COMPUTING DELTAS OF CALLABLE LIBOR EXOTICS IN FORWARD LIBOR MODELS

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Abstract. Callable Libor exotics is a class of single-currency interest-rate contracts that are Bermuda-style exercisable into underlying contracts consisting of fixed-rate, floating-rate and option legs. The most common callable Libor exotic is a Bermuda swaption. Other, more complicated examples include callable inverse floaters and callable range accruals. Because of their non-trivial dependence on the volatility structure of interest rates, these instruments need a flexible multi-factor model, such as a forward Libor model, for pricing. Only Monte-Carlo based methods are generally available for such models. Being able to obtain risk sensitivities from a model is a prerequisite for its successful application to a given class of products. Computing risk sensitivities in a Monte-Carlo simulation is a difficult task. Monte-Carlo valuation is generally quite slow and noisy. Additionally, numerical noise is amplified when computing risk sensitivities by a "bump-and-revalue" method. Various methods have been proposed to improve accuracy and speed of risk sensitivity calculations in Monte-Carlo for European-type options. Building on previous work in this area, most notably Glasserman and Zhao's [GZ99], we propose a novel extension of some of these methods to the problem of computing deltas of Bermuda-style callable Libor exotics. The method we develop is based on a representation of deltas of a callable Libor exotic as functionals of the optimal exercise time and deltas of the underlying coupons. This representation is obtained by deriving a recursion for the deltas of "nested" Bermuda-style options.

The proposed method saves computational effort by computing all deltas at once in the same simulation in which the value is computed. In addition, it produces significantly more stable and less noisy deltas using only a fraction of the number of paths required by the standard "bump and revalue" approach.

1. Introduction

Interest rate derivatives have become increasingly more complex over the years. A class of *callable Libor exotics* has emerged as the most challenging to price and risk-manage among all pure (i.e. non-hybrid) interest rate derivatives. This class of derivative contracts is loosely defined by the provision that the holder has a Bermuda-style (i.e. multiple-exercise) option to exercise into various underlying interest rate instruments. The instruments into which one can exercise can be, for instance, interest rate swaps (for Bermuda swaptions), interest rate caps (for captions, callable capped floaters, callable inverse floaters), collections of digital call and put options on Libor rates (callable range accruals), collections of options on spreads between various CMS rates (callable CMS coupon diffs), and so on.

The market in callable Libor exotics has developed in response to an increasing sophistication of corporate and institutional clients in tailoring their interest rate exposures to specific views and objectives. Also, an appetite for above-market current yield, especially in the

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falling interest rate environment with few attractive investment alternatives, has prompted clients to sell increasingly more complex (and more valuable) options to interest rate dealers. In the most general form, a callable Libor exotic comes in a form of a swap where an interest rate dealer pays a high coupon to a customer for a number of years (”lockout period”) and then a coupon linked to Libor or CMS rates in a way that is tailored to client’s expectations about future dynamics of interest rates. To compensate for a high up-front coupon, a client sells an option to cancel this swap on each coupon date, after the lockout period, back to the dealer.

From a modeling prospective, callable Libor exotics are extremely difficult to handle. Through non-linear payoffs in underlying Libor or CMS rates, a callable Libor exotic depends on volatilities of all those rates. Being a Bermuda-style option (essentially a “best of” type of option) it depends on correlations between instruments into which it can be exercised (instruments themselves being collections of interest rate options, of course). Typically, the dependence on the volatility structure of interest rates is mind-bogglingly complex. To top it off, all options involved typically have strikes that are deep in or out of the money, so volatility skews and smiles cannot be ignored.

Let us give an example. A caption, a Bermuda style options to enter into an interest rate cap, clearly depends on volatilities of all Libor rates (an interest rate cap is a collection of options on Libor rates). Because of the Bermuda-style exercise feature it also depends on volatilities of swap rates. (This is most easily seen in the limit when strikes of the underlying caps are set to zero). Correlations between various swap and Libor rates should also be taken into account as they play an important role in assigning relative values to the choices of exercising the contract or holding it to the next date. Forward volatilities, i.e. volatilities of various rates observed at future times, also determine how valuable different exercise choices are on future exercise dates and thus also enter the equation.

It is well-known (see for example [Reb98] for an excellent discussion) that a simple, low dimensional interest rate model is limited in the amount of volatility/correlation information it can faithfully reproduce. Typically for such a model, there are not enough “moving parts” to match required volatility/correlation information even for relatively simple callable Libor exotics (assuming one wanted to do so). Of course one may try to identify the “most important” set of volatilities/correlations to match and calibrate a low-dimensional model to those, hoping that other “less important” volatilities/correlations are not that far off. Problems that such an approach brings are too numerous to list (and well known to many who pursued it). In our opinion, for instruments as complex as callable Libor exotics, a multi-factor model, properly calibrated to all, or a large subset of, market volatility information, is a must.

Forward Libor, or BGM-type, models are among the most popular and well-studied models that are flexible enough to handle callable Libor exotics. Flexibility comes at a price, however – typically only a Monte-Carlo method is available for valuation. Recent advances in pricing American-style options in Monte-Carlo (see [LS98], [And99]) make the application of BGM-type models to callable Libor exotics possible and practical.

Being able to price instruments is only one part of the puzzle. Being able to compute risk sensitivities to various market inputs is a prerequisite for using any model in practice. It is well-known that computing risk sensitivities in a Monte-Carlo simulation is a complicated task. A Monte-Carlo simulation is typically quite slow. Each risk sensitivity calculation usually requires a complete revaluation of the instrument with one of the inputs slightly changed. A number of risk sensitivities that needs to be computed for an interest rate derivative is usually quite large. As a bare minimum, one needs to compute first and second order derivatives (deltas and gammas) to all rates comprising the interest rate curve, and first-order derivatives to volatilities (vegas) of all market instruments involved. Moreover, Monte-Carlo errors (numerical noise) are much worse for computing sensitivities than for computing values, requiring even larger number of simulated paths.

These considerations justify keen interest in various methods used to speed up risk calculations, or make them more accurate. A large body of work exists addressing these issues (see "Literature Review"). Of particular relevance is work by Glasserman and Zhao [GZ99] that proposes a number of strategies for computing risk sensitivities via Monte-Carlo in the context of forward Libor models. Their methods and results are powerful and important, and we use them throughout. Glasserman and Zhao develop their methods and present their results for *European-style* options. Handling *Bermuda-style* optionality in callable Libor exotics requires more effort. This is the step we make in this paper.

In this paper we explore a particular structure of Bermuda-style exercisable options to derive formulas for computing deltas in the same simulation as the value, at little additional cost. Our main idea, and the main theoretical contribution, is based on deriving a recursion for various deltas from a recursion on values of "nested" Bermuda-style options. We presented this idea first in [Pit03] where it was applied to vanilla Bermuda swaptions in the context of PDE-based models. In this paper we apply and specialize the idea for Monte-Carlo based forward Libor models. In addition we consider a much more general class of instruments. Our main result, Theorem 10.3, provides a particularly elegant representation for deltas of a callable Libor exotic as an expected value of a sum of deltas of coupons one receives under the optimal exercise.

Tests of the proposed method demonstrate that it is far superior to the standard "bump and revalue" approach. Not only do we get all deltas at a significantly reduced computational cost by computing them in the same simulation as the value, they turn out to be a lot more stable as well, requiring even fewer Monte-Carlo paths for the same level of accuracy. In the test cases presented, comparable accuracy is achieved with only 1/32 to 1/64 of number of paths required by the standard approach.

Our main focus is on pathwise deltas as defined by Glasserman and Zhao in [GZ99]. Pathwise deltas are known to be significantly less noisy and more stable than the other kind explored in [GZ99], the likelihood ratio deltas. Moreover, no extra work is required to obtain likelihood ratio deltas for callable Libor exotics beyond what was done by Glasserman and Zhao.

The paper is organized as follows. First, we put our contribution in the context of available research. Then we specify notations we use, and specify the forward Libor model we use throughout. In Section 5, we review the definitions and available results on pathwise deltas and how they can be simulated in the context of forward Libor models. Section 6 is devoted to defining various types of callable Libor exotics. The main recursion for computing values of callable Libor exotics, and the starting point for our derivation for the deltas, is presented in Section 7. In Section 8 we review the valuation algorithm for callable Libor exotics in a Monte-Carlo based model. The main recursion for deltas of callable Libor exotics, as well as the expression of these deltas in terms of deltas of the coupons and the optimal exercise time, is derived in Section 10. A discussion on computing pathwise deltas of the coupons is presented in Section 11, where some useful approximations are also proposed. Test results are presented next. Finally, the Appendix contains a number of results used and referenced throughout the paper.

2. Literature review

Our paper is concerned with computing derivatives of Bermuda-style interest rate instruments in a forward Libor model. Forward Libor models (also known as Libor market models or BGM models) are extensively covered in the literature. We follow the exposition of [AA98] where skew extensions of forward Libor models were first introduced. The reader can consult that paper for the review of literature on forward Libor models; we do not reproduce it here.

The problem of pricing Bermuda-style and American-style derivatives in Monte-Carlo simulation has seen some recent and impressive advances. The problem of obtaining lower bounds on prices of such derivatives is addressed in [LS98], [And99] and [BG97].The much more challenging problem of obtaining upper bounds for American-style derivatives in Monte-Carlo simulation is tackled in [AB01] and [Rog01].

Computing risk sensitivities of derivatives in a Monte-Carlo simulation is a challenging problem. A number of approaches has been proposed in the literature. For a good review see [BBG97]. Methods based on the change of measure idea, either via Malliavin calculus approach or via likelihood ratio approach, are presented in [FLLT99], [Ber99] and [BG96]. Differentiating payoffs along paths, another approach to the same problem, is well explained in [GZ99].

The paper by Glasserman and Zhao [GZ99] comes the closest to ours in scope. They discuss various methods of computing Greeks of interest rate derivatives valued by Monte-Carlo simulation in forward Libor models. Both measure-based and path-based methods are considered. The latter are found better performing although more limited in their applicability. They only consider applications of those methods to European-style derivatives.

Our paper bridges the gap between the body of literature concerning with computing risk sensitivities in Monte-Carlo simulation (most notably [GZ99]) and the body of literature devoted to valuing American and Bermuda-style options in Monte Carlo. Our contributions can be seen as extending the results of Glasserman and Zhao from European to Bermuda-style derivatives, a nontrivial leap. Alternatively, we show that the algorithms designed for valuing Bermuda-style derivatives in Monte-Carlo can be "leveraged" to provide building blocks needed for computing deltas in the same simulation.

3. Notations

Actual/Actual day counting convention is assumed for simplicity, i.e. all day counting fractions are equal to the period length as a fraction of a calendar year. A zero coupon bond paying one dollar at time T, as observed at time t, $t \leq T$, is denoted by $P(t,T)$. A forward Libor rate for the period $[T,M]$, as observed at time t, is defined by

$$F(t,T,M) \triangleq \frac{1-P(T,M)}{(M-T)\,P(T,M)}.$$

A probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is chosen, together with a sigma-algebra filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$.

For any $\mathcal{F}_T$-measurable payoff $X = X(\omega)$, $\omega \in \Omega$, we define its value at time t by

$$\pi_t(X).$$

4. Forward Libor model

We start with a review of forward Libor models. For most of our results the exact specification of the model is of no importance. For concreteness we use a skew-extended forward Libor model as presented in [AA98]. The original lognormal BGM model is a special case.

Let

$$\begin{aligned} 0 &= t_0 < t_1 < \cdots < t_M, \\ \tau_n &= t_{n+1} - t_n, \end{aligned}$$

be a tenor structure, i.e. a collection of approximately equally spaced (three or six months is common) maturities.

We define the n-th forward Libor rate $F_n(t)$ (n-th "primary" Libor rate) by the expression

$$F_n(t) \triangleq F(t, t_n, t_{n+1}) = \frac{P(t, t_n) - P(t, t_{n+1})}{\tau_n P(t, t_{n+1})\,.}, \quad 0 \le n < M.$$

We impose the following dynamics on each of the forward Libor rates,

$$dF_n(t) = \lambda_n(t)\,\phi(F_n(t))\, dW^{T_{n+1}}(t)\,, \quad n = 1, \ldots, M-1, \quad t \in [0, t_n]\,, \tag{4.1}$$

Here $\lambda_n(\cdot)$ is a deterministic function of time $\mathbb{R}_+ \to \mathbb{R}^n$, the skew function $\phi(\cdot)$ is common to all Libor rates and satisfies certain regularity and growth conditions (see [AA98, Theorem 1]), and $dW^{T_{n+1}}(\cdot)$ is a one-dimensional Brownian motion under the T_{n+1}-forward measure. (We consider a one-dimensional case for brevity only. All our results are applicable in the multi-dimensional case as well.) It is known that this specification leads to a valid HJM model.

