Mathematical proofs + Synthetic Simulations

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1 Progress

I have proved u^*_{MF}, x^*_{MF}, u^*_{MB} and x^*_{MB} for my model. I also see how u^*_{MF} causes sub-row stochasticity but that is okay because λ-connectivity remains. x^*_{MB} was found using KKT (I watched some videos about it on YouTube). I want to prove theorem 3, but I am finding it quite difficult, I am still working on it.

2 Reminder of cost function

$$\theta_m(x(t), u(t)) = \|x(t) - u(t)1_n\|_2^2 + a(n) \cdot \|u(t)\|^2 \cdot e^{-\lambda(t-t_c)} \tag{1}$$
$$a(n) = \rho \cdot n \tag{2}$$

3 Finding x^*_{MF} (and u^*_{MF})

3.0.1 Finding u^*_{MF}

The model-free approach solves:

$$u^*_{MF}(t) = \arg\min_{u\in[0,1]} \theta_m(x(t), u) \tag{3}$$

Expanding θ_m:

$$\theta_m(x(t), u(t)) = \sum_{i=1}^{n} (x_i(t) - u(t))^2 + \rho \cdot n \cdot u(t)^2 \cdot e^{-\lambda(t-t_c)} \tag{4}$$

To find the minimum:

$$\frac{\partial \theta_m}{\partial u} = -2\sum_{i=1}^{n}(x_i(t) - u(t)) + 2\rho \cdot n \cdot u(t) \cdot e^{-\lambda(t-t_c)} = 0 \tag{5}$$

Simplifying:

$$-2\sum_{i=1}^{n} x_i(t) + 2n \cdot u(t) + 2\rho \cdot n \cdot u(t) \cdot e^{-\lambda(t-t_c)} = 0 \tag{6}$$

$$2n \cdot u(t) \cdot (1 + \rho \cdot e^{-\lambda(t-t_c)}) = 2\sum_{i=1}^{n} x_i(t) \tag{7}$$

Solving for $u(t)$ gives $u^*_{MF}(t)$

$$u^*_{MF}(t) = \frac{\sum_{i=1}^{n} x_i(t)}{n(1 + \rho \cdot e^{-\lambda(t-t_c)})} \tag{8}$$

3.0.2 Finding x^*_{MF}

System dynamics:

$$x(t+1) = Ax(t) + Bu(t) + \Lambda x(0) \tag{9}$$

Substituting in u^*_{MF}:

$$x(t+1) = Ax(t) + \frac{B \cdot 1_n^T}{n(1+\rho \cdot e^{-\lambda(t-t_c)})} x(t) + \Lambda x(0) \tag{10}$$

This can be rewritten as:

$$x(t+1) = (I_n - \Lambda)Fx(t) + \Lambda x(0) \tag{11}$$

where:

$$F = W + \frac{w_{rec} 1_n^T}{n(1+\rho \cdot e^{-\lambda(t-t_c)})} \tag{12}$$

3.0.3 F matrix analysis

The F matrix can be seen as an adjacency graph. It is sub-row stochastic so satisfies . The sub-stochastic is a direct effect of the mitigation of extreme opinions.

3.0.4 Finding x^*_{MF} (continued)

At steady state, $x(t+1) = x(t) = x^*_{MF}$:

$$x^*_{MF} = (I_n - \Lambda)Fx^*_{MF} + \Lambda x(0) \tag{13}$$

Rearranging:

$$(I_n - (I_n - \Lambda)F)x^*_{MF} = \Lambda x(0) \tag{14}$$

Substituting F and then replacing with $A = (I_n - \Lambda)W$ and $B = (I_n - \Lambda)w_{rec}$:

$$\left(I_n - A - \frac{B \cdot 1_n^T}{n(1+\rho \cdot e^{-\lambda(t-t_c)})}\right) x^*_{MF} = \Lambda x(0) \tag{15}$$

So:

$$x^*_{MF} = \left(I_n - A - \frac{B \cdot 1_n^T}{n(1+\rho \cdot e^{-\lambda(t-t_c)})}\right)^{-1} \Lambda x(0) \tag{16}$$

