Measure theory

Andreas Slaattelid

Contents

These are lecture notes based on MAT4400 – Linear Analysis with Applications, they are mainly based on [1], as well as Tom Lindstrøm's excellent notes himself from classes.

1 An introduction to measure theory

1.1 Sigma-algebras and measures

Definition 1 (sigma-algebra) *Assume that X is a non-empty set, a family $\mathcal{A}$ of subsets of X is called a sigma-algebra if the following holds:*

i) $\emptyset \in \mathcal{A}$

ii) If $A \in \mathcal{A}$, *then* $A^C \in \mathcal{A}$

iii) If $A_n \in \mathcal{A}$ *for all* $n \in \mathbb{N}$, *then* $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

Proposition 1 *Assume that $\mathcal{A}$ is a sigma-algebra on X, then the following holds:*

i) $X \in \mathcal{A}$

ii) if $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{A}$, *then* $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$

iii) if $A_1, \dots A_n \in \mathcal{A}$, *then* $\bigcup_{k=1}^n A_k \in \mathcal{A}$ *and* $\bigcap_{k=1}^n A_k \in \mathcal{A}$

iv) if $A, B \in \mathcal{A}$, *then* $A \setminus B \in \mathcal{A}$

This is a useful proposition, and are not included in the definition since it actually follows from the definition.

Definition 2 (measure) *Assume that X is a non-empty set, and that $\mathcal{A}$ is a σ-algebra on X.*
A measure μ on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+ = [0, \infty) \cup \{\infty\}$ such that:

i) $\mu(\emptyset) = 0$

ii) if $\{A_n\}_{n\in\mathbb{N}}$ *is a pairwise disjoint sequence, then:* $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$

We call the triplet $(X, \mathcal{A}, \mu)$ a measure space.

When we know what a measure and a sigma-algebra is, then we are ready for examples of measures, and see why in fact we can call them measures:

Example 1 (Counting measure) *This is the counting measure, also known as the cardinality of a set: let $X = \{x_1, \dots, x_n\}$ be a finite set, and let $\mathcal{A} = \mathcal{P}(X)$, then: $\mu(A) = |A|$ is a measure on X.*
Why is this a measure? First of all we have that the set X is finite, so for all sets $A \in \mathcal{P}(X)$ we will have $|A| < \infty$, as well as $|A| \geq 0$. this means that μ takes values in $[0, \infty)$ which is a requirement for a measure. And by definition we also have that $\mu(\emptyset) = |\emptyset| = 0$ (the empty set has zero elements).
We also have that for the disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$: $\mu\left(\bigcup_{n\in\mathbb{N}}\right) = \left|\bigcup_{n\in\mathbb{N}} A_n\right| = \sum_{n\in\mathbb{N}} |A_n| = \sum_{n\in\mathbb{N}} \mu(A_n)$. Hence the counting measure is indeed a measure.

Example 2 (Dirac measure) *We here have one rather simple measure, but actually one of the most important ones, and it looks like this:*

$$\mu(A) = \delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

We have that the measure takes only 1 or 0 as values,
so we have that $\mu(A) \geq 0, \forall A \in \mathcal{A}$, we also have that $\mu(\emptyset) = 0$, this is because by definition $x \notin \emptyset$.
Before we generalize, lets assume the following:

$$X = \{a_1, a_2\} \;\; \mathcal{P}(X) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\} = \{A_1, A_2, A_3, A_4\}$$
$$\textit{our disjoint sets here are: } A_1, A_2, A_3 \textit{ so this means that:}$$
$$x \in \bigcup_{k=1}^{3} A_k = \{x \in A_1 \textit{ or } x \in A_2 \textit{ or } x \in A_3\}$$
$$x \in \bigcup_{k=1}^{3} A_k \implies \mu\left(\bigcup_{k=1}^{3} A_k\right) = 1 = \sum_{k=1}^{3} \mu(A_k) = 0 + 1 + 0 \;\; (x \in A_2)$$

And with the above intuition we can generalize: let $\{A_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence, for some $m \in \mathbb{N}$, we will have $x \in A_m$ and for $n \neq m$, we will have $x \notin A_n$, which means that:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 1 = \sum_{n\in\mathbb{N}} \mu(A_n) = 0 + 0 + \cdots + \mu(A_m) + 0 + \cdots = 1$$

