Digitale Signalverarbeitung

Zusammenfassung

Andreas Ming / Quelldateien

Inhaltsverzeichnis

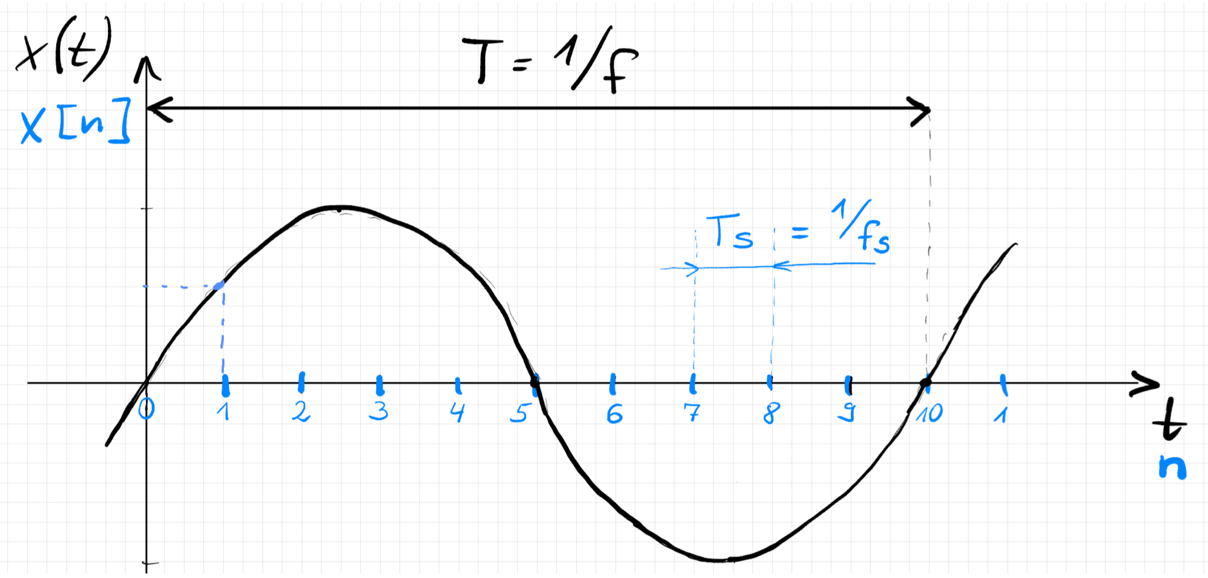
Digital Signals in the Time Domain

Signal Analysis

Sampling of Analog Signals

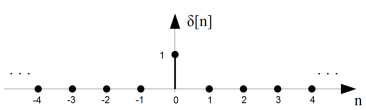
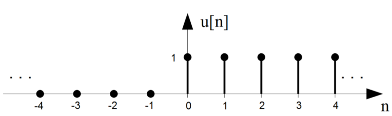
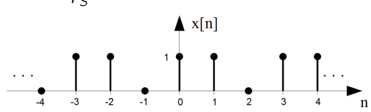
By sampling $x(t)$ in the interval of T_S we get the sequence of signal values $x[n]$ with $-\infty \leq n \leq +\infty$

$$x(n \cdot T_S) = x[n]$$



Signal	Property
causal	$x[n] = 0$ for $n < 0$
real	$x[n]$ Real
complex	Re & Im or Amplitude & Phase

Basic Digital Signals

unit impulse	unit step	periodical signal
$\delta[n] = \begin{cases} 0 : n \neq 0 \\ 1 : n = 0 \end{cases}$	$u[n] = \begin{cases} 0 : n < 0 \\ 1 : n \geq 0 \end{cases}$	$x[n] = x\left[n + \frac{T_0}{T_S}\right]$ with $\frac{T_0}{T_S} = k$
		

There is also a **complex hamonic** sequence with the period duration of $T_0 = \frac{1}{f_0}$