For convenience we define

$$F_n(t) = F_n(t_n)\,, \quad t > t_n.$$

A special numeraire is usually chosen. We define a discrete money-market numeraire B_t by

$$\begin{aligned} B_{t_0} &= 1, \\ B_{t_{n+1}} &= B_{t_n} \times (1 + \tau_n F_n(t_n))\,, \quad 1 \le n < M, \\ B_t &= P(t, t_{n+1})\, B_{t_{n+1}}, \quad t \in [t_n, t_{n+1}]\,. \end{aligned}$$

The dynamics of all forward Libor rates under the same measure, the measure associated with B_t, are given by

$$\begin{aligned} dF_n(t) &= \lambda_n(t)\,\phi(F_n(t)) \sum_{j=1}^{n} 1_{\{t<T_j\}} \frac{\tau_j \phi(F_j(t))}{1 + \tau_j F_j(t)} \lambda_j(t)\, dt + \lambda_n(t)\,\phi(F_n(t))\, dW(t)\,, \\ & n = 1, \ldots, M-1. \end{aligned} \tag{4.2}$$

where dW is a Brownian motion under this measure.

The measure $\mathbf{P}$ is assumed to be the probability measure associated with the numeraire B_t. The filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ is assumed to be generated by the Brownian motion W_t (and properly augmented with the zero-probability events of $\mathbf{P}$).

If a payoff X is $\mathcal{F}_T$-measurable then its value at time t, $t \le T$, is computed by

$$\pi_t(X) = B_t \mathbf{E}_t\left(B_T^{-1} X\right).$$

The vector-valued process

$$\bar{F}(t) = (F_0(t), F_1(t), \dots, F_{M-1}(t))$$

is Markov. For each t, $t \geq 0$, we define another sigma-algebra of events generated by the Libor rates process *at that time* t,

$$\mathcal{G}_t \triangleq \sigma\left(\bar{F}(t)\right).$$

The model defined by (4.2) is of HJM type, as explained in [MR97]. In particular, in this model zero coupon discount bonds satisfy the following SDE under the risk-neutral measure,

$$dP(t,T) = r(t)\, P(t,T)\; dt + \Sigma(t,T)\, P(t,T)\; dW(t),$$

for some bond volatility process $\{\Sigma(\cdot,T), 0 \leq T < \infty\}$. It is also known that we can choose the bond volatility process in such a way that

$$\Sigma(t, t_n) \equiv 0 \text{ for } t \in [t_{n-1}, t_n]$$

for any n, $1 \leq n \leq M$. We adopt this specification. In particular it implies

$$P(t, t_n) = P(t_{n-1}, t, t_n) \text{ for } t \in [t_{n-1}, t_n], \tag{4.3}$$

for any n, $1 \leq n \leq M$.

5. Pathwise deltas

A delta of an interest rate instrument is defined as its first-order derivative with respect to a particular shock of the interest rate curve. An interest rate instrument with a payment at a final date T_N is exposed to changes in all points of the interest rate curve up to the final payment date T_N. Deltas are usually *bucketed* and reported via the following procedure. A time line between zero and the final payment date T_N is split into time intervals (buckets) of common length, for example three months. Forward interest rates that correspond to all buckets are shocked in turn. A shocked interest rate curve is fed into the model of choice, and the interest rate instrument in question is revalued. The sensitivity of the instrument value to this shock is then computed by subtracting a base value of the instrument from its shocked value, and normalizing the difference by the size of the shock.

In our setting the time line is naturally split into intervals of equal length by the tenor structure $\{t_m\}_{m=0}^{M}$ specified by the forward Libor model. Therefore, we define the m-th bucketed delta as a change in value of an instrument with respect to a shock of the rate $F_m(0)$, $m = 0, \dots, M-1$. For any random payoff X we define the m-th delta by

$$\Delta_m X \triangleq \frac{\partial X}{\partial F_m(0)}.$$

Glasserman and Zhao discuss ways to differentiate European-style payoffs. Suppose $X = V\left(\bar{F}(T)\right)$ for a (reasonably smooth) deterministic function $V(\mathbf{x})$, $\mathbf{x} = (x_0, \dots, x_{M-1})$. Then

$$\Delta_m V\left(\bar{F}(T)\right) = \sum_{i=0}^{M-1} \frac{\partial V\left(\bar{F}(T)\right)}{\partial x_i} \cdot \Delta_m F_i(T),$$

and

$$\begin{aligned}\pi_0(X) &= \mathbf{E}_0\left(B_T^{-1}X\right),\\ \Delta_m\pi_0(X) &= \mathbf{E}_0\left(\Delta_m\left(B_T^{-1}V\left(\bar{F}(T)\right)\right)\right)\\ &= \mathbf{E}_0\left(\Delta_m\left(B_T^{-1}\right)V\left(\bar{F}(T)\right)\right)\\ &\quad +\mathbf{E}_0\left(B_T^{-1}\Delta_m V\left(\bar{F}(T)\right)\right).\end{aligned}$$

It is clear from these formulas that all we need to compute pathwise deltas is to be able to simulate $\Delta_m F_i(T)$, the deltas of Libor rates with respect to their initial values, along with the values of the Libor rates themselves. By differentiating (4.2) through we obtain the following system of SDE's for Libor rates and their deltas. If we denote the drift for the n-th Libor rate, as a function of the vector of Libor rates $\bar{F}(t)$, by $\mu_n(t,\mathbf{x})$, $\mathbf{x}=(x_0,\ldots,x_{M-1})$, then for both the Libor rates and their derivatives ($i=1,\ldots,M-1$, $m=0,\ldots,M-1$)

$$dF_i(t) = \mu_i\left(t,\bar{F}(t)\right)\,dt + \lambda_i(t)\,\phi\left(F_i(t)\right)\,dW(t), \tag{5.1}$$

$$d\left(\Delta_m F_i(t)\right) = \sum\frac{\partial\mu_i\left(t,\bar{F}(t)\right)}{\partial x_j}\cdot\Delta_m F_j(t)\,dt + \lambda_i(t)\cdot\phi'\left(F_i(t)\right)\cdot\Delta_m F_i(t)\,dW(t),$$

with the initial conditions for deltas given by

$$\Delta_m F_j(0) = 1_{\{m=j\}}.$$

Glasserman and Zhao prove that this system of SDE's admits a solution in the lognormal case $\phi(x)=x$, see [GZ99, Section 7]. Extending the result to other "well-behaved" local functions $\phi(\cdot)$ is trivial and we do not dwell on such technicalities here. Glasserman and Zhao also propose to use the following simplified system of SDE's for simulating values and pathwise deltas of forward Libor rates,

$$dF_i(t) = \mu_i\left(t,\bar{F}(t)\right)\,dt + \lambda_i(t)\,\phi\left(F_i(t)\right)\,dW(t), \tag{5.2}$$

$$d\left(\Delta_m F_i(t)\right) = \sum\frac{\partial\mu_i\left(t,\bar{F}(0)\right)}{\partial x_j}\cdot\Delta_m F_j(0)\,dt + \lambda_i(t)\cdot\phi'\left(F_i(t)\right)\cdot\Delta_m F_i(t)\,dW(t).$$

Note that the simplification is based on evaluating drifts μ_i for the forward values $\bar{F}(0)$, instead of the actual values $\bar{F}(t)$ at time t (just for the deltas, not for the actual Libor rates). They show that the loss of accuracy in this approximation is very small, but speed gains are substantial (observations supported by our own testing). This is so because the drift in the equation for deltas of forward Libor rates can be precomputed before the simulation.

6. Callable Libor exotics

In this section we define a universe of instruments we will be dealing with. We will simplify some of the definitions for brevity.

6.1. **The underlying instrument.** First we specify the underlying instrument for the Bermuda-style option. The underlying instrument is a stream of payments (sometimes called coupons in what follows) $\{X_i\}_{i=1}^{N-1}$. Each X_i is $\mathcal{G}_{T_i}$-measurable (in financial parlance, T_i is a *fixing date* for X_i). The coupon X_i pays at time T_{i+1} (so T_{i+1} is a *payment date* for X_i).

The fact that X_i is $\mathcal{G}_{T_i}$-measurable means that the value of the payment is completely determined by the interest rate curve at time T_i.

Note that the contract-specific tenor structure $\{T_i\}_{i=1}^{N}$ is potentially different from the model's tenor structure (defined in the previous section) $\{t_i\}_{i=0}^{M}$.

A callable Libor exotic is a Bermuda-style option to enter the underlying instrument on any of the dates $\{T_i\}_{i=1}^{N-1}$. If the option is exercised at time T_n, then the option goes away and the holder receives all coupons X_i with $i \geq n$ (i.e. all payments with fixing dates on or after the exercise date). Note that even though we say the holder receives all payments after a certain date, some of the payments can be negative, which means he has to pay those amounts to the counterparty.

We denote by $E_n(t)$ the n-th exercise value, i.e. the value of all payments one enters into if the callable Libor exotic is exercised at time T_n. Clearly

$$E_n(t) = B_t \sum_{i=n}^{N-1} \mathbf{E}_t \left(B_{T_{i+1}}^{-1} X_i \right).$$

For completeness we set

$$E_N(t) \equiv 0.$$

If a callable Libor exotic is exercised on T_n, the holder receives the remaining part of the underlying $E_n(T_n)$.

6.2. **The callable structure.** For future considerations it is important to define a whole family of "nested" callable contracts. By $H_n(t)$ we denote the value, at time t, of a callable Libor exotic that has only the dates $\{T_{n+1}, \ldots, T_{N-1}\}$ as exercise opportunities. In particular, $H_0(0)$ is the value of the callable contract we are interested in at time zero. Necessarily

$$H_0(t) \geq H_1(t) \geq \cdots \geq H_{N-2}(t).$$

We will speak of a "Callable H_n" as a shorthand for "the callable Libor exotic whose value at time t is equal to $H_n(t)$.".

6.3. **Examples.** Here we present a few examples of callable Libor exotics. They differ by the type of payments X_i the underlying instrument is comprised of. We denote

$$\delta_i = T_{i+1} - T_i.$$

As will be clear from the examples, underlying instruments for most callable Libor exotics can be described as streams of European style options on some reference rates (either Libor or CMS).

6.3.1. *A Bermuda swaption.* A simplest example is a Bermuda swaption. The underlying instrument is a plain vanilla fixed-for-floating swap. In particular, each coupon is of the form

$$X_i = \delta_i \times (F(T_i, T_i, T_{i+1}) - c),$$

an exchange of a simple payment based on a floating rate for a one based on a fixed rate c.

6.3.2. *A callable capped floater.* In a callable capped floater, the underlying instrument is an exchange of a floating rate capped from above for a regular floating rate (with a spread). If the cap is c and the spread is s, the i-th coupon X_i is given by

$$X_i = \delta_i \times (\min[F(T_i, T_i, T_{i+1}), c] - [F(T_i, T_i, T_{i+1}) + s]).$$

6.3.3. *A callable inverse floater.* In a callable inverse floater, the underlying instrument is an exchange of a payment based on an inverse of a floating rate (capped and floored) for a payment based on a regular floating rate (with a spread). If k is the strike, f is the floor and c is the cap of the inverse floating payment and the spread is s, the i-th coupon X_i is given by

$$X_i = \delta_i \times \left(\min\left[\max\left[c - F\left(T_i, T_i, T_{i+1}\right), f\right], c\right] - \left[F\left(T_i, T_i, T_{i+1}\right) + s\right]\right).$$

6.3.4. *A callable range accrual.* In a callable range accrual, a payment is based on a number of days that a reference rate (most often a Libor rate) is within a certain range. While the range observations are typically performed daily, for notational simplicity we assume that there is only one range observation on the fixing date. The underlying instrument for a callable range accrual consists of an exchange of payments based on range observations for the payments based on a floating rate. In particular we have,

$$X_i = \delta_i \times \left(c \cdot 1_{\{F(T_i,T_i,T_{i+1}) \in [l,b]\}} - \left[F\left(T_i, T_i, T_{i+1}\right) + s\right]\right).$$

Here c is the fixed rate for a range accrual payment, l is the lower range bound, b is the upper range bound, and s is the spread on the floating rate.

6.3.5. *A callable capped CMS floater, a callable inverse CMS floater, a callable CMS range accrual.* These are variations of the contracts discussed above except the payoffs are based on swap rates and not on Libor rates. For example, if $S_i(t)$ is the forward swap rate to which the i-th payoff is linked, then a callable capped CMS floater will have payments of the form

$$X_i = \delta_i \times \left(\min\left[S_i\left(T_i\right), c\right] - \left[F\left(T_i, T_i, T_{i+1}\right) + s\right]\right).$$

6.3.6. *A callable CMS spread.* The underlying instrument for this contract consists of payments linked to a spread between two different forward swap rates. If $S_{i,1}(t)$ is one such rate (for example a 10 year CMS rate) and $S_{i,2}(t)$ is another such rate (for example a 2 year CMS rate), then the coupon X_i is given by

$$X_i = \delta_i \times \left(\max\left[\min\left[S_{i,1}\left(T_i\right) - S_{i,2}\left(T_i\right), c\right], f\right] - \left[F\left(T_i, T_i, T_{i+1}\right) + s\right]\right).$$

Here c and f are a cap and a floor on the spread between two CMS rates $S_{i,1}(T_i) - S_{i,2}(T_i)$, and s is a spread on the floating rate.

