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Author(s): Robert Geske and H. E. Johnson

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The American Put Option Valued Analytically

ROBERT GESKE and H. E. JOHNSON*

ABSTRACT

An analytic solution to the American put problem is derived herein. The hedge ratio and other derivatives of the solution are presented. The formula derived implies an exact duplicating portfolio for the American put consisting of discount bonds and stock sold short. The formula is extended to consider put options on stocks paying cash dividends. A polynomial expression is developed for evaluating these formulae. Values and hedge ratios for puts on both dividend and nondividend paying stocks are calculated, tabulated, and compared with values derived by numerical integration and binomial approximation. As with European options, evaluating an analytic formula is more efficient than approximating the stock price process or the partial differential equation by binomial or finite difference methods. Finally, applications of this American put solution are discussed.

MERTON [16] SHOWED THAT American put options are more difficult to value than European puts because at every instant for American puts there is a positive probability of premature exercise. Black and Scholes [1] derived a formula for a European put when the stock price follows geometric Brownian motion. For this same stock price distribution, Merton [16] derived a formula for a perpetual American put. Brennan and Schwartz [2], Cox and Rubinstein [4], and Parkinson [17] have developed numerical solutions for the value of a finite-lived American put. However, numerical solutions are expensive and do not offer the intuition which the comparative statics of an analytic solution provide. An analytic approximation has been developed by Johnson [15], but it does not handle dividends or hedge ratios and there is no way to make this approximation arbitrarily accurate.

This paper presents an analytic formula which satisfies the partial differential equation and boundary conditions that characterize the American put valuation problem. Since at every instant there is a positive probability of premature exercise, this situation is equivalent to an infinite sequence of options on options, or compound options. Geske [8] originally showed how to value an option on an option. Using this solution technique, the American put formula is derived. The hedge ratio and all other derivatives of the formula with respect to its parameters are presented. Furthermore, it is easy to show that the formula satisfies the partial differential equation.

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While formula evaluation is generally straightforward, this is not the case for most option formulae, and American options are more difficult than European. In order to evaluate our equation directly, it would be necessary to compute an infinite series of conditional exercise terms. Instead, we show that, since our formula is exact in the limit, arbitrary accuracy can be achieved by considering puts which can only be exercised at a few discrete dates and then using their prices to extrapolate to the price of a put which can be exercised at any date. This approach provides an efficient way to evaluate the American put formula and its derivatives.

Section I presents the analytic formula and its comparative statics, and extends the solution to the case of stocks paying cash dividends. Section II demonstrates that the formula can be evaluated to arbitrary accuracy by a polynomial expression similar to that used to evaluate the Black-Scholes European put and call option formulae. It is also shown that the hedge ratio and cash dividend adjustments can be similarly computed. Section II also notes that the same solution and evaluation techniques can be used to value other complex contracts, such as currency options, options on futures, coupon bonds, or warrants on dividend paying stocks. Section III summarizes the paper.

I. The Formulae

A. The Solution

Let S, σ^2, r, T, X, and P be the underlying stock price, the variance of the rate of return on the stock, the risk-free rate, the time to maturity of the put, the exercise price, and the American put price, respectively. Following Black and Scholes [1], we assume perfect markets, constant r and σ, no dividends, and geometric Brownian motion for the stock price. We set current time to zero.

The stochastic process for stock price changes is assumed to be:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

where μ is the expected return on the stock and dz is the differential of a Gauss-Wiener process. Since the only parameters which are assumed variable are the stock price and time, and the stock price is stochastic, put price changes can be characterized using Itô's lemma. Then, by constructing a self-financing, risk-free hedge between the put, the stock, and a riskless security, the put's equilibrium price path can be described by the familiar Black-Scholes partial differential equation:

$$\frac{\partial P}{\partial t} = rP - rS\frac{\partial P}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}. \tag{2}$$

Because the American put can be exercised at any instant and because the stock price boundary which triggers exercise is not constant, the problem is termed a free boundary problem.

The free boundary condition which the American put must satisfy at every instant is

$$P(S, T) \geq \max(0, X - S) \tag{3}$$

for all $T \geq 0$. Except for numerical methods, which do not produce a formula satisfying partial differential equation (2) above, the solution to this free boundary problem has been considered intractable.

Here we show that there does exist an analytic solution[1] to this partial differential equation subject to the free boundary condition. A key to our solution is the assumption that each exercise decision is a discrete event. Thus, the formula derived is a continuous time solution to partial differential equation (2), subject to the free boundary condition (3) applied at an infinite number of discrete instants. Like the Black-Scholes equation, our formula is easier to use than the numerical methods applied to approximate the solution to the partial differential equation. Our formula would be obtainable by traditional methods for solving partial differential equations, but such procedures would be less tractable. However, since a riskless hedge can be formed, the Cox-Ross [3] technique can be used in conjunction with Geske's [8] compound option valuation, thus circumventing the usual transformation solution to the partial differential equation.

