Binary Search Trees

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Maps

Definition

A **record** is a key-value pair: (k, v)

A **map** is an abstract data type for maintaining a set of records

- No two records can have the same key
- However, two records can have same values though
- Association of keys to values define a **mapping**: $f(\text{key}) = \texttt{value}$

Examples of maps

- UNF maintains a map of `(N#, student information)` records
- A social media company maintains a map of `(email address, user account information)` records
- An assembler maintains a symbol table (a map) of `(opcode, hex)` records
- A text-editor maintains a map of `(color, RGB representation)` records

How to implement a map?

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

Approach 1: maintain a sorted list of records

- **Insertion.** will take $O(n)$ time for figuring out the correct spot for the incoming record; then $O(n)$ time for shifting items to the right to accommodate the new record; total time taken is $O(n) + O(n) = O(n)$
- **Retrieval.** will take $O(\log n)$ time using binary search
- **Deletion.** will take $O(\log n)$ time to locate it using a binary search; then then $O(n)$ time for left shifting items to kill the empty spot; total time taken $O(\log n) + O(n) = O(n)$

How to implement a map?

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

Approach 2: maintain an unsorted list of records

- **Insertion.** will take $O(1)$ time (add the new record at the end)
- **Retrieval.** will take $O(n)$ time using a linear search; may need to search the whole list in the worst-case
- **Deletion.** will take $O(n)$ time to locate it using a linear search; then $O(n)$ time for left shifting items to kill the empty spot; total time taken $O(n)+O(n) = O(n)$

Our aim

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

☞ To accomplish the above three tasks in $O(\log n)$ time each

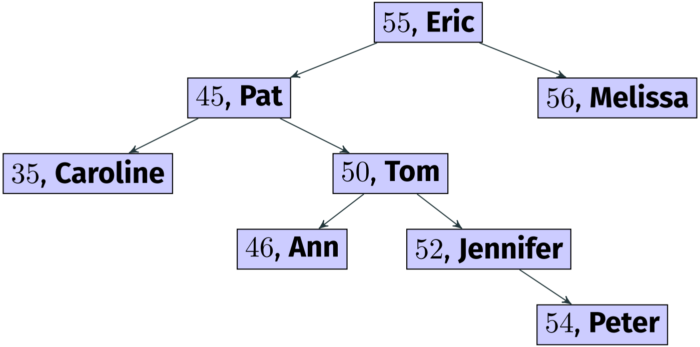
☞ **Balanced** binary search trees is the solution; stay tuned ...

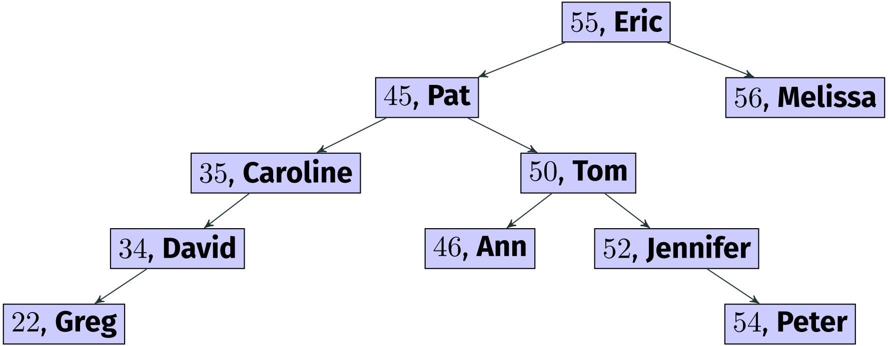
The map ADT

```java
public interface MapADT<K,V> {
   boolean put(K key, V value); // adds a new record with key 'key' and value 'value'
   V remove(K key); // removes the record having key 'key'
   V get(K key); // return the value part of the record whose key is 'key'
   V updateValue(K key, V newValue); // updates the value part of the record whose key is 'key' with a new value
   int size(); // returns the number of records stored in the map
   void clear(); //Removes all records from the map
}
```

What is a Binary Search Tree?

- It is a **binary tree** where every node contains a **<key, value>** pair (a **record**); keys must be **comparable** but the values don't need to be
- Moreover, for every node p in the tree, the following 2 properties hold
  1. Keys stored in the left subtree of p are $<$ the key stored at p
  2. Keys stored in the right subtree of p are $>$ the key stored at p



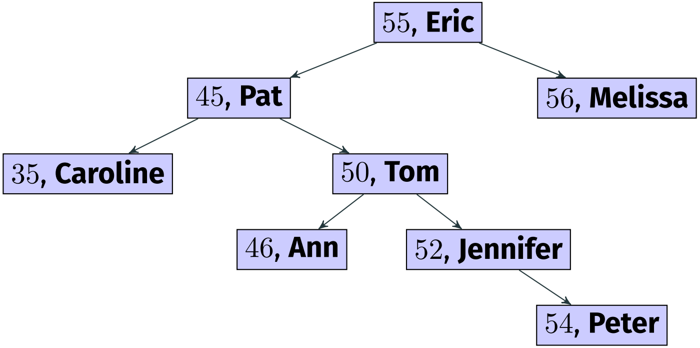


Use

BSTs can be used to implement **maps** and are commonly used for fast searching (typically need far less comparisons than lists)

An important property of BSTs

An inorder traversal of a BST always gives the sorted sequence based on the keys



Inorder traversal

35, **Caroline**; 45, **Pat**; 46, **Ann**; 50, **Tom**; 52, **Jennifer**; 54, **Peter**; 55, **Eric**; 56, **Melissa**

