CH table

programming	logic
programs	proofs
types	propositions
functions	implications
product types	conjunctions
sum types	disjunction
simply typed λC STLC	intuitionistic logic (GND)
polymorphism	forall, 2nd order logic
control-flow	classical
combinator calculus	Hilbert ← more axioms, less inference rules
dependent types	FOL

LL :

with, plus, par, top, one, zero

$$\tau ::= \tau_1 \,\&\, \tau_2 \mid \tau_1 \otimes \tau_2 \mid \tau_1 \oplus \tau_2 \mid \tau_1 \parr \tau_2 \mid \top \mid 1 \mid \bot \mid 0$$

multiplicative and

additive and/or

multiplicative disj./or

program error

simply typed λC when you erase the program.

$$\frac{}{\Gamma, \tau \vdash \tau} \text{ Axiom, id}$$

$$\frac{\Gamma \vdash \tau \to \tau' \quad \Gamma \vdash \tau}{\Gamma \vdash \tau'} \text{ app (elim impl) } \to E$$

$$\frac{\Gamma, \tau \vdash \tau'}{\Gamma \vdash \tau \to \tau'} \text{ (intro impl) } \to I$$

Linear Logic — LL is a resource-sensitive logic

$\tau_1, \tau_2, \dots, \tau_n \vdash \tau_1', \tau_2', \dots, \tau_n'$

if all $\tau_1, \dots, \tau_n$ are true then one ~~of~~ at least $\tau_1', \dots, \tau_n'$ is true

Δ is a list in which comma says disjunction

Γ is a list in which comma says conjunction

R: "," is and on the left of "⊢" and or on the right

Admissible structural rules

$$\frac{\Gamma \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ weakening left} \qquad \frac{\Gamma, \tau, \tau \vdash \tau'}{\Gamma, \tau \vdash \tau'} \text{ contraction left}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \tau, \Delta} \text{ weakening right} \qquad \frac{\Gamma \vdash \tau, \tau, \Delta}{\Gamma \vdash \tau, \Delta} \text{ contraction right}$$

Gentzen Sequent Calculus

weakening means more exposure

linear logic starts by rejecting this

& is "with"

⊗ is "times"

they is "I will make both!"

with & "I could make either but I will only make one"

$$\frac{\Gamma, \tau_1 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \&L_1 \qquad \frac{}{\tau \vdash \tau} Ax$$

$$\frac{\Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \& \tau_2 \vdash \Delta} \&L_2 \qquad \frac{\Gamma, \tau_1, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \otimes \tau_2 \vdash \Delta} \otimes L$$

$$\frac{\Gamma, \tau_1 \vdash \Delta \quad \Gamma, \tau_2 \vdash \Delta}{\Gamma \vdash \tau_1 \& \tau_2, \Delta} \&R \qquad \frac{\Gamma_1 \vdash \tau_1, \Delta_1 \quad \Gamma_2 \vdash \tau_2, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \tau_1 \otimes \tau_2, \Delta_1, \Delta_2} \otimes R$$

Conjunction for L

Linear Logic | R rules make, L rules use

$$\frac{}{\tau \to \tau}\ \text{Id Ax}$$

cut connects makers and users, common!

$$\frac{\Gamma \vdash \tau_1, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta}\ \oplus R_1 \qquad \frac{\Gamma, \tau_1 \vdash \Delta \quad \Gamma, \tau_2 \vdash \Delta}{\Gamma, \tau_1 \oplus \tau_2 \vdash \Delta}\ \oplus L \qquad \frac{\Gamma \vdash \tau, \Delta \quad \Gamma', \tau \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{cut}$$

$$\frac{\Gamma \vdash \tau, \Delta}{\Gamma \vdash \tau_1 \oplus \tau_2, \Delta}\ \oplus R_2$$

