MAT4720 - Lectures + excercises

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These lecture notes are based on what we have discussed in class, as well as material from: [2], [1]

1 Lecture 4 - Kolmogorov's continuity theorem + BM

In this lecture we discussed different theoretical properties of stochastic processes, what shall it mean for a process to be continuous, as well as measurable ++.

Definition 1 (Trajectory/path of a stochastic process) *The trajectory of a stochastic process, is the evolution of a process if we fix one scenario. This means that we fix $\omega \in \Omega$, and look how this develops in time, i.e we look at:*

$$[0, \infty) \ni t \mapsto X_t(\omega) \in \mathbb{R} \;\; (\mathbb{R}^d, \dots)$$

Definition 2 (Continuous process) *A process $X = X_t, t \geq 0$ is continuous whenever P. a-s its trajectories are continuous i.e:*

$$P(\{\omega \in \Omega : X_\bullet(\omega) \;\; \textit{is continuous}\}) = 1$$

Here $X_\bullet(\omega)$ represents the fact that we have fixed ω, and look at $t \mapsto X_t(\omega)$

We have that continuity is not in general preserved by modifications, as we saw in an earlier example, where X, Y were modifications, but X was not continuous.

Definition 3 (Measurable process) *A stochastic process $X = X_t, t \geq 0$ is measurable, if the mapping*

$$[0, \infty) \times \Omega \ni (t, \omega) \mapsto X_t(\omega) \in \mathbb{R}$$

is measurable. This means that for $\forall B \in \mathcal{B}(\mathbb{R})$ *we have:*

$$X^{-1}(B) = \{(t, \omega) \in [0, \infty) \times \Omega : X_t(\omega) \in B\} \in \mathcal{B}([0, \infty)) \otimes \mathcal{F}$$

We need some more explanation:

$$\mathcal{B}([0, \infty)) \otimes \mathcal{F} = \sigma(\{A \times F : A \in \mathcal{B}([0, \infty)), F \in \mathcal{F}\})$$

Definition 4 (Progressively measurable process) *A stochastic process* $X = X_t, t \geq 0$ *is progressively measurable w.r.t* $\mathbb{F}$ *if* $\forall t$*, and* $s \leq t$ *the map:*

$$[0, t] \times \Omega \ni (s, \omega) \mapsto X_s(\omega) \in \mathbb{R} \ \ (\mathbb{R}^d, \dots)$$

is measurable w.r.t $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$

Proposition 1 (Adapted and right-continuous process is progressively measurable) *If* $X = X_t, t \geq 0$ *is* $\mathbb{F}$*-adapted and right-continuous then* X *is progressively measurable.*

We note that any progressively measurable process is measurable.

Proof 1 *We start by creating an appropriate approximation of* X*, we create an approximating process* $X^{(n)} = X_s^{(n)}, s \in [0, \infty)$ *are defined as:*

$$X_s^{(n)}(\omega) = \begin{cases} X_{\frac{(k+1)}{2^n} t} & s \in \left[\frac{k}{2^n} t, \frac{k+1}{2^n} t\right) \\ X_t(\omega), & s = t \end{cases}$$

$X_s^{(n)}, s \geq 0$ *is right-continuous.*

We now want to show that $(X^{(n)})_n$ *is approximating* X *in the appropriate way. We have* $s \leq t$*: for all* n*,* $\exists k_n$ *such that* $s \in \left[\frac{k_n}{2^n} t, \frac{k_n+1}{2^n} t\right)$*, if we denote* $s_n = \frac{k_n+1}{2^n} t$*, then* $s < s_n$*, furthermore* $s_n \downarrow s$*, i.e converges from above. This means that we get a pointwise convergence for the rv's i.e*

$$\begin{gathered} X_s^n(\omega) = X_{\frac{(k_n+1)}{2^n} t}(\omega) = X_{s_n}(\omega) \\ \Downarrow \\ \lim_{n \to \infty} X_{s_n}(\omega) = X_s(\omega) \end{gathered}$$

