AIP Conference Proceedings

RESEARCH ARTICLE | MAY 15 2015

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AIP Conference Proceedings 1660, 050084 (2015)
https://doi.org/10.1063/1.4915717

Review on the Structural Approach of the Black-Scholes Model

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Abstract. Black-Scholes model developed in 1973 has become one of the important concepts in modern financial theory. This model is regarded as one of the best ways in determining fair prices of the options. Many studies have been done to improve the performance of the Black-Scholes model since this model is built with few limitations. Thus, the objective of this review paper is to discuss on the Black-Scholes model. The aim of this review paper is to present the derivation of Black-Scholes, Merton and KMV-Merton models. Besides, it provides a literature review on the modifications done by the researchers on the Black-Scholes model.

Keywords: Black-Scholes model, Merton model, KMV-Merton model.
PACS: 02.50.Ey, 05.40.Jc

INTRODUCTION

Credit risk nowadays has become a crucial problem in the banking sector. It defines as a risk that relates to the default by borrowers that fail to make a required payment to the debt holders. According to Benos and Papanastasopoulos [1] there are two categories of approach in measuring credit risk, which are structural and theoretical. Structural approach adopts contingency claim analysis while theoretical approach adopts accounting ratio-based approach. However, the focus of this paper is to review on the structural approach only. The origin of this approach goes back to Fischer Black and Myron Scholes.

Fischer Black and Myron Scholes [2] in their seminal paper propose the Black-Scholes model (B-S model) to value options in terms of the price of the stocks. Despite its popularity, the B-S model is actually built with some unrealistic assumptions about the market such as the underlying asset follow a lognormal random walk, volatility and interest rate are constant, the options can be exercised at maturity date only, the stock pays no dividends during the life of the option and etc, as discussed by Willmott [3] and Teneng [4]. Due to those unrealistic assumptions, many models have been proposed to tackle these problems. The most well-known among these are Merton model and KMV-Merton model.

In 1974, Merton [4] together with Black and Scholes enhanced the original B-S model and claimed that this model can be used to develop a pricing theory of corporate liabilities. The analysis of their study is also extending to include the callable bonds. The assumptions of Black-Scholes Merton model (Merton model) are in line with the original B-S model.

KMV-Merton was introduced by the KMV Company in 1989. They enhance the Merton's model ideas with a little bit different in determining the risk of a portfolio.

The aim of this paper is to derive the B-S model and to review on the study done by the academicians to the B-S model which consists of Merton model and KMV-Merton model.

The outlines of this paper are as follows: section 2 presents the derivation of B-S formula for a call option, section 3 discusses on the modifications done to the B-S model and in chapter 4, we conclude.

International Conference on Mathematics, Engineering and Industrial Applications 2014 (ICoMEIA 2014)
AIP Conf. Proc. 1660, 050084-1–050084-8; doi: 10.1063/1.4915717

BLACK-SCHOLES MODEL

This section presents the derivation of the B-S model, followed by the derivation of Merton and KMV-Merton models. By referring to Black and Scholes [2], Coelen [6], and Wilmott [3], we derive the B-S model. Denote the option values as

$$V(S,t;\sigma,\mu;T;r). \tag{1}$$

The semi-colons are used to separate the different types of variables and parameters:

- The underlying stocks prices S and time t are variables;
- Volatility σ and drift rate μ are parameters related to asset price. The drift rate measures the average rate of growth in the stocks and the volatility measures the standard deviation of the returns;
- The strike price E and maturity date T are parameters related to the details of the option;
- The risk-free interest rate r is a parameter related to the currency in which the asset is quoted.

However in this paper, $V(S,t)$ is used to denote the option value.

Any changes in time dt, the underlying stock prices will change by an amount dS. By assuming that the underlying stock prices follow a lognormal random walk, we have

Punkt in GL

$$dS = \mu S\,dt + \sigma S\,dX. \tag{2}$$

SDGL?