Using the Cox-Ross approach, we can price the American put as the discounted expected value of all future cash flows. The cash flows arise because the put can be exercised at the next instant, dt, or the following instant, $2dt$, if not previously exercised, $\cdots$, ad infinitum. Since the assumption of geometric Brownian motion implies that the stock price at any future date is a lognormally distributed random variable, the correlation coefficient between the overlapping Brownian increments at times t_1 and t_2 ($t_2 > t_1$) is given by

$$\rho_{12} = \frac{\mathrm{Cov}(\Delta z_1, \Delta z_2)}{[\mathrm{Var}(\Delta z_1)\mathrm{Var}(\Delta z_2)]^{1/2}} = (t_1/t_2)^{1/2} \tag{4}$$

where $\Delta z_1 = z(t_1) - z(0)$ and $\Delta z_2 = z(t_2) - z(0)$. At each instant, we will exercise the put if (a) the put has not already been exercised and (b) the payoff from exercising the put equals or exceeds the value of the put if it is not exercised. This implies a "critical stock price," $\bar{S}$, at which exercise occurs. The critical stock price is independent of the current stock price and is determined from the free boundary given in condition (3), whenever the exercise proceeds equal the American put value, or $X - \bar{S} = P(\bar{S}, T)$, for some $S = \bar{S}$ and any T. (This can also be expressed as a derivative boundary condition, $\partial P/\partial S = -1$.) At the first instant, there is no probability that the put will already have been exercised, so we just integrate the exercise price less the future stock price over all stock prices less than $\bar{S}_{dt}$, the critical stock price at this date, and then discount to the present. This yields two terms, one being simply the discounted exercise price times the probability that the stock price will be below $\bar{S}_{dt}$. At the next instant, we perform a similar integration up to $\bar{S}_{2dt}$, the new critical stock price, but we

[1] For a definition of "analytic solution," see James and James [13, p. 11]. The use of the word analytical or numerical is tricky in the option context since in the end all methods require numerical procedures.

must exclude all those cases where the put will be exercised at the first date. Again we obtain two terms, one being the discounted exercise price times the probability that the stock price at the first instant will be above the first critical stock price, $\overline{S}_{dt}$, and that the stock price at the second instant will be below the second critical stock price, $\overline{S}_{2dt}$.

The procedure for the third instant is similar, except now there are trivariate normals instead of bivariates. The correlation coefficient is negative between the argument for the last instant and the arguments for the previous ones, but positive between the arguments for the previous times. The intuition here is that the put will be exercised at this instant if the stock price is below the critical stock price for this instant, given that it was not exercised at all previous instants because the stock price was always above the critical stock price. Proceeding in this way, we obtain the following solution for the value of an American put option:

$$P = Xw_2 - Sw_1 \tag{5}$$

where the weights w_1 and w_2 are[2]

$$\begin{aligned} w_1 = \{&N_1(-d_1(\overline{S}_{dt}, dt) \\ &+ N_2(d_1(\overline{S}_{dt}, dt), -d_1(\overline{S}_{2dt}, 2dt); -\rho_{12}) \\ &+ N_3(d_1(\overline{S}_{dt}, dt), d_1(\overline{S}_{2dt}, 2dt), -d_1(\overline{S}_{3dt}, 3dt); \\ &\quad \rho_{12}, -\rho_{13}, -\rho_{23}) + \cdots\} \\ w_2 = \{&e^{-rdt}N_1(-d_2(\overline{S}_{dt}, dt)) \\ &+ e^{-r2dt}N_2(d_2(\overline{S}_{dt}, dt), -d_2(\overline{S}_{2dt}, 2dt); -\rho_{12}) \\ &+ e^{-r3dt}N_3(d_2(\overline{S}_{dt}, dt), d_2(\overline{S}_{2dt}, 2dt), -d_2(\overline{S}_{3dt}, 3dt); \\ &\quad \rho_{12}, -\rho_{13}, -\rho_{23}) + \cdots\} \end{aligned}$$

and where in general

$$d_1(q, \tau) = \frac{\ln(S/q) + (r + ½\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2(q, \tau) = d_1 - \sigma\sqrt{\tau}$$

for any q (i.e., critical stock price or exercise price) and τ (i.e., time interval). The correlation coefficients are, sequentially, as follows:

$$\rho_{12} = 1/\sqrt{2}$$

$$\rho_{13} = 1/\sqrt{3}$$

[2] This is simply the sum of a series of options on options or compound options. The equation is the solution to an optimal stopping problem. Although the equation contains an infinite series of terms, it is an exact solution of the partial differential equation, subject to an infinite number of discrete exercise, or stopping, boundaries. This is notably different than Fourier, or other infinite series approximations to the partial differential equation.

$$\rho_{23} = \sqrt{2/3}$$

N_1, N_2, and N_3 are the standard cumulative univariate, bivariate, and trivariate normals, respectively. There is an infinite number of terms (each a higher order multivariate normal integral) in the solution, ending when the final instant has been reached.

