Cluster set

From Encyclopedia of Mathematics

$C(f, z_0; S)$ *of a function* $f: G \rightarrow \Omega$*, defined on a domain* $G \subset \mathbf{C}$ *with values in the Riemann sphere* Ω*, at a point* $z_0 \in \overline{G}$ *with respect to a set* $S \subset G$*,* $z_0 \in \overline{S}$

The set of values $\alpha \in \Omega$ for which there exists a sequence of points $\{z_n\}_{n=1}^{\infty}$, $z_n \in S$, $\lim_{n \rightarrow \infty} z_n = z_0$, such that

$$\lim_{n \rightarrow \infty} f(z_n) = \alpha .$$

Every number $\alpha \in C(f, z_0; S)$ is called a cluster value of f at z_0 with respect to S. The theory of cluster sets is a branch of function theory in which boundary properties of functions are studied in terms of topological and metric properties of various cluster sets.

If the entire domain G is taken for S, one obtains the full cluster set $C(f, z_0; G) = C(f, z_0)$; if the inclusion $S \subset G$ is strict, the corresponding set $C(f, z_0; S)$ is sometimes called a partial cluster set. A full cluster set $C(f, z_0)$ is closed; if f is continuous on a set S that is locally connected at $z_0 \in \overline{S}$, then the cluster set $C(f, z_0; S)$ is either degenerate, i.e. consists of a single point, or is a non-degenerate continuum. If $C(f, z_0; S)$ coincides with Ω, then it is called a total cluster set. A number $\alpha \in \Omega$ belongs to the set of recurrent values $R(f, z_0; S)$ of f at z_0 with respect to S if there is a sequence $\{z_n\}$ of points $z_n \in S$, $n = 1, 2, \ldots$, $\lim_{n \rightarrow \infty} z_n = z_0$, such that $\alpha = f(z_n)$, $n = 1, 2, \ldots$. One always has $R(f, z_0; S) \subset C(f, z_0; S)$. If for some $\alpha \in \Omega$ there is a path L: $z = z(t)$, $0 \leq t < 1$, in G ending at a point z_0, $z_0 \in \overline{G}$, $\lim_{t \rightarrow 1} z(t) = z_0$, and such that $\lim_{t \rightarrow 1} f(z(t)) = \alpha$, then α is called an asymptotic value of f at z_0 (along L). The asymptotic set $A(f, z_0; G)$ is the set of all asymptotic values of f at z_0.

The notion of a cluster set was clearly formulated for the first time by P. Painlevé in 1895 (he called it the "region of indeterminacy" , cf. [1]) in connection with studying an analytic function near one of its singular points and with classifying singularities of such functions. At that time one basically studied three, geometrically most simple, cases in the theory of cluster sets: a) z_0 is an isolated point of the boundary ∂G or an interior point of G; b) $G = D = \{z: |z| < 1\}$ is the unit disc or, in general, a Jordan domain, and z_0 is a point on the boundary $\Gamma = \partial D$; and c) the boundary $E = \partial G$ is an everywhere-discontinuous compactum in the plane (i.e. a totally-disconnected compact set) and $z_0 \in E$. A number of classical results in complex function theory have a formulation in terms of cluster sets. E.g., the Sokhotskii theorem, in a somewhat stronger form, states: If z_0 is an isolated point of an everywhere-discontinuous compactum $E \subset G$ and f is a meromorphic function on $G \setminus E$, then the cluster set $C(f, z_0; G \setminus E)$ is either degenerate or total. The Picard theorem, supplementing it, states that if $C(f, z_0; G \setminus E)$ is total, i.e. if z_0 is an essential singular point, then the set $CR(f, z_0; G \setminus E) = \Omega \setminus R(f, z_0; G \setminus E)$ contains at most two different values. Also, in this case

$$CR(f, z_0; G \setminus E) \subset A(f, z_0; G \setminus E)$$

(the Iversen theorem).