Term X in dX representing the Brownian motion. Itô's lemma act as a "chain rule" in stochastic calculus. Then, by using Itô's lemma, equation (1) becomes

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt. \tag{3}$$

We use Π to denote the value of the portfolio. The portfolio contains of one long option position, $V(S,t)$ and short position in some quantity Δ, delta, of the underlying stock, S. Thus, the value of the portfolio is

$$\Pi = V(S,t) - \Delta S. \tag{4}$$

The change of the portfolio from time t to $t+dt$ is due partly to the change in the option value and partly to the change in the underlying stock

$$d\Pi = dV - \Delta\,dS$$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS. \tag{5}$$

The right hand side of equation (5) contains two types of terms. The first type of terms are called deterministic terms (those with the dt) and the second type of terms are called random terms (those with the dS). The random terms are the risk in our portfolio. To reduce the risk, we choose $\Delta = \frac{\partial V}{\partial S}$ and equation (5) becomes

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{6}$$

We then apply the arbitrage-free principle which states that there is no situation of taking advantage in a price difference between two or more markets. Thus

$$d\Pi = r\Pi dt. \tag{7}$$

Plugging equations (4) and (5) into (7), we have the B-S equation as shown below

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{8}$$

The payoff function for call option is

$$\text{Payoff}\,(S) = \max\,(S - E, 0).$$

Then, the option pricing equation can be written as:

$$E = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{9}$$

where $d_1 = \dfrac{\log\left(S/E\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \dfrac{\log\left(S/E\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$. (10)

Merton Model

Merton Model uses the same formula as B-S formula. However, in Merton model the equity value, E_T of a company is a European call option on the asset of the company, A_T with the strike price equal to the value of the debt, D at maturity. The payment to shareholders at maturity is given by

$$V = \max\ (A_T - D, 0). \tag{11}$$

Based on (9), the value of equity at time $T-t$ is

$$E_T = A_T N(d_1) - De^{-r(T-t)}N(d_2) \tag{12}$$

where

$$d_1 = \frac{\log\left(A_T/D\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{13}$$

and

$$d_2 = \frac{\log\left(A_T/D\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{14}$$

Here $N(d_i)$ where $i = 1,2$ is the cumulative standard normal distribution and $De^{-r(T-t)}$ is the present value of the principal debt due at maturity.

KMV-Merton Model

The KMV-Merton model uses two important equations. The first equation is the Merton's equation (12), while the second equation relates to the volatility of the firm's value, σ_V to the volatility of its equity, σ_E. Following the Merton's assumptions, the value of equity is a function of firm's value and time. Thus, from Itô lemma, we have

$$\sigma_E = \frac{\partial E}{\partial V}.\sigma_V.\frac{V}{E} \tag{15}$$

where $\frac{\partial E}{\partial V} = N(d_1)$ as shown in Merton's model and d_1 is as defined in equation (13) above. Hence, the volatilities of the firms and its equity E are related by

$$\sigma_E = N(d_1)\sigma_V \frac{V}{E}. \tag{16}$$

The distance-to-default d can be calculated using the formula d_2 in equation (14),

$$d = \frac{\log\left(A_T/D\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{17}$$

where μ is stand for the expected return of firm's asset. The formula to estimate the expected default frequency P_t, is defined as

$$P_t = N\left(-\left[\frac{\log\left(A_T/D\right) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]\right) \tag{18}$$

REVIEW ON BLACK-SCHOLES MODEL

In the past four decades, various attempts have been made to modify the original B-S model. Amongst them are Merton [5], Geske [7], Tudela and Young [8], Cathcard and El-Jahel [9], Bharath and Shumway [10] and others. The modifications by them are then summarized in table 1 below, followed by the details of each modification done in the next paragraph.