7. Recursion for callable Libor exotics

If a callable contract H_0 has not been exercised up to and including time T_n, ("still alive at time T_n") then it is worth exactly the same as the callable contract H_n (hence 'H' for "hold value"). If the callable contract is exercised at time T_n its value is equal to $E_n(T_n)$ (hence 'E' for "exercise value"). Assuming optimal exercise, the value of the callable Libor exotic H_0 at time T_n is then the maximum of the two,

$$\max\left\{H_n\left(T_n\right), E_n\left(T_n\right)\right\}.$$

The value of this payoff at time T_{n-1} is then

$$\pi_{T_{n-1}} \max\left\{H_n\left(T_n\right), E_n\left(T_n\right)\right\}.$$

Clearly this is the value of the Bermuda swaption that can only be exercised at times T_n and beyond, i.e. of the Bermuda swaption H_{n-1}. These considerations define a recursion

$$H_{n-1}(T_{n-1}) = \pi_{T_{n-1}} \max\{H_n(T_n), E_n(T_n)\}, \quad n = N-1, \ldots, 1, \qquad (7.1)$$
$$H_{N-1} \equiv 0.$$

The recursion starts at the final time $n = N-1$ and progresses backward in time. For $n = 1$ we obtain the value $H_0(0)$, the value of the callable that we are after.

This is of course nothing more than a well-known algorithm for pricing American-style options in a backward induction. This recursion, however, is our starting point in deriving an efficient algorithm for computing Greeks.

Let us define the exercise region at time T_n by R_n, $R_n \subset \Omega$,

$$R_n = \{\omega \in \Omega : H_n(T_n, \omega) \leq E_n(T_n, \omega)\}, \quad 1 \leq n \leq N-1. \qquad (7.2)$$

Let $\eta = \eta(\omega)$ be the index of the first time that the exercise region is hit (or N if it is never hit),

$$\eta(\omega) = \min\{n \geq 1 : \omega \in R_n\} \wedge N.$$

The callable contract value can be re-written as

$$\begin{aligned} H_0(0) &= \mathbf{E}_0\left(B_{T_\eta}^{-1} E_\eta(T_\eta)\right) \\ &= \mathbf{E}_0\left(\sum_{n=\eta}^{N-1} B_{T_{n+1}}^{-1} X_n\right). \end{aligned}$$

8. Valuing callable Libor exotics in a forward Libor model

The problem of pricing American-style options in Monte-Carlo has been considered in [LS98] and [And99]. In the latter, an algorithm for pricing Bermuda swaptions in a forward Libor model was explicitly presented. Extending both algorithms to price callable Libor exotics is a trivial exercise (in theory, not in practice). In this paper we adopt a framework of [LS98]; everything should work with the scheme proposed in [And99] just as well.

We quickly review the Longstaff-Schwartz (LS) valuation scheme, adapting it to our notations and setup.

Suppose we have an estimate of the exercise regions $\tilde{R}_n$, $n = 1, \ldots, N-1$. Then we can define an estimate of the optimal exercise time index

$$\tilde{\eta}(\omega) = \min\left\{n \geq 1 : \omega \in \tilde{R}_n\right\} \wedge N.$$

Then, a lower bound on the value of a callable contract can be computed in a standard Monte-Carlo via the formula

$$\begin{aligned} H_0(0) &\geq \tilde{H}_0(0), \\ \tilde{H}_0(0) &= \mathbf{E}_0\left(\sum_{n=\tilde{\eta}}^{N-1} B_{T_{n+1}}^{-1} X_n\right). \end{aligned} \qquad (8.1)$$

The closer the estimated exercise region $\tilde{R}_n$ to the actual one, the tighter the lower bound on the value will be.

The LS algorithm (as well as the Andersen algorithm from [And99]) provides a way to estimate the exercise region from a collection of pre-simulated paths.

For each n, $1 \leq n \leq N-1$, we choose a p-dimensional $\mathcal{G}_T$-measurable "explanatory" random vector

$$\bar{V}(T_n) = \{V_m(T_n)\}_{m=1}^{p} = \{V_m(T_n,\omega)\}_{m=1}^{p}.$$

Also, for each n, $1 \leq n \leq N-1$, we select two parametric families of $\mathbb{R}$-valued functions $f_n(v;\alpha)$ and $g_n(v;\beta)$, $v \in \mathbb{R}^p$, $\alpha, \beta \in \mathcal{A} \subset \mathbb{R}^q$, $q \geq 1$. Without loss of generality we assume that

$$\begin{aligned} f_n(x;0) &\equiv 0, \\ g_n(x;0) &\equiv 0. \end{aligned}$$

We choose special values of the parameters α and β, denoted by $\hat{\alpha}_n$ and $\hat{\beta}_n$, such that the function f_n is a good approximation for the hold value at time T_n as a function of the explanatory vector $\bar{V}(T_n)$,

$$H_n(T_n,\omega) \approx f_n\left(\bar{V}(T_n,\omega), \hat{\alpha}_n\right)$$

and the function g_n is a good approximation for the exercise value at time T_n as a function of the explanatory vector $\bar{V}(T_n)$,

$$E_n(T_n,\omega) \approx g_n\left(\bar{V}(T_n,\omega), \hat{\beta}_n\right).$$

In particular, the Longstaff-Schwartz estimate of R_n will be of the form

$$\tilde{R}_n = \left\{\omega \in \Omega : f_n\left(\bar{V}(T_n,\omega), \hat{\alpha}_n\right) \leq g_n\left(\bar{V}(T_n,\omega), \hat{\beta}_n\right)\right\}, \quad 1 \leq n \leq N-1. \tag{8.2}$$

This is similar to (7.2) except that the real hold and exercise values H_n and E_n are replaced by their proxies $f_n\left(\bar{V}(T_n)\right)$ and $g_n\left(\bar{V}(T_n)\right)$.

Let us describe the algorithm for choosing the values of parameters $\hat{\alpha}_n$, $\hat{\beta}_n$, $n = 1,\ldots,N-1$, used in (8.2). To get the best possible estimate of the exercise region $\tilde{R}_n$ for each n, we need to approximate the hold value $H_n(T_n)$ as close as possible with one of the functions from the family $f_n\left(\bar{V}(T_n,\omega), \alpha_n\right)$, and we need to approximate the exercise value $E_n(T_n)$ as close as possible with one of the functions from the family $g_n\left(\bar{V}(T_n,\omega), \beta_n\right)$. We use this is an optimality condition to find $\hat{\alpha}_n$, $\hat{\beta}_n$. We optimize the choice of α_n and β_n over a set of Monte-Carlo paths pre-simulated for that purpose.

Let ω_k, $k = 1,\ldots,K$, be a collection of Monte-Carlo simulated paths. For any random variable X, we denote its k-th simulated value by $X(\omega_k)$. We choose the optimal fit value $\hat{\beta}_n$ from the condition (a non-linear regression of the n-th exercise value on $\left\{g_n\left(\bar{V}(T_n,\omega_k);\beta\right)\right\}_{k=1}^{K}$),

$$\hat{\beta}_n = \arg\min_{\beta} \sum_{k=1}^{K} \left(B_{T_n}(\omega_k) \sum_{i=n}^{N} B_{T_i}^{-1}(\omega_k) X_i(\omega_k) - g_n\left(\bar{V}(T_n,\omega_k);\beta\right)\right)^2 \tag{8.3}$$

for $n = 1,\ldots,N-1$.

The optimal fit variables $\hat{\alpha}_n$, $n = 1,\ldots,N-1$ for the hold value are obtained in backward induction. We set

$$\alpha_{N-1} = 0$$

(so that $f_N \equiv 0$ which is consistent with the fact that $H_{N-1} \equiv 0$ by definition).Then, for each n, having computed $\hat{\alpha}_{n+1}$, $\hat{\beta}_{n+1}$ on the previous step, we obtain $\hat{\alpha}_n$ from

(8.4)

$$\hat{\alpha}_n = \arg\min_{\alpha} \sum_{k=1}^{K} \Big(B_{T_n}(\omega_k) B^{-1}_{T_{n+1}}(\omega_k) \max\Big\{ f_{n+1}\left(\bar{V}(T_{n+1},\omega_k), \hat{\alpha}_{n+1}\right), g_{n+1}\left(\bar{V}(T_{n+1},\omega_k), \hat{\beta}_{n+1}\right) \Big\}$$
$$- f_n\left(\bar{V}(T_n,\omega_k);\beta\right)\Big)^2.$$

In practice the parametric families $f_n(\cdot,\alpha)$ and $g_n(\cdot,\beta)$ are usually chosen to depend linearly on parameters α and β. This makes solving for $\hat{\alpha}_n$ and $\hat{\beta}_n$ from (8.3) and (8.4) as simple as running a linear regression.

The output of the Longstaff-Schwartz pre-simulation step is an estimate of a set of exercise regions $\left\{\tilde{R}_n\right\}_{n=1}^{N-1}$ and an estimate of the optimal exercise time index $\tilde{\eta}$. These are used in obtaining the value (a lower bound) of the contract in Monte-Carlo simulation via (8.1). They will also be crucial for computing deltas, as will be clear shortly.

9. Likelihood ratio deltas for callable Libor exotics

Likelihood ratio deltas (see [GZ99] for definition) were found to be inferior to pathwise deltas in [GZ99]. However, sometimes one has no choice but to use it, in particular if coupon payoffs are not smooth enough. Digital payoffs, for example, can only be handled with likelihood deltas.

Computing likelihood deltas of callable Libor exotics is no more difficult that computing likelihood deltas of non-callable Libor contracts. This can be seen from the following representation of the value of a callable Libor exotic,

$$\tilde{H}_0(0) = \mathbf{E}_0\left(\sum_{n=\tilde{\eta}}^{N-1} B^{-1}_{T_{n+1}} X_n\right).$$

Here $\tilde{\eta}$ is our estimate of an optimal exercise time index obtained during pre-simulation, as explained in Section 8. The optimal exercise time index $\tilde{\eta}$ is defined in terms of exercise regions which in turn are defined in terms of values of market instruments (the explanatory vector $\bar{V}(\cdot)$) which are functions of Libor rates. The money market account $B_\cdot$ and the payoffs X_n, $n = 1,\ldots,N-1$ are functions of Libor rates as well. Therefore, we can write the callable Libor exotic value as a function of primary Libor rates observed at various times

$$H_0(0) = f\left(\bar{F}(T_1),\ldots,\bar{F}(T_{N-1})\right).$$

This representation allows us to use the results of Glasserman and Zhao directly and write

$$\begin{aligned}
\Delta_m H_0(0) &= \mathbf{E}_0\left(W_m\left(\bar{F}(T_1),\ldots,\bar{F}(T_{N-1})\right)\right) \\
&= \mathbf{E}_0\left(W_m \sum_{n=\tilde{\eta}}^{N-1} B^{-1}_{T_{n+1}} X_n\right)
\end{aligned}$$

for weights W_m computed in [GZ99].

Pathwise deltas are treated in the next section.

10. Pathwise deltas of callable Libor exotics via optimal exercise time and deltas of coupons

Pathwise deltas are superior to likelihood ratio deltas and can be computed for almost all (but not all) important types of callable Libor exotics. While likelihood ratio deltas for callable Libor exotics followed easily from the results of Glasserman and Zhao, we are not so fortunate for pathwise deltas. In this section we derive the main representation result that allows us to write a pathwise delta of a callable Libor exotic as an expectation of a functional of an optimal exercise time.

The following result follows by "differentiating through" the expected value operator.

Proposition 10.1. *If a payoff X is $\mathcal{F}_T$-measurable then at time t, $t \leq T$, the m-th pathwise delta Δ_m can be carried under the expectation operator,*

$$\Delta_m \left(B_t^{-1} \pi_t \left(X\right)\right) = \mathbf{E}_t \left(\Delta_m \left(B_T^{-1} X\right)\right).$$

The proof is trivial and follows from the linearity of the expectation operator.

Let us recall the main recursion for the callable contract. We have, slightly rewriting (7.1), that

$$B_{T_{n-1}}^{-1} H_{n-1} \left(T_{n-1}\right) = \mathbf{E}_{T_{n-1}} B_{T_n}^{-1} \max \left\{H_n \left(T_n\right), E_n \left(T_n\right)\right\}.$$

Using Proposition 10.1 we obtain,

$$\Delta_m \left(B_{T_{n-1}}^{-1} H_{n-1} \left(T_{n-1}\right)\right) = \mathbf{E}_{T_{n-1}} \Delta_m \left(B_{T_n}^{-1} \max \left\{H_n \left(T_n\right), E_n \left(T_n\right)\right\}\right). \tag{10.1}$$

Carrying out the differentiation under the expectation operator, we obtain our first major result (see the Appendix for the proof).

