Stetige Wavelet Transformation

Prof. Dr. Andreas Müller

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Plan

1. Skalarprodukt

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2. Transformationen: Translation und Dilatation

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3. Stetige Wavelet-Transformation

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4. Zulässigkeitsbedingung

Skalarprodukt

Abgetastete Signale:

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Definition

Die quadratintegrierbaren Funktionen $L^2 = \{f\colon \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(t)|^2\,dt < \infty\}$ bilden einen Vektorraum mit dem Skalarprodukt (1).

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Chauchy-Schwarz-Ungleichung

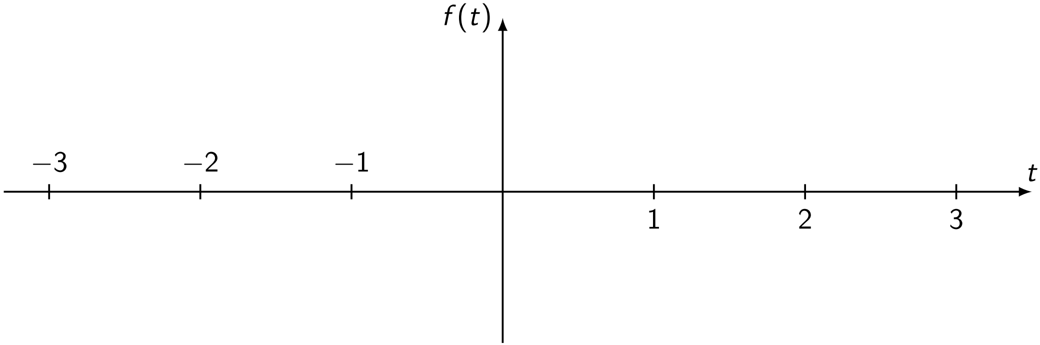
Für $f, g \in L^2(\mathbb{R})$ gilt

$$\langle f, g\rangle \le \|f\| \cdot \|g\|$$

mit Gleichheit genau dann wenn f und g linear abhängig sind.

Gabor-Wavelet

Sinus-Signal lokalisiert mit Exponentialfunktion



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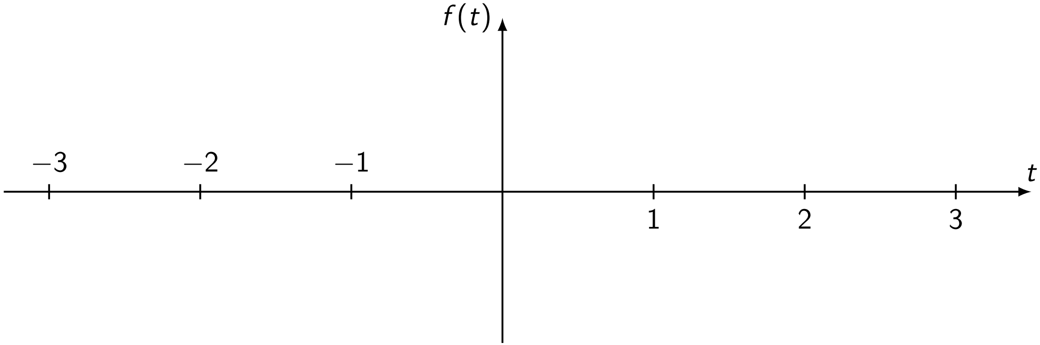
$$\psi(t) = \sin(8t) \cdot e^{-t^2}$$



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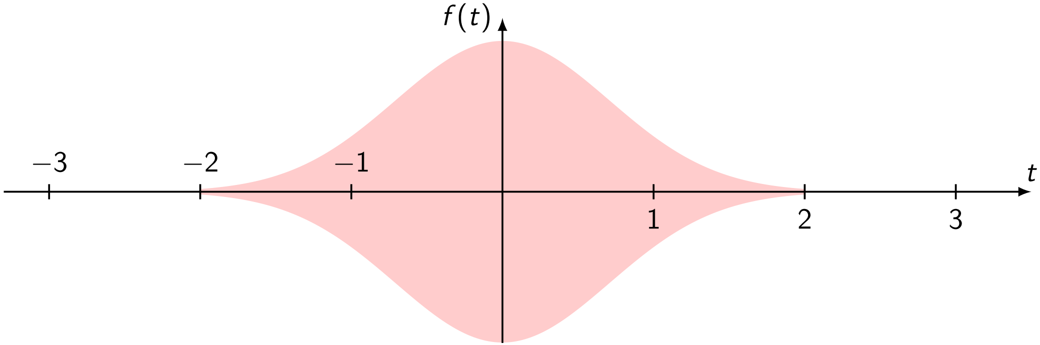

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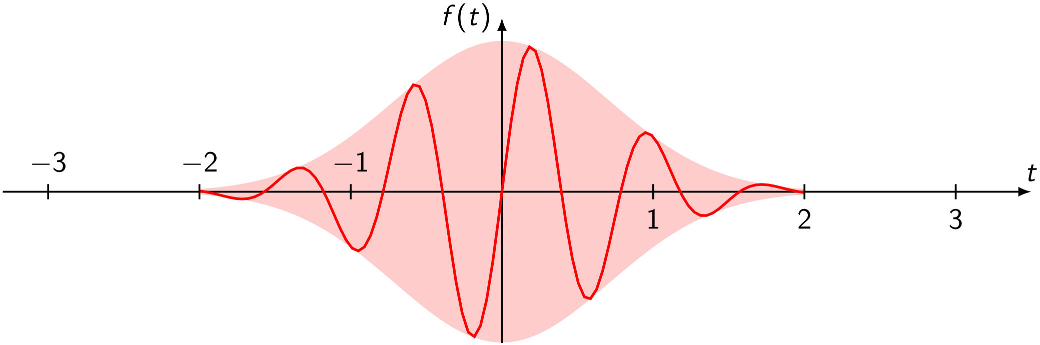
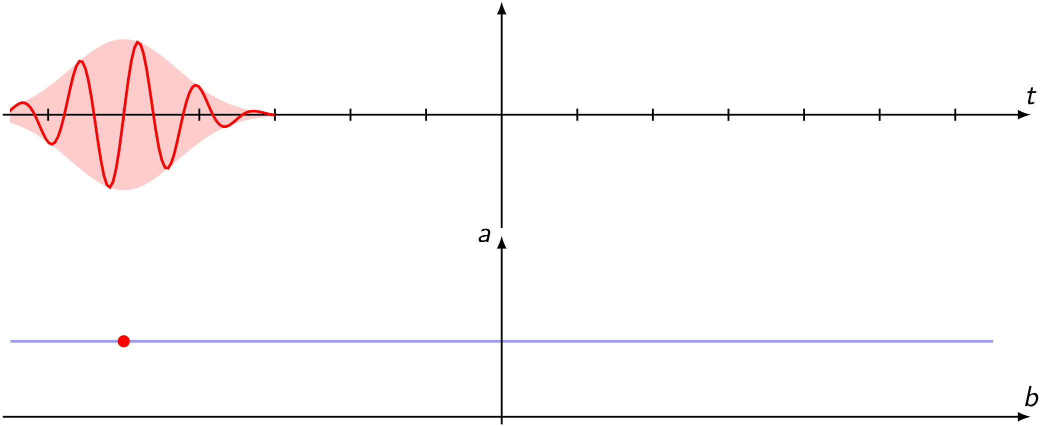


Translation

$$T_b\psi(t) = \quad T_{-5.0}\psi(t) = \psi(t+5.0)$$



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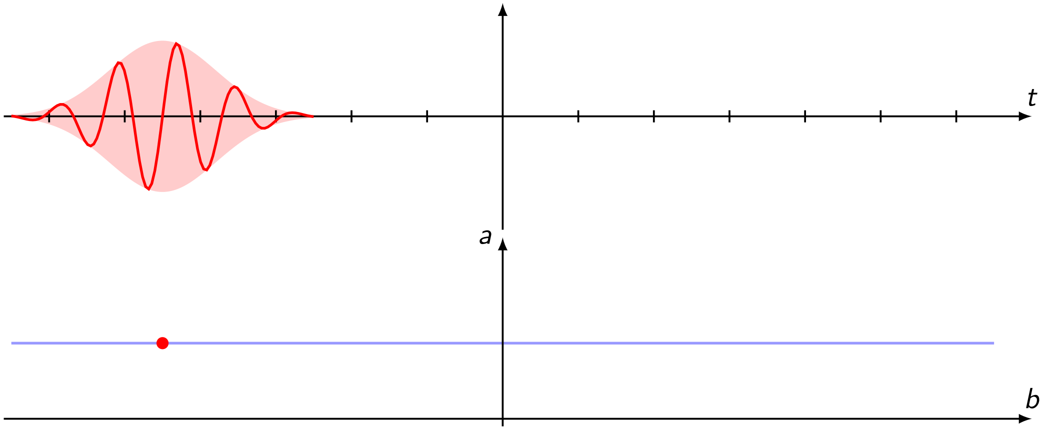
$$T_b\psi(t) = \quad T_{-4.5}\psi(t) = \psi(t+4.5)$$



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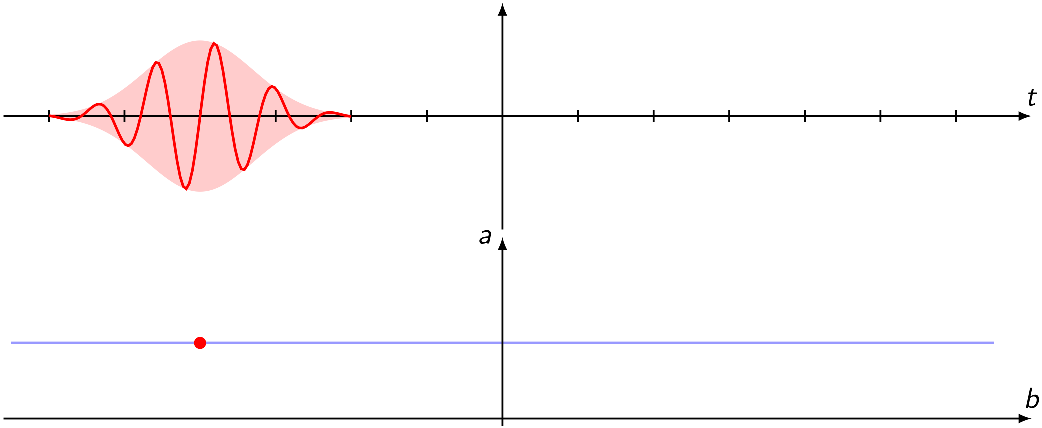

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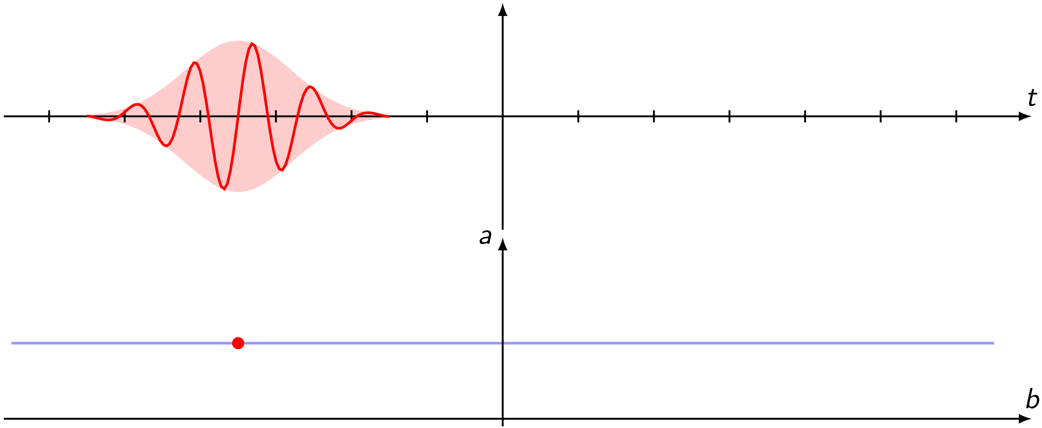

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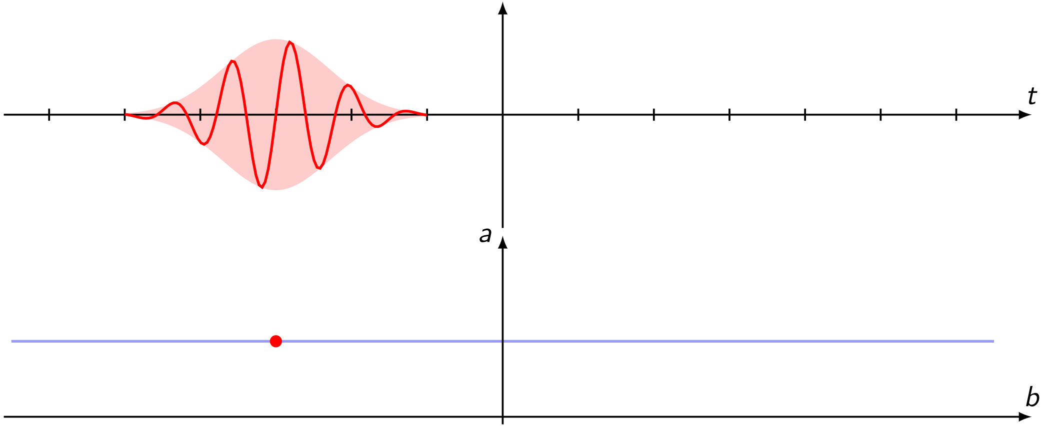

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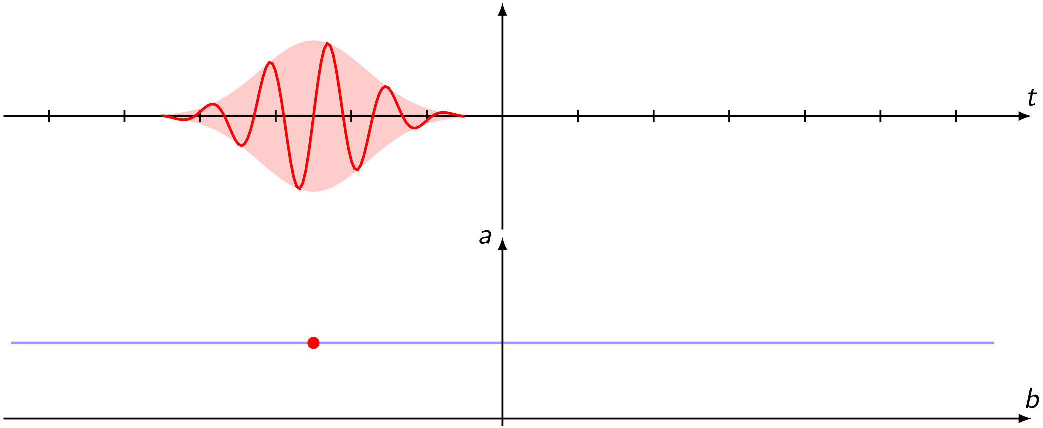

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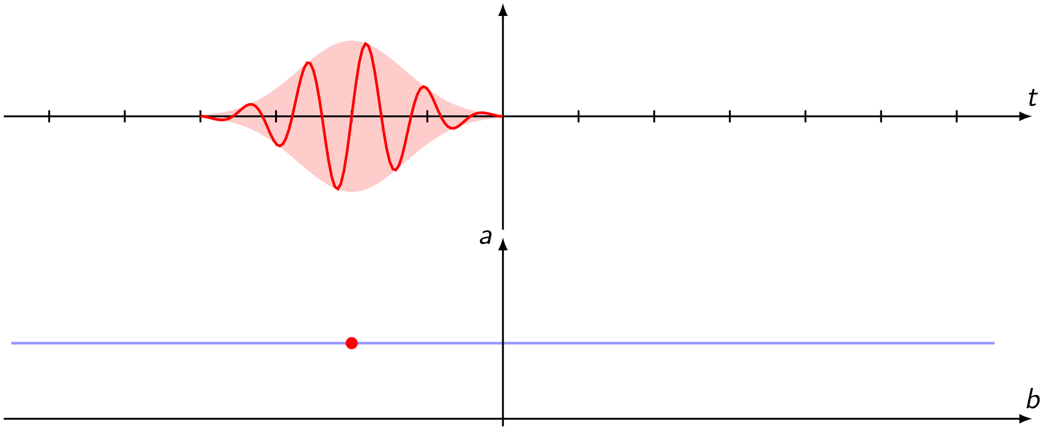

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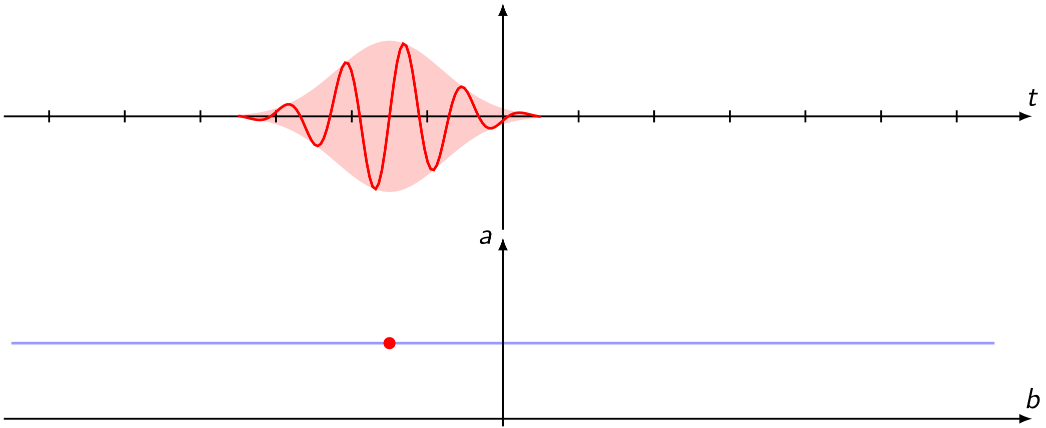

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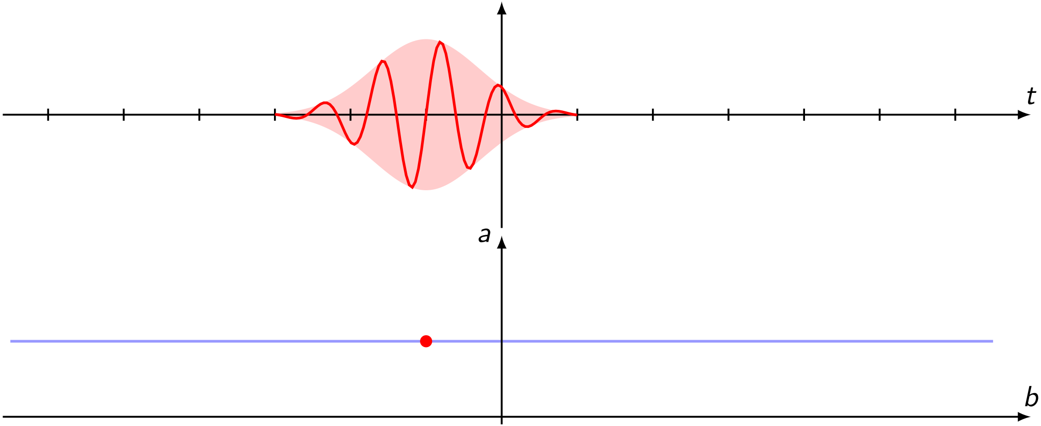

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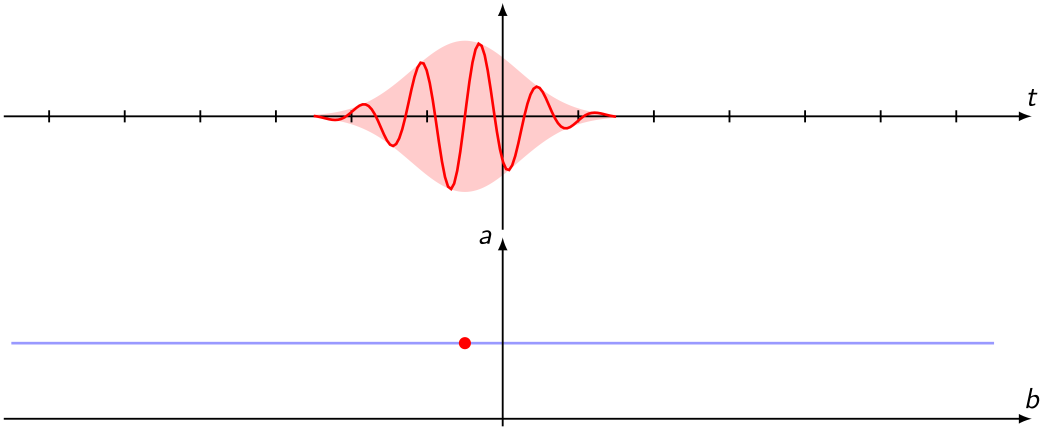

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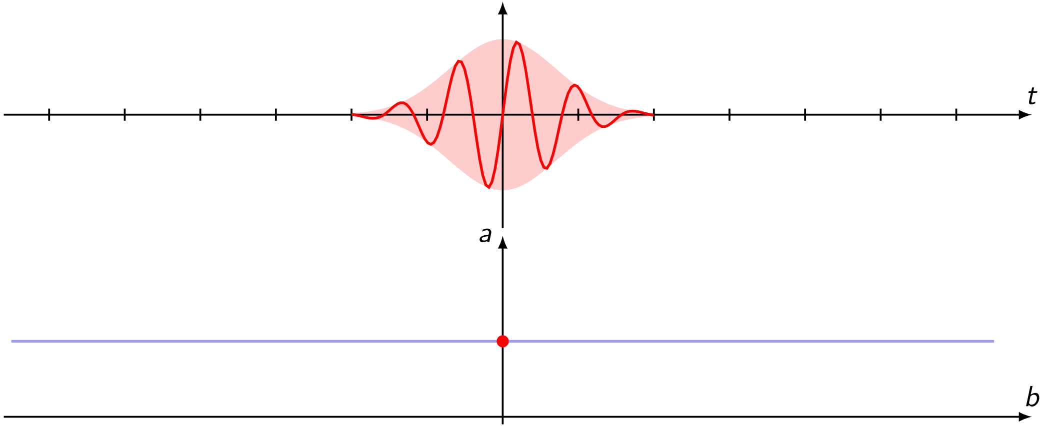
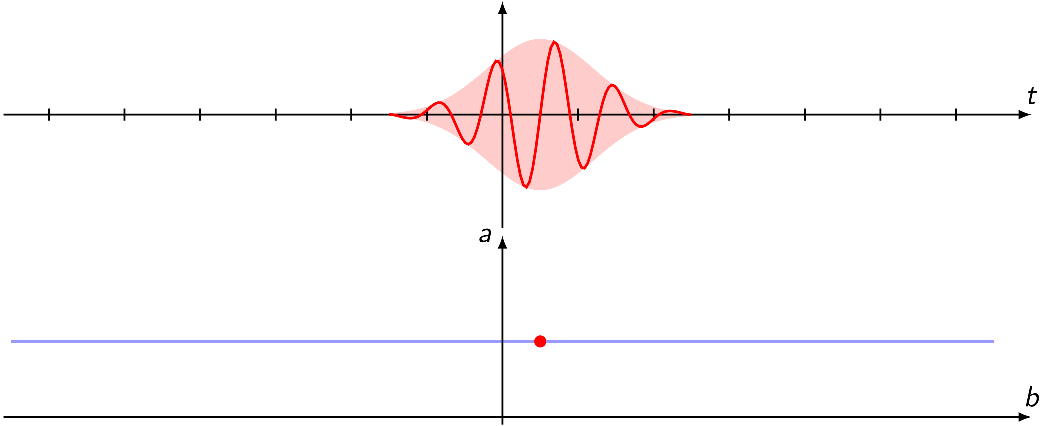


Translation

$$T_b\psi(t) = \quad T_{0.5}\psi(t) = \psi(t-0.5)$$



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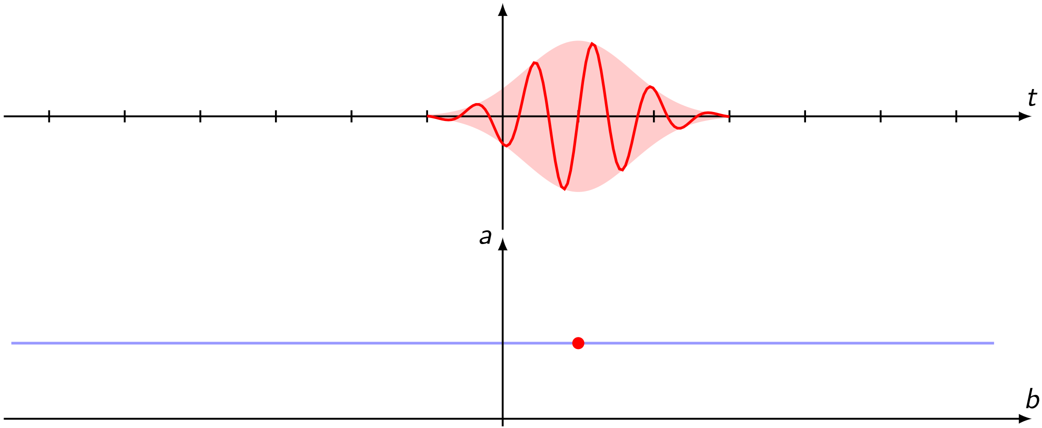

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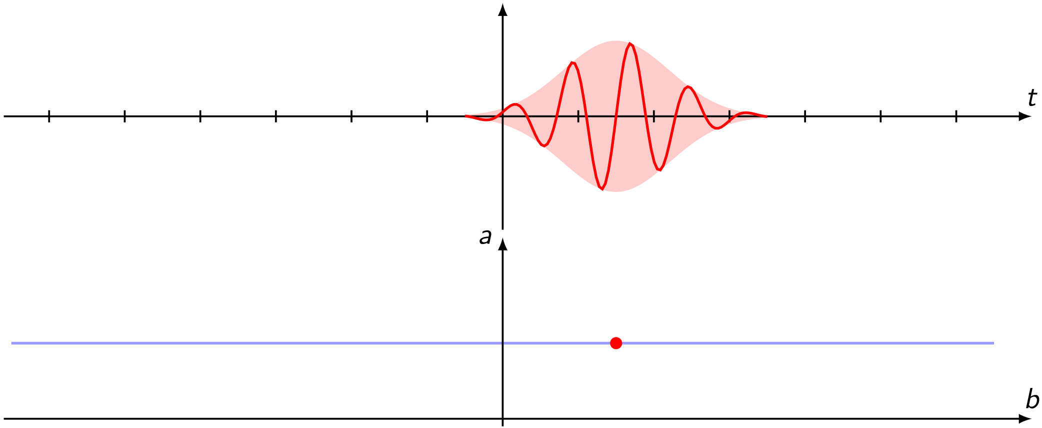

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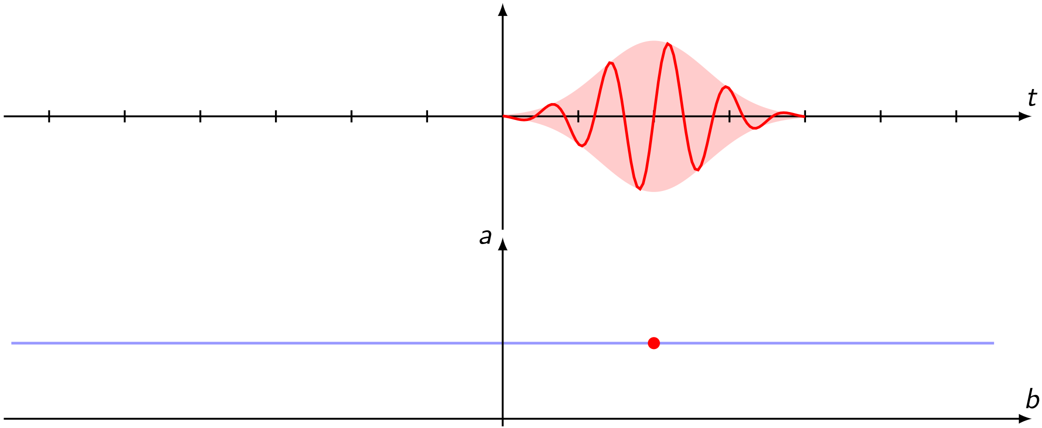

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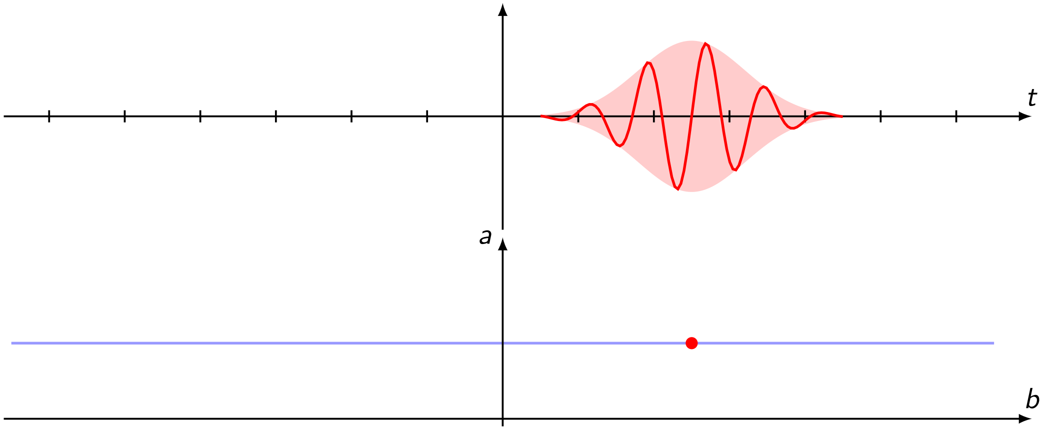

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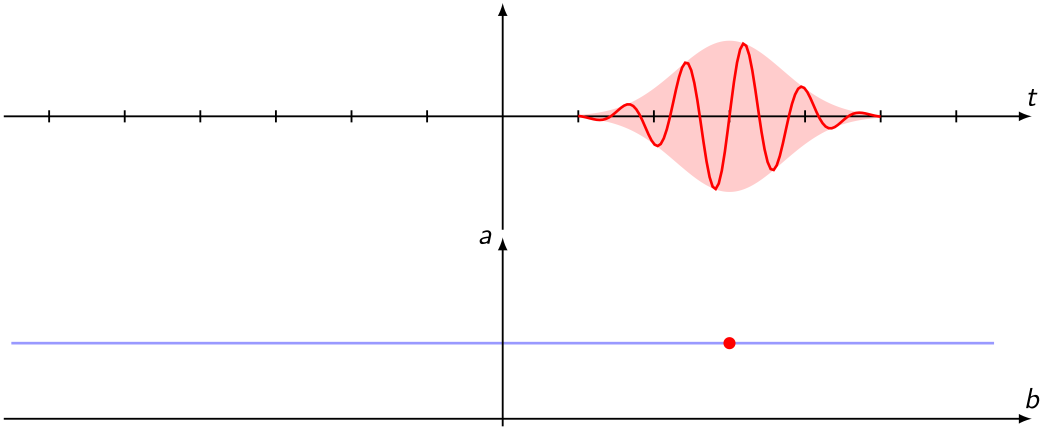