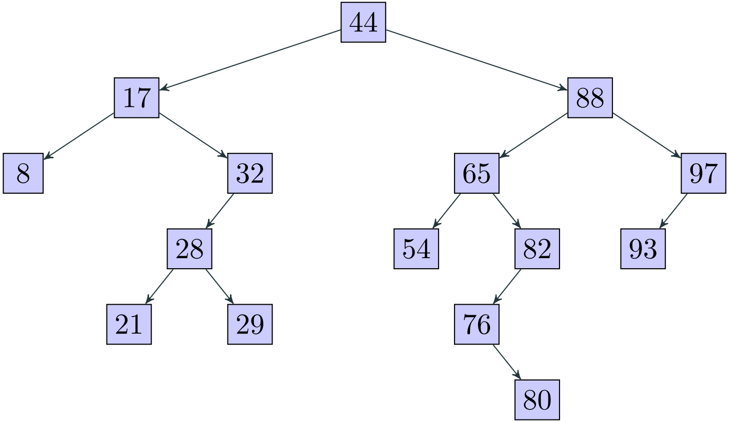
Let's say you need to look for the record that has the key k; how will you do this?

Algorithm

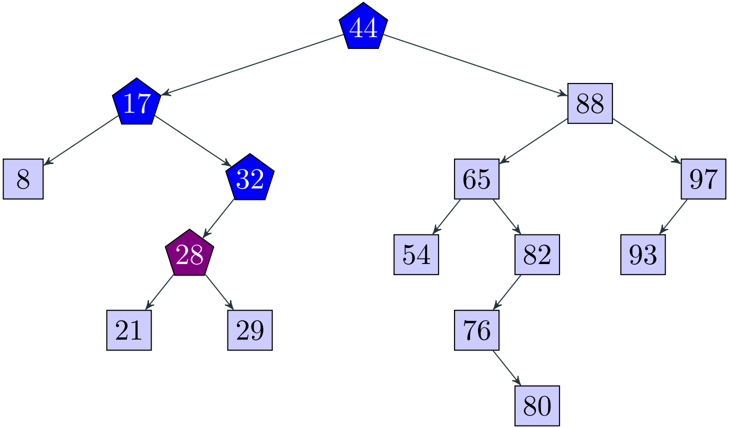
- Start at the root
- If the root's key is k, then search is successful
- If $k <$ root's key, search recursively (or iteratively) in the left subtree of the root
- Otherwise, search recursively (or iteratively) in the right subtree of the root
- If we have reached a `null` link, no record exists in the tree with key k

To save space in the figures, we will write only the record keys inside the nodes and avoid the corresponding values

Example

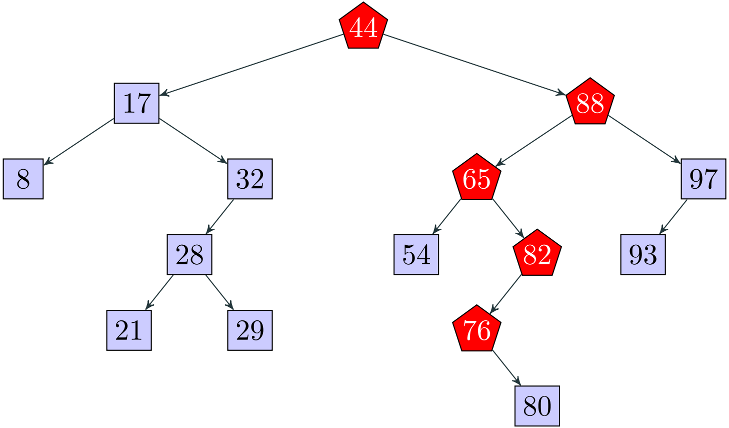


Example: search for 28



Found!

Example: search for 68



Not found; a null link is reached (the left link of 76)

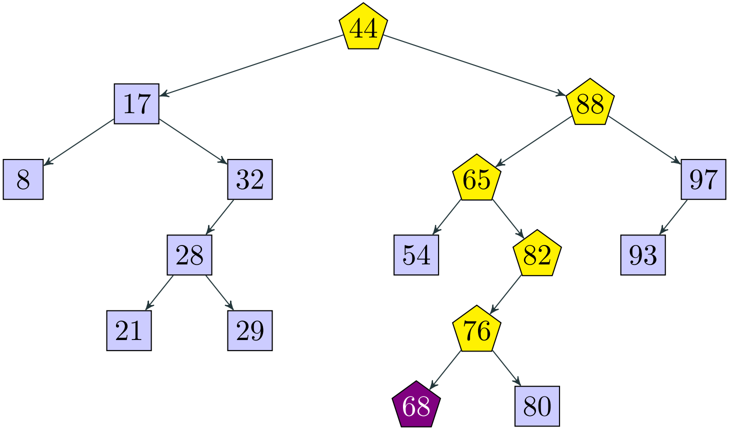
$O(h)$, where h is the height of the BST under consideration, since, for searching, we need to traverse a path whose length is h in the worst-case

How to insert a record into a BST having key k?

