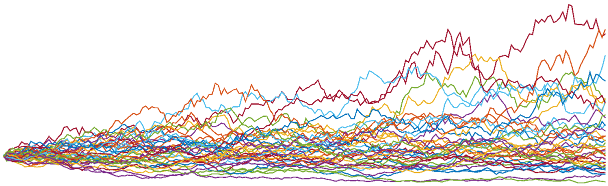

Fast Delta-Estimates for American Options by Adjoint Algorithmic Differentiation

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Outline

Motivation

- In addition to the option price, sensitivities are of particular interest
- Sensitivities play an important role in terms of hedging and risk management
- Numerical approximation is very expensive and inaccurate
- Improve the computation of sensitivities by adjoint algorithmic differentiation

Options and Greeks

"An American option gives the holder the right, but not the obligation, to trade an underlying financial asset S at a previously defined strike price K during a certain period of time until date T."[1]

- Payoff function with stock price S_t and strike price K

$$v(S_t, K) = \begin{cases} \max(K - S_t, 0) & \text{(put)} \\ \max(S_t - K, 0) & \text{(call)} \end{cases}$$

- Greeks of the option price V
  - Measurement for risks and stability

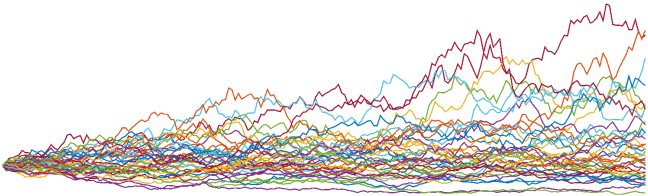
$$\Delta = \frac{\partial V}{\partial S_0} \qquad \nu = \frac{\partial V}{\partial \sigma}$$

$$\Gamma = \frac{\partial^2 V}{\partial S_0^2} \qquad \text{Vanna} = \frac{\partial^2 V}{\partial S_0 \sigma}$$

Path Generation

- Path generation of stock prices S_t at time t with Black-Scholes formula [2] for given risk-free interest rate r, volatility σ, time of maturity T, number of time steps N_T and standard normal random numbers Z

$$
\begin{aligned}
S_t &= S_{t-1} \cdot \exp\left(\left(r - 0.5\sigma^2\right)\frac{T}{N_T} + \sigma Z_t\right) \\
&= S_0 \cdot \exp\left(\left(r - 0.5\sigma^2\right)\frac{T}{N_T} t + \sigma \sum_{i=1}^{t} Z_i\right)
\end{aligned}
$$



Longstaff-Schwartz Algorithm

- Longstaff-Schwartz algorithm to estimate American option price V
- Monte-Carlo simulation with least-squares approach (LSA) [3]

for $p = 1$ to N_P **do**
    $v_p = max(K - S_{T,p}, 0)$
end for
for $t = N_T - 1$ to 1 **do**
    Identify set of in-the-money paths I
    Least squares method for all $p \in I$ to estimate the exercise boundary b
    for all $p \in I$ **do**
        if $K - S_{t,p} < b$ **then**
            $v_p = v_p \cdot \exp(-r\frac{T}{N_T})$
        else
            $v_p = K - S_{t,p}$
        end if
    end for
end for
$V = \exp(-r\frac{T}{N_T}) \cdot \frac{1}{N_P} \sum_{p=1}^{N_P} v_p$

Code Analysis

- For given exercise times $\tilde{t}_p$ the LSA computes the option price as

$$V = \frac{1}{N_P}\sum_{p=1}^{N_P}\left[(K - S_{\tilde{t}_p,p}) \cdot \exp\left(-r\frac{T}{N_T}\tilde{t}_p\right)\right]$$

$$= \frac{1}{N_P}\sum_{p=1}^{N_P}\left[K \cdot \exp\left(-r\frac{T}{N_T}\tilde{t}_p\right) - S_0 \cdot \exp\left(-0.5\sigma^2\frac{T}{N_T}\tilde{t}_p + \sigma\sum_{i=1}^{\tilde{t}_p} Z_i\right)\right]$$