We must now show the progressively measurable property of our approximating process, i.e:

$$\begin{aligned} &\{(s, \omega) \in [0, t] \times \Omega : X_s^{(n)}(\omega) \in B\} \\ &= \bigcup_{k=0}^{2^n - 1} \left[\frac{k}{2^n} t, \frac{k+1}{2^n} t\right) \times \{X_{\frac{(k+1)}{2^n} t} \in B\} \cup \{t\} \cup \{X_t \in B\} \end{aligned}$$

Now: $\left[\frac{k}{2^n} t, \frac{k+1}{2^n} t\right) \in \mathcal{B}([0, t])$*, furthermore* $\{t\} \in \mathcal{B}([0, t])$*, as well as* $\{X_t \in B\} \in \mathcal{F}_t$ *since* X *is* $\mathbb{F}$*-adapted. Now as mentioned* X *is* $\mathbb{F}$*-adapted, meaning*

$u \leq t \implies \mathcal{F}_u \subseteq \mathcal{F}_t$, we have $\frac{(k+1)}{2^n}t \leq t$ and $\mathcal{F}_{\frac{(k+1)}{2^n}t} \subseteq \mathcal{F}_t$, leaving us with the conclusion that $\{X_{\frac{(k+1)}{2^n}t} \in B\} \in \mathcal{F}_t$. Putting all this together leaves us with:

$$\{(s,\omega) \in [0,t] \times \Omega : X_s^{(n)}(\omega) \in B\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t$$

We would like some sort of result that guarantees that a stochastic process is continuous, or at least have a continuous modification. This is exactly what Kolmogorov's continuity theorem gives us:

Theorem 1 (Kolmogorov's continuity theorem) *Let $D \subseteq \mathbb{R}^m$, i.e the domain for the stochastic process, let $X = X_\theta, \theta \in D$ be a stochastic process. Furthermore assume that there exists $\alpha > 0$, $\beta > 0$ and $C > 0$, such that:*

$$E[|X_{\theta_1} - X_{\theta_2}|] \leq C|\theta_1 - \theta_2|^{m+\alpha} \ \ \forall \theta_1, \theta_2 \in D$$

Then there exists a continuous modifications $\widetilde{X}_\theta, \theta \in D$ of X i.e

$$P(\{\omega \in \Omega : \widetilde{X}_\theta = X_\theta\}) = 1, \ \ \forall \theta \in D$$

Furthermore $\widetilde{X}$ is actually Hölder-continuous with exponent $\gamma < \frac{\alpha}{\beta}$ on all compact subsets of D.

Definition 5 (Hölder continuity) *Let $K \subseteq D$ be a compact set, then there exist $M > 0$ such that:*

$$|\widetilde{X}_{\theta_1} - \widetilde{X}_{\theta_2}| \leq M|\theta_1 - \theta_2|^\gamma, \ \ \gamma < \frac{\alpha}{\beta}, \ \ \theta_1, \theta_2 \in K$$

Corollary 1 (Specification of Kolmogorov's continuity theorem for $m = 1$) *Let $X = X_t, t \geq 0$ be a stochastic process. If there exits $\alpha > 0$, $\beta > 0$ and $C > 0$ such that:*

$$E[|X_t - X_s|^\beta] \leq C|t - s|^{1+\alpha}, \ \ s, t \in [o, \infty)$$

then there exists a continuous modification $\widetilde{X}$ of X, furthermore $\widetilde{X}$ is Hölder continuous on the compact subsets of $[0, \infty)$.

We will now turn to the definition of Brownian motion, this is what allows us to model, the uncertainty to some extent.