Algorithm

- Start at the root
- If the root is `null`, replace empty tree with a new tree with the new record as the root, and signal **SUCCESS**
- If k equals root's key, signal **FAILURE** since a record with key k already exists
- If $k <$ root's key, insert recursively (or, iteratively) in the left subtree of the root
- Otherwise, insert recursively (or, iteratively) in the right subtree of the root

Example: insert 68



Time complexity of insertion

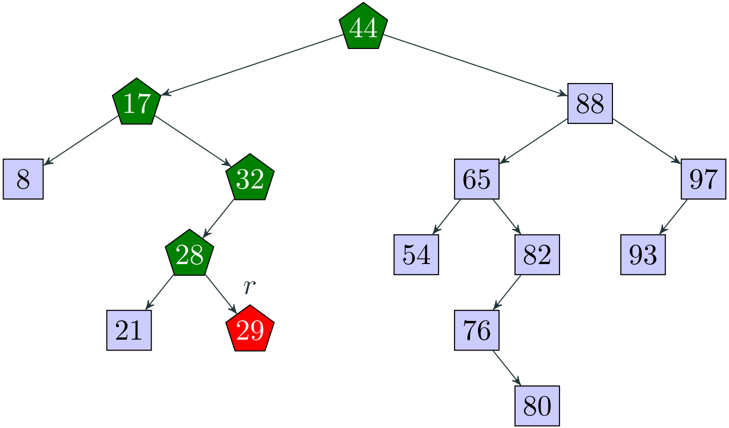
$O(h)$, where h is the height of the BST under consideration, since, for inserting a new record, we need to traverse a path whose length is h in the worst-case

How to delete the record from a BST having key k?

Idea

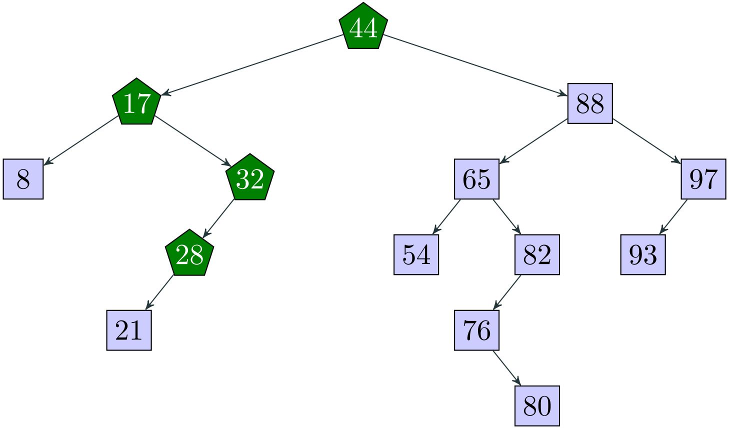
- Traverse through the tree to find the record having key k
- If the record cannot be found, no action is needed
- Now assume that we have found the record that has key k at node r
  1. r is a leaf node (no child)
  2. r has exactly one child (either left or right)
  3. r has two children (both left and right)