- Differentiating with respect to the initial stock price S_0 yields

$$\frac{\partial V}{\partial S_0} = \frac{1}{N_P}\sum_{p=1}^{N_P}\left[-\exp\left(-0.5\sigma^2\tilde{t}_p\frac{T}{N_T} + \sigma\sum_{i=1}^{\tilde{t}_p} Z_i\right)\right]$$

- Exercise decision is not differentiable at $K - S_{t,p} = b$
- From the viewpoint of AD the option price V is independent of the boundary b and therefore the exercise times $\tilde{t}_p$ behave like constants
- AD will compute some second-order Greeks to be zero, e.g. $\Gamma = 0$

Pathwise Adjoint LSA

- Run the primal of the LSA and store exercise times $\tilde{t}_p$ for each path
- Compute the option price V with

  for $p = 1$ to N_P **do**
  $\quad S_{\tilde{t}_p,p} = S_0 \cdot \exp\left(\left(r - 0.5\sigma^2\right) \frac{T}{N_T} \tilde{t}_p + \sigma \sum_{i=1}^{t} Z_i\right)$
  $\quad v_p = (K - S_{\tilde{t}_p,p}) \cdot \exp(-r \frac{T}{N_T} \tilde{t}_p)$
  end for
  $V = \frac{1}{N_P} \sum_{p=1}^{N_P} v_p$

- Adjoint computation and path loop can be interchanged due to

$$\frac{\partial \frac{1}{N_P} \sum v_p}{\partial x} = \frac{1}{N_P} \sum \frac{\partial v_p}{\partial x}$$

- Embarrassingly parallel
- Pathwise approach yields the the same values for the Greeks as the other AD approaches

Smoothing

- Discontinuous function

$$f(x) = \begin{cases} f_1(x) & \text{for} \quad x < x_0 \\ f_2(x) & \text{else} \end{cases}$$

- smooth transition between f_1 and f_2:

$$f(x) = [1 - \sigma_s(x)]\, f_1(x) + \sigma_s(x) f_2(x)$$

- with sigmoid function

$$\sigma_s = \frac{1}{1 + \exp(-(x - x_0)/\alpha)}$$

- and transition width α

Smoothing of LSA

- Apply smoothing to exercise decision in LSA

  1: **if** $K - S_{t,p} < b$ **then**
  2: $\quad v_p = v_p \cdot \exp(-r\frac{T}{N_T})$
  3: **else**
  4: $\quad v_p = K - S_{t,p}$
  5: **end if**

- Exercise decision is replaced by

  1: $\sigma = 1/(1 + \exp(-(K - S_{t,p} - b)/\alpha))$
  2: $v_p = (1 - \sigma) \cdot (v_p \cdot \exp(-r\frac{T}{N_T})) + \sigma \cdot (K - S_{t,p})$

Setup

- 5 active inputs:
  - initial stock price $S_0 = 1.0$
  - strike price $K = 1.0$
  - time of maturity $T = 1.0$
  - volatility $\sigma = 0.2$
  - risk-free interest rate $r = 0.04$
- 1 active output:
  - option price V
- smoothing parameter $\alpha = 0.005$
- Computation of gradient and two columns of Hessian

Results: First-order Greeks

- Compare Greeks obtains with numerical (N) and algorithmic (A) differentiation with Greeks of the smoothed LSA (S)

- Analytical reference value $\Delta = -0.416$ [4]

N_P	N_T	Δ_{N}	Δ_{A}	Δ_{S}	ν_{N}	ν_{A}	ν_{S}
100000	100	-0.4186	-0.4186	-0.4150	0.3761	0.3761	0.3785
	200	454.9442	-0.4193	-0.4149	0.3763	0.3763	0.3813
	500	-0.4236	-0.4236	-0.4202	0.3762	0.3762	0.3801
	1000	-0.4266	-0.4266	-0.4196	0.3762	0.3762	0.3870
500000	100	28.5414	-0.4189	-0.4170	0.3759	0.3759	0.3786
	200	-0.4211	-0.4211	-0.4209	0.3765	0.3765	0.3795
	500	49.1352	-0.4238	-0.4196	0.3764	0.3764	0.3814
	1000	-10.2527	-0.4253	-0.4207	0.3760	0.3760	0.3833

