Game Semantics

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1 Denotational Semantics

vs. Operational Semantics

$$t, c \to t', c' \qquad \text{(Operational)}$$

Meta Language SD:

$$[\![-]\!] : \texttt{PL} \to \texttt{SD} \qquad \text{(Denotational)}$$

$$[\![k\ t_0, \ldots, t_i]\!] = f([\![t_0]\!], \ldots, [\![t_i]\!])$$

1.0.1 Example

Regular Language - Kleeve Algebras - FSA

1.0.2 Equivalence in PL's

- OS: $t \to t' \quad t \equiv t'$
  But, $\forall C[-].C[t] \stackrel{?}{\equiv} C[t']$ Not universally.
- DS: $[\![C[t]]\!] = [\![t]\!] \circ [\![C]\!]$
  $[\![t]\!] = [\![t']\!] \iff \forall C.[\![C[t]]\!] = [\![C[t']]\!]$

1.1 Relating OS-DS

1. Termination: $t \Uparrow \sim [\![t]\!] = \bot$ or $t \Downarrow \sim [\![t]\!] = \top$
2. Observational Equivalence: $t_0 \equiv t_1 \iff \forall C[-].(C[t_0] \Downarrow \iff C[t_1] \Downarrow)$

1.1.1 Property 1: Soundness

$$[\![t_0]\!] = [\![t_1]\!] \Rightarrow t_0 \equiv t_1 \qquad \text{(Soundness)}$$

1.1.2 Property 2: Adequacy

$$[\![t]\!] = \top \iff t \Downarrow \qquad \text{(Adequacy)}$$

1.1.3 Property 3: Definability

$$\forall \tau \in \texttt{SD}\ \exists t \in \texttt{PL}\ [\![t]\!] = \tau \tag{definablility}$$

1.1.4 Full Abstraction

$$[\![t_0]\!] = [\![t_1]\!] \iff t_0 \equiv t_1 \tag{Full Abstraction}$$

1.1.5 Theorem

$$1 + 2 + 3 \Rightarrow \texttt{FA}$$

1.1.6 Proof

Assume: $[\![t_0]\!] \neq [\![t_1]\!]$
$\exists^* \tau \in \texttt{SD}$ s.t. $[\![t_0]\!] \circ \tau = \top \neq \bot = [\![t_1]\!] \circ \tau$
By Definability: $\exists C.[\![C]\!] = \tau$ s.t. $[\![t_0]\!] \circ \tau = [\![t_0]\!] \circ [\![C]\!] = [\![C[t_0]]\!] = \top$
By Adequacy: $C[t_0] \Downarrow$ and $C[t_0] \Uparrow$ which is a contradiction.

2 Game Semantics

D.S., Interaction-Based
Two Protagonists - P (Proponent or Popeye) and O (Opponent or Olive Oil)
These protagonists interact via "moves" (actions): Questions and Answers

	O	P
Q		
A		

2.1 Example

0 :

2.2 Example

$\lambda x.x : \texttt{Nat} \rightarrow \texttt{Nat}$

- play = one interaction
- all plays = strategy $\rightarrow$ SD

2.3 Definition: Arena $\langle M, Q, O, I, \vdash \rangle$

- M is a set of moves
- $Q \subseteq M \quad A = M \backslash Q$
- $O \subseteq M \quad P = M \backslash O$
- $I \subseteq O \cap Q$ initial moves

- $\vdash \subseteq Q \times M$ enabling
  $m \vdash n \iff (m \in O \iff n \in P)$

2.4 Example

$\texttt{Nat} = \langle 1 + \mathbb{N}, 1, 1, 1, 1 \times \mathbb{N} \rangle$
$I = \langle 0, 0, 0, 0, 0 \rangle$

2.4.1 Composite Arenas

$A \times B = \langle M_A + M_B, Q_A + Q_B, O_A + O_B, I_A + I_B, \vdash_A + \vdash_B \rangle$
$A \to B = \langle M_A + M_B, Q_A + Q_B, P_A + O_B, \texttt{inl}(I_B), \vdash_A + \vdash_B \cup (\texttt{inr}(I_B) \times \texttt{inl}(I_A)) \rangle$

2.4.2 Theorem

$(A \times B) \to C \cong A \to (B \to C)$
$1 \times A \cong A \times I \cong A$

2.4.3 Definition: Plays

$\lambda x.x : \texttt{Nat} \to \texttt{Nat}$
Plays = Sequence + Pointers
$q\ q_1'\ m'\ q_2'\ n'\ p$
$p \to q;\ n' \to q_2';\ q_2' \to q;\ m' \to q_1';\ q' \to q$
$q(a\langle b \rangle).q'(b\langle c \rangle).m'(c).q'(b\langle d \rangle).n'(d).p.b$

2.4.4 Definition: Play

A play is a pointer sequence in arena A s.t.

- $p'.m(a) \subseteq p$ then $\exists q \in Q_A$ s.t. $g(b\langle a \rangle) \in p'$ and $q \vdash_A m$
- If $q(a\langle b \rangle) \subseteq p$ then $q \in I_A$

2.4.5 Definition: Strategy

A strategy $T : A$ is a set of plays s.t. $\forall p \in T$

- $p' \subseteq p$ then $p' \in T$
- $p \cdot m \in \texttt{Play}_A$ and $m \in O_A$ then $p \cdot m \in T$
- $\forall$ permutation $\Pi : A \to A$ $\Pi \cdot p \in T$

Π defined as follows:
$\Pi \cdot \varepsilon = \varepsilon$
$\Pi \cdot (p \cdot m(a(b))) = (\Pi \cdot p) \cdot m(\Pi(a)(\Pi(b)))$
$\Pi \cdot (p \cdot m(a)) = (\Pi \cdot p) \cdot m(\Pi(a))$