$$x[n] = \hat{X} \cdot e^{j 2\pi f_0 n T_S}$$

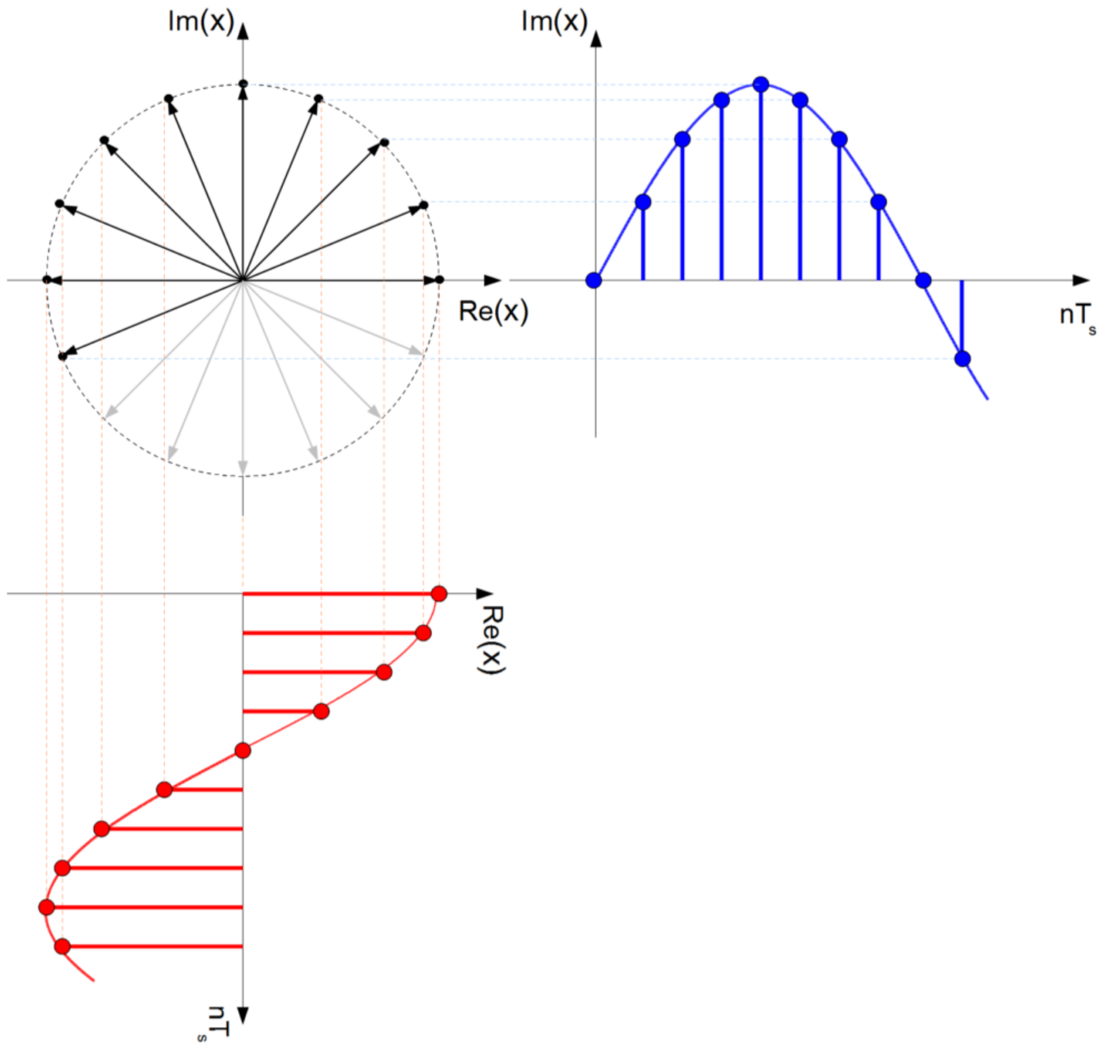


Abbildung 1: Complex hamronic sequence with period duration $T_0 = 16 \cdot T_S$

Statistical Signal Parameters

Stochastic signals must be qualified by *statistical signal parameters* within the **observation interval** $T = N \cdot T_S$.

expected / mean	quadratic mean	variance
DC-component	average power *(w/ DC)*	average power *(w/o DC)*
$\mu_x = \frac{1}{N}\sum_{i=0}^{N-1} x[i]$	$\rho_x^2 = \frac{1}{N}\sum_{i=0}^{N-1} x[i]^2 = P_{avg}$	$\sigma_x^2 = \frac{1}{N}\sum_{i=0}^{N-1} (x[i] - \mu_x)^2 = P_{AC}$

Signal Operations

Correlation

	cross-correlation	auto-correlation
Static	$R = \frac{1}{N}\sum_{i=0}^{N-1} x[i]y[i]$	$R = \frac{1}{N}\sum_{i=0}^{N-1} x[i]x[i]$

	cross-correlation	auto-correlation
Linear	$r_{xy}[n] = \sum_{i=-\infty}^{\infty} x[i]y[i+n]$	$r_{xx}[n] = \sum_{i=-\infty}^{\infty} x[i]x[i+n] = P_{avg}$

For **linear correlation** the resulting length of r_{xy} equals

$$N_{xy} = N_x + N_y - 1$$

and the range of shifts for the computation is given by

$$-N_x + 1 \leq n \leq N_y - 1$$

For signals differing in length, zero-padding can be applied.

Convolution

The *Convolution* involves folding the time-displaces signal around the point $n = 0$

$$z[n] = \sum_{i=-\infty}^{\infty} x[i]y[-i+n] \tag{0.1}$$

A convolution equals a polynomial multiplication.

The range of shifts for the computation is given by

$$0 \leq n \leq N_x + N_y - 2$$

The Convolution described in Gleichung 0.1 is called a **linear convolution** and can be applied to two signals of different length

$$z[n] = x[n] * y[n] = y[n] * x[n]$$

```
z = conv(x, y)
```

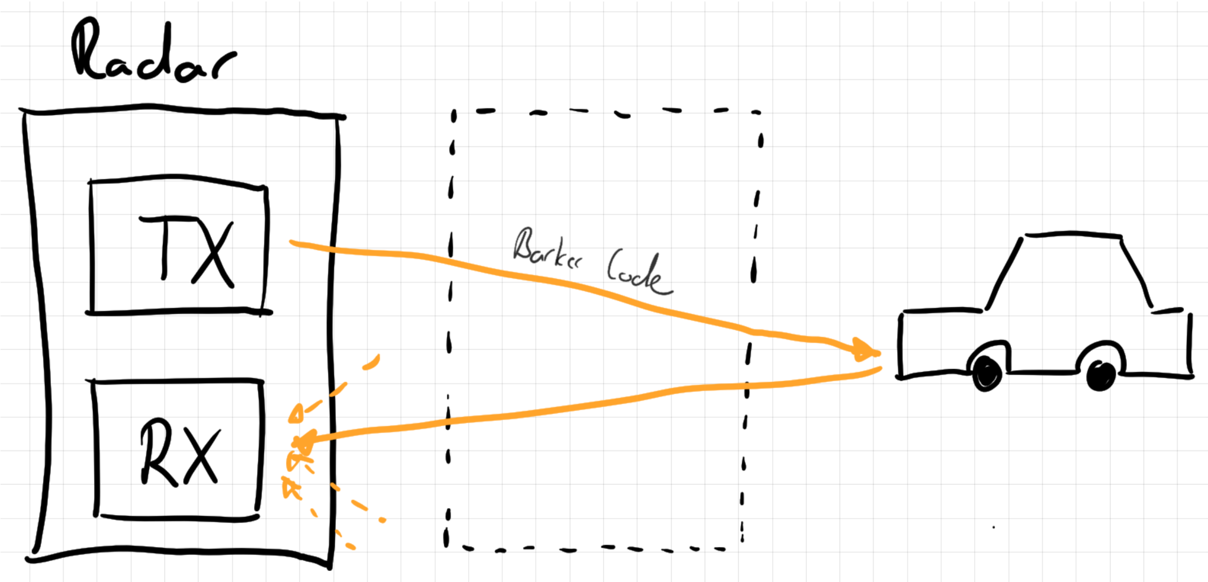
There is also the **circular convolution** which requires both signals to be of equal length N. If necessary, *zero padding* can be applied. The resulting signal then also is of length N.

$$z[n] = x[n] \circledast_N y[n] = y[n] \circledast_N x[n]$$

The circular convolution corresponds to matrix multiplication In order to compute $x[n] \circledast_N y[n]$, the NN-matrix constructed from circular shifting y must be multiplied with vector x.

```
z = convmtx(x,y)
```

Anwendung: Radar



Um bei einem Radar nur auf das gewünschte Signal zu reagieren, also auf das eigene, wird vom Radar ein Barker-Code ausgesendet. Über Korrelation kann so die Laufzeit eindeutig zugeordnet werden.

> **Barker-Code**
>
> Es können auch andere Codes ausgesendet werden, die verwendeten Signale müssen jedoch sehr gute Autokorrelationseigenschaften aufweisen.