$\underline{R}$ an elim is a combination of elim & plus

$\underline{R}$ Disjunction either has a common of responses, or it doesn't

$$\frac{\Gamma_1, \tau_1 \vdash \Delta \qquad \Gamma_2, \tau_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \tau_1 ⅋ \tau_2 \vdash \Delta_1, \Delta_2}\ ⅋L \qquad \frac{\Gamma \vdash \tau_1, \tau_2}{\Gamma \vdash \tau_1 ⅋ \tau_2, \Delta}\ ⅋R$$

binarys : $\oplus, \otimes, \&, ⅋$

nullarys : Top, 1, Bottom, 0

$$\frac{}{\Gamma \vdash \top, \Delta}\ \top R \qquad \frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ 1L \qquad \frac{}{\bot \vdash \bullet}\ \bot L \qquad \frac{}{\Gamma, 0 \vdash \Delta}\ 0L$$

empty $\top L$

$$\frac{}{\bullet \vdash 1}\ 1R \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot R$$

empty $0R$

mix:

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Mix}$$

Some super tree

[illegible]

$$\dfrac{\dfrac{\vdots}{\text{beef, bun} \vdash \text{hamburger}} \qquad \dfrac{\vdots}{\text{potato, oil} \vdash \text{fries}}}{\text{beef, bun, potato, oil} \vdash \text{burger} \otimes \text{fries}}$$

$$\dfrac{\text{beef, oil} \vdash \text{cfs} \qquad \text{potatoform} \vdash \text{potatosandwich}}{\text{beef, oil, potatoform} \vdash \text{cfs} \otimes \text{potatosandwich}}$$

in LL $\dfrac{}{\Gamma, \Gamma \vdash C}$ is ~~not~~ it be case you can't squander Γ resources

"soup of the day"
a name

$$\dfrac{\dfrac{\vdots}{\text{tomato, cream} \vdash \text{tomSoup}}}{\text{tomato, cream} \vdash \text{tomato} \oplus \text{chNoodleSoup}} \quad \circ \circ \circ \quad \dfrac{\vdots}{\text{chicken, noodle} \vdash \text{chNoodleSoup}}$$

$$\dfrac{\dfrac{\vdots}{\text{strawberries, banana} \vdash \text{fruitSalad}} \qquad \dfrac{\vdots}{\text{strawb, banana} \vdash \text{smoothie}}}{\text{strawb, banana} \vdash \text{fruitSalad} \;\&\; \text{smoothie}}\;\&$$

names:
Main = burger $\otimes$ fries
soup = tomSoup $\oplus$ ChxNoodle
dessert = fruitSalad & Smoothie
lunch = main $\otimes$ soup $\otimes$ dessert
ingred = beef, bun, potato, oil, tomato, cream, strawberry
ingred' = " " " " chicken, noodle, "

$$\dfrac{\vdots}{\text{ingred} \vdash \text{lunch}}\otimes R^2 \qquad \dfrac{\vdots}{\text{ingred}' \vdash \text{lunch}}\otimes R^2$$

are two perfectly good restaurant programs depending on what ingredients happen to be available.

$$\dfrac{\dfrac{\text{ingred} \vdash \text{lunch, lunch}}{\text{ingred, ingred} \vdash \text{lunch, lunch}}\text{mix}}{\text{ingred, ingred} \vdash \text{lunch} \parr \text{lunch}}\parr R$$

independent processes in parallel

Scope is resources

Ftro 12
Downen

pg4

$(\tau_1 \parr \tau_2)^\perp$ negation is defined inductively on rules

$= \tau_1^\perp \oplus \tau_2^\perp$ demorgan but switches between additive and multiplicative

$(\tau_1 \oplus \tau_2)^\perp$

$= \tau_1^\perp \parr \tau_2^\perp$

$(\tau_1 \otimes \tau_2)^\perp = \tau_1^\perp \parr \tau_2^\perp$

$(\tau_1 \parr \tau_2)^\perp = \tau_1^\perp \otimes \tau_2^\perp$

$\top^\perp = 0$ $\qquad \bot^\perp = 1$

$0^\perp = \top$ $\qquad 1^\perp = \bot$

$!\tau$ "of corse" means copyable! nonlinear

Th $\tau = (\tau^\perp)^\perp$

Pf by induction

Th duality

if $\Gamma \vdash \Delta$ is derivable

then $\Delta^\perp \vdash \Gamma^\perp$ is too

$$\frac{\Gamma, \tau \vdash \Delta}{\Gamma \vdash \tau^\perp, \Delta}\ invL \qquad \frac{\Gamma \vdash \tau, \Delta}{\Gamma, \tau^\perp \vdash \Delta}\ invR$$

$$\frac{}{\vdash \tau, \tau^\perp}\ id \qquad \frac{\vdash \tau, \Delta \quad \vdash \tau^\perp, \Delta'}{\vdash \Delta, \Delta'}\ cut$$