Case 1: r is a leaf node, delete 29

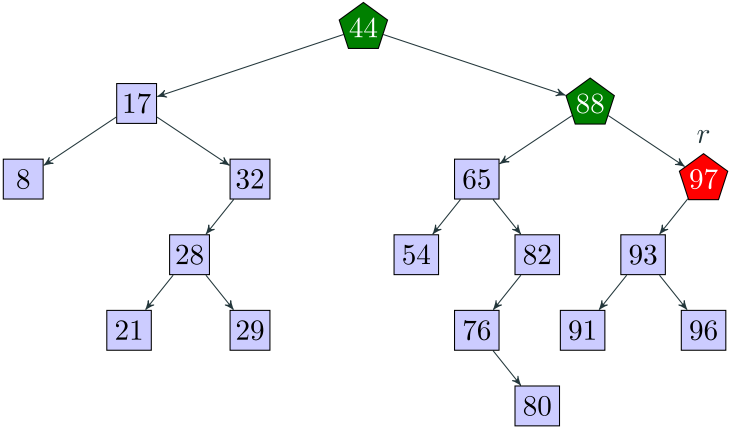


Easy! just delete it right away

Case 1: r is a leaf node, delete 29

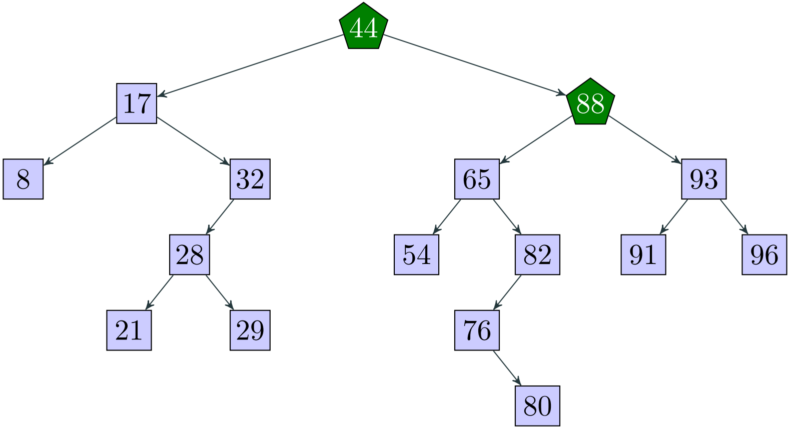


Case 2: r has one child, delete 97

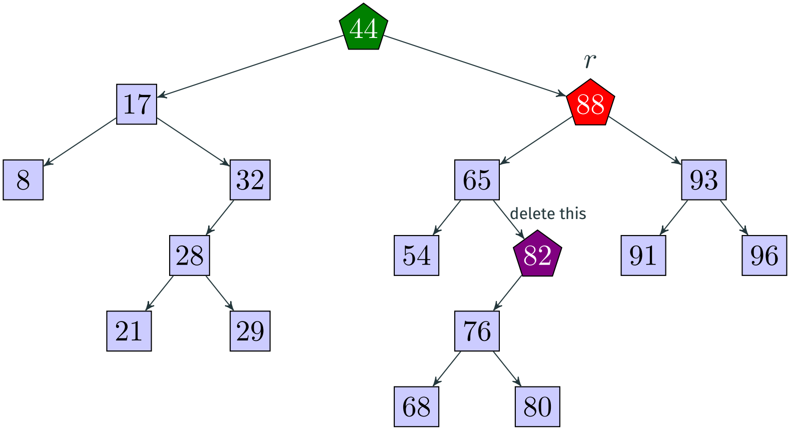


Make the parent of r point to the only child of r instead of r

Case 2: r has one child, delete 97

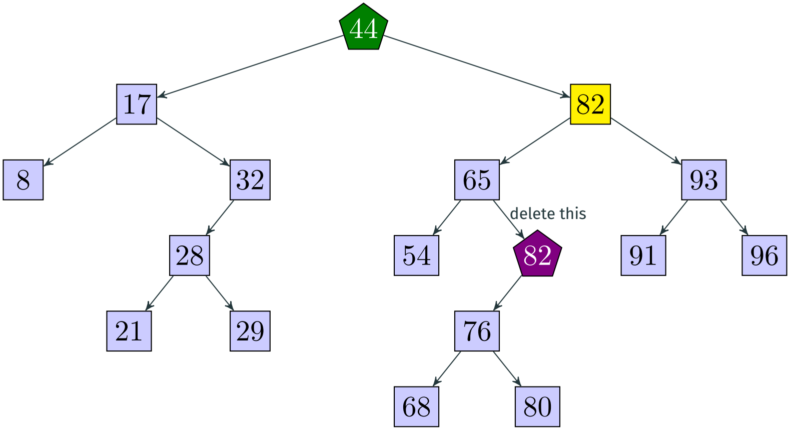


Case 3: r has two children, delete 88



Replace the record at r with the record at the inorder predecessor p of n; then delete p using Case 1 or 2, depending on the number of children of p. Note that the inorder predecessor is either a leaf node or has a left child only (no right child).

Case 3: r has two children, delete 88



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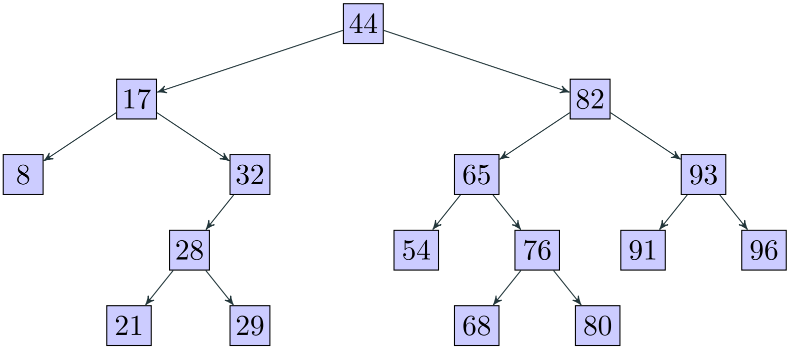
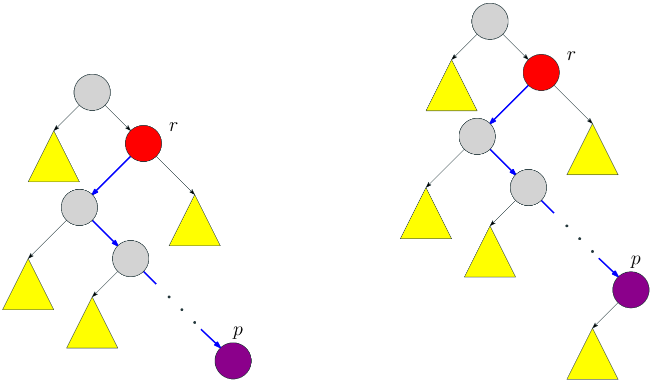


Replace the record at r with the record at the inorder predecessor p of n; then delete p using Case 1 or 2, depending on the number of children of p. Note that the inorder predecessor is either a leaf node or has a left child only (no right child).

Finding the inorder predecessor in Case 3



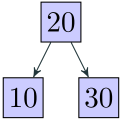
The inorder predecessor p of node r is either a leaf node (left figure) or has a left child but no right child (right figure). So, to find p, go left of r and then keep on moving right as long as possible. It takes $O(h)$ time to find p, where h is the height of the tree. The yellow triangles represent subtrees (some of them could be empty).

Time complexity of deletion

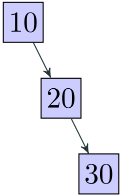
- It takes $O(h)$ time for locating the record to be deleted, where h is the height of the BST
- Case 1 takes $O(1)$ time to delete a record
- Case 2 takes $O(1)$ time to delete a record
- Case 3 takes $O(h)$ time to delete a record since we need to locate the inorder predecessor p of node r in $O(h)$ time and then delete p in $O(1)$ time using Case 1 or 2
- So, a single deletion takes $O(h)$ time in the worst-case

See the class `TreeMapBST`

BSTs are not unique



Insertion sequence: $20, 10, 30$



Insertion sequence: $10, 20, 30$

Their structures really depend on the insertion sequence of records

An application

Problem

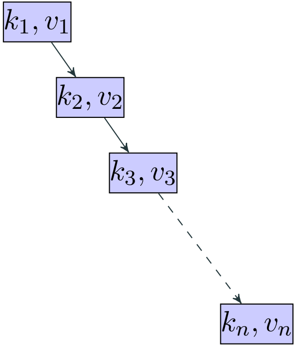
Given a text, find out the unique words in it along with their counts. We also need to output the distinct words with their count.