Analog-to-Digital & Digital-to-Analog Conversion

Steps of A/D- and D/A-Conversion

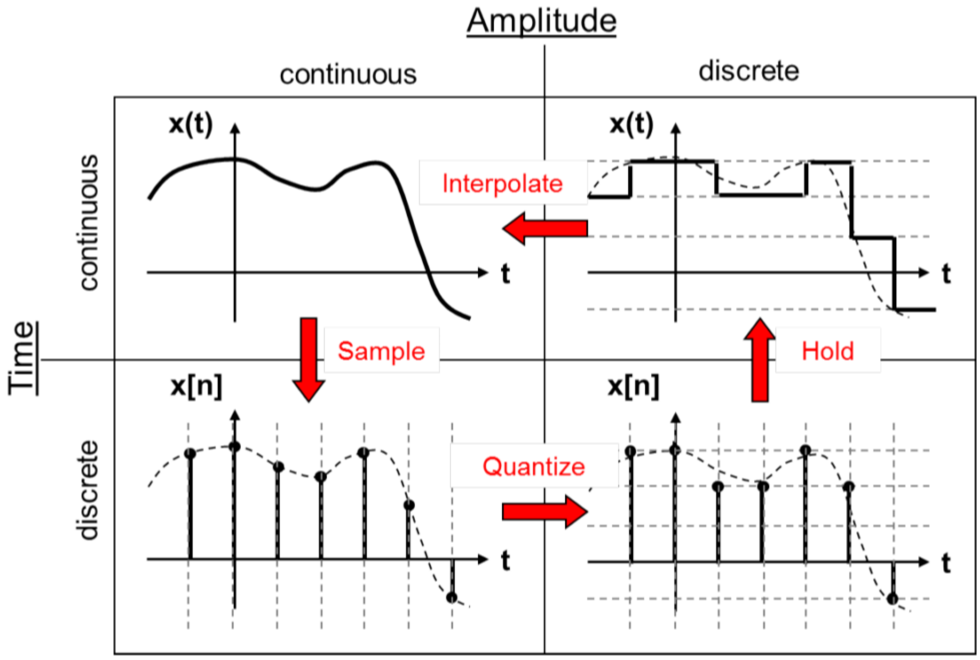


Abbildung 2: Signal classification in a/d- and d/a-conversion

A/D

Sample: Signal values are recorded at sampling rate f_S. This yields a train of pulses.

Quantize: The discrete signal values are mapped to a given number of quantization levels.

Code: The quantified values can be stored in a coded way. DSPs most often store the quantified values.

D/A

Decode: The coded samples are converted back into a suitable representation for the digital-to-analog conversion method used.

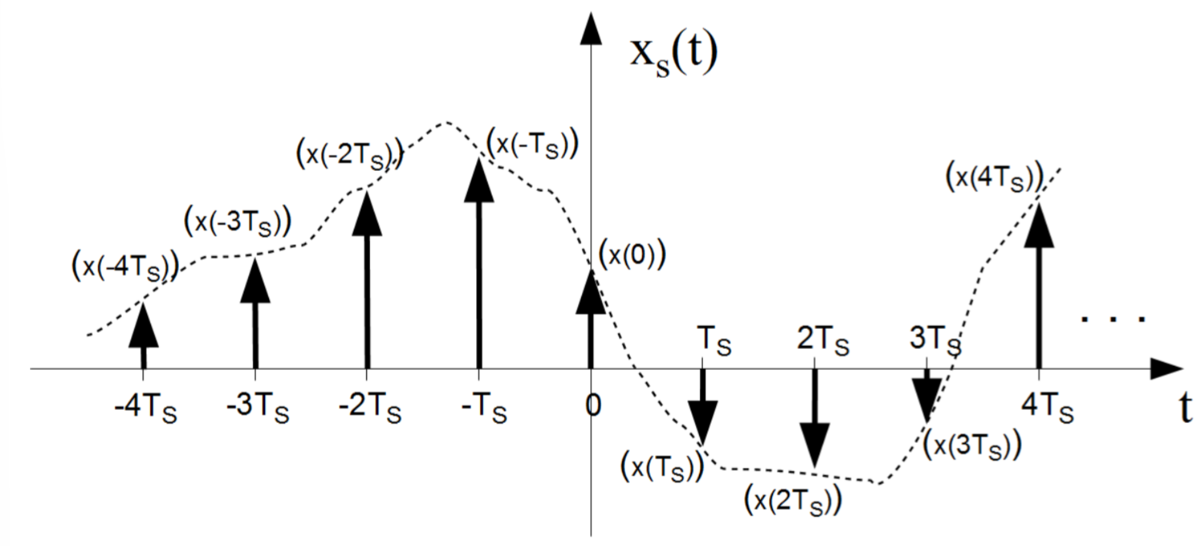
Hold: A momentary discrete signal value is constant over the sample period T_S.

Interpolate: The continuous staircase signal form is smoothed by a low-pass-filter.

Sampling and Aliasing

Sampling a time-continuous signal $x(t)$ corresponds to a multiplication with a Dirac impulse series. The resulting signal $x_S(t)$ can be regarded as a train of weighted Dirac impulses.

$$x_S(t) = \sum_{n=-\infty}^{\infty} x(t) \cdot \delta(t - nT_S)$$



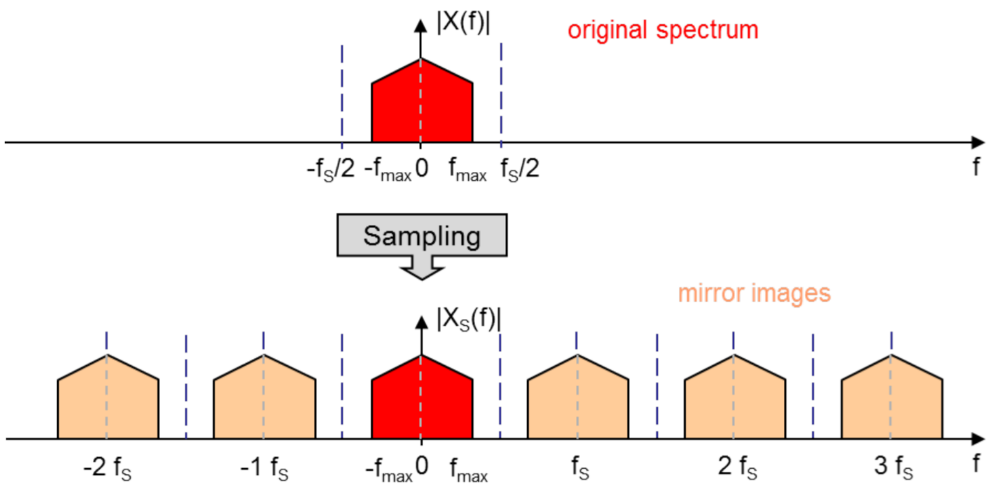
Through the application of the Fourier property $x(t)e^{j2\pi f_0 t} \circ\!\!-\!\!\bullet X(f - f_0)$ we obtain the frequency spectrum of the sampled signal as

$$X_S(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} X(f - kf_S)$$

! Observation

The frequency of the analog signal $x(t)$ consists of the original spectrum $X(f)$ superimposed (*überlagert*) by mirror images of the spectrum