Definition 6 (Brownian Motion) *Assume that we are on $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F}$. We denote the Brownian motion by $B = B_t, t \geq 0$, which is $\mathbb{F}$-adapted. We say that B is a brownian motion if:*

i) $B_0 = 0$.

ii) $B_t - B_s \sim \mathcal{N}(0, t - s)$ *for* $s \leq t$

iii) *Have independent increments, i.e for* $0 \leq t_0 < t_1 < \cdots < t_n$ *then* $B_{t_n} - B_{t_{n-1}}, B_{t_{n-1}} - B_{t_{n-2}}, \ldots, B_{t_1} - B_{t_0}$ *are independent.*

We start by some immediate remarks:

Remark 1 *Some consequences of the BM:*

1. $B_t - B_s$ *is independents of* $\mathcal{F}_s$*, this means that for all* $F \in \mathcal{F}_s$ *the random variable* $\mathbf{1}_F(\omega)$ *is independent of* $(B_t - B_s)$
2. $\mathcal{F}_t^B = \sigma(B_s : s \leq t) = \sigma(B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_1} - B_{t_0}),\ \forall\ 0 \leq t_0 < t_1 < \cdots < t_n \leq t$
3. $(B_t - B_s)$ *is independent of* B_r *for* $r \leq s$*.*
4. *If* B *is a BM w.r.t* $\mathbb{F}$*, then it's also a BM w.r.t any sub-filtration* $\mathbb{K} \subseteq \mathbb{K}$
5. *It's not true that a BM* B *keeps being a BM w.r.t* $\mathbb{G}$ *with* $\mathcal{F}_t \subseteq \mathcal{G}_t$

If we think of remark number 5, here this means that for instance when it comes to inside trading, there could be that one could not model the data with for instance Black-Scholes as with their information $\mathbb{G}$, there could be that B is not a BM under this filtration.

Furthermore, it's not always that the smallest filtration $\mathcal{F}_t^B$ is the best filtration to work with. Let's say you have a complex SDE, where you for instance model relationships between assets in different markets, we would then have to work with the larger filtration, wich encaptures this total information.

Proposition 2 (Application of Kolmogorov's continuity theorem on BM) *The BM has a continuous modification, furthermore this modification is Hölder continuous with exponent* $\gamma < \frac{1}{2}$*.*

Proof 2 *We must show that if there exists* $\alpha > 0$*,* $\beta > 0$ *and* $C > 0$*, then:*

$$E[|B_t - B_s|^\beta] \leq C|t - s|^{1+\alpha}$$

We have that $B_t - B_s \sim \mathcal{N}(0, t - s)$ $(s \leq t)$*.*

$$B_t - B_s \overset{d}{=} B_{t-s} \overset{d}{=} (t - s)^{1/2} Z, \quad Z \sim \mathcal{N}(0, 1)$$

Leaving us with:

$$E[|B_t - B_s|^\beta] = E[|t - s|^{\frac{\beta}{2}} |Z|^\beta] = |t - s|^{\frac{\beta}{2}} E[|Z|^\beta] = |t - s|^{\frac{\beta}{2}} C$$

Now: $1 + \alpha = \frac{\beta}{2}$*, and* $\alpha > 0$*, leaving us with:* $\alpha = \frac{\beta}{2} - 1 > 0 \iff \beta > 2$*.*
For the Hölder continuity part we have by the theorem that
$\gamma < \frac{\alpha}{\beta} = (\frac{\beta}{2} - 1)\frac{1}{\beta} = \frac{1}{2} - \frac{1}{\beta} \in (0, \frac{1}{2})$*.*

In this course we will work with the continuous modification of the BM.