A solution

- Use a BST T where the key-type is `String` and the value-type is `Integer`. This means every node will store a word from the text and its count in the same text.
- For every word w in the text, first check if a node exists in T, where the stored key is w
  - If such a node does not exist, insert a new node in T with key w and value 1
  - If such a node exists in T, increment the stored value (essentially a counter) by 1

See the class `WordCounter`

Worst case scenario: skewed binary trees



In this case, $k_1 < k_2 < k_3 < \ldots < k_n$ and the tree is right-skewed

Note that when $k_n < k_{n-1} < \ldots < k_1$, the tree will be left-skewed (try!)

So, in the worst-case, $h = n - 1 = O(n)$

Therefore, searching, insertion, deletion take $O(h) = O(n)$ time each!

This is as bad as using singly linked-lists!

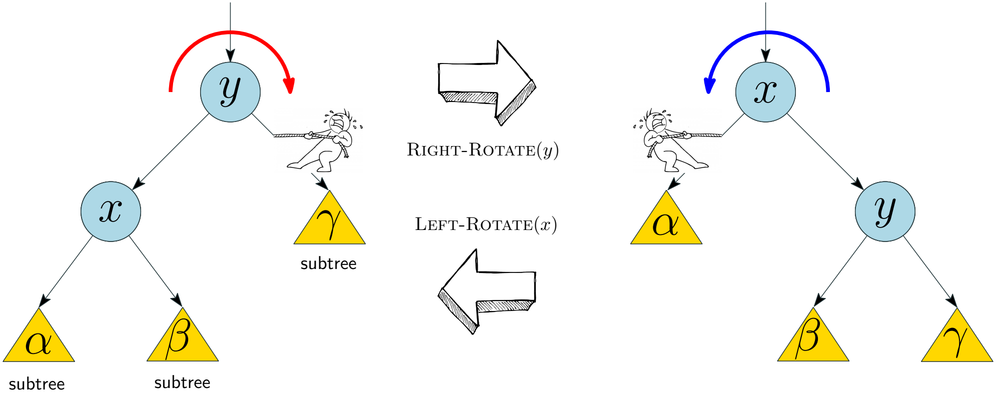
Idea

Do something so that h remains under control (logarithmic)
We aim for $h = O(\log n)$

A solution. use **Red-Black** trees

Red-Black trees are never skewed or close to being skewed unlike the plain binary search trees we just talked about

Rotation on BSTs

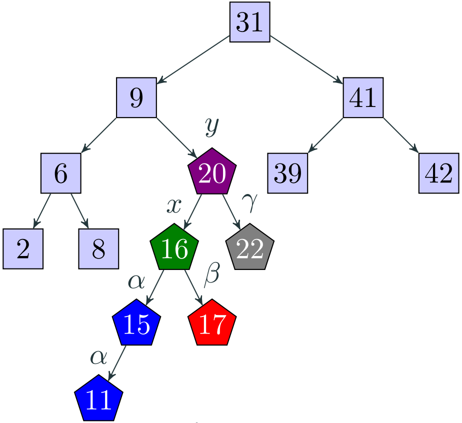


Rotations help in **reducing** height of BSTs; this means faster operations on BSTs

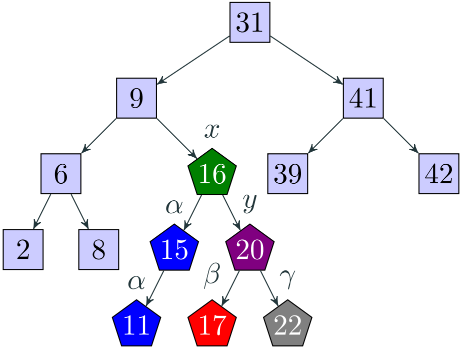
A single rotation can be done in $O(1)$ time

☞ Some of the three subtrees (α, β, γ) can be empty (devoid of nodes).

An example of rotation



Height: 5

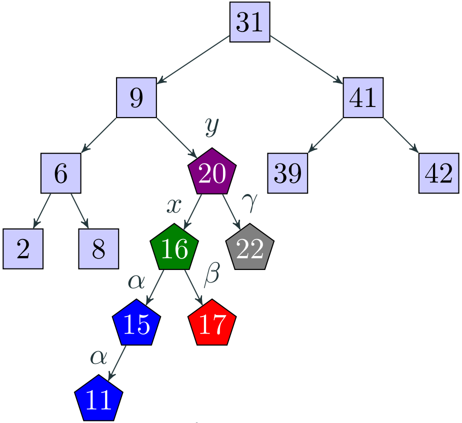


Height: 4

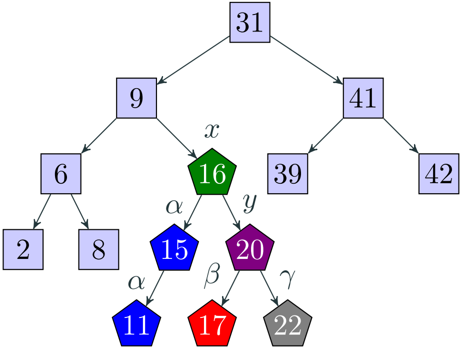
☞ If a **right** rotation is performed at node y in the left tree, we get the tree on the right.

☞ If a **left** rotation is performed at node x in the right tree, we get back the tree on the left.

An example of rotation



Height: 5



Height: 4

Note that rotations never alter the inorder traversal sequences. It also means that after a rotation, the resulting binary tree is still a binary search tree.

