UiO : Department of Mathematics
University of Oslo

# MSc Thesis

## Advancements in Risk-Free Reference Rates and ESG-linked interest rate swaps

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# Introduction/Motivation

- Risk-Free Reference rates (RFRs): The transition from IBOR to RFRs is ongoing/almost fully completed. Market participants need a better understanding of these special rates, how they differ, and available products.
- ESG: Environmental, Social and Governance. EU taxonomy, regulatory requirements, and increased demand for ESG-linked products.

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# Interest rate theory

# Framework

- $(\Omega, \mathcal{F}, \mathbb{F}, P)$ probability space.
- $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$, right-continous and augmented, $T$ a fixed time horizon.
- $W = \{W(t), t \in [0, T]\}$ a one-dimensional Brownian Motion.
- Let $\gamma \in M^2_{loc}([0, T])$ and define: $\mathcal{E} = \{\mathcal{E}_t(\gamma \bullet W), t \in [0, T]\}$, where:

$$\mathcal{E}_t(\gamma \bullet W) := \exp\left(\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma^2(s)ds\right)$$

- We assume that the Novikov condition holds:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\gamma(s)|^2 ds\right)\right] < \infty$$

# Framework

- This means that:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := \mathcal{E}_T(\gamma \bullet W)$$

Is well-defined, furthermore $\mathcal{E}$ is a martingale, and from Girsanov's theorem, we have that: $W^Q = \left\{ W^Q(t) = W(t) - \int_0^t \gamma(s)ds, t \in [0, T] \right\}$ is a $Q$-Brownian Motion defined on $(\Omega, \mathcal{F}, Q)$

# Financial market

- $r = \{r(t), t \in [0, T]\}$ short-rate process such that $\int_0^t r(s)ds$ is well-defined, with $Q$-dynamics given by:

$$dr(t) = b(t, r(t))dt + \sigma(t, r(t))dW^Q(t)$$
$$r(0) = r_0 \in \mathbb{R}$$

  $b(t, x)$ and $\sigma(t, x)$ are assumed to be so that the above SDE has a strong solution and that the solution is strongly unique.
- $B = \{B(t), t \in [0, T]\}$ the money market account, defined as:

$$B(t) = \exp\left(\int_0^t r(s)ds\right)$$

- $P(s, t)$ market value at time $s$ for a zero-coupon bond (ZCB) maturing at time $t$, $0 \leq s \leq t \leq T$. Guarantees its holder one unit of account at maturity.

# Term Structure

- A collection $\{P(s,t), 0 \leq s \leq t \leq T\}$, is called a term structure.
- Relationship between the short-rate and the ZCB:

$$P(t,T) = \mathbb{E}_Q \left[ \exp\left( -\int_t^T r(s)ds \right) \middle| \mathcal{F}_t \right]$$

- Assume that there exists a smooth function:

$$g : \{(s,t) \in [0,T] \times [0,T] : s \leq t\} \times \mathbb{R} \to \mathbb{R}$$

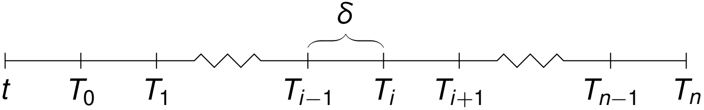
such that $P(s,t) = g(s,t,r(s))$. If:

$$g(t,s,r(s)) = \exp\left(-A(s,t) - B(s,t)r(s)\right)$$

we say that we have an affine term structure (ATS). ($A, B$ smooth $C^1$-functions).

# Interest rate swap

An Interest Rate Swap (IRS) is a financial agreement between parties to exchange interest payments on a specific nominal $N$. The idea is to exchange floating and fixed rates.




- $N$ nominal value.
- $0 < T_0 < T_1 < \cdots < T_n$ sequence of future dates.
- $\delta := T_i - T_{i-1}$ fixed leg between payments.
- $\kappa$ fixed rate
- $F(T_{i-1}, T_i)$ floating rate over $[T_{i-1}, T_i]$.

The floating rate will reset at $T_0, \dots, T_{n-1}$ and the fixed rate will be paid at $T_1, \dots, T_n$.

