Probability density functions (PDFs)

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1 Definition

A probability density function is defined as "probability per interval" (a PDF therefore has units!). Mathematically we write

$$\psi(x) = \frac{\mathrm{d}p}{\mathrm{d}x},$$

i.e. the probability that x is in the range x to $x + \mathrm{d}x$ is $\psi(x)\,\mathrm{d}x$. Any probability density function has to be normalised,

$$\int \psi(x)\,\mathrm{d}x = 1.$$

The Planck law is also just a probability density function, telling how many photons there are in any frequency/wavelength interval.

2 Stellar initial mass function

The stellar initial mass function (IMF) tells the probability of finding a star of initial mass mass M_{ini} in the range M_{ini} to $M_{\mathrm{ini}} + \mathrm{d}M_{\mathrm{ini}}$ and is often described as a power law function (e.g. Salpeter mass function),

$$\xi(M_{\mathrm{ini}}) = \frac{\mathrm{d}p}{\mathrm{d}M_{\mathrm{ini}}} = AM_{\mathrm{ini}}^{\gamma}$$

with $\gamma = -2.35$. The normalisation constant A follows from

$$\int_{M_l}^{M_u} \xi(M_{\mathrm{ini}})\,\mathrm{d}M_{\mathrm{ini}} = 1,$$

i.e.

$$A = \frac{\gamma + 1}{M_u^{\gamma+1} - M_l^{\gamma+1}}.$$

Often one just defines $\Gamma = \gamma + 1$. This is also very useful if we are to define the mass function not on a linear ($\mathrm{d}M_{\mathrm{ini}}$) but on a logarithmic scale, e.g. as the probability to find a star of mass M_{ini} in the interval from $\log M_{\mathrm{ini}}$ to $\log M_{\mathrm{ini}} + \mathrm{d}\log M_{\mathrm{ini}}$. From the chain rule we have

$$\mathrm{d}\log x = \frac{\mathrm{d}\ln x}{\ln 10} = \frac{\mathrm{d}x}{x \ln 10}$$

and hence

$$\xi(\log M_{\mathrm{ini}}) = \frac{\mathrm{d}p}{\mathrm{d}\log M_{\mathrm{ini}}} = M_{\mathrm{ini}} \ln 10 \frac{\mathrm{d}p}{\mathrm{d}M_{\mathrm{ini}}} = \ln 10 \cdot A \cdot M_{\mathrm{ini}}^{\Gamma}.$$

With these re-formulations the normalisation is guaranteed again, i.e.

$$\int_{M_l}^{M_u} \xi(M_{\mathrm{ini}})\,\mathrm{d}M_{\mathrm{ini}} = \int_{\log M_l}^{\log M_u} \xi(\log M_{\mathrm{ini}})\,\mathrm{d}\log M_{\mathrm{ini}} = 1$$

as can be easily checked with the above formulae.

3 A population of stars

Now imagine we want to simulate a population of stars and want to ensure that the distirbution of their initial masses follows a Salpeter IMF. There are two ways to do that

1. Randomly draw N stars from the PDF describing the IMF
2. Set up a grid in mass and compute the probability of finding a star in the different mass intervals. We can then convert to absolute numbers by multiplying the probabilities by the total number of stars N.

The latter approach is more efficient in lower dimensional problems and ensures that we really take every possible mass into account because we sample the whole mass range. In the former case this is not ensured and only true for large enough N!

