Homework 1 - Underwater Acoustics

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1 Arctic Ocean

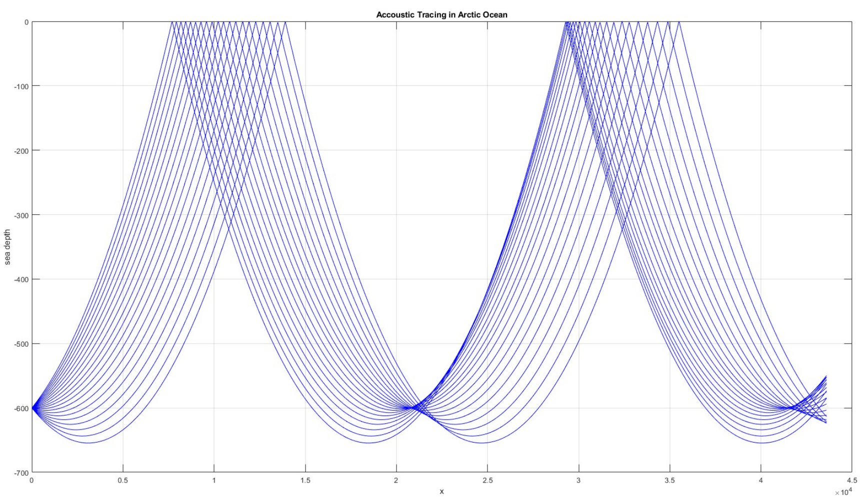
The simplest model for the velocity profile in this ocean is an affine function of depth:

$$c(z) = c_0 + \gamma_0 |z|$$

where $c_0 = 1450$ ms^{-1} and $\gamma_0 = 1.63 \times 10^{-2} s^{-1}$ with seafloor at depth $h = 3.50$ km.

1.1 Question 1

Let us consider a sonar placed at the depth $z_s = 600m$, emitting a beam of angular width 4° around the horizontal direction. With the help of the software, plot the trajectory of the rays constituting the beam over a period of 30 s. What is the maximum depth reached by this beam?
Answer:
Using Matlab code, the acoustic beam trajectory tracing will be shown in the picture below:



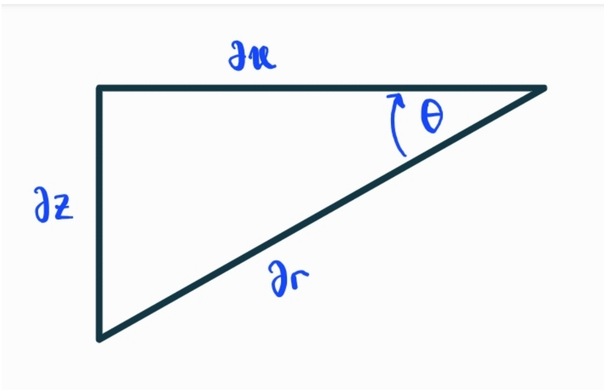
The maximum depth reached by this beam is 654.5874 metres.

1.2 Question 2

What is the maximum depth reached by a sound ray as a function of z_s and of the incidence angle θ_0. Check the formula with the software for the example of question 1.

Answer:

To find the maximum depth, we have to evaluate the velocity profile at $\theta = 0$. Geometrically, we can also derive the relation between the angle of incidence θ with depth and travel distance, as shown in equation (1.1):



$$\tan\theta = \frac{\partial z}{\partial x} = \frac{\sin\theta}{\cos\theta} \tag{1.1}$$

We will also consider Snell's Law to find the maximum depth. Snell's Law will be shown in equation (1.2):

$$\frac{\cos\theta}{c} = \frac{\cos\theta_s}{c_s} \tag{1.2}$$

We are going to combine Snell's Law and the Velocity Profile Gradient. But first, we need to modify equation (1.1) so it will only contain $\cos\theta$.

$$\tan\theta = \sqrt{\frac{1}{\cos^2\theta} - 1} \tag{1.3}$$

We then modify Snell's Law as shown in equation (1.4):

$$\cos^2\theta = \frac{c^2\cos^2\theta_s}{c_s^2} \tag{1.4}$$

And then we will substitute equation (1.1) and equation (1.4) into equation (1.3) and we get a new equation:

$$\frac{\partial z}{\partial x} = \sqrt{\frac{c_s^2}{c^2 \cos^2 \theta_s} - 1} \tag{1.5}$$

With $\frac{\partial z}{\partial x} = 0$, continuing equation (1.5), we will get:

$$c = \frac{c_s}{\cos \theta_s} \tag{1.6}$$

We can then decompose c and c_s, and move every component on the left side except z_{max} to the right side. We will get the z_{max} equation as shown in equation (1.8)

$$c_0 + \gamma_0 z_{max} = \frac{c_0 + \gamma_0 z_s}{\cos \theta_s} \tag{1.7}$$

$$z_{max} = \frac{1}{\gamma_0} \left(\frac{c_0 + \gamma_0 z_s}{\cos \theta_s} - c_0 \right) \tag{1.8}$$

Solving the z_{max} equation with $c_0 = 1450 \text{ ms}^{-1}$, $z_s = 600$ m, $\gamma_0 = 1.63 \times 10^{-2} s^{-1}$, and $\theta = 2^\circ$,

$$z_{max} = \frac{1}{0.0163} \left(\frac{1450 + 0.0163 \times 600}{\cos 2^\circ} - 1450 \right)$$

we find that the maximum depth using z_{max} equation is 654.5889 metres.