$$f_k = k \cdot \frac{f_S}{N}$$

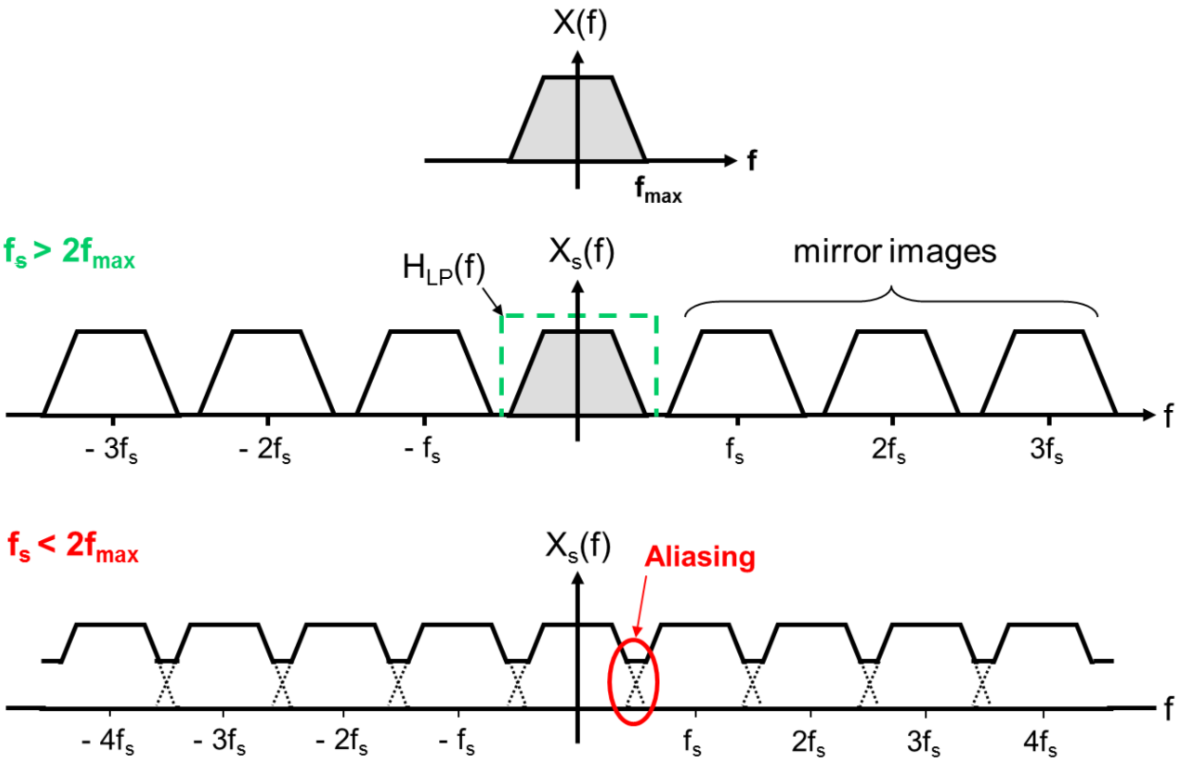


Aliasing

Sampling Theorem

An analog signal $x(t)$ with $X(f) = 0$ for $|f| > |f_{max}|$ is uniquely defined by its sample values $x[n] = x(nT_S)$, if for the sampling frequency $F_S = \frac{1}{T_S}$ holds:

$$f_s > 2 \cdot f_{max}$$



Band-Pass Sampling

$x(t)$ can be perfectly reconstructed if an integer $N \geq 0$ exists, such that $X(f) = 0$ holds for all frequencies f outside

$$-\frac{N+1}{2} f_S \leq f \leq -\frac{N}{2} f_S \qquad \text{and} \qquad \frac{N}{2} f_S \leq f \leq \frac{N+1}{2} f_S$$

For a given band-pass signal with given limits f_{min} and f_{max} it can be checked if the sampling frequency f_S can be used ($N \geq 1$)