# Interest rate swap

Perspective from payer IRS, at each instance $T_i, i = 1, \dots n$:

- Pay $\kappa\delta N$ (-)
- Receive $F(T_{i-1}, T_i)\delta N$ (+)

Time-$t$ value $t \leq T_0$ of total cashflow:

$$\pi(t) = N[P(t, T_0) - P(t, T_n)] - \kappa\delta N \sum_{i=1}^{n} P(t, T_i)$$

Now, the fixed swap-rate $\kappa = R_{Swap}(t)$, should be chosen such that $\pi(t) = 0$, namely:

$$\kappa = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}$$

# RFR’s/SOFR

# LIBOR scandal

- LIBOR: Interbank rate provided by panel banks for multiple tenors.
- 2012: International-investigation. Collusion and rate-rigging: Barclays, UBS, Royal Bank of Scotland + others.
- Reliability, transparency and trust were needed, hence the demand for new alternative reference rates.
- Solution: US: Secured Overnight Financing Rate (SOFR), UK: Sterling Overnight Index Average (SONIA). Similar alternatives exist in other countries as well.

# SOFR

- SOFR: overnight rate, developed by ARRC and managed by New York Fed.
- Based upon overnight Repo-market, backed up by US treasury securities.
- Volume-weighted median.

**Definition (Discrete overnight, backward-looking avarage)**

$$R_{d_i}(T_i) = \frac{1}{d_i}\left(\frac{1}{P(T_i, T_i + d_i)} - 1\right)$$

$$R^B(S, T) = \frac{1}{T - S}\left(\prod_{i=1}^{N}[1 + d_i R_{d_i}(T_i)] - 1\right)$$

$d_i$: day count fraction.

# SOFR

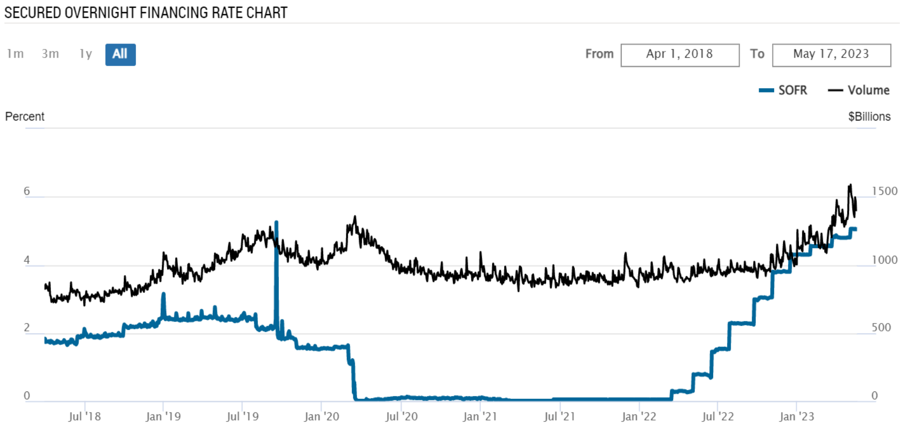



Figure: O/N-SOFR and Repo volume

Source: https://www.newyorkfed.org/markets/reference-rates/sofr

# SOFR-futures

- SOFR, overnight rate, outlook: 24 hours.
- CME (Chicago Mercantile Exchange), publish term SOFR.
- tenors: 1M, 3M, 6M and 12M.
- Inferred from the futures market.

**Definition (1M- and 3M-SOFR futures)**

$$f^{1M}(t,S,T) = \frac{1}{T-S}\mathbb{E}_Q\left[\int_S^T r(s)ds \middle| \mathcal{F}_t\right] \quad \text{(arithmetic)}$$

$$f^{3M}(t,S,T) = \frac{1}{T-S}\left(\mathbb{E}_Q\left[e^{\int_S^T r(s)ds} \middle| \mathcal{F}_t\right] - 1\right) \quad \text{(geometric)}$$

# Hedging 3M-arithmetic rate

- Loan of 30 Millon dollars over a 3M-period.
- 3M-arithmetic floating-rate $X^{3M_A}(S,T)$ is to be paid, over this period.

$$X^{3M_A}(S,T) = \frac{1}{T-S}\int_S^T r(u)du$$

- Available products in market: 1M-SOFR futures and 3M-SOFR futures.
- Short-rate $r = \{r(t), t \in [0,T]\}$ follows Vasicek-dynamics:

$$dr(t) = \alpha[m - r(t)]dt + \sigma dW^Q(t)$$

# Hedge 1

- Hedge possibility number one: using the available 3M-SOFR futures:

**Proposition (Hedge 1: 3M-Geometric situation)**

$$\underset{a_t \in \mathbb{R}}{\arg\min} \mathbb{E}_Q \left[ \left( X^{3M_A}(S,T) - a_t f^{3M}(t,S,T) \right)^2 \middle| \mathcal{F}_t \right] = \frac{\int_S^T \mathbb{E}_Q[r(u)|\mathcal{F}_t] du}{(T-S) f^{3M}(t,S,T)}$$

# Hedge 1

- $r$ ATS: $dr(t) = [b(t) + \beta(t)r(t)]dt + \sqrt{a(t) + \alpha(t)r(t)}dW^Q(t)$

## Proposition (Hedge 1: 3M-ATS setting)

$$\underset{a_t \in \mathbb{R}}{\arg\min} \mathbb{E}_Q \left[ \left( X^{3M_A}(S,T) - a_t f^{3M}(t,S,T) \right)^2 \middle| \mathcal{F}_t \right]$$
$$= \frac{r(t)(T-S) + \int_S^T \int_t^u b(s)dsdu + \int_S^T \int_t^u \beta(s)g(s)dsdu}{(T-S)f^{3M}(t,S,T)}$$

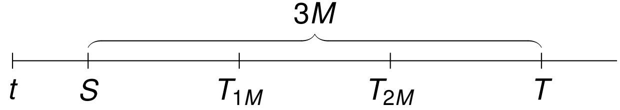
*Where:*

$$g(s) = \exp\left( \int_t^s \beta(v)dv \right) \left( \int_t^s e^{-\int_t^w \beta(v)dv} b(w)dw + \mathbb{E}_Q[r(t)] \right)$$

# Hedge 2

- Hedge possibility number two: use available 1M-SOFR futures:

$$
\begin{aligned}
&Y(a_t, b_t, c_t) \\
&:= a_t f^{1M}(t, S, T_{1M}) + b_t f^{1M}(t, T_{1M}, T_{2M}) + c_t f^{1M}(t, T_{2M}, T)
\end{aligned}
$$




The hedge looks like this:

$$
\underset{(a_t, b_t, c_t) \in \mathbb{R}^3}{\arg\min} \ \mathbb{E}_Q \left[ \left( X^{3M_A}(S, T) - Y(a_t, b_t, c_t) \right)^2 \middle| \mathcal{F}_t \right]
$$

# Hedge 2

- $\mathbf{x}_t = (a_t, b_t, c_t)$, weightings in 1M-SOFR future rates.
- SOFR-future rates:

$$\left(f^{1M}(t, S, T_{1M}), f^{1M}(t, T_{1M}, T_{2M}), f^{1M}(t, T_{2M}, T)\right) = (\alpha_t, \beta_t, \gamma_t)$$

- $G(\mathbf{x}_t) := \mathbb{E}_Q\left[\left(X^{3M_A}(S, T) - Y(a_t, b_t, c_t)\right)^2 \middle| \mathcal{F}_t\right]$

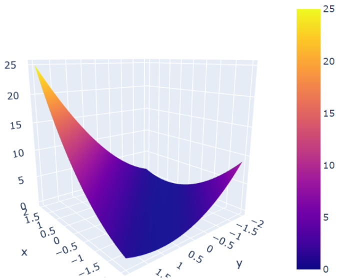



Figure: level Curve $G(\mathbf{x}_t) = k$, where all constants are set to one

# Hedge 2

- If $\det(M) \neq 0$, where:

$$M = \begin{bmatrix} \alpha_t^2 & \alpha_t\beta_t & \alpha_t\gamma_t \\ \beta_t^2 & \alpha_t\beta_t & \beta_t\gamma_t \\ \gamma_t^2 & \alpha_t\gamma_t & \beta_t\gamma_t \end{bmatrix}$$

- Optimal weight $\hat{\mathbf{x}}_t$ given by:

$$\hat{\mathbf{x}}_t = M^{-1}\mathbb{E}_Q\left[X^{3M_A}(S,T)\middle|\mathcal{F}_t\right]\begin{bmatrix} \alpha_t \\ \beta_t \\ \gamma_t \end{bmatrix}$$