Although we cannot use Equation (5) to compute actual numbers for the American put values, it does offer intuition about the portfolio which duplicates the American put payoffs. Also, the formula can be differentiated to yield intuition regarding the sensitivity to all specified parameters, and to simplify the computation of the hedge ratio. The American put can be thought of as an infinite series of contingent payoffs. At each date, there is a payoff if and only if the stock price is below the critical stock price for that date and it was not below any critical stock price at any previous date. The American put's contingent payoffs can be duplicated by an infinite series of risk-free, discount bonds, and a short position in the stock. Since we have assumed that the risk-free rate is known and constant, the portfolio of bonds represented by Xw_2 is equivalent to investing the same amount in a risk-free bond of any maturity. However, if one were to introduce uncertainty about future interest rates, then term structure effects could be important. Note that our formula implies the intuitive result that the duplicating portfolio for out-of-the-money puts is skewed toward longer maturity bonds, while for in-the-money puts it is skewed toward shorter maturities.

The European put option is simply a special case of an American put option with only one exercise boundary. Thus, formula (5) above will reduce to the European put formula when boundary condition (3) only holds at $T = 0$. Although the critical stock prices are unknown, they can be determined at any instant for all future dates prior to expiration. The critical stock price is a time-dependent path of stock prices that separates the exercise from the no exercise region in such a way as to maximize the value of the American put. Just at the point where the stock price is equal to the critical stock price, the put value would decrease one dollar for a one dollar increase in the stock price (i.e., $\partial P/\partial S(S = S_c) = -1$). As the stock price approaches the critical stock price, the sensitivity of the American put with respect to time to expiration approaches zero (i.e., $\partial P/\partial T(S \rightarrow S_c) \rightarrow 0$). Also, at this exercise point, the interest rate effect on the American put exactly offsets the variance effect. Although these points may not be intuitively obvious, they are clarified in the next section, where the derivatives of the American put formula are presented.

B. Comparative Statics

Here the sensitivity of the American put option formula to changes in each of its arguments is presented. Note that because the option value is linearly homogeneous with respect to the stock price and exercise price, either partial derivative implies the other by Euler's Theorem. Also note that because time to maturity only appears in the formula multiplied by the interest rate or the variance rate, any two of the three partial derivatives with respect to r, σ, and T imply the third.

The partial derivative of the valuation formula (5) with respect to the stock price is the hedge ratio, or number of shares of stock to options in a perfectly hedged portfolio:

$$\frac{\partial P}{\partial S} = -w_1 < 0. \tag{6}$$

The negative sign indicates that as the stock price rises the put price falls. The hedge ratio can be thought of as either the negative amount of stock (sold short) to which the put is equivalent, or, under risk neutrality, as the discounted expected cash outflow (divided by the stock price) that the put holder will experience.

As the exercise price rises, the put value rises:

$$\frac{\partial P}{\partial X} = w_2 > 0. \tag{7}$$

This can similarly be considered the expected cash inflow (divided by the exercise price).[3]

As the interest rate rises, the American put value falls:

$$\frac{\partial P}{\partial r} = -X\, dt[e^{-rdt}N_1(') + 2e^{-2rdt}N_2(') + \cdots] \le 0. \tag{8}$$

This is because the present value of the bonds in the duplicating portfolio decreases as the interest rate increases.

As the variance rate rises, the put value rises:

$$\frac{\partial P}{\partial \sigma^2} = X\, \frac{\sqrt{T}}{2\sigma}\, w_2' \ge 0. \tag{9}$$

An increase in volatility increases the probability of both high and low stock prices, and with the asymmetry of an option's contingent payoffs, increases the option value.

As time to expiration increases, the American put value rises:

$$\frac{\partial P}{\partial T} = -Xr\,\frac{dt}{T}\,[e^{-rdt}N_1(') + 2e^{-2rdt}N_2(') \cdots + X\,\frac{\sigma}{2\sqrt{T}}\,w_2' \ge 0. \tag{10}$$

The first four partial derivatives are functionally different from their corresponding European counterparts, but they do have the same sign and similar intuition.[4] Although it is not obvious, the partial derivative with respect to time to expiration

[3] It may appear that Equations (7)–(10) require taking derivatives of the critical stock prices, but in fact all such terms cancel.