$$\frac{2 \cdot f_{min}}{N} \geq f_S \geq \frac{2 \cdot f_{max}}{N+1}$$

For sampling with $N =$ **even** we get the mirror spectrums

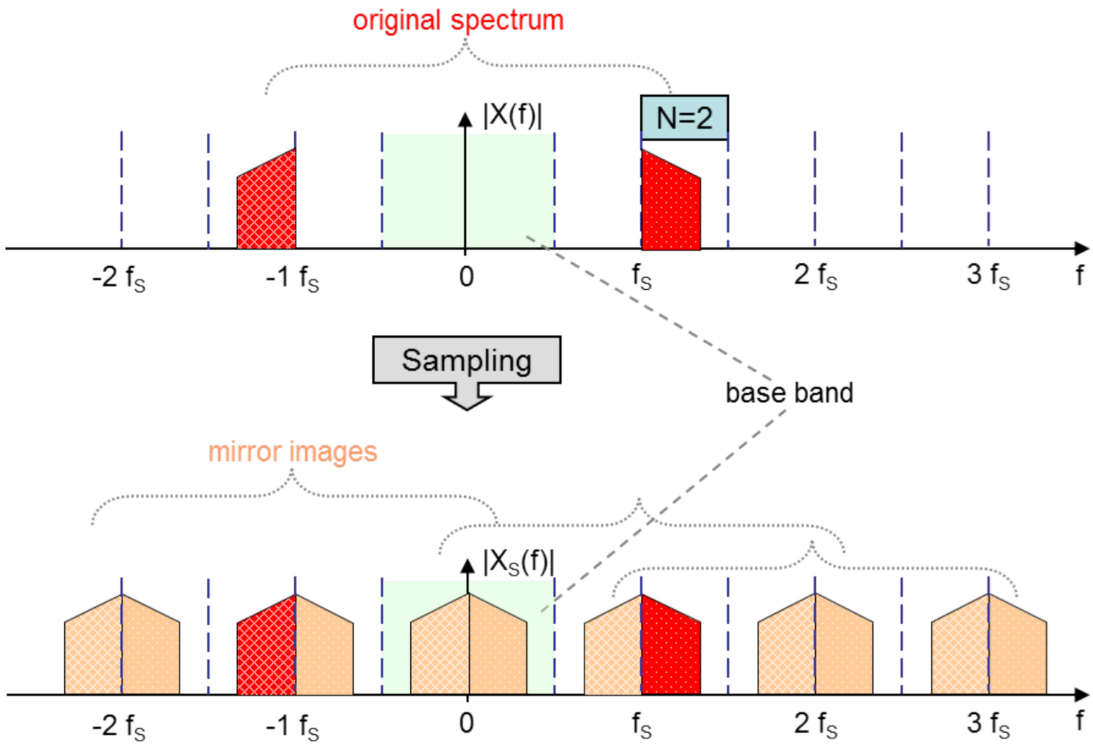


Abbildung 3: Band-pass sampling for even N

For sampling with $N =$ **odd** we get the mirror spectrums

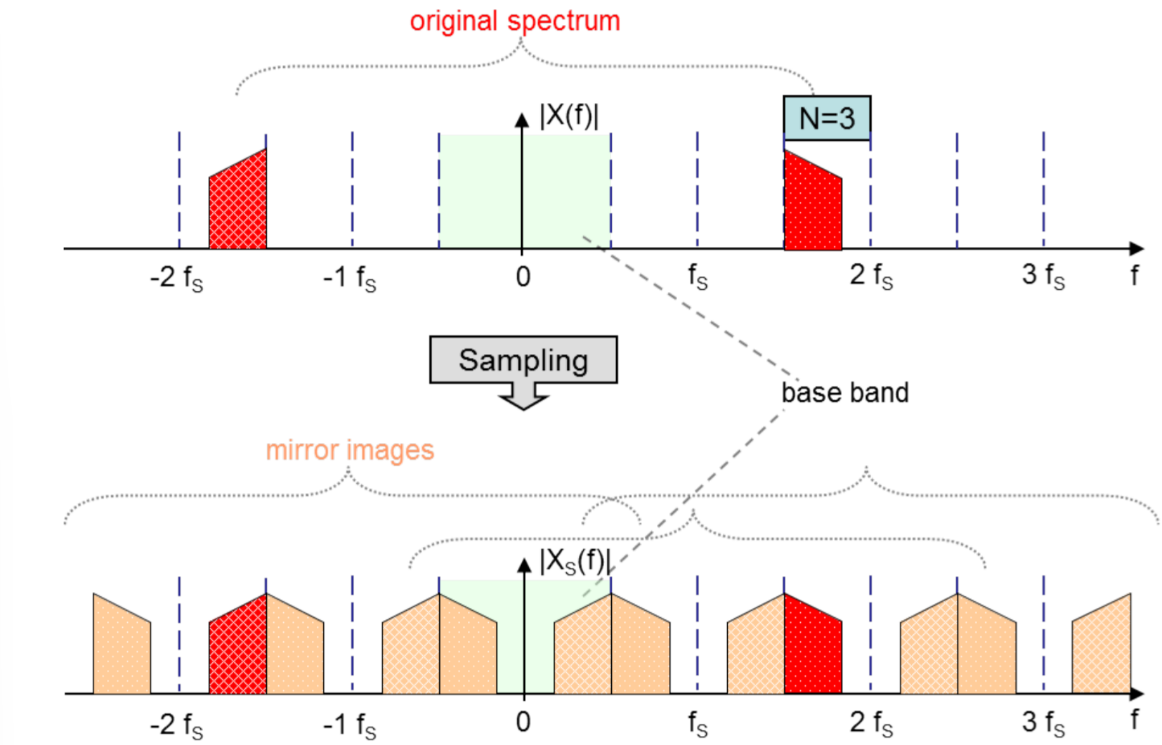


Abbildung 4: Band-pass sampling for odd N

! Spectrum Correction

Note that for N odd, the original spectrum appears "inverted" in the base band. The original structure of the spectrum can be re-obtained by changing the sign of every second sample of the time-domain sequence, i.e.

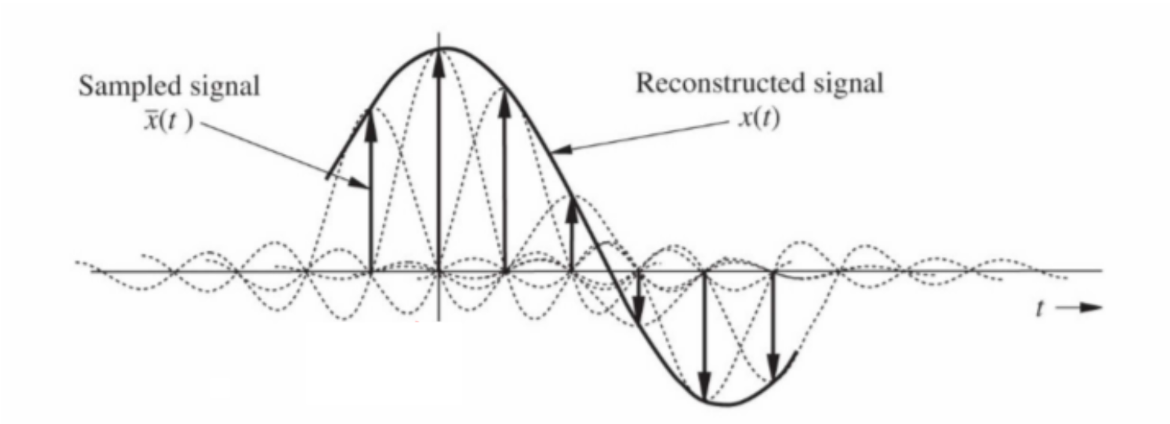
$$\tilde{x} = (-1)^n \cdot x[n]$$

Reconstruction

Ideal Reconstruction

Sampled signals w/o Aliasing can be *theoretically* reconstructed error-free. For this all mirror-spectra must be eliminated by a ideal low-pass filter. Because of the property $\text{rect}\left(\frac{t}{T}\right) \circ\!\!-\!\!\bullet |T| \cdot \text{si}(\pi T f)$ the ideal interpolation equals

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S) \cdot \text{sinc}(\pi f_S (t - nT_S))$$



i Ideal values

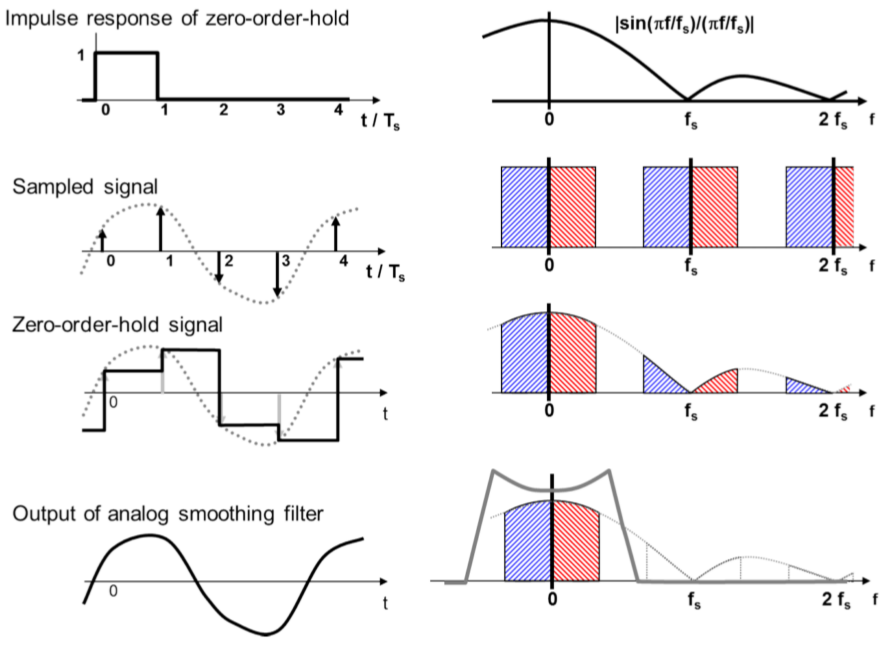
At the points $t = nT_S$ all values of except of $x(nT_S)$ equal 0. Thus at every point of $x(nT_S)$ the signal reaches the right value.
Caution! because of the infinit sum of sinc-pulses, the values between $x(nT_S)$ aren't particularly correct. Also the further to the "edge" of x you get, the more inaccurate it gets.