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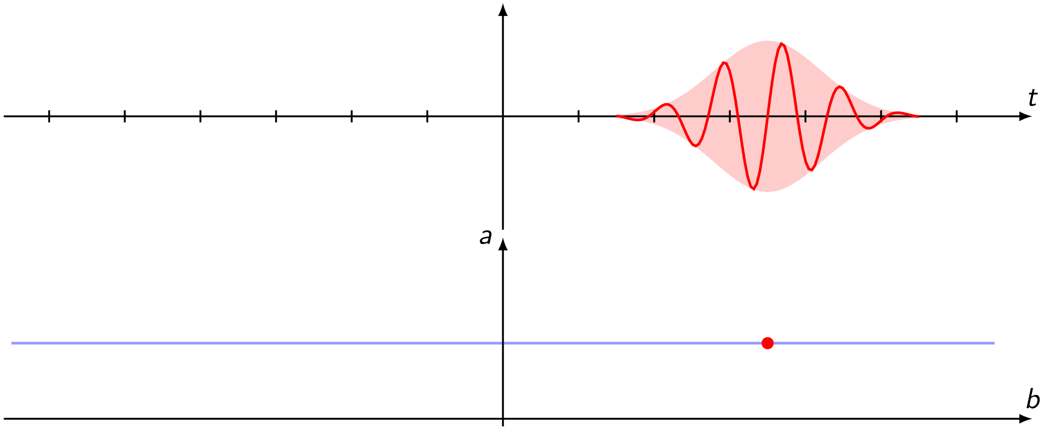

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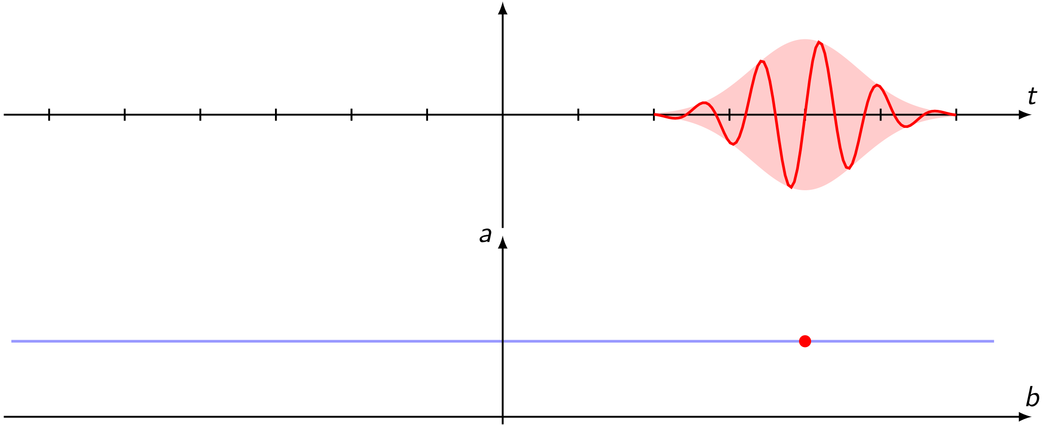

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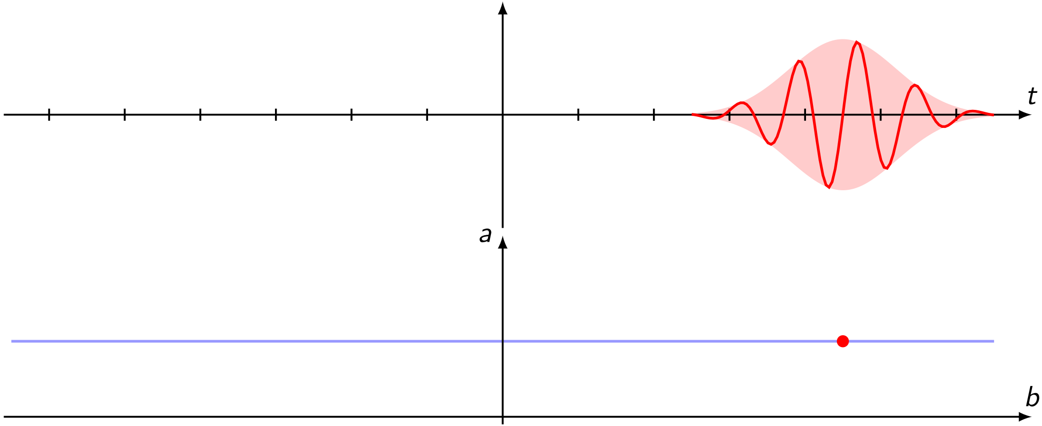

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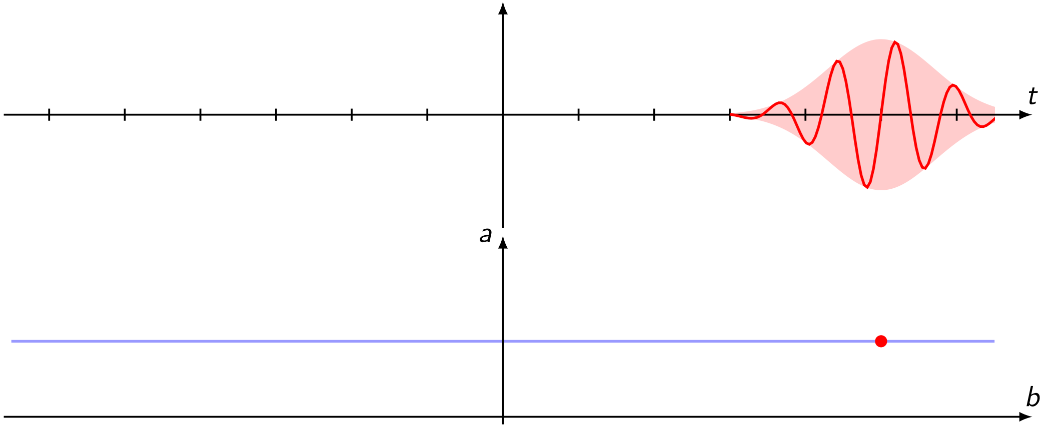
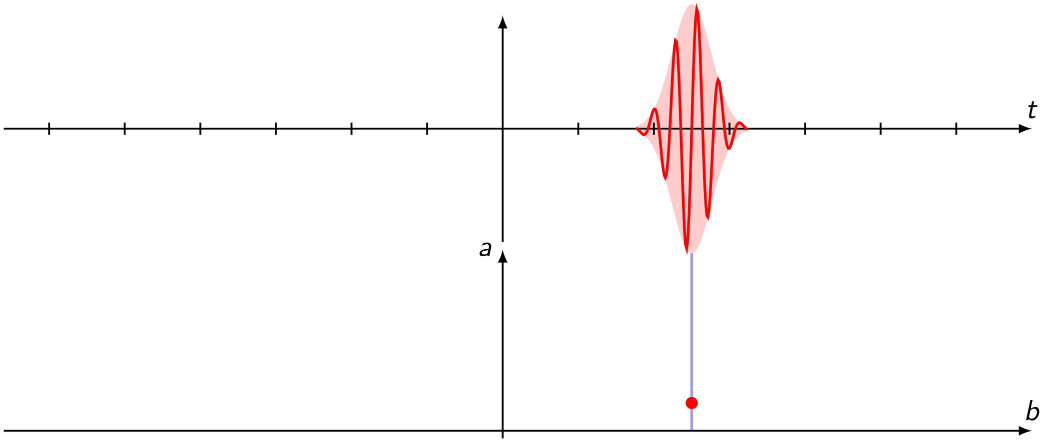


Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{0.36798}\psi(t) = \psi((t-2.5)/0.36798)$$



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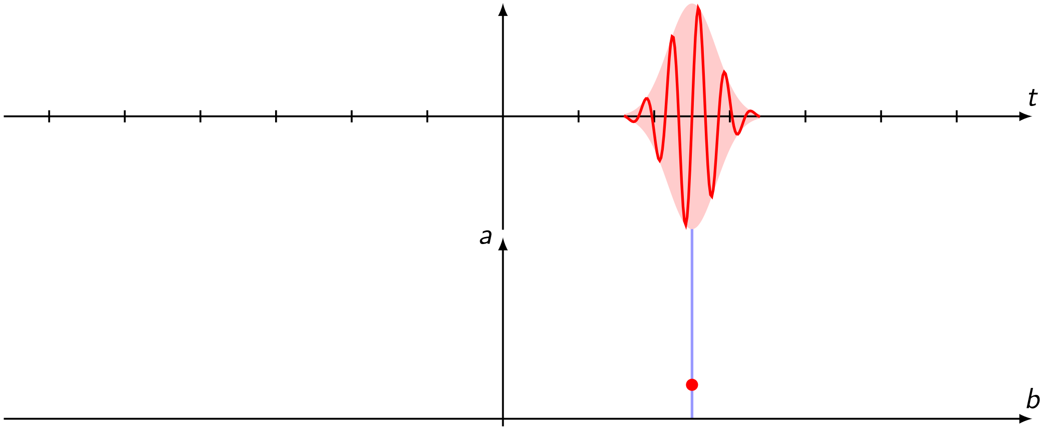
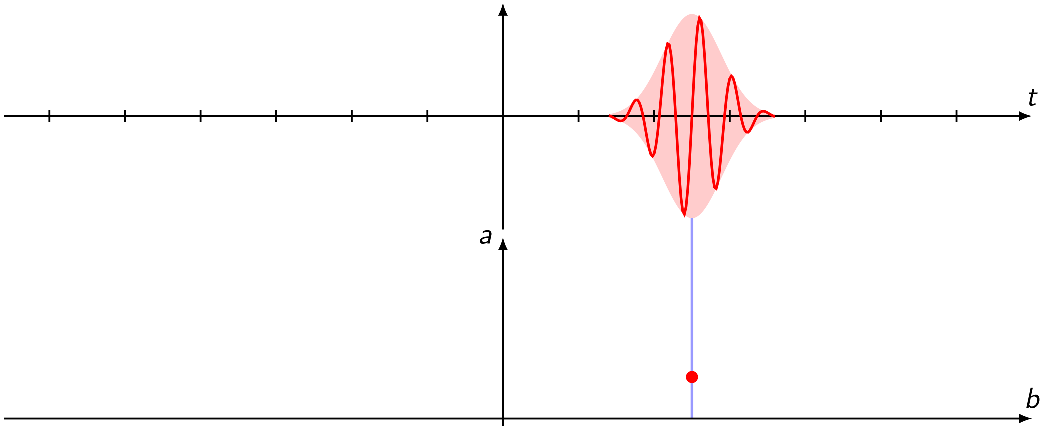
$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{0.4494}\psi(t) = \psi((t-2.5)/0.4494)$$



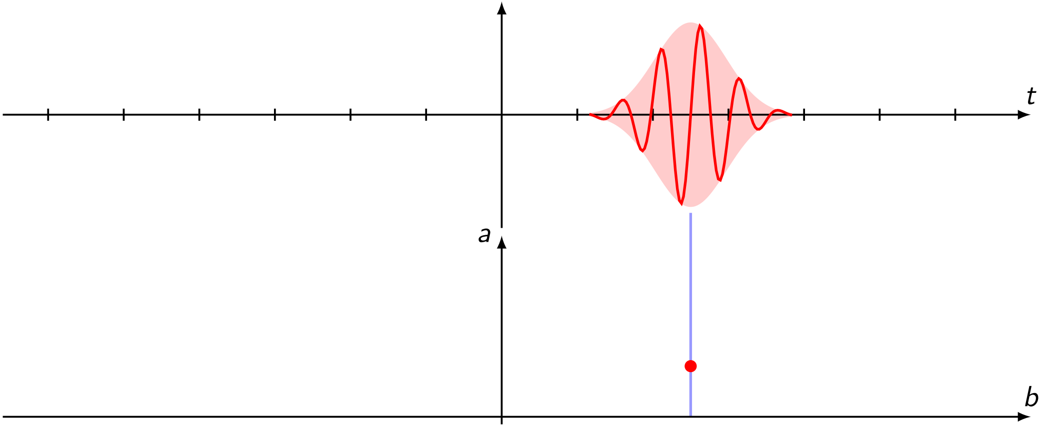
Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{0.54892}\psi(t) = \psi((t-2.5)/0.54892)$$



Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{0.67046}\psi(t) = \psi((t-2.5)/0.67046)$$



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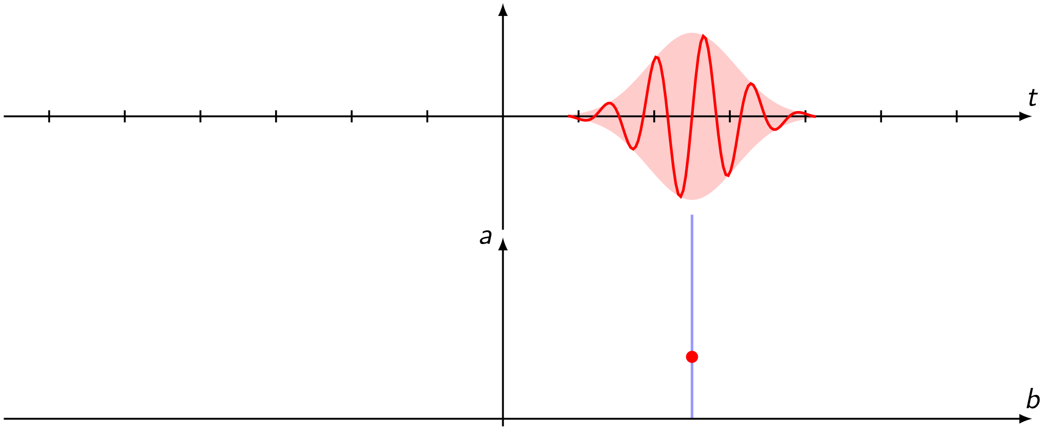
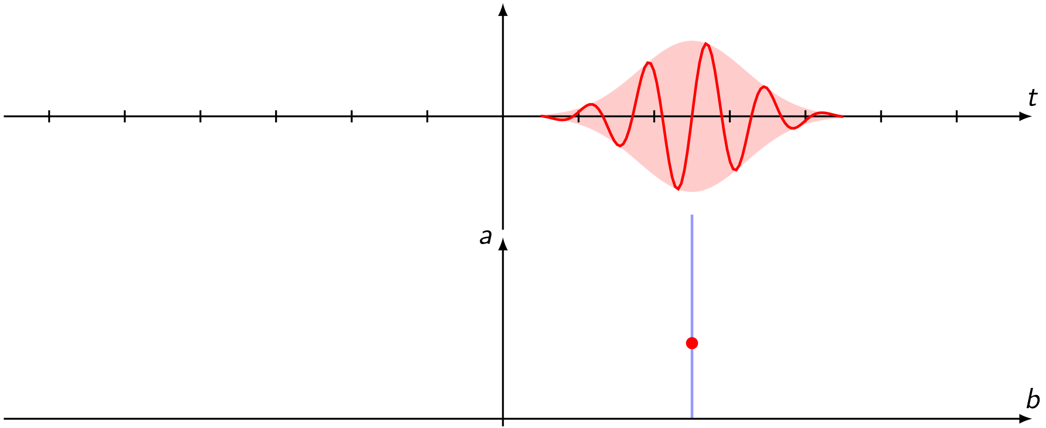
$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{0.8189}\psi(t) = \psi((t-2.5)/0.8189)$$



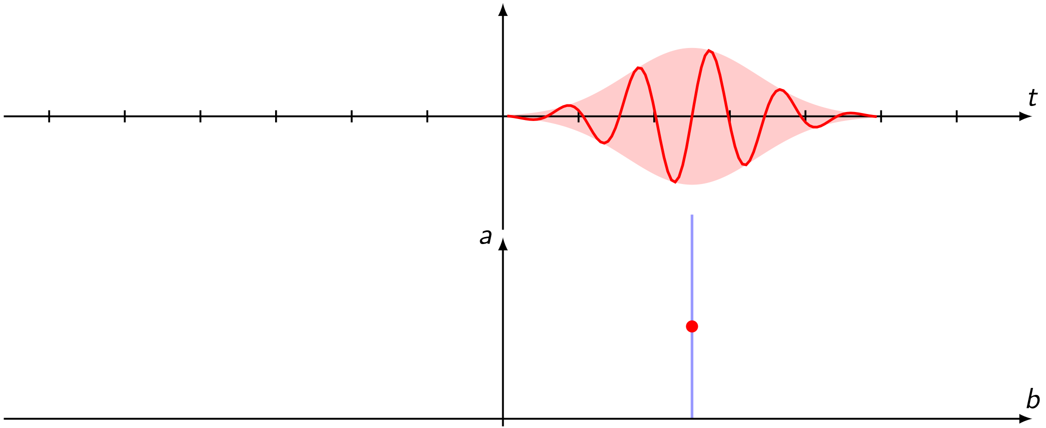
Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_1\psi(t) = \psi(t-2.5)$$



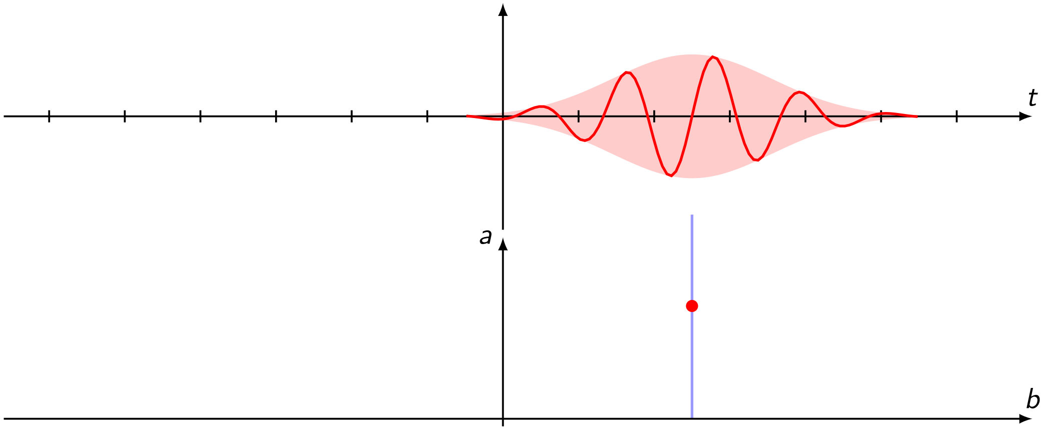
Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{1.22137}\psi(t) = \psi((t-2.5)/1.22137)$$



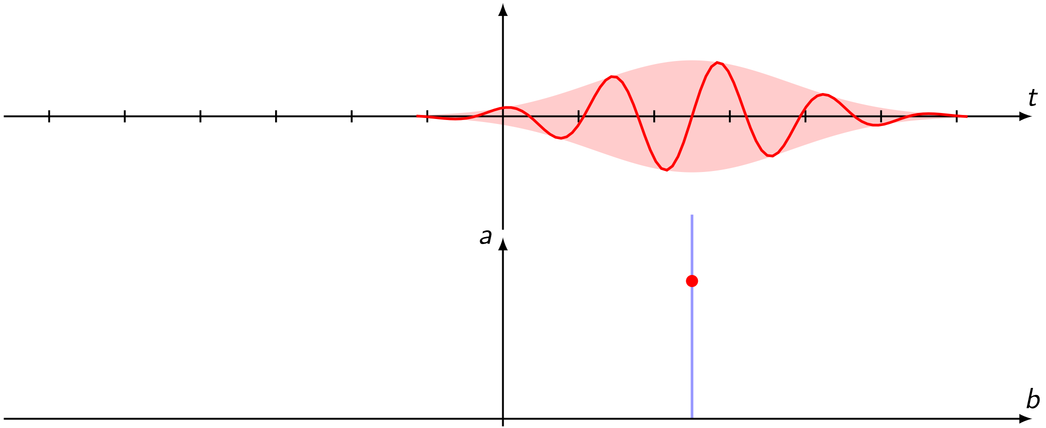
Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{1.49176}\psi(t) = \psi((t-2.5)/1.49176)$$



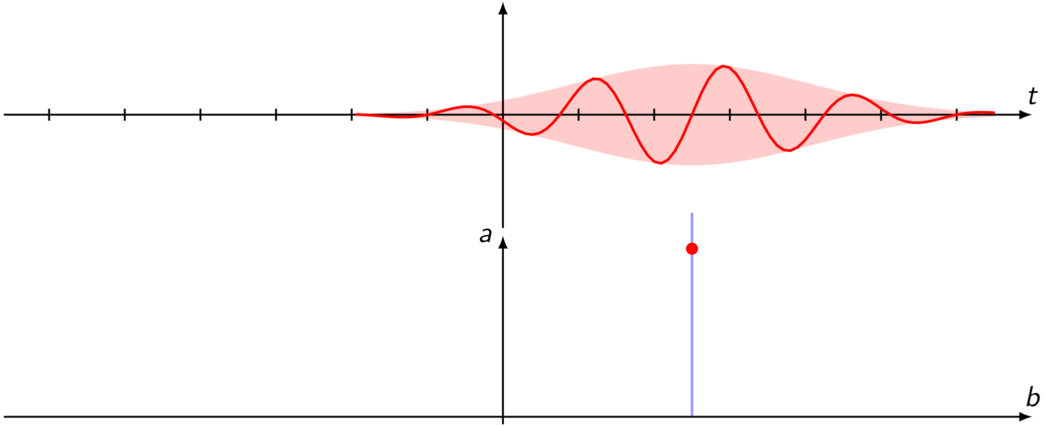
Dilatation

$$T_{2.5}D_a\psi(t) = \quad T_{2.5}D_{1.822}\psi(t) = \psi((t-2.5)/1.822)$$



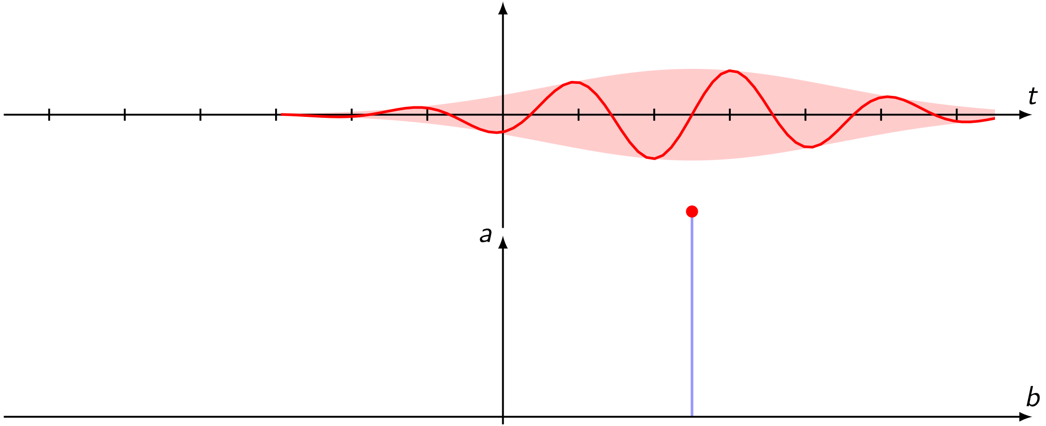
Dilatation

$$T_{2.5} D_a \psi(t) = \quad T_{2.5} D_{2.22542} \psi(t) = \psi((t-2.5)/2.22542)$$



Dilatation

$$T_{2.5} D_a \psi(t) = \quad T_{2.5} D_{2.71806} \psi(t) = \psi((t-2.5)/2.71806)$$



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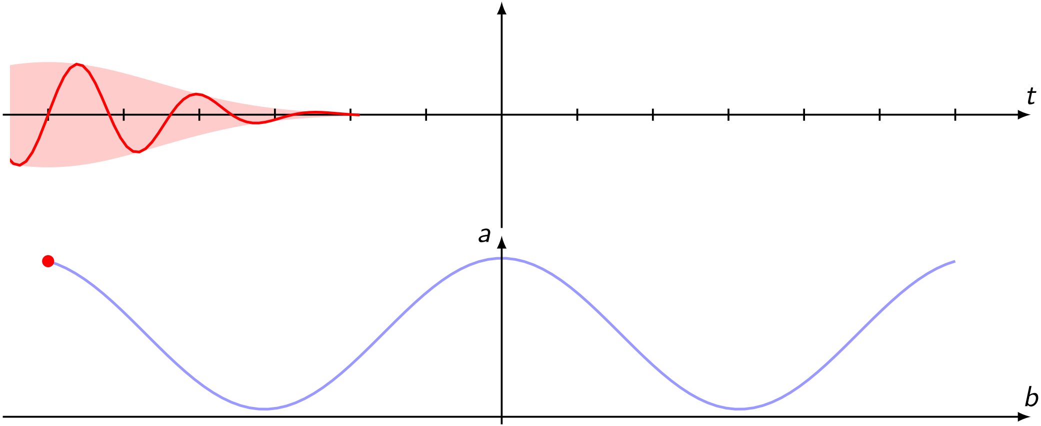

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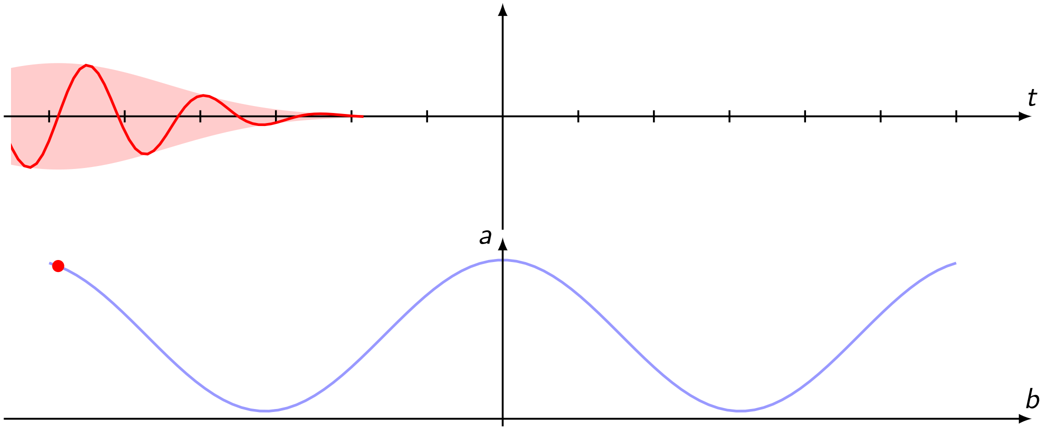

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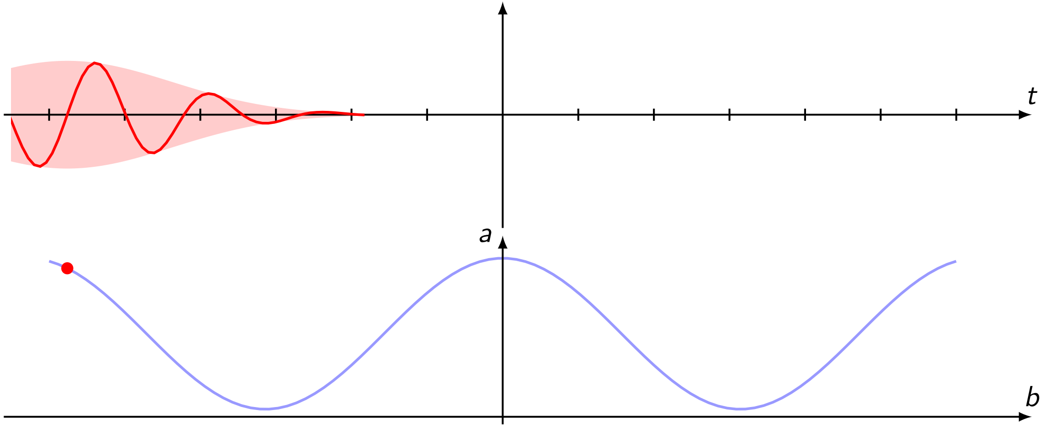

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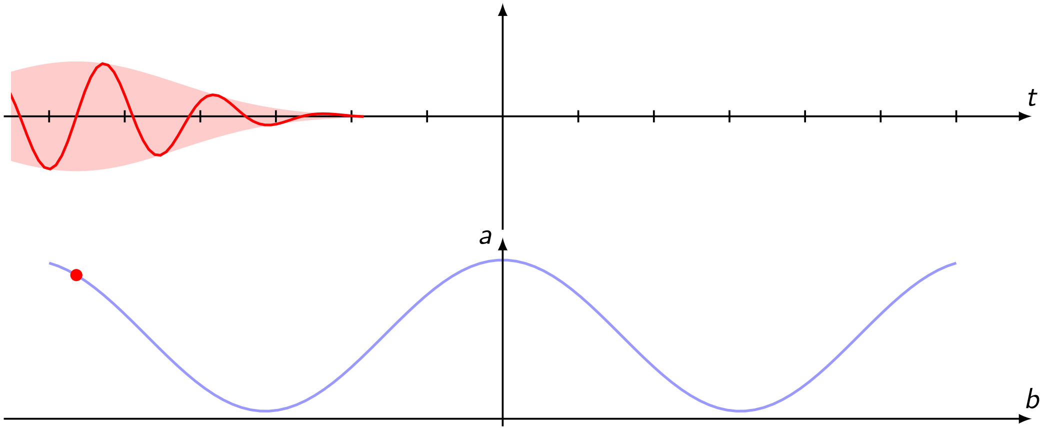

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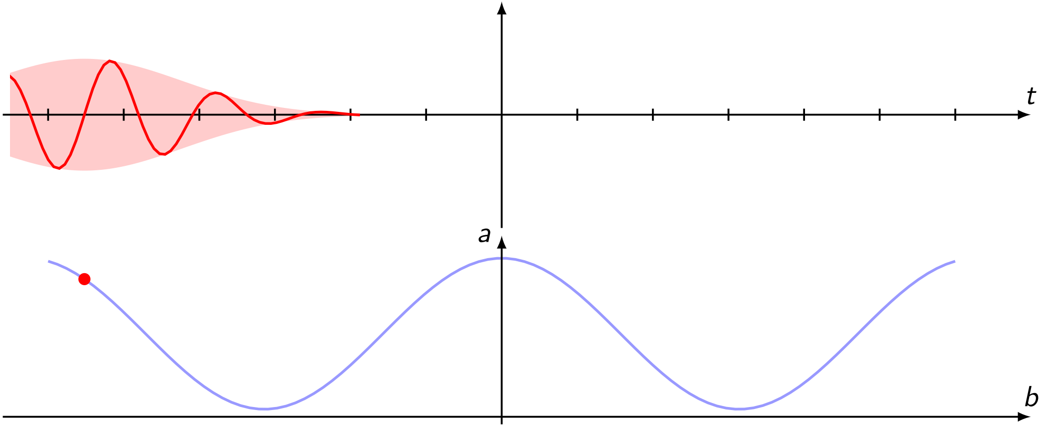

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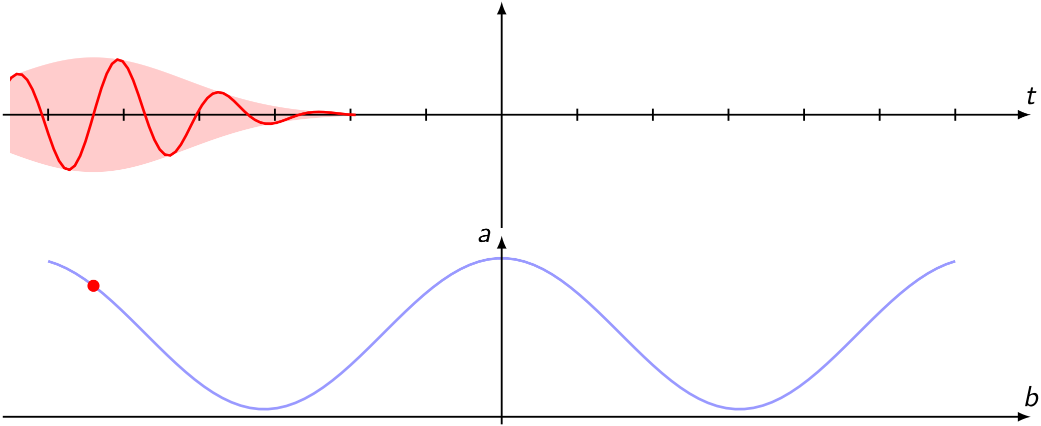