[4] In Equation (8), it may be useful to think of dt as T/n and then take the limit as $n \to \infty$. This relates the derivative to the European put derivative. In Equations (9) and (10), w_2' is the sum of the probability density functions derived from the cumulative distribution functions.

is strictly positive for the American put (provided $S > \bar{S}$), while its sign is ambiguous for the European put. This ambiguity for the European put is obvious because more time helps if the put is out-of-the-money, but hurts if the holder wants to exercise immediately. The strictly positive sign for the American put is intuitive because extending the life gives the holder more options (i.e., choices). As the stock price approaches the critical stock price and $\partial P/\partial T$ approaches zero, the interest rate effect offsets the variance effect.

When the American put is alive (i.e., $S > \bar{S}$), the partial derivatives of formula (5), given in Equations (6)–(10), satisfy the original partial differential equation (2). This demonstrates that American put formula (5) is the solution to partial differential equation (2) subject to its boundary conditions.

C. *Dividends*

The majority of listed put options are traded on stocks paying cash dividends. Geske and Shastri [10] demonstrated for American puts that dividends significantly reduce the probability of early exercise. This diminishes the difference between American and European put option values, and consequently one might conclude that dividends simplify the valuation problem. Although this conclusion is correct in the sense that the errors from using the European formula for American put options would be smaller, dividends complicate the exact valuation of American puts.

Roll [19] developed a formula for valuing American call options on stocks paying a single cash dividend. He avoided the discontinuity in the stock price at the ex-dividend date by escrowing the dividend from the stock price and employing the "net-of-dividends stock price" in the valuation problem. In order to incorporate multiple dividends, Geske's [9] adjustment to Roll's dividend procedure is used here for valuing American puts. Although the form of the solution is similar to Equation (5), the next section demonstrates that the evaluation of the formula for American puts on stocks paying dividends is more complex because dividends disrupt the critical stock price path.

II. Formula Evaluation and Applications

In this section, we show how to evaluate the American put formula (5) with a polynomial expression based upon an extrapolation from only a small number of exercise points to the infinite limit. The evaluation is very efficient because we are approximating an exact solution rather than the partial differential equation or the stock price process itself. Arbitrary accuracy can be obtained by adding exercise points. However, we show that only a few (cf. three) critical stock prices need to be computed in order to obtain penny accuracy. We stress that unlike previous authors (Brennan and Schwartz [2] or Parkinson [17]), we do not approximate the partial differential equation or the stock price process (Cox and Rubinstein in [4]). Instead, we evaluate our formula, which is an exact solution to the partial differential equation subject to the (discrete) free exercise boundary.

Let P_1 be the price of a put that can only be exercised at time T (i.e., at expiration); this option is just the European put, and we can write $P_1 = p$, the

European put value. Let P_2 be the value of a put that can only be exercised at time $T/2$ (i.e., halfway to expiration) or at time T. Then

$$\begin{aligned} P_2 = {} & Xe^{-rT/2}N_1[-d_2(\bar{S}_{T/2}, T/2)] - SN_1[-d_1(\bar{S}_{T/2}, T/2)] \\ & + Xe^{-rT}N_2[d_2(\bar{S}_{T/2}, T/2), -d_2(X, T); -1/\sqrt{2}] \\ & - SN_2[d_1(\bar{S}_{T/2}, T/2), -d_1(X, T); -1/\sqrt{2}]. \end{aligned} \tag{11}$$

The critical stock price, $\bar{S}_{T/2}$, solves

$$S = X - p(S, X, T/2, r, \sigma) = \bar{S}_{T/2}. \tag{12}$$

Let P_3 be the value of a put that can only be exercised at time $T/3$, time $2T/3$, or time T. Then

$$\begin{aligned} P_3 = {} & Xe^{-rT/3}N_1[-d_2(\bar{S}_{T/3}, T/3)] - SN_1[-d_1(\bar{S}_{T/3}, T/3)] \\ & + Xe^{-2rT/3}N_2[d_2(\bar{S}_{T/3}, T/3), -d_2(\bar{S}_{2T/3}, 2T/3); -1/\sqrt{2}] \\ & - SN_2[d_1(\bar{S}_{T/3}, T/3), -d_1(\bar{S}_{2T/3}, 2T/3); -1/\sqrt{2}] \\ & + Xe^{-rT}N_3[d_1(\bar{S}_{T/3}, T/3), d_1(\bar{S}_{2T/3}, 2T/3), -d_1(X, T); \\ & \quad 1/\sqrt{2}, -1/\sqrt{3}, -\sqrt{2/3}] \\ & - SN_3[d_2(\bar{S}_{T/3}, T/3), d_2(\bar{S}_{2T/3}, 2T/3), -d_2(X, T); \\ & \quad 1/\sqrt{2}, -1/\sqrt{3}, -\sqrt{2/3}] \end{aligned} \tag{13}$$

and the critical stock prices $\bar{S}_{T/3}$ and $\bar{S}_{2T/3}$ solve

$$S = X - P_2(S, X, 2T/3, r, \sigma) = \bar{S}_{T/3} \tag{14}$$

and

$$S = X - p(S, X, T/3, r, \sigma) = \bar{S}_{2T/3}, \tag{15}$$

respectively.