1.3 Question 3

Let us consider an echosounder measuring the depth h of the ocean by measuring the travel time of short pulses. Determine the expression for the round-trip time t of a sound wave emitted under vertical incidence from the surface. What is the error on the depth if it is estimated from the measurement of t assuming that the velocity is uniform, equal to c_0.
Answer:

To get the travel time of the short pulses, we can divide the travel distance of the pulses by their velocity. Using geometric approach similar to the answer for Question 2, we will get equation (1.9).

$$\partial t = \frac{\partial r}{c} = \frac{\partial z}{c \sin \theta} \tag{1.9}$$

Because the sound wave emitted vertically from the surface, means that the initial angle θ is

90°, we will get:

$$\partial t = \frac{\partial z}{c} \tag{1.10}$$

Then we expand the c and we will get:

$$\partial t = \frac{\partial z}{c_0 + \gamma_0 z_0} \tag{1.11}$$

We will then integrate both sides of the equation, also at the same time applying the boundary for the sound wave to travel from the source to the seabed, we will get:

$$\int_0^{t_{\frac{1}{2}}} dt = \int_{z_s}^{z_seabed} \frac{1}{c_0 + \gamma_0 z_0} dz \tag{1.12}$$

$$[t + K_1]_0^{t_{\frac{1}{2}}} = \left[\frac{1}{\gamma_0} \ln (c_0 + \gamma_0 z_0) + K_2\right]_{z_s}^{z_seabed} \tag{1.13}$$

Completing the process, we will get the equation for the sound wave travel time from the source to the seabed as shown in equation (1.14)

$$t_{\frac{1}{2}} = \left[\frac{1}{\gamma_0} \ln\left(\frac{c_0 + \gamma_0 z_{seabed}}{c_0 + \gamma_0 z_s}\right)\right] \tag{1.14}$$

For the round-trip time, we simply need to multiply the equation (1.14) by 2, thus we will get:

$$t = 2t_{\frac{1}{2}} = 2\left[\frac{1}{\gamma_0} \ln\left(\frac{c_0 + \gamma_0 z_{seabed}}{c_0 + \gamma_0 z_s}\right)\right] \tag{1.15}$$

For the case in which sound wave velocities are uniform for each depth, we have:

$$t_{\frac{1}{2}uniform} = \frac{z}{c} \tag{1.16}$$

Then for the round-trip time, we simply need to multiply the equation (1.16) by 2. We will also replace the term z and c to z_{seabed} and c_0 respectively.

$$t_{uniform} = 2\frac{z_{seabed}}{c_0} \tag{1.17}$$

Then we compute t using equation (1.15) by applying the available parameters to the equation

$$t = 2\left[\frac{1}{0.0163} \ln\left(\frac{1450 + 0.0164 \times 3500}{1450 + 0.0163 \times 600}\right)\right]$$

and we find that the round-trip time for the sound wave is 3.910 s.

We will also compute the $t_{uniform}$ using equation (1.17)

$$t_{uniform} = 2 \times \frac{3500}{1450}$$

and we find that the round-trip time for the sound wave with the sound velocities uniform at all depths is 4.828 s.

To compute the error, we will take the absolute value of the difference between t and $t_{uniform}$. The error will be:

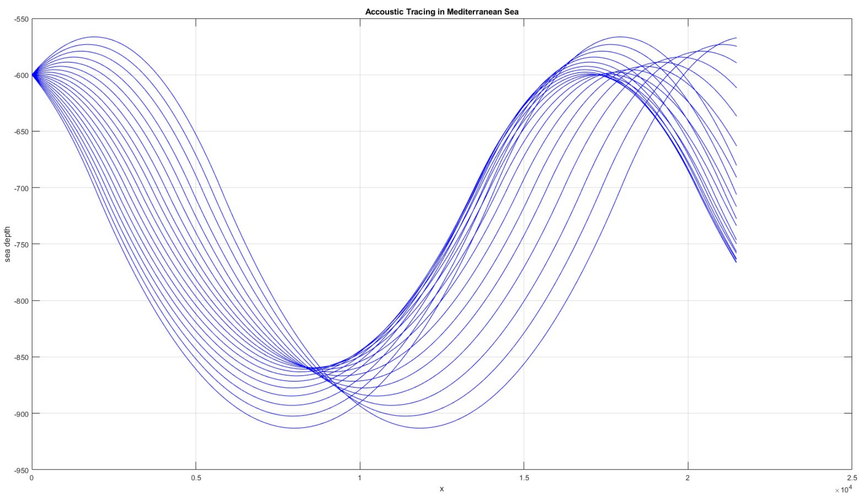
$$error = |t - t_{uniform}| = |3.910 - 4.828| = 0.918$$

2 Mediterranean Sea

Because of the higher water temperature at the surface, we observe a decrease of the sound velocity over 700 hundred meters, with -0.026 s^{-1} gradient, before returning to a pressure-driven behavior, as described above for the Arctic Ocean ($\gamma_0 = +1.63 \times 10^{-2} s^{-1}$).

2.1 Question 4

As in question 1, using the software, plot the trajectory of the rays constituting the beam over a period of 15 s. The transmitter remains the same and is still located at 600 m depth.
Answer:
Using Matlab code, the acoustic beam trajectory tracing will be shown in the picture below:



The maximum depth reached by this beam is 913.1468 metres.

2.2 Question 5

It is planned to establish a communication with another antenna located at 10 km. At which depth should this antenna be located to receive a signal?

Answer:

Based on the obtained results, at 10 km (10,000 metres), the antenna is best placed at depths between **844.511 metres** to **893.465 metres** to ensure that the antenna will receive a signal.