The main result related to the theory of the behaviour of meromorphic functions near "thin" boundaries (the Painlevé theory) is (cf. [1], [2]): If a set $E \subset G$ has linear Hausdorff measure zero, $\mu(E) = \mu_1(E) = 0$, and the function f is meromorphic in $G \setminus E$, then for every point $z_0 \in E$ the cluster set $C(f, z_0; G \setminus E)$ is either degenerate or total; moreover, in the first case f is also meromorphic at z_0. Thus, a point $z_0 \in E$ for which the cluster set $C(f, z_0; G \setminus E)$ is degenerate is a removable singular point of f; the study of removable sets of various function classes can be regarded as a branch of the theory of cluster sets.

Golubev's theorem is an important strengthening of the theorem of Picard: If $E \subset G$, $\mu(E) = 0$ and f is meromorphic in $G \setminus E$, then the set $CR(f, z_0; G \setminus E)$ has analytic capacity zero at every essential singular point $z_0 \in E$ (hence its plane measure $\mu_2(CR) = 0$).

The work of P. Fatou (1906) on boundary values of functions $f(z)$ holomorphic in the unit disc $D = \{z : |z| < 1\}$ was the starting point for the theory of cluster sets in the case of continuous boundaries. If such a function f is bounded in D, then almost-everywhere (in the sense of the Lebesgue measure) on the circle $\Gamma = \{z : |z| = 1\}$ it has radial and angular (non-tangential) boundary values (Fatou's theorem). Let $\zeta = e^{i\theta} \in \Gamma$ be an arbitrary point; denote by $h(\zeta, \phi)$ the chord of D ending at ζ and forming with the radius at ζ an angle ϕ, $-\pi/2 < \phi < \pi/2$. Let $\Delta(\zeta, \phi_1, \phi_2)$ be the angular domain with vertex $\zeta \in \Gamma$, consisting of those points of D lying between the chords

$$h(\zeta, \phi_1) \quad \text{and} \quad h(\zeta, \phi_2), \qquad -\frac{\pi}{2} < \phi_1 < \phi_2 < \frac{\pi}{2}.$$

A point $\zeta \in \Gamma$ is called a Fatou point, and belongs to the set $F(f)$, if the union

$$\bigcup C(f, \zeta; \Delta(\zeta, \phi_1, \phi_2))$$

over all angular domains $\Delta(\zeta, \phi_1, \phi_2)$ consists of a single value $f(e^{i\theta})$, which is called the angular boundary value of f at ζ. Another formulation of Fatou's theorem: For a bounded holomorphic function f in D the decomposition $\Gamma = F(f) \cup E$, $\text{mes}\, E = 0$, holds. This result is supplemented by the F. and M. Riesz uniqueness theorem (1916): If f is holomorphic and bounded in D and if on some set $M \subset F(f)$, $\text{mes}\, M > 0$, it has angular boundary values $f(\zeta) = \alpha$, $\zeta \in M$, then $f(z) \equiv \alpha$. This statement was proved, independently, by N.N. Luzin and I.I. Privalov (1919), who obtained an essential generalization of it to the case of arbitrary meromorphic functions. In the same year they published a boundary uniqueness theorem for the case of radial boundary values: If a function f, holomorphic in D, has the same radial boundary value $\alpha \in \Omega$ on a set M of the second category and metrically dense on some arc $\gamma \subset \Gamma$, i.e. if $\lim_{r \to 1} f(re^{i\theta}) = \alpha$, $e^{i\theta} \in M$, then $f(z) \equiv \alpha$.

Privalov, in 1936, noted that the statement $f(z) \equiv \text{const}$ remains true also when the values $\alpha_\zeta = \lim_{r \to 1} f(re^{i\theta})$ are not necessarily equal at the points $\zeta = e^{i\theta} \in M$, but belong to a set of (logarithmic) capacity zero. The basic idea and the elements of the proof of the Luzin–Privalov theorem are applicable in the general case of continuous mappings f of D, which was subsequently used in many papers.