TABLE (1). Overview on Related Works of Black-Scholes Model

Publications	**Model**			**Work Done**	**Results**
	Merton	**KMV-Merton**	**Bharath & Shumway**		
Geske (1979)	✓			Develop a theory and propose a new formula for pricing of compound options	Results obtain are used to correct some biases of the B-S model
Tudela & Young (2003)	✓			Propose a model by assuming that default can happen anytime up to maturity	Provides information of failure event one year in advance before its occurrence
Cathcart & El-Jahel (2004)	✓			Extend the Merton's model to measure multiple defaults	Produce a unified framework in

					calculation of single and multiple defaults probabilities
Bharath & Shumway (2004)		✓		Introduce a simpler alternative, called naive alternative ‘probability’ in measuring firm’s asset and its volatility	Conclude that KMV-Merton model does not produce enough statistic for the probability of default
Majumder (2006)	✓			Modify the Merton’s model by considering the source of uncertainty in equity price includes the uncertainty in the firm itself plus uncertainty in market sentiment	Modified version widen the scope, for example from the strict efficient market to the inefficient market
Feng & Xiao (2009)	✓			Three modifications done to the original Merton’s model: (a) equity price change because of uncertainty in firm itself plus uncertainty in market sentiment, (b) allow cash dividend payment and (c) default boundary is not constant	The modified version can perform well in non-efficient market in evaluating credit risk
Kung & Lee (2009)	✓			Derive new formula by assuming that short rate are not constant during the life of the option	Based on numerical result, the authors indicates that the original Black- Scholes model overvalues for out-of-the-money calls, moderately overvalues for at-the-money calls and slightly overvalues for in-the-money calls
Xu (2012)	✓			Adjust the default point of KMV-Merton model from short-term debt plus half long-term debt to short-term debt plus 10% of long-term debt	The new assumption of default point is more suitable in Chinese cement industry and provide a way on how to calculate the value of the limit sell shares
Norliza & Maheran (2012)		✓		To consider cases like when one of the value such as the firm’s asset, the firm’s liability or firm’s asset volatility equal to zero	Formulae obtain are tested and the results produce are equivalent with the analysis done by the KMV-Merton model
Charitou et al. (2013)			✓	Extend the Bharath and Shumway’s idea by estimating volatility directly from market-observable returns on firm value	The B-S model performs better than previous modified models

Compound options or options on options is a method in valuing many corporate liabilities. Thus, Geske [7] develops a theory and derives the new formula for call and put in pricing the compound options. Besides, the author also introduces the leverage effects into these new formulas. In dealing with compound options, the variance rate depends on the firm's value. Hence, the assumption of the B-S model that variance rate is constant cannot be used. The results obtained show that the B-S model underpriced out-of-the-money and overpriced in-the-money options.

The original Merton model assumes that default occurs only on the maturity date. Tudela and Young [8] disagree with this assumption by doing a modification on the current Merton model and suggest that the default may happen at anytime up to maturity. It means that default can happen at the first time when the value of the firm's debt is greater than the value of the firm's asset. As an alternative, the authors use the concept of "Barrier option" instead of call option in measuring the firm's value. Barrier option is an option where the value of the payoff depends on whether or not the price of the underlying assets reached a certain level of price during a certain period of time. In measuring the probability of default, authors use only publicly available information on market prices and time series estimates of parameters. Three types of test that have been done are by evaluating their model based on the actual default, comparing the performance of their model with other default models and measuring the model using statistical method.

Most of the studies used structural approach to evaluate the probability of default and credit default swap spread. Cathcart and El-Jahel [9] extend the structural approach and produce a model that can measure the multiple defaults. The most important thing that must take into consideration in measuring the multiple defaults is the correlation between the involved firms. Among the advantages of this model are a new framework is proposed in calculating the single and multiple default probabilities, results produce can be a benchmark to other generalizations model and etc. The formula for calculating the default correlations and multiple default probability is proposed by Zhou [11]. Many applications in asset pricing and risk management use default correlation analysis.

Under KMV-Merton model, the firm's asset and its volatility are not directly observed. These values can be accessed from the equity's value, its volatility and other observable variables by solving two nonlinear simultaneous equations, for example equation (12) and (15) from above derivation. Bharath and Shumway [10] introduce naive alternative in their study. The authors construct the naive approach with two objectives such as naive alternative can perform as well as KMV-Merton and naive alternative can be a simple alternative 'probability' that does not require solving the simultaneous equation. Using naive alternative, option variables are observable through the market. The value of the firm is estimated as a sum of the market value of equity and the face value of debt. Hence, the volatility is estimated based on the specific relation between the volatility of debt and equity. They use naive probability in the KMV-Merton structure for calculating distance-to-default and expected default frequency.