# Numerical example

- Vasicek parameters:

$$\alpha = 0.25,\ m = 0.035, \sigma = 0.02, r_0 = 0.0425$$

- SOFR futures rates:

$$f^{3M}(0, S, T) = 0.0423 \text{ and } \begin{bmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{bmatrix} = \begin{bmatrix}0.0423\\ 0.0421\\ 0.0420\end{bmatrix}$$

- $\hat{a}_t^{3M}$: optimal weiging in 3M-SOFR futures rates.
- $(\hat{a}_t^{1M}, \hat{b}_t^{1M}, \hat{c}_t^{1M})$: optimal weights in 1M-SOFR futures rates.
- We choose to look at the following to benchmark the hedges:

$$ER_1(t) := X^{3M_A}(S, T) - \hat{a}_t^{3M} f^{3M}(t, S, T)$$
$$ER_2(t) := X^{3M_A}(S, T) - Y(\hat{a}_t^{1M}, \hat{b}_t^{1M}, \hat{c}_t^{1M})$$

# Numerical example

- 1 Million simulations:

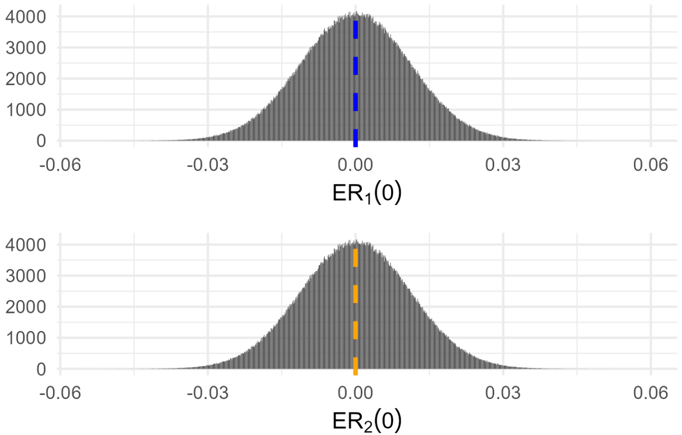



Figure: $ER_1(0)$ and $ER_2(0)$

# ESG

# ESG

- ESG: Environmental, Social and Governance
- EU: taxonomy, carbon neutral by 2050.
- Partial Solution: Sustainable Finance, ESG-linked interest rate swap.
- ISDA: SBM Offshore and ING, the world's first sustainability improvement derivative [1].
- SBM pays fixed and receives floating.
- ESG-part: If SBM meets certain ESG criteria, a 5-10 bp discount is applied to the fixed rate. Conversely, a penalty of 5-10 bp is added.

# ESG fixed rate

Consider the following setup:

- $d$: basis points added or subtracted to fixed-rate $\kappa$.
- $\{A_i\}_{i=1}^{n}$ sequence of events, where: $A_i = \{X_{T_i} \leq C_{T_i}^{ESG}\}$ i.e. the sequence of events measuring if the ESG-risk score at time $T_i$: $X_{T_i}$, is below the ESG-criteria $C_{T_i}$ or not.

**Definition (ESG-fixed rate process)**

Let $K^{ESG} = (K_i^{ESG}(\omega))_{i=1}^{n}$ denote the ESG fixed rate process, we define it recursively as:

$$K_i^{ESG}(\omega) = (K_{i-1}^{ESG}(\omega) - d)\mathbb{1}_{A_i}(\omega) + (K_{i-1}^{ESG}(\omega) + d)\mathbb{1}_{A_i^C}(\omega), \quad i \geq 2$$

Where:

$$K_1^{ESG}(\omega) = (\kappa - d)\mathbb{1}_{A_1}(\omega) + (\kappa + d)\mathbb{1}_{A_1^C}(\omega)$$