Code for rotation

Right rotation at node y

```
private void rightRotateAt(Node<K,V> y) {
   Node<K,V> x = y.left;
   y.left = x.right;

   if( x.right != null )
      x.right.parent = y;

   x.parent = y.parent;

   if( y.parent == null )
      root = x;
   else if( y == y.parent.right )
      y.parent.right = x;
   else
      y.parent.left = x;

   x.right = y;
   y.parent = x;
}
```

Left rotation at node x

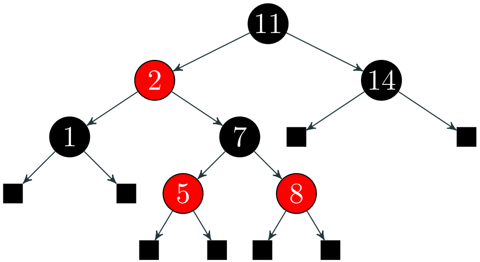
```
private void leftRotateAt(Node<K,V> x) {
   Node<K,V> y = x.right;
   x.right = y.left;

   if( y.left != null )
      y.left.parent = x;

   y.parent = x.parent;

   if( x.parent == null )
      root = y;
   else if( x == x.parent.left )
      x.parent.left = y;
   else
      x.parent.right = y;

   y.left = x;
   x.parent = y;
}
```

A single rotation (left/right) takes $O(1)$ time

Definition

A **red**-**black** tree is a self-balancing BST (height never gets too bad) that has the following properties:

1. Every node is either **red** or **black**
2. The root is **black**
3. If a node is **red**, then both its children are **black**. The `null` links (shown using black squares in the figure) of the leaves are **black**.
4. The number of **black** nodes in any path from the root to a leaf is the same



How to maintain the colors?

Since the colors are bichromatic, a boolean variable `color` is enough to specify the color of a node. Set `color` to `false`, when the node is painted **RED**, or else, set it to `true` to specify the color is **BLACK**.

☞ There is no magic behind these two colors; feel free to use any two colors

A node class implementing for RB-tree

```java
public class TreeMapRBTree<K extends Comparable<K>, V> {
   private static final boolean RED = false, BLACK = true;

   private static class Node<K, V> {
      final private K key;
      private V val;

      private Node<K, V> left, right, parent;

      private boolean color;

      public Node(K k, V v) {
         key = k; val = v;
         left = right = parent = null;
         color = RED;
      }

      public String toString() {
         String colorString = (color == RED)? "RED" : "BLACK";
         if (val != null) return "<" + key.toString() + ", " + val.toString() + ", " + colorString + ">" ;
         else return key.toString();
      }

      // other variables and methods
}
```

Height of a red-black tree

- A RB-tree containing n records has height $\leq 2\log_2(n+1) = O(\log n)$
- **Intuition.** By constraining the node colors on any simple path from the root to a leaf (property 4), it can be ensured that no such path is more than twice as long as any other so that the tree's height is always logarithmic.

Implication

When $n = 1,000,000$, $h \leq 2\log_2(n+1) < 40$.
To search for a record, at most 41 comparisons are required in this case.

In contrast, when a plain binary search tree is used, $h \leq 999,999$.
For searching for a record, at most $1,000,000$ comparisons are required in this case.

This clearly demonstrates the advantage of using RB-trees over plain BSTs and also shows that RB-trees can never be skewed.

☞ *Searching, insertion, and deletion in RB-trees take $O(h) = O(\log n)$ time each.*

1. **Search**. same as the search operation for plain BSTs; takes $O(h) = O(\log n)$ time (note that RB-trees are also BSTs, so the same search algorithm works here too!)
2. **Insertion**. we **will** discuss this; takes $O(\log n)$ time
3. **Deletion**. we **won't** discuss this; takes $O(\log n)$ time

☞ The `TreeMap` class in Java implements RB-Tree

`https://docs.oracle.com/en/java/javase/17/docs/api/java.base/java/util/TreeMap.html`

Inserting a new node z into a RB-tree

1. Color z **red** and insert it as you would in a plain BST
2. If necessary, start fixing the tree (using rotations and recoloring) as long as you see z's parent is **red** (z may change as we climb up the tree); see the cases next

```
while( z.parent != null && z.parent.color == RED ){
   // deal with the cases inside this loop
}
```

3. At the end, color the root using **black**

```
root.color = BLACK;
```

Uncle of a node

The uncle of a node is the sibling of its parent. In some cases, it could be a null link if there is no such sibling node.

```
private boolean isLeftChild( Node<K,V> node ) {
   return node.parent != null && node.parent.left == node;
}

private Node<K,V> uncle( Node<K,V> node ){
   if( node == null ) return null;
   return ( isLeftChild(node.parent) ) ? node.parent.parent.right : node.parent.parent.left;
}
```

Grandparent of a node

The grandparent of a node is the parent of its parent. In some cases, it could be a null link if there is no such grandparent node.

```
private Node<K,V> grandParent( Node<K,V> node ){
   return node.parent.parent;
}
```

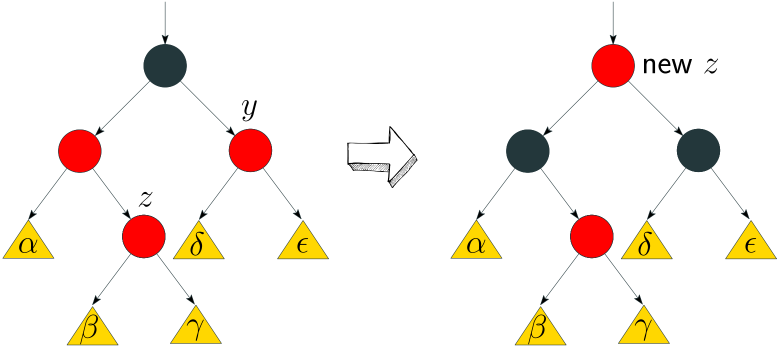
The six cases of insertion

- (Case 1a) z's uncle y is **RED** and z is a right child
- (Case 1b) z's uncle y is **RED** and z is a left child
- (Case 2a) z is a right child, z's parent is a left child, and z's uncle y is **BLACK**
- (Case 2b) z is a left child, z's parent is a right child, and z's uncle y is **BLACK**
- (Case 3a) z is a left child, z's parent is left child, and z's uncle y is **BLACK**
- (Case 3b) z is a right child, z's parent is a right child, and z's uncle y is **BLACK**

