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Downer
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Logical Relations & Termination

- STLC "Reducibility"
- System F "Reducibility candidates"

$b ::= \text{True} \mid \text{False}$

$e ::= x \mid \lambda x.e \mid e_1\, e_2 \mid b \mid \text{if } e_1 \text{ then } e_2 \text{ else } e_3$

$\tau ::= \text{bool} \mid \tau_1 \to \tau_2$

$(\lambda x.e)e' \mapsto e[e'/x]$

$\text{if true then } e_1 \text{ else } e_2 \mapsto e_1$

$\text{if false then } e_1 \text{ else } e_2 \mapsto e_2$

$$\frac{e_1 \mapsto e_1'}{e_1\, e_2 \mapsto e_1'\, e_2}$$

$$\frac{e_1 \mapsto e_1'}{\text{if}(e_1; e_2; e_3) \mapsto \text{if}(e_1'; e_2, e_3)}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x.e : \tau \to \tau'}$$

$$\frac{}{\Gamma \vdash \text{True} : \text{Bool}} \qquad \frac{}{\Gamma \vdash \text{False} : \text{Bool}}$$

$$\frac{\Gamma \vdash e_1 : \text{Bool} \quad \Gamma \vdash e_2 : \text{Bool} \quad \Gamma \vdash e_3 : \text{Bool}}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau}$$

<u>closed</u>

D: e is closed when $FV(e) = \{\}$

D: e is <u>well-typed</u> when there is a Γ and τ s.t. $\Gamma \vdash e : \tau$ is derivable.

D: e is <u>terminating</u> when there is an e' s.t. $e \mapsto^* e'$ and $e' \not\mapsto$

T: if e is closed and well-typed then e is well-typed

ToShow

I. if $\cdot \vdash e : \tau$ is derivable then there is an e' s.t. $e \mapsto^* e' \not\mapsto$

Pf by induction on the derivation $\mathcal{D}$ of $\cdot \vdash e : \tau$

$\mathcal{D} = \cdot \vdash e : \tau$: doesn't happen

$\mathcal{D} = \dfrac{}{\Gamma \vdash \text{true} : \text{Bool}}$; $\text{True} \mapsto^* \text{True} \not\mapsto$

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\cdot \vdash e_1 : \text{Bool}} \quad \overset{\mathcal{D}_2}{\cdot \vdash e_2 : \tau} \quad \overset{\mathcal{D}_3}{\cdot \vdash e_3 : \tau}}{\cdot \vdash \text{if}(e_1; e_2; e_3) : \tau}$$

IH: for $i = 1,2,3$, there is some e_i' s.t. $e_i \mapsto^* e_i' \not\mapsto$

if e_1 then e_2 else $e_3 \mapsto^*$ if e_1' then e_2 else e_3

suppose $e_1' = b$: $\text{if}(e_1; e_2; e_3) \mapsto^* e_j \mapsto^* e_j'$ ($j = 2$ if b = true else $j =$

where $e_j \mapsto e_j' \not\mapsto$

else $e_1' \neq b$ ① $\text{if}(e_1'; e_2; e_3)$

$\not\mapsto$ when $e_1' \neq b$

② is because type safety can't happen

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{x : \tau \vdash e : \tau'}}{\cdot \vdash \lambda x. e : \tau \to \tau'}$$

$\lambda x.e \mapsto^* \lambda x.e \not\mapsto$

IH is vacuous

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\cdot \vdash e_1 : \tau \to \tau'} \quad \overset{\mathcal{D}_2}{\cdot \vdash e_2 : \tau}}{\cdot \vdash e_1\, e_2 : \tau'}$$

IH: for $k = 1,2$ $e_k \mapsto e_k' \not\mapsto$

Suppose $e_1' = \lambda x. e'$: $e_1 e_2 \mapsto^* (\lambda x. e') e_2 \mapsto e'[e_2/x]$

we don't know enough about e'!

Lesson: this proof strategy won't work.