A point $\zeta \in \Gamma = \{z : |z| = 1\}$ is called a Plessner point, and belongs to the set $I(f)$, if the intersection

$$\bigcap C(f, \zeta; \Delta(\zeta, \phi_1, \phi_2))$$

over all angular domains $\Delta(\zeta, \phi_1, \phi_2)$ with vertex ζ coincides with Ω. A.I. Plessner proved (1927) that for a meromorphic function f in D almost-all points of the boundary Γ belong either to $F(f)$ or to $I(f)$, i.e. $\Gamma = F(f) \cup I(f) \cup E$, $\text{mes}\, E = 0$. A point $\zeta \in \Gamma$ is called a Meier point, and belongs to $M(f)$, if $C(f, \zeta; D) \neq \Omega$ and if the intersection of the chordal cluster sets, $\cap C(f, \zeta; h(\zeta, \phi))$, over all chords drawn at ζ, coincides with $C(f, \zeta; D)$. K. Meier established (1961) the following analogue of Plessner's theorem in terms of Baire categories: If f is meromorphic in D, then all points of the boundary Γ, with the possible exception of a set E of the first category, belong to the union $M(f) \cup I(f)$. A more precise statement of Meier's theorem has been obtained, in which E is a set of the first category and of type F_σ (cf. [12]–[14], in which generalizations of Plessner's and Meier's theorems have been obtained, and in which a converse of Meier's theorem and a characterization of $M(f)$ have been given).

The work of Fatou served as an original source for the development of fundamental research on boundary properties of analytic functions. The studies of F. and M. Riesz, Luzin, Privalov, R. Nevanlinna, Plessner, V.I. Smirnov, and others were conducted independently of the ideas of Painlevé, and the use of methods related to measure and integration theory, including the notion of Baire categories, is characteristic for them (cf. [4]–[9]).

The basic objects of study for F. Iversen and W. Gross were meromorphic functions f in domains D with a Jordan boundary $\Gamma = \partial D$. At an arbitrary point $\zeta_0 \in \Gamma$, the boundary cluster set $C(f, \zeta_0; \Gamma)$ is defined as follows: If M_r denotes the closure of the union $\cup\, C(f, \zeta; D)$ over all points

$$\zeta \in (\Gamma \setminus \{\zeta_0\}) \bigcap \{z : |z - \zeta_0| < r\},$$

then $C(f, \zeta_0; \Gamma) = \cap_{r>0} M_r$. One of the main theorems obtained, independently, by them asserts that, under the conditions stated, the set

$$C_i(f, \zeta_0; D) = C(f, \zeta_0; D) \setminus C(f, \zeta_0; \Gamma)$$

is open (for any $\zeta_0 \in \Gamma$), and all values $a \in C_i(f, \zeta_0; D)$, with possibly two exceptions, belong to the set of recurrent values $R(f, \zeta_0; D)$. Moreover, every exceptional value (if existing) is an asymptotic value of f at ζ_0.

The research of Iversen and Gross obtained a further development in the work of A. Beurling, W. Seidel (who in 1932 also introduced the term "cluster set") and others (cf. [5]–[9]). They basically considered the case when ζ_0 belongs to a "small" set E on the boundary Γ, having zero linear measure or zero capacity, and studied the cluster set $C(f, \zeta_0; \Gamma \setminus E)$, defined analogously to $C(f, \zeta_0; \Gamma)$. Methods of potential theory are also used in these studies.