The values $P_1, P_2, P_3, \cdots$ define a sequence, the limit of which is the American put value. Many techniques are available for computing such limits. One method is Richardson extrapolation (see, e.g., Dahlquist and Bjorck [5, p. 269]). This method permits the determination of the limiting value of some quantity as the "step length," h, approaches zero. In our case, the quantity to be determined is the American put price for a particular set of values of S, X, T, r, and σ. The step length is the time between points at which exercise is permitted. The version of Richardson extrapolation we use is developed in the Appendix, and leads to the following equation;

$$P = P_3 + 7/2(P_3 - P_2) - 1/2(P_2 - P_1). \tag{16}$$

This polynomial can be used to determine American put values and hedge ratios.

Table I presents our formula values, which are given in the next to last column, while the last column gives values found by numerical integration in Parkinson [17] and by binomial approximation in Cox and Rubinstein [4] (with 150 time

Table I

Comparison of American and European Values and Hedge Ratios[a]

r	X	σ	T	$\frac{\partial p}{\partial S}$ (European)	$\frac{\partial P}{\partial S}$ (Analytic)	p (European)	P (Analytic)	P (Numerical)
0.1250	1.0	0.5	1.0000	−0.309	−0.359	0.1327	0.1476	0.148
0.0800	1.0	0.4	1.0000	−0.345	−0.381	0.1170	0.1258	0.126
0.0450	1.0	0.3	1.0000	−0.382	−0.407	0.0959	0.1005	0.101
0.0200	1.0	0.2	1.0000	−0.421	−0.436	0.0694	0.0712	0.071
0.0050	1.0	0.1	1.0000	−0.460	−0.467	0.0373	0.0377	0.038
0.0900	1.0	0.3	1.0000	−0.326	−0.385	0.0761	0.0859	0.086
0.0400	1.0	0.2	1.0000	−0.382	−0.416	0.0600	0.0640	0.064
0.0100	1.0	0.1	1.0000	−0.440	−0.455	0.0349	0.0357	0.036
0.0800	1.0	0.2	1.0000	−0.309	−0.393	0.0442	0.0525	0.053
0.0200	1.0	0.1	1.0000	−0.401	−0.434	0.0304	0.0322	0.033
0.1200	1.0	0.2	1.0000	−0.242	−0.387	0.0317	0.0439	0.044
0.0300	1.0	0.1	1.0000	−0.363	−0.418	0.0263	0.0292	0.03
0.0488	35.0	0.2	0.0833	−0.008	−0.008	0.0062	0.0062	0.01
0.0488	35.0	0.2	0.3333	−0.088	−0.090	0.1960	0.1999	0.20
0.0488	35.0	0.2	0.5833	−0.128	−0.134	0.4170	0.4321	0.43
0.0488	40.0	0.2	0.0833	−0.460	−0.470	0.8404	0.8528	0.85
0.0488	40.0	0.2	0.3333	−0.421	−0.443	1.5222	1.5807	1.58
0.0488	40.0	0.2	0.5833	−0.396	−0.427	1.8813	1.9905	1.99
0.0488	45.0	0.2	0.0833	−0.974	−1.000	4.8399	4.9985	5.00
0.0488	45.0	0.2	0.3333	−0.794	−0.888	4.7805	5.0951	5.09
0.0488	45.0	0.2	0.5833	−0.694	−0.805	4.8402	5.2719	5.27
0.0488	35.0	0.3	0.0833	−0.051	−0.052	0.0771	0.0774	0.08
0.0488	35.0	0.3	0.3333	−0.171	−0.174	0.6867	0.6969	0.70
0.0488	35.0	0.3	0.5833	−0.206	−0.213	1.1890	1.2194	1.22
0.0488	40.0	0.3	0.0833	−0.464	−0.470	1.2991	1.3100	1.31
0.0488	40.0	0.3	0.3333	−0.428	−0.442	2.4376	2.4817	2.48
0.0488	40.0	0.3	0.5833	−0.406	−0.425	3.0636	3.1733	3.17
0.0488	45.0	0.3	0.0833	−0.898	−0.926	4.9796	5.0599	5.06
0.0488	45.0	0.3	0.3333	−0.691	−0.726	5.5290	5.7012	5.71
0.0488	45.0	0.3	0.5833	−0.608	−0.651	5.9725	6.2365	6.24
0.0488	35.0	0.4	0.0833	−0.106	−0.106	0.2458	0.2466	0.25
0.0488	35.0	0.4	0.3333	−0.222	−0.226	1.3298	1.3450	1.35
0.0488	35.0	0.4	0.5833	−0.247	−0.254	2.1129	2.1568	2.16
0.0488	40.0	0.4	0.0833	−0.463	−0.467	1.7579	1.7679	1.77
0.0488	40.0	0.4	0.3333	−0.426	−0.437	3.3338	3.3632	3.38
0.0488	40.0	0.4	0.5833	−0.403	−0.418	4.2475	4.3556	4.35
0.0488	45.0	0.4	0.0833	−0.823	−0.835	5.2362	5.2855	5.29
0.0488	45.0	0.4	0.3333	−0.627	−0.646	6.3769	6.5093	6.51
0.0488	45.0	0.4	0.5833	−0.556	−0.580	7.1656	7.3831	7.39