Results: Second-order Greeks

N_P	N_T	Γ_N	Γ_A	Γ_S	$\partial\Delta/\partial\sigma_\mathrm{N}$	$\partial\Delta/\partial\sigma_\mathrm{A}$	$\partial\Delta/\partial\sigma_\mathrm{S}$
100000	100	-17777.4563	0.0000	0.5063	-148.4161	0.3761	0.4460
	200	-20023.3616	0.0000	0.7151	-1615.1344	0.3763	0.3963
	500	-37873.6509	0.0000	1.0385	-894.4365	0.3762	0.3754
	1000	-66975.9573	0.0000	1.0045	2414.4518	0.3762	0.4312
500000	100	-10547.4894	0.0000	0.7742	441.0091	0.3759	0.4075
	200	-18052.8577	0.0000	1.0584	-11.8954	0.3765	0.4306
	500	-44991.0654	0.0000	1.2196	-146.1275	0.3764	0.3343
	1000	-81752.7066	0.0000	1.1069	-246.8543	0.3760	0.4247

Results: Wall Time

- Wall time comparison of the numerical differentiation, an adjoint method and the pathwise adjoint method
- Adjoint method with an equidistant checkpointing on the time loop

		Wall time in seconds			
N_P	N_T	Pricer	Numerical Differentiation	Checkpoint Adjoint	Pathwise Adjoint
100000	100	1	39 (39.0)	18 (18.0)	1 (1.0)
	200	3	77 (25.7)	35 (11.7)	3 (1.0)
	500	8	196 (24.5)	81 (10.1)	8 (1.0)
	1000	14	391 (27.9)	172 (12.3)	15 (1.1)
500000	100	8	213 (26.6)	92 (11.5)	8 (1.0)
	200	17	505 (29.7)	189 (11.1)	18 (1.1)
	500	38	1107 (29.1)	479 (12.6)	39 (1.0)
	1000	77	2215 (28.8)	1057 (13.7)	79 (1.0)

Results: Memory Requirements

N_P	N_T	Memory requirements in gigabyte			
		Pricer	Numerical Differentiation	Checkpoint Adjoint	Pathwise Adjoint
100000	100	0.08	0.08	1.14	0.10
	200	0.16	0.16	2.14	0.18
	500	0.38	0.38	5.15	0.40
	1000	0.75	0.75	10.16	0.89
500000	100	0.40	0.40	5.59	0.41
	200	0.77	0.77	10.50	0.78
	500	1.89	1.89	25.25	1.90
	1000	3.75	3.75	49.83	3.78

Conclusion

- Pathwise approach computes the first-order sensitivities at the computational cost of a single pricing calculation
- Local non-differentiability of the exercise decision yields e.g. $\Gamma = 0$
- Approximation of second-order sensitivities by a sigmoidal smoothing of the exercise decision

Outlook

- Use another stochastic differential equation for the stock price evolution (e.g. local volatility)
- Check accuracy of the exercise boundary by using another set of random numbers
- Analysis of the smoothing with another function and of the smoothing parameters should be considered

For Further Reading I

[1] Wilmott, P. (2007). *Paul Wilmott introduces quantitative finance*. John Wiley & Sons.

[2] Black, F., and Scholes, M. (1973). *The pricing of options and corporate liabilities*. The journal of political economy, 637-654.

[3] Longstaff, F. A., and Schwartz, E. S. (2001). *Valuing American options by simulation: a simple least-squares approach*. Review of Financial studies, 14(1), 113-147.

[4] Geske, R., and Johnson, H. E. (1984). *The American put option valued analytically*. Journal of Finance, 1511-1524.