Bauer 1

Algebraic effects & handlers
Math → programming

1. Algebraic theories
2. Programming language
3. Reasoning
4. Applications

historically math → programming
Power, Plotkin (cat theory, lots of math)

math can inform PL design

① algebraic theories "universal algebra"

Example: a group $(G, \overset{u \in G}{u}, \overset{G \times G \to G}{\cdot}, \overset{G \to G}{{}^{-1}})$:

$$u \cdot x = x = x \cdot u$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x \cdot x^{-1} = u = x^{-1} \cdot x$$

important form: operations
equations

operations everywhere defined

alternative defn $(G, u, \cdot)$ monoid

$+ \ \forall x. \exists y. \ x \cdot y = u = y \cdot x$

(equivalent defn, need prove y is unique) difference in form

$\forall \exists$ complicates matters

equations simpler

in general to have theory ops + eqn's

op w/ how many args

terms, what can be on either side =

Def. a signature $\Sigma = \{(op_i, n_i)\}_{i \in I}$

where op_i are operation symbols
(can be anything)

and $n_i \in \mathbb{N}$ is the arity of op_i.

Def. a term in context $x_1, \dots, x_k$ is

- one of the variables x_i
- one of the operations applied to terms

$op_i(t_1, \dots, t_{n_i})$ where $t_1, \dots, t_{n_i}$ are terms

in context $x_1, \dots, x_k$

(read as inductive definition)

Def. An (equational) algeraic theory (almost Lawvere theory)

T is $(\Sigma_T, \mathcal{E}_T)$ where Σ_T is a signature

$\mathcal{E}_T$ is a set of equations

An equation is

$$x_1, \ldots, x_k \mid l = r$$

where l, r are Σ_T terms in $x_1, \ldots, x_k$

Example Group.

signature $\Sigma_{Group} = \{(u,0), (m,2), (i,1)\}$

$$x \mid m(u(), x) = x$$
$$x \mid m(x, i(x)) = u()$$
$$\vdots$$

Examples others:

- ring ✓
- empty theory

"math is study of pathological examples"

Example Pointed set

- signature $(\cdot, 0)$
- no equations

Example semilattice

- signature $\{(\bot, 0), (\vee, 2)\}$
- axioms: $\bot \vee x = x$ $\quad x \vee y = y \vee x$

$$x \vee (y \vee z) = (x \vee y) \vee z \qquad x \vee x = x$$

· field $+, -, \times, {}^{-1}, 0, 1,$

problematic

not defined everywhere, particularly 0^{-1} undefined

interpretation of models

axioms / theory of group

examples of group (theory)

(special case of 1st order logic, model theory)

(2) Interpretations & models

Suppose T (algebraic) theory

An interpretation I of Σ_T is given by

- a carrier set $|I|$
- for each $(op_i, n_i) \in \Sigma_T$ a map

$$[\![op_i]\!]_I : \underbrace{|I| \times \cdots |I|}_{n_i} \to |I|$$

$|I|^{n_i}$

$X^m = \underbrace{X \times \cdots \times X}_{m}$

$X^0 = \{()\} = 1$

0-tuple

Each term $x_1, \ldots, x_k \mid t$ is interpreted as a map

$$[\![x_1, \ldots, x_k \mid t]\!] : |I|^k \to |I|$$

as follows:

- $[\![x_1, \ldots, x_k \mid x_i]\!]_I : |I|^k \to |I|$

π_i i-th projection

- $[\![x_1, \dots, x_k \mid op_i(t_1, \dots, t_{n_i})]\!]_I$ is composition

$$|I|^k \xrightarrow{([\![t_1]\!]_I, \dots, [\![t_n]\!])} |I|^{n_i} \xrightarrow{[\![op_i]\!]_I} |I|$$

<u>Def</u> A <u>T-model</u> is an interpretation M of theory T such that for every

$$x_1, \dots, x_k \mid l = r \quad \text{in } \mathcal{E}_T$$

the maps

$$[\![x_1, \dots, x_k \mid l]\!]_M : |M|^k \to |M| \quad \text{and}$$

$$[\![x_1, \dots, x_k \mid r]\!]_M : |M|^k \to |M|$$

are equal

<u>Examples</u> A model M of the theory of a pointed set

- a carrier set $|M|$
- a map $[\![\cdot]\!]_M : |M|^0 \to |M|$

$$1 \to |M| \quad \text{(picks out exactly one element)}$$

Isomorphically: (S, s) where S is a set and $s \in S$.