Current Merton's model assumes that the factors affect the firm's equity is the uncertainty in the firm itself such as rise/fall of sale, firm's performance and etc. However, this assumption only holds when market is efficient. Unfortunately, in real situation, this assumption is not true since markets are far from being efficient. Therefore, using Merton's model in a non-efficient market will generate misleading results. To capture this problem, Majumder [12] makes an extension of the Merton's model. He considers the factors that affect the firm's equity value are not only the uncertainty in the firm itself, but it includes the factors from market condition. Through this study, the author assumes that the firm's equity is affected by combinations of uncertainty from the firm plus uncertainty from market sentiment. The modified version will necessarily widen the scope of equity based, from efficient market to non-efficient market. We agree with this modified version since in a real world situation, market condition is not predictable. Thus, in getting the correct value, all factors should be considered.

Later, the study by [12] followed by Feng and Xiao [13]. In addition, [13] make other modifications such as by allowing for cash dividend payment, and assuming that the default time may happen at any time. The result based on modified Merton shows that the expected probability of default is closed to the actual default probability.

During the option valuation, the value of short rate is assumed to be constant. However, the value of the short rate may change from time to time and this assumption seems to be unrealistic. To capture this type of problem, the modification on Merton's model has been done by Kung and Lee [14]. The authors derive new formula in valuing European call and put options and claim that Black-Scholes model prices when compared to market prices are overvalues for out-of-the-money calls, moderately overvalues for at-the-money calls and slightly overvalues for in-the-money calls. They conclude that this method can be extended to American calls on no-dividend-paying stocks and European puts by virtue on put-call parity. Further research should be done regarding this modification since another derivation by Cui and Mcleish [15] produces different result from [14], and conclude that B-S model undervalues for out-of-the money, at-the-money and in-the-money calls.

Default point suggested by KMV-Merton model happens at short term debts plus half of long term debts. Xu [16] proves that this suggestion is not suitable to be applied in cement industries. He suggests that the default point for this industry should be short term debts plus 10% of long term debts. Besides, this paper also considers the changes that can happen if the non-tradable shares change to limit sell shares. Based on the analysis done, the author gives few suggestions such as, a model to calculate the distance-to-default of the non-listed firms should be proposed, the improvement should be done on the way to calculate the non-tradable shares and the KMV-Merton model should be modified since the actual volatility might not fit in the current model since the original assumption is that the volatility of the stock's price obeys normal distribution. More data should be applied since the result might not be accurate if they only apply it to four firms.

KMV-Merton model is known as one of the widely used model in calculating the distance-to-default of the firms. Nonetheless, there is some unusual case that cannot be determined using KMV-Merton model like when one of the value such as the firm's asset, the firm's liability or firm's asset volatility equal to zero. Norliza and Maheran [17] address this problem in their study by doing a modification on the original KMV-Merton model. The formulae are then tested on two Malaysian companies and the results produced are the same as the analysis done by KMV-Merton model. Hence it can be concluded that these formulae are true.

Charitou et al. [18] extend the idea recognized by [10] in estimating volatility. Using Bharath and Shumway model, the volatility is estimated based on specific relation between equity and the volatilities of debts. However, based on [18] the volatility is estimated directly from the market. Results show that it is not necessary to solve the Merton simultaneous equations in getting accurate predict default. Besides, this model shows an improvement in its ability to forecast default.

CONCLUSION

The studies on the B-S model have been the motivation behind this review paper. This paper provides an overview of the B-S model focusing on the derivation and modifications done by the researchers. Since the B-S model is still relevant in finance world today, it is a major challenge for researchers to look for new solutions or models that considers all the limitations involved in the current B-S model in producing better estimation or analysis. To assist them, this review paper presents a gap analysis for researchers to come out with new improvement in the B-S model.

ACKNOWLEDGMENT

Our sincere appreciation is directed to the Ministry of Education, Malaysia and Universiti Teknologi MARA, Shah Alam, Selangor, Malaysia. This research is funded by Fundamental Research Grant Scheme 600-RMI/FRGS 5/3 (65/2013).

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