MAT4770: Recap of curriculum

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The following content is heavily based upon the wonderful lecture notes of Fred Espen Benth in the course MAT4770-Stochastic Modelling in Energy and Commodity Markets. Please check out the course STK4530- Interest Rate Modelling via SPDE's, the course consists of many of the same methodologies.

1 Temperature markets

let $p \in \mathbb{N}$, $A \in \mathbb{R}^{p \times p}$ and $B(t)$ a 1-dim BM. $\boldsymbol{e}_p$ is the p-th unit vector.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \dots & -\alpha_1 \end{bmatrix}$$

The model we will be interested in is:

$$dX(t) = AX(t)dt + \sigma \boldsymbol{e}_p dB(t)$$

Proposition 1 *The solution to the above SDE, for $s \geq t$ is given by:*

$$X(s) = e^{A(s-t)}X(t) + \sigma \int_t^s e^{A(s-u)} \boldsymbol{e}_p dB(u)$$

Proof 1 *We start off by the usual way, and apply a "OU/Vasicek"-trick:*

$$\begin{aligned} d(e^{-At}X(t)) &= -Ae^{-At}X(t)dt + e^{-At}dX(t) \\ &= -Ae^{-At}X(t)dt + e^{-At}\left[AX(t)dt + \sigma \boldsymbol{e}_p dB(t)\right] \\ &= \sigma e^{-At} \boldsymbol{e}_p dB(t) \\ &\Downarrow \\ X(s) &= e^{A(s-t)}X(t) + \sigma \int_t^s e^{A(s-u)} \boldsymbol{e}_p dB(u) \end{aligned}$$

Let $b' = (b_0, b_1, \ldots, b_q, 0, \ldots, 0) \in \mathbb{R}^p$, with $b_q = 1$ for $q < p$, we then define:

Definition 1 (CARMA(p,q)) *The CARMA(p,q) process $Y(t)$ is defined by:*

$$CARMA(p,q) = Y(t) := b'X(t)$$

Definition 2 (CAR(p)) *The CAR(p) process is defined by:*

$$CAR(p) = CARMA(p,0) := Y(t) = \boldsymbol{e}_1' X(t) = X_1(t)$$

We will be interested in $CAR(p)$-processes, as these will be a part of modelling temperature dynamics.

1.1 CAT-futures

We define the CAT-future as follows:

$$F_{CAT}(t, \tau_1, \tau_2) = \mathbb{E}_Q \left[\int_{\tau_1}^{\tau_2} T(s) ds | \mathcal{F}_t \right]$$

Where:

$$T(s) = \Lambda(s) + \boldsymbol{e}_1' X(s)$$

The temperature follows a deterministic seasonal-function Λ and a $CAR(p)$-process. We model under Q, and would therefore need a change of measure:

$$\left. \frac{dQ^\theta}{dP} \right|_{\mathcal{F}_t} = \exp\left(\int_0^t \theta dB_s - \frac{1}{2} \int_0^t \theta^2 ds \right)$$

And from Girsanov, we get: $dB^\theta(t) = dB(t) - \theta dt$ is a Q^θ-BM.

$$X(s) = e^{A(s-t)} X(t) + \sigma \int_t^s e^{A(s-u)} \boldsymbol{e}_p \theta du + \sigma \int_s^t e^{A(s-u)} \boldsymbol{e}_p dB^\theta(u) \ \ (Q^\theta)$$

Proposition 2 (dynamics of CAT-futures) *The CAT-futures has the following dynamics:*

$$\begin{aligned} dF_{CAT}(t, \tau_1, \tau_2) &= \Sigma_{CAT}(t, \tau_1, \tau_2) dB^\theta(t) \\ \Sigma_{CAT}(t, \tau_1, \tau_2) &= \sigma \boldsymbol{e}_1' A^{-1} \left(e^{A(\tau_2 - t)} - e^{A(\tau_1 - t)} \right) \boldsymbol{e}_p \\ &= \sigma \boldsymbol{e}_1' \int_{\tau_1}^{\tau_2} e^{A(s-t)} ds \boldsymbol{e}_p \end{aligned}$$

1.2 Call option on CAT-futures

$$C(t,\tau,K,\tau_1,\tau_2) = e^{-r(\tau-t)}\mathbb{E}_Q\left[(F_{CAT}(\tau,\tau_1,\tau_2)-K)^+ \,|\mathcal{F}_t\right]$$

We start off by decomposing F_{CAT}:

$$F_{CAT}(\tau,\tau_1,\tau_2) = F_{CAT}(t,\tau,\tau_1,\tau_2) + \int_{\tau_1}^{\tau_2} \Sigma_{CAT}(s,\tau_1,\tau_2)dB^\theta(s)$$

Here the first part is $\mathcal{F}_t$-measurable and the second part is $\mathcal{F}_t$-independent.

Furthermore, we exploit the following:

$$\int_{\tau_1}^{\tau_2} \Sigma_{CAT}(s,\tau_1,\tau_2)dB^\theta(s) \sim \mathcal{N}\left(0, \int_{\tau_1}^{\tau_2} \Sigma^2(s,\tau_1,\tau_2)ds\right)$$

Let $c_1 = \sqrt{\int_{\tau_1}^{\tau_2} \Sigma^2(s,\tau_1,\tau_2)ds}$:

$$\begin{aligned}\mathbb{E}_Q\left[(F_{CAT}(\tau,\tau_1,\tau_2)-K)^+ \,|\mathcal{F}_t\right] &= \mathbb{E}_Q\left[\left(x+\int_{\tau_1}^{\tau_2} \Sigma_{CAT}(s,\tau_1,\tau_2)dB^\theta(s)\right)^+\right]\Bigg|_{x=F_{CAT}(t,\tau_1,\tau_2)} \\ &= \mathbb{E}_Q\left[(x+c_1Z)^+\right]\Big|_{x=F_{CAT}(t,\tau_1,\tau_2)}\end{aligned}$$

Now: $x + c_1 Z > 0 \implies Z > \frac{K-x}{c_1} := -d$

$$\begin{aligned}\mathbb{E}_Q\left[(x+c_1Z)^+\right]\Big|_{x=F_{CAT}(t,\tau_1,\tau_2)} &= \int_{-d}^{\infty}(x-K+c_1z)\varphi(z)dz \\ &= (x-K)\int_{-d}^{\infty}\varphi(z)dz + c_1\int_{-d}^{\infty}z\varphi(z)dz\end{aligned}$$

Now we will use the symmetry of the Normal distribution, as well as the fact: $\varphi'(z) = -z\varphi(z)$:

$$\begin{aligned}(x-K)\int_{-d}^{\infty}\varphi(z)dz + c_1\int_{-d}^{\infty}z\varphi(z)dz &= (x-K)\Phi(d) - c_1\int_{-d}^{\infty}\varphi'(z)dz \\ &= (x-K)\Phi(d) - c_1[0-\varphi(-d)] \\ &= (x-K)\Phi(d) + c_1\varphi(d)\end{aligned}$$

Which the finally gives:

$$C(t,\tau,K,\tau_1,\tau_2) = e^{-r(\tau-t)}\left([F_{CAT}(t,\tau_1,\tau_2)-K]\Phi(d) + \sqrt{\int_{\tau_1}^{\tau_2}\Sigma^2(s,\tau_1,\tau_2)ds}\;\varphi(d)\right)$$

$$d = \frac{F_{CAT}(t,\tau_1,\tau_2)-K}{\sqrt{\int_{\tau_1}^{\tau_2}\Sigma^2(s,\tau_1,\tau_2)ds}}$$

1.3 HDD/CDD-futures

These are contracts traded on CME, these represent demand for heating and cooling, they are defined as:

$$CDD(t, \tau_1, \tau_2) = \max(T(t) - c, 0)$$
$$HDD(t, \tau_1, \tau_2) = \max(c - T(t), 0)$$

The price of a forward at time t with measurement period $[\tau_1, \tau_2]$ is:

$$F_{CDD}(t, \tau_1, \tau_2) = \mathbb{E}_Q\left[\int_{\tau_1}^{\tau_2} CDD(s)ds \middle| F_t\right]$$

Meaning that one in general is interested in: $\mathbb{E}_Q[CDD(s)|\mathcal{F}_t]$, meaning that one needs a model for the temperature $T(s)$ and a measure change $Q \sim P$.

$$T(s) = \Lambda(s) + \boldsymbol{e}_1' X(s)$$

2 Esscher transform

We start off by the following:

$$f(t,\tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t]$$

Where does this relation come from: All tradable assets should be Q-martingales after discounting. Furthermore futures are free to enter, giving:

$$\begin{aligned}
\pi_{future}(t) = 0 = e^{-r(\tau-t)}\mathbb{E}_Q[S(\tau) - f(t,\tau)|\mathcal{F}_t] \\
\Downarrow \\
f(t,\tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t]
\end{aligned}$$

We have that power spot is not tradable, hence the completeness of the market is gone, thus giving us many pricing measures, we should choose Q so that:

1. the spot S, can be categorized under Q
2. the expected value can be computed.