The most recent results in this direction are stated below for the case of the disc $D = \{z : |z| < 1\}$. Suppose a set E on an arc γ of the boundary Γ of D having $\operatorname{mes} E = 0$ is fixed, and let $\zeta_0 \in E$. To every point $\zeta \in \gamma \setminus E$ one assigns a Jordan arc $\Lambda_\zeta \subset D$ ending at ζ. Let M_r^* be the closure of the union $\cup\, C(f, \zeta; \Lambda_\zeta)$ over all points

$$\zeta \in (\gamma \setminus E) \bigcap \{z : |z - \zeta_0| < r\}$$

and suppose

$$C^*(f, \zeta_0; \Gamma \setminus E) = \bigcap_{r>0} M_r^* .$$

Then the set

$$S(\zeta_0) = C(f, \zeta_0; D) \setminus C^*(f, \zeta_0; \Gamma \setminus E)$$

is open, the set $S(\zeta_0) \setminus R(f, \zeta_0; D)$ has capacity zero, and every value $a \in S(\zeta_0) \setminus R(f, \zeta_0; D)$ is an asymptotic value of f either at ζ_0 or at every point of some sequence $\{\zeta_n\}$, $\zeta_n \in \Gamma$, $n = 1, 2, \dots,$ $\lim_{n\to\infty} \zeta_n = \zeta_0$. If E has capacity zero, then for every connected component $S_k(\zeta_0)$, $k = 1, 2, \dots,$ of

$S(\zeta_0)$ the set $S_k(\zeta_0) \setminus R(f, \zeta_0; D)$ consists of at most two distinct values.

Lindelöf's theorem has been proved using normal families (cf. Normal family): If a holomorphic function f is bounded in D and has asymptotic value α at $\zeta_0 \in D$, then it has at this point α as angular boundary value. Normality of a family $F = \{f(z)\}$ of meromorphic functions $f(z)$ in a domain G can be characterized in terms of the so-called spherical derivative

$$\rho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

To be precise, F is a normal family if and only if the spherical derivatives $\rho(f(z)), f \in F$, are uniformly bounded inside G, i.e. if for every compactum $K \subset G$ there is a constant $C = C(K)$ such that

$$\rho(f(z)) \leq C(K), \quad z \in K, \quad f \in F.$$

However, the most important occurrence of normal families in the theory of cluster sets is in the notion of a normal function. A function $f(z)$, meromorphic in a simply-connected domain G, is called a normal function in G if the family $\{f(S(z))\}$, where S runs through the family of all conformal automorphisms of G, is normal; $f(z)$ is normal in a multiply-connected domain G if it is normal on the universal covering surface of G. A function $f(z)$, meromorphic in D, is normal if and only if there is a constant $C = C(f)$, $0 < C < \infty$, such that

$$\rho(f(z))|dz| \leq C\frac{|dz|}{1 - |z|^2}.$$

Here, the left-hand side is the line element in the so-called chordal metric on the Riemann sphere Ω for the mapping $w = f(z)$, while the expression $d\sigma(z) = |dz| / (1 - |z|^2)$ is the hyperbolic metric of D. Bounded holomorphic functions and meromorphic functions not taking three distinct values are normal, and certain properties of functions of the classes indicated carry over to arbitrary normal functions. E.g., the conclusion of Lindelöf's theorem holds for arbitrary normal functions. The class of all normal meromorphic functions in D has some resemblance to the class of functions of bounded characteristic (cf. Function of bounded characteristic). There are, however, essential differences. E.g., there exist normal meromorphic functions without asymptotic values, hence without radial boundary values, a fact which cannot hold for functions of bounded characteristic. G.R. MacLane [7], [9] conducted important studies on asymptotic values. MacLane's theory allows one to obtain new proofs of already known properties of normal functions. E.g., the set of points $\zeta \in \Gamma$ at which a normal holomorphic function $f(z)$ has asymptotic values, hence angular boundary values, is dense on Γ.