In the other case, i.e where $x \notin A_n$ $\forall n \in \mathbb{N}$ we have:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 0 = \sum_{n\in\mathbb{N}} \mu(A_n)$$

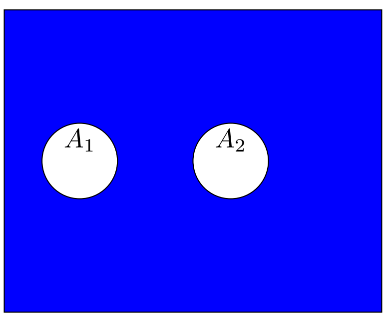
Hence the Dirac measure is a measure

Example 3 (sigma-algebra of countable sets and measure) *Let X be an uncountable set, and define the sigma-algebra*
$\mathcal{A} = \{A \in \mathcal{P}(X) : A \textit{ is countable or } A^C \textit{ is countable}\}$ *as well as the measure:*

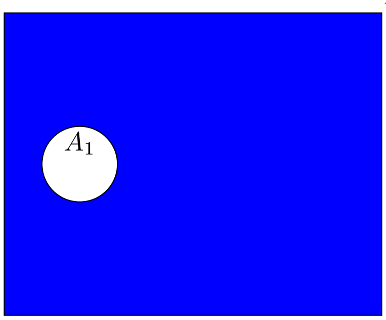
$$\mu(A) = \begin{cases} 0 & \textit{if } A \textit{ is countable} \\ 1 & \textit{if } A^C \textit{ is countable} \end{cases}$$

We can start by verifying that $\mathcal{A}$ is a σ-algebra: We have that $\emptyset \in \mathcal{P}(X)$, and by definition we have that the $\emptyset$ is countable, hence $\emptyset \in \mathcal{A}$.
We also want to have that $A \in \mathcal{A} \implies A^C \in \mathcal{A}$: so assume that A is countable, i.e an element of $\mathcal{A}$ and let $B = A^C$, we will then have that
$B^C = (A^C)^C = A$, hence we must have that A^C is an element of $\mathcal{A}$, since it's complement is countable.
We must also show that: $A_1, \ldots A_n \in \mathcal{A} \implies \bigcup_{k=1}^n A_k \in \mathcal{A}$. let's start with the simplest case where all A_k is countable, then we have by definition that $\bigcup_{k=1}^n A_k$ is countable, hence also in $\mathcal{A}$.
In our second case it could be that they are not countable, i.e maybe all or some of them, but assume that there exist $i \in \mathbb{N}$ s.t A_i^C is countable. Then we will have: $(\bigcup A_n)^C = \bigcap A_n^C \subseteq A_i^C$. This is easiest understood by venn diagrams:

$A_1^C \cap A_2^C$



A_1^C



The idea here is that the blue area in the last figure is larger than the blue area in the first figure, hence $A_1^C \cap A_2^C \subseteq A_1^C$ Now we have finally shown that $\mathcal{A}$ is a sigma-algebra, we now only have to show that μ is indeed a measure.

$\mu(\emptyset) = 0$, since $\emptyset$ is countable.
Now again let all sets $A_1, \ldots, A_n$ be countable pairwise disjoint. hence:
$\mu\left(\bigcup_{k=1}^n A_k\right) = 0 = \sum_{k=1}^n \mu(A_k)$, again this holds since the union is countable.
Now we look at a less ideal situation, namely maybe not all sets are countable. Let $A = \bigcup_{n\in\mathbb{N}} A_n$, and assume that there exist atleast one $m \in \mathbb{N}$, such that A_m^C is countable. So we have then that $A^C \subseteq A_m^C$, this is easiest understood by drawings, but as A_m^C is just one countable set, we will have that the union $A^C = (\bigcup_{n\in\mathbb{N}} A_n)^C$ is a smaller set than A_m^C. But this means that A^C is a countable set, as it is a subset of a countable set, hence $A \in \mathcal{A}$ (since $A^C \in \mathcal{A}$).
We still assume that $\{A_n\}_{n\in\mathbb{N}}$ is a disjoint sequence, this means that:
$A_n \subseteq A_m^C$, $n \neq m$. So for $n \neq m$ we have that A_n is countable.