Proposition 3 (Other way's of checking if we are dealing with a BM) *If $B = B_t, t \geq 0$ is a BM, then:*

i) $B_0 = 0$ P*-a.s*

ii) for $0 \leq t_1 < t_2 < \cdots < t_m$*, the vector* $(B_{t_1}, B_{t_2}, \ldots, B_{t_m}) \sim \mathcal{N}(\mathbf{0}, \Gamma)$

iii) $E[B_s B_t] = s \wedge t = \min(s, t)$

Proof 3 *By assumption B is a BM, so $B_0 = 0$ is obvious. Now, we only need to prove $iii)$ as this determines Γ in $ii)$, but with some more, we will see that at the end of the proof. Assume $s \leq t$ for simplicity:*

$$B_s B_t = B_s(B_s + (B_t - B_s)) = B_s^2 + B_s(B_t - B_s)$$

$$\Downarrow$$

$$E[B_s B_t] = E[B_s^2] + B_s(B_t - B_s)] = E[B_s^2] + E[B_s]E[B_t - B_s] = E[B_s^2] + 0 = s$$

From $iii)$ we actually get that $\Gamma_{ij} = t_i \wedge t_j$.

This means that when we need to show that something is a BM, it could be easier to show that $i)$ and $iii)$ in proposition 3 are satisfied. $ii)$ is checked basically by taking the expectation, and then in $iii)$ find the covariance.

1.1 Characteristic functions and Gaussian processes

If we go back to earlier courses we learned about moment generating functions, i.e if X is a random variable, then $M_X(t) = E[e^{tX}]$ is it moment generating function, with the nice property that $E[X^r] = M_X^{(r)}(0)$, furthermore there are known explicit formulas for different types of distributions.

However, the e^{tX} could lead to integrability formulas, we could get in trouble i.e $E[e^{tX}] = \infty$, this is where characteristic functions comes to play, somehow a integrability saviour.

Definition 7 (Characteristic function) *The charateristic function [2] of a random variable X is defiend as:*

$$\phi(t) = E[e^{itX}]$$

I will just state some nice properties of the characteristic function:

Proposition 4 (Properties of the characteristic function) *These illustrate why one often would prefer characteristic functions instead of moment-generating functions*

i) $\phi(0) = 1$

ii) $|\phi(t)| \leq 1$

iii) $\phi^{(n)}(0) = i^n E[X^n]$

iv) X, Y *independent:* $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

v) $\phi_{\mathcal{N}(\mu,\sigma^2)}(t) = \exp\left(-\frac{\sigma^2 t^2}{2} + it\mu\right)$

2 Lecture 5: paths of BM

We want to study the trajectories of a BM on $\mathbb{R}$.

Definition 8 (p-variation) *Given a function $f : \mathbb{R} \to \mathbb{R}$ we call the p-variation of f on $[a,b]$, the quantity:*

$$V^p_{[a,b]}(f) = \sup_{\pi} \sum_{i=0}^{n-1} |f(t_{i+1} - f(t_i)|^p$$

Here π denotes the partitions of $[a,b]$ with $a = t_0 < t_1 < \cdots < t_n = b$. We denotes it's mesh $|\pi|$, by $|\pi| = \max_{i=0,\ldots,n-1} |t_{i+1} - t_i|$

If $V^p_{[a,b]}(f) < \infty$, we say that f has finite p-variation. For $p = 1$ we speak about the total variation, and for $p = 2$, we speak about the quadratic variation.

Result 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz function, i.e $\exists\ L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$, $\forall x, y \in \mathbb{R}$. Then for any $[a,b] \subseteq \mathbb{R}$ and $\pi : a = t_0 < t_1 < \cdots < t_n = b$ we have:*

$$\sum_{i=0}^{n-1} |f(t_{i+1} - f(t_i)| < \infty \implies V^p_{[a,b]}(f) < \infty$$

Proof 4 *We got that f is Lipschitz, furthermore $\{t_i\}$ is a partition of $[a,b]$:*

$$\begin{aligned}\sum_{i=0}^{n-1} |f(t_{i+1} - f(t_i)| &\leq \sum_{i=0}^{n-1} L|t_{i+1} - t_i| \\ &= L\sum_{i=0}^{n-1} |t_{i+1} - t_i| = L(b-a)\end{aligned}$$