Case 1

(Case 1a) z's uncle y is **RED** and z is a right child
(Case 1b) z's uncle y is **RED** and z is a left child

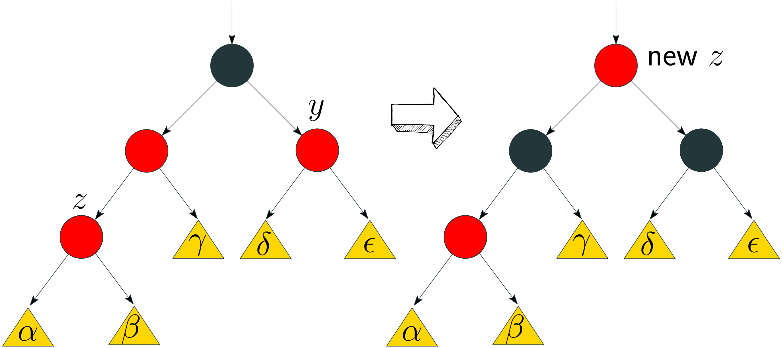
Case 1a



z's uncle y is **red** and z is a right child; recoloring is needed but no rotation; takes $O(1)$ time; **continue fixing up** using the new z node

- `z.parent.color =` **BLACK**; `y.color =` **BLACK**; `grandParent(`z`).color =` **RED**;
- `z = grandParent(z); // continue fixing up using the new 'z'`

Case 1b



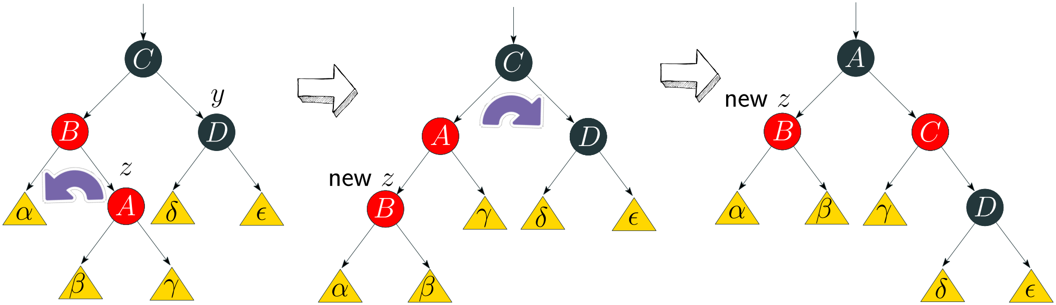
z's uncle y is **red** and z is a left child; recoloring is needed but no rotation; takes $O(1)$ time; **continue fixing up** using the new z node

- `z.parent.color = BLACK; y.color = BLACK; grandParent(z).color = RED;`
- `z = grandParent(z); // continue fixing up using the new 'z'`

Case 2

(Case 2a) z is a right child, z's parent is a left child, and z's uncle y is **BLACK**
(Case 2b) z is a left child, z's parent is a right child, and z's uncle y is **BLACK**

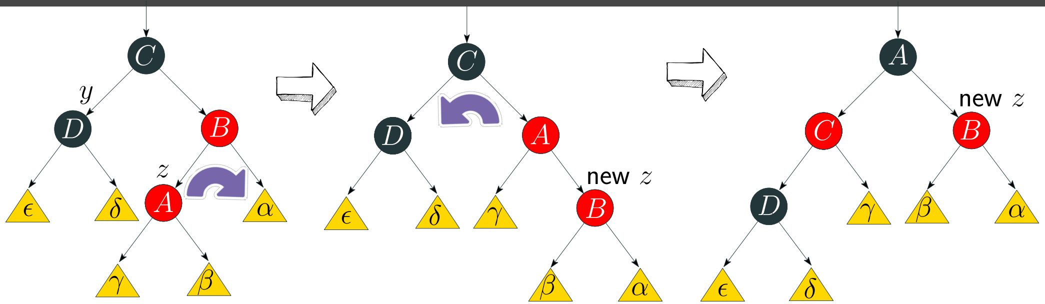
Case 2a



z is a right child, z's parent is a left child, and z's uncle y is **black**; recolorings + 2 rotations are needed; the fixing-up process **terminates** since z's parent is **black** after the two rotations; takes $O(1)$ time

- `z = z.parent; leftRotateAt(z);`
- `z.parent.color = BLACK; grandParent(z).color = RED; rightRotateAt(grandParent(z));`

Case 2b



z is a left child, z's parent is a right child, and z's uncle y is **black**; recolorings + 2 rotations are needed; the fixing-up process **terminates** since z's parent is **black** after the two rotations; takes $O(1)$ time