Practical Reconstruction

In practice Reconstruction is very often done with a simple *zero-order-holder* (ZHO). Such operation holds each sample value constant over a subsequent sample interval T_S. This results in a stair-case waveform, thus making a very poor low-pass filter. For this reason a analog low-pass filter is usually implemented.

Without analog filtering the **SNR** can be approximated as

$$\text{SNR} \approx 6dB \cdot \log_2\left(\frac{f_S}{f_0}\right) - 11dB$$

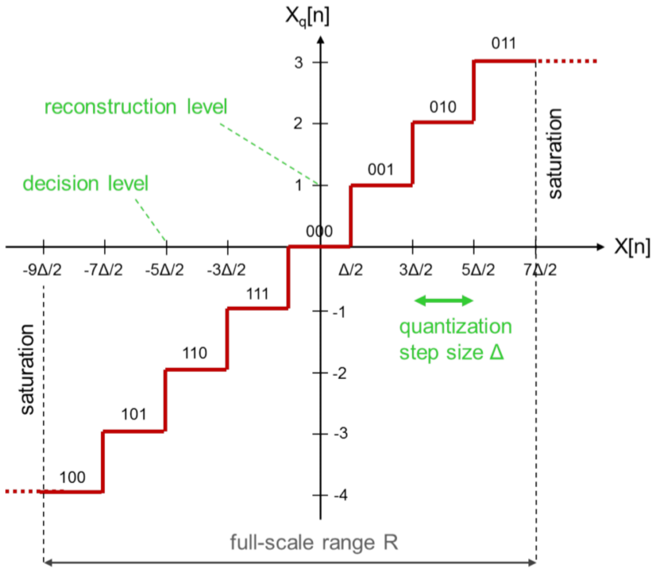
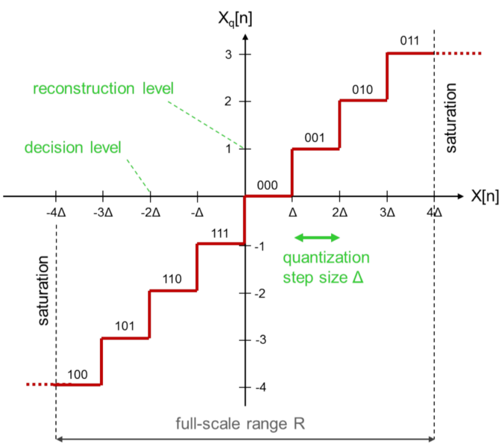


Quantization of Signals

Uniform Quantization

When quantizing sample values with W bits, the dynamic range R of the sampled signal $x[n]$ can be divided into 2^W intervals of equal width. Thus, the width of one **quantization step** is given by

$$\Delta = \frac{R}{2^W}$$

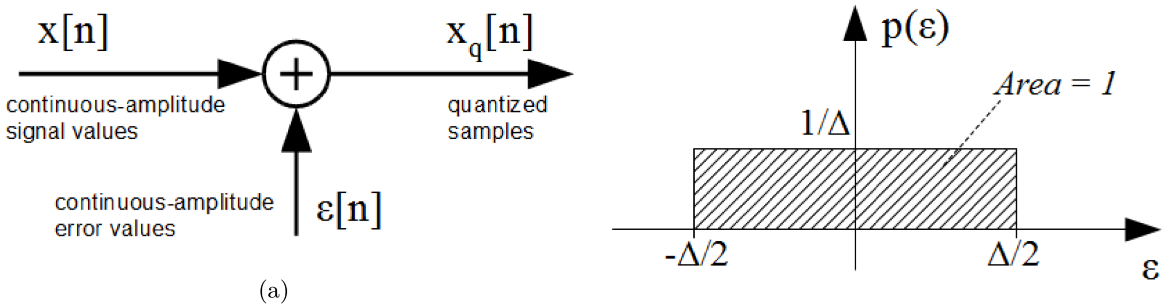
Mid-Tread	Mid-Rise
Decision levels at $\pm 0.5\Delta, \pm 1.5\Delta, \pm 2.5\Delta, \ldots$	Decision levels at $0, \pm\Delta, \pm 2\Delta, \ldots$
	
$-\frac{\Delta}{2} < \epsilon \leq \frac{\Delta}{2}$	$-\Delta < \epsilon \leq 0$

Furthermore **Clipping** occurs if the signal values of $x[n]$ are outside of the full-scale range R.

Quantization noise

The quantization error ϵ manifests itself as noise superimposed to the quantized signal

$$\epsilon[n] = x_q[n] - x[n]$$



(a) (b)

Abbildung 5: Model of quantization noise (Abbildung 5a) and probability density function (Abbildung 5b)

The power of the quantization noise signal is

$$P_\epsilon = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \epsilon^2 d\epsilon = \frac{\Delta^2}{12}$$

The signal-to-noise ratio expressed in dB yields

$$SNR_{dB} = 6 \cdot W + 10 \cdot \log_{10}\left(\frac{12 \cdot P_x}{R^2}\right)$$

For Harmonic- & Full-Scale-Signals

$$SNR_{dB} = 6 \cdot W + 1.76 \approx 6 \cdot W$$

For every additional Bit, the SNR can be sixfold in $[dB]$.