Linear Implication

$$\frac{\dfrac{}{\tau \vdash \tau}\ id}{\vdash \tau \multimap \tau}\ \multimap R$$

$$\frac{\dfrac{\dfrac{}{\tau^{\perp\perp} \vdash \tau}\ id}{\cdot \vdash \tau^\perp, \tau}\ Inv}{\cdot \vdash \tau^\perp \parr \tau}\ \parr R$$

$\tau_1 \multimap \tau_2 = \tau_1^\perp \parr$

Feb 13 Dorm pg. 5

Linear Logic

Linear Lambda Calculus (λ + resource management)

$\lambda x:\tau. \langle x, x\rangle : \tau \to \tau \otimes \tau$ has to be ruled out because it copies,

so we rule out par ⅋

we also rule out $\bot$ because its arbitrary.

"of course" return types must look like "exactly one"

$\tau ::= \top \mid 0 \mid 1 \mid \tau_1 \multimap \tau_2 \mid\ !\tau$

$\mid \tau_1 \& \tau_2 \mid \tau_1 \oplus \tau_2 \mid \tau_1 \otimes \tau_2 \mid$

$\vdash \_$ exactly one thing.

therefor, no par and no $\bot$

typing judgment

$\Gamma ; \Delta \vdash e : \tau$

nonlinear linear

convention

Δ is unordered list of linear resource scope

Γ " " list of nonlinear resource scope

just FVs, no restrictions

assign the correct resource (type) to the correct expression

$$\frac{\Gamma ; \Delta \vdash e : \tau \qquad \Gamma ; \Delta', x:\tau \vdash e' : \tau'}{\Gamma ; \Delta, \Delta' \vdash let(e' ; x.e) : \tau'}\ cut$$

$$\frac{}{\Gamma ; x:\tau \vdash x : \tau}\ Id^{lnr} \qquad \frac{}{\Gamma, x:\tau ; \cdot \vdash x : \tau}\ Id^{Nlnr}$$

$$\frac{\Gamma ; \Delta \vdash e : \tau}{\Gamma ; \Delta \vdash l \cdot e : \tau \oplus \tau_2}\ I\oplus_L$$

$$\frac{\Gamma ; \Delta \vdash e : \tau}{\Gamma ; \Delta \vdash r \cdot e : \tau \oplus \tau_2}\ I\oplus_R$$

$$\frac{\Gamma ; \vdash e : \tau_1 \oplus \tau_2 \qquad \Gamma ; \Delta', x:\tau_1 \vdash e_1 : \tau \qquad \Gamma ; \Delta', y:\tau_2 \vdash e_2 : \tau}{\Gamma ; \Delta ; \Delta' \vdash \text{case } e \text{ of } \begin{array}{l} l \cdot x \Rightarrow e_1 \\ r \cdot y \Rightarrow e_2 \end{array} : \tau}$$

Δ should share across both cases so that it gets consumed.

here a Δ is copied syntactically but not duplicated semantically

but there is also some splitting

tensor product

Δ_1 and Δ_2 are disjoint in variables, not necessarily in resources

$$\frac{\Gamma ; \Delta_1 \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta_2 \vdash e_2 : \tau_2}{\Gamma ; \Delta_1, \Delta_2 \vdash \langle e_1, e_2 \rangle : \tau_1 \otimes \tau_2}\ I\otimes$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \& \tau_2}{\Gamma ; \Delta \vdash e \cdot l : \tau_1}\ E\&_L$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \& \tau_2}{\Gamma ; \Delta \vdash e \cdot r : \tau_2}\ E\&_R$$

$$\frac{\Gamma ; \Delta \vdash e : \tau_1 \otimes \tau_2 \qquad \Gamma ; \Delta', x:\tau_1, y:\tau_2 \vdash e' : \tau'}{\Gamma ; \Delta, \Delta' \vdash \text{case } e \text{ of } \langle x, y \rangle \Rightarrow e' : \tau'}\ E\otimes$$

but so... ca...