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \mu(A_n) &= \sum_{n\neq m} \mu(A_n) + \mu(A_m) \\
&= 0 + 1 = 1 \ \ (A_m^C \textit{ is countable}) \\
\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= 1 \\
\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= 1 = \sum_{n\in\mathbb{N}} \mu(A_n)
\end{aligned}$$

Proposition 2 *Let $(X, \mathcal{A}, \mu)$ be a measure space, then we have:*

i) (Finite addativity) if $A_1, \ldots, A_m$ are disjoint sets in $\mathcal{A}$, then:

$$\mu\left(\bigcup_{n=1}^m A_n\right) = \sum_{n=1}^m \mu(A_n)$$

ii) (Monotinicity) if $A, B \in \mathcal{A}$, and $A \subseteq B$, with $\mu(B) < \infty$, then: $\mu(A) \leq \mu(B)$

iii) if $A, B \in \mathcal{A}$, with $A \subseteq B$ and $\mu(B) < \infty$, then: $\mu(A \setminus B) = \mu(B) - \mu(A)$

iv) (Subaddativity) let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$, not necessarily disjoint, then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$$

The idea again, is to use the definition of the measure to verify these, so for $i)$ we just supply the sequence with empty-sets. $iii)$ is a consequence of $ii)$ and $iv)$ is a consequence of defintion again:

Proof 1 *We start with ii):*

$$\begin{aligned} B &= A \cup (B \setminus A) \text{ } (disjoint) \\ \mu(B) &= \mu(A) + \mu(B \setminus A) \\ &\geq \mu(A) \end{aligned}$$

iii) follows directly from above, just reorder the equation.
iv): Let $\{A_n\}$ be the sequence in $\mathcal{A}$ not necessarily disjoint. We want to create a disjoint set of these sets namely $\{B_n\}$

$$B_1 = A_1, \;\; B_2 = A_2 \setminus B_1, \;\; B_3 = A_3 \setminus (B_1 \cup B_2), \ldots, B_n = A_n \setminus (\bigcup_{k=1}^{n-1} B_k)$$

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n) \text{ } (by\ monotinicity:\ B_n \subseteq A_n)$$

Proposition 3 (Continuity of measure) *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets in $(X, \mathcal{A}, \mu)$, then we have:*

i) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is an increasing sequence, i.e that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then:

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

ii) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence, i.e that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, and that $\mu(A_1) < \infty$ then:

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof 2 *We start by proving i): let $A = \bigcup_{n\in\mathbb{N}} A_n$ and lets define a new disjoint sequence $\{B_n\}_{n\in\mathbb{N}}$ by: $B_1 = A_1$, $B_2 = A_2 \setminus A_1, \ldots B_n = A_n \setminus A_{n-1}$. This means that:*

$$\mu(A) = \mu(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \mu(B_n) = \lim_{m\to\infty} \sum_{n=1}^{m} \mu(B_n) = \lim_{m\to\infty} \mu\left(\bigcup_{n=1}^{m} B_n\right) = \lim_{m\to\infty} \mu(A_m)$$

For part two, we again want to define a disjoint sequence: let $B_n = A_1 \setminus A_n$ so $\{B_n\}_{n\in\mathbb{N}}$ is an increasing sequence. And let $A = \bigcap_{n\in N} A_n$. We will then have that $\bigcup_{n\in\mathbb{N}} B_n = A_1 \setminus \bigcap_{n\in\mathbb{N}} A_n = A_1 \setminus A$

Example 4 (Cont of measure) *Let μ be a measure on $\mathbb{R}$ such that $\mu([-1/n, 1/n]) = 1 + 2/n, n \in \mathbb{N}$, then we have $\mu(\{0\}) = 1$.*
Define $A_n = [-1/n, 1/n]$ then we have that the sequence $\{A_n\}_{n\in\mathbb{N}}$ is decreasing ($A_{n+1} \subseteq A_n$. $\mu(A_1) = 1 + 2/1 = 3 < \infty$. As well as $\{0\} = \bigcap_{n\in\mathbb{N}} A_n$. Now since all the requirements for continuity of measure is fulfilled, we thus have:

$$\mu(\{0\}) = \mu(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} 1 + 2/n = 1$$

Example 5 (When requirements does not hold) *Let μ be the Lesbegue measure on $\mathbb{R}$, i.e: $\mu([a,b]) = b - a$. Let's put $A_n = [n, \infty)$ so this means that $\mu(A_n) = \infty$, $\forall n \in \mathbb{N}$, thus $\lim_{n\to\infty} \mu(A_n) = \infty$. We have that $\{A_n\}$ is a decreasing sequence, with $\mu(A_1) = \infty$. We can also look at the intersection: $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} [n, \infty) = \emptyset$. Which again yields:*
$\mu(\bigcap_{n\in\mathbb{N}} A_n) = \mu(\emptyset) = 0 \neq \lim_{n\to\infty} \mu(A_n)$