ToShow attempt 2

$\underline{T}$ if $\bullet \vdash e:\tau$ is derivable then there is an e' s.t. $e \mapsto^* e' \not\mapsto$

To Show $\bullet \vdash e:\tau \Rightarrow e \in [\![\tau]\!] \subseteq$ terminal where $[\![\tau]\!]$ is "all programs of type τ", "the meaning of τ"

$$[\![bool]\!] = \{true, false\}^* \quad \text{where for expressions } \mathbb{C},\ \mathbb{C}^* = \{e \mid \exists e' \in \mathbb{C}\ e \mapsto^* e'\}$$

properties of expansion:

- if $\mathbb{C} \subseteq term$, $\mathbb{C}^* \subseteq term$
- $\mathbb{C} \subseteq \mathbb{C}^*$
- $\mathbb{C}^{**} = \mathbb{C}^*$

$$[\![\tau_1 \to \tau_2]\!] = \{e \in Term \mid \forall e' \in [\![\tau]\!],\ e\,e' \in [\![\tau_2]\!]\}$$

$$[\![\Gamma]\!] = \{\sigma \in Subst \mid \forall (x:\tau) \in \Gamma\ \ x[\sigma] \in [\![\tau]\!]\}$$

$$[\![\Gamma \vdash e:\tau]\!] = \forall \sigma \in [\![\Gamma]\!]\ \ e(\sigma) \in [\![\tau]\!]$$

$\underline{L}$ "fundamental lemma" aka adequacy, soundness.

if $\Gamma \vdash e:\tau$ is derivable then $[\![\Gamma \vdash e:\tau]\!]$ true

Proof is by induction on the given derivation of $\Gamma \vdash e:\tau$

$\mathcal{D} = \dfrac{}{\Gamma, x:\tau \vdash x:\tau}$: Show $[\![\Gamma, x:\tau \vdash x:\tau]\!]$ is true

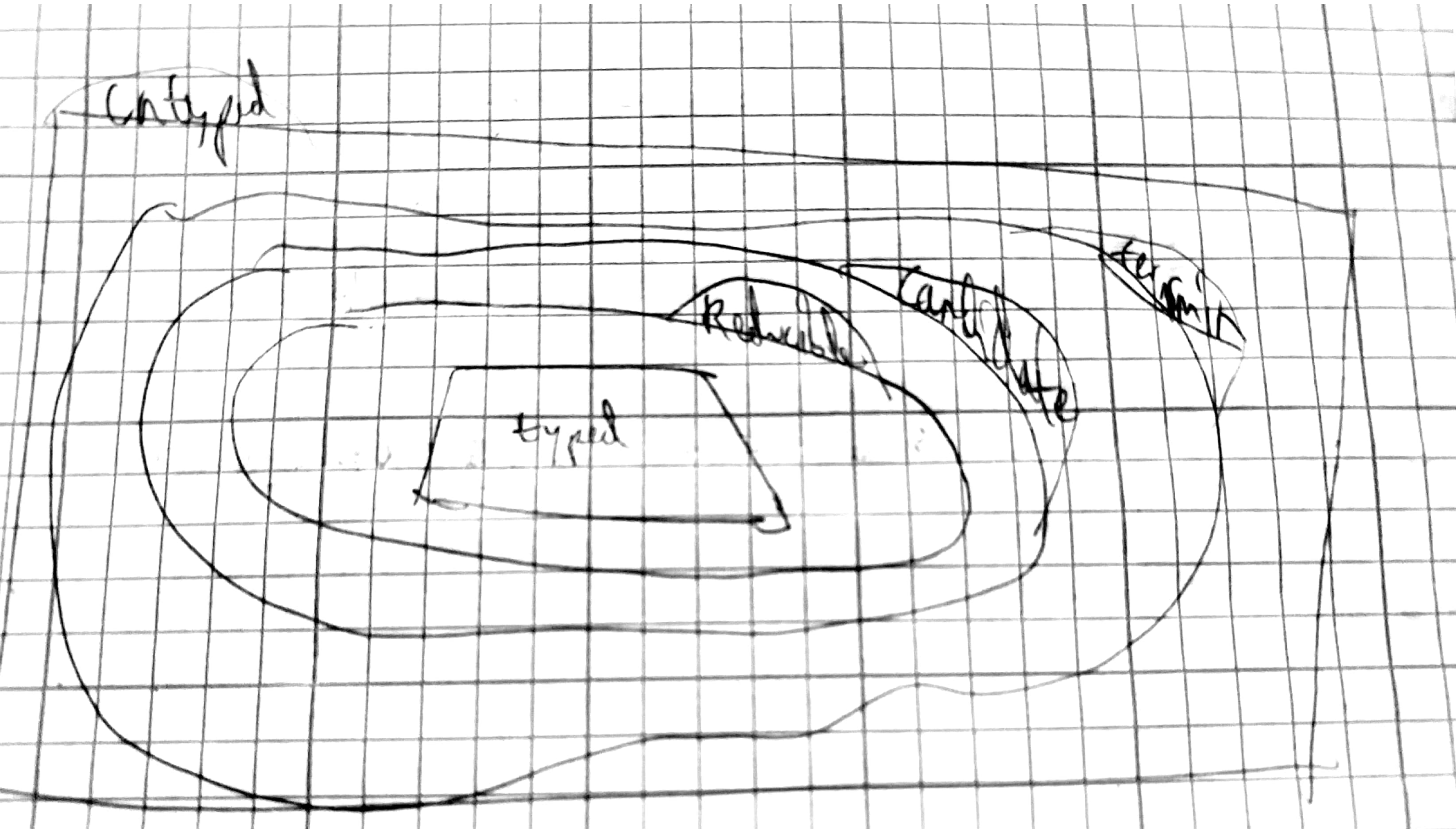
Suppose $\sigma \in [\![\Gamma, x:\tau]\!]$. show $x[\sigma] \in [\![\tau]\!]$