$$\frac{\Gamma ; \Delta \vdash e_1 : \tau_1 \qquad \Gamma ; \Delta \vdash e_2 : \tau_2}{\Gamma ; \Delta \vdash \{ l \Rightarrow e_1, r \Rightarrow e_2 \} : \tau_1 \& \tau_2}$$

Intro 13 Downen pg 6

Linear λCalc

$$\frac{\Gamma ; \Delta, x : \tau \vdash e : \tau'}{\Gamma ; \Delta \vdash \lambda(x:\tau).e : \tau \multimap \tau'} \; I\multimap$$

$$\frac{\Gamma ; \Delta \vdash e : \tau \multimap \tau' \qquad \Gamma ; \Delta' \vdash e' : \tau}{\Gamma ; \Delta, \Delta' \vdash e \; e' : \tau'} \; E\multimap$$

$$\frac{\Gamma ; \bullet \vdash e : \tau}{\Gamma ; \bullet \vdash \text{many } e : !\tau} \; I!$$

$$\frac{\Gamma ; \Delta \vdash e : !\tau \qquad \Gamma, x : \tau ; \Delta' \vdash e : \tau'}{\Gamma, x : \tau ; \Delta, \Delta' \vdash \text{case of many } x \Rightarrow e : \tau'}$$

constants

$$\frac{}{\Gamma ; \Delta \vdash \{\} : \top}$$

$$\frac{\Gamma ; \Delta \vdash e : 0}{\Gamma ; \Delta \vdash \text{case } e \text{ of } \{\} : \tau} \; \text{absurd,}$$

$$\frac{}{\Gamma ; \bullet \vdash \langle\rangle : 1}$$

$$\frac{\Gamma ; \Delta \vdash e : 1 \qquad \Gamma ; \Delta' \vdash e' : \tau}{\Gamma ; \Delta, \Delta' \vdash \text{case } e \text{ of } \langle\rangle \Rightarrow \tau} \; \text{unit}$$

DYNAMICS mostly CBV

$$V ::= \langle\rangle \mid \langle V_1, V_2\rangle \mid l \cdot V \mid r \cdot V \mid \text{many } V$$
$$\mid \lambda(x:\tau).e \mid \{l \Rightarrow e, r \Rightarrow e\} \mid \{\}$$

$$\text{Let}(e ; x.V) \mapsto e[V/x]$$

$$(\lambda x.e)\, V \mapsto e[V/x]$$

$$\{l \Rightarrow e_1, r \Rightarrow e_2\}.l \mapsto e_1$$

$$\text{case } l \cdot V \text{ of } \{l \cdot x \Rightarrow e_1, \; r \cdot y \Rightarrow e_2\} \mapsto e_1[V/x]$$

$$\text{case } \langle V_1, V_2\rangle \text{ of } \langle x, y\rangle \Rightarrow e \mapsto e[V_1/x, V_2/y]$$

$$\text{case } \langle\rangle \text{ of } \langle\rangle \Rightarrow e \mapsto e$$

$$\text{many } V \text{ of } x \Rightarrow e \mapsto e[V/x]$$

Substitution

Linear: if $\Gamma ; \Delta \vdash e : \tau$ and $\Gamma ; \Delta', x : \tau \vdash e' : \tau'$

then $\Gamma ; \Delta, \Delta' \vdash e'[e/x] : \tau'$

NonLinear: if $\Gamma ; \bullet \vdash e : \tau$ and

$\Gamma, x : \tau ; \Delta' \vdash e' : \tau'$

then $\Gamma, \Delta' \vdash e'[e/x] : \tau'$

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