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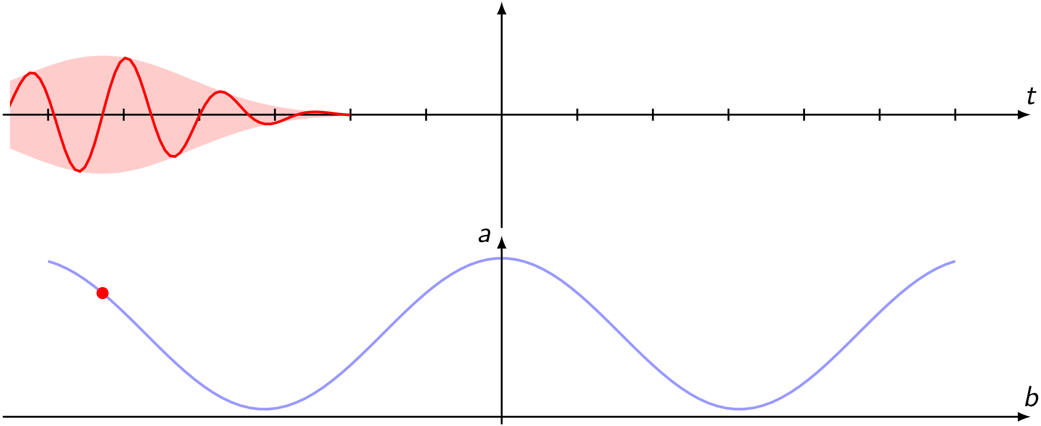

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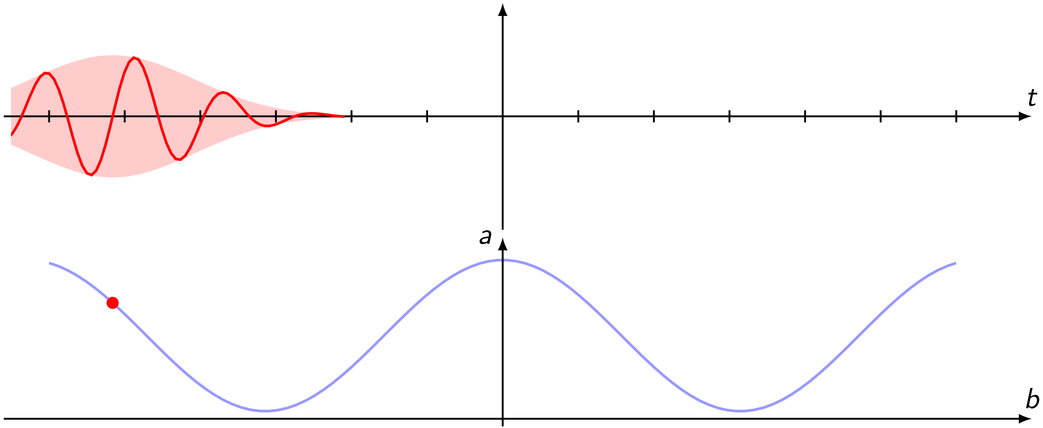

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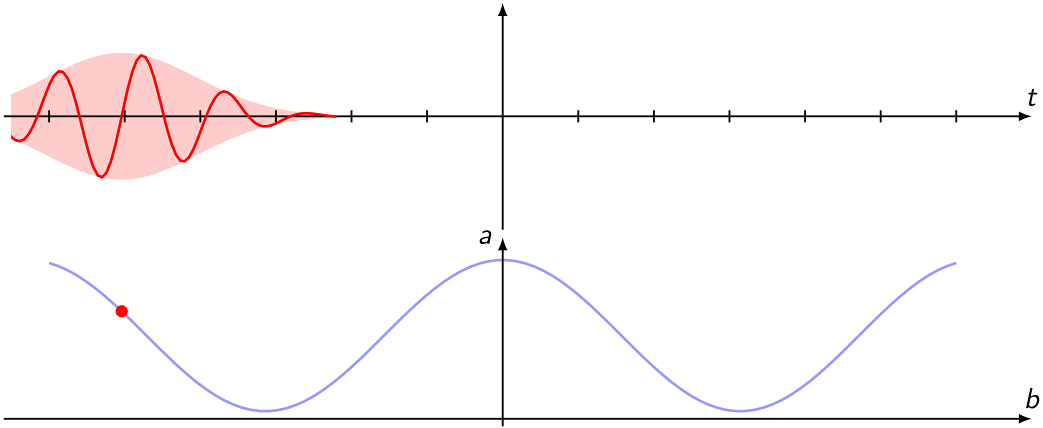

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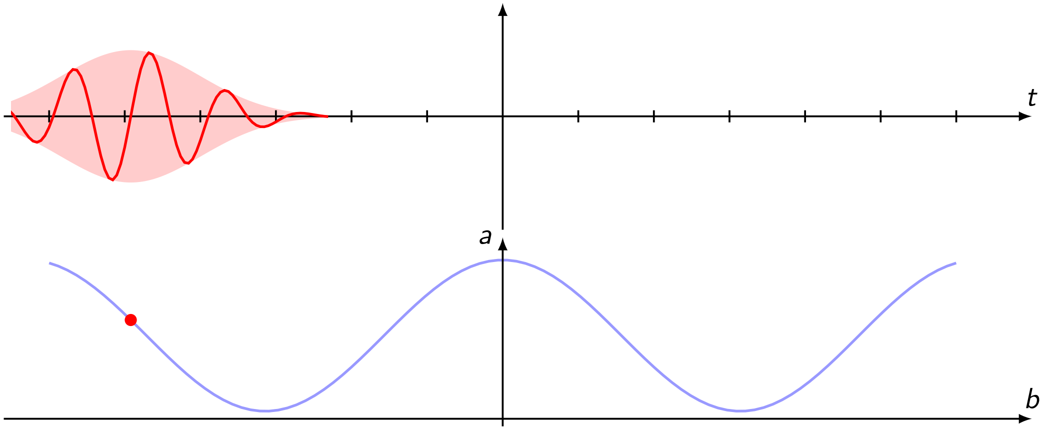

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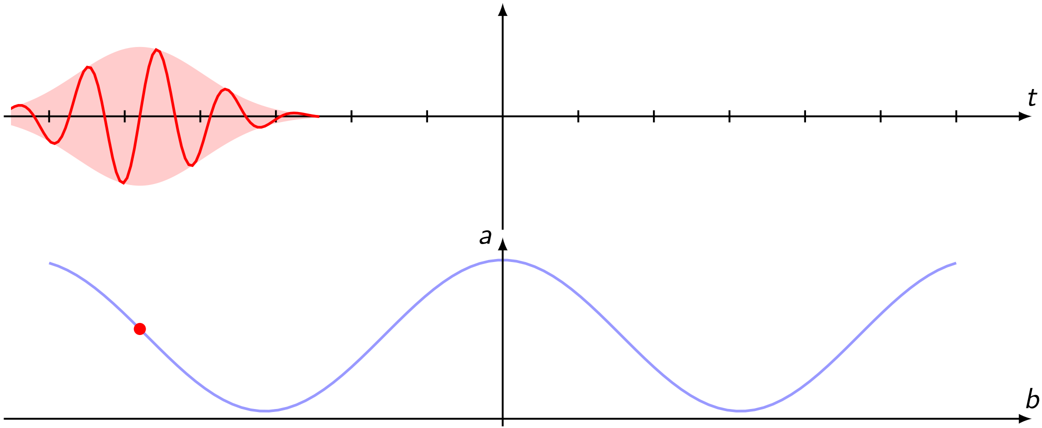

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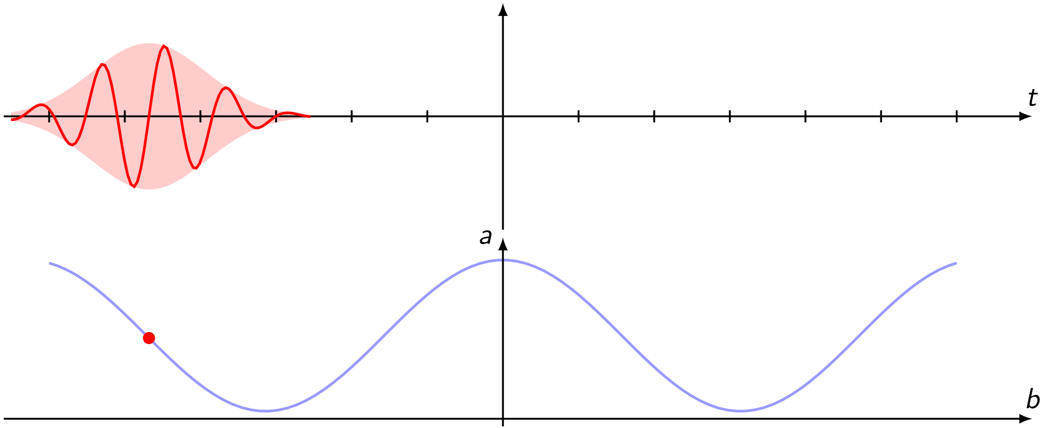

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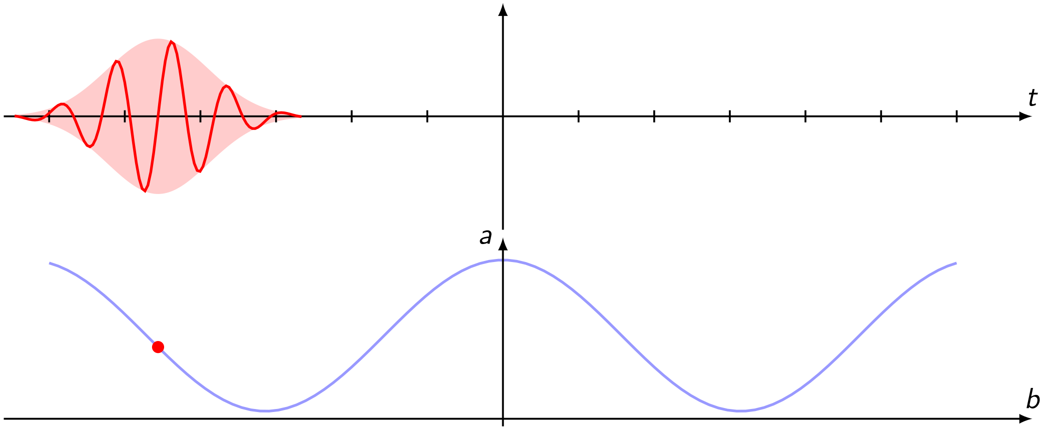

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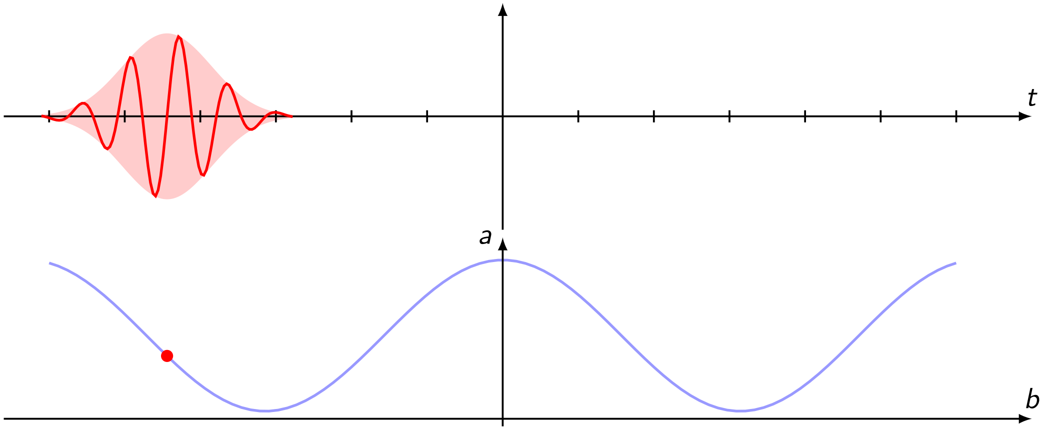

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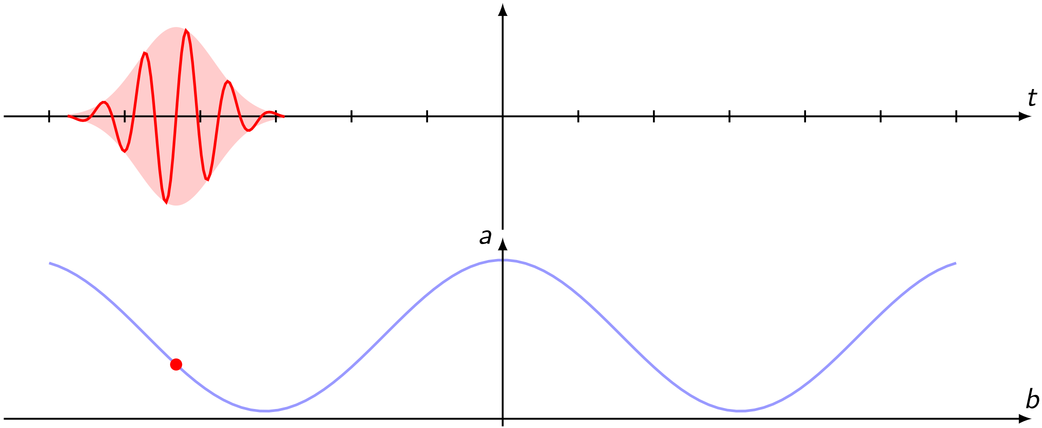

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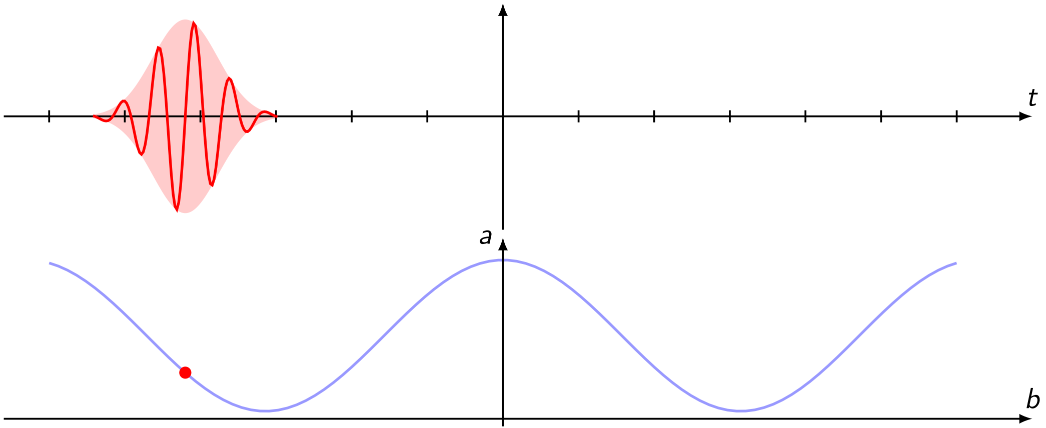

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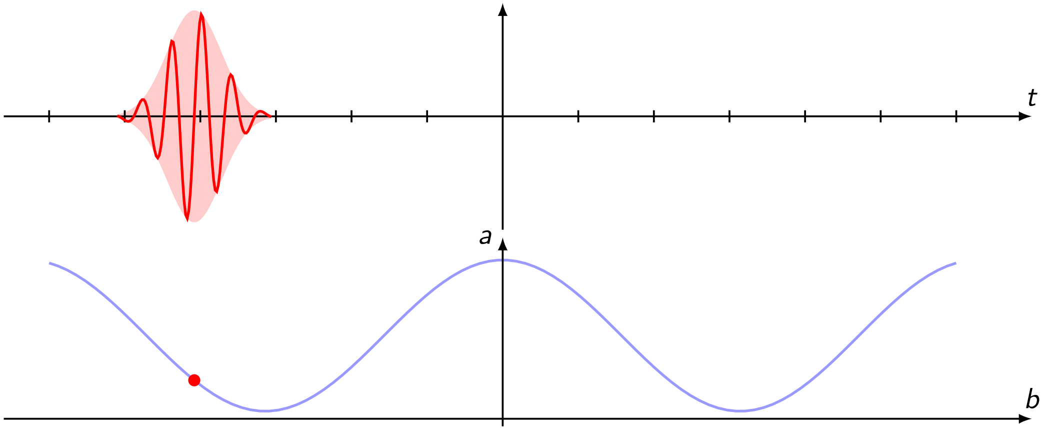

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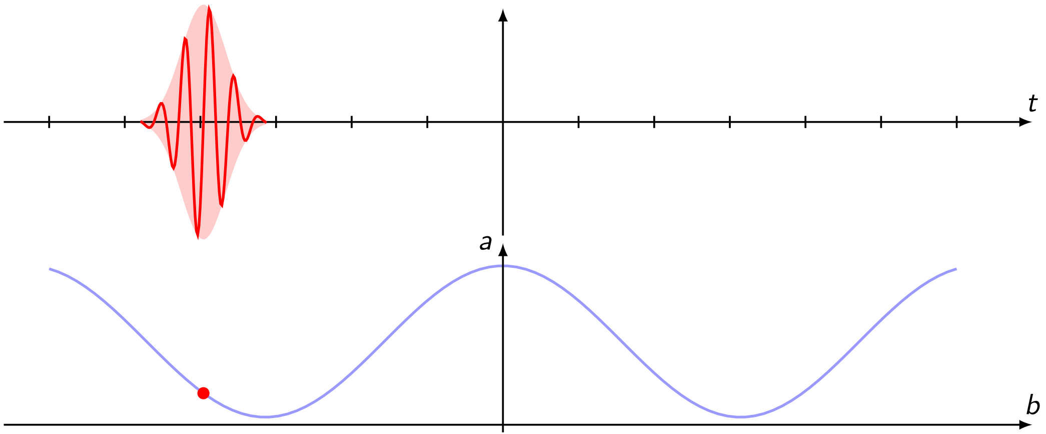

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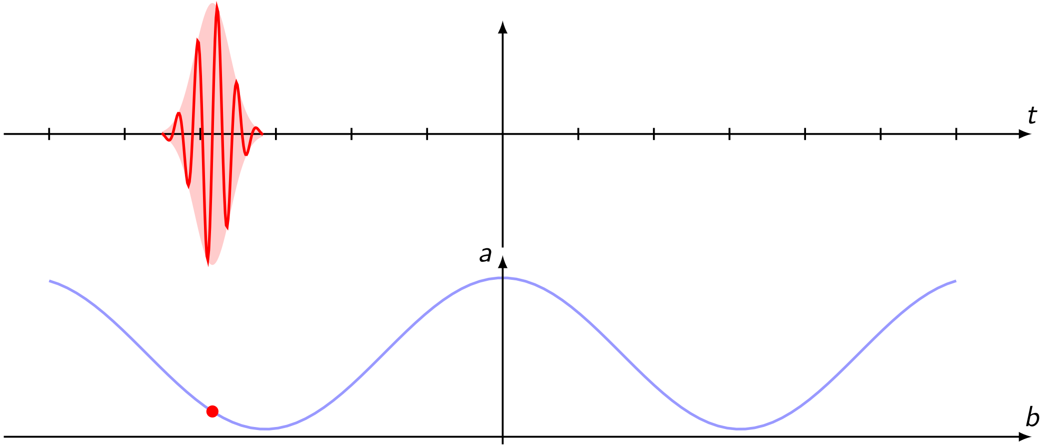
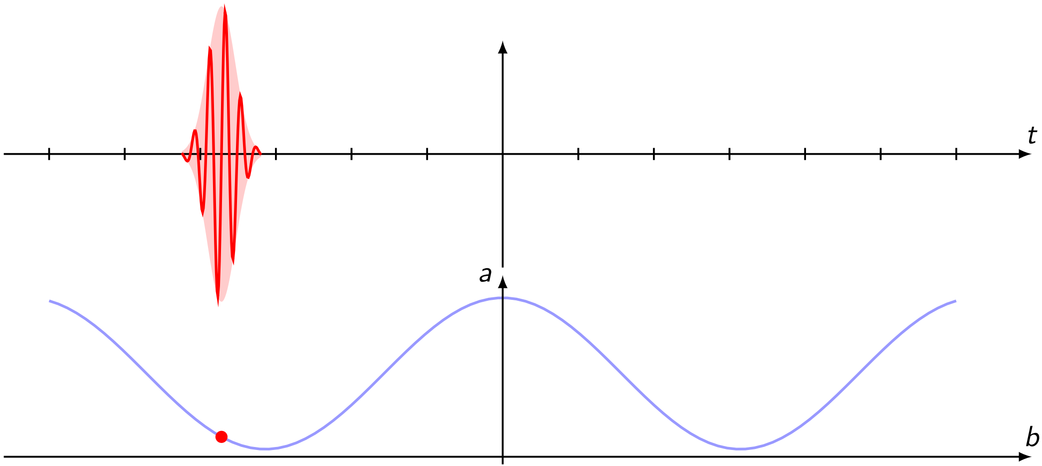


Translation und Dilatation



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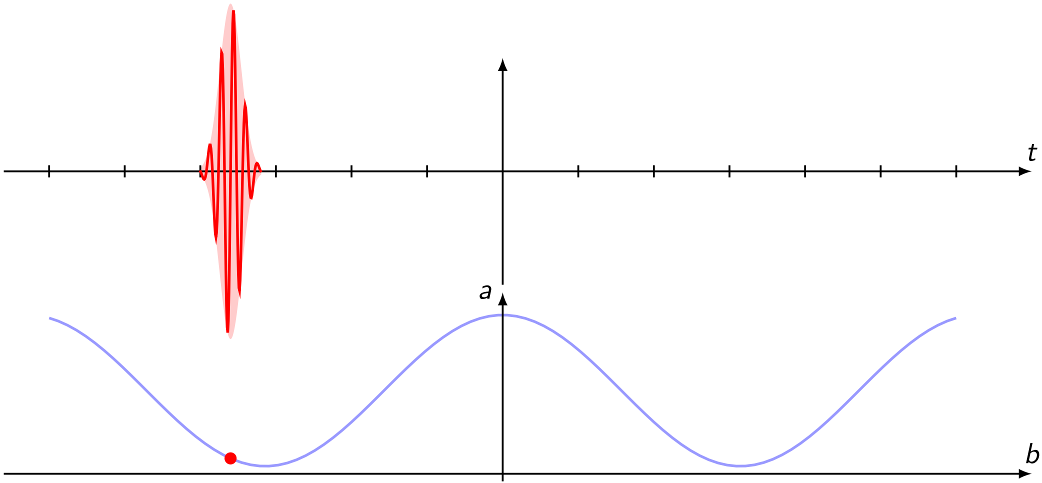

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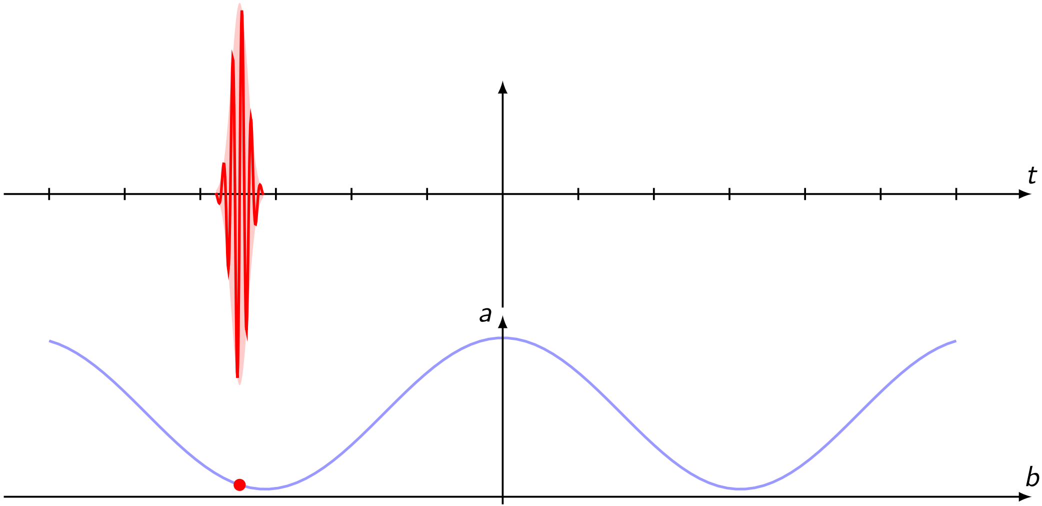

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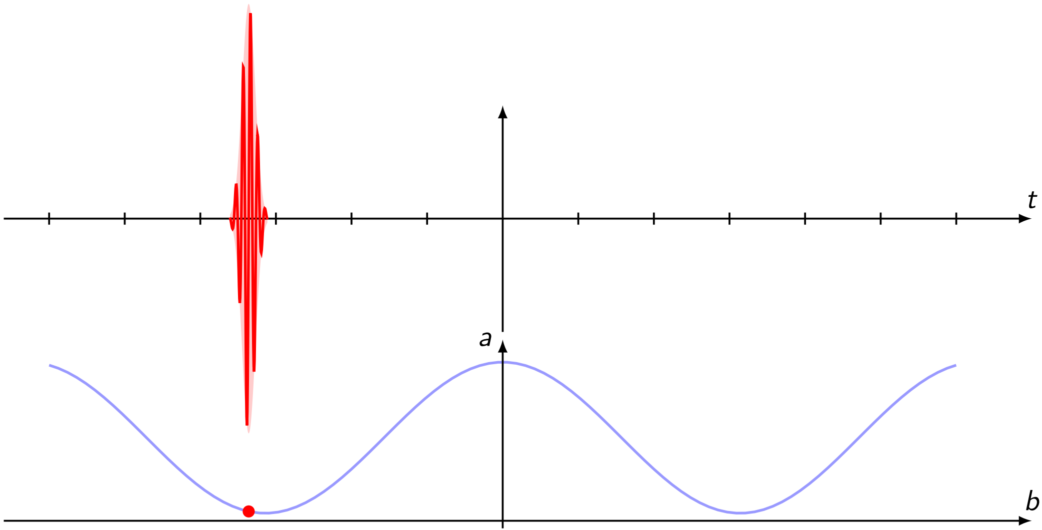

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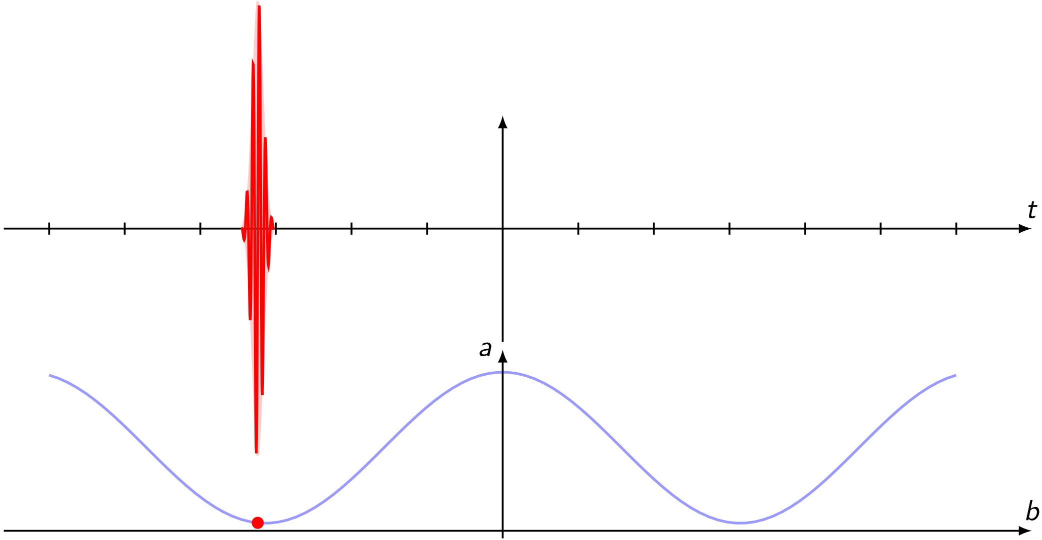

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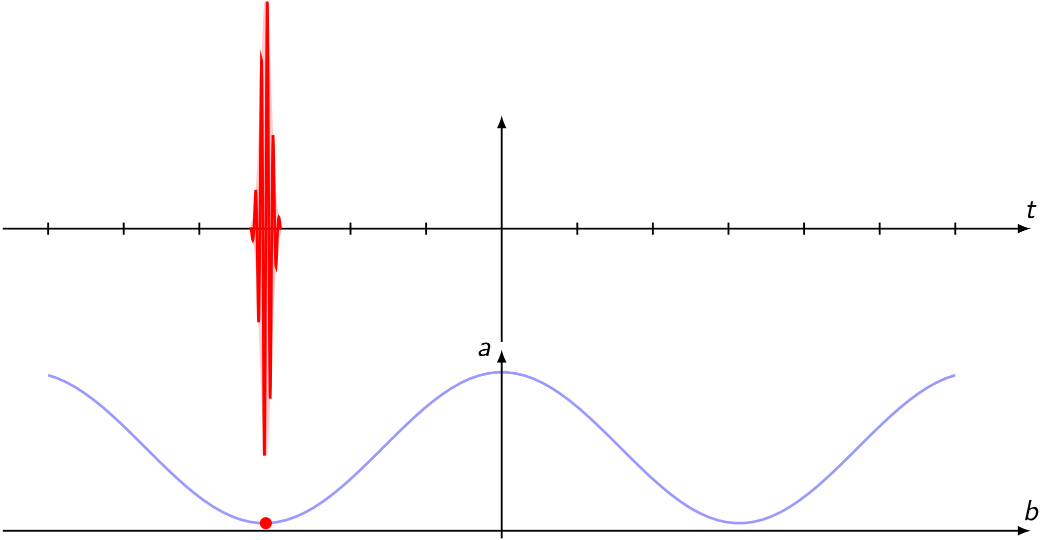

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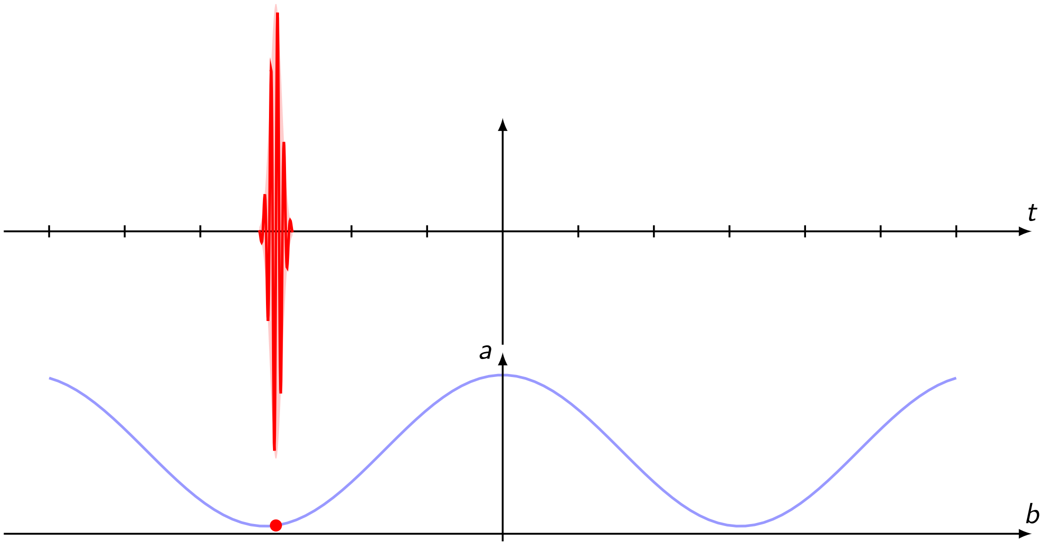