true by definition of $[\![\Gamma, x:\tau]\!]$

$\mathcal{D} = \dfrac{\Gamma \vdash e_1 : \tau \to \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}$ IH: $[\![\Gamma \vdash e_1 : \tau \to \tau']\!]$ is true and $[\![\Gamma \vdash e_2:\tau]\!]$ is true

Show: $[\![\Gamma \vdash e_1\, e_2 : \tau']\!]$

Suppose $\sigma \in [\![\Gamma]\!]$ show $(e_1\, e_2)[\sigma] \in [\![\tau']\!] = e_1[\sigma]\ e_2[\sigma]$, that similarly $e_1[\sigma] \in [\![\tau_1 \to \tau$ $e_2[\sigma] \in [\![\tau_1]\!]$


Untyped
Reducible
Candidate
typed

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contd

$$D = \frac{\Gamma, x:\tau \vdash e:\tau'}{\Gamma \vdash \lambda x.e : \tau \to \tau'}$$

IH: $[\![\Gamma, x:\tau \vdash e, \tau']\!]$ is true

show $[\![\Gamma \vdash \lambda x.e : \tau \to \tau']\!]$

Suppose $\sigma \in [\![\Gamma]\!]$, show $(\lambda x.e)\sigma$
$= \lambda x.\, e(\sigma)$
$\in [\![\tau \to \tau']\!]$

Suppose $e_2 \in [\![\tau]\!]$ show $(\lambda x.\, e(\sigma))\, e_2 \in [\![\tau']\!]$
$\mapsto e[\sigma, {}^{e_2}/_x]$ $\qquad \sigma, {}^{e_2}/_x \in [\![\Gamma, x:\tau]\!]$

$\underline{\underline{L^2}}$ $[\![\tau]\!] \subseteq$ termin

$\underline{\underline{L^3}}$ (expansion) : $e \mapsto^* e'$ and $e' \in [\![\tau]\!]$ then $e \in [\![\tau]\!]$ ("e is reducible") i.e. $[\![\tau]\!]^* \subseteq [\![\tau]\!]$

$\underline{\underline{C}}$ if $\cdot \vdash e:\tau$ is derivable then $e \in [\![\tau]\!]$ and therefore e is terminating.

$\underline{\underline{C}}$ if $\cdot \vdash e:$ bool then there is a boolean $b =$ True, False s.t. $e \mapsto^* b$

$$[\![\forall\alpha.\tau]\!] = \{ e \in \text{Termin.} \mid \forall\, \tau' \in \text{Typ.}\ e\,\tau' \in [\![\tau[\tau'/\alpha]]\!] \}$$

this is too circular! the subtree $\tau[\tau'/\alpha]$ requires you to already be done in order to get started.

this is the nature of <u>impredicative polymorphism</u>

$\underline{\underline{D}}$ a <u>reducibility candidate</u> is any set of expressions $\mathbb{C}$ s.t. $\mathbb{C}^* \subseteq \mathbb{C} \subseteq$ termin.