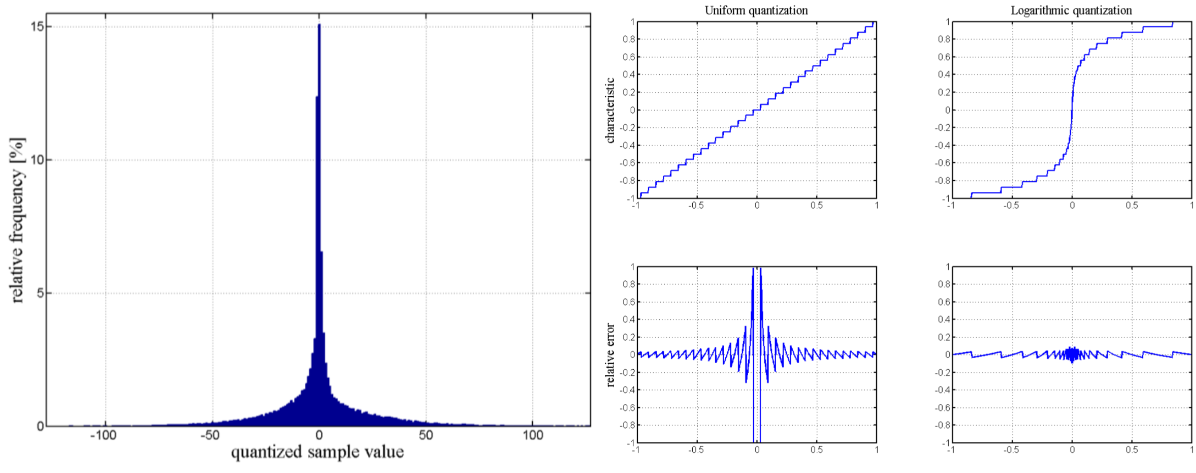
Logarithmic Quantization

One way to increase the SNR associated with quantization, is to adapt the quantizer characteristics to the statistical properties of the signal being quantized. One kind of signal with these properties are voice signals were very small amplitude values are orders of magnitude more likely than large amplitude values (Abbildung 6a).

There are several standards for implementing Logarithmic Quantization. One such standard is the μ-law algorithm

$$f_\mu(x) = \operatorname{sgn}(x) \cdot \frac{\ln(1 + \mu \cdot |x|)}{\ln(1 + \mu)} \qquad -1 \le x \le 1$$

With this applied the relative error can be significantly improved (Abbildung 6b).



(a) (b)

Abbildung 6: Comparison of uniform and logarithmic quantization

Digital Signals in the Frequency Domain

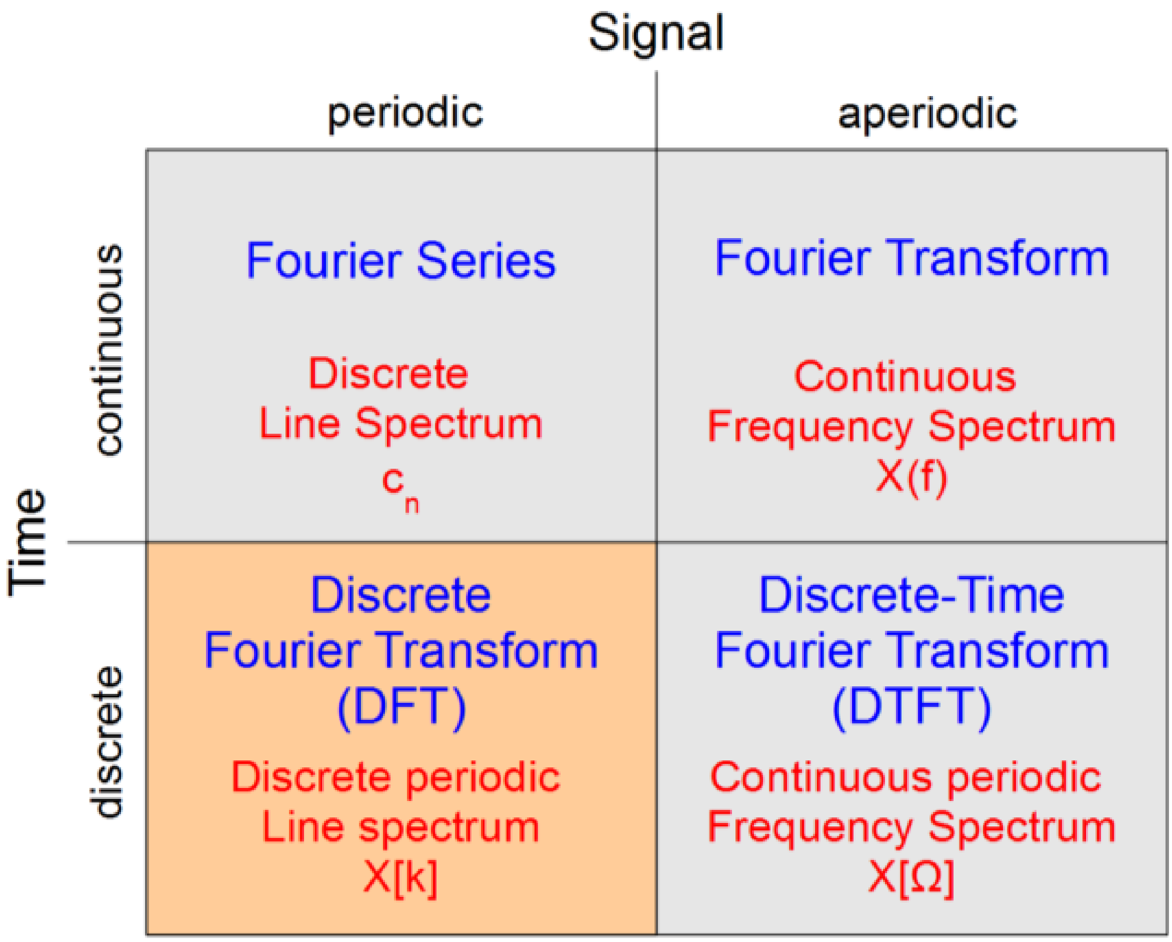


Abbildung 7: Comparison of Fourier methods

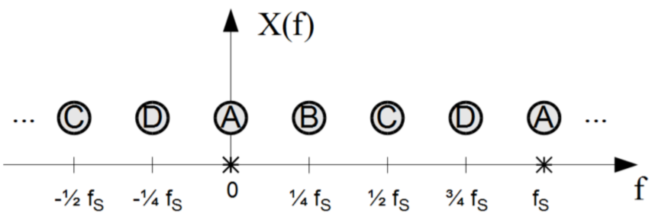
Fourier in Discrete Time

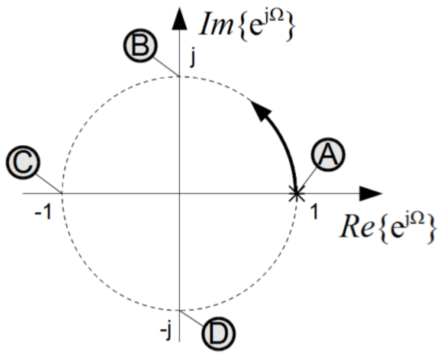
Discrete Time

Substituteing the continuous time t and integration with the discrete sample points nTs and summation, we get the frequency spectrum of the sampled signal $x_S(t)$. Thourgh the *normalized angular frequency*, we obtain the **Discrete-Time Fourier Transform (DTFT)**.

$$\underbrace{X_S(f) = \sum_{n=-\infty}^{\infty} x(nT_S)e^{-j2\pi f nT_S}}_{\text{Continuous Fourier Transform}} \overset{\Omega=2\pi f T_S=2\pi\frac{f}{f_S}}{=} \underbrace{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}}_{\text{Discrete Fourier Transform}}$$

The **DTFT** produces a 2π-periodic, continuous spectrum. This is due to the mapping of the linear frequency axis onto the unit circle at the sampling rate of f_S.