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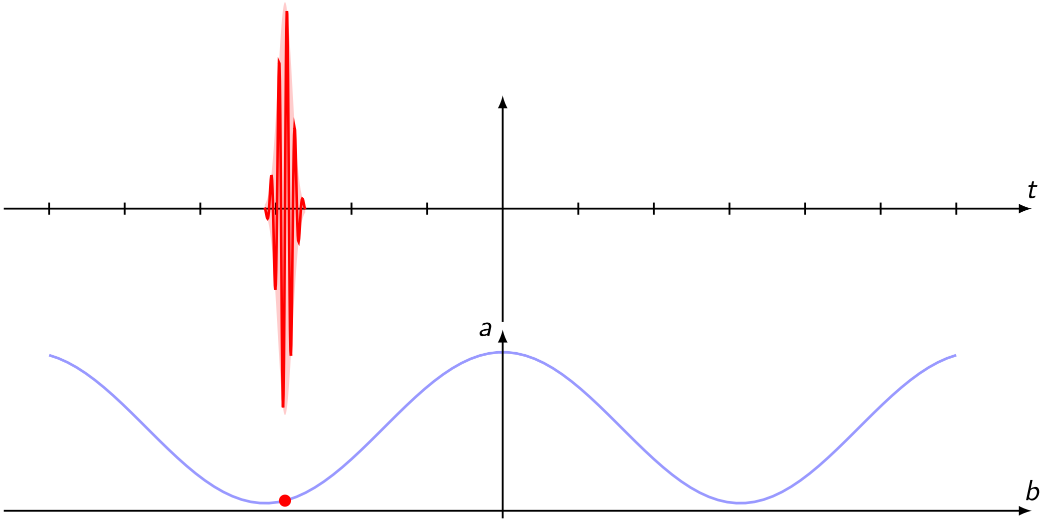

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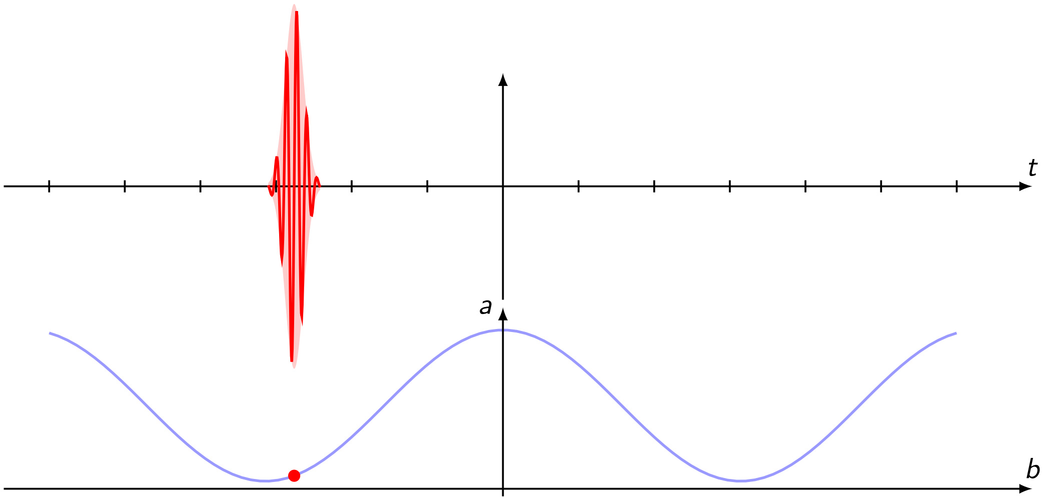

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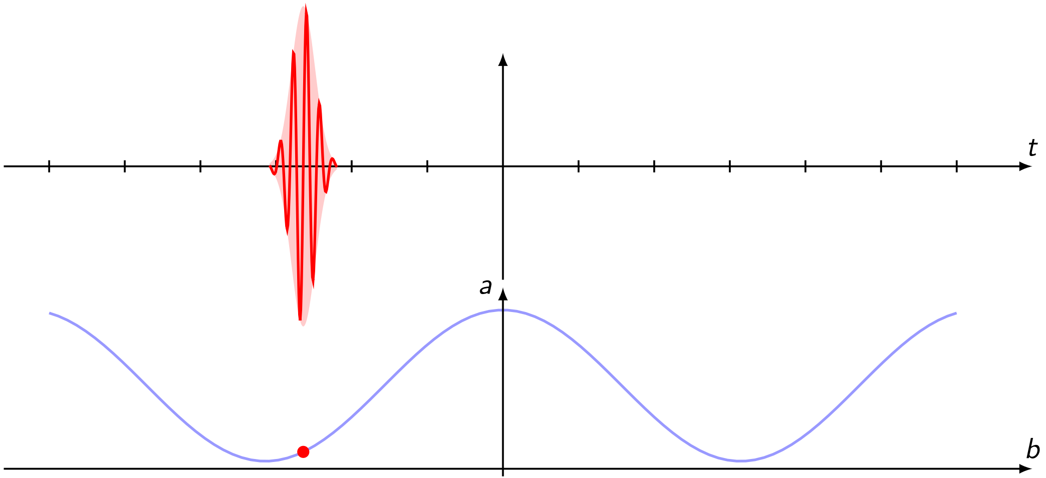

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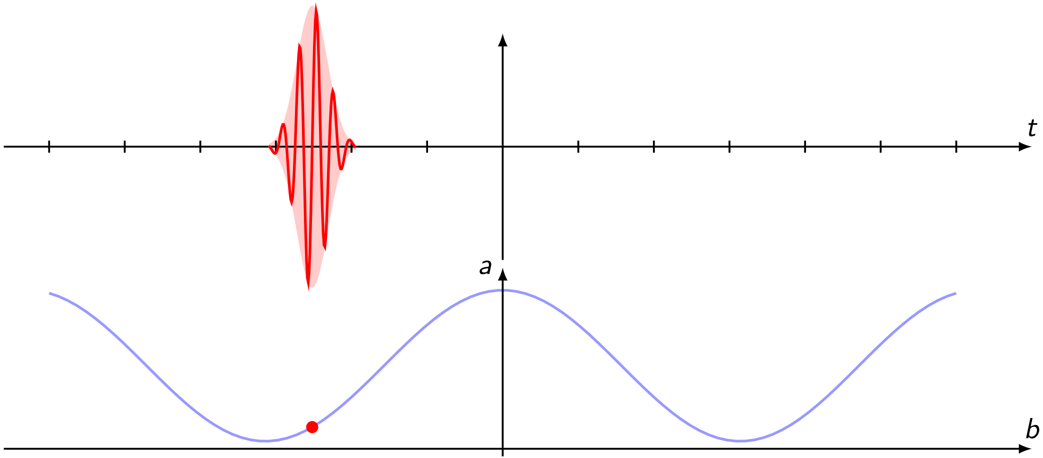

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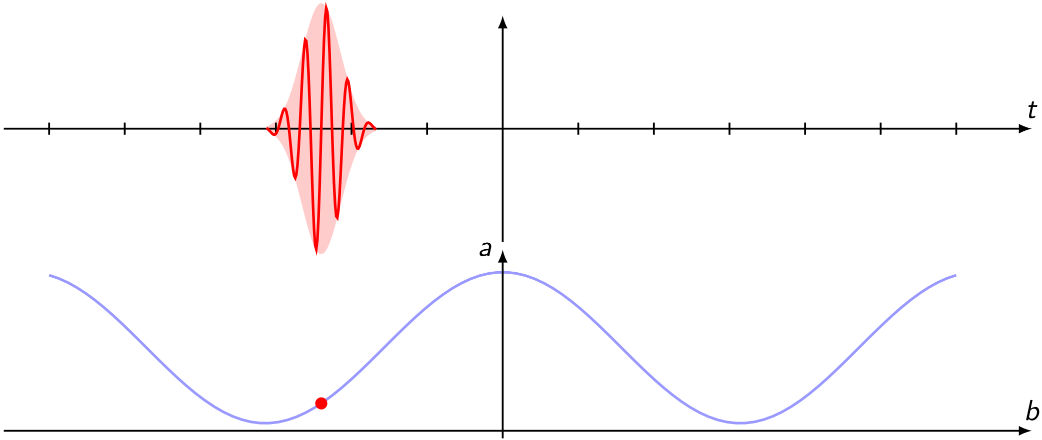

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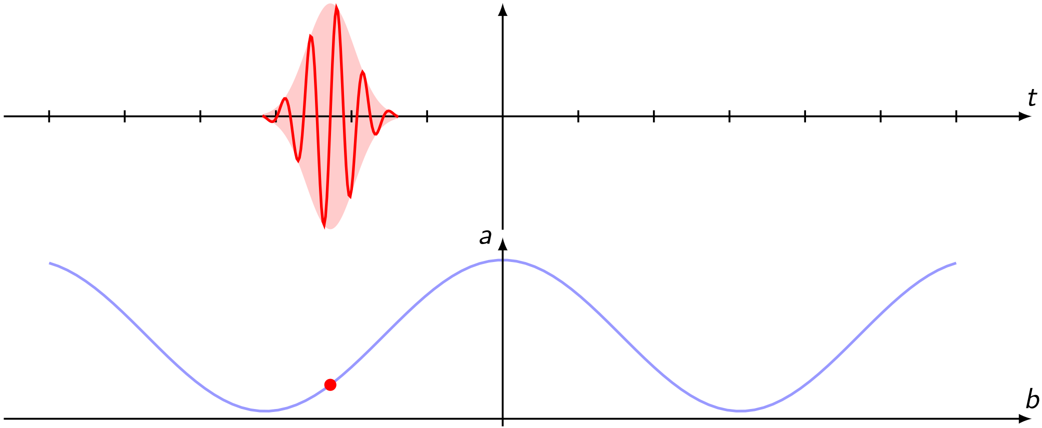

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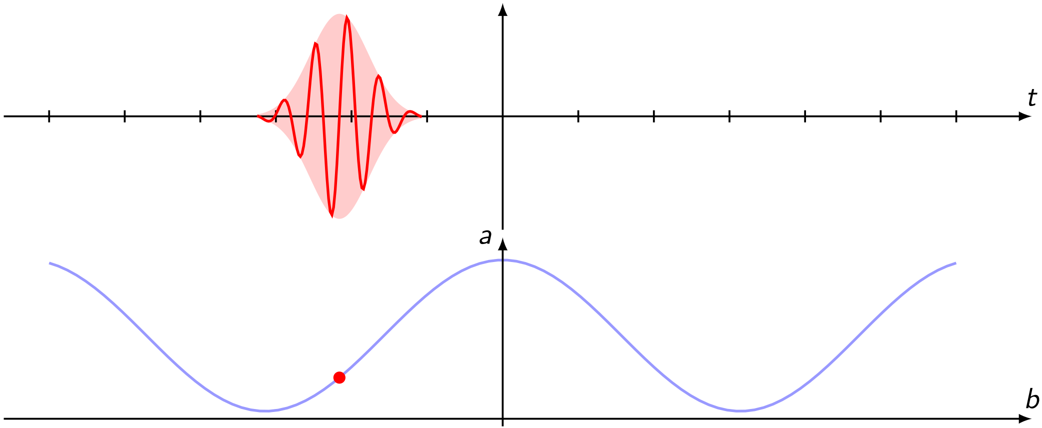

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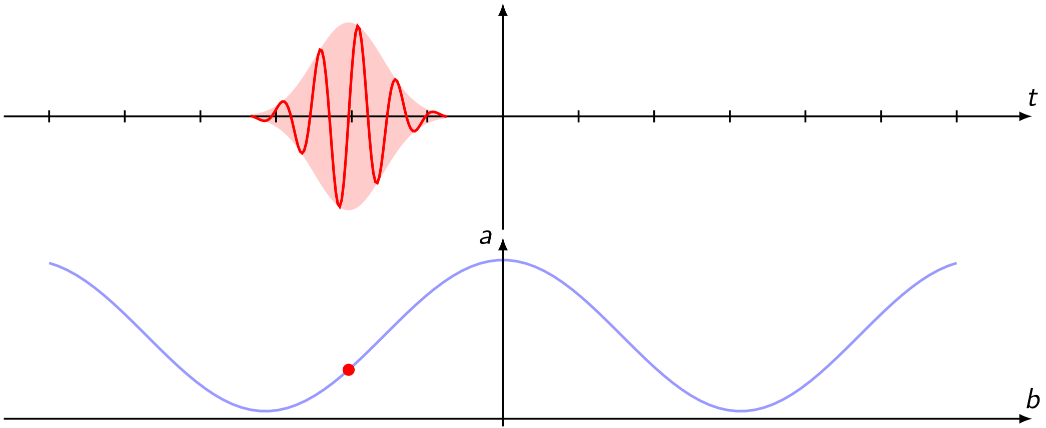

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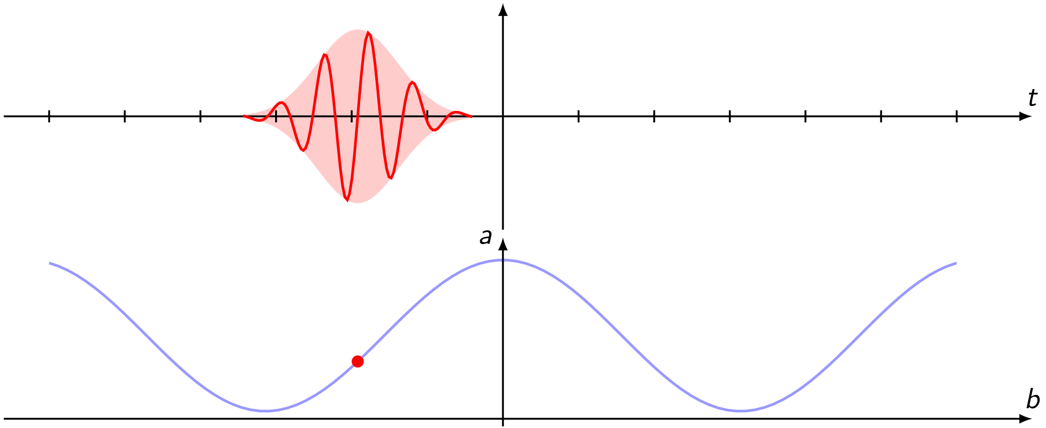

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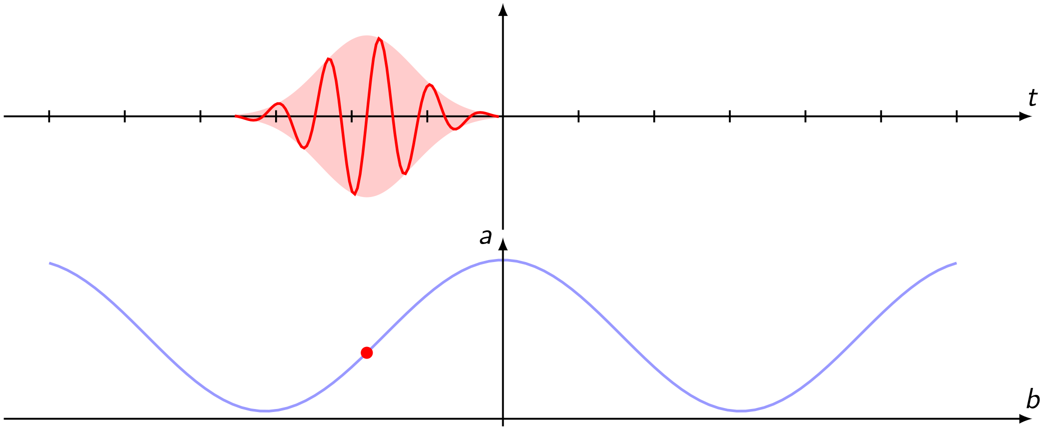

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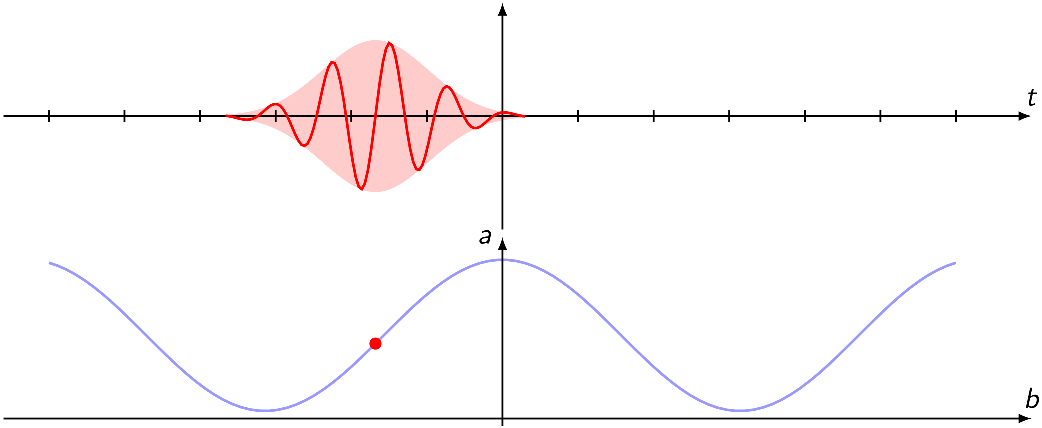

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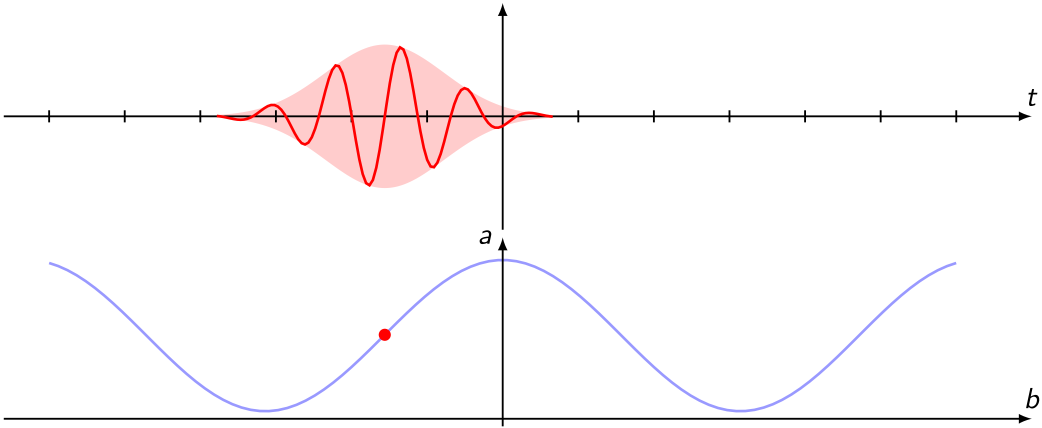

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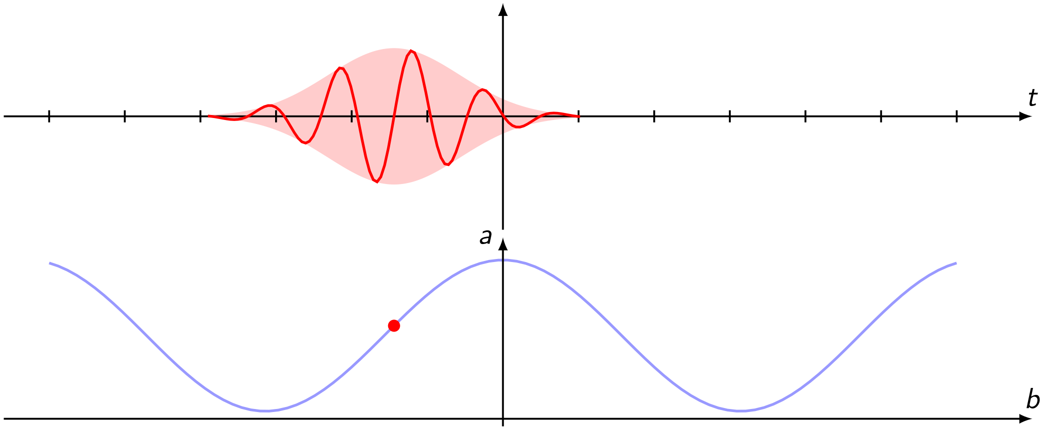

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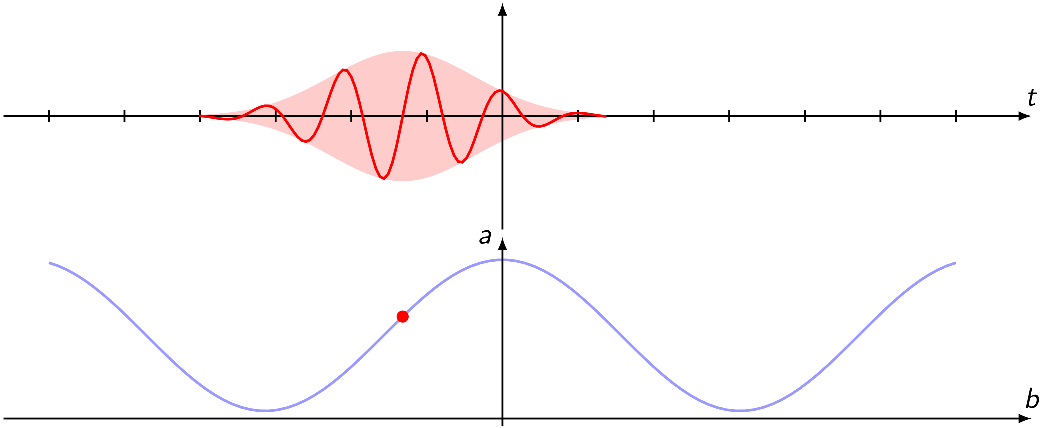

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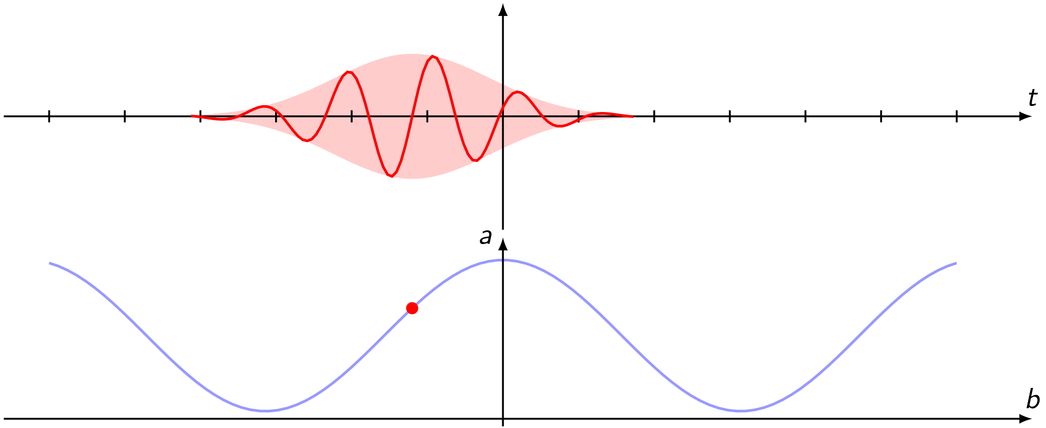

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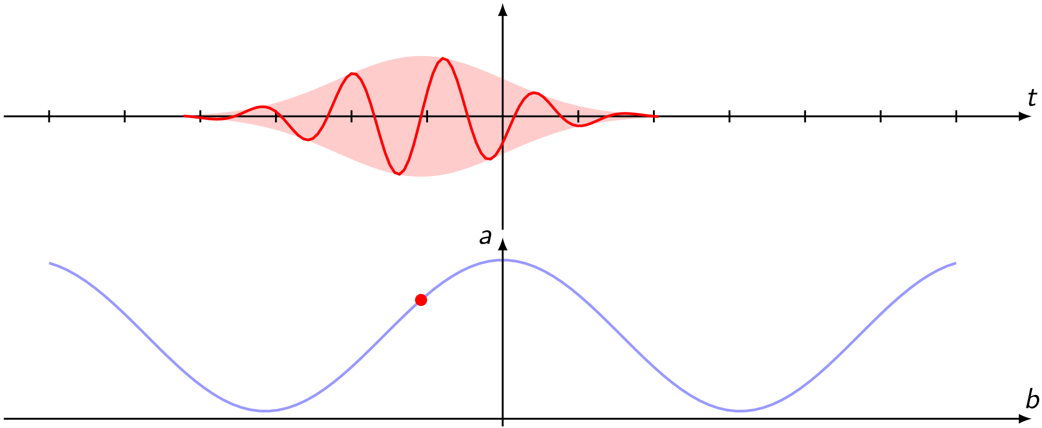

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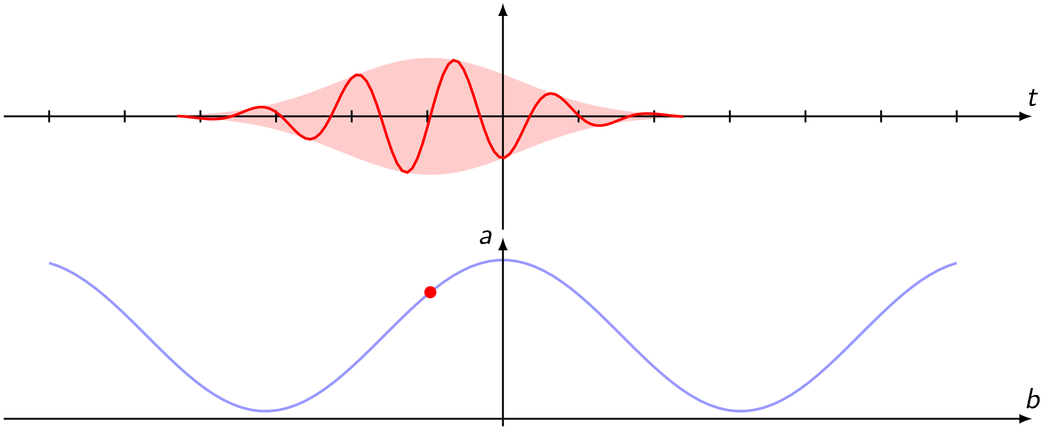

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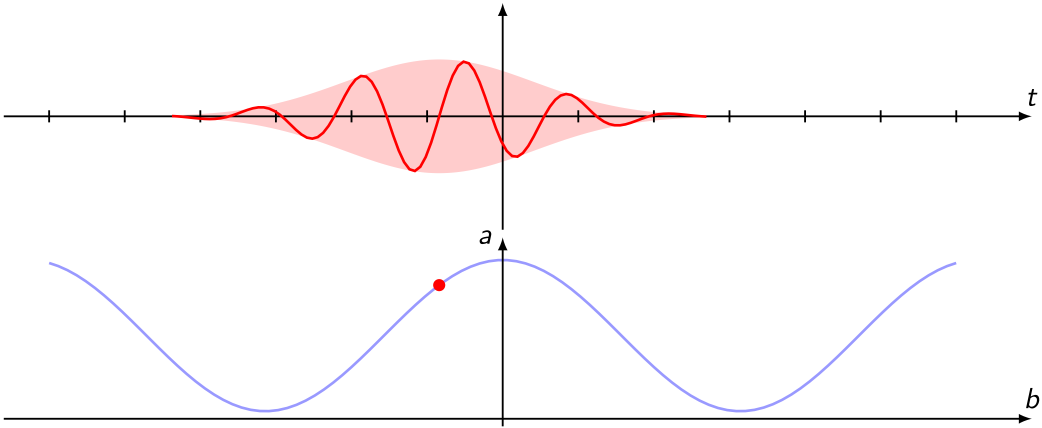

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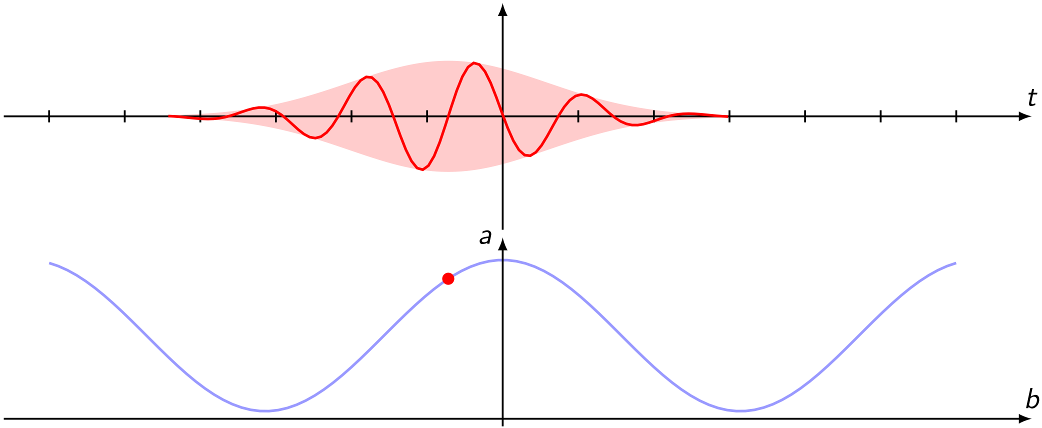

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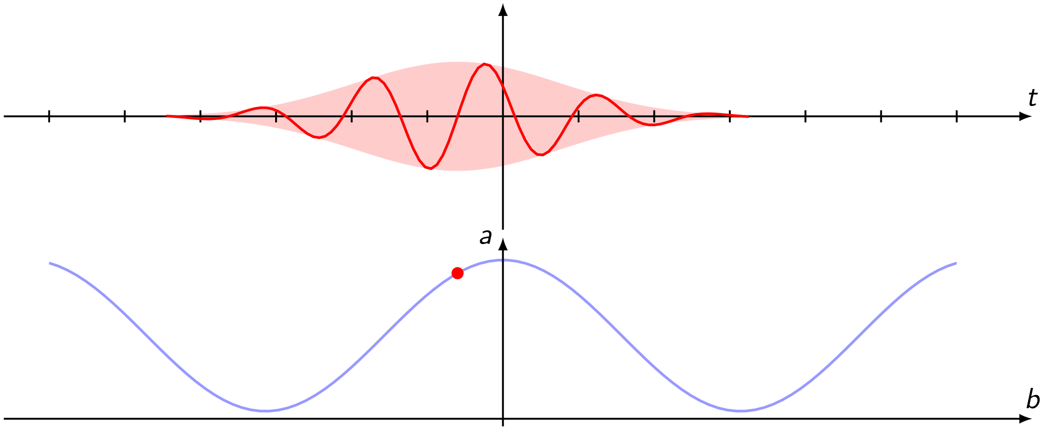

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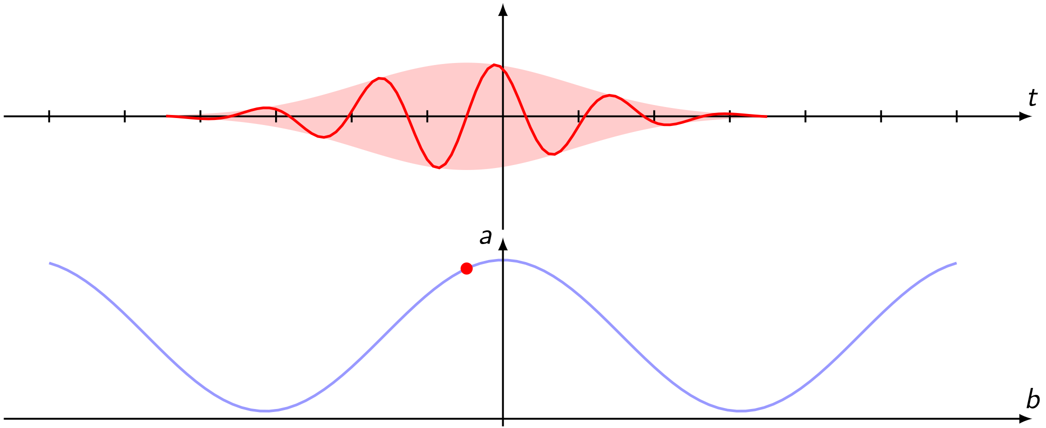