$\underline{\underline{L^4}}$ for every τ the $\underbrace{\text{meaning of } \tau}_{[\![\tau]\!]}$ is a reducibility candidate

remember $\underline{N}$ $[\![\alpha]\!]_\theta = \theta(\alpha)$ where θ is a context on level of type variables

<u>attempt 2</u> $[\![\forall\alpha.\tau]\!] = \{ e \in \text{Termin} \mid \forall \tau' \in \text{Typ}\ \ e\,\tau' \in [\![\tau]\!]_{\theta, \tau'/\alpha} \}$

$\underline{N}$ $[\![\text{bool}]\!]_\theta$ is a set of expressions satisfying certain properties

θ : TypeVar $\to$ CR where CR is set of all reducibility candidates

$\underline{N}$ $(\theta, \mathbb{C}/\alpha)(\alpha) = \mathbb{C}$
$(\theta, \mathbb{C}/\alpha)(\beta) = \theta\beta$ where $\alpha \neq \beta$

corrected

correct attempt!

$$[\![\forall\alpha.\tau]\!]_\theta = \{ e \in \text{term in} \mid \forall \tau' \in \text{Type}.\ \forall C \in CR,\ e\,\tau' \in [\![\tau]\!]_{\theta, C/\alpha} \}$$

$$[\![\Theta]\!] = \{ \theta \in \text{TypeVar} \to CR \}$$

$$[\![\Theta;\Gamma]\!]_\theta = \{ \sigma \in \text{Subst} \mid (\forall (x:\tau) \in \Gamma.\ x[\sigma] \in [\![\tau]\!]_\theta) \wedge (\forall \alpha \in \Theta.\ \alpha[\sigma] \in \text{Type}) \}$$

$$[\![\Theta;\Gamma \vdash e:\tau]\!] = \forall\theta \in [\![\Theta]\!], \forall\sigma \in [\![\Theta;\Gamma]\!]_\theta.\ e[\sigma] \in [\![\tau]\!]_\theta \qquad [\![\Theta \vdash \tau : \star]\!] = \forall\theta \in [\![\Theta]\!].\ [\![\tau]\!]_\theta \in CR$$

L5 if $\Theta \vdash \tau : \star$ is derivable than $[\![\Theta \vdash \tau : \star]\!]$ is true

R we are in a single-kinded theory. i.e., $\star$ is unique. Many-kinded theories would have $\star_1, \star_2, \ldots$

Pf L5

$$\mathcal{D} = \frac{\Theta;\Gamma \vdash e : \forall\alpha.\tau \qquad \Theta \vdash \tau' : \star}{\Theta;\Gamma \vdash e\,\tau' : \tau[\tau'/\alpha]}$$

IH: $[\![\Theta;\Gamma \vdash e : \forall\alpha.\tau]\!]$ and $[\![\Theta \vdash \tau' : \star]\!]$

show $[\![\Theta;\Gamma \vdash e\,\tau : \tau[\tau'/\alpha]]\!]$

$\therefore$ suppose $\theta \in [\![\Theta]\!]$ and $\sigma \in [\![\Theta;\Gamma]\!]_\theta$

Say $(e\,\tau')[\sigma] = e[\sigma]\,\tau'[\sigma] \in [\![\tau[\tau'/\alpha]]\!]_\theta$

L6 $[\![\tau[\tau'/\alpha]]\!]_\theta = [\![\tau]\!]_{\theta, [\![\tau']\!]_\theta/\alpha}$

$$\mathcal{D} = \frac{\Theta,\alpha;\Gamma \vdash e:\tau}{\Theta;\Gamma\ \Lambda\alpha.e : \forall\alpha.\tau}$$

IH: $[\![\Theta,\alpha;\Gamma \vdash e:\tau]\!]$ is true

show $[\![\Theta;\Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau]\!]$ is true

Suppose $\theta \in [\![\Theta]\!]$ and $\sigma \in [\![\Theta;\Gamma]\!]_\theta$

show that $(\Lambda\alpha.e)[\sigma] = \Lambda\alpha.e[\sigma] \in [\![\forall\alpha.\tau]\!]_\theta$

Suppose τ' and $C \in CR$ $(\Lambda\alpha.e[\sigma])\,\tau' \mapsto e[\sigma, \tau'/\alpha]$

$[\![\tau]\!]_{\theta, C/\alpha}$ $\quad \sigma, \tau'/\alpha \in [\![\Theta,\Gamma]\!]$

R

Free theorems FT

in System F: "consider $Unit = \forall\alpha, \alpha \to \alpha$"

FT

if $\vdash e : \forall\alpha, \alpha \to \alpha$ then $e =_{\beta\eta} \Lambda\alpha. \lambda x : \alpha. x$

Proof from L1 (fundamental) we know that $e \in [\![\forall\alpha, \alpha \to \alpha]\!]_{\varepsilon}$, $\{x\}^{*} \in CR$