Proposition 5 *Consider $X = X_t, t \geq 0$ and let it be a BM, furthermore let $\pi : s = t_0 < t_1 < \cdots < t_n$ be a partition of $[s,t]$, denote $|\pi| = \max_{i=0,\ldots,n-1} |t_{i+1} - t_i|$, and consider the sum:*

$$S_\pi^2(\omega) = \sum_{i=0}^{n-1} |X_{t_{i+1}}(\omega) - X_{t_i}|^2, \quad \omega \in \Omega$$

$$\lim_{|\pi|\downarrow 0} S_\pi^2(\omega) \stackrel{L^2}{=} (t-s)$$

This means that:

$$\lim_{|\pi|\downarrow 0} E\left[\left(S_\pi^2 - (t-s)\right)^2 = 0\right]$$

Furthermore the trajectories of the X does not have finite variation on any interval $[s,t]$ P-a.s

Proof 5 *We will prove this for the L^2 limit, the strategy will be to decompose sums smart, and exploit independent increments of the BM.*

$$\begin{aligned}
\left[S_\pi^2 - (t-s)\right] &= \sum_{i=0}^{n-1} \left[(X_{t_{i+1}}(\omega) - X_{t_i}(\omega))^2 - (t_{i+1} - t_i)\right] \\
E\left[\left(S_\pi^2 - (t-s)\right)^2\right] &= \sum_{i=0}^{n-1} E\left[\left[(X_{t_{i+1}}(\omega) - X_{t_i}(\omega))^2 - (t_{i+1} - t_i)\right]^2\right] \\
&+ \sum_{i,j=0, i\neq j}^{n-1} E[\left((X_{t_{i+1}} - X_{t_i})^2 - (t_{i+1} - t_i)\right)\left((X_{t_{j+1}} - X_{t_j})^2 - (t_{j+1} - t_j)\right)]
\end{aligned}$$

In the last sum, we have that the two expressions are independent, furthermore:

$$\begin{aligned}
E[(X_{t_{i+1}} - X_{t_i})^2 - (t_{i+1} - t_i)] &= E[(X_{t_{i+1}} - X_{t_i})^2] - (t_{i+1} - t_i) \\
&= (t_{i+1} - t_i) - (t_{i+1} - t_i) = 0
\end{aligned}$$

This leaves us with:

$$E\left[\left(S_\pi^2 - (t-s)\right)^2\right] = \sum_{i=0}^{n-1} E\left[\left[(X_{t_{i+1}}(\omega) - X_{t_i}(\omega))^2 - (t_{i+1} - t_i)\right]^2\right]$$

We need some observations:

$$(X_{t_{i+1}} - X_{t_i})^2 - (t_{i+1} - t_i) = (t_{i+1} - t_i)\left[\left(\frac{X_{t_{i+1}} - X_{t_i}}{\sqrt{t_{i+1} - t_i}}\right)^2 - 1\right]$$

This fits us perfect, as:

$$Z = \frac{X_{t_{i+1}} - X_{t_i}}{\sqrt{t_{i+1} - t_i}} \sim \mathcal{N}(0,1)$$

This leavs us with:

$$\begin{aligned}
E\left[\left(S_\pi^2 - (t-s)\right)^2\right] &= \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 E[(Z^2 - 1)^2] \\
&= C \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 \\
&\leq C \sum_{i=0}^{n-1} |\pi|(t_{i+1} - t_i) = C|\pi| \sum_{i=0}^{n-1} (t_{i+1} - t_i) \\
&= C|\pi|(t-s)
\end{aligned}$$

Thus:

$$\lim_{|\pi| \downarrow 0} C|\pi|(t-s) = 0$$

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[2] J.B. Walsh. *Knowing the Odds: An Introduction to Probability*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN: 9780821890325. URL: https://books.google.no/books?id=4uC0uEXpvyoC.