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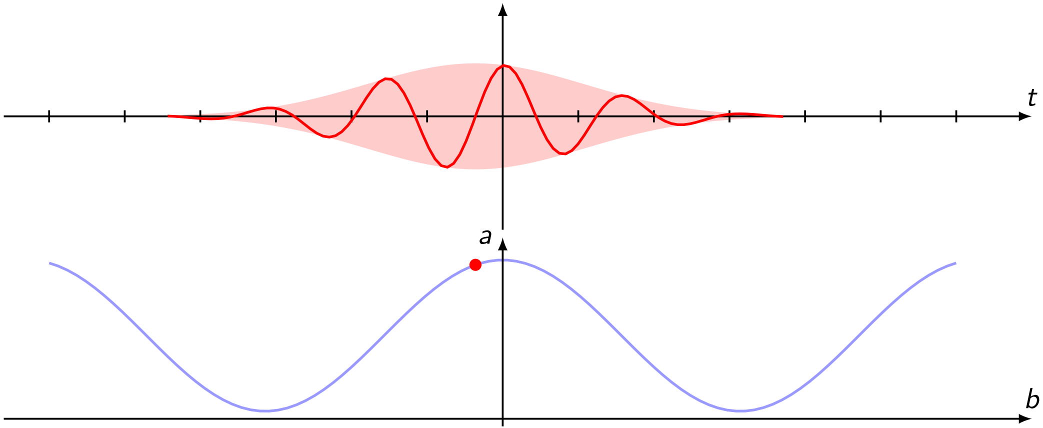

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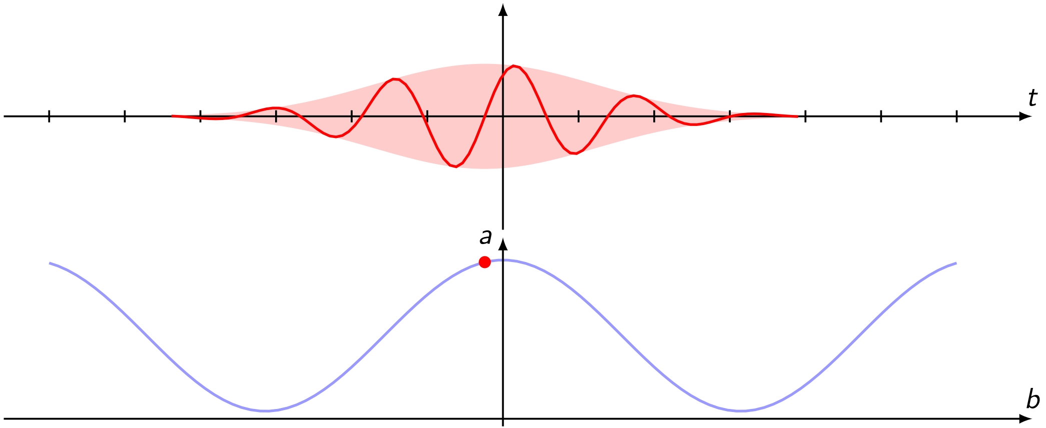

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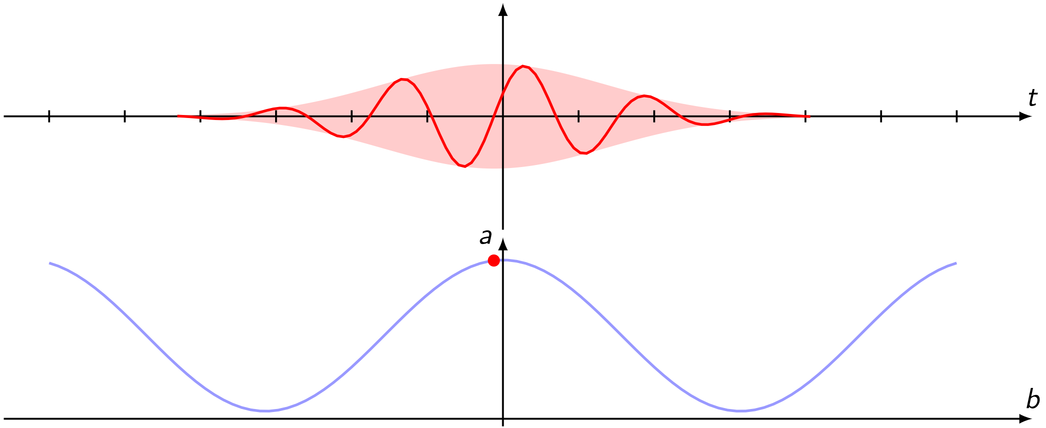

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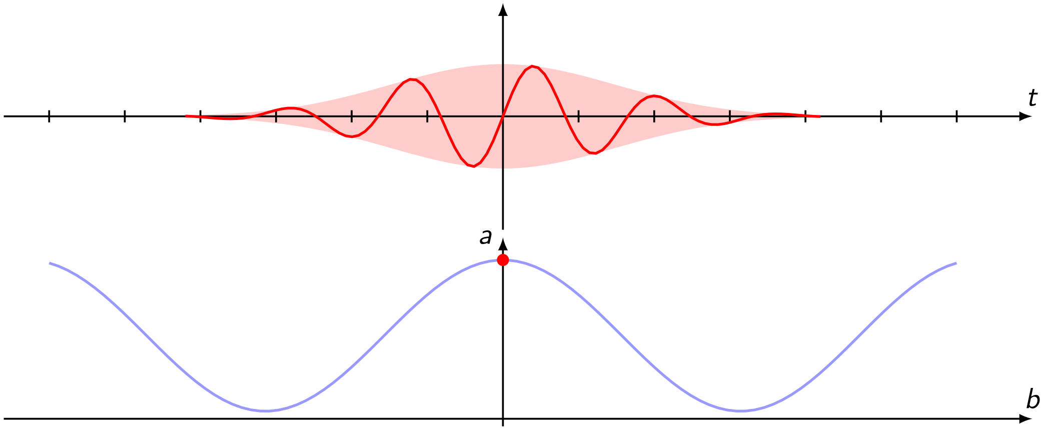

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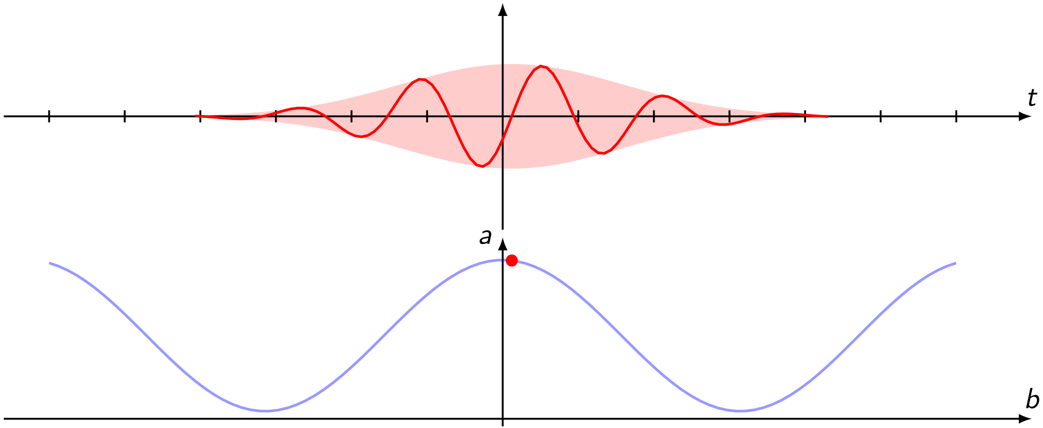

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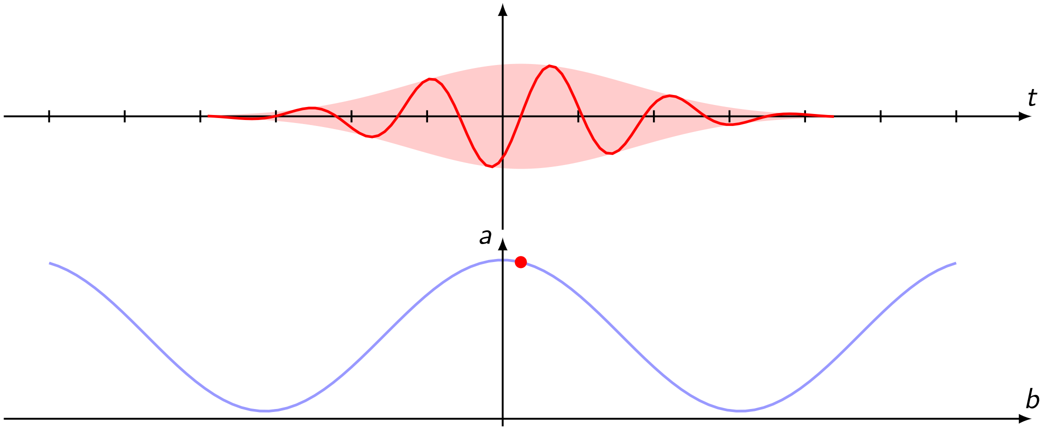

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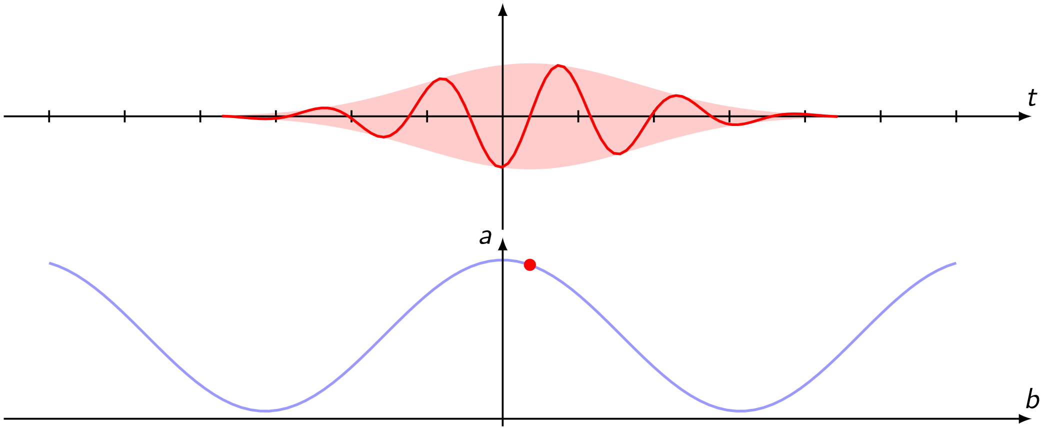

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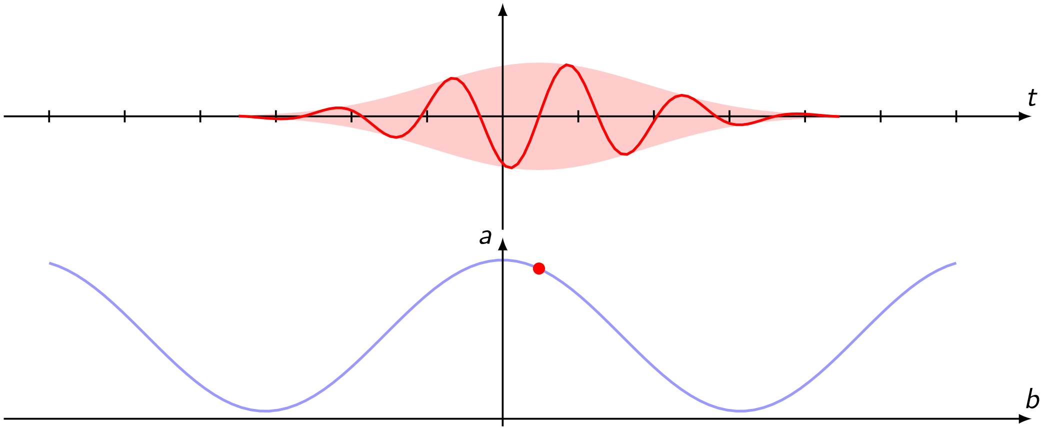

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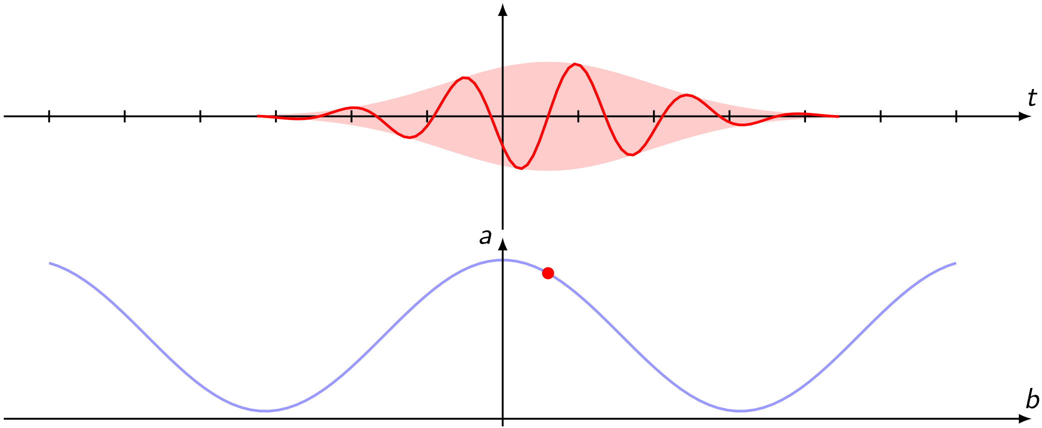

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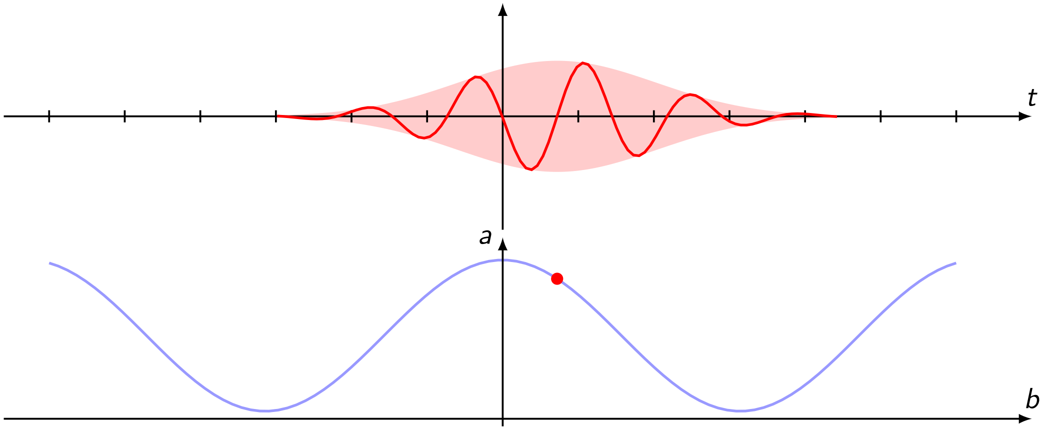

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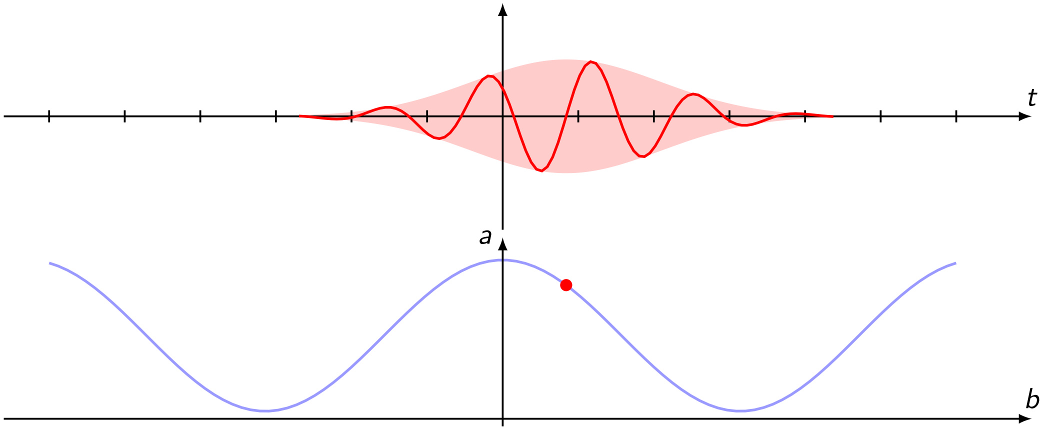

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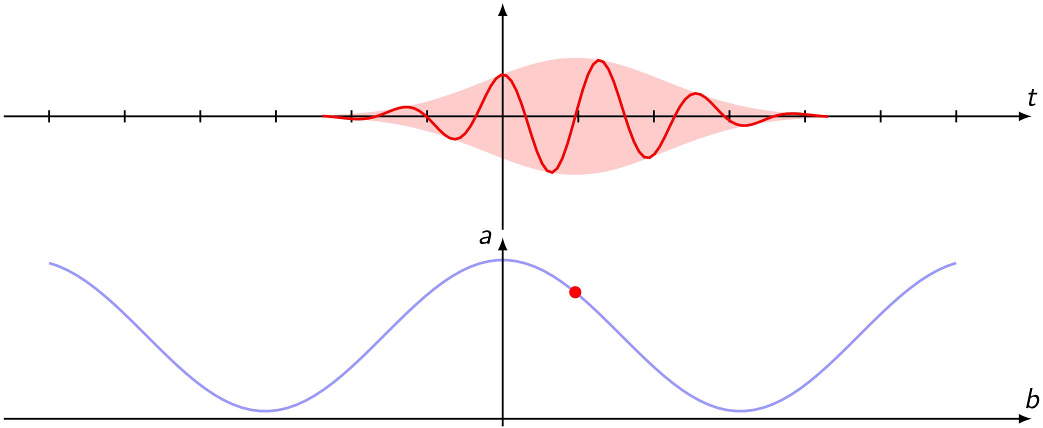

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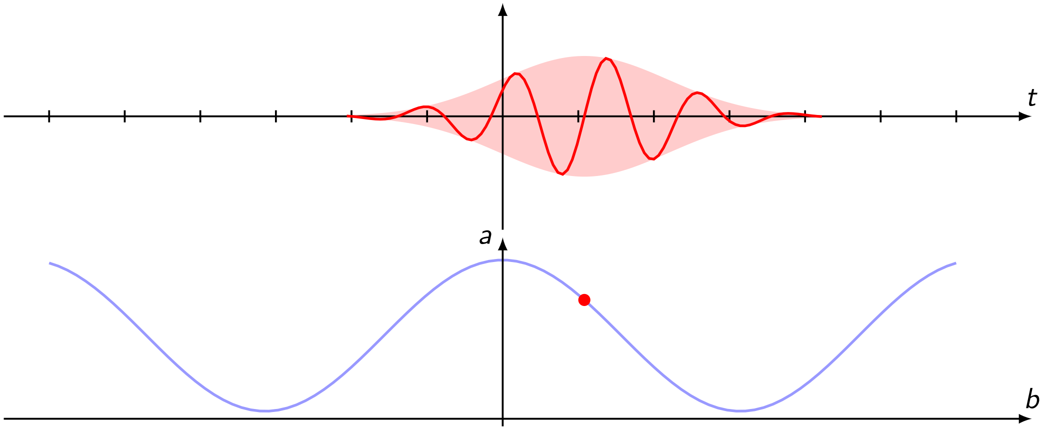

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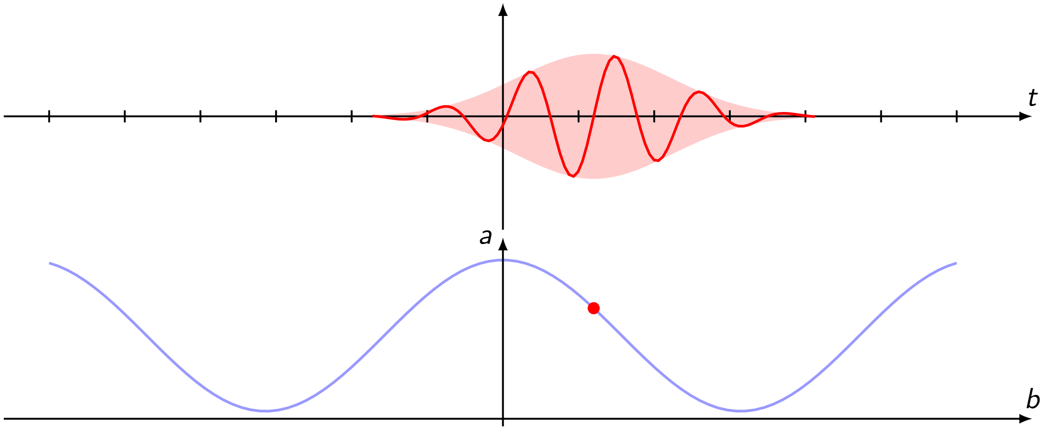

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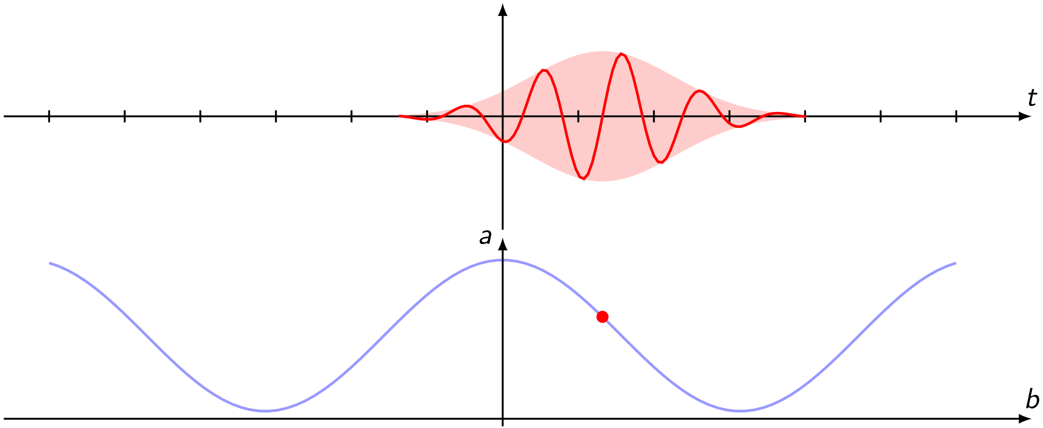

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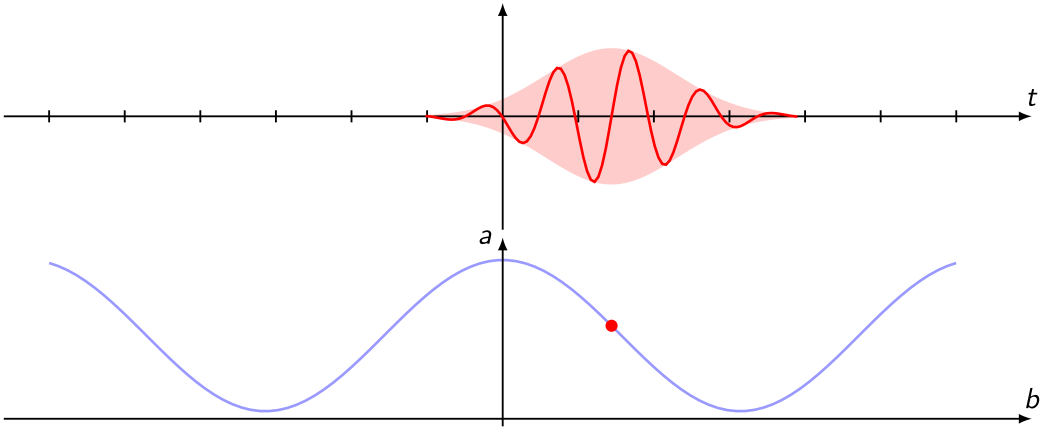

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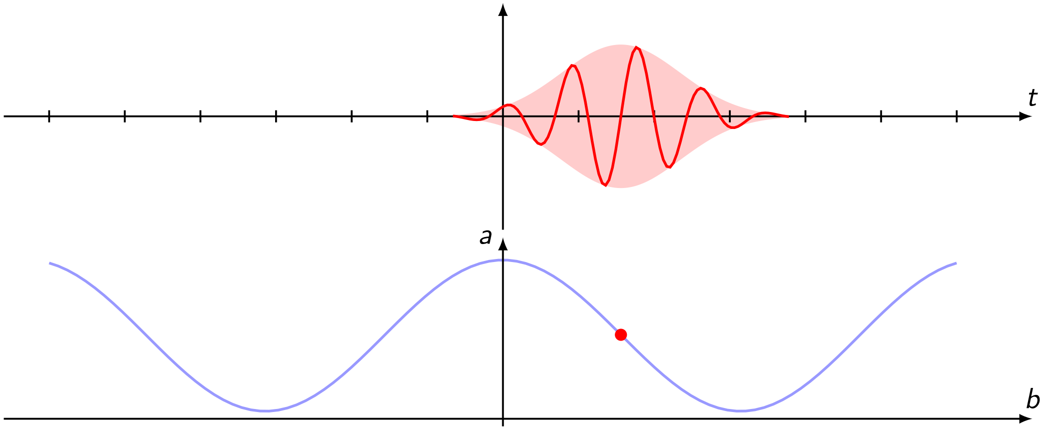

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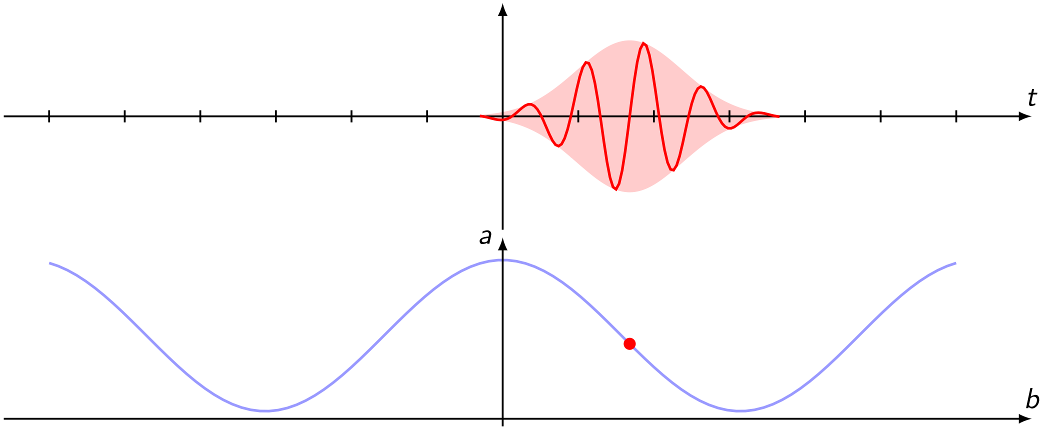

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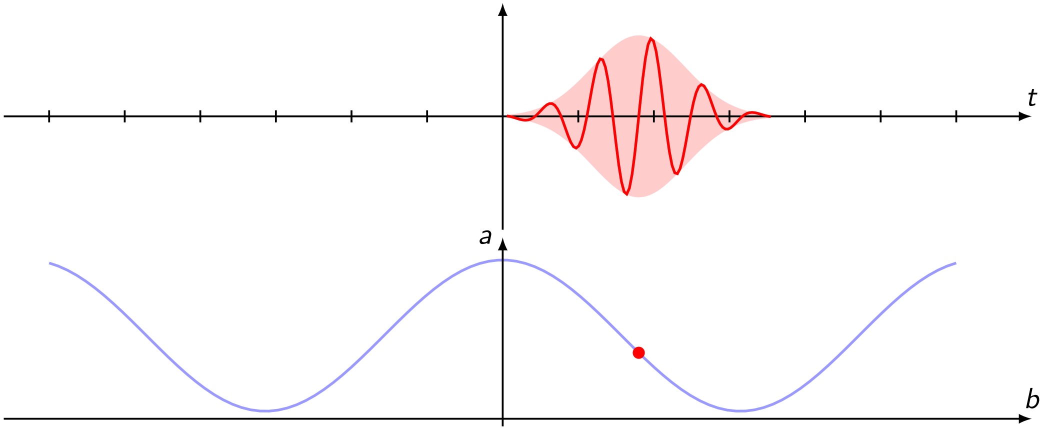

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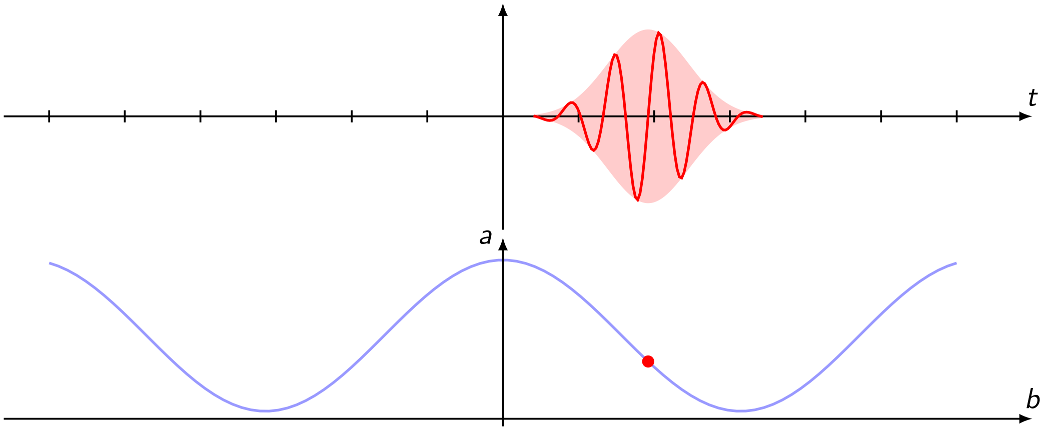

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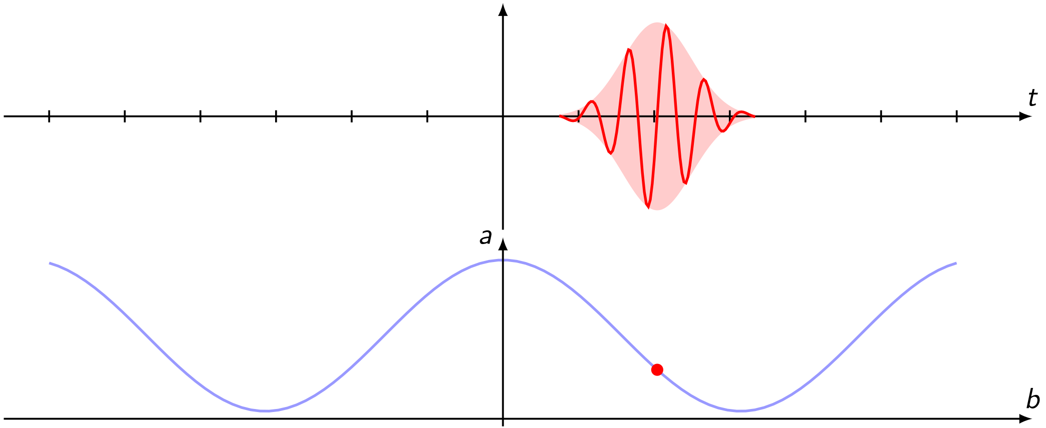

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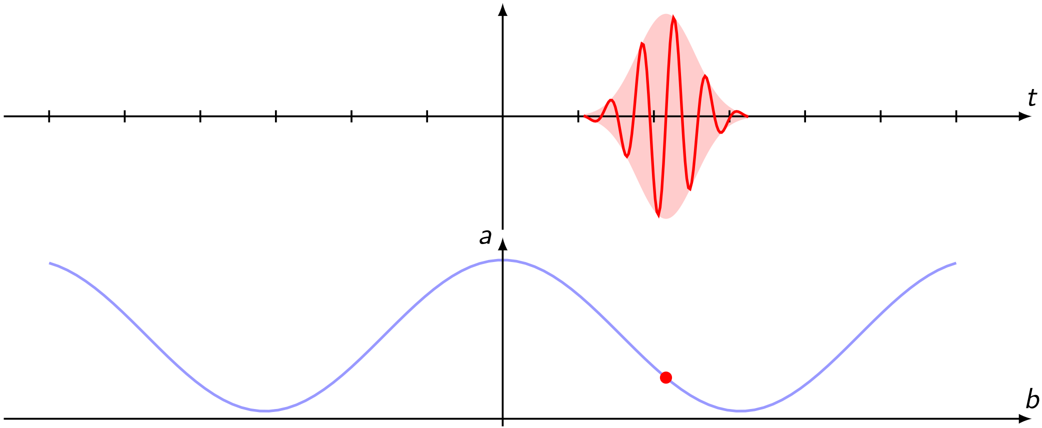