# ESG fixed rate

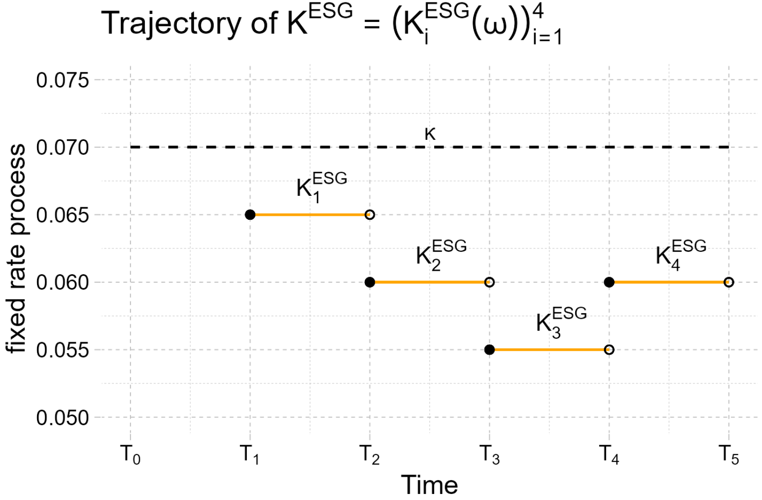



Figure: ESG-fixed rate trajectory

# ESG fixed rate

## Notation

Let $\mathcal{I} = \{k_1, \dots, k_n\}$ represent an index set. Let $\mathcal{H} \subseteq \mathcal{I}$, we then define:

$$\left(\bigcap_{i\in\mathcal{I}} A_i\right)^{\mathcal{H}} := \left(\bigcap_{i\in\mathcal{H}} A_i^C\right) \cap \left(\bigcap_{i\in\mathcal{I}\setminus\mathcal{H}} A_i\right)$$

## Example

$\mathcal{I} = \{1, 2, 3, 4, 5\}$ and $\mathcal{H} = \{1, 3, 4\}$, this gives us:

$$\left(\bigcap_{i=1}^{5} A_i\right)^{\{1,3,4\}} = A_1^C \cap A_2 \cap A_3^C \cap A_4^C \cap A_5$$

# ESG fixed rate

- Let $n \in \mathbb{N}$, $\mathcal{I}_n := \{1, \ldots, n\}$, $\mathcal{I}_{2n}^{Even} := \{2, \ldots, 2n\}$
- $(A_i)_{i \in \mathcal{I}_n}$ sequence of events measuring whether or not ESG-criteria is met:

$$A_i = \{X_{T_i} \leq C_{T_i}^{ESG}\}$$

- ESG fixed rate process: $K^{ESG} = (K_r^{ESG}(\omega))_{r \in \mathcal{I}_n}$

## Observation (Explicit form)

$$K_r^{ESG}(\omega) = [\kappa - d \cdot r]\mathbb{1}\left[\bigcap_{i \in \mathcal{I}_r} A_i\right](\omega) +$$

$$\sum_{\alpha \in \mathcal{I}_{2r}^{Even}} \left( [\kappa - d \cdot (r - \alpha)]\mathbb{1}\left[\bigcup_{j_1 \neq \ldots \neq j_{|\mathcal{I}_\alpha^{Even}|} \in \mathcal{I}_r} \left(\bigcap_{i \in \mathcal{I}_r} A_i\right)^{\{(j_1, \ldots, j_{|\mathcal{I}_\alpha^{Even}|})\}}\right]\right)(\omega)$$

# ESG swap-rate process

**Proposition (Swap rate process $\kappa_t^{ESG} = (\kappa_t^{ESG}(i))_{i \in \mathcal{I}_n}$)**

*Denote* $\kappa_t^{ESG}(i) := \mathbb{E}_Q[K_i^{ESG}(\omega) | \mathcal{F}_t]$. *Then for* $(t \leq T_0)$ *we have:*

$$\kappa_t^{ESG}(i) = \kappa - d \cdot D(i)$$

*Where:*

$$D(i) = i \cdot \mathbb{E}_Q\left[\prod_{l=1}^{i} \mathbb{1}(A_l) \middle| \mathcal{F}_t\right] + \sum_{\alpha \in \mathcal{I}_{2i}^{Even}} [i - \alpha] \sum_{j_1 \neq \ldots \neq j_{|\mathcal{I}_\alpha^{Even}|} \in \mathcal{I}_i} \mathbb{E}_Q\left[\left(\prod_{l=1}^{i} \mathbb{1}(A_l)\right)^{\{(j_1, \ldots, j_{|\mathcal{I}_\alpha^{Even}|})\}} \middle| \mathcal{F}_t\right]$$

