow |I|$$

Each term $x_1, \ldots, x_k \mid t$ is interpreted as a map $[\![x_1, \ldots, x_k \mid t]\!] : |I|^k \longrightarrow |I|$ as follows:

From now on, write $X^m = \underbrace{X \cdots X}_{m \text{ times}}$

$X^0 = \{()\} = 1$

- $[\![x_1, \ldots, x_k \mid x_i]\!] : |I|^k \to |I|$

 (π_i = i^{th} projection)

- $[\![x_1, \ldots, x_k \mid op_i(t_1, \ldots, t_{n_i})]\!]$:

$$|I|^k \xrightarrow{[\![t_1]\!]_I \cdots [\![t_{n_i}]\!]} |I|^{n_i} \xrightarrow{[\![op_i]\!]} |I|$$

Def: A T-model is an interpretation M of theory T s.t. for every $x_1, \ldots, x_k \mid l = r$ in $\mathcal{E}_T$ the maps

$$[\![x_1, \ldots, x_k \mid l]\!]_M : |M|^k \longrightarrow |M| \quad \text{and}$$

$$[\![x_1, \ldots, x_k \mid r]\!]_M : |M|^k \longrightarrow |M|$$

are equal.

~~Example~~ Convention: Every theory has the trivial model M

$|M| = 1 \quad [\![op_i]\!]_M : 1^{n_i} \to 1$

3 Free Models

Suppose T is a theory and X is a set. Say that a T-model M with a map $\eta : X \to |M|$ is **freely generated by X** when $\forall$ T-models L
$\forall f : X \to |L|$
$\exists! h : |M| \to |L|$ hom.
s.t. $h \circ \eta = f$

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "f"'] & {|M|} \arrow[d, dashed, "\exists! h"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & {|L|}
\end{tikzcd}
$$

A T-homomorphism $f : M \to L$ is a map from $|M|$ to $|L|$ s.t.
$f\, [\![op_i]\!]_M(a_1, \ldots, a_{n_i}) = [\![op_i]\!]_L(f(a_1), \ldots, f(a_{n_i}))$

Terminology: $(M, \eta : X \to |M|)$ is $\left\{\begin{array}{l}\text{free model over } X \\ \text{or free model gen by } X\end{array}\right.$

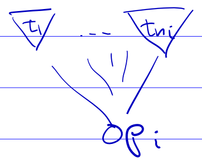
Example: (Free semilattice)

Define $\mathcal{P}_{<\omega}(X) = \{ S \subseteq X \mid S \text{ finite} \}$

Claim: $\langle \mathcal{P}_{<\omega}(X), \emptyset, \cup \rangle$ the free SL generated by X.
(with $\eta : X \to \mathcal{P}_{<\omega}(X)$)
$\eta : x \mapsto \{x\}$

Let L be a S.L.
and $f : X \to L$ $\qquad L = \langle |L|, \bot, \vee \rangle$
Define $h : \mathcal{P}_{<\omega}(X) \to L$ as follows:
Let $h(\emptyset) = \bot$
and $h(\eta(x_i)) = f(x_i)$ i.e. $h(\{x_i\}) = f(x_i)$.
Extend h to all of $\mathcal{P}_{<\omega}(X)$.
check: $h(S_1 \cup S_2) = h(S_1) \vee h(S_2)$.

Free Model Construction $\mathrm{Free}_T(X)$

1. $\mathrm{Tree}_T(X)$ = set of well founded trees
(no infinite paths)
defined inductively.
   - for every $x \in X$, there is a tree called "return x", i.e. return $x \in \mathrm{Tree}_T(X)$
   - If $(op_i, n_i) \in \Sigma_T$ and $t_1, \ldots, t_{n_i} \in \mathrm{Tree}_T(X)$
then 

$op_i(t_1, \ldots, t_{n_i}) \in \mathrm{Tree}_T(X)$.

2. Define an equivalence relation $\simeq_T$ on $\mathrm{Tree}_T(X)$ to be the least equiv. rel. s.t. ...
a. ...
b. ...
} (Homework) congruence (wrt ops) and it validates the eqns $\mathcal{E}_T$.