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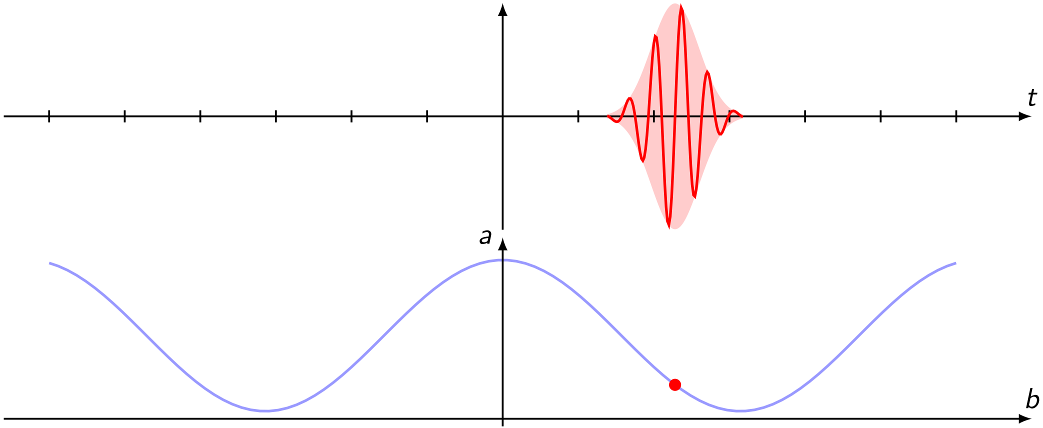

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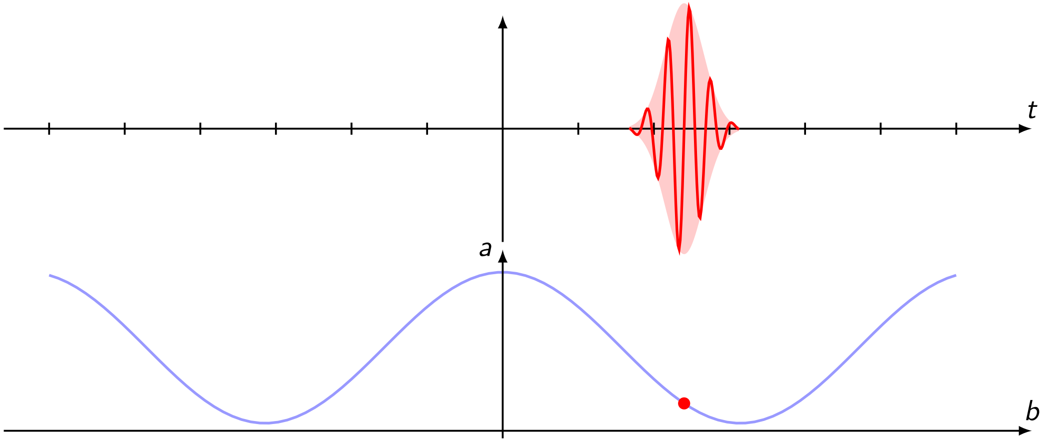

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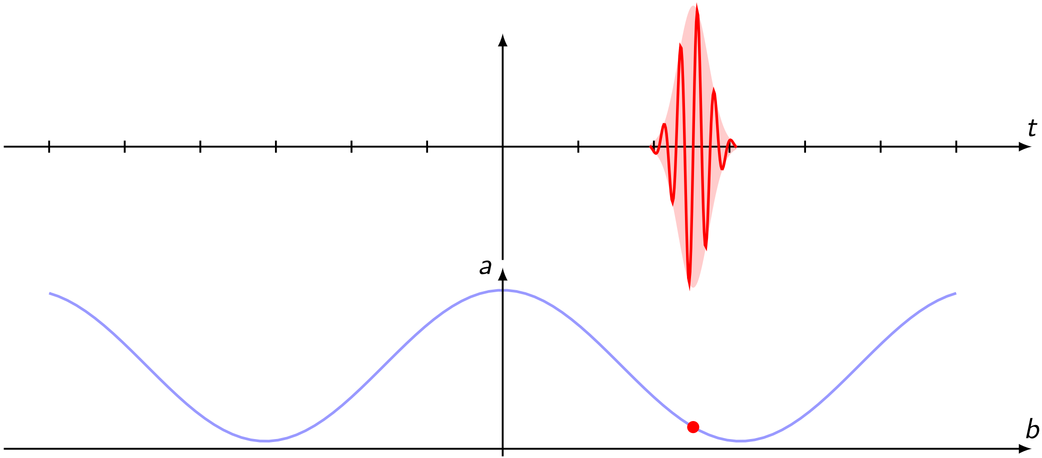

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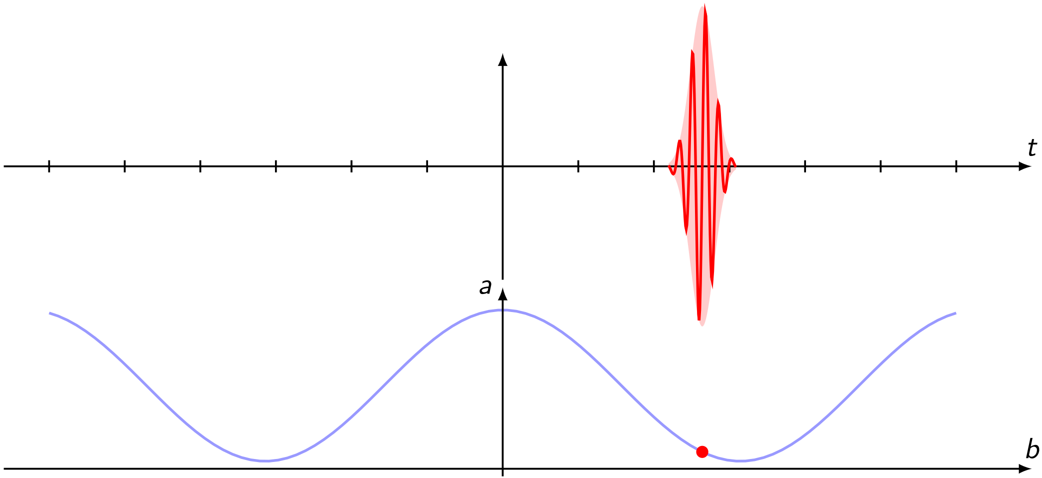

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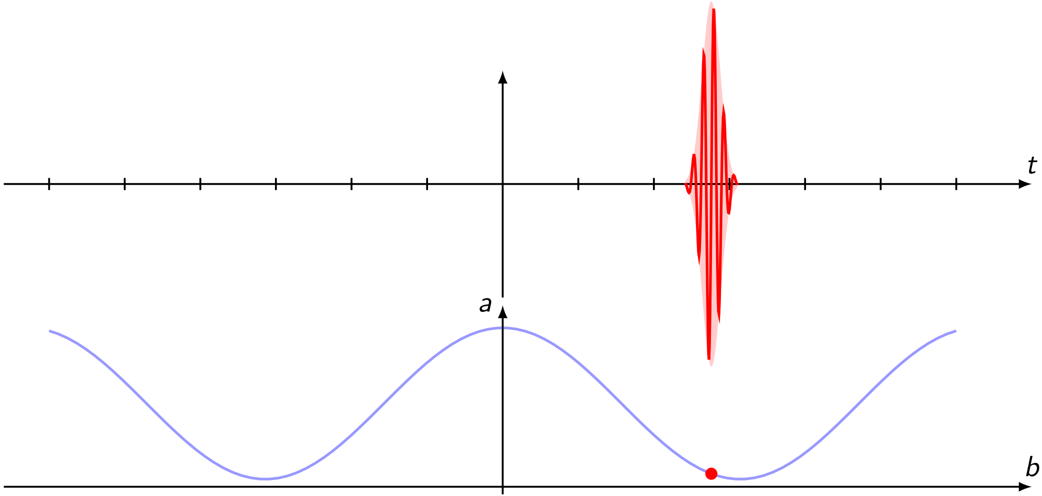

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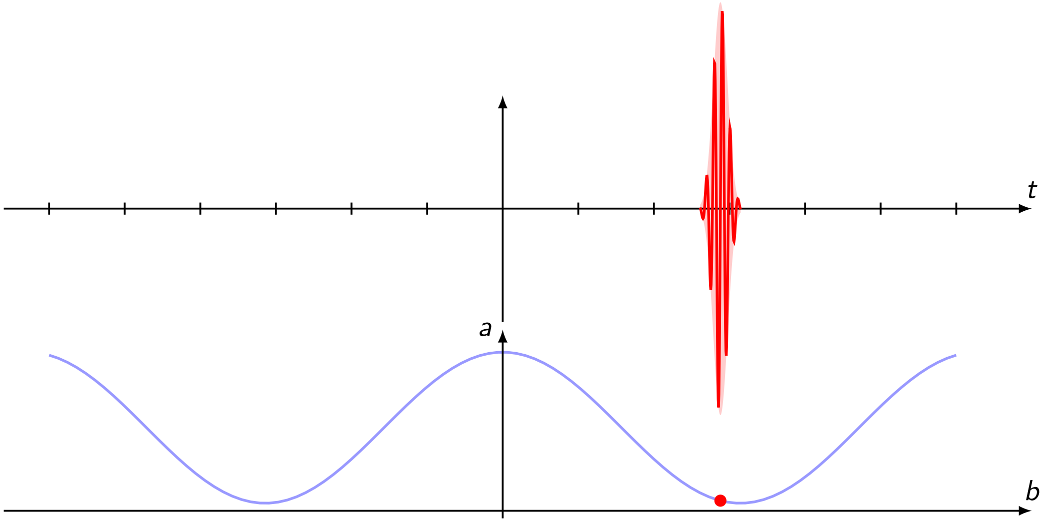

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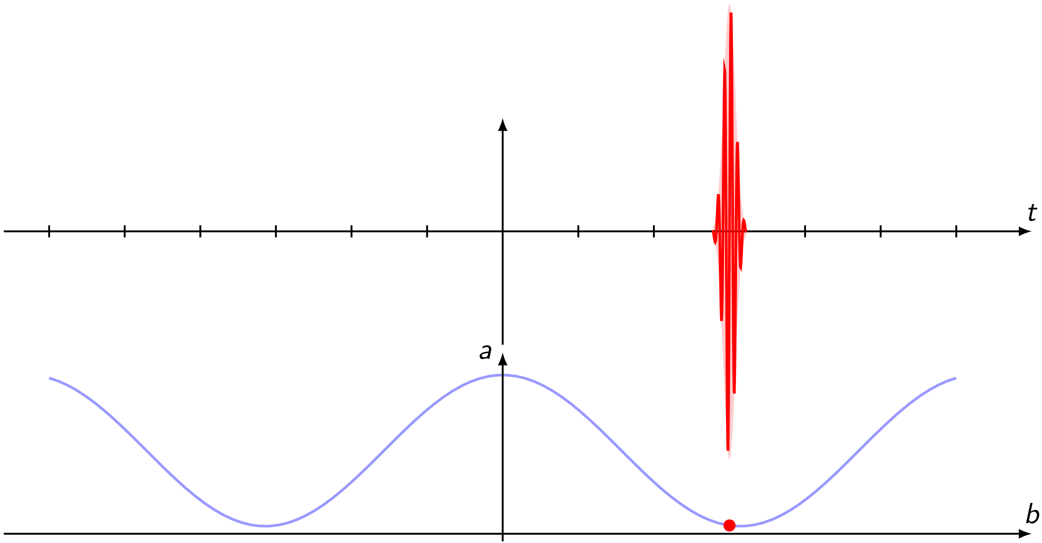

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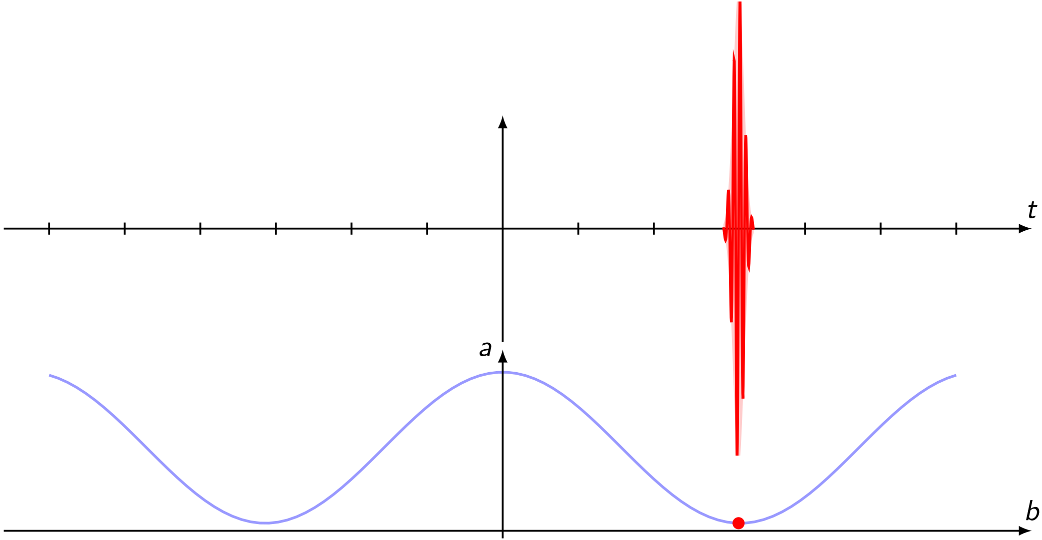

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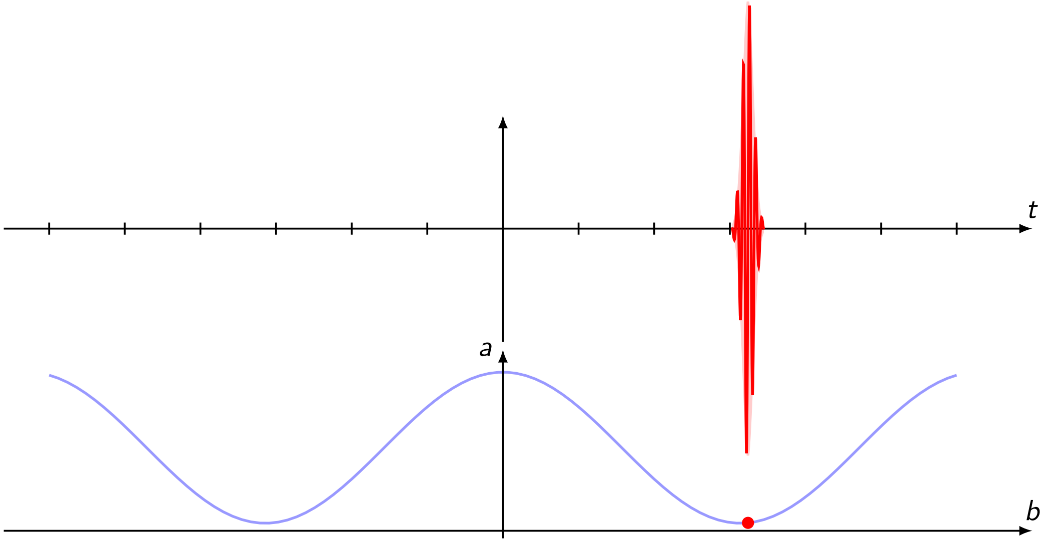

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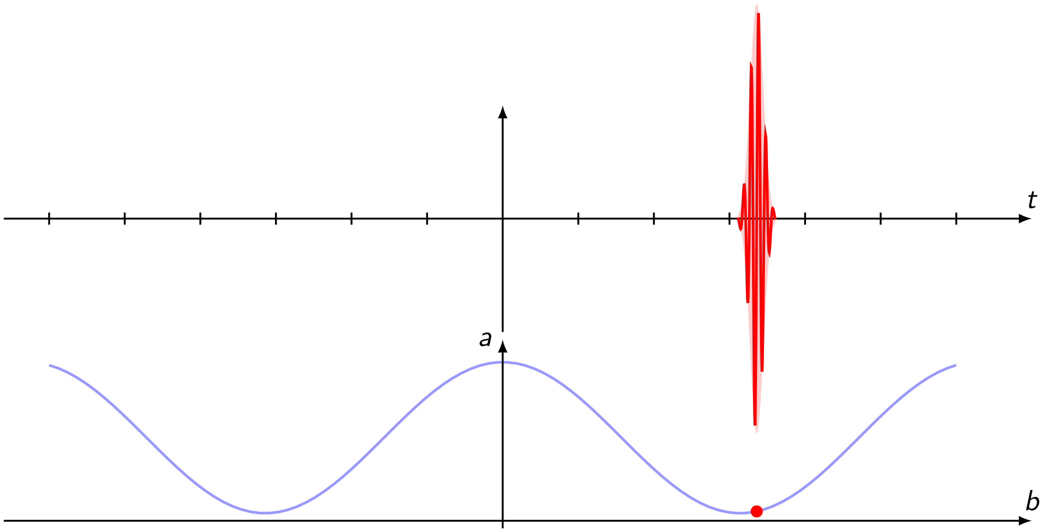

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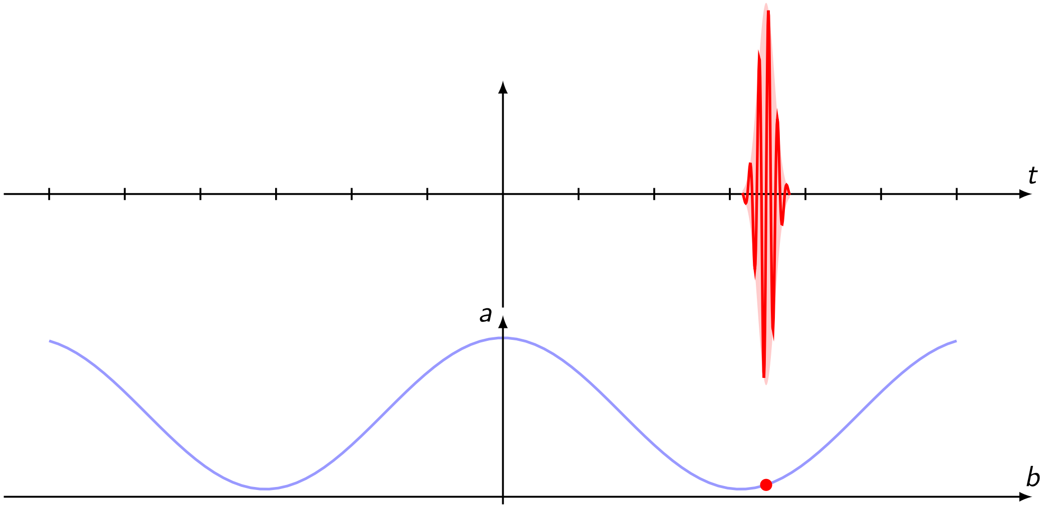

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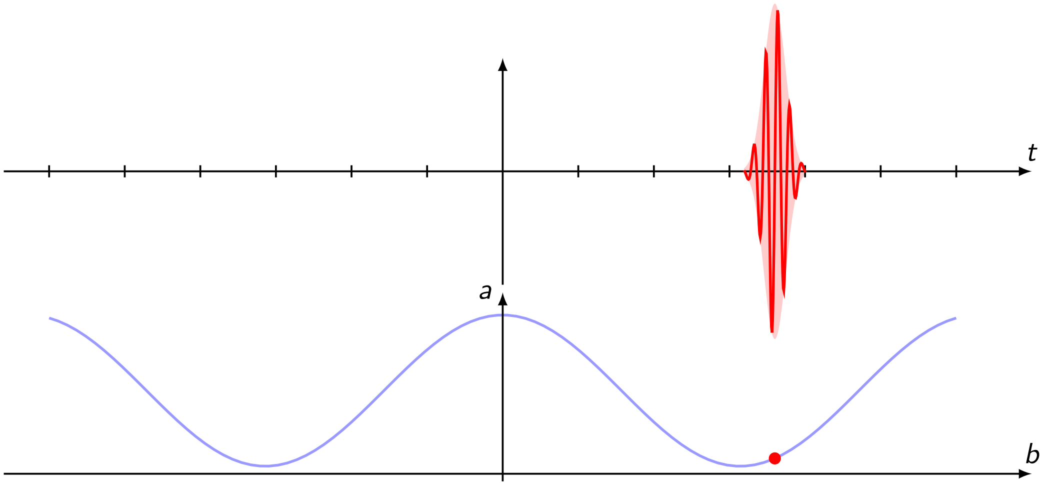

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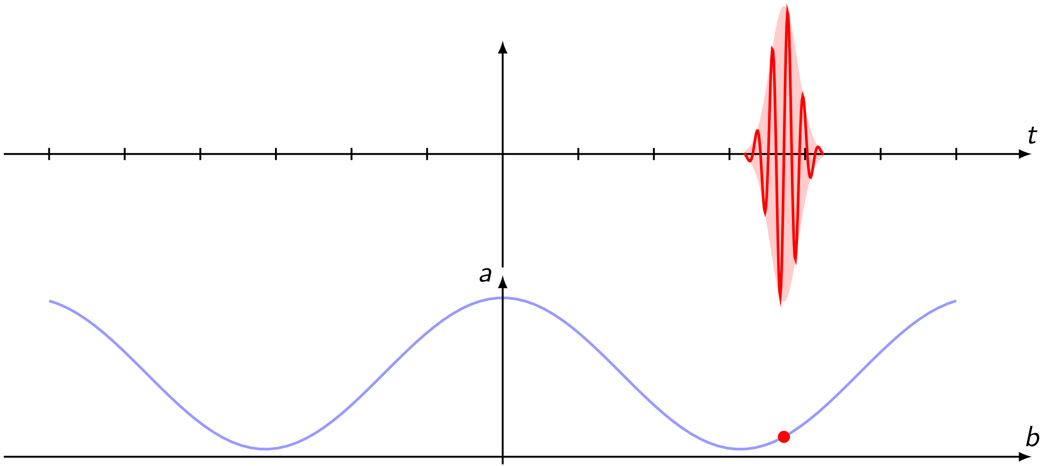

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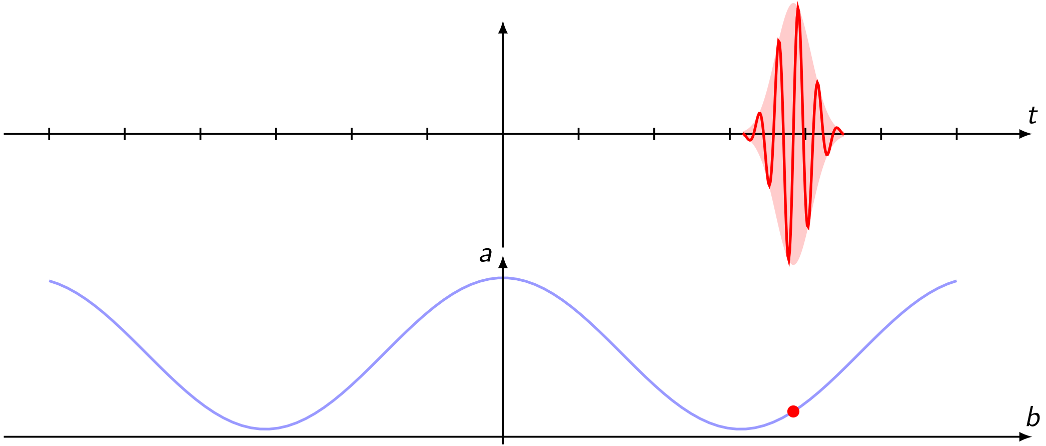

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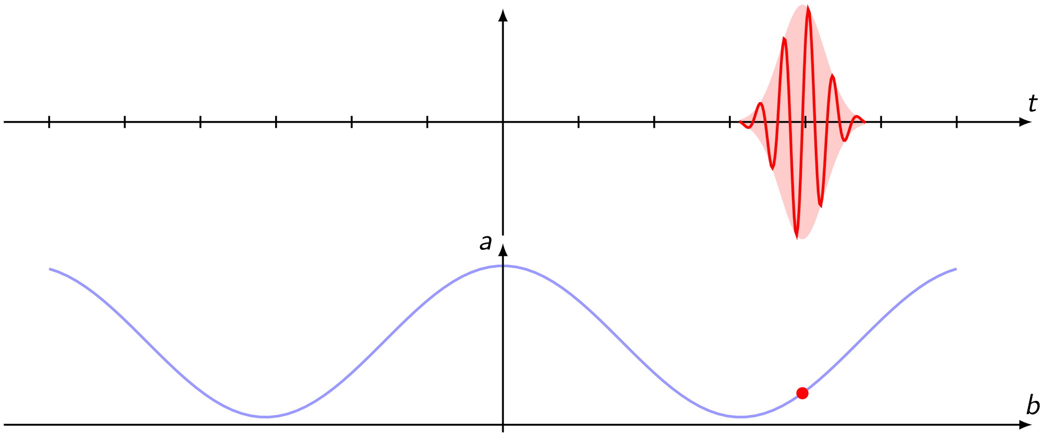

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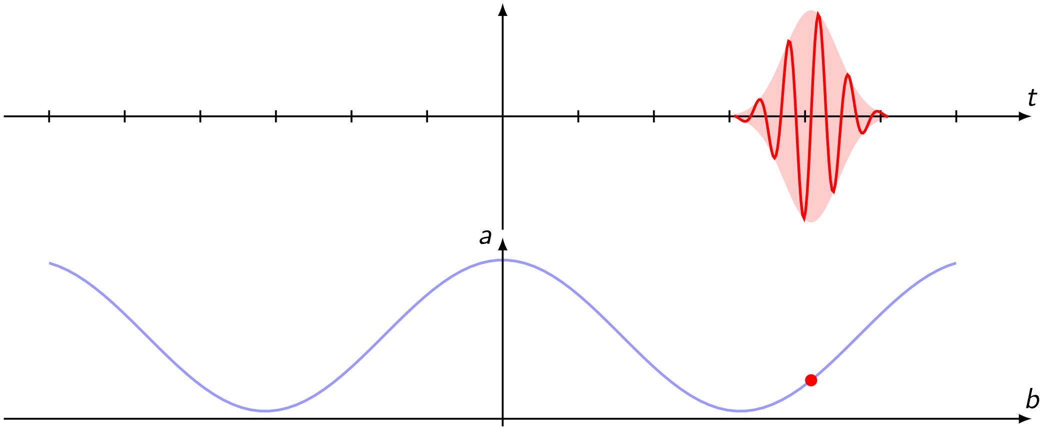

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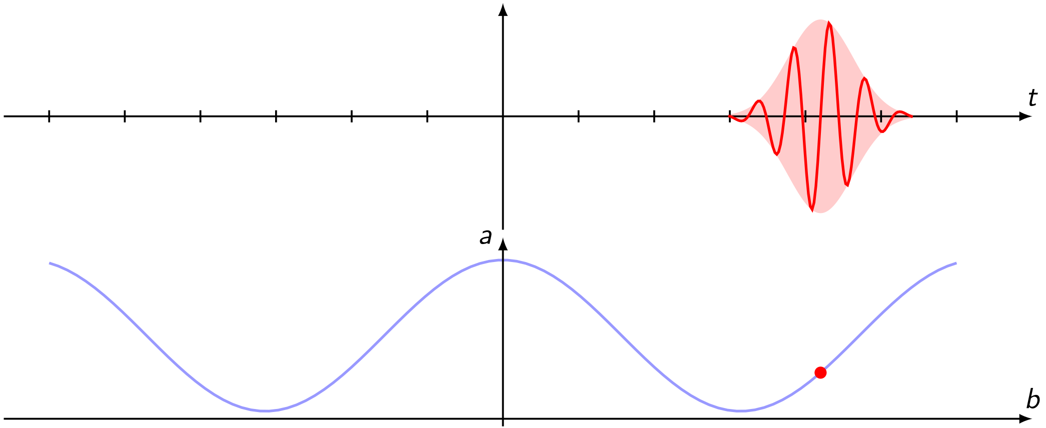

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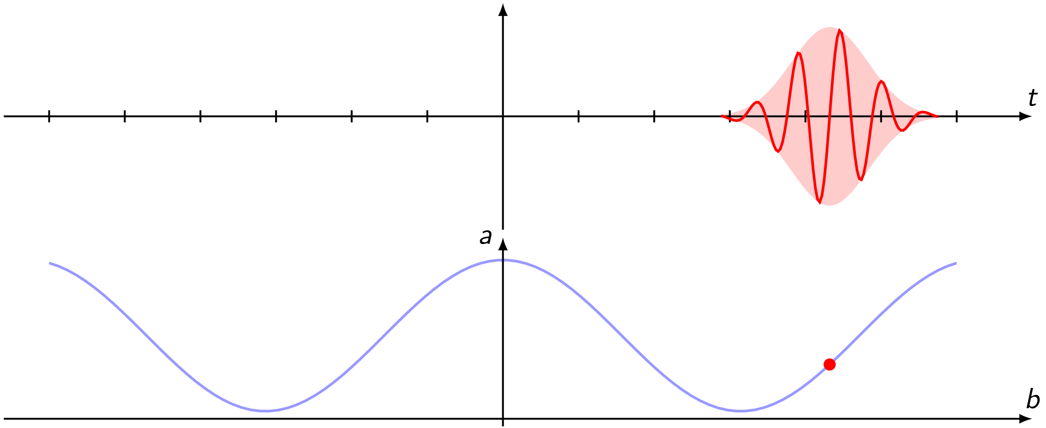

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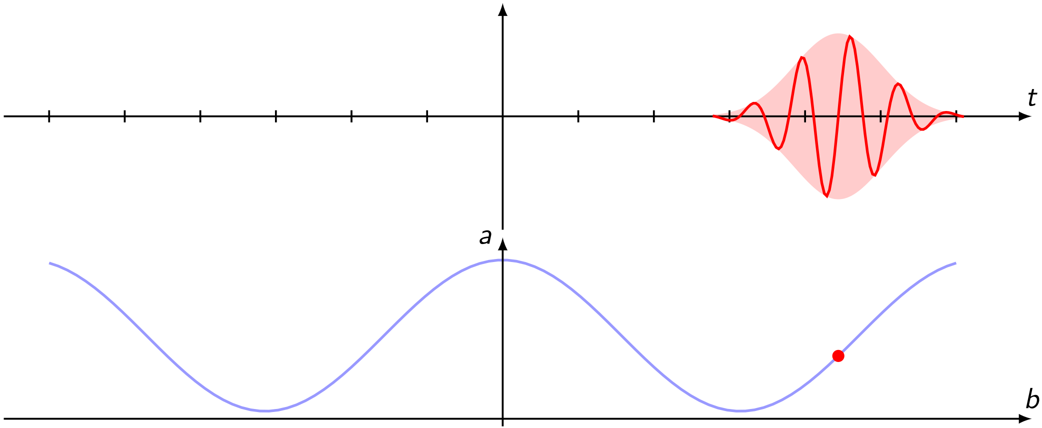