[a] Numerical values in far right column are from Parkinson [17] for $X = 1$ and from Cox and Rubinstein [4] for $X \neq 1$. Analytic values in next to last column are from evaluating Equation (5). European values are from the Black-Scholes equation. The stock price equals one dollar for Parkinson and 40 dollars ($S = \$40$) for Cox and Rubinstein.

steps).[5] The analytic and numerical solutions yield values within a penny of each other. Note that the European values are close to the American values for the parameters presented (these parameters were chosen to match the other established values). Of course the American feature will be most valuable relative to the European when the option is more likely to be exercised early. In Table I the absolute difference is largest, for example, when the exercise price is \$45.00. Since this is only \$5.00 in-the-money (or about 12.5% when S = \$40.00), more significant differences would be observed for options further in-the-money, or where early exerise is more likely.

Preliminary evidence indicates that the analytic formula evaluation tabulated is faster to compute, by a factor of 10 times, than the numerical methods.[6] This is because the binomial and finite difference methods compute n critical stock prices ($n = 150$ in Cox and Rubinstein) while our three point extrapolation computes only three, and the tabulated four point method computes six. (The three point extrapolation is about twice as fast as the four point.)

The hedge ratio for the American put is the change in the put value with respect to a change in the stock price. This partial derivative is given analytically in Equation (6) as $\partial P/\partial S = -w_1$. In the absence of an analytic formula the hedge ratio is numerically approximated by computing two put values for two different stock prices and then using a difference equation to approximate the partial derivative at an intermediate stock price. Hedge ratios computed from Equation (6) are given in Column 5 of Table I.

Our technique can also be used for valuing American puts adjusted for cash dividends. However, introducing dividends changes the critical stock price path (or, equivalently, the probability of exercise at different dates) so radically that an extrapolation based on a few exercise points equally spaced through time could result in large errors. Rather than using a large number of exercise points and then extrapolating, we use the following device which finesses the problem in an intuitive way. First, note that any dividend precludes exercise for a period preceding the ex-dividend date (see Johnson [14, p. 42]). Let D' be that dividend which is just big enough to preclude exercise for the entire period between dividends.[7] If the actual dividend, D, is greater than D', then we can put all the

[5] There are many routines available for calculating multivariate normal integrals. Here, the univariate and bivariate normals are calculated using the IMSL routines, MDNORD, and a double precision version of MDBNOR. The trivariate normal integral is evaluated as

$$N_3(h, k, j; \rho_{12}, \rho_{13}, \rho_{23}) = \int_{-\infty}^{j} F'(z)N_2\left(\frac{k - \rho_{23}z}{(1-\rho_{23}^2)^{1/2}}, \frac{h - \rho_{13}z}{(1-\rho_{13}^2)^{1/2}}; \frac{\rho_{12} - \rho_{13}\rho_{23}}{(1-\rho_{13}^2)^{1/2}(1-\rho_{23}^2)^{1/2}}\right) dz$$

where $F'(z)$ is the standard normal density function. The integral can be done using the IMSL routine, DCADRE. The critical stock prices are computed by a Newton-Raphson gradient scheme.

[6] The tabulated values are actually computed from a more accurate four point extrapolation developed similarly to the three point method described above. The resulting four point formula is:

$$P = P_4 + {}^{29}\!/_3(P_4 - P_3) - {}^{23}\!/_6(P_3 - P_2) + {}^{1}\!/_6(P_2 - P_1).$$

The higher order integrals can be evaluated using an integral reduction given in Geske [7]. The comparative times come from the numerical methods presented in Geske and Shastri [11].