Theorem 10.2. *For any n, $1 \leq n \leq N-1$, and for any m, $m = 0, \ldots, M-1$, we have that*

$$\begin{aligned}\Delta_m \left(B_{T_{n-1}}^{-1} H_{n-1} \left(T_{n-1}\right)\right) &= \mathbf{E}_{T_{n-1}} \left(1_{\{E_n(T_n) > H_n(T_n)\}} \Delta_m \left(B_{T_n}^{-1} E_n \left(T_n\right)\right)\right) \\ &\quad + \mathbf{E}_{T_{n-1}} \left(1_{\{H_n(T_n) > E_n(T_n)\}} \Delta_m \left(B_{T_n}^{-1} H_n \left(T_n\right)\right)\right).\end{aligned}$$

Theorem 10.2 provides us with a recursive relationship (in n, the exercise date index) between $\Delta_m \left(B_{T_{n-1}}^{-1} H_{n-1} \left(T_{n-1}\right)\right)$ and $\Delta_m \left(B_{T_n}^{-1} H_n \left(T_n\right)\right)$. The next theorem "unwraps" this recursion to give us formulas for $\Delta_m H_0$ (the proof is in the Appendix)

Theorem 10.3. *Let η be the index of the optimal exercise time. Then, for any m, $0 \leq m \leq M-1$,*

$$\Delta_m H_0 \left(0\right) = \mathbf{E}_0 \left(\sum_{n=\eta}^{N-1} \Delta_m \left(B_{T_{n+1}}^{-1} X_n\right)\right). \tag{10.2}$$

It is interesting to compare the expression for the value of a callable Libor exotic with the one for its delta,

$$\begin{aligned} H_0 \left(0\right) &= \mathbf{E}_0 \left(\sum_{n=\eta}^{N-1} B_{T_{n+1}}^{-1} X_n\right), \\ \Delta_m H_0 \left(0\right) &= \mathbf{E}_0 \left(\sum_{n=\eta}^{N-1} \Delta_m \left(B_{T_{n+1}}^{-1} X_n\right)\right). \end{aligned} \tag{10.3}$$

Somewhat surprisingly, it appears that one can compute the delta Δ_m by differentiating the sum in (10.3) while pretending that optimal exercise time index η does not depend on $F_m(0)$. Clearly the distribution of η does depend on the initial interest rate curve – is there a contradiction here?

This seeming contradiction can be resolved with the help of the following (rather non-rigorous) argument. For an arbitrary stopping time ζ, define $h(\zeta, X)$ by

$$h(\zeta, X) = \mathbf{E}_0 \left(\sum_{n=\zeta}^{N} B_{T_{n+1}}^{-1} X_n \right),$$

(X in the argument of $h(\zeta, X)$ represents all coupons X_n and all numeraire factors $B_{T_n}^{-1}$). Loosely, we can think of $h(\zeta, X)$ as the value of a barrier option with the barrier defined by the stopping time ζ. Note that $h(\zeta, X)$ is equal to $H_0(0)$ for $\zeta = \eta$. Formally differentiating with respect to $F_m(0)$,

$$\Delta_m h(\zeta, X) = \frac{\partial}{\partial \zeta} h(\zeta, X) \times \Delta_m \zeta + \frac{\partial}{\partial X} h(\zeta, X) \times \Delta_m X. \tag{10.4}$$

Let us plug $\zeta = \eta$ into the last equation. The critical observation we make is that

$$\left. \frac{\partial}{\partial \zeta} h(\zeta, X) \right|_{\zeta=\eta} = 0.$$

Why is it so? Because η by definition is the *optimal* stopping time that maximizes the value of a callable Libor exotics over all stopping times. Therefore the first term in (10.4) drops out and we are left with

$$\Delta_m H_0 = \Delta_m h(\eta, X) = \frac{\partial}{\partial X} h(\eta, X) \times \Delta_m X.$$

The expression on the right hand side can be interpreted as the partial derivative of the sum in (10.3) with η held constant.

Writing the delta in the form (10.2) makes it clear how it can be approximated in the Longstaff-Schwartz framework. We can replace the optimal stopping time η with its estimate $\tilde{\eta}$. This gives us our main approximation formula.

Corollary 10.4. *A pathwise delta $\Delta_m H_0(0)$ of a callable Libor exotic contract can be approximated with $\tilde{\Delta}_m H_0(0)$ computed as follows,*

$$\tilde{\Delta}_m H_0(0) = \mathbf{E}_0 \left(\sum_{n=\tilde{\eta}}^{N-1} \Delta_m \left(B_{T_{n+1}}^{-1} X_n \right) \right), \tag{10.5}$$

where $\tilde{\eta}$ is an estimate of the optimal exercise time index computed during Longstaff-Schwartz pre-simulation, as outlined in Section 8.

This elegant formula for deltas of a callable Libor exotic is easy to implement in practice. The estimate of the optimal exercise time, $\tilde{\eta}$, comes "for free" from the pre-simulation step of the valuation. Once $\tilde{\eta}$ is estimated (usually in the form of a collection of exercise regions, as explained in Section 8), deltas are computed by

1. Running a forward simulation, for each path ω determining the optimal exercise time index $\tilde{\eta}(\omega)$;
2. For each path, computing deltas of all coupons X_n, $n = 1, \ldots, N-1$ (as well as the deltas of the numeraire $B_{\cdot}^{-1}$) along the path;

3. Adding up deltas $\Delta_m\left(B_{T_{n+1}}^{-1}X_n\right)$ for those coupons that occur after the exercise index $\tilde{\eta}(\omega)$;
4. Averaging the result over all paths.

11. Pathwise deltas of coupons at exercise times

To apply the formula (10.5) we need to be able to compute

$$\Delta_m\left(B_{T_{n+1}}^{-1}X_n\right), \quad n = 1, \ldots, N-1,$$

for each simulated path. Applying a product rule we obtain

$$\Delta_m\left(B_{T_{n+1}}^{-1}X_n\right) = B_{T_{n+1}}^{-1}\Delta_m(X_n) + \Delta_m\left(B_{T_{n+1}}^{-1}\right)X_n.$$

Deltas $\Delta_m\left(B_{T_{n+1}}^{-1}\right)$ of the inverse numeraire (discrete money market account) are computed in Appendix B.

Most coupons X_n can be represented as an absolutely continuous function applied to a market rate (a Libor or a CMS rate), see Section 6. Their deltas can be computed via a chain rule. Let us present a simple example. Let

$$X_n = q(Y(T_n)),$$

where $q(\cdot)$ is an absolutely continuous function (an example would be $q(y) = \max(y, c)$ for a floored payoff) and $Y(t)$ is some forward Libor or CMS rate observed at time t. Then

$$\Delta_m(X_n) = q'(Y(T_n)) \times \Delta_m(Y(T_n)).$$

Furthermore,

$$\Delta_m(Y(T_n)) = \sum_{v=1}^{M}\frac{\partial Y(T_n)}{\partial F_v(T_n)} \cdot \Delta_m F_v(T_n).$$

The quantities $\Delta_m F_v(T_n)$ are directly simulated in the model, see (5.1) or (5.2). What can we say about $\frac{\partial Y(T_n)}{\partial F_v(T_n)}$? These are sensitivities of a market rate Y, as observed at time T_n, to primary Libor rates F_v also observed at time T_n. These sensitivities are determined by the way an interest rate curve at future time T_n is constructed from simulated primary Libor rates $\{F_v(T_n)\}_{v=1}^{M}$. (One possible such construction is presented in Appendix A.) Having specified the curve construction algorithm, the quantity $\frac{\partial Y(T_n)}{\partial F_v(T_n)}$ can be obtained via a chain rule. The rate Y can be expressed as a function of zero coupon discount bonds which in turn are functions of primary Libor rates.

An approximation can be employed to speed up calculations of the quantities $\frac{\partial Y(T_n)}{\partial F_v(T_n)}$. The approximation we have in mind is motivated by swap rate volatility approximations usually employed in BGM calibration, see e.g. [AA98]. As explained in Appendix C, those (generally excellent) approximations are based on the assumption that the quantities

$$\frac{\partial Y(t)}{\partial F_v(t)} \times \frac{\phi(F_v(t))}{\phi(Y(t))}$$

are constant in time t, $t \in [0, T_n]$. Making this assumption we obtain an approximation

$$\frac{\partial Y(T_n)}{\partial F_v(T_n)} \approx \frac{\partial Y(0)}{\partial F_v(0)} \times \frac{\phi(F_v(0))}{\phi(Y(0))} \times \frac{\phi(Y(T_n))}{\phi(F_v(T_n))}.$$

An added advantage of this approximation is that the initial deltas $\frac{\partial Y(0)}{\partial F_v(0)}$ can be computed once before the simulation, and then reused.

12. Test results

12.1. **Bermuda swaption.** Bermuda swaptions are by far the most prevalent and important type of callable Libor exotics. We look at a particular Bermuda swaption. The underlying swap is a 5% receiver (we receive 5% fixed rate and pay Libor) and has a 9y year tenor. The Bermuda swaption gives us a right to enter this swap every year starting at year 1 ("10-no-call-1" in industry parlance).

For the tests, we used the EUR-LIBOR interest rate curve and swaption volatilities for March 07, 2003. We calibrated a two-factor log-normal BGM model to the whole swaption volatility grid. The model was based on Libor rates with one-year tenor (for simplicity). We computed deltas to all Libor rates that span the Bermuda swaption, both using the standard "bump" method and our "fast" method. For the "fast" method, we used the approximation (5.2). For valuation we used a Longstaff-Schwartz scheme with the explanatory variable being the remaining part of the underlying swap on each of the exercise dates, and polynomials of second order as parametric families $f_n(\cdot;\cdot)$ and $g_n(\cdot;\cdot)$ (see Section 8 for notations used).

The number of paths used is reported as N/M, where N is the number of paths used in pre-simulation and M is the number of paths used in the actual simulation.

Values for deltas are given in the units of Euro per 1 basis point change in rates for a swap notional of 100,000,000 Euros. The Bermuda swaption value is 8,400,000 Euros.

The main conclusions that we reached are as follows.

- Deltas computed by the two methods converge to each other as the number of paths is increased.
- "Bump" deltas show significant dependence on the amount of the actual bump that is applied to each Libor rate, while "fast" deltas are obviously free from this problem.
- "Fast" deltas show a remarkably higher stability as the number of paths is increased, unlike their "bump" counterparts. "Fast" deltas stabilize with 1024/2048 paths. For "bump" deltas, 65,536/131,072 paths is required.

12.1.1. *Delta convergence.* The first graph, Figure 1, presents "fast" and "bump" deltas for all Libor rates spanning the Bermuda swaption. "Bump" deltas are computed with three different bump sizes, 1, 5, and 10 basis points. A bigger bump leads to more stable deltas but may pick up more of the second order effects. The "fast" deltas and "bump" deltas are close, comparable to the variability within the "bump" deltas themselves. The table below gives the total delta for different methods.

Method	**Total delta, Euros per 1 bp**
"Fast" deltas	42,740
1 bp "bump" deltas	42,000
5 bp "bump" deltas	42,400
10 bp "bump deltas	42,600

12.1.2. *Bump size dependence for "bump" deltas.* As one can see in Figure 1, even for extremely high number of paths (65,536/131,072), the differences between "bump" deltas computed using bumps of different sizes is quite significant. It is even more pronounced for lower number of paths. This obviously makes any using any particular bump size hard to justify. For “fast” deltas, since all differentiations are done analytically, such an issue does not arise.

12.1.3. *Stability and convergence.* Among different bump sizes, 10 basis-point-bump deltas show the highest amount of stability as we increase the number of paths. Even so, the convergence and stability are quite bad, see Figure 2. Even for 16,384/32,768 paths, the total delta is off by 3,000, or about 7.5%.

The picture is much better for “fast” deltas, see Figure 3. Except for the very low, 256/512, number of paths, all delta profiles are right on top of each other. What it means is that with “fast” deltas, we can use considerably fewer number of paths than required for the “bump” method (in this case, by the order of 64!) to achieve desired accuracy.

12.2. **Callable Inverse Floater.** We performed similar tests for a callable inverse floater, another popular type of a callable Libor exotic. Its underlying swap is composed of annual coupons

$$\max\left(7.00\% - F, 3.50\%\right),$$

where F is a Libor rate with 1 year tenor that we receive, and Libor rate payments (also 1 year tenor) that we pay in return. The swap starts in 1 year and goes on for 6 years. The callable inverse floater has a first exercise one year from today and then annually for 6 years.

We used the same market data and the same model as in the Bermuda swaption tests. The callable inverse floater value was 5,530,000 Euros.