Our model framework is:

$$\Lambda(t) + X(t) + Y(t) = \begin{cases} \ln(S(t)) & \text{Geometric} \\ S(t) & \text{Arithmetic} \end{cases}$$

With the following dynamics:

$$\begin{aligned}
dX(t) &= [\mu - \alpha X(t)]dt + \sigma dB(t) \\
dY(t) &= [\delta - \beta Y(t)]dt + \eta dI(t)
\end{aligned}$$

Consider X: here we change measure by Girsanov: let $\hat{\theta}$ be a constant, and define:

$$\hat{Z}(t) = \exp\left(\hat{\theta}B(t) - \frac{1}{2}\hat{\theta}^2 t\right)$$
$$d\hat{B}_t = dB(t) - \hat{\theta}dt$$

Here $t \mapsto \hat{Z}(t)$ is a $(P, \mathcal{F})$-martingale. We get the Radon-nikodym derivative:

$$\left.\frac{dQ^{\hat{\theta}}}{dP}\right|_{\mathcal{F}_t} = \hat{Z}(t)$$

From Girsanov's thm: $Q^{\hat{\theta}}$ is a probability measure, and $\hat{B}$ is a $Q^{\hat{\theta}}$-BM.
We state the dynamics under $Q^{\hat{\theta}}$:

$$\begin{aligned} dX(t) &= [\mu - \alpha X(t)]dt + \sigma dB(t) \\ &= [\mu - \alpha X(t)]dt + \sigma[d\hat{B}_t + \hat{\theta}]dt \\ &= \left[(\mu + \sigma\hat{\theta}) - \alpha X(t)\right]dt + \sigma d\hat{B}(t) \end{aligned}$$

2.1 Esscher transform

Let $\widetilde{\theta} \in \mathbb{R}$, and define:

$$\widetilde{Z}(t) = \exp\left(\widetilde{\theta}I(t) - \varphi(\widetilde{\theta})t\right)$$

Here:

$$I(t) = \sum_{k=1}^{N_t} J_k$$
$$\varphi(\widetilde{\theta}) = \ln \mathbb{E}[e^{\widetilde{\theta}I(1)}]$$

We will study $\varphi(\widetilde{\theta})$ a bit further, later on.

In order for $\widetilde{Z}(t)$ to be well defined, we must have that $\mathbb{E}[e^{\widetilde{\theta}I(t)}]$ exist:

$$\begin{aligned}
\mathbb{E}[e^{\widetilde{\theta}I(t)}] &= \mathbb{E}\left[e^{\widetilde{\theta}\sum_{k=1}^{N_t} J_k}\right] \\
&= \mathbb{E}\left[\mathbb{E}[e^{\widetilde{\theta}\sum_{k=1}^{N_t} J_k}|N_t]\right] \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}[e^{\widetilde{\theta}\sum_{k=1}^{N_t} J_k}|N_t]P(N_t = n) \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}[\prod_{k=1}^{n} e^{\widetilde{\theta}J_k}]e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n\in\mathbb{N}_0} \left(\prod_{k=1}^{n} \mathbb{E}[e^{\widetilde{\theta}J_k}]\right) e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n\in\mathbb{N}_0} \left(\mathbb{E}[e^{\widetilde{\theta}J_1}]\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\sum_{n\in\mathbb{N}_0} \frac{(\mathbb{E}[e^{\widetilde{\theta}J_1}]\lambda t)^n}{n!} \\
&= e^{-\lambda t}e^{\mathbb{E}[e^{\widetilde{\theta}J_1}]\lambda t} \\
&= \exp\Big(\lambda t(\mathbb{E}[e^{\widetilde{\theta}J_1}] - 1)\Big)
\end{aligned}$$

Hence we must require $\mathbb{E}[e^{\widetilde{\theta}J_1}] < \infty$

2.2 Understanding $\varphi(\widetilde{\theta})$:

We recall that we defined:

$$\varphi(\widetilde{\theta}) = \ln \mathbb{E}[e^{\widetilde{\theta}I(1)}]$$

From above, we just found out that:

$$\mathbb{E}[e^{\widetilde{\theta}I(t)}] = \exp\Big(\lambda t(\mathbb{E}[e^{\widetilde{\theta}J_1}] - 1)\Big)$$

$$\Downarrow$$

$$\mathbb{E}[e^{\widetilde{\theta}I(1)}] = \exp\Big(\lambda(\mathbb{E}[e^{\widetilde{\theta}J_1}] - 1)\Big)$$

$$\Downarrow$$

$$\varphi(\widetilde{\theta}) = \ln \mathbb{E}[e^{\widetilde{\theta}I(1)}] = \Big(\lambda(\mathbb{E}[e^{\widetilde{\theta}J_1}] - 1)\Big)$$

Now:

$$\varphi(\widetilde{\theta})t = \Big(\lambda t(\mathbb{E}[e^{\widetilde{\theta}J_1}] - 1)\Big) = \ln \mathbb{E}[e^{\widetilde{\theta}I(t)}]$$

$$\Downarrow$$

$$e^{\varphi(\widetilde{\theta})t} = \mathbb{E}[e^{\widetilde{\theta}I(t)}]$$

Proposition 3 *$\widetilde{Z}(t) = \exp(\widetilde{\theta} I(t) - \varphi(\widetilde{\theta})t)$ is a $(P, \mathcal{F})$-martingale*

Proof 2 *As I is a Levy-process, it has independent and stationary increments: $I_t = I_s + (I_t - I_s)$,*

$$\begin{aligned}
\mathbb{E}[\widetilde{Z}(t)|\mathcal{F}_s] &= \mathbb{E}[e^{\widetilde{\theta}I(t)-\varphi(\widetilde{\theta})t}|\mathcal{F}_s] \\
&= e^{-\varphi(\widetilde{\theta})t}\mathbb{E}[e^{\widetilde{\theta}(I_s+(I_t-I_s))}|\mathcal{F}_s] \\
&= e^{-\varphi(\widetilde{\theta})t}e^{\widetilde{\theta}I(s)}\mathbb{E}[e^{\widetilde{\theta}(I(t)-I(s))}] \\
&= e^{-\varphi(\widetilde{\theta})t}e^{\widetilde{\theta}I(s)}\mathbb{E}[e^{\widetilde{\theta}(I(t-s))}] \\
&= e^{-\varphi(\widetilde{\theta})t}e^{\widetilde{\theta}I(s)}e^{\varphi(\widetilde{\theta})(t-s)} \\
&= e^{\widetilde{\theta}I(s)-\varphi(\widetilde{\theta})s} = \widetilde{Z}(s)
\end{aligned}$$

We define $Q^{\widetilde{\theta}}$ such that for $A \in \mathcal{F}_t$, we get:

$$Q^{\widetilde{\theta}}(A) = \mathbb{E}[\widetilde{Z}(t)\mathbf{1}_A]$$
$$\left.\frac{dQ^{\widetilde{\theta}}}{dP}\right|_{\mathcal{F}_t} = \widetilde{Z}(t), \quad Q^{\widetilde{\theta}} \sim P$$

Proposition 4 *$I(t)$ is a CPP under $Q^{\widetilde{\theta}}$, meaning that the Esscer transform is structure preserving.*

Proposition 5 (Characteristic function of I under $\widetilde{Q}$)

$$\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{ixI(t)}\right] = \exp\left([\varphi(\tilde{\theta}+ix) - \varphi(\tilde{\theta})]t\right)$$

Proof 3 *This relies all upon Bayes theorem:*

$$\begin{aligned}
\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{ixI(t)}\right] &= \frac{\mathbb{E}\left[e^{ixI(t)}\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}\right]}{\underbrace{\mathbb{E}\left[\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}\right]}_{P-mart}} = \mathbb{E}\left[e^{ixI(t)}\widetilde{Z}(t)\right] \\
&= \mathbb{E}\left[e^{ixI(t)}e^{(\tilde{\theta}I(t)-\varphi(\tilde{\theta})t)}\right] \\
&= e^{-\varphi(\tilde{\theta})t}\mathbb{E}\left[e^{ixI(t)+\tilde{\theta}I(t)}\right] \\
&= e^{-\varphi(\tilde{\theta})t}\underbrace{\mathbb{E}\left[e^{(\tilde{\theta}+ix)I(t)}\right]}_{:=e^{\varphi(\tilde{\theta}+ix)t}} \\
&= \exp\left([\varphi(\tilde{\theta}+ix) - \varphi(\tilde{\theta})]t\right)
\end{aligned}$$