[7] Assuming that the stock price drops by the dividend, D, on the ex-dividend date, Johnson [14] shows that there will be no early exercise provided $D \geq X(e^{rT} - 1)$.

exercise points in the period between the last ex-dividend date and the maturity date of the option. If $D < D'$, we find the put value from linear interpolation:

$$P(D) = P(0) + D/D'[P(D') - P(0)]$$

where $P(0)$, $P(D)$, and $P(D')$ are the values of puts when there is no dividend, a dividend D, and a dividend D', respectively. Table II indicates that penny accuracy is obtained with this procedure.

The methods discussed here can also be used to simplify the evaluation of certain contracts, such as coupon bonds, warrants, and currency options. For example, suppose a 30-year bond pays semiannual coupons. Evaluating the analytic formula for this compound option would be very tedious because 60 exercise points involving at least 120 terms of higher order integrals would have to be computed. In fact, it is usually more efficient to evaluate such formulae by

Table II

Comparison of American and European Puts Adjusted for Cash Dividends[a]
(S = \$40.00, r = 0.0488 percent annually)

X	σ	T	p (European)	P (Interpolate)	P (Numerical)
35.0	0.2	0.0833	0.01	0.0116	0.01
35.0	0.2	0.3333	0.30	0.3071	0.31
35.0	0.2	0.5833	0.65	0.6580	0.66
40.0	0.2	0.0833	1.09	1.1079	1.11
40.0	0.2	0.3333	1.98	2.0120	2.01
40.0	0.2	0.5833	2.54	2.5717	2.58
45.0	0.2	0.0833	5.33	5.4209	5.41
45.0	0.2	0.3333	5.60	5.69	5.67
45.0	0.2	0.5833	5.93	6.03	6.02
35.0	0.3	0.0833	0.11	0.1073	0.11
35.0	0.3	0.3333	0.88	0.8837	0.88
35.0	0.3	0.5833	1.53	1.5454	1.55
40.0	0.3	0.0833	1.55	1.5590	1.56
40.0	0.3	0.3333	2.88	2.9072	2.91
40.0	0.3	0.5833	3.71	3.7435	3.74
45.0	0.3	0.0833	5.43	5.4996	5.50
45.0	0.3	0.3333	6.24	6.3089	6.29
45.0	0.3	0.5833	6.92	6.9977	6.99
35.0	0.4	0.0833	0.30	0.3049	0.31
35.0	0.4	0.3333	1.57	1.5798	1.58
35.0	0.4	0.5833	2.51	2.5277	2.52
40.0	0.4	0.0833	2.00	2.0120	2.01
40.0	0.4	0.3333	3.78	3.8033	3.81
40.0	0.4	0.5833	4.88	4.9116	4.92
45.0	0.4	0.0833	5.65	5.7015	5.70
45.0	0.4	0.3333	7.02	7.0774	7.07
45.0	0.4	0.5833	8.02	8.0914	8.10

[a] Values in far right column are from Cox and Rubinstein [4]. Values in the next to last column are from a version of Equation (5). The European option values are from the Black-Scholes equation adjusted by reducing the stock price by the scheduled dividend. A 50-cent quarterly dividend is paid in ½, 3½, and 6½ months. Thus one-, four-, and seven-month options (T = 0.0833, 0.3333, 0.5833) have one, two, and three scheduled dividend payments, respectively.

finite differences for integral dimensions greater than ten. Richardson extrapolation provides a simple alternative. By extrapolating from three or four payment dates to 60, an efficient computation results. Also, with American currency options and options on futures, since both calls and puts may be exercised early (independent of dividends), this evaluation procedure is applicable. (See Grabbe [12] or Garman and Kolhagen [6] for currency option formulae, Ramaswamy and Sundaresan [18] for options on futures formula, and Shastri and Tandon [20] for an application of this evaluation technique to American currency options.)

III. Conclusion

This paper presents an analytic solution to the American put problem. To our knowledge this is the first formula presented with its comparative statics that satisfies this partial differential equation, subject to the free boundary condition. We attribute the solution of this mathematical problem to simplifications afforded by the economic interpretation. First, the risk-free hedge allows economists to avoid the complex transformations usually required for solutions to partial differential equations. Second, compound option theory provides a straightforward method for interpreting the infinite series of interrelated probability integrals arising from the free boundary condition. A key to the solution is that each exercise decision is considered as a discrete event. Thus, the formula derived is a continuous time solution to the partial differential equation, subject to the free boundary condition applied at an infinite number of discrete instants. The formula adds to our intuition because it implies an exact duplicating portfolio for the American put, consisting of specific positions in discount bonds and stock sold short. All the comparative statics of the American put formula are presented and discussed. The solution is extended to stocks paying cash dividends.