# Numerical Simulation

- $X(t)$: process representing the ESG-risk score in ESG-linked IRS. Proposal:

$$X(t) = 100 \exp(-Z(t))$$
$$dZ(t) = -\beta Z(t)dt + \sigma dW^Q(t) + dI^Q(t)$$

- $I^Q(t)$ is a CPP:

$$I^Q(t) = \sum_{k=1}^{N(t)} J_k, \quad J_k \sim Exp(\mu), \quad N(t) \sim Pois(\lambda t)$$

# Numerical Simulation

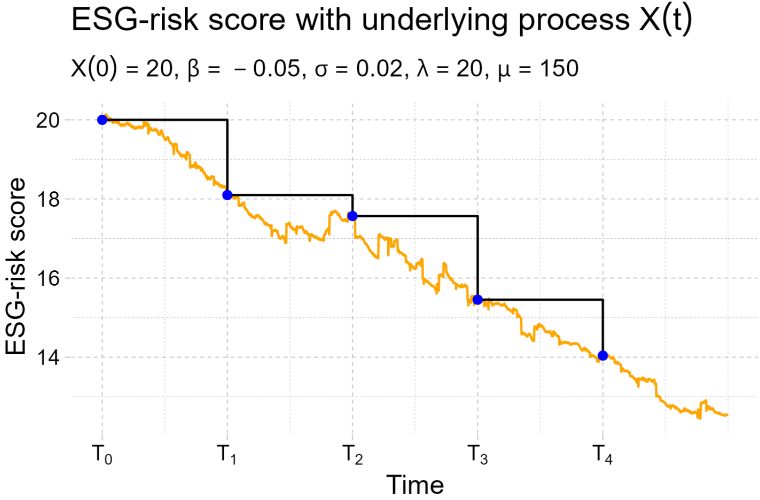



Figure: ESG-risk score with underlying process $X(t)$

# Specifications

- $C^{ESG} = (C^{ESG}_{T_i})_{i \in \mathcal{I}_4}$, $\mathcal{F}_0$-measurable.
- $\kappa^{ZCB}_t = 0.07$, the fixed-rate from the original IRS.
- $d = 0.005$ constant penalty/discount.
- $\delta := T_i - T_{i-1} = 1$

We will showcase some scenarios where we consider "reasonable" and "unreasonable" ESG-criteria $C^{ESG}_{T_i}$.