The value distribution of meromorphic functions is closely connected with the notion of normality. A sequence $\{z_n\}$ of points z_n in D with $\lim_{n \to \infty} |z_n| = 1$ is called a P-sequence for a meromorphic function $f(z)$ in D if for every infinite subsequence $\{z_{n_k}\}$ and every $\epsilon > 0$ the set

$$\Omega \setminus R\left(f, z; \bigcup_{k=1}^{\infty} \{z \in D : \sigma(z, z_{n_k}) < \epsilon\}\right)$$

contains at most two values. It has been proved that f has at least one P-sequence if and only if

$$\limsup_{|z| \to 1} q_f(z) = +\infty, \quad q_f(z) = (1 - |z|^2)\rho(f(z)).$$

Thus, the value distribution of the meromorphic function $f(z)$ is related to the structure of the cluster set of the continuous function $q_f(z)$.

Substantial progress has been made on the theory of cluster sets of general mappings $f: D \rightarrow \Omega$, $D = \{z: |z| < 1\}$. Already in 1955 the ambiguous point theorem was proved: Let $f: D \rightarrow \Omega$ be an arbitrary mapping; then the points $\zeta \in \Gamma$ at which one can draw two continuous curves L_ζ^1 and L_ζ^2 such that

$$C(f, \zeta; L_\zeta^1) \neq C(f, \zeta; L_\zeta^2),$$

form a set that is at most countable. Collingwood's maximality theorem: Let L_0 be an arbitrary continuum in D such that $L_0 \cap \Gamma = \{z=1\}$, let L_θ be the continuum obtained from L_0 by rotation over θ around the coordinate origin and let $f: D \rightarrow \Omega$ be an arbitrary mapping; then the points $\zeta = e^{i\theta} \in \Gamma$ at which

$$C(f, \zeta; L_\theta) \neq C(f, \zeta; D),$$

form a set of the first category on Γ. A point $\zeta \in \Gamma$ is said to belong to the set $C(f)$ if the cluster set $C(f, \zeta; D)$ coincides with the intersection

$$\bigcap C(f, \zeta; \Delta(\zeta, \phi_1, \phi_2))$$

over all angular domains with vertex ζ. It has been proved [10] that

$$\Gamma = C(f) \bigcup E$$

for an arbitrary mapping $f: D \rightarrow \Omega$, where E is a set of the first category of type F_σ. Conversely, for an arbitrary set $E \subset \Gamma$ of the first category and of type F_σ there exists a function f, holomorphic and bounded in D, for which $E = \Gamma \setminus C(f)$. The set $C(f)$ is a subset of the set $K(f)$ of all $\zeta \in \Gamma$ at which

$$C(f, \zeta; \Delta(\zeta, \phi_1, \phi_2)) = C(f, \zeta; \Delta(\zeta, \phi_1', \phi_2'))$$

for any two angular domains $\Delta(\zeta, \phi_1, \phi_2)$ and $\Delta(\zeta, \phi_1', \phi_2')$. Let $E \subset \Gamma$ and $\zeta \in \Gamma$. For a given $\epsilon > 0$ let $r(\zeta, \epsilon, E)$ denote the length of the largest open arc on Γ contained in the ϵ-neighbourhood $\left\{e^{i\theta}: |\theta - \arg \zeta| < \epsilon\right\}$ of ζ and not having points in common with E; if such an arc does not exist, $r(\zeta, \epsilon, E) = 0$. A set E is called porous on Γ if for any point $\zeta \in E$,

$$\limsup_{\epsilon \rightarrow 0} \frac{r(\zeta, \epsilon, E)}{\epsilon} > 0;$$

a σ-porous set is a union of at most countably many porous sets. Every σ-porous set is of the first category and has linear measure zero. The equality $\Gamma = K(f) \cup E$ is valid for any mapping $f: D \rightarrow \Omega$, where E is a σ-porous set of type $G_{\delta\sigma}$. Conversely, for an arbitrary σ-porous set E there exists a function f, holomorphic and bounded in D, such that $\Gamma \setminus K(f) \supset E$.

About the theory of cluster sets of functions of several complex variables see, e.g., [15]–[17].

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Comments

For the notions of linear Hausdorff measure and plane measure cf. Hausdorff measure; for the chordal metric (also called spherical metric) cf. Extended complex plane.