2.3 Jump intensity and distribution under $Q^{\widetilde{\theta}}$:

$$\begin{aligned}
\varphi(\widetilde{\theta}+ix)-\varphi(\widetilde{\theta}) &= \ln \mathbb{E}[e^{\widetilde{\theta}+ix}I(1)] - \ln \mathbb{E}[e^{\widetilde{\theta}I(1)}] \\
&= \lambda(\mathbb{E}[e^{(\widetilde{\theta}+ix)J}]-1) - \lambda(\mathbb{E}[e^{\widetilde{\theta}J}]-1) \\
&= \lambda\mathbb{E}[e^{(\widetilde{\theta}+ix)J}] - \lambda\mathbb{E}[e^{\widetilde{\theta}J}] \\
&= \lambda\mathbb{E}[e^{\widetilde{\theta}J}e^{ixJ} - e^{\widetilde{\theta}J}] \\
&= \lambda\mathbb{E}[e^{\widetilde{\theta}J}(e^{ixJ}-1)] \\
&= \lambda\int_{\mathbb{R}}(e^{ixy}-1)e^{\widetilde{\theta}y}P_J(dy)
\end{aligned}$$

Before we move on, we recall the characteristic function of a CPP with intensity λ, and law $P_J(dy)$

$$\mathbb{E}[e^{iuI(t)}] = \exp\left(\lambda t\int_R (e^{iuy}-1)P_J(dy)\right)$$

We continue:

$$\begin{aligned}
\lambda\int_{\mathbb{R}}(e^{ixy}-1)e^{\widetilde{\theta}y}P_J(dy) &= \lambda\int_{\mathbb{R}}e^{\widetilde{\theta}y}P_J(dy)\int_{\mathbb{R}}(e^{ixy}-1)\frac{e^{\widetilde{\theta}y}P_J(dy)}{\int_{\mathbb{R}}e^{\widetilde{\theta}y}P_J(dy)} \\
&= \lambda\mathbb{E}[e^{\widetilde{\theta}J}]\int_{\mathbb{R}}(e^{ixy}-1)P_J^{\widetilde{\theta}}(dy)
\end{aligned}$$

Where we have defined:

$$P_J^{\widetilde{\theta}}(A) = \int_A \frac{e^{\widetilde{\theta}y}}{\mathbb{E}[e^{\widetilde{\theta}J}]}P_J(dy)$$

In conclusion: $I(t)$ is a CPP undet $Q^{\widetilde{\theta}}$ with intensity $\lambda_{\widetilde{\theta}} = \lambda\mathbb{E}[e^{\widetilde{\theta}J}]$ and jump distribution:

$$P_J^{\widetilde{\theta}}(dy) = \frac{e^{\widetilde{\theta}y}P_j(dy)}{\mathbb{E}[e^{\widetilde{\theta}J}]}$$

2.4 Tricks on calculating integrals of CPP's

The first one we will look at is:

$$\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} dI(s)\right]$$

Before we start doing things, we recall somethings:

Proposition 6 (Characteristic function)

$$\mathbb{E}\left[e^{i\theta \int_0^t g(s)dI(s)}\right] = \exp\left(\int_0^t \Psi(\theta g(s))ds\right)$$

where:

$$\begin{aligned}\Psi(x) &= \lambda \int_{\mathbb{R}} (e^{ixy} - 1) P_J(dy) \\ &= \lambda \left(\mathbb{E}[e^{ixJ}] - 1\right)\end{aligned}$$

Proposition 7 (Relation between characteristic function and expectation)

$$\mathbb{E}[X^k] = i^{-k} \phi_X^{(k)}(\theta)\Big|_{\theta=0}$$

Where:

$$\phi_X^{(k)}(\theta) = \frac{d}{d\theta^k}\mathbb{E}[e^{i\theta X}]$$

We will now use this proposition, in combination with the relationship

$$\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} dI(s)\right] = i^{-1}\frac{d}{d\theta}\left[\mathbb{E}[e^{i\theta \int_t^T e^{-\beta(T-s)} dI(s)}]\right]\Bigg|_{\theta=0}$$

Furthermore:

$$\begin{aligned}\mathbb{E}\left[e^{i\theta \int_t^T e^{-\beta(T-s)} dI(s)}\right] &= \exp\left(\int_t^T \Psi(\theta e^{-\beta(T-s)})ds\right) \\ &= \exp\left(\int_0^{T-t} \Psi(\theta e^{-\beta s})ds\right)\end{aligned}$$

We are now ready to take the derivative:

$$\begin{aligned}\frac{d}{d\theta}\exp\left(\int_0^{T-t} \Psi(\theta e^{-\beta s})ds\right) &= \exp\left(\int_0^{T-t} \Psi(\theta e^{-\beta s})ds\right)\int_0^T \frac{d}{d\theta}\Psi(\theta e^{-\beta s})ds \\ &= \exp\left(\int_0^{T-t} \Psi(\theta e^{-\beta s})ds\right)\int_0^T \Psi'(\theta e^{-\beta s})e^{-\beta s}ds\end{aligned}$$

Here we have:

$$\begin{aligned}\Psi'(x) &= \frac{d}{dx}\lambda\left(\mathbb{E}[e^{ixJ}]-1\right) = \lambda\frac{d}{dx}\mathbb{E}[e^{ixJ}]\\ &= \lambda\mathbb{E}\left[\frac{d}{dx}e^{ixJ}\right]\\ &= \lambda\mathbb{E}\left[e^{ixJ}iJ\right]\\ &= \lambda i\mathbb{E}[e^{ixJ}J]\end{aligned}$$

Evaluating the above at $\theta = 0$, gives us:

$$\frac{d}{d\theta}\exp\left(\int_0^{T-t}\Psi(\theta e^{-\beta s})ds\right)\Big|_{\theta=0} = \exp\left(\int_0^{T-t}\Psi(0)ds\right)\int_0^{T-t}\Psi'(0)e^{-\beta s}ds$$

We have $\Psi(0) = 0$ and $\Psi'(0) = \lambda i\mathbb{E}[J]$, leaving us with:

$$\begin{aligned}\frac{d}{d\theta}\exp\left(\int_0^{T-t}\Psi(\theta e^{-\beta s})ds\right)\Big|_{\theta=0} &= \int_0^{T-t}\lambda i\mathbb{E}[J]e^{-\beta s}ds\\ &= \frac{\lambda}{\beta}i\mathbb{E}[J](1-e^{-\beta(T-t)})\end{aligned}$$

And finally we can compute:

$$\begin{aligned}\mathbb{E}\left[\int_t^T e^{-\beta(T-s)}dI(s)\right] &= i^{-1}\frac{d}{d\theta}\left[\mathbb{E}[e^{i\theta\int_t^T e^{-\beta(T-s)}dI(s)}]\right]\Big|_{\theta=0}\\ &= \frac{\lambda}{\beta}\mathbb{E}[J](1-e^{-\beta(T-t)})\end{aligned}$$

2.5 Trick number 2: compensated CPP

Let's say we still want to compute:

$$\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} dI(s)\right]$$

We can also use martingale-theory to compute this quantity, namely by looking at the compensated CPP, meaning that we look at:

$$\tilde{I}(t) := I(t) - \mathbb{E}[I(t)]$$

We start off by computing: $\mathbb{E}[I(t)]$

$$\mathbb{E}[I(t)] = i^{-1} \frac{d}{dx} \mathbb{E}\left[e^{ixI(t)}\right]\Big|_{x=0}$$

We use the same procedure as earlier:

$$\begin{aligned}
\frac{d}{dx}\mathbb{E}\left[e^{ixI(t)}\right] &= \frac{d}{dx}\exp\left(\Psi(x)t\right) \\
&= \exp\left(\Psi(x)t\right)\Psi'(x)t \\
&= \exp\left(\Psi(x)t\right)\lambda i \mathbb{E}[e^{ixJ}J]t \\
&\Downarrow \\
\frac{d}{dx}\mathbb{E}\left[e^{ixI(t)}\right]\Big|_{x=0} &= \lambda i \mathbb{E}[J]t
\end{aligned}$$

This leaves us with:

$$\mathbb{E}[I(t)] = \lambda \mathbb{E}[J]t$$

We have that $\tilde{I}$ is a Levy-process and a martingale (it can be shown).

We write down the differential form of $\tilde{I}$:

$$\begin{aligned}
d\tilde{I}(t) &= dI(t) - d(\mathbb{E}[I(t)]) \\
&= dI(t) - d(\lambda \mathbb{E}[J]t) \\
&= dI(t) - \lambda \mathbb{E}[J]dt
\end{aligned}$$

We now get:

$$\begin{aligned}
\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} dI(s)\right] &= \mathbb{E}\left[\int_t^T e^{-\beta(T-s)} (d\tilde{I}(s) + \lambda\mathbb{E}[J]ds)\right] \\
&= \mathbb{E}\left[\int_t^T e^{-\beta(T-s)} d\tilde{I}(s)\right] + \lambda\mathbb{E}[J]\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} ds\right] \\
&= 0 + \lambda\mathbb{E}[J]\mathbb{E}\left[\int_t^T e^{-\beta(T-s)} ds\right] \\
&= \lambda\mathbb{E}[J]\left(1 - e^{-\beta(T-t)}\right)
\end{aligned}$$

Why is the integral w.r.t the martingale zero?