Finite Measurement Interval

The lowest frequency we can capture in the measurement interval T, with a finite amount of sample points N is

$$f_1 = \frac{1}{T} = \frac{1}{N \cdot T_S} = \frac{f_S}{N}$$

! Resolvable frequencies

From the equation we learn that we can resolve lower frequencies if we increase N, on the other hand the highest frequency we can cover is f_S. The following generally applies

$$f_k = k \cdot \frac{f_S}{N}$$

With this in mind we let the discrete-time-index n only run from 0 to $N-1$. We replace the $\frac{f}{f_S}$ in Ω with $\frac{k}{N}$ and get the **Discrete Fourier Transform (DFT)**

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi n\frac{k}{N}} \qquad k = 0, 1, 2, \ldots, N-1$$

i Observation

We get exactly as many *Fourier coefficients* $X[k]$ back, as we provide *samples* $x[n]$.
The DFT produces a discrete and periodic line spectrum. The frequency resolution is $\frac{f_S}{N}$, i.e. we get the spectral values at the frequency points

$$0, \frac{f_S}{N}, 2\frac{f_S}{N}, \ldots, (N-1)\frac{f_S}{N}$$

Inverse Discrete Fourier Tranfsorm (IDFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n \frac{k}{N}} \qquad n = 0, 1, 2, \ldots, N-1$$

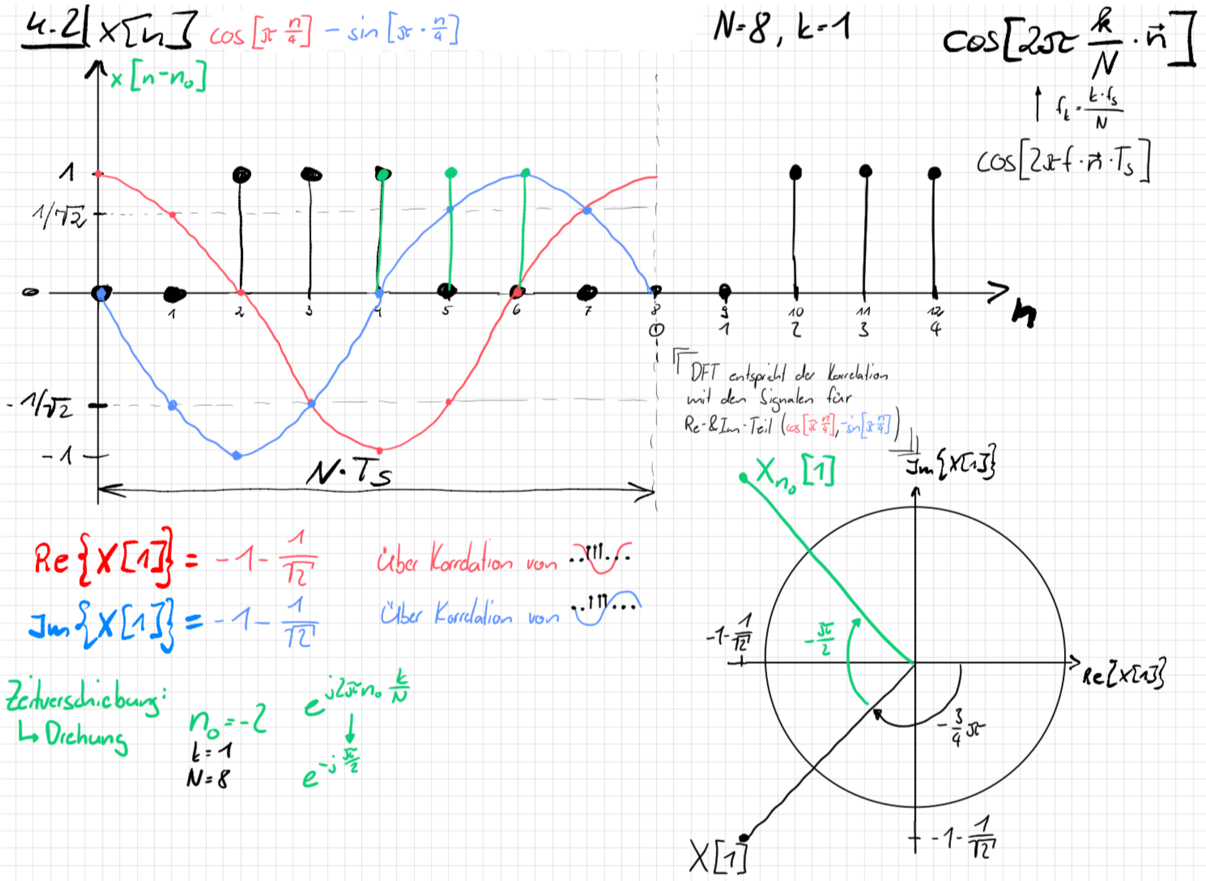


Abbildung 8: A intuitive approach to the DFT, also check chapter 4.2 of the main script by Wassner

Porperties of the DFT

Tabelle 6: Periodicity

The *DFT* is f_S-periodic	The *IDFT* is periodic with $T = NT_S$
$X[k] = X[k+N]$	$x[n] = x[n+N]$

Tabelle 7: Symmetry

DFT of a *real-valued signal* is symmetric around $k = \frac{N}{2}$
$X\left[\frac{N}{2}+m\right] = X^*\left[\frac{N}{2}+m\right]$

Tabelle 8: Time/Frequency Shifting

Shifting by n_0 corresponds to a linear phase offset to all spectral values of the original time sequence	Multiplying the time sequence with a complex exponential, results in a constant frequency shift of the original spectrum
$x[n+n_0] \quad ○\!\!-\!\!\!-\!\!● \quad e^{j2\pi n_0 \frac{k}{N}} \cdot X[k]$	$e^{j2\pi k_0 \frac{n}{N}} \cdot x[n] \quad ○\!\!-\!\!\!-\!\!● \quad X[k-k_0]$

Tabelle 9: Modulation

A direct consequence of the frequency shifting property
$\cos(2\pi k_0 \frac{n}{N} \cdot x[n] \quad ○\!\!-\!\!\!-\!\!● \quad \frac{1}{2}(X[k+k_0]+X[k-k_0])$

Tabelle 10: Parseval Theorem

Between the signal samples $x[n]$ and the Fourier coefficients $X[k]$ following relationship exists
$\frac{1}{N}\sum_{n=0}^{N-1} x[n]^2 = \sum_{k=0}^{N-1} \left\lvert \frac{X[k]}{N} \right\rvert^2$

Digital LTI Systems

Design of Digital Filters

Fourier Analysis of analog Signals

DFT Inside

The z-Transform