The evaluation of the formula is a separate problem. At first blush the American put formula might be considered intractable due to the infinite series of integrals. However, we demonstrate that because the formula is exact in the limit, arbitrary accuracy can be obtained by extrapolating from a small sequence of terms to the actual solution containing an infinite series. This formula evaluation procedure leads to a polynomial expression similar to that used to evaluate the integral terms in the Black-Scholes European put option formula.

Before a formula existed that satisfied this boundary value problem, numerical procedures (such as the binomial or finite difference methods) were used for approximating the values of American puts. We demonstrate that our formula allows the use of an evaluation technique resulting in a significant reduction in the number of critical stock price computations necessary for penny accuracy and thus enhances computational efficiency. We also demonstrate that this formula evaluation technique can be used to calculate analytic hedge ratios derived by differentiating the formula. Furthermore, it allows the determination of American put values on stocks that pay cash dividends. A version of this evaluation method can be used to simplify the valuation of American currency options, options on futures, coupon bonds, and many other complex problems.

The technique might also be combined with finite differences and extended in order to evaluate two-dimensional stochastic problems.

Appendix

Let $F(h)$ be the value of the function of interest when a step size of h is used. We wish to find $F(0)$. Suppose that $F(h)$ takes the form

$$F(h) = F(0) + a_1 h^p + a_2 h^r + 0(h^s)$$

where $s > r > p$.

Then we can also write

$$F(kh) = F(0) + a_1 (kh)^p + a_2 (kh)^r + 0(h^s)$$

and

$$F(qh) = F(0) + a_1 (qh)^p + a_2 (qh)^r + 0(h^s)$$

where $q > k > 1$. Substituting for a_1 and a_2 and solving for $F(0)$ yields

$$F(0) = F(h) + \frac{A}{C}[F(h) - F(kh)] - \frac{B}{C}[F(kh) - F(qh)] \tag{A1}$$

where

$$A = q^r - q^p + k^p - k^r$$

$$B = k^r - k^p$$

$$C = q^r(k^p - 1) - q^p(k^r - 1) + k^r - k^p.$$

Using $P_1 = F(qh)$, $P_2 = F(kh)$, and $P_3 = F(h)$, we have $q = 3$ and $k = 3/2$. If we expand $F(h)$ in a Taylor series around $F(0)$ and drop terms of third order or higher, we have $p = 1$ and $r = 2$. Substitution into (16) gives

$$P = P_3 + 7/2(P_3 - P_2) - 1/2(P_2 - P_1). \tag{A2}$$

There is some error in (A2) from dropping the higher order terms.

REFERENCES

1. F. Black and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May–June 1973), 637–59.
2. M. Brennan and E. Schwartz. "The Valuation of American Put Options." *Journal of Finance* 32 (May 1977), 449–62.
3. J. Cox and S. Ross. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics* 3 (January/March 1976), 145–66.
4. ——— and M. Rubinstein. *Options Markets.* Englewood Cliffs, N.J.: Prentice-Hall, forthcoming, 1984.
5. G. Dahlquist and A. Bjorck. *Numerical Methods.* Englewood Cliffs, N.J.: Prentice-Hall, 1974.
6. M. Garman and S. Kohlhagen. "Foreign Currency Option Values." *Journal of International Money and Finance* 2 (December 1983), 231–37.
7. R. Geske. "The Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis* 12 (November 1977), 541–52.

8. ———. "The Valuation of Compound Options." *Journal of Financial Economics* 7 (March 1979), 63–81.
9. ———. "A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends." *Journal of Financial Economics* 7 (December 1979), 375–80.
10. ——— and K. Shastri. "The Early Exercise of American Puts." forthcoming, *Journal of Banking and Finance* (1985).
11. ——— and ———. "Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques." forthcoming, *Journal of Financial and Quantitative Analysis* (1985).
12. J. Grabbe. "The Pricing of Call and Put Options on Foreign Exchange." *Journal of International Money and Finance* 2 (December 1983), 239–53.
13. R. James and E. James. *Mathematical Dictionary.* 4th Edition, New York: Van Nostrand Reinhold, 1982.
14. H. E. Johnson. "Three Topics in Option Pricing." Ph.D. thesis, UCLA, 1981.
15. ———. "An Analytic Approximation for the American Put Price." *Journal of Financial and Quantitative Analysis* 18 (March 1983), 141–48.
16. R. Merton. "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring 1973), 141–83.
17. M. Parkinson. "Option Pricing: The American Put." *Journal of Business* 50 (January 1977), 21–36.
18. K. Ramaswamy and S. Sundaresan. "The Valuation of Options on Futures Contracts." Columbia University Working Paper, 1984.
19. R. Roll. "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends." *Journal of Financial Economics* 5 (November 1977), 251–58.
20. K. Shastri and K. Tandon. "The Valuation of American Options on Foreign Currency." University of Pittsburgh Working Paper, 1984.