In tests for the callable inverse floater we reached the same conclusions as for the Bermuda swaptions in the previous section. The most striking feature, again, was the fact that “fast” deltas were significantly more stable for a much lower number of simulated paths.

To check the last point we carried out an additional test. We computed deltas, both “bump” and “fast”, multiple times with different initial random seeds, and computed standard deviations of deltas. We found that the “bump” deltas computed with 65,536/131,072 paths had standard deviations comparable to “fast” deltas computed with 2,048/4,096 paths, implying a reduction in the number of paths required by a factor of about 32.

12.2.1. *Delta convergence.* Figure 4, presents “fast” and “bump” deltas for all Libor rates spanning the callable inverse floater. “Bump” deltas are computed with three different bump sizes, 1, 5, and 10 basis points. “Fast” deltas and “bump” deltas are close, the differences are comparable to the variability within the “bump” deltas themselves. The table below gives the total delta for different methods.

Method	**Total delta, Euros per 1 bp**
“Fast” deltas	38,500
1 bp “bump” deltas	39,200
5 bp “bump” deltas	39,200
10 bp “bump deltas	38,800

12.2.2. *Bump size dependence for “bump” deltas.* As one can see in Figure 4, shows the differences between “bump” deltas computed using bumps of different sizes even for very high number of paths (65,536/131,072). For “fast” deltas, since all differentiations are done analytically, such an issue does not arise.

12.2.3. *Stability and convergence.* Among different bump sizes, 10 basis-point-bump deltas show the highest amount of stability as we increase the number of paths. Even so, the convergence and stability are not very good, see Figure 5.

The picture is much better for "fast" deltas, see Figure 6. Except for the very low, 256/512, number of paths, all delta profiles are right on top of each other. What it means is that with "fast" deltas, we can use considerably fewer number of paths than required for the "bump" method to achieve desired accuracy.

12.2.4. *Standard deviation of deltas.* We estimated simulation errors of deltas (both "bump" and "fast") by performing multiple delta calculation runs using different initial seeds (seeds were generated randomly), and calculating standard deviations of resulting time series of deltas. We used 16 runs. We found that the standard deviations of "bump" deltas computed using 65,536/131,072 paths were roughly the same (actually somewhat worse) as the standard deviations of "fast" deltas computed using 1/32 the number of paths, i.e. 2,048/4,096 paths. Figure 7 shows the results for all buckets (Libor rates to which sensitivities were computed). The standard deviation of the total delta (i.e. the sum of all bucketed deltas) was 363 for the "fast" method and 506 for the "bump" method.

Figures 8 and 9 plot "bump" and "fast" bucketed deltas, respectively, from all 16 runs on the same graph to visually demonstrate the amount of dispersion among then.

As a consistency check we compared the standard deviations for the "fast" deltas as computed above against estimates of standard mean errors for deltas computed during a single run. The two estimates were broadly consistent across all buckets.

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Appendix A. Constructing a full interest rate curve from primary Libor rates observed at a future time

Suppose that at a future time t, $t \geq 0$, we have a collection of simulated primary Libor rates $\{F_n(t), n = 0, \ldots, M\}$. Since the payment and other important dates of underlying instruments do not necessarily fall on Libor tenor dates t_n, $n = 0, \ldots, M$, we need to construct a full interest rate curve (i.e. all discount factors) as observed on date t,

$$\{P(t,T), T \geq t\},$$

from these Libor rates. This is basically just a matter of proper interpolation of *all* zero coupon discount bonds from a *discrete* collection of Libor rates. This interpolation scheme should be consistent with what is used for computing Libor rates from the initial interest rate curve.

For simplicity, we will use a “constant simple rate” interpolation.

Suppose m, $1 \leq m \leq M$, is chosen such that

$$t_{m-1} \leq t < t_m.$$

The date t_m is the first Libor date after t. We define all discount factors $P(t,T)$, $T \geq t$, in three stages:

1. First, we define discount actors $P(t,T)$ for T prior to the first Libor date t_m, $T \leq t_m$;
2. Second, we define all discount factors corresponding to Libor dates $P(t,t_i)$ for $i \geq m$;
3. Finally, we define all remaining discount factors.

To compute discount factors $P(t,T)$ for $T \leq t_m$ we assume that the simple rate between t and T is the same as the simple rate between t_{m-1} and t_m,

$$F(t,t,T) = F_{m-1}(t) = F_{m-1}(t_{m-1}).$$

(the last equality is a definition of $F_{m-1}(T)$ for T past the fixing date t_{m-1}). From this equation we obtain

$$\begin{aligned} \frac{1}{T-t}\frac{1-P(t,T)}{P(t,T)} &= F_{m-1}(t_{m-1}), \\ P(t,T) &= \frac{1}{1+(T-t)F_{m-1}(t_{m-1})}. \end{aligned}$$

In particular,

$$P(t,t_m) = 1 + (t_m - t)F_{m-1}(t_{m-1}). \tag{A.1}$$

For any t_i, $i \geq m$, we use the following recursion

$$P(t,t_i) = P(t,t_{i-1}) \times \frac{1}{1+\tau_{i-1}F_{i-1}(t)},$$

together with the starting condition (A.1).

Finally, suppose we have a time T such that

$$t_i \leq T < t_{i+1}$$

for some $i > m$. Then

$$P(t,T) = P(t,t_i) \times P(t,t_i,T).$$

To compute $P(t, t_{i-1}, T)$ from primary Libor rates we again use constant interpolation of simple rates,

$$\begin{aligned} F(t, t_i, T) &= F_i(t), \\ \frac{1}{(t - t_i)} \frac{P(t, t_i) - P(t, T)}{P(t, t_i)} &= F_i(t). \end{aligned}$$

Then

$$P(t, t_i, T) = \frac{1}{1 + (t - t_i) F_i(t)}$$

and finally

$$P(t, T) = P(t, t_i) \times \frac{1}{1 + (t - t_i) F_i(t)},$$

for an arbitrary T.

Appendix B. Pathwise deltas of the money market account

Recall that the discrete money market account B_T, the numeraire used in our model, is defined by (we assume that $t_n \leq T < t_{n+1}$),

$$B_T = \left(\prod_{i=0}^{n} (1 + \tau_i F_i(t_i)) \right) \times P(T, t_{n+1}).$$

As explained in Section 4, bond volatility structure in our model is chosen in such a way that

$$P(T, t_{n+1}) = P(t_n, T, t_{n+1}),$$

see (4.3).To express the quantity $P(t_n, T, t_{n+1})$ in terms of primary Libor rates F_j, we need to choose an interpolation scheme, just like in Appendix A. As before, we assume constant interpolation of simple rates and write

$$\begin{aligned} F(t_n, T, t_{n+1}) &= F(t_n, t_n, t_{n+1}), \\ \frac{1}{(t_{n+1} - T)} \frac{1 - P(t_n, T, t_{n+1})}{P(t_n, T, t_{n+1})} &= F_n(t_n), \\ P(t_n, T, t_{n+1}) &= \frac{1}{1 + (t_{n+1} - T) F_n(t_n)}. \end{aligned}$$

Then we have

$$B_T = \left(\prod_{i=0}^{n-1} (1 + \tau_i F_i(t_i)) \right) \times \frac{1 + \tau_n F_n(t_n)}{1 + (t_{n+1} - T) F_n(t_n)}.$$

We are interested in computing

$$\Delta_m \left(B_T^{-1} \right)$$

for some m. We have,

$$B_T^{-1} = \left(\prod_{i=0}^{n-1} \frac{1}{1 + \tau_i F_i(t_i)} \right) \times \frac{1 + (t_{n+1} - T) F_n(t_n)}{1 + \tau_n F_n(t_n)}.$$

Then

$$
\begin{aligned}
\Delta_m \left(B_T^{-1}\right) &= \sum_{j=0}^{n} \frac{\partial B_T^{-1}}{\partial F_j\left(t_j\right)} \times \Delta_m F_j\left(t_j\right) \\
&= B_T^{-1} \times \sum_{j=0}^{n-1} \left(1+\tau_j F_j\left(t_j\right)\right) \times \frac{\partial}{\partial F_j\left(t_j\right)} \frac{1}{1+\tau_j F_j\left(t_j\right)} \times \Delta_m F_j\left(t_j\right) \\
&\quad + B_T^{-1} \times \frac{1+\tau_n F_n\left(t_n\right)}{1+\left(t_{n+1}-T\right) F_n\left(t_n\right)} \times \frac{\partial}{\partial F_n\left(t_n\right)} \frac{1+\left(t_{n+1}-T\right) F_n\left(t_n\right)}{1+\tau_n F_n\left(t_n\right)} \times \Delta_m F_n\left(t_n\right) \\
&= -B_T^{-1} \times \sum_{j=0}^{n-1} \frac{\tau_j}{1+\tau_j F_j\left(t_j\right)} \times \Delta_m F_j\left(t_j\right) \\
&\quad - B_T^{-1} \times \frac{T-t_n}{\left(1+\left(t_{n+1}-T\right) F_n\left(t_n\right)\right) \cdot \left(1+\tau_n F_n\left(t_n\right)\right)} \times \Delta_m F_n\left(t_n\right).
\end{aligned}
$$

Appendix C. Libor and swap rate approximations in forward Libor models

In this Appendix we present the results on approximate dynamics of non-primary Libor rates and swap rates in a forward Libor model (4.2). The results are taken from [AA98, Section 5] and presented here for completeness.

Let $Y(\cdot)$ be a Libor or a swap rate in the model (4.2). Under the appropriate forward or swap measure, $Y(\cdot)$ is a martingale. Under such a measure, the expression for dY does not have a dt term; in particular, we have

$$
\begin{aligned}
dY(t) &= \sum_{n=1}^{M-1} \frac{\partial Y(t)}{\partial F_n(t)} \, dF_n(t) \\
&= \left(\sum_{n=1}^{M-1} \frac{\partial Y(t)}{\partial F_n(t)} \, \phi\left(F_n(t)\right) \lambda_n(t) \right) d\hat{W}(t)
\end{aligned}
$$

(here $d\hat{W}$ is a driftless Brownian motion under the measure used).

Following [AA98, Section 5] we approximate

$$
\begin{aligned}
\frac{\partial Y(t)}{\partial F_n(t)} &\approx \frac{\partial Y(0)}{\partial F_n(0)}, \\
\phi\left(F_n(t)\right) &\approx \phi\left(Y(t)\right) \frac{\phi\left(F_n(0)\right)}{\phi\left(Y(0)\right)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
dY(t) &\approx \phi\left(Y(t)\right) \left(\sum_{n=1}^{M-1} \frac{\partial Y(0)}{\partial F_n(0)} \times \frac{\phi\left(F_n(0)\right)}{\phi\left(Y(0)\right)} \times \lambda_n(t) \right) d\hat{W}(t) \\
&= \phi\left(Y(t)\right) \left(\sum_{n=1}^{M-1} w_n \lambda_n(t) \right) d\hat{W}(t),
\end{aligned}
$$

where

$$
w_n = \frac{\partial Y(0)}{\partial F_n(0)} \times \frac{\phi\left(F_n(0)\right)}{\phi\left(Y(0)\right)}, \quad n = 1, \ldots, M-1.
$$

Appendix D. Proofs

D.1. **Proof of Theorem 10.210.2.** We have from (10.1),

$$\begin{aligned}\Delta_m\left(B_{T_{n-1}}^{-1}H_{n-1}\left(T_{n-1}\right)\right) &= \mathbf{E}_{T_{n-1}}\Delta_m\left(B_{T_n}^{-1}\max\left\{H_n\left(T_n\right),E_n\left(T_n\right)\right\}\right)\\ &= \mathbf{E}_{T_{n-1}}\Delta_m\left(\max\left\{B_{T_n}^{-1}H_n\left(T_n\right),B_{T_n}^{-1}E_n\left(T_n\right)\right\}\right).\end{aligned}$$

The function $x\longmapsto\max\{x,c\}$ is absolutely continuous and thus is equal to the integral of its almost-everywhere defined derivative. Its derivative is equal to $1_{\{x>c\}}$. Hence, we can differentiate $\max\left\{H_n\left(T_n\right),E_n\left(T_n\right)\right\}$ under the expectation sign to obtain

$$\begin{aligned}\mathbf{E}_{T_{n-1}}\left(\Delta_m\left\{B_{T_n}^{-1}H_n\left(T_n\right),B_{T_n}^{-1}E_n\left(T_n\right)\right\}\right) &= \mathbf{E}_{T_{n-1}}\left(1_{\{E_n(T_n)>H_n(T_n)\}}\Delta_m\left(B_{T_n}^{-1}E_n\left(T_n\right)\right)\right)\\ &\quad+\mathbf{E}_{T_{n-1}}\left(1_{\{H_n(T_n)>E_n(T_n)\}}\Delta_m\left(B_{T_n}^{-1}H_n\left(T_n\right)\right)\right).\end{aligned}$$

Combining the last two equalities we obtain the statement of the theorem.