1.2 Complete measures

We have until now defined what a sigma-algebra and a measure is, but the definition does not capture everything. Assume that $(X, \mathcal{A}, \mu)$ is a measurespace and let $B \in \mathcal{A}$ be such that $\mu(B) = 0$, further assume that $N \subseteq B$, a question that arises is then: will $\mu(N) = 0$? If $N \in \mathcal{A}$, then we will have: $\mu(N) = 0$. This follows from the monotonicity property of the measure. The problem is when $N \notin \mathcal{A}$, our defintion of the measure does not say anything about this. So in general we will have that N does not need to be in $\mathcal{A}$.

Definition 3 (Null set) *A set $N \subseteq X$ is called a null set, if there is a set $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$.*

Definition 4 (Complete measure space) *A measure space $(X, \mathcal{A}, \mu)$ is called complete if all null sets belongs to $\mathcal{A}$.*

The goal of this section is to turn an arbitrary measure space into a complete measure space. Before we continue, we denote $\mathcal{N}$ the collection of all null sets.

Lemma 1 *If $N_n \in \mathcal{N}$, then: $\bigcup_{n\in\mathbb{N}} N_n \in \mathcal{N}$*

Proof 3 *Since N_n is a null set, we have that there is a $B_n \in \mathcal{A}$ s.t $N_n \subseteq B_n$, $\forall n \in \mathbb{N}$, with $\mu(B_n) = 0$. We will therefore have: $\bigcup_{n\in\mathbb{N}} N_n \subseteq \bigcup_{n\in\mathbb{N}} B_n$, and by subadditivity of measure:*

$$\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n\in\mathbb{N}} \mu(B_n) = 0$$

Hence $\bigcup_{n\in\mathbb{N}} N_n \in \mathcal{N}$

Proposition 4 (Smallest sigma-algebra containg $\mathcal{A}$ and $\mathcal{N}$) *Let $\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A} \text{ and } N \in \mathcal{N}\}$, then $\overline{\mathcal{A}}$ is the smallest σ-algebra containg $\mathcal{A}$ and $\mathcal{N}$.*

Proof 4 *We start by showing that $\overline{\mathcal{A}}$ contains $\mathcal{A}$ and $\mathcal{N}$:*
$\mathcal{A} \subseteq \overline{\mathcal{A}}$:

$$A \in \mathcal{A} : A = A \cup \emptyset \in \overline{\mathcal{A}} \; (A \in \mathcal{A}, \; \emptyset \in \mathcal{N})$$
$$N \in \mathcal{N} : N = \emptyset \cup N \in \overline{\mathcal{A}} \; (\emptyset \in \mathcal{A}, \; N \in \mathcal{N})$$

If we assume that $\overline{\mathcal{A}}$ is a σ-algebra, then it must be the smallest one containg $\mathcal{A}$ and $\mathcal{N}$ because any other σ-algebra $\mathcal{G}$ must have $A \cup N$ as an element, hence $\overline{\mathcal{A}} \subseteq \mathcal{G}$. Now the last thing we need to show is that $\overline{\mathcal{A}}$ actually is a σ-algebra.

$$\begin{aligned}
\emptyset \in \overline{\mathcal{A}} : \emptyset = \emptyset \cup \emptyset \in \overline{\mathcal{A}} \\
A \cup N \in \overline{\mathcal{A}} \implies (A \cup N)^C \in \overline{\mathcal{A}} : \\
(A \cup N)^C = (A \cup B)^C \cup (B \setminus N) \in \overline{\mathcal{A}} \\
(A_n \cup N_n) \in \overline{\mathcal{A}} \, \forall n \in \mathbb{N} \implies \bigcup_{n \in \mathbb{N}} (A_n \cup N_n) \in \overline{\mathcal{A}} : \\
\bigcup_{n \in \mathbb{N}} (A_n \cup N_n) = \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cup \left(\bigcup_{n \in \mathbb{N}} N_n \right) \in \overline{\mathcal{A}}
\end{aligned}$$

Thus $\overline{\mathcal{A}}$ is the smallest σ-algebra containg $\mathcal{A}$ and $\mathcal{N}$.