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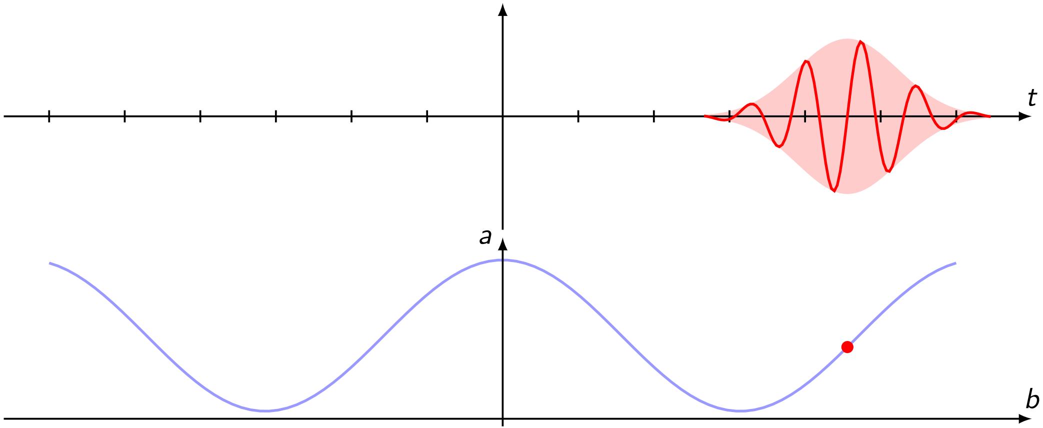

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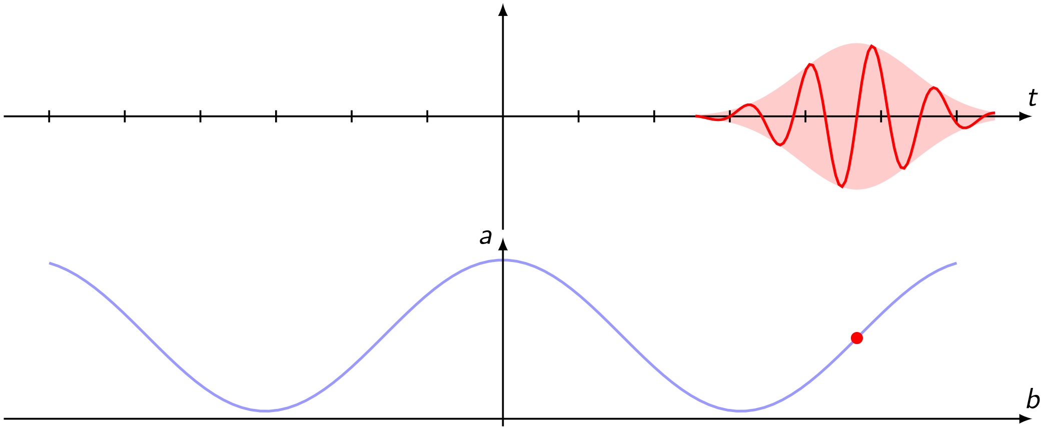

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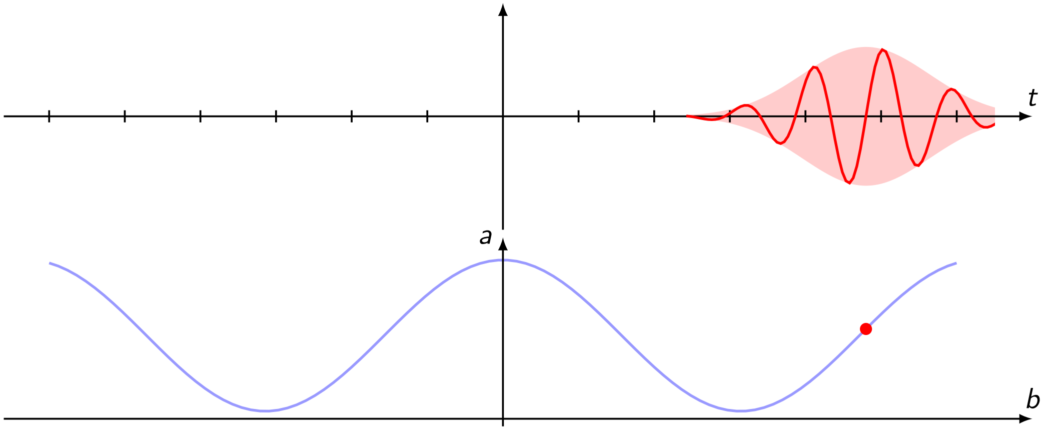
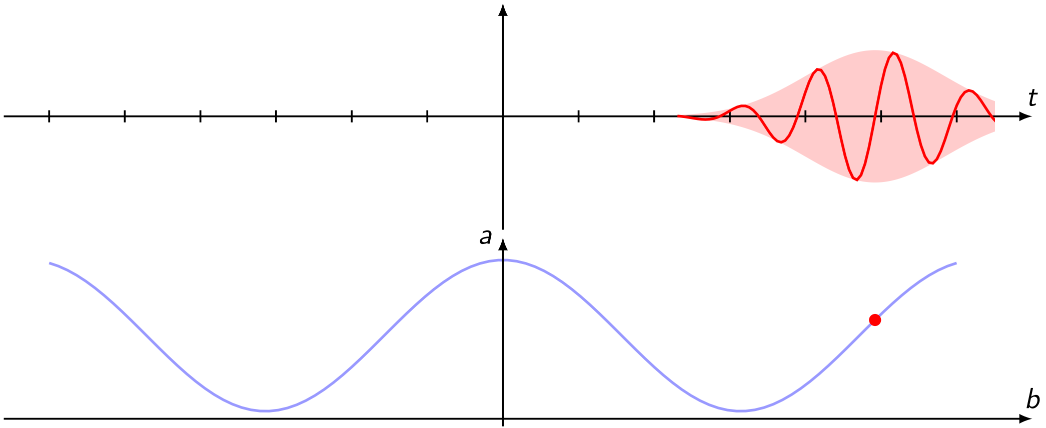


Translation und Dilatation



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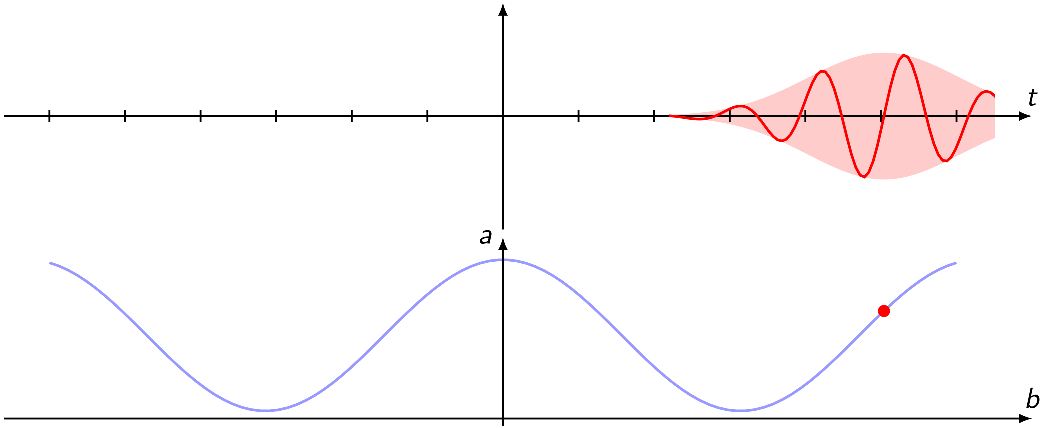

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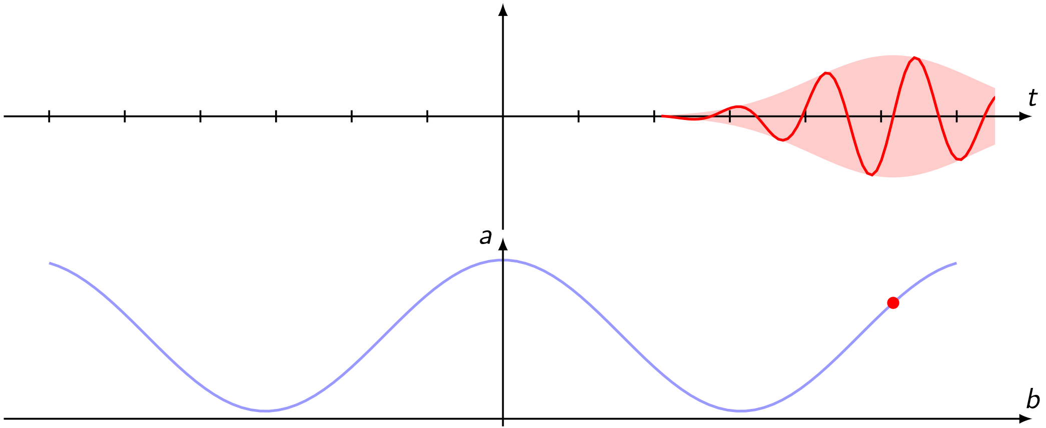

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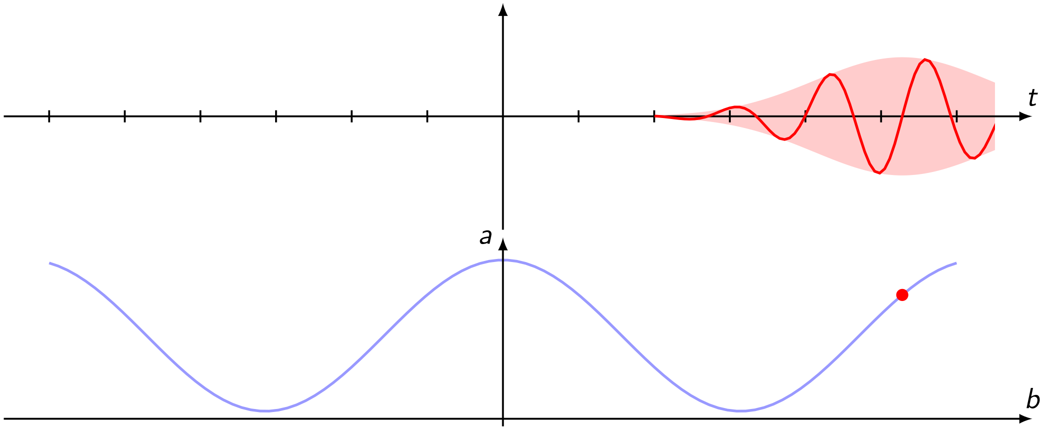

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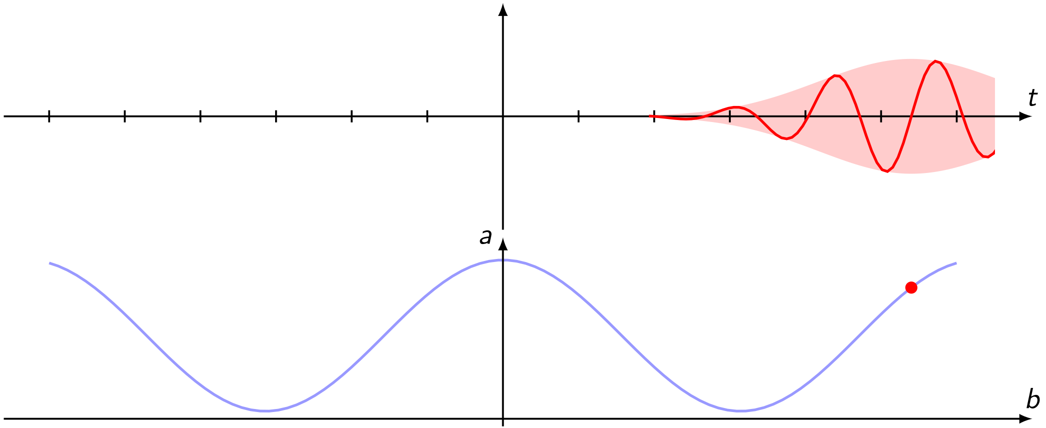

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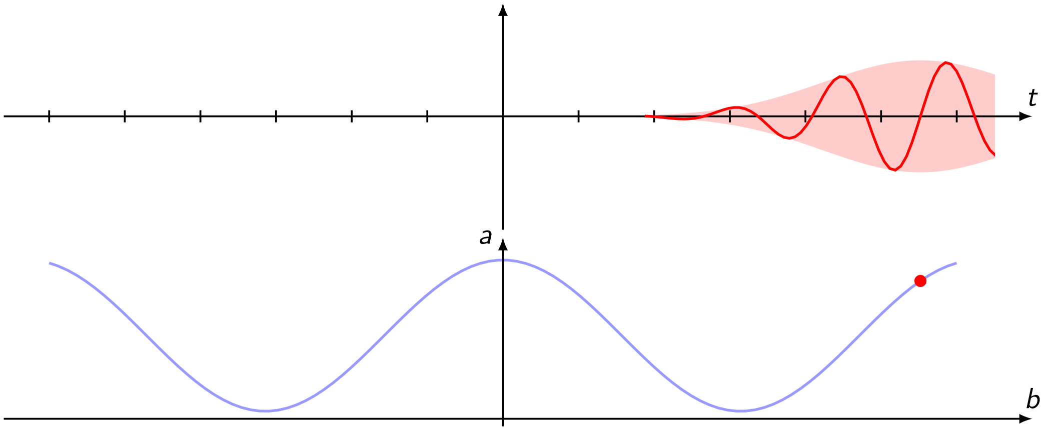

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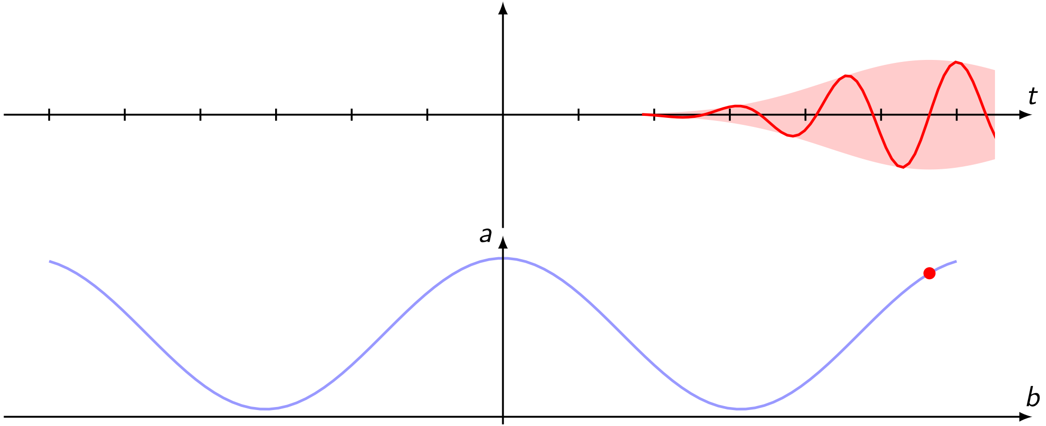

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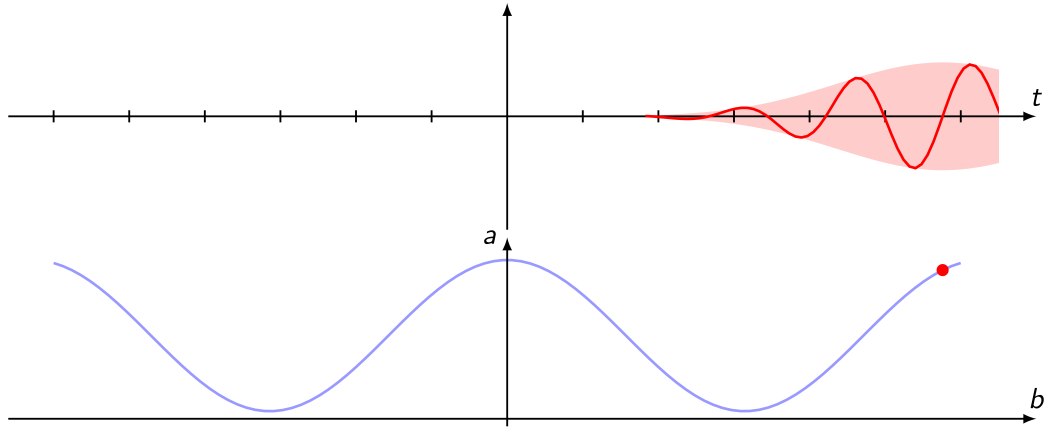

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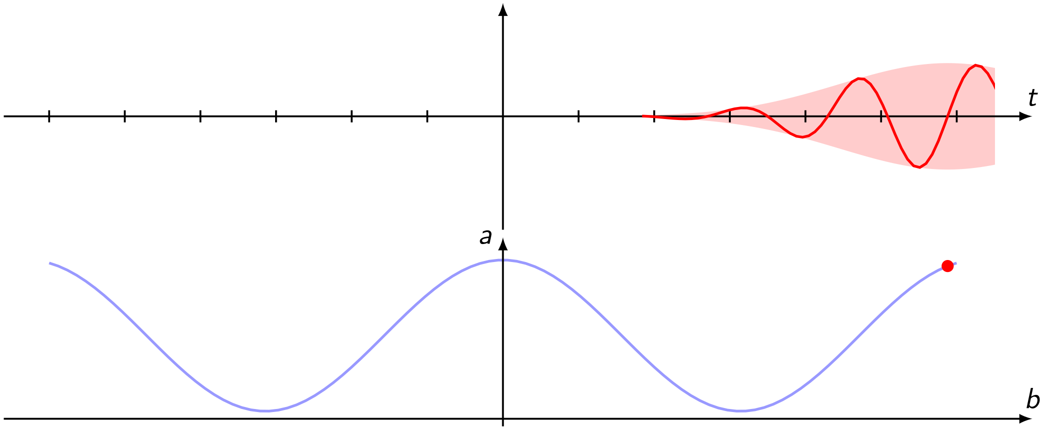

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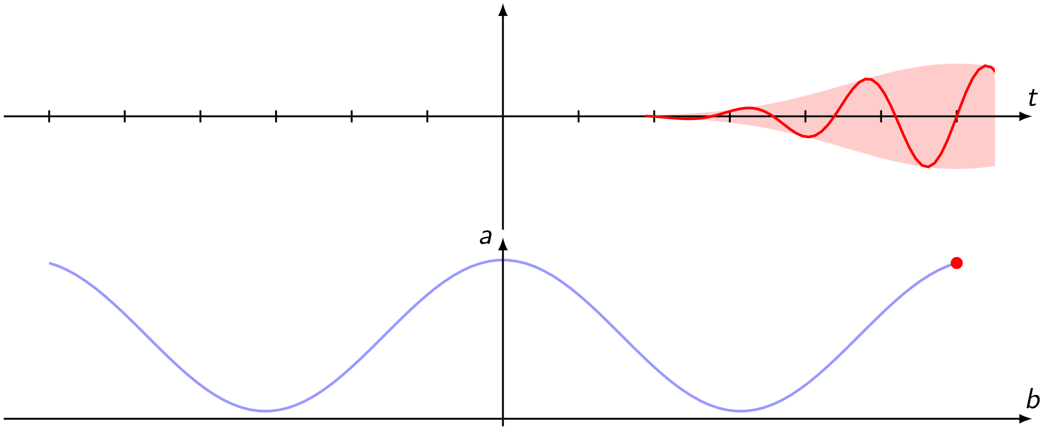

Fourier-Transformation

Definition (Fourier-Transformation)

Für $f \in L^1(\mathbb{R})$ gilt

$$\begin{aligned}(\mathcal{F}f)(\omega) = \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \\ &= \langle f, e_\omega \rangle, \qquad \text{mit} \quad e_\omega(t) = e^{i\omega t}\end{aligned}$$

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&= \langle f, e_\omega \rangle, \qquad \text{mit} \quad e_\omega(t) = e^{i\omega t}
\end{aligned}
$$

Plancherel-Formel

Die Fourier-Transformation ist eine Isometrie:

$$
\int_{-\infty}^{\infty} f(t)\bar{g}(t)\, dt = \langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle = \int_{-\infty}^{\infty} \hat{f}(\omega)\bar{\hat{g}}(\omega)\, d\omega
$$

Fourier-Transformation und T_b

$$\widehat{T_b f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_b f(t)\, e^{-i\omega t}\, dt$$

Fourier-Transformation und T_b

$$\widehat{T_bf}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T_bf(t)\,e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\underbrace{t-b}_{=t'})\,e^{-i\omega t}\,dt$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t')\, e^{-i\omega(t'+b)}\, dt'$$

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&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')\,e^{-i\omega(t'+b)}\,dt' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t')\,e^{-i\omega t'}\,dt'\cdot e^{-i\omega b}
\end{aligned}
$$

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&= e^{-i\omega b} \hat{f}(\omega)
\end{aligned}$$

Satz

Die Fouriertransformierte der verschobenen Funktion ist

$$\widehat{T_b f} = e^{-i\omega b} \hat{f}$$

Fourier-Transformation und T_b

$$\widehat{T_bf}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} T_bf(t)\,e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\underbrace{t-b}_{=t'})\,e^{-i\omega t}\,dt$$

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$$= e^{-i\omega b}\hat{f}(\omega)$$

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Die Fouriertransformierte der verschobenen Funktion ist

$$\widehat{T_bf} = e^{-i\omega b}\hat{f} =: M_{e^{-i\omega b}}\hat{f}$$

Fourier-Transformation und T_b

$$\widehat{T_bf}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T_bf(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\underbrace{t-b}_{=t'})\, e^{-i\omega t}\, dt$$

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$$\widehat{T_bf} = e^{-i\omega b}\hat{f} =: M_{e^{-i\omega b}}\hat{f}$$

$$\widehat{M_{e^{i\omega b}}f} = T_b\hat{f}$$

Fourier-Transformation und D_a

$$\widehat{D_af}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D_af(t)\, e^{-i\omega t}\, dt$$

Fourier-Transformation und D_a

$$\widehat{D_a f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D_a f(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(\underbrace{t/a}_{=t'})\, e^{-i\omega t}\, dt$$

Fourier-Transformation und D_a

$$\widehat{D_af}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D_af(t)\,e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(\underbrace{t/a}_{=t'})\,e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(t')e^{i\omega at'}\,|a|\,dt'$$

Fourier-Transformation und D_a

$$
\begin{aligned}
\widehat{D_a f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D_a f(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(\underbrace{t/a}_{=t'})\, e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(t') e^{i\omega a t'} |a|\, dt' = \sqrt{|a|} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') e^{i(a\omega)t'}\, dt'
\end{aligned}
$$

Fourier-Transformation und D_a

$$\widehat{D_a f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D_a f(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} f(\underbrace{t/a}_{=t'})\, e^{-i\omega t}\, dt$$

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$$= (D_{1/a}\hat{f})(\omega)$$

Fourier-Transformation und D_a

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$$= (D_{1/a}\hat{f})(\omega)$$

Satz

Die Fouriertransformierte der gestreckten Funktion ist

$$\widehat{D_a f}(\omega) = (D_{1/a}\hat{f})(\omega)$$

Stetige Wavelet Transformation (CWT)

Analyse mit verschobenen und gestreckten Kopien von ψ:

$$\psi_{a,b}(t) = T_b D_a \psi(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$$

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- Translation um b

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Definition (Stetige Wavelet Transformation)

Für ein Wavelet ψ ist die Stetige Wavelet-Transformation eines Signals $f(t)$ die Funktion von zwei Variablen $(a, b) \in \mathbb{R}^* \times \mathbb{R}$

$$\mathcal{W}f(a, b) = \mathcal{W}_\psi f(a, b) = \langle f, \psi_{a,b} \rangle = \langle f, T_b D_a \psi \rangle$$

Stetige Wavelet Transformation (CWT)

Analyse mit verschobenen und gestreckten Kopien von ψ:

$$\psi_{a,b}(t) = T_b D_a \psi(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right)$$

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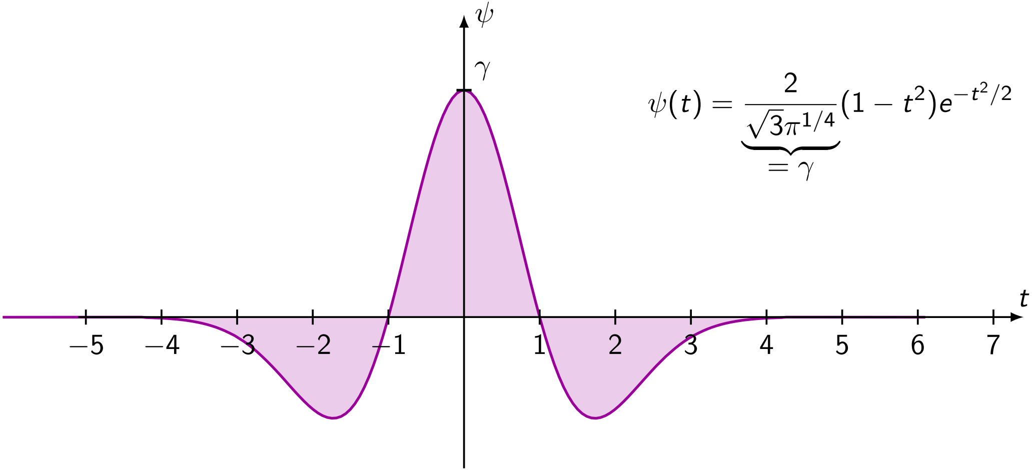
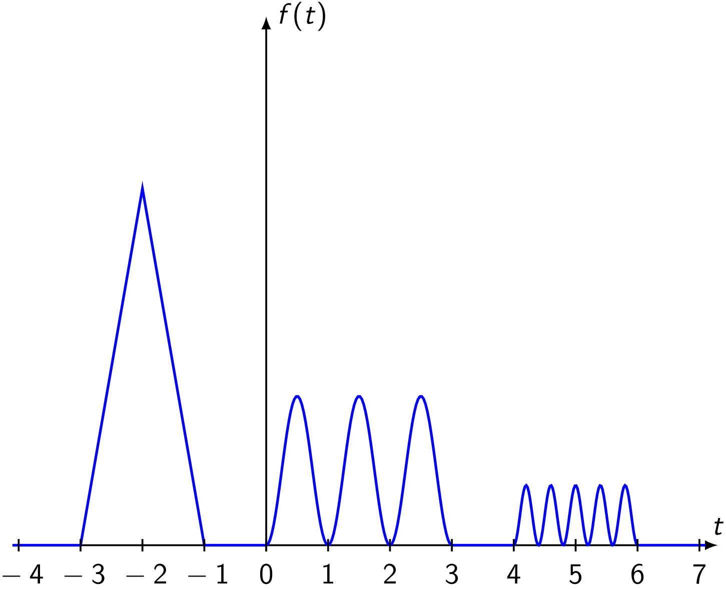
$$\begin{aligned}\mathcal{W}f(a,b) = \mathcal{W}_\psi f(a,b) = \langle f, \psi_{a,b}\rangle &= \langle f, T_b D_a \psi\rangle \\ &= \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)}\, dt.\end{aligned}$$