D.2. **Proof of Theorem 10.3.** Unwrapping the recursive statement of Theorem 10.2 we obtain that

$$\Delta_m H_0\left(0\right)=\sum_{n=1}^{N-1}\mathbf{E}_0\left(\prod_{i=1}^{n-1}1_{\{H_i(T_i)>E_i(T_i)\}}\times 1_{\{E_n(T_n)>H_n(T_n)\}}\times\Delta_m\left(B_{T_n}^{-1}E_n\left(T_n\right)\right)\right).$$

If η is the optimal exercise time index, then

$$1_{\{\eta=n\}}=\prod_{i=1}^{n-1}1_{\{H_i(T_i)>E_i(T_i)\}}\times 1_{\{E_n(T_n)>H_n(T_n)\}}.$$

Therefore,

$$\Delta_m H_0\left(0\right)=\sum_{n=1}^{N-1}\mathbf{E}_0\left(1_{\{\eta=n\}}\times\Delta_m\left(B_{T_n}^{-1}E_n\left(T_n\right)\right)\right).$$

From Proposition 10.1 and the expression for the exercise value

$$B_{T_n}^{-1}E_n\left(T_n\right)=\sum_{i=n}^{N-1}\mathbf{E}_{T_n}\left(B_{T_{i+1}}^{-1}X_i\right)$$

we obtain

$$\Delta_m\left(B_{T_n}^{-1}E_n\left(T_n\right)\right)=\sum_{i=n}^{N-1}\mathbf{E}_{T_n}\left(\Delta_m\left(B_{T_{i+1}}^{-1}X_i\right)\right).$$

Hence

$$\Delta_m H_0\left(0\right)=\sum_{n=1}^{N-1}\mathbf{E}_0\left(1_{\{\eta=n\}}\times\sum_{i=n}^{N-1}\mathbf{E}_{T_n}\left(\Delta_m\left(B_{T_{i+1}}^{-1}X_i\right)\right)\right).$$

The event $\{\eta = n\}$ is $\mathcal{F}_{T_n}$-measurable because η is a stopping time. Thus we can carry the indicator $1_{\{\eta=n\}}$ inside the expectation $\mathbf{E}_{T_n}$ to get

$$\begin{aligned}
\Delta_m H_0(0) &= \sum_{n=1}^{N-1} \mathbf{E}_0 \mathbf{E}_{T_n} \left(1_{\{\eta=n\}} \times \sum_{i=n}^{N-1} \left(\Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right) \right) \\
&= \sum_{n=1}^{N-1} \mathbf{E}_0 \left(1_{\{\eta=n\}} \times \sum_{i=n}^{N-1} \left(\Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right) \right) \\
&= \mathbf{E}_0 \left(\sum_{n=1}^{N-1} \sum_{i=n}^{N-1} 1_{\{\eta=n\}} \left(\Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right) \right).
\end{aligned}$$

Changing the order of summation we obtain

$$\Delta_m H_0(0) = \mathbf{E}_0 \left(\sum_{i=\eta}^{N-1} \left(\Delta_m \left(B_{T_{i+1}}^{-1} X_i \right) \right) \right),$$

and the theorem is proved.

Appendix E. Figures

All figures appear at the end of the paper.

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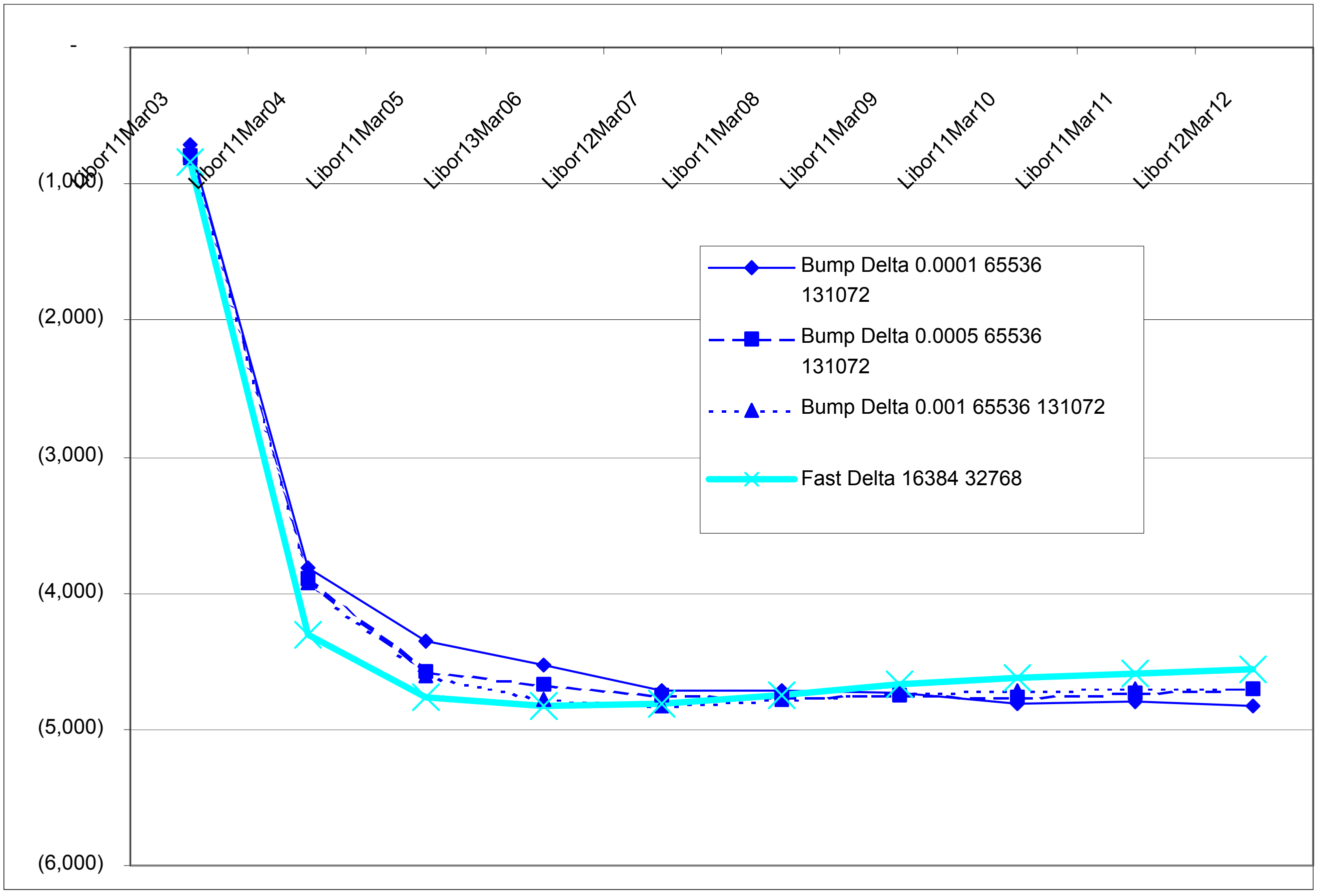

FIGURE 1. Bermuda swaption deltas computed using the "fast" method versus those computed by a "bump" method with bump sizes of 0.0001, 0.0005 and 0.001, for all relevant Libor rates. "Bump" deltas are computed with 65,536/131,072 paths and "fast" deltas are computed with 16,384/32,768 paths.

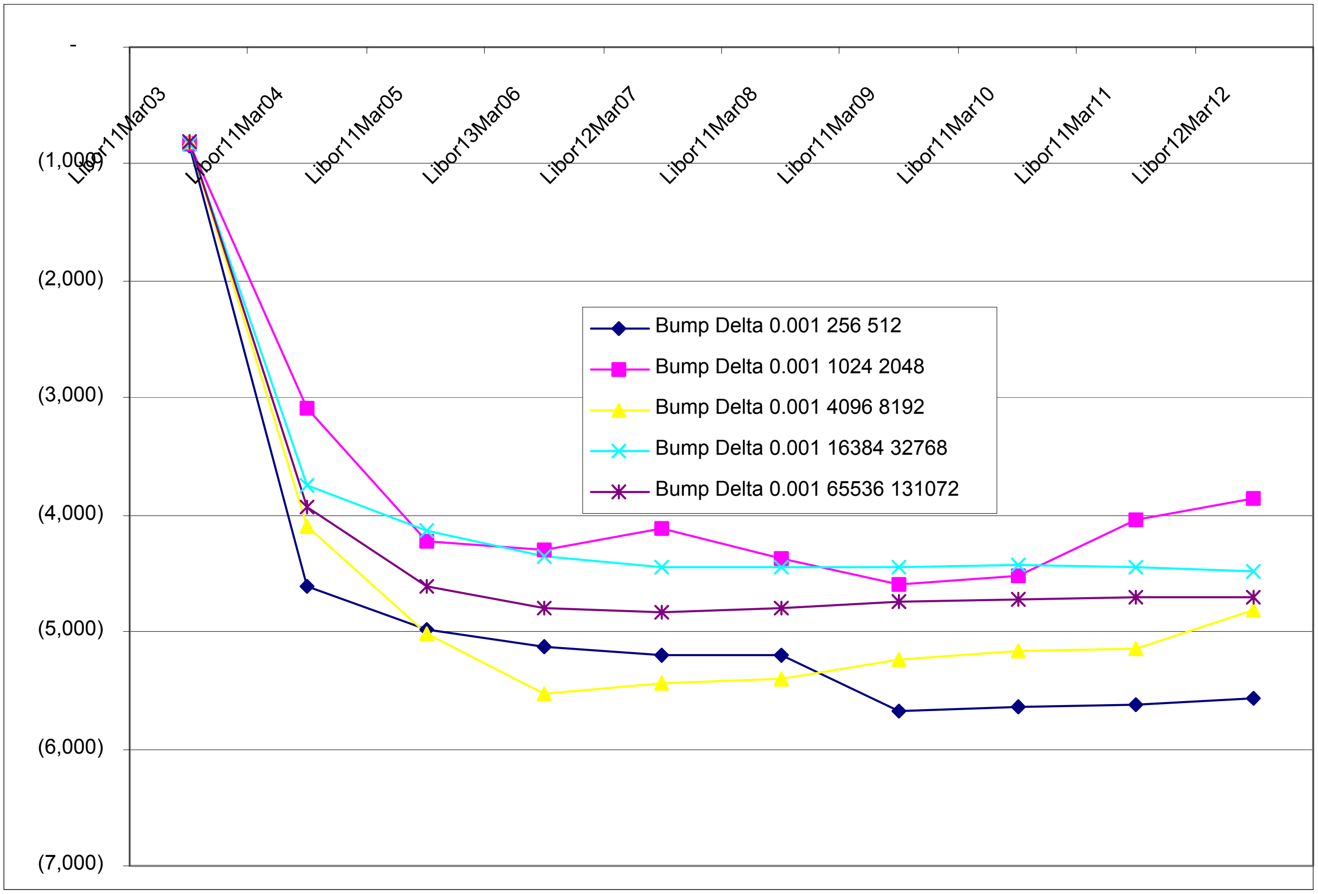

FIGURE 2. Bermuda swaption "bump" deltas (with bump size = 0.001) to all underlying Libor rates for different number of paths used.