The intuition comes when we look at simple functions $g_n \to g$:

$$\int_t^T g_n(s) d\tilde{I}(s) = \sum_i g_n(s_i)[\tilde{I}(s_{i+1}) - \tilde{I}(s_i)]$$

$$\Downarrow$$

$$\mathbb{E}\left[\int_t^T g_n(s) d\tilde{I}(s)\right] = \sum_i g_n(s_i)\mathbb{E}[\tilde{I}(s_{i+1}) - \tilde{I}(s_i)] = 0$$

2.6 Trick number 3: using characteristic function of I under Q

From earlier we have:

$$
\begin{aligned}
\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{ixI(t)}\right] &= \exp\left([\varphi(\widetilde{\theta}+ix)-\varphi(\widetilde{\theta})]t\right) \\
&= \exp\left(\int_0^t [\varphi(\widetilde{\theta}+ix)-\varphi(\widetilde{\theta})]ds\right)
\end{aligned}
$$

The key observation here is the following:

$$
\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{ixaI(t)}\right] = \mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{ix\int_0^t a dI(s)}\right] \qquad = \exp\left(\int_0^t [\varphi(\widetilde{\theta}+ixa)-\varphi(\widetilde{\theta})]ds\right)
$$

Let's say we want to calculate:

$$
\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right] = \mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{i(-i)\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]
$$

So here $x=-i$ and $a=\eta e^{-\beta(\tau-s)}$, giving us:

$$
\begin{aligned}
\mathbb{E}_{Q^{\tilde{\theta}}}\left[e^{i(-i)\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right] &= \exp\left(\int_t^\tau [\varphi\left(\widetilde{\theta}+i*(-i)*(\eta e^{-\beta(\tau-s)})\right)-\varphi(\widetilde{\theta})]ds\right) \\
&= \exp\left(\int_t^\tau [\varphi(\widetilde{\theta}+\eta e^{-\beta(\tau-s)})-\varphi(\widetilde{\theta})]ds\right) \\
&= \exp\left(\int_0^{\tau-t} [\varphi(\widetilde{\theta}+\eta e^{-\beta s})-\varphi(\widetilde{\theta})]ds\right)
\end{aligned}
$$

Now let's say we want to calculate the following:

$$
\mathbb{E}_Q\left[\int_t^T e^{-\alpha(T-s)}dI(s)\right]
$$

$$
\begin{aligned}
\mathbb{E}_Q[Y^k] &= i^{-k}\phi_{Q,Y}^{(k)}(\theta)\Big|_{\theta=0} \\
&= i^{-k}\frac{d}{d\theta^k}\mathbb{E}_Q[e^{i\theta Y}]\Big|_{\theta=0}
\end{aligned}
$$

3 Pricing of forwards: Geometric case

Let us assume the geometric spot price model:

$$S(t) = \exp(\Lambda(t) + X(t) + Y(t))$$

furthermore we also assume that we have a pricing measure $Q \sim P$ defined by:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \hat{Z}(t)\tilde{Z}(t)$$

Where:

$$\hat{Z}(t) = \exp\left(\hat{\theta}B(t) - \frac{1}{2}\hat{\theta}^2 t\right)$$
$$\tilde{Z}(t) = \exp\left(\widetilde{\theta}I(t) - \varphi(\widetilde{\theta})t\right)$$

$(\widetilde{\theta}, \hat{\theta}) \in \mathbb{R}^2$ are parameters so that $\varphi(\widetilde{\theta})$ is well defined, namely $\mathbb{E}[e^{\widetilde{\theta}J}] < \infty$

Goal: find:

$$f(t, \tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t]$$

We start with how X looks like under P:

$$X(\tau) = e^{-\alpha(\tau-t)}X(t) + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(\tau-t)}\right) + \sigma\int_t^\tau e^{-\alpha(\tau-s)}dB(s)$$

Now from Girsanov's thm we have: $d\widehat{B}(t) = dB(t) - \widehat{\theta}dt$ $\widehat{B}$ is a $Q^{\widehat{\theta}}$-BM, giving us:

$$\widehat{X}(\tau) = e^{-\alpha(\tau-t)}X(t) + \frac{\mu + \sigma\widehat{\theta}}{\alpha}\left(1 - e^{-\alpha(\tau-t)}\right) + \sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)$$

We also recall how Y looks like under P:

$$Y(\tau) = e^{-\beta(\tau-t)}Y(t) + \frac{\delta}{\beta}\left(1 - e^{-\beta(\tau-t)}\right) + \eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)$$

Under P we have that I is a CPP with intensity λ and jump-dist $P_J(dy)$, under $Q^{\widetilde{\theta}}$ we have that I is a CPP with intensity $\lambda_{\widetilde{\theta}} = \lambda\mathbb{E}[e^{\widetilde{\theta}J}]$ and jump-distribution:

$$P_J^{\widetilde{\theta}}(dy) = \frac{e^{\widetilde{\theta}y}}{\mathbb{E}[e^{\widetilde{\theta}J}]}P_J(dy)$$

$$\begin{aligned}\mathbb{E}_Q\left[e^{X(\tau)+Y(\tau)}\middle|\mathcal{F}_t\right] &= \exp\left(e^{-\alpha(\tau-t)}X(t)+e^{-\beta(\tau-t)}Y(t)\right)\\ &\quad\times\exp\left(\frac{\mu+\sigma\widehat{\theta}}{\alpha}(1-e^{-\alpha(\tau-t)})+\frac{\delta}{\beta}(1-e^{-\beta(\tau-t)}\right)\\ &\quad\times\mathbb{E}_Q\left[e^{\sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)}e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]\end{aligned}$$

Where $\widehat{B}$ and I are independent, giving us:

$$\mathbb{E}_Q\left[e^{\sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)}e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]=\mathbb{E}_{\widehat{Q}}\left[e^{\sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)}\right]\mathbb{E}_{\widetilde{Q}}\left[e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]$$

We will now focus on how to compute:

$$\mathbb{E}_{\widetilde{Q}}\left[e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]$$

Actually from earlier we have:

$$\begin{aligned}\mathbb{E}_{\widetilde{Q}}\left[e^{i\widetilde{\theta}\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right] &= \exp\left(\int_t^\tau\Psi_{\widetilde{Q}}(\widetilde{\theta}\eta e^{-\beta(\tau-s)})ds\right)\\ &= \exp\left(\int_0^{\tau-t}\Psi_{\widetilde{Q}}(\widetilde{\theta}\eta e^{-\beta s})ds\right)\end{aligned}$$

Hence the one we are looking for is when $\widetilde{\theta}=-i$, giving us:

$$\mathbb{E}_{\widetilde{Q}}\left[e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right]=\exp\left(\int_0^{\tau-t}\Psi_{\widetilde{Q}}(-i\eta e^{-\beta s})ds\right)$$

Now for the other expectation, we have:

$$\sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)\sim\mathcal{N}\left(0,\frac{\sigma^2}{2\alpha}\left[1-e^{-2\alpha(\tau-t)}\right]\right)$$

This yields:

$$\mathbb{E}_{\widehat{Q}}\left[e^{\sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)}\right]=\exp\left(\frac{\sigma^2}{4\alpha}\left[1-e^{-2\alpha(\tau-t)}\right]\right)$$

We can then summarize our findings:

$$\begin{aligned} f(t,\tau) &= \mathbb{E}_Q[S(\tau)|\mathcal{F}_t] \\ &= \exp\left(\Lambda(\tau) + e^{-\alpha(\tau-t)}X(t) + e^{-\beta(\tau-t)}Y(t) + \Theta(\tau-t)\right) \end{aligned}$$

Where:

$$\Theta(x) = \frac{\mu+\sigma\widehat{\theta}}{\alpha}(1-e^{-\alpha x}) + \frac{\delta}{\beta}(1-e^{-\beta x}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha x}) + \int_0^x \Psi_{\widetilde{Q}}(-i\eta e^{-\beta s})ds$$

From Fred's lecture notes he has:

$$\Theta(x) = \frac{\mu+\sigma\widehat{\theta}}{\alpha}(1-e^{-\alpha x}) + \frac{\delta}{\beta}(1-e^{-\beta x}) + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha x}) + \int_0^x \left[\varphi(\widetilde{\theta}+\eta e^{-\beta s}) - \varphi(\widetilde{\theta})\right] ds$$

The reason for that is that he uses the trick I learned about later on in the semester:

$$\mathbb{E}_Q\left[e^{ix\int_0^t g(s)dI(s)}\right] = \exp\left(\int_0^t [\varphi(\theta+ixg(s)) - \varphi(\theta)]ds\right)$$