Beispiel: Analyse mit Mexikanerhut



$$\psi(t) = \underbrace{\frac{2}{\sqrt{3}\pi^{1/4}}}_{=\gamma}(1-t^2)e^{-t^2/2}$$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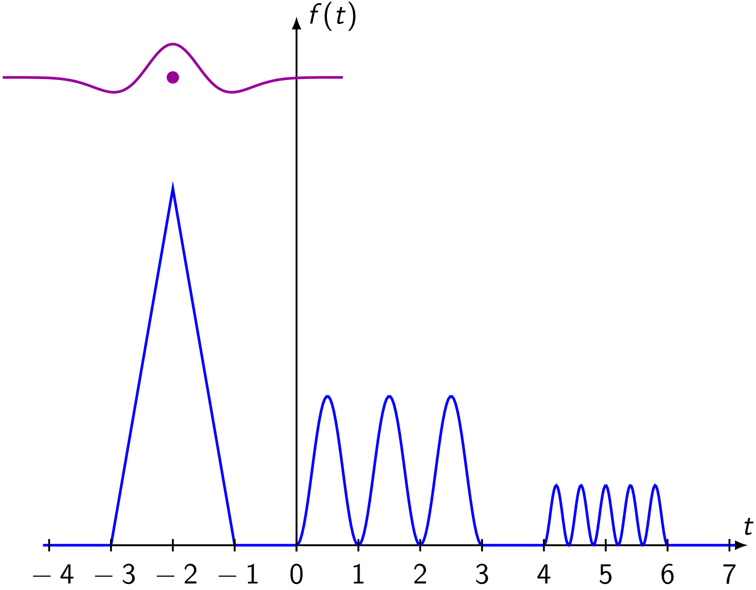
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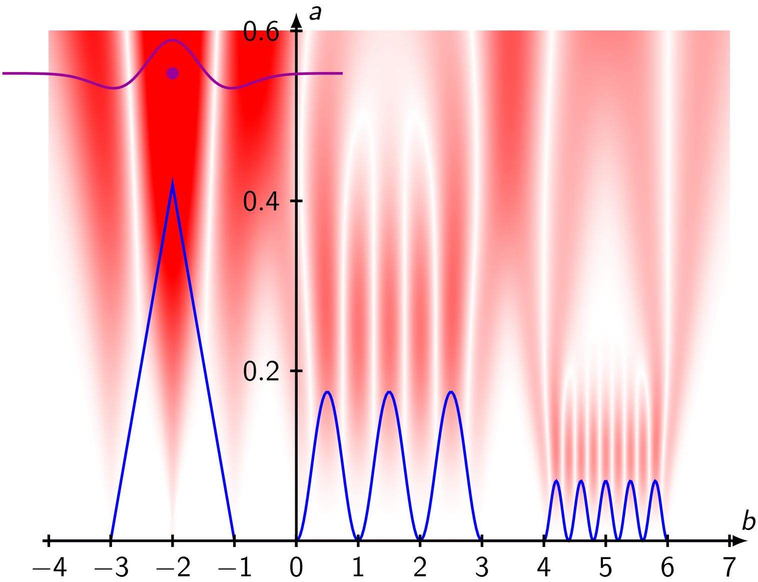
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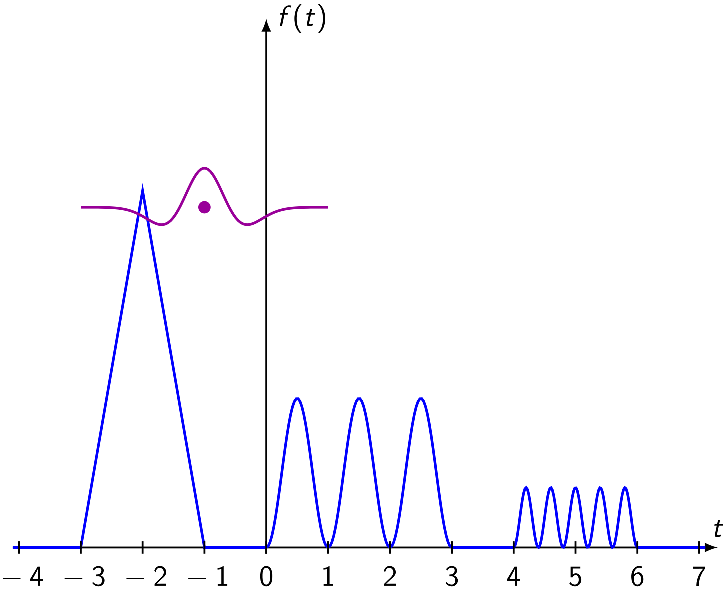
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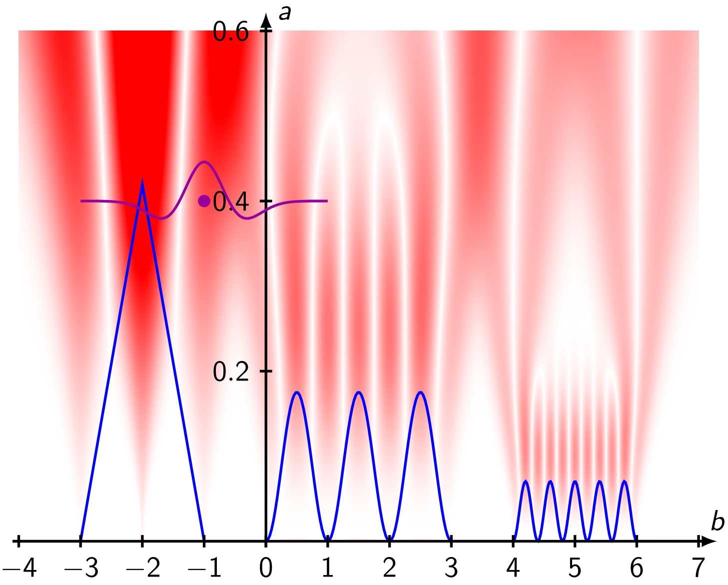
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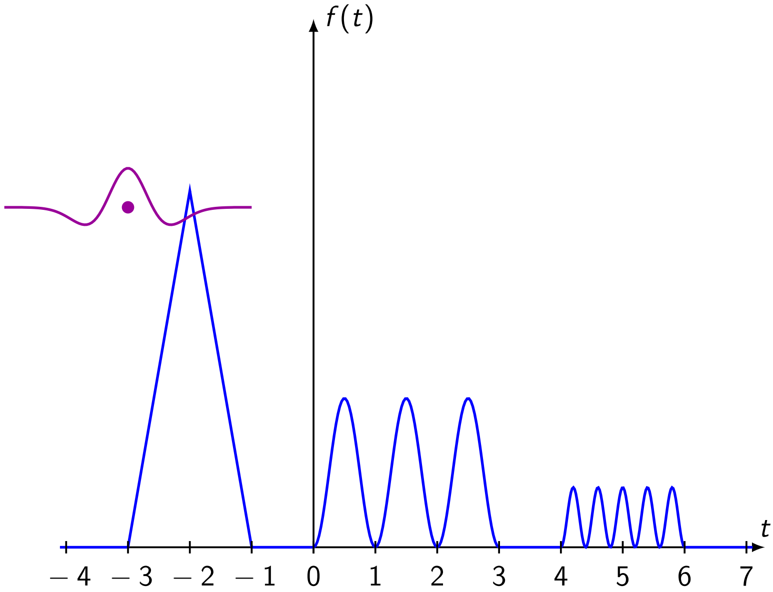
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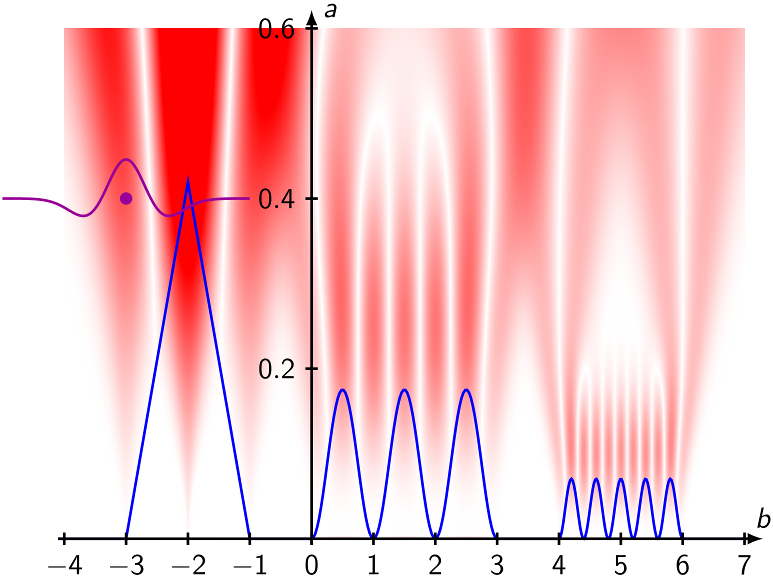
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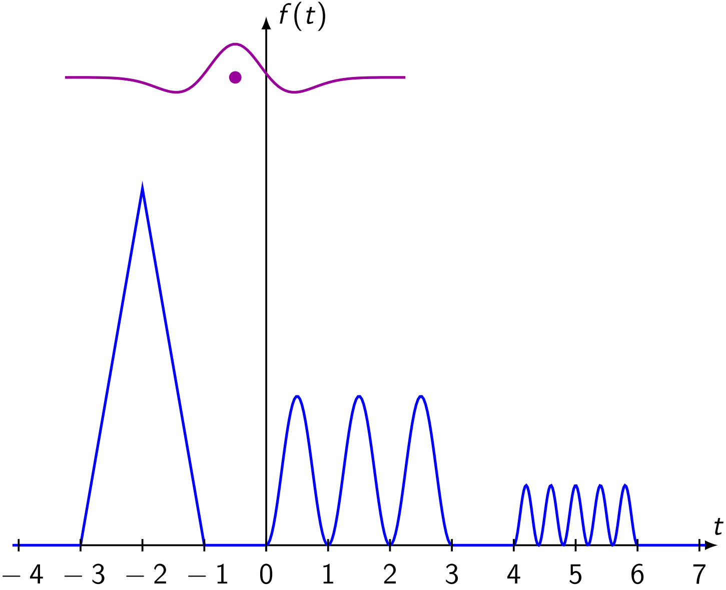
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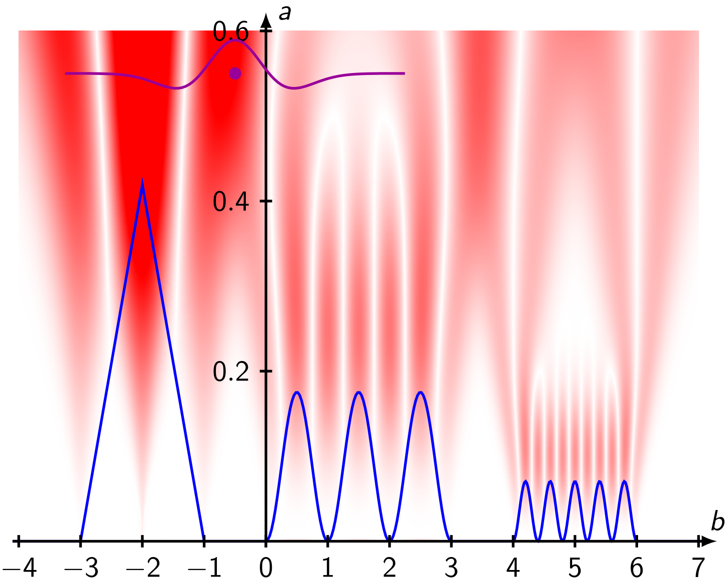
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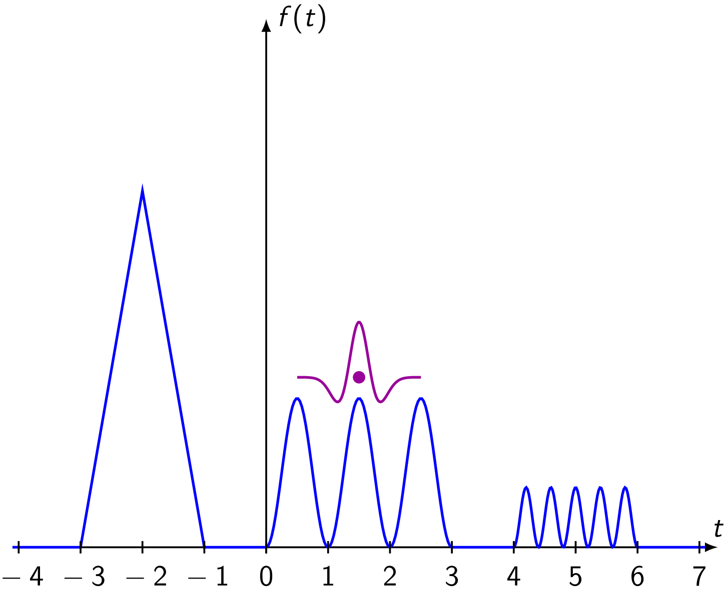
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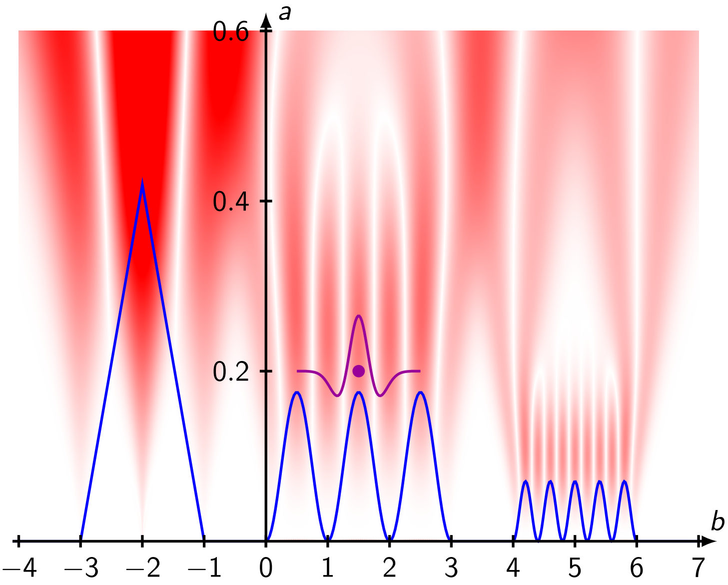
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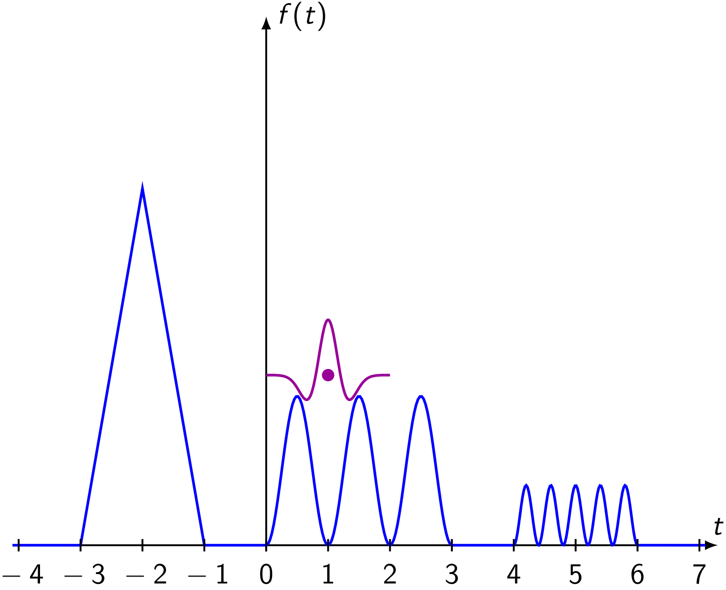
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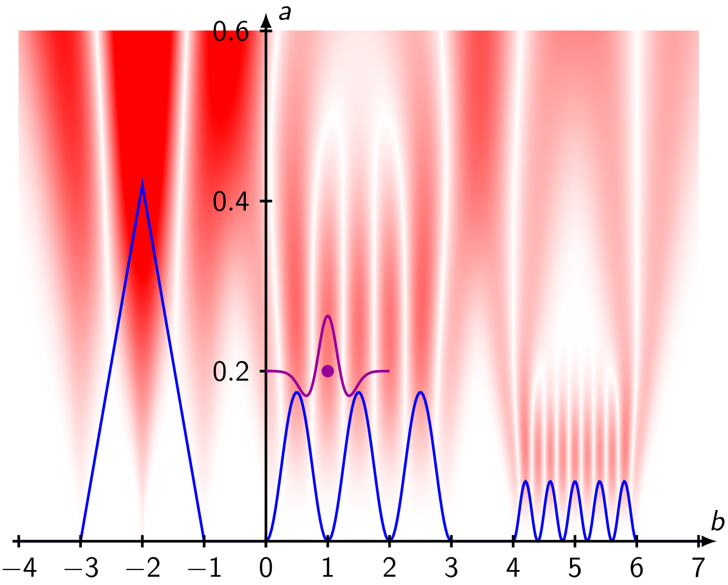
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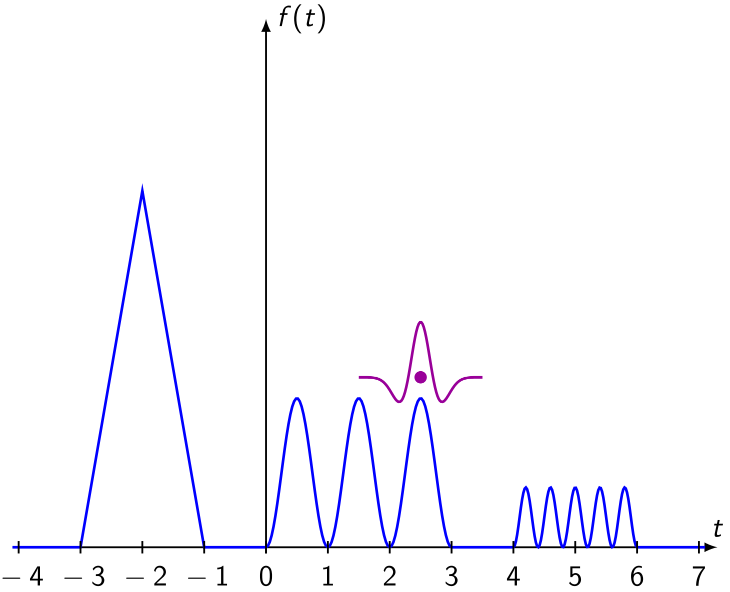
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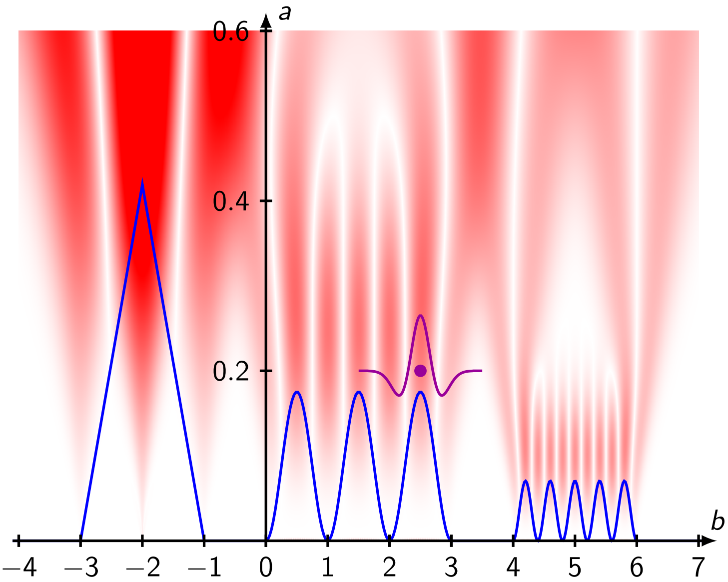
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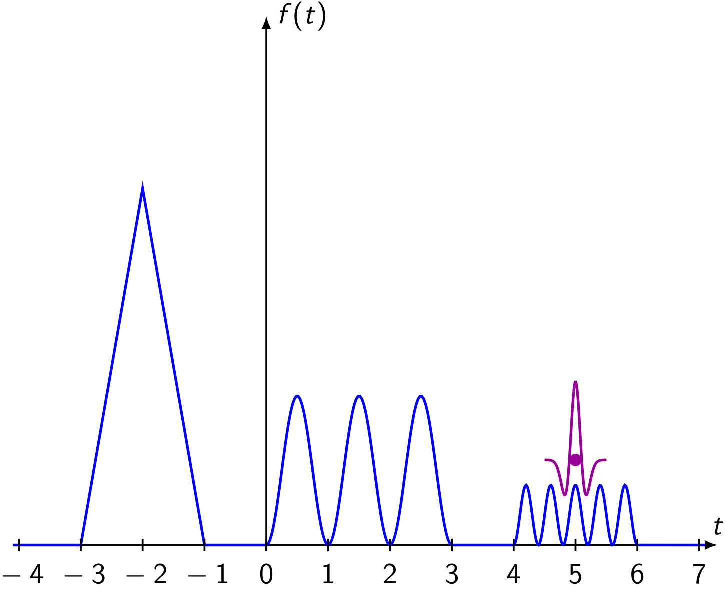
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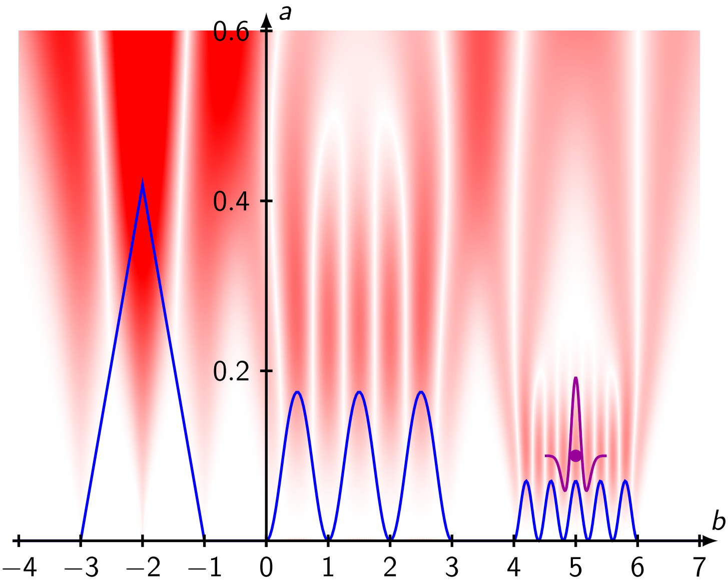
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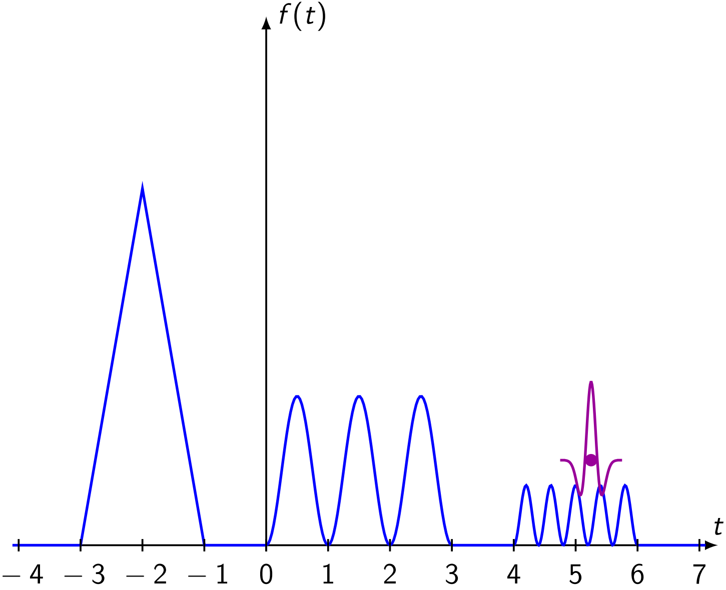
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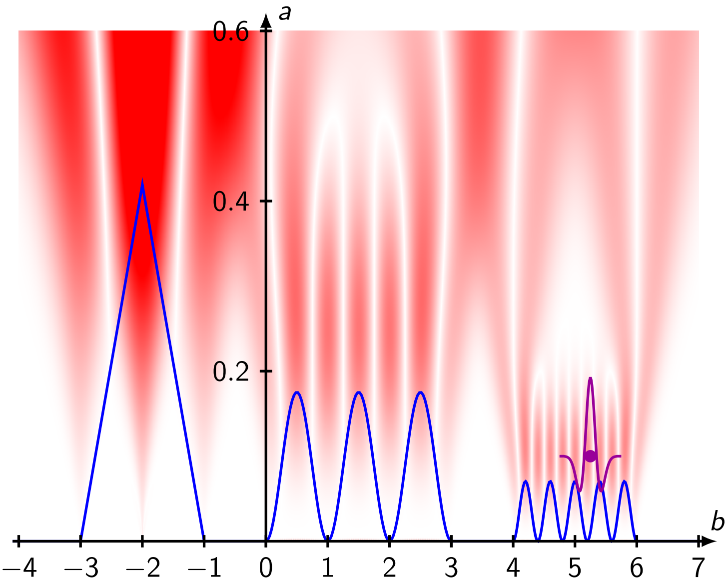
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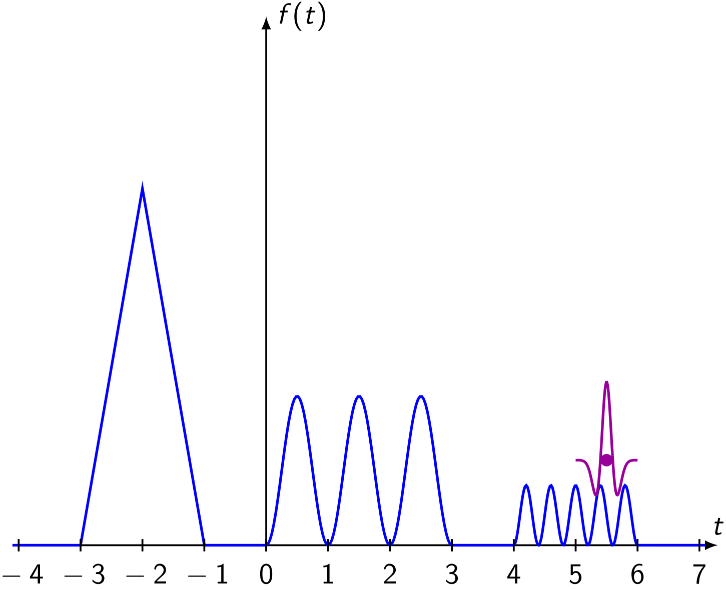
$f(t)$
t
-4
-3
-2
-1
0
1
2
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6
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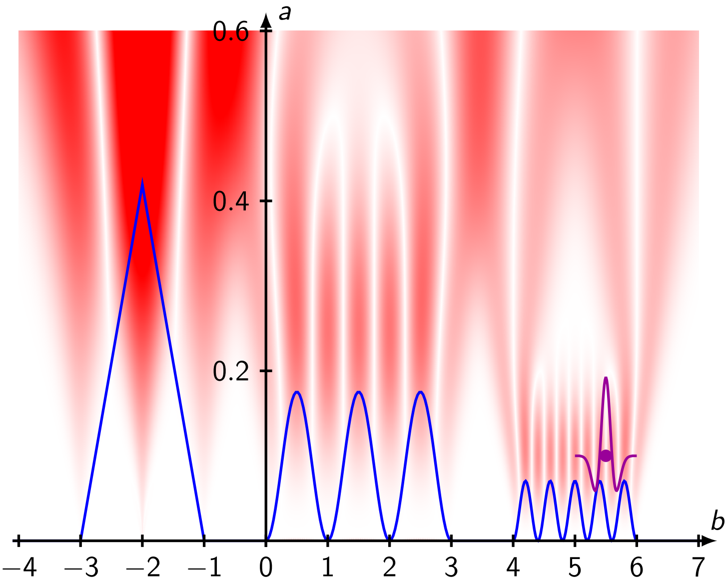
a
0.6
0.4
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b
−4
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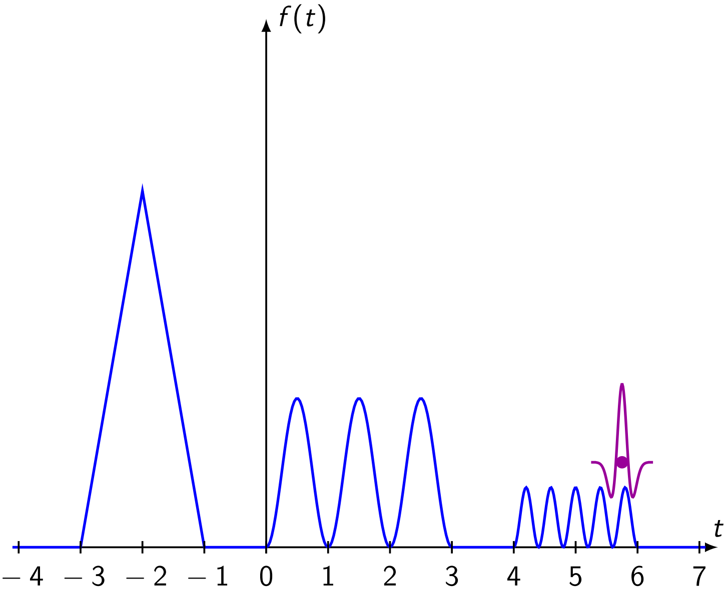
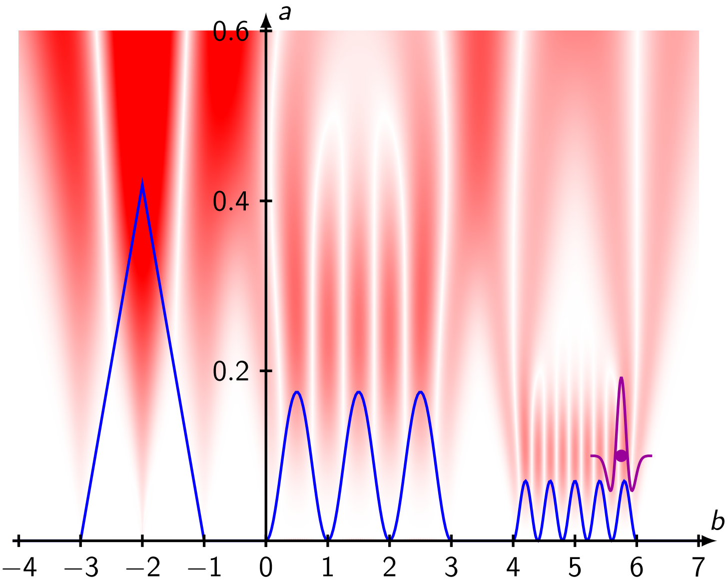

$f(t)$
t
-4 -3 -2 -1 0 1 2 3 4 5 6 7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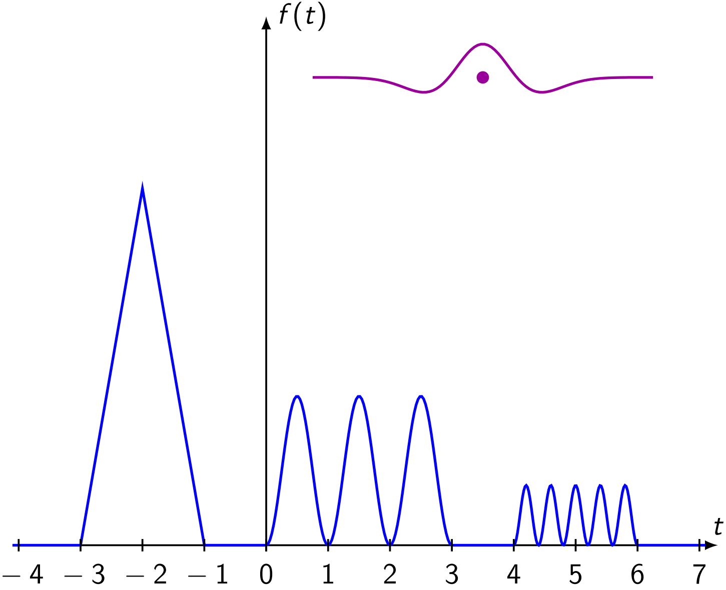

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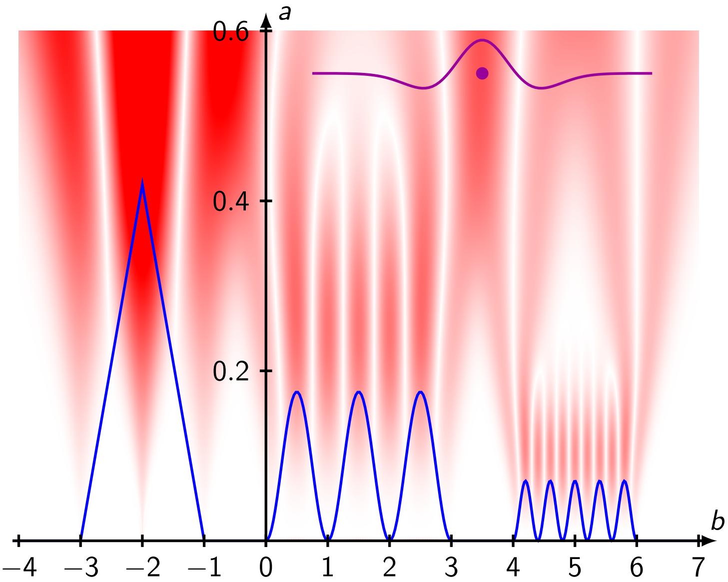
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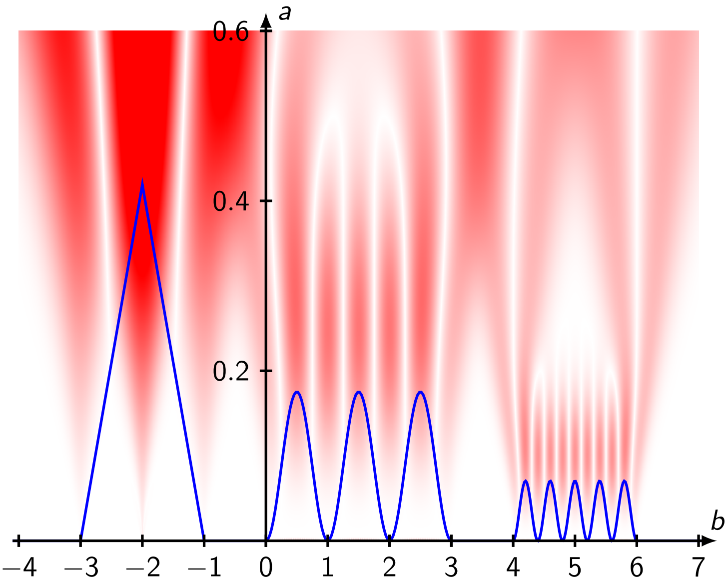
a
0.6
0.4
0.2
b
−4
−3
−2
−1
0
1
2
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4
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6
7

Analyse eines Sweep mit Morlet-Wavelet

Wavelet:

$$\psi(t) = e^{-t^2/2} \cdot e^{5it}$$

Signal:

$$f(t) = \sin(t \cdot (4 + 0.2t))$$

Beobachtungen

Eigenschaften von $\mathcal{W}$

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- $\mathcal{W}$ ist linear

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$$\mathcal{W}(f+g) = \mathcal{W}f + \mathcal{W}g$$

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Eigenschaften von $\mathcal{W}$

- $\mathcal{W}$ ist linear:

$$
\begin{aligned}
\mathcal{W}(f+g) &= \mathcal{W}f + \mathcal{W}g \\
\mathcal{W}(\lambda f) &= \lambda \mathcal{W} f
\end{aligned}
$$

Beobachtungen

Eigenschaften von $\mathcal{W}$

- $\mathcal{W}$ ist linear:

$$\begin{aligned}\mathcal{W}(f+g) &= \mathcal{W}f + \mathcal{W}g\\ \mathcal{W}(\lambda f) &= \lambda\mathcal{W}f\end{aligned}$$

- $\mathcal{W}$ ist injektiv

Beobachtungen

Eigenschaften von $\mathcal{W}$

- $\mathcal{W}$ ist linear:

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- $\mathcal{W}$ ist injektiv $\Rightarrow$ Umkehrformel?

Beobachtungen

Eigenschaften von $\mathcal{W}$

- $\mathcal{W}$ ist linear:

$$\begin{aligned} \mathcal{W}(f+g) &= \mathcal{W}f + \mathcal{W}g \\ \mathcal{W}(\lambda f) &= \lambda \mathcal{W}f \end{aligned}$$

- $\mathcal{W}$ ist injektiv $\Rightarrow$ Umkehrformel?
- $\mathcal{W}$ ist nicht surjektiv

Wavelet

Definition

Ein *Mutter-Wavelet* oder *Wavelet* ist eine Funktion $\psi\colon \mathbb{R} \to \mathbb{C}$ mit

$$\psi \in L^2(\mathbb{R}) \qquad \text{und} \qquad \|\psi\| = 1,$$

welche zudem die Zulässigkeitsbedingung erfüllt.

Zulässigkeitsbedingung

$\psi \in L^2(\mathbb{R})$ heisst zulässig, wenn

$$C_\psi = \int_{\mathbb{R}^*} \frac{|\hat{\psi}(\omega)^2|}{|\omega|}\, d\omega < \infty$$

Die Zulässigkeitsbedingung wird benötigt für die Umkehrformel.