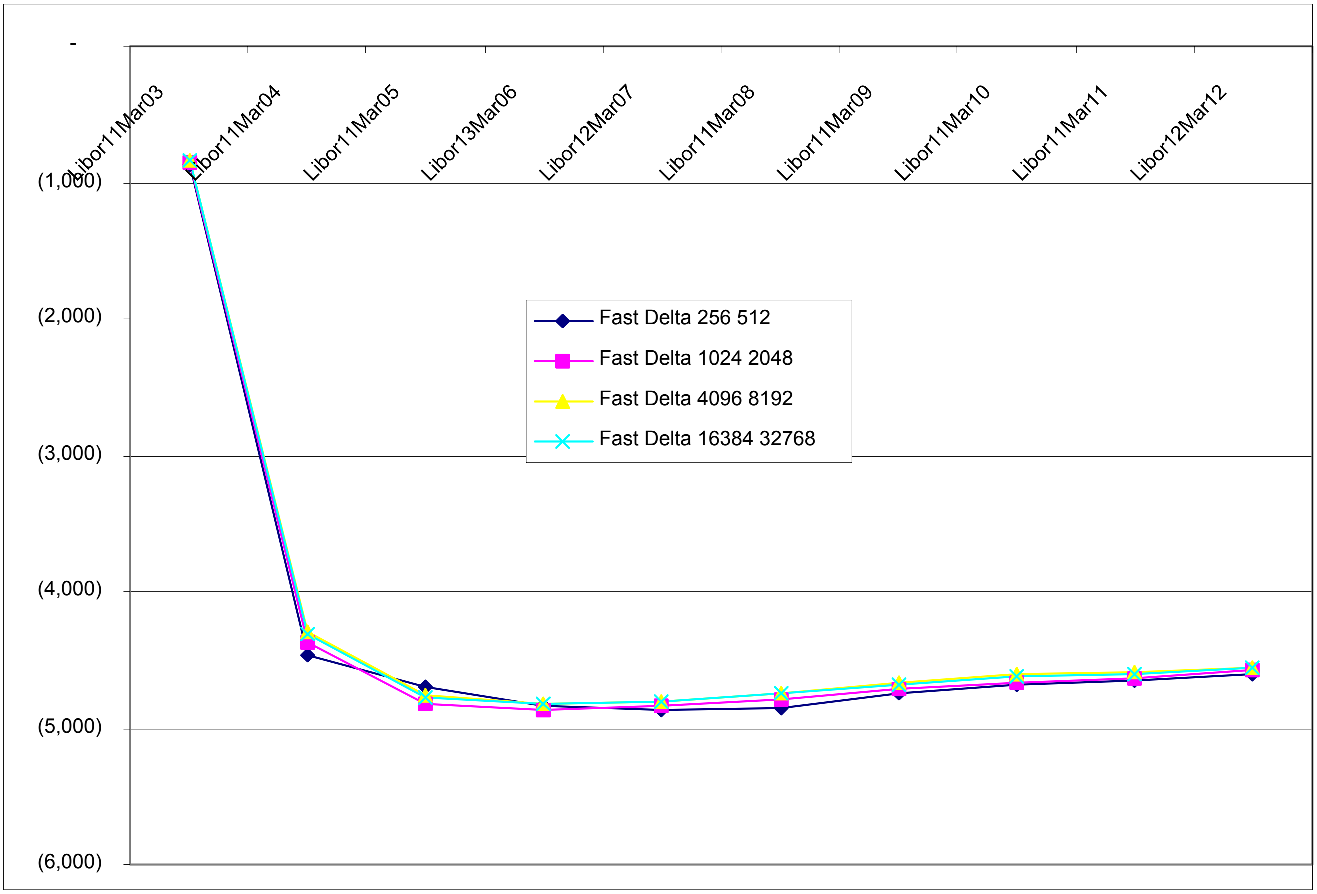

FIGURE 3. Bermuda swaption "fast" deltas to all underlying Libor rates for different number of paths used.

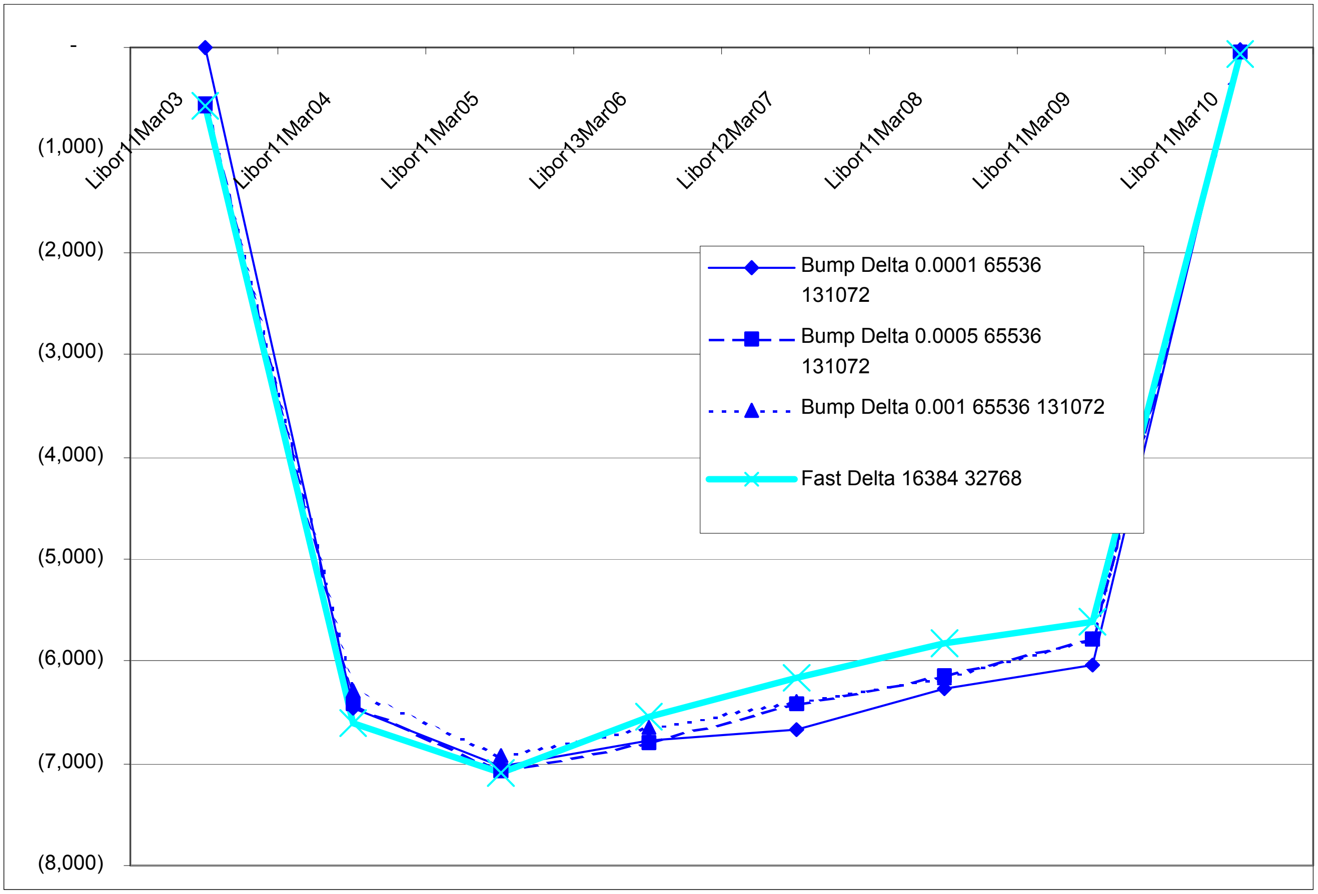

FIGURE 4. Callable inverse floater deltas computed using the "fast" method versus those computed by a "bump" method with bump sizes of 0.0001, 0.0005 and 0.001, for all relevant Libor rates. "Bump" deltas are computed with 65,536/131,072 paths and "fast" deltas are computed with 16,384/32,768 paths.

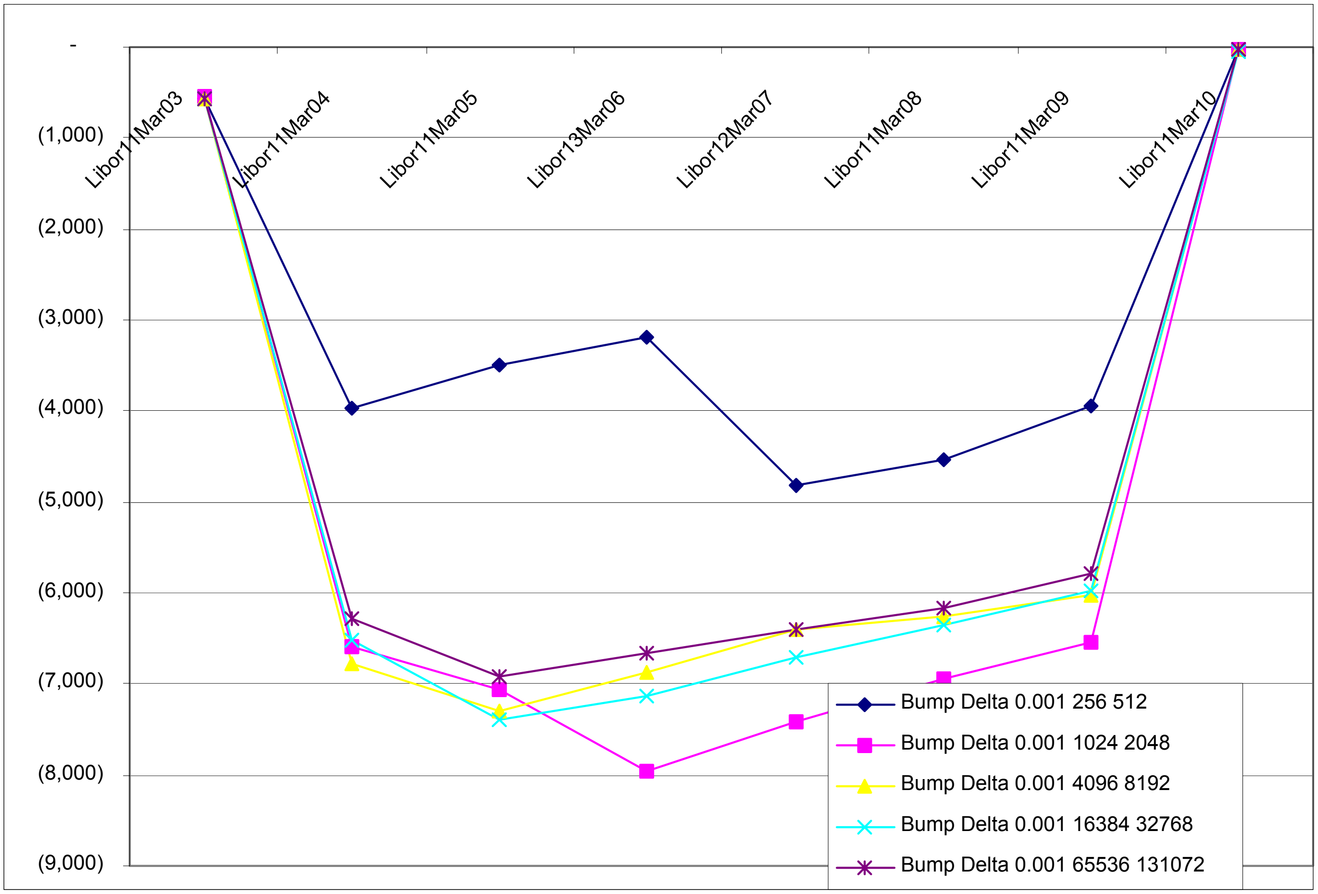

FIGURE 5. Callable inverse floater "bump" deltas (with bump size = 0.001) to all underlying Libor rates for different number of paths used.

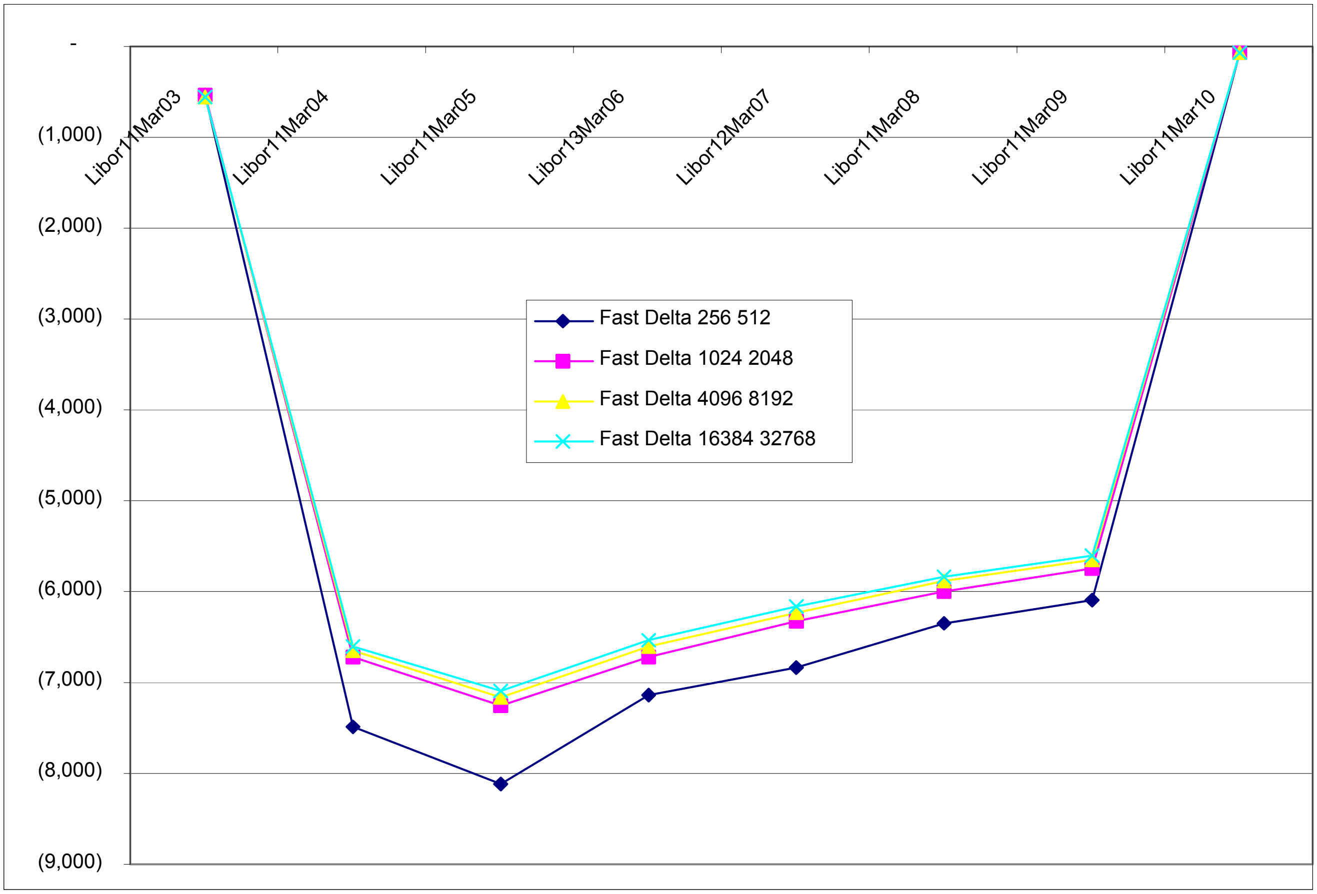

FIGURE 6. Callable inverse floater "fast" deltas to all underlying Libor rates for different number of paths used.