Translating this to our setting:

$$\begin{aligned} \mathbb{E}_Q\left[e^{\eta\int_t^\tau e^{-\beta(\tau-s)}dI(s)}\right] &= \mathbb{E}_Q\left[e^{i*(-i)\int_t^\tau \eta e^{-\beta(\tau-s)}dI(s)}\right] \\ &= \exp\left(\int_t^\tau [\varphi(\theta+i*(-i)*\eta e^{-\beta(\tau-s)}) - \varphi(\theta)]ds\right) \\ &= \exp\left(\int_0^{\tau-t} [\varphi(\theta+\eta e^{-\beta s}) - \varphi(\theta)]ds\right) \end{aligned}$$

3.1 Analysis of risk premium

The risk premium is defined as:

$$\text{Risk premium} := f(t,\tau) - \mathbb{E}[S(\tau)|\mathcal{F}_t]$$

We note here that we use the ordinary expectation, this corresponds to $(\widehat{\theta},\widetilde{\theta}) = (0,0)$. The goal here is to study the sign of this risk premium, we can to this by analyzing:

$$\ln(f(t,\tau)) - \ln(\mathbb{E}[S(\tau)|\mathcal{F}_t])$$

We can look at the logarithm as it's an increasing function, kinda the same approach as one uses in likelihood theory.

$$\begin{aligned}
&\ln(f(t,\tau)) - \ln(\mathbb{E}[S(\tau)|\mathcal{F}_t]) = \\
&\Lambda(\tau) + e^{-\alpha(\tau-t)}X(t) + e^{-\beta(\tau-t)}Y(t) + \Theta(\tau-t) - \Lambda(\tau) + e^{-\alpha(\tau-t)}X(t) + e^{-\beta(\tau-t)}Y(t) + \Theta(\tau-t)_{(\widehat{\theta},\widetilde{\theta})=(0,0)} \\
&= \Theta(\tau-t) - \Theta(\tau-t)_{(\widehat{\theta},\widetilde{\theta})=(0,0)} \\
&= \frac{\mu+\sigma\widehat{\theta}}{\alpha}(1-e^{-\alpha(\tau-t)}) + \int_0^{\tau-t}\left[\varphi(\widetilde{\theta}+\eta e^{-\beta s}) - \varphi(\widetilde{\theta})\right]ds - \frac{\mu}{\alpha}(1-e^{-\alpha(\tau-t)}) - \int_0^{\tau-t}\left[\varphi(\eta e^{-\beta s})\right]ds \\
&= \frac{\sigma\widehat{\theta}}{\alpha}[1-e^{-\alpha(\tau-t)}] + \int_0^{\tau-t}[\varphi(\widetilde{\theta}+\eta e^{-\beta s}) - \varphi(\widetilde{\theta}) - \varphi(\eta e^{-\beta s})]ds
\end{aligned}$$

Summary:

$$\ln(f(t,\tau)) - \ln(\mathbb{E}[S(\tau)|\mathcal{F}_t]) = \underbrace{\frac{\sigma\widehat{\theta}}{\alpha}[1-e^{-\alpha(\tau-t)}]}_{(a)} + \int_0^{\tau-t}[\underbrace{\varphi(\widetilde{\theta}+\eta e^{-\beta s}) - \varphi(\widetilde{\theta}) - \varphi(\eta e^{-\beta s})}_{(b)}]ds$$

We start with (a), if $\widehat{\theta} > 0$, then (a) is positive, if $\widehat{\theta} < 0$, then (a) is negative. Let's do some algebra on (b):

$$\begin{aligned}
\varphi(\widetilde{\theta}+\eta e^{-\beta s}) - \varphi(\widetilde{\theta}) - \varphi(\eta e^{-\beta s}) &= \lambda(\mathbb{E}[e^{(\widetilde{\theta}+\eta e^{-\beta s})J} - 1) \\
&- \lambda(\mathbb{E}[e^{\widetilde{\theta}}] - 1) \\
&- \lambda(\mathbb{E}[e^{\eta e^{-\beta s}J} - 1) \\
&= \lambda + \lambda(\underbrace{\mathbb{E}[e^{\widetilde{\theta}J}e^{\eta e^{-\beta s}J} - e^{\widetilde{\theta}J} - e^{\eta e^{-\beta s}J}]}_{\xi(\widetilde{\theta})}
\end{aligned}$$

$$
\begin{aligned}
\xi(0) &= \lambda\mathbb{E}[e^{\eta e^{-\beta s}J} - e^{\eta e^{-\beta s}J}] = 0 \\
\xi'(\widetilde{\theta}) &= \lambda\mathbb{E}\left[\frac{d}{d\theta}e^{(\widetilde{\theta}+\eta e^{-\beta s})J} - Je^{\widetilde{\theta}J}\right] \\
&= \lambda\mathbb{E}\left[Je^{(\widetilde{\theta}+\eta e^{-\beta s})J} - Je^{\widetilde{\theta}J}\right] \\
&= \lambda(\mathbb{E}[Je^{\widetilde{\theta}J}(e^{\eta e^{-\beta s}J} - 1)])
\end{aligned}
$$

We assume $J > 0$ and $\eta > 0$, this gives us that $\xi'(\theta) > 0$, meaning that $\widetilde{\theta} \mapsto \xi(\widetilde{\theta})$ is increasing. Furthermore $\xi(0) = 0$, this makes the analysis of sign easier for (b). As then for $\widetilde{\theta} > 0$ we get (b) positive, and for $\widetilde{\theta} < 0$ we get (b) negative.

- $\widehat{\theta} > 0$ and $\widetilde{\theta} > 0$ yields positive risk premium.
- $\widehat{\theta} < 0$ and $\widetilde{\theta} < 0$ yields negative risk premium.
- mixed signs means that we must look into the size of parameters. i.e $\widehat{\theta} > 0$ and $\widetilde{\theta} < 0$ for instance.

4 Pricing of forwards Arithmetic case

The arithmetic model is given by:

$$S(t) = \Lambda(t) + X(t) + Y(t)$$

We will now model directly under $Q^{\widehat{\theta}}$ and $Q^{\widetilde{\theta}}$:

$$\begin{aligned}
dX(t) &= [(\mu + \sigma\widehat{\theta}) - \alpha X(t)]dt + \sigma d\widehat{B}(t) && (Q^{\widehat{\theta}}) \\
dY(t) &= [\delta - \beta Y(t)]dt + \eta dI^{\widetilde{\theta}}(t) && (Q^{\widetilde{\theta}})
\end{aligned}$$

Where $I^{\widetilde{\theta}}$ is a CPP with intensity $\lambda_{\widetilde{\theta}} = \lambda\mathbb{E}[e^{\widetilde{\theta}J}]$ and jump distribution $P_J^{\widetilde{\theta}}$

$$\widehat{X}(\tau) = e^{-\alpha(\tau-t)}X(t) + \frac{\mu + \sigma\widehat{\theta}}{\alpha}\left(1 - e^{-\alpha(\tau-t)}\right) + \sigma\int_t^\tau e^{-\alpha(\tau-s)}d\widehat{B}(s)$$

$$\widetilde{Y}(\tau) = e^{-\beta(\tau-t)}Y(t) + \frac{\delta}{\beta}\left(1 - e^{-\beta(\tau-t)}\right) + \eta\int_t^\tau e^{-\beta(\tau-s)}dI^{\widetilde{\theta}}(s)$$

We need to calculate: $\mathbb{E}_{\widehat{Q}}[\widehat{X}(\tau)|\mathcal{F}_t]$ and $\mathbb{E}_{\widetilde{Q}}[\widetilde{Y}(\tau)|\mathcal{F}_t]$, the first one is straight forward, as the Ito-integral here has expectation zero:

$$\mathbb{E}_{\widehat{Q}}[\widehat{X}(\tau)|\mathcal{F}_t] = e^{-\alpha(\tau-t)}X(t) + \frac{\mu + \sigma\widehat{\theta}}{\alpha}\left(1 - e^{-\alpha(\tau-t)}\right)$$

The other one requires some more work, as we need to calculate:

$$\mathbb{E}_{\widetilde{Q}}[\eta\int_t^\tau e^{-\beta(\tau-s)}dI^{\widetilde{\theta}}(s)]$$

Here we will use trick number two, i.e where we smuggle in the compensated CPP under $\widetilde{Q}$, which is a martingale, i.e:

$$I^{\widetilde{\theta}}(s) - \mathbb{E}_{\widetilde{Q}}[I^{\widetilde{\theta}}(s)] := M^{\widetilde{\theta}}(s)$$