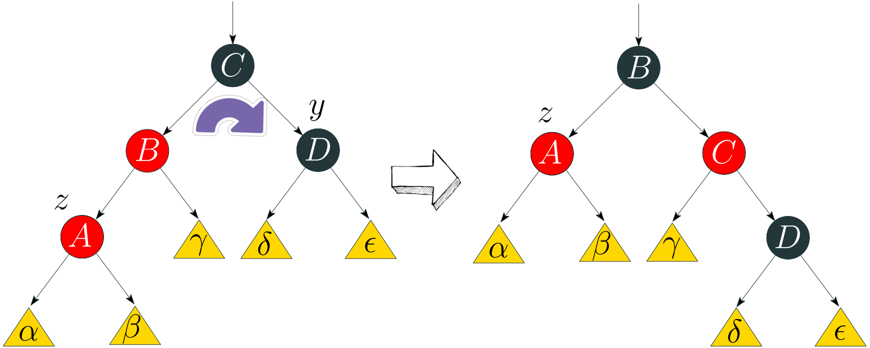
- `z = z.parent; rightRotateAt(z);`
- `z.parent.color =` **`BLACK`**`; grandParent(z).color =` **`RED`**`; leftRotateAt(grandParent(z));`

Case 3

(Case 3a) z is a left child, z's parent is left child, and z's uncle y is **BLACK**

(Case 3b) z is a right child, z's parent is a right child, and z's uncle y is **BLACK**

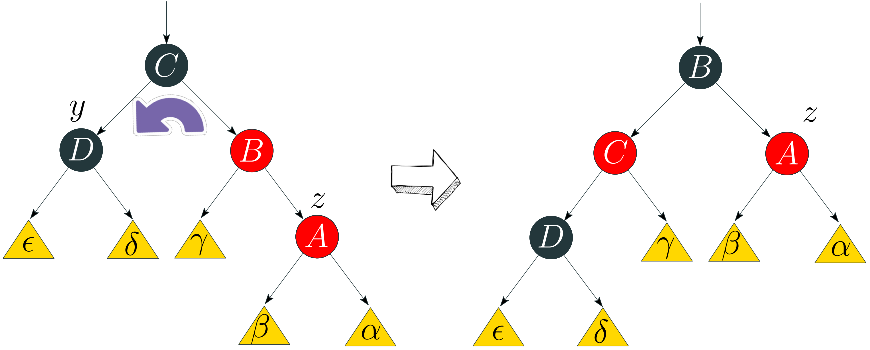
Case 3a



z is a left child, z's parent is left child, and z's uncle y is **black**; recolorings + 1 right rotation are needed; the fixing-up process **terminates** since z's parent is **black** after the rotation; takes $O(1)$ time

- parent(z).color = **BLACK**; grandParent(z).color = **RED**; rightRotateAt(grandParent(z));

Case 3b



z is a right child, z's parent is a right child, and z's uncle y is **black**; recolorings + 1 left rotation are needed; the fixing-up process **terminates** since z's parent is **black** after the rotation; takes $O(1)$ time

- parent(z).color = **BLACK**; grandParent(z).color = **RED**; leftRotateAt(grandParent(z));

The second rotation of Case 2a is exactly an execution of Case 3a
Similarly, the second rotation of Case 2b is exactly an execution of Case 3b

See the class `TreeMapRBTree`

$$41, 38, 31, 12, 19, 8$$

The value parts of the above records are ignored in this example

Item	Action
41	The only node, just color it **black**
38	Insert it to the left of 41; its parent is **black**, so, no action is needed
31	Case 3a; 31's uncle (a null reference) is **black**; right rotate at 41
12	Case 1b; a simple recoloring is enough
19	Case 2a; two rotations are needed
8	Case 1b; a simple recoloring is enough

`https://www.cs.csubak.edu/~msarr/visualizations/RedBlack.html`

Observations

- Time taken for inserting a new node is $O(h) = O(\log n)$; after this fix-up may be needed
- During fix-up, we climb up the tree using Case 1, which only recolors but never rotates
- If we ever use Case 2 (uses 2 rotations) or 3 (uses 1 rotation), we are done (fix-up process terminates)!
- This means at every insertion of a new item, at most 2 rotations are needed
- At most h executions of Case 1 are needed plus 1 execution of Case 2/3, each taking $O(1)$ time
- So, the total time taken for fix-up is $(h+1) \times O(1) = O(\log n) \times O(1) = O(\log n)$, since, for RB-trees, $h = O(\log n)$
- Total time taken for one insertion equals time taken for inserting a new node plus total time taken for fix-up $= O(\log n) + O(\log n) = 2 \times O(\log n) = O(\log n)$
- Time complexity of searching is $O(h) = O(\log n)$, since, for RB-trees, $h = O(\log n)$
- Deletion also takes $O(h) = O(\log n)$ time but we are not going to discuss it

Plain BSTs vs RB-Trees: tree creation time (using n insertions)

n	RB-tree	Plain BST
10	1	1
100	1	1
1000	3	13
10000	7	214
100000	20	17249

When the input records were already **sorted** in ascending order, RB-trees could easily beat plain BSTs since the heights of the plain BSTs were exactly $n-1$ everywhere (much worse than the logarithmic heights of RB-trees).

n	RB-tree	Plain BST
10	1	1
100	1	1
1000	3	2
10000	7	8
100000	72	49

When the input records were in **random** order, plain BSTs performed quite as fast as RB-trees since the heights of plain BSTs were not $n-1$ or even close (in fact, they were almost logarithmic like RB-trees).

☞ Times are reported in milliseconds

Are plain BSTs completely useless?

- Plain BSTs perform **terribly** when the inputs are sorted (or, almost sorted) in ascending or descending order
- But, BSTs are found to perform great on randomly ordered inputs
- In those cases, h is found to be much less than $n - 1$ and is almost logarithmic
- Consequently, we get to see fast searching, insertion, and deletion times
- **Example.** when $n = 5000$, heights of plain BSTs are around 30 if the input is randomly ordered. Note that this is much less than 4999 (worst case height)

Reading

https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/BST.html