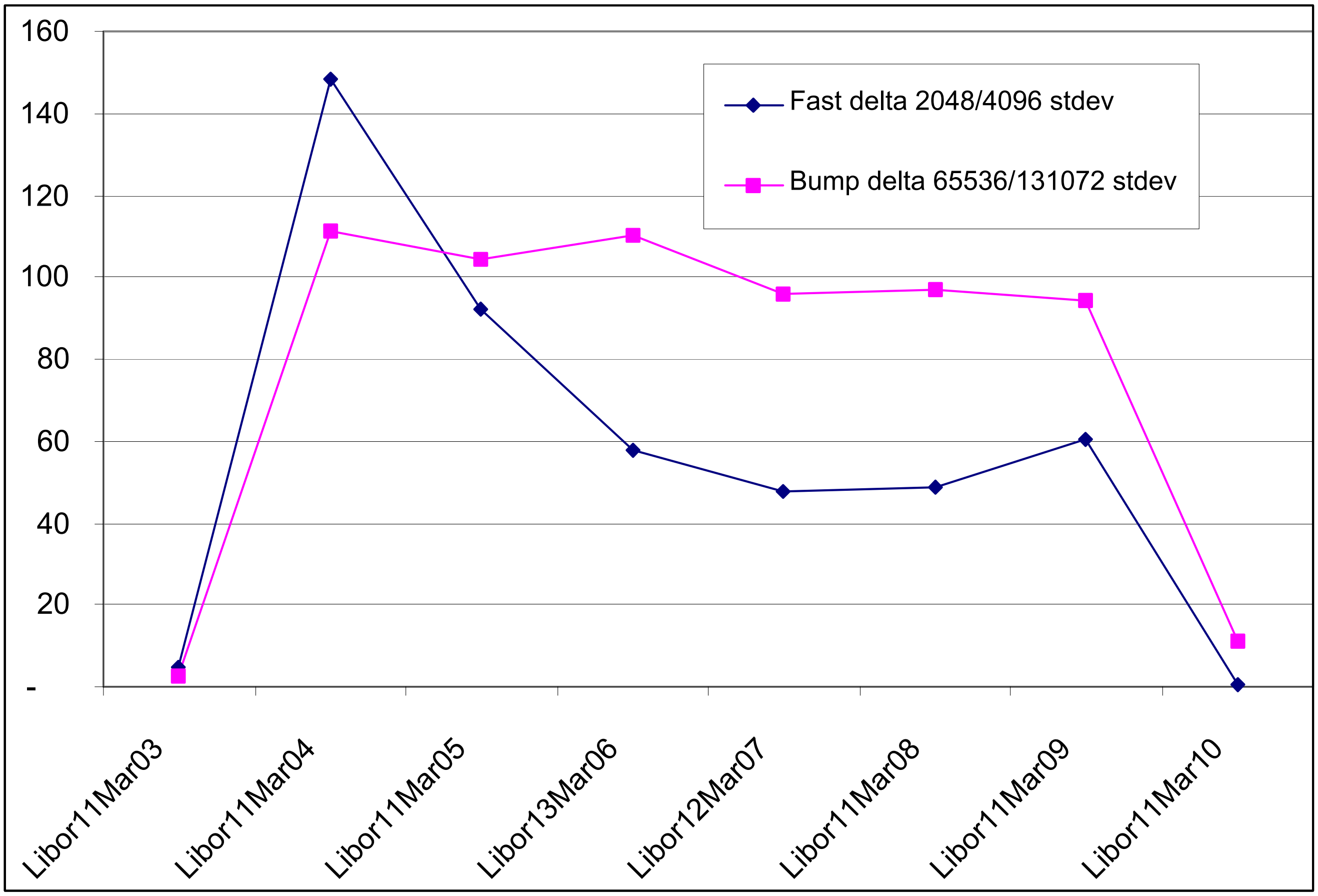

FIGURE 7. Standard deviations of "fast" deltas (2,048/4,096 paths) versus "bump" deltas *65,536/131,072 paths) for different buckets, from a sample of 16 runs each.

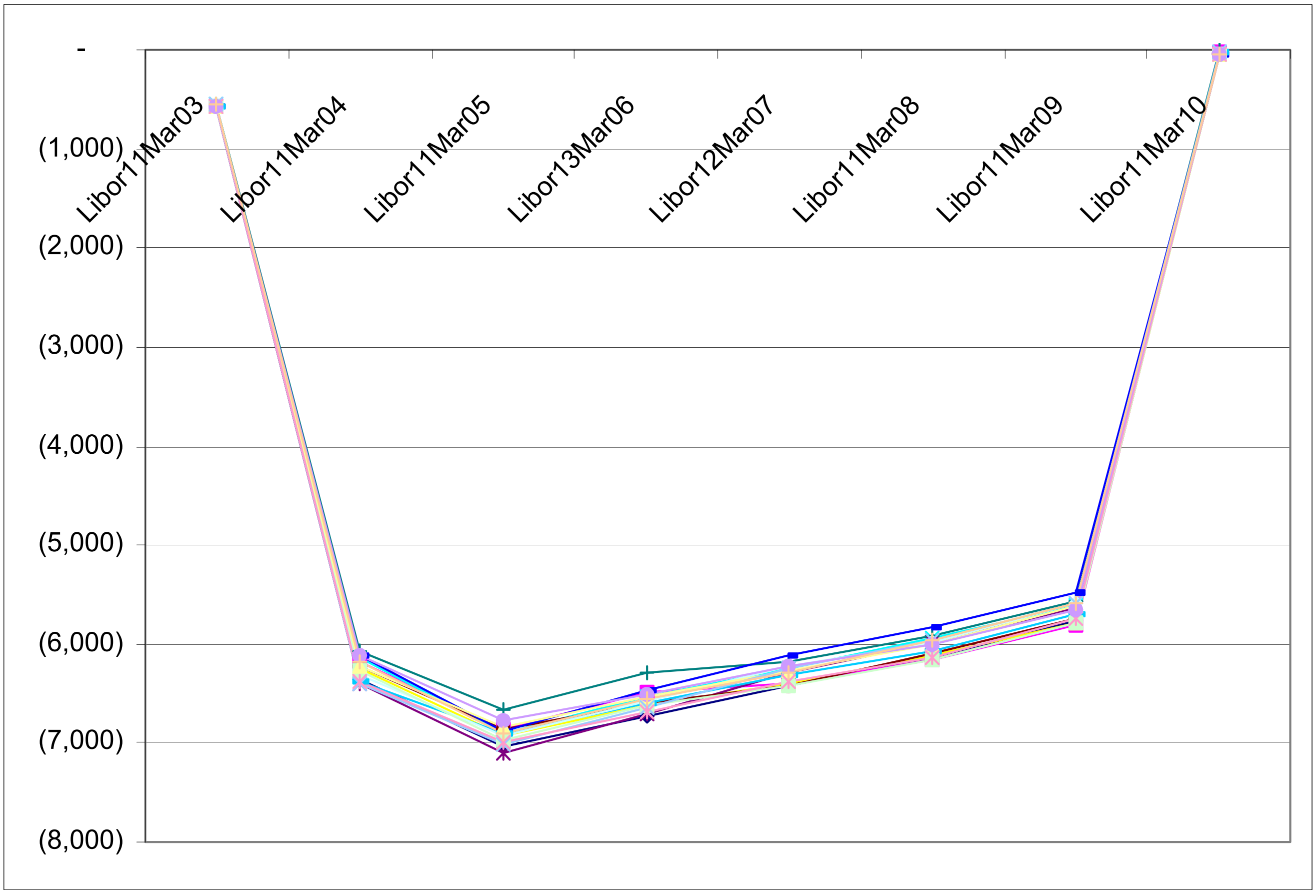

Figure 8. “Bump” deltas from 16 runs using different (random) initial seeds.

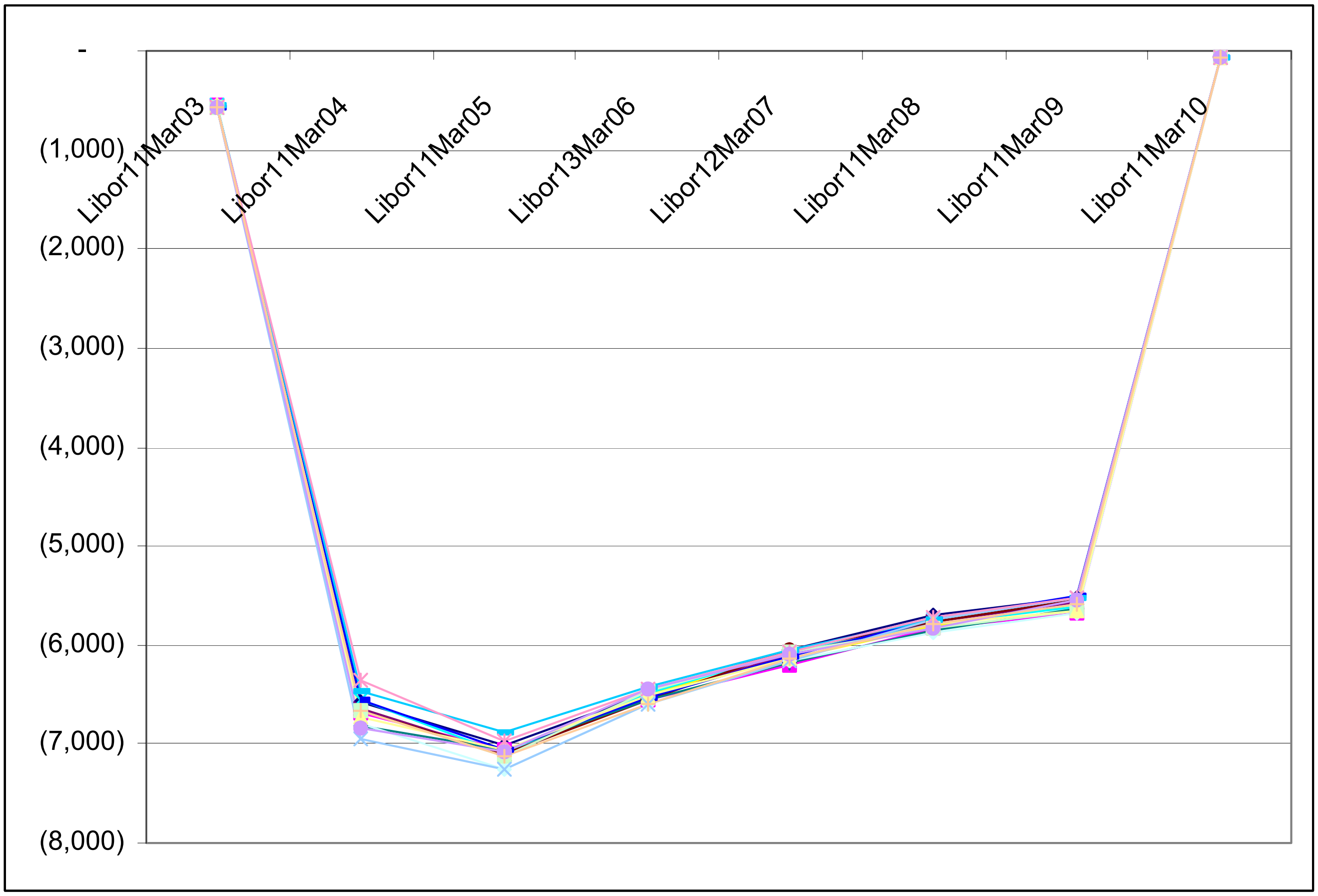

Figure 9. "Fast" deltas from 16 runs using different (random) initial seeds