Meaning that we need

$$\mathbb{E}_{\widetilde{Q}}[I^{\widetilde{\theta}}(s)]$$

$$\mathbb{E}_{\widetilde{Q}}[I^{\widetilde{\theta}}(s)] = i^{-1}\frac{d}{dx}\mathbb{E}_{\widetilde{Q}}\left[e^{ixI^{\widetilde{\theta}}(s)}\right]\Big|_{x=0}$$

$$\begin{aligned}
\mathbb{E}_{\widetilde{Q}}\left[e^{ixI^{\widetilde{\theta}}(s)}\right] &= \exp\left(\Psi_{\widetilde{Q}}(x)s\right) \\
&\Downarrow \\
\frac{d}{dx}\mathbb{E}_{\widetilde{Q}}\left[e^{ixI^{\widetilde{\theta}}(s)}\right] &= \frac{d}{dx}\exp\left(\Psi_{\widetilde{Q}}(x)s\right) = \exp\left(\Psi_{\widetilde{Q}}(x)s\right)\Psi'_{\widetilde{Q}}(x)s
\end{aligned}$$

Here:

$$\begin{aligned}
\Psi_{\widetilde{Q}}(x) &= \lambda_{\widetilde{\theta}}\int_{\mathbb{R}}(e^{ixy}-1)P_J^{\widetilde{\theta}}(dy) \\
&\Downarrow \\
\Psi'_{\widetilde{Q}}(x) &= \lambda_{\widetilde{\theta}}\int_{\mathbb{R}} e^{ixy} iy P_J^{\widetilde{\theta}}(dy) = \lambda_{\widetilde{\theta}} i\int_{\mathbb{R}} e^{ixy} y P_J^{\widetilde{\theta}}(dy)
\end{aligned}$$

This yields:

$$\begin{aligned}
\frac{d}{dx}\mathbb{E}_{\widetilde{Q}}\left[e^{ixI^{\widetilde{\theta}}(s)}\right]\Big|_{x=0} &= \lambda_{\widetilde{\theta}} i\mathbb{E}_{\widetilde{Q}}[J]s \\
&= \lambda\mathbb{E}[e^{\widetilde{\theta}J}]\mathbb{E}_{\widetilde{Q}}[J]s
\end{aligned}$$

Meing that:

$$\begin{aligned}
M^{\widetilde{\theta}}(s) &= I^{\widetilde{\theta}}(s) - \lambda\mathbb{E}[e^{\widetilde{\theta}J}]\mathbb{E}_{\widetilde{Q}}[J]s \\
&\Downarrow \\
dM^{\widetilde{\theta}}(s) &= dI^{\widetilde{\theta}}(s) - \lambda\mathbb{E}[e^{\widetilde{\theta}J}]\mathbb{E}_{\widetilde{Q}}[J]ds
\end{aligned}$$

Which then again gives us:

$$\begin{aligned}
\mathbb{E}_{\widetilde{Q}}[\eta\int_t^\tau e^{-\beta(\tau-s)}dI^{\widetilde{\theta}}(s)] &= \mathbb{E}_{\widetilde{Q}}[\eta\int_t^\tau e^{-\beta(\tau-s)}dM^{\widetilde{\theta}}(s)] + \eta\lambda\mathbb{E}[e^{\widetilde{\theta}J}]\mathbb{E}_{\widetilde{Q}}[J]\int_t^\tau e^{-\beta(\tau-s)}ds \\
&= \frac{\eta\lambda_{\widetilde{\theta}}\mathbb{E}_{\widetilde{Q}}[J]}{\beta}\left(1-e^{-\beta(\tau-t)}\right)
\end{aligned}$$

Which the finally gives:

$$\mathbb{E}_{\widetilde{Q}}[\widetilde{Y}(\tau)|\mathcal{F}_t] = e^{-\beta(\tau-t)}Y(t) + \frac{\delta}{\beta}\left(1-e^{-\beta(\tau-t)}\right) + \frac{\eta\lambda_{\widetilde{\theta}}\mathbb{E}_{\widetilde{Q}}[J]}{\beta}\left(1-e^{-\beta(\tau-t)}\right)$$

Also the actual gangster we were going to compute:

$$
\begin{aligned}
f(t,\tau) &= \mathbb{E}_Q[S(\tau)|\mathcal{F}_t] \\
&= \Lambda(\tau) + \frac{\mu + \sigma\widehat{\theta}}{\alpha}\left(1 - e^{-\alpha(\tau-t)}\right) + \frac{\eta\lambda_{\widetilde{\theta}}\mathbb{E}_{\widetilde{Q}}[J]}{\beta}\left(1 - e^{-\beta(\tau-t)}\right) \\
&+ e^{-\alpha(\tau-t)}X(t) + e^{-\beta(\tau-t)}Y(t)
\end{aligned}
$$

5 HJM-modelling

It turns out that one smart way to model forwards, is to model them directly under Q, meaning that we state $f(t,\tau)$ directly:

$$f(t,\tau) = f(0,\tau)\exp\left(\int_0^t a(u,\tau)du + \int_0^t \sigma(u,\tau)dW(u) + \int_0^\tau \eta(u,\tau)dI(u)\right)$$

We look into some technical conditions:

1.st set of conditions

$$\int_0^\tau |a(u,\tau)|du < \infty \;\; Q\text{-a.e}$$
$$\int_0^\tau |\sigma^2(u,\tau)|du < \infty \;\; Q\text{-a.e}$$
$$\sum_{k=1}^{N_t} |\eta(s_i,\tau)||J_k| < \infty \;\; \omega\text{-a.e}$$

2nd set of conditions

$$\begin{aligned}
\mathbb{E}_Q[f(t,\tau)] &< \infty \;\; t \le \tau \\
\mathbb{E}_Q[f(t,\tau)] &= f(0,\tau)\exp\left(\int_0^t a(u,\tau)du\right)\mathbb{E}_Q\left[\exp\left(\int_0^t \sigma(u,\tau)dW(u)\right)\right] \\
&\quad \times \mathbb{E}_Q\left[\exp\left(\int_0^t \eta(u,\tau)dI(u)\right)\right] \\
&\Downarrow
\end{aligned}$$

$$\mathbb{E}_Q\left[\exp\left(\int_0^t \sigma(u,\tau)dW(u)\right)\right] < \infty \;\; \mathbb{E}_Q\left[\exp\left(\int_0^t \eta(u,\tau)dI(u)\right)\right] < \infty$$

3rd set of conditions
We need f to be a $(Q,\mathcal{F})$-martingale:

$$\begin{aligned}
\mathbb{E}_Q[f(t,\tau)|\mathcal{F}_s] = f(0,\tau)&\exp\left(\int_0^t a(u,\tau)du\right)\exp\left(\int_0^s \sigma(u,\tau)dW(u)\right)\exp\left(\int_0^s \eta(u,\tau)dI(u)\right) \\
&\times \mathbb{E}_Q\left[\exp\left(\int_s^t \sigma(u,\tau)dW(u)\right)\exp\left(\int_s^t \eta(u,\tau)dI(u)\right)\right]
\end{aligned}$$

Now I and W are independent, meaning that we can split up the expectation into two:

$$\begin{aligned}
\mathbb{E}_Q\left[\exp\left(\int_s^t \sigma(u,\tau)dW(u)\right)\exp\left(\int_s^t \eta(u,\tau)dI(u)\right)\right] &= \mathbb{E}_Q\left[\exp\left(\int_s^t \sigma(u,\tau)dW(u)\right)\right] \\
&\quad \times \mathbb{E}_Q\left[\exp\left(\int_s^t \eta(u,\tau)dI(u)\right)\right]
\end{aligned}$$

Furthermore we have:

$$\mathbb{E}_Q\left[\exp\left(\int_s^t \sigma(u,\tau)dW(u)\right)\right] = \exp\left(\frac{1}{2}\int_s^t \sigma^2(u,\tau)du\right)$$
$$\mathbb{E}_Q\left[\exp\left(\int_s^t \eta(u,\tau)dI(u)\right)\right] = \exp\left(\int_s^t \varphi_Q(\eta(u,\tau))du\right)$$

Meaning that we have:

$$\mathbb{E}_Q[f(t,\tau)|\mathcal{F}_s] = f(s,\tau)\exp\left(\int_s^t a(u,\tau)du\right)\exp\left(\frac{1}{2}\int_s^t \sigma^2(u,\tau)du\right)\exp\left(\int_s^t \varphi_Q(\eta(u,\tau))du\right)$$

Meaning that we must have:

$$\int_s^t a(u,\tau)du = -\frac{1}{2}\int_s^t \sigma^2(u,\tau)du - \int_s^t \varphi_Q(\eta(u,\tau))du$$

We can take the derivative w.r.t t, and then we get:

$$a(t,\tau) = -\frac{1}{2}\sigma^2(t,\tau) - \varphi_Q(\eta(t,\tau))$$

6 Black-76 formula

We have a forward price given by GBM, with no jumps, and our goal is to price a European-call.

$$df(t,\tau) = f(t,\tau)\sigma(t,\tau)dW_t \ \ (Q)$$

Result 1 *The price of a European call written on a forward is:*

$$C(t,T,K,\tau) = e^{-r(T-t)}\mathbb{E}_Q\left[f(t,\tau)\Phi(d_1) - K\Phi(d_2)\right]$$