Example Every theory T has the **trivial model**

M : $|M| = 1$

$[\![op_i]\!]_M : 1^{n_i} \to 1$

good thing (ring w/ 0 diff from 1 destroys algebraic structure of ring)

Example: If M and L are T-models

$|M \times L| = |M| \times |L|$

$[\![op_i]\!]_{M \times L} : (|M| \times |L|)^{n_i} \longrightarrow |M| \times |L|$

$[\![op_i]\!]_{M \times L}(a_1, \dots, a_{n_i}) =$

$([\![op_i]\!]_M(\pi_1 a_1, \dots, \pi_1 a_{n_i}), [\![op_i]\!]_L(\pi_2 a_1, \dots, \pi_2 a_{n_i}))$

do all ops component-wise
all hold component-wise

③ Free Models

Given theory T and set X

say that a T model M ~~is freely~~ ~~ge~~ with a map $\eta : X \to |M|$ is freely generated by X

when

$$\forall f \; \exists! \bar{f} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & {|M|} \arrow[d, "\bar{f}"] \\ & {|L|} \end{tikzcd}$$

- best way of doing
- better than any other way of doing it
- spse had another way of doing it is better than it

where $\bar{f} : M \to L$ is a T-homomorphism

A <u>T-homomorphism</u> $f : M \to L$ (map that preserves operations)

For every op_i in T:

$$f(\llbracket op_i \rrbracket_M (a_1, \dots, a_{n_i}))$$
$$= \llbracket op_i \rrbracket_L (f(a_1), \dots, f(a_{n_i}))$$

<u>Terminology</u>

$(M, \eta : X \to |M|)$ is $\begin{cases} \text{free model over } X \\ \text{free model generated by } X \end{cases}$

[$\exists$ 2 such models, isomorphic in a unique way]

Example: Define

$P_{<\omega}(X) := \{ S \subseteq X \mid S \text{ finite} \}$

Claim: $(P_{<\omega}(X), \emptyset, \cup)$ is the free semilattice generated by X

with $\eta : X \to P_{<\omega}(X)$,
$\eta : x \mapsto \{x\}$

$$\begin{tikzcd} X \arrow[r, "\{-\}"', "\text{singleton}"] \arrow[dr, "f"'] & P_{<\omega}(X) \arrow[d, "\bar{f}"] \\ & L \end{tikzcd} \qquad \bot, \vee$$

$=$

$\bar{f}$ homomorphism

$\begin{cases} \bar{f}(\emptyset) = \bot \\ \bar{f}(S_1 \cup S_2) = \bar{f}(S_1) \vee \bar{f}(S_2) \end{cases}$

$\bar{f}(\{x\}) = f(x)$ (diagram commutes)

does $\bar{f}$ exist?
if so, how many are there?

every algebraic theory has a free model gen by any set

computation trees

trivial model believes all equations

smallest } w/ respect elements & equations
most economical)

Free Model construction

2 steps

$Free_T(X)$

1. $Free_T(X)$ set of <u>well-founded</u> trees

(<u>no infinite paths</u>
inductively defined)

defined inductively:

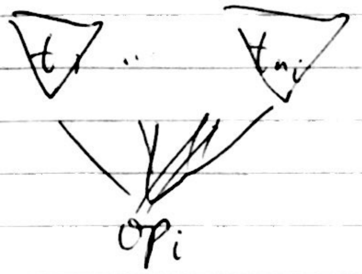
- for every $x \in X$
  there is tree

  · leaf labeled w/ x
  · label is arbitrary

  return $x \in Free_T(X)$

way of building new trees using operations

- if $(op_i, n_i) \in \Sigma_T$
  and $t_1, \dots t_{n_i} \in Free_T(X)$
  then



$op_i(t_1, \dots, t_{n_i}) \in Free_T(X)$

Scanned with CamScanner

not done bec. equations

I have right elements

but some may have to be

considered equal

quotient by those equiv

2. Define equivalence relation

Define $\approx_T$ on $\mathrm{Tree}_T(X)$ to be

the least equivalence relation such that

(need to make congruence relation
as well as theory equations)

congruence: equiv which
commutes w/ operations

behaves nicely w/ resp to ops)

Homework: write two conditions

- it's a congruence w/ respect to
forming trees (tree formation)
- it validates the equations of the theory