First we have to discretise our problem by defining the mass resolution. Let's say we want to look into a population of stars with masses between $1\,\mathrm{M}_\odot$ and $100\,\mathrm{M}_\odot$. Because of the power law form of the IMF it is instructive to work with a logarithmic grid spacing, i.e.

$$
\begin{aligned}
\log M_l &= 0, \\
\log M_u &= 2, \\
\Delta \log M &= \frac{\log M_u - \log M_l}{N_M} = 0.02
\end{aligned}
$$

for a mass resolution of $N_M = 100$. The probability to find a star with mass M in the interval $\log M$ to $\log M + \Delta \log M$ is then given by

$$
\Delta p = \int_{\log M}^{\log M + \Delta \log M} \xi(\log M)\,\mathrm{d}\log M \approx \xi(\log M)\,\Delta \log M
$$

for small enough $\Delta \log M$ (Taylor expansion! Try it out and see how small $\Delta \log M$ has to be in order that the normalisation condition $\Sigma_{\log M} \xi(\log M)\,\Delta \log M = 1$ is met and to which accuracy).

Now in your case, there are PDFs not only for the dimension in initial mass but also in distance and age. These PDFs tell you what the probability is to find a star in a certain distance interval and age interval, respectively. **As one exercise for your thesis you should write down how to derive the PDF for distance and age.** Age is probably simplest because we want that all ages are equally probable, i.e. that the PDF is constant (dp/dage=const.). The distance is a bit more complicated but your program already contains one correct representation although I would still like you to derive a "nicer" representation for the case that $\mathrm{d}p/\mathrm{d}V = \mathrm{const.}$ with V being the volume, i.e. stars are found at all places in the volume with the same probability. Rewrite the PDF as a function of distance r only ($\mathrm{d}V = \ldots$) and normalise it properly for specific integration bounds.

So now let's have a look at the whole problem. Let $\phi(r) = \mathrm{d}p/\mathrm{d}r$ be the PDF for the distance r and $\theta(t) = \mathrm{d}p/\mathrm{d}t$ the PDF for the ages t. Let us further assume that you specified the following resolutions: a spacing in distance of Δr and in age of Δt. The probability to find a star of mass M in the mass interval $\log M$ to $\log M + \Delta \log M$, at a distance between r and $r + \Delta r$ and of age t in the interval t to $t + \Delta t$ is then given by

$$
\begin{aligned}
\Delta p &\approx \xi(\log M)\Delta \log M \cdot \phi(r)\Delta r \cdot \theta(t)\Delta t \\
&\equiv \Psi(\log M, r, t)\Delta V
\end{aligned}
$$

for small enough $\Delta \log M$, Δr and Δt. The last line is just to compactify the equation and to show the similarity to the one-dimensional case of masses only ($\Psi(\log M, r, t) = \frac{\mathrm{d}p}{\mathrm{d}\log M\,\mathrm{d}r\,\mathrm{d}t} = \xi(\log M)\phi(r)\theta(t)$ and $\Delta V = \Delta \log M \Delta r \Delta t$ being the volume of the phase space). By the way: the above equation is only valid if the individual PDFs for mass, distance and age are independent of each other, i.e. if the probability to find a star of a certain mass does not depend on distance and/or age and so on.

Now back to your problem: you discretised the phase volume and now you loop over the phase space and compute the probabilities Δp to find stars in each cell/volume-element of your phase space. You then check whether the star is visible (magnitude cut) and if so you read off the fractional main-sequence (MS) age of that star and make a histogram in fractional MS age by adding the individual Δp's to the appropriate fractional MS age bin. For easier thinking, imagine there are N_{tot} stars in the whole volume. The number of stars in a cell is then given by $\Delta N = N_{\mathrm{tot}} \Delta p$ and you would then add up number of stars in the individual fractional MS age bins. I hope this makes sense to you! Please do not hesitate to ask if anything is unclear.

The advantage of this approach can be its flexibility. For example, we may pick a different PDF for the distances such that it is more likely to find stars in certain distances. In the above approach, this would now be very simple because only one function needs to be changed!

In the end you compute the PDF of fractional MS ages of stars in a certain volume with a certain magnitude cut. If you normalised all the above PDFs properly, also the PDF of fractional MS ages is normalised correctly (why?). This will be ultimate check whether we did no mistakes in the whole process. Also, you should plot a probability density afterwards, i.e. $\mathrm{d}p/\mathrm{d}\tau$ with τ being the fractional MS age.