4 Finding x^*_{MB} (and u^*_{MB})

The model-based approach solves:

$$\begin{align}
(x^*_{MB}, u^*_{MB}) = \arg\min_{x,u} \theta_m(x,u) \tag{17}\\
\text{subject to} \quad x = Ax + Bu + \Lambda x(0), \tag{18}\\
u \in [0,1] \tag{19}
\end{align}$$

where $\theta_m(x,u) = \|x - u\mathbf{1}_n\|_2^2 + \rho n \|u\|^2 e^{-\lambda(t-t_c)}$

4.0.1 Lagrangian

$\mathcal{L}(x,u,\mu,\alpha,\beta) = \theta_m(x,u) + \mu^T(x - Ax - Bu - \Lambda x(0)) + \alpha(u-1) + \beta(-u)$

4.0.2 Gradients

$$\nabla_x \theta_m(x,u) = 2(x - u\mathbf{1}_n) \tag{20}$$
$$\nabla_u \theta_m(x,u) = -2\mathbf{1}_n^T x + 2nu + 2\rho nu \cdot e^{-\lambda(t-t_c)} \tag{21}$$

4.0.3 Stationarity Conditions

$$\nabla_x \mathcal{L} = 2(x - u\mathbf{1}_n) + \mu - A^T\mu = 0 \tag{22}$$
$$\nabla_u \mathcal{L} = -2\mathbf{1}_n^T x + 2nu + 2\rho nu \cdot e^{-\lambda(t-t_c)} - B^T\mu + \alpha - \beta = 0 \tag{23}$$

4.0.4 From First Stationarity Condition

$\mu = (I_n - A^T)^{-1}(-2(x - u\mathbf{1}_n))$

4.0.5 From Equality Constraint

$$x = Ax + Bu + \Lambda x(0) \tag{24}$$
$$x = (I_n - A)^{-1}(Bu + \Lambda x(0)) \tag{25}$$

$$v = (I_n - A)^{-1}B \tag{26}$$
$$z = (I_n - A)^{-1}\Lambda x(0) \tag{27}$$

Therefore: $x = vu + z$

4.0.6 Substituting into Second Stationarity Condition

Substituting μ and redefined x into $\nabla_u \mathcal{L}$:

$$\nabla_u \mathcal{L} = -2\mathbf{1}_n^T(vu + z) + 2nu + 2\rho nu \cdot e^{-\lambda(t-t_c)} - B^T(I_n - A^T)^{-1}(-2((vu + z) - u\mathbf{1}_n)) + \alpha - \beta \tag{28}$$

4.0.7 Simplifying equation

$B^T(I_n - A^T)^{-1}$ can also be defines as:

$$v^T = B^T(I_n - A^T)^{-1} \tag{29}$$

This can be found by applying properties of transpose matrices. This allows to simplify further:

$$\nabla_u \mathcal{L} = -2\mathbf{1}_n^T vu - 2\mathbf{1}_n^T z + 2nu + 2\rho nu \cdot e^{-\lambda(t-t_c)} + 2v^T vu + 2v^T z - 2v^T u\mathbf{1}_n + \alpha - \beta \tag{30}$$

4.0.8 Collecting Terms with u

$$\nabla_u \mathcal{L} = [-2\mathbf{1}_n^T v + 2n + 2\rho n \cdot e^{-\lambda(t-t_c)} + 2v^T v - 2v^T\mathbf{1}_n]u + 2v^T z - 2\mathbf{1}_n^T z + \alpha - \beta \tag{31}$$

For simplicity:

$$k = e^{-\lambda(t-t_c)} \tag{32}$$
$$q = -\mathbf{1}_n^T v + n + v^T v - v^T\mathbf{1}_n \tag{33}$$
$$s = -(v^T z - \mathbf{1}_n^T z) \tag{34}$$

Giving:

$$\nabla_u \mathcal{L} = 2(q + \rho nk)u - 2s + \alpha - \beta \tag{35}$$

$$\tag{36}$$

Since $\nabla_u \mathcal{L} = 0$, we can find an equation defining u.

$$u = \frac{2s + \beta - \alpha}{2(q + \rho nk)} \tag{37}$$

4.0.9 Interior Solution$(0 < u < 1)$

$$\alpha = 0, \beta = 0 \tag{38}$$

$$u^*_{MB} = \frac{s}{q + \rho nk} \tag{39}$$

So:

$$x^*_{MB} = vu^*_{MB} + z \tag{40}$$

5 Next Steps

- Finish theorem 3 (MB convergence)
- Try simulations with many users (100+)
- Make different simulations to get different results
- Improve the BEP paper based on all new progress
- re-do the presentation