Lemma 2 *Assume that $A_1, A_2 \in \mathcal{A}$ and $N_1, N_2 \in \mathcal{N}$ and that: $A_1 \cup N_1 = A_2 \cup N_2$, then: $\mu(A_1) = \mu(A_2)$*

Proof 5 *Since N_2 is a null set, we have that $B_2 \in \mathcal{A}$ with $N_2 \subseteq B_2$ and $\mu(B_2) = 0$. But then:*

$$\begin{aligned}
A_1 \subseteq A_1 \cup N_1 = A_2 \cup N_2 \subseteq A_2 \cup B_2 \\
\mu(A_1) \leq \mu(A_2 \cup B_2) \leq \mu(A_2) + \mu(B_2) = \mu(A_2) \\
A_2 \subseteq A_2 \cup N_2 = A_1 \cup N_1 \subseteq A_1 \cup B_1 \\
\mu(A_2) \leq \mu(A_1 \cup B_1) \leq \mu(A_1) + \mu(B_1) = \mu(A_1)
\end{aligned}$$

Hence $\mu(A_1) = \mu(A_2)$.

It may seem strange why we included this lemma, but the idea is that we want to define a measure $\overline{\mu} : \overline{\mathcal{A}} \to \overline{R}_+$ such that: $\overline{\mu}(A \cup N) = \mu(A)$, so if we have two equal unions: $A_1 \cup N_1 = A_2 \cup N_2$, then: $\overline{\mu}(A_1 \cup N_1) = \overline{\mu}(A_2 \cup N_2) = \mu(A_1) = \mu(A_2)$ this lemma ensures us that if we have two equal unions, then they have the same measure.

Theorem 1 (Complete measure space with complete measure) *Assume that $(X, \mathcal{A}, \mu)$ is a measure space, and let: $\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A} \text{ and } N \in \mathcal{N}\}$, define $\overline{\mu} : \overline{\mathcal{A}} \to \overline{\mathbb{R}}_+$ by:*

$$\overline{\mu}(A \cup N) = \mu(A), \;\; \forall A \in \mathcal{A}$$

Then $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space extending $(X, \mathcal{A}, \mu)$.

Proof 6 *We first want to check that $\overline{\mu}$ is a measure*

$$\begin{aligned}
&\overline{\mu}(\emptyset) = 0 : \\
&\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0
\end{aligned}$$

Now let $\{C_n\}_{n\in\mathbb{N}}$ be a disjoint sequence in $(X, \overline{\mathcal{A}}, \overline{\mu})$, with $C_n = A_n \cup N_n$ we must show that : $\overline{\mu}(\bigcup_{n\in\mathbb{N}} C_n) = \sum_{n\in\mathbb{N}} \overline{\mu}(C_n)$. First we notice that since the C_n's are disjoint we must have that the A_n's are disjoint.

$$\sum_{n\in\mathbb{N}} \overline{\mu}(C_n) = \sum_{n\in\mathbb{N}} \overline{\mu}(A_n \cup N_n) = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \overline{\mu}\left(\bigcup_{n\in\mathbb{N}} (A_n \cup N_n)\right) = \overline{\mu}\left(\bigcup_{n\in\mathbb{N}} C_n\right)$$

We therefore have that $\overline{\mu}$ is a measure. But why is this new measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ complete? There could be that we have some new null sets in $\overline{\mathcal{A}}$, so we must therefore show that it is actually a complete measuresapce, i.e all null sets belongs to $\overline{\mathcal{A}}$.
To prove that this new measure space is complete, we must show that if M is a $\overline{\mu}$ null set, then $M \in \overline{\mathcal{A}}$, which yields $\overline{\mu}(M) = 0$.
M a $\overline{\mu}$ null set: let $C \in \overline{\mathcal{A}}$ such that $M \subseteq C$ with $\overline{\mu}(C) = 0$. Since $C \in \overline{\mathcal{A}}$ we have that C is on the form: $C = A \cup N$, with $N \in \mathcal{N}$. But since $N \in \mathcal{N}$, we know that there is a $B \in \mathcal{A}$ such that $N \subseteq B$ with $\mu(B) = 0$. This gives us:

$$\begin{aligned} M \subseteq C = A \cup N \subseteq A \cup B \\ A \cup B \in \mathcal{A} \implies \overline{\mu}(A \cup B) = \mu(A \cup B),\ (\textit{by definition}) \\ 0 = \overline{\mu}(C) \leq \overline{\mu}(A \cup B) = \mu(A \cup B) \leq \mu(A) + \mu(B) = 0 \\ \mu(A) = 0 : \overline{\mu}(C) = \overline{\mu}(A \cup N) = \mu(A) \implies \overline{\mu}(C) = \mu(A) = 0 \end{aligned}$$

This means that $M \in \mathcal{N} \subseteq \overline{\mathcal{A}}$, hence the new measure $\overline{\mu}$ encaptures all null sets.