Where:

$$d_{1,2} = \frac{\ln\left(\frac{f(t,\tau)}{K}\right) \pm \frac{1}{2}\int_t^T \sigma^2(u,\tau)du}{\sqrt{\int_t^T \sigma^2(u,\tau)du}}$$

Proof 4 *We start off by applying Ito's formula on* $\ln f(t,\tau)$*, thus* $g(t,x) = \ln(x)$*, meaning that* $\partial_x g = 1/x$ *and* $\partial_{xx} g = -1/x^2$*,* $(df(t,\tau))^2 = f^2(t,\tau)\sigma^2(t,\tau)dt$*, leaving us with:*

$$\begin{aligned}
d\ln f(t,\tau) &= \frac{1}{f(t,\tau)}df(t,\tau) - \frac{1}{2}\frac{1}{f^2(t,\tau))}f^2(t,\tau)\sigma^2(t,\tau)dt \\
&= \sigma(t,\tau)dW_t - \frac{1}{2}\sigma^2(t,\tau)dt \\
&\Downarrow \\
\ln f(T,\tau) &= \ln f(t,\tau) + \int_t^T \sigma(u,\tau)dW_u - \frac{1}{2}\int_t^T \sigma^2(u,\tau)du
\end{aligned}$$

$\int_t^T \sigma(u,\tau)dW_u \sim \mathcal{N}\left(0, \int_t^T \sigma^2(u,\tau)du\right)$*, hence* $\int_t^T \sigma(u,\tau)dW_u \stackrel{d}{=} \sqrt{\int_t^T \sigma^2(u,\tau)du}Z$ *where* $Z \sim \mathcal{N}(0,1)$

Now working with the price of the call:

$$C(t,T,K,\tau) = e^{-r(T-t)}\mathbb{E}_Q\left[(f(T,\tau) - K)^+|\mathcal{F}_t\right]$$

We will simplify a bit:

$$(f(T,\tau) - K)^+ = \left(f(t,\tau)\exp\left(\sqrt{\int_t^T \sigma^2(u,\tau)du}Z - \frac{1}{2}\int_t^T \sigma^2(u,\tau)du\right) - K\right)\mathbb{1}_{\{f(T,\tau)>K\}}$$

We will now work with the set $\{f(T,\tau) > K\}$:

$$f(T,\tau) > K$$
$$\Updownarrow$$
$$\exp\left(\sqrt{\int_t^T \sigma^2(u,\tau)du}Z - \frac{1}{2}\int_t^T \sigma^2(u,\tau)du\right) > \frac{K}{f(t,\tau)}$$
$$Z > \frac{\ln\left(\frac{K}{f(t,\tau)}\right) + \frac{1}{2}\int_t^T \sigma^2(u,\tau)du}{\sqrt{\int_t^T \sigma^2(u,\tau)du}} := -d_2$$

We have $\mathcal{F}_t$-independent parts, furthermore we use the freezing lemma leading to:

$$\mathbb{E}_Q\left[(f(T,\tau)-K)^+\right] = \int_{-d_2}^{\infty}\left[x\exp\left(\sqrt{\int_t^T \sigma^2(u,\tau)du}z - \frac{1}{2}\int_t^T \sigma^2(u,\tau)du\right) - K\right] f_{\mathcal{N}(0,1)}(z)dz$$

Before we move on, we will expand the square properly:

$$\begin{aligned}
-\frac{z^2}{2} + \sqrt{\int_t^T \sigma^2(u,\tau)du}\, z - \frac{1}{2}\int_t^T \sigma^2(u,\tau)du &= -\frac{1}{2}\left(z^2 - 2z\sqrt{\int_t^T \sigma^2(u,\tau)du} + \int_t^T \sigma^2(u,\tau)du\right) \\
&= -\frac{1}{2}\left(z - \sqrt{\int_t^T \sigma^2(u,\tau)du}\right)^2
\end{aligned}$$

This leaves us with:

$$\mathbb{E}_Q\left[(f(T,\tau)-K)^+\right] = x\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z-\sqrt{\int_t^T \sigma^2(u,\tau)du}\right)^2}dz - K\int_{-d_2}^{\infty} f(z)dz$$

Let's work with the first integral, first let $u = z - \sqrt{\int_t^T \sigma^2(u,\tau)du}$, then $\frac{du}{dz} = 1$, and $z = -d_2$ leads to $u = -d_2 - \sqrt{\int_t^T \sigma^2(u,\tau)du} := -d'$

Furthermore we exploit the symmetry of the normal-distribution: $P(Z > -d) = P(Z \leq d)$

$$\begin{aligned}
\mathbb{E}_Q\left[(f(T,\tau)-K)^+\right] &= x\int_{-d'}^{\infty} f_Z(u)du - K\int_{-d_2}^{\infty} f_Z(z)dz \\
&= f(t,\tau)\Phi(d') - K\Phi(d_2)
\end{aligned}$$

Here we have:

$$d' := d_1 = \frac{\ln\left(\frac{f(t,\tau)}{K}\right) + \frac{1}{2}\int_t^T \sigma^2(u,\tau)du}{\sqrt{\int_t^T \sigma^2(u,\tau)du}}$$

We can then finally conclude on the price:

$$C(t, T, K, \tau) := e^{-r(T-t)} \left[f(t, \tau)\Phi(d_1) - K\Phi(d_2) \right]$$

Where:

$$d_{1,2} = \frac{\ln\left(\frac{f(t,\tau)}{K}\right) \pm \frac{1}{2}\int_t^T \sigma^2(u, \tau) du}{\sqrt{\int_t^T \sigma^2(u, \tau) du}}$$

7 Pricing of options when we have jumps

We recall from the HJM-framework the following:

$$f(t,\tau) = f(0,\tau)\exp\left(\int_0^t a(u,\tau)du + \int_0^t \sigma(u,\tau)dW_u + \int_0^t \eta(u,\tau)dI(u)\right)$$

σ, η are deterministic, W is a Q-BM and I is a Q-CPP. The martingale condition on a:

$$a(t,\tau) = -\frac{1}{2}\sigma^2(t,\tau) - \varphi_Q(\eta(t,\tau)$$

where:

$$\varphi_Q(z) = \lambda_Q(\mathbb{E}_Q[e^{zJ}] - 1)$$

We will simplify the notation a bit:

$$f(t,\tau) = h(t,\tau)\exp\left(\int_0^t \sigma(u,\tau)dW_u + \int_0^t \eta(u,\tau)dI(u)\right)$$
$$h(t,\tau) = f(0,\tau)\exp\left(\int_0^t a(u,\tau)du\right)$$

And form mathematical finance theory, we still have:

$$C(t,T,K,\tau) := e^{-r(T-t)}\mathbb{E}_Q\left[(f(T,\tau)-K)^+|\mathcal{F}_t\right]$$

When we are dealing with jumps, our approach goes out on taking a round with some Fourier analysis:

7.1 Fourier Analysis

Definition 3 (Fourier transform) *Let $g \in L^1(\mathbb{R})$, then the fourier transform $\widehat{g}$ of g is:*

$$\widehat{g}(y) := \int_{\mathbb{R}} g(x)e^{-ixy}dx$$

Definition 4 (Inverse Fourier transform) *If $\widehat{g} \in L^1(\mathbb{R})$, then the inverse Fourier transform is given by:*

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(y)e^{ixy}dy$$

Let X be a r.v, objects we will be interested in are:

$$g(X) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(y)e^{iXy}dy \text{ and } \mathbb{E}_Q[g(X)|\mathcal{F}_t] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(y)\mathbb{E}_Q[e^{iXy}|\mathcal{F}_t]dy$$

Let

$$X := \int_0^T \sigma(u,\tau)dW_u + \int_0^T \eta(u,\tau)dI(u)$$

and $g(x) := (h(T,\tau)e^x - K)^+$, an immediate problem here is that $g \notin L^1(\mathbb{R})$, as for $x > \ln(K/h(T,\tau))$ we get that it behaves as an exponential.