# Reasonable criteria

- $C^{ESG} = (17.8, 16.8, 15.8, 14.8)$

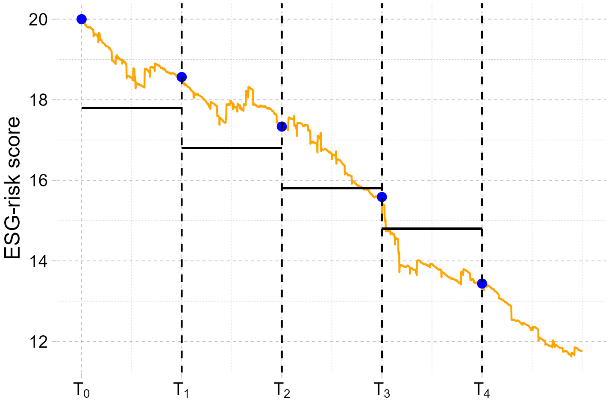



Figure: ESG-risk score where ESG-criteria is reasonable

# Reasonable criteria

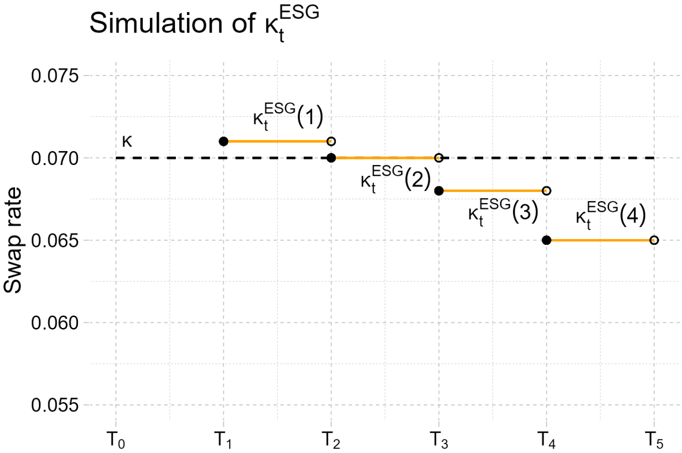



Figure: ESG-swap rate when ESG-criteria is reasonable

# Unreasonable Criteria

Always managing to meet criteria: $C^{ESG} = (24, 23, 22, 21)$

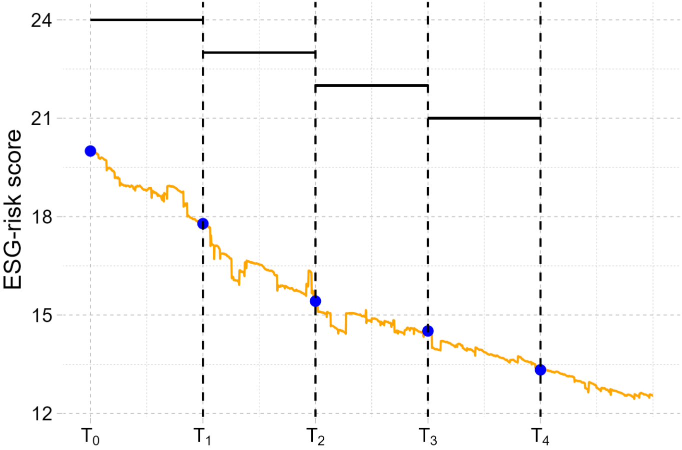



Figure: ESG-risk score where ESG-criteria is always met

# Unreasonable Criteria

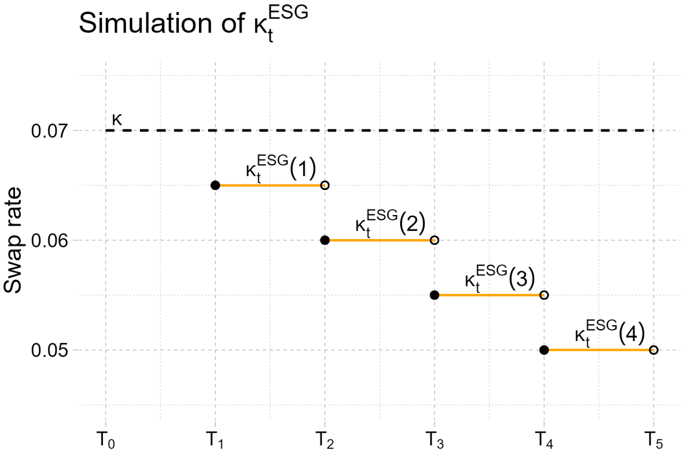



Figure: ESG-swap rate, when ESG-criteria is always met

# Conclusions and further work

# SOFR: Conclusion and further work

- SOFR: overnight, thus 1-tenor: 24h.
- futures market: market view on future rates.
- CME term-SOFR: inferred from the futures market.
- Further work: look more into the modelling of term-SOFR, closely related to EFFR (Effective Federal Funds Rate), as discussed in: [3].
- Hedge: we used a one-factor model and operated under $Q$. More reasonable: multi-factor models and suitable $P$ dynamics. Following the author's approach in [2] (Skov and Skovmand, 2020)

# ESG: Conclusion and further work

- ESG-criteria:

$$A_i = \{X_{T_i} \leq C_{T_i}^{ESG}\}$$

- $C^{ESG}$: $\mathcal{F}_{T_{i-1}}$-measurable instead of $\mathcal{F}_0$.
- Choice of process $X = (X(t))_{t\in[0,T]}$ important. Which one to choose?
- ESG-score/risk-score will be observed under $P$, not $Q$. Complex transformations.
- Data accessibility, standardization of score $\mathcal{S}$:

$$\mathcal{S} := \sum_{j=1}^{m} w_j X_j$$

- how to choose the weights $w_j$'s and which metrics to $X_j$'s should one choose? Different agencies can give different scores to the same company, further discussed in [4] (Billio et al, 2021)

# ESG: Conclusion and further work

- Closed expression for $K_n^{ESG}(\omega)$, quite complex, requires the Monte-Carlo approach. However, it gives flexibility for modelling ESG scores/risk scores.

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