We therefor introduce a trick, namely define

$$g_\alpha(x) := e^{-\alpha x} g(x), \quad \alpha > 1$$

Here we see why $\alpha > 1$:

$$\begin{aligned} g_\alpha(x) &= e^{-\alpha x} g(x) \\ &= \left(h(T,\tau)e^{(1-\alpha)x} - Ke^{-\alpha x}\right) \mathbb{1}(e^x > K/h(T,\tau)) \end{aligned}$$

So $\alpha > 1 \implies g_\alpha(x) \in L^1(\mathbb{R})$
We also define:

$$\widehat{g}_\alpha(y) := \int_{\mathbb{R}} g_\alpha(x) e^{-ixy} dx$$

Goal: calculate $\widehat{g}(y)$

$$\begin{aligned} \widehat{g}_\alpha(y) &= \int_{\mathbb{R}} g_\alpha(x) e^{-ixy} dx \\ &= \int_{\ln(\frac{K}{h(T,\tau)})}^\infty \left(h(T,\tau)e^{(1-\alpha)x} - Ke^{-\alpha x})e^{-ixy}\right) dx \\ &= h(T,\tau) \int_{\ln(\frac{K}{h(T,\tau)})}^\infty e^{(1-\alpha-iy)x} dx - K \int_{\ln(\frac{K}{h(T,\tau)})}^\infty e^{-(\alpha+iy)x} dx \\ &= \frac{h(T,\tau)}{\alpha - 1 + iy} e^{(1-\alpha-iy)\ln(\frac{K}{h})} - \frac{K}{\alpha+iy} e^{-(\alpha+iy)\ln(K/h)} \\ &= e^{(-\alpha+iy)\ln(K/h)} \left[c_1 e^{\ln(K/h)} - c_2\right] \\ &= e^{(-\alpha+iy)\ln(K/h)} \left[\frac{K}{\alpha-1+iy} - \frac{K}{\alpha+iy}\right] \\ &= Ke^{(-\alpha+iy)\ln(K/h)} \left[\frac{1}{\alpha-1+iy} - \frac{1}{\alpha+iy}\right] \\ &= \frac{K}{(\alpha-1+iy)(\alpha+iy)} e^{-(\alpha+iy)\ln\left(\frac{K}{h(T,\tau)}\right)} \end{aligned}$$

It can be shown that $\widehat{g}_\alpha \in L^1(\mathbb{R})$

$$
\begin{aligned}
g(x) :&= e^{\alpha x} g_\alpha(x) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\alpha(y) e^{(\alpha+iy)x} dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{K}{(\alpha - 1 + iy)(\alpha + iy)} e^{-(\alpha+iy)\ln\left[\left(\frac{K}{h(T,\tau)}\right)+x\right]} dy
\end{aligned}
$$

Let's start with calculating the option price:

$$
\begin{aligned}
\mathbb{E}_Q\left[(f(T,\tau) - K)^+ \middle| \mathcal{F}_t\right] &= \mathbb{E}_Q\left[\left(h(T,\tau)e^X - K\right)^+ \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}_Q[g(X)|\mathcal{F}_t]
\end{aligned}
$$

Now:

$$
\begin{aligned}
\mathbb{E}_Q\left[g(X) \middle| \mathcal{F}_t\right] &= \mathbb{E}_Q\left[\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\alpha(y) e^{(\alpha+iy)X} dy \middle| \mathcal{F}_t\right] \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\alpha(y) \mathbb{E}_Q\left[e^{(\alpha+iy)X} \middle| \mathcal{F}_t\right] dy
\end{aligned}
$$

Remeber W and I are independent, so:

$$
\begin{aligned}
\mathbb{E}_Q\left[e^{(\alpha+iy)X} \middle| \mathcal{F}_t\right] &= \mathbb{E}_Q\left[e^{(\alpha+iy)\int_0^T \sigma(u,\tau)dW_u} \middle| \mathcal{F}_t\right] \times \mathbb{E}_Q\left[e^{(\alpha+iy)\int_0^T \eta(u,\tau)dI(u)} \middle| \mathcal{F}_t\right] \\
&= e^{(\alpha+iy)\left[\int_0^t \sigma(u,\tau)dW_u + \int_0^t \eta(u,\tau)dI_u\right]} \times \mathbb{E}_Q\left[e^{(\alpha+iy)\int_t^T \sigma(u,\tau)dW_u}\right] \mathbb{E}_Q\left[e^{(\alpha+iy)\int_t^T \eta(u,\tau)dI(u)}\right]
\end{aligned}
$$

Here we used the measurability and independent increments of Levy-processes.
Hence:

$$
\begin{aligned}
\mathbb{E}_Q\left[e^{(\alpha+iy)\int_t^T \sigma(u,\tau)dW_u}\right] &= e^{\frac{1}{2}(\alpha+iy)^2 \int_t^T \sigma^2(u,\tau)du} \\
\mathbb{E}_Q\left[e^{(\alpha+iy)\int_t^T \eta(u,\tau)dI(u)}\right] &= e^{\int_t^T \varphi_Q((\alpha+iy)\eta(u,\tau))du}
\end{aligned}
$$

Summarizing our findings:

$$
\begin{aligned}
\mathbb{E}_Q[g(X)|\mathcal{F}_t] &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\alpha(y) e^{(\alpha+iy)\left[\int_0^t \sigma(u,\tau)dW_u + \int_0^t \eta(u,\tau)dI_u\right]} e^{\frac{1}{2}(\alpha+iy)^2 \int_t^T \sigma^2(u,\tau)du} e^{\int_t^T \varphi_Q((\alpha+iy)\eta(u,\tau))du} dy \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\alpha(y) \left(\frac{f(t,\tau)}{f(0,\tau)}\right)^{(\alpha+iy)} e^{\frac{1}{2}(\alpha+iy)^2 \int_t^T \sigma^2(u,\tau)du} e^{\int_t^T \varphi_Q((\alpha+iy)\eta(u,\tau))du} dy
\end{aligned}
$$

Which means that:

$$
C(t,T,K,\tau) = e^{-r(T-t)} \mathbb{E}_Q[g(X)|\mathcal{F}_t]
$$

7.2 Fourier methods in practice

Let's consider a put option:

$$g(x) := \max(K - e^x, 0)$$

We are asked to find the inverse Fourier transform of g:

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(y)e^{ixy}dy$$

The problem is that $g \notin L^1(\mathbb{R})$:

$$g(x) = (K - e^x)\mathbf{1}\{x < \ln(K)\}$$

Consider $g_\alpha(x) = e^{\alpha x}g(x)$:

$$\begin{aligned} g_\alpha(x) &= e^{\alpha x}(K - e^x)\mathbf{1}\{x < \ln(K)\} \\ &= (Ke^{\alpha x} - e^{(\alpha+1)x})\mathbf{1}\{x < \ln(K)\} \in L^1(\mathbb{R}) \end{aligned}$$

In order for us to find $g(x)$ we need $\widehat{g}$:

$$\begin{aligned} \widehat{g}_\alpha(y) &= \int_{\mathbb{R}} g_\alpha(x)e^{-ixy}dx \\ &= \int_{-\infty}^{\ln(K)} e^{\alpha x}(K - e^x)e^{-ixy}dx \\ &= \int_{-\infty}^{\ln(K)} (K - e^x)e^{(\alpha-iy)x}dx \\ &= K\int_{-\infty}^{\ln(K)} e^{(\alpha-iy)x}dx - \int_{-\infty}^{\ln(K)} e^{(\alpha-iy+1)x}dx \\ &= \frac{K}{\alpha - iy}[e^{(\alpha-iy)\ln(K)}] - \frac{1}{\alpha - iy + 1}[e^{(\alpha-iy+1)\ln(K)}] \end{aligned}$$

We notice that:

$$e^{(\alpha-iy+1)\ln(K)} = e^{(\alpha-iy)\ln(K)}e^{\ln(K)} = e^{(\alpha-iy)\ln(K)}K$$

Leaving us with:

$$\begin{aligned} \widehat{g}_\alpha(y) &= \frac{K}{\alpha - iy}e^{(\alpha-iy)\ln(K)} - \frac{K}{\alpha - iy + 1}e^{(\alpha-iy)\ln(K)} \\ &= Ke^{(\alpha-iy)\ln(K)}\left[\frac{1}{\alpha - iy} - \frac{1}{\alpha - iy + 1}\right] \\ &= \frac{Ke^{(\alpha-iy)\ln(K)}}{(\alpha - iy)(\alpha - iy + 1)} \end{aligned}$$

Do we have that $\widehat{g}_\alpha(y) \in L^1(\mathbb{R})$?:

$$\begin{aligned}\left|\frac{Ke^{(\alpha-iy)\ln(K)}}{(\alpha-iy)(\alpha-iy+1)}\right| &= \frac{|Ke^{\alpha\ln(K)}||e^{-iy\ln(K)}|}{|(\alpha-iy)||(\alpha-iy+1)|} \\ &= \frac{K^{\alpha+1}*1}{\sqrt{\alpha^2+y^2}\sqrt{(\alpha+1)^2+y^2}} \sim \frac{K^{\alpha+1}}{c_2+y^2} \in L^1(\mathbb{R})\end{aligned}$$

In this case we have $g(x) = e^{-\alpha x} g_\alpha(x)$, this yields:

$$\begin{aligned}g(x) &= e^{-\alpha x}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}_\alpha(y)e^{ixy}dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}_\alpha(y)e^{(iy-\alpha)x}dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{K^{\alpha+1}}{(\alpha-iy)(\alpha-iy+1)}e^{(iy-\alpha)x}dy\end{aligned}$$