The main takeaway from this theorem is that we now have a way of turning arbitrary measure spaces $(X, \mathcal{A}, \mu)$ into complete measure spaces $(X, \overline{\mathcal{A}}, \overline{\mu})$.

Example 6 (Measure space that is not complete) *Let $X = \{0, 1, 2\}$ and $\mathcal{A} = \{\emptyset, \{0, 1\}, \{2\}, X\}$. We have the measure: $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ defined by: $\mu(\emptyset) = \mu(\{0, 1\}) = 0, \mu(\{2\}) = \mu(X) = 1$. We start by showing that $\mathcal{A}$ is a sigma-algebra:*

$$\begin{aligned} \emptyset \in \mathcal{A} : \textit{holds by defintion} \\ A \in \mathcal{A} \implies A^C \in \mathcal{A} : \\ A_1 = \emptyset \implies A_1^C = X \in \mathcal{A} \\ A_2 = \{0, 1\} \implies A_2^C = \{2\} \in \mathcal{A} \\ A_3 = \{2\} \implies A_3^C = \{0, 1\} \in \mathcal{A} \\ \{A_n\}_{n\in\mathbb{N}} \in \mathcal{A} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A} : \\ \bigcup_{n=1}^{4} A_n = X \in \mathcal{A} \end{aligned}$$

Thus $\mathcal{A}$ is a sigma-algebra. We also get that μ is a measure. But why is then the measure space $(X, \mathcal{A}, \mu)$ not complete? Notice that $\mu(\{0,1\}) = 0$ so if we call $B = \{0,1\}$ we have $\mu(B) = 0$, but here $N_1 = \{0\} \in \mathcal{N}$ and $N_2 = \{1\} \in \mathcal{N}$ and $N_1, N_2 \notin \mathcal{A}$, thus $(X, \mathcal{A}, \mu)$ is not complete. So how can we transform this into a complete measure space? By definition $\overline{\mathcal{A}} = \{A \cup N : A \in \mathcal{A}$ and $N \in \mathcal{N}\}$, here are some of the elements:

$$\emptyset \cup \{0\} = \{0\},\ \emptyset \cup \{1\} = \{1\}$$
$$\{2\} \cup \{0\} = \{0,2\},\ \{2\} \cup \{1\} = \{1,2\}$$

So we end up with:

$$\mathcal{A} = \{\emptyset, \{0,1\}, \{2\}, \{0,1,2\}\}$$
$$\overline{\mathcal{A}} = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{2\}, \{2,0\}, \{2,1\}, \{0,1,2\}\} = \mathcal{P}(X)$$

The requirements for the measure yields: $\overline{\mu}(A \cup N) = \mu(A)$ and we have $\mathcal{N} = \{N_1, N_2\} = \{\{0\}, \{1\}\}$, using the above:

$$\overline{\mu}(\{2,0\}) = \mu(\{2\}) \implies \mu(N_1) = \mu(\{0\}) = 0$$
$$\overline{\mu}(\{2,1\}) = \mu(\{2\}) \implies \mu(N_2) = \mu(\{1\}) = 0$$

Example 7 *Assume that $(X, \mathcal{A}, \mu)$ is a complete measure space, i.e $\mathcal{A} = \overline{\mathcal{A}}$, so $\mathcal{A}$ contains all the null sets. Further we let $A, B \in \mathcal{A}$ with $\mu(A) = \mu(B) < \infty$. If $A \subseteq C \subseteq B \implies C \in \mathcal{A}$. So why is this true? The idea is to see what figure 1 can give us:*

$$B = A \cup (B \setminus A)\ (disjoint)$$
$$\mu(B) = \mu(A) + \mu(B \setminus A) = \mu(A) \implies \mu(B \setminus A) = 0$$

We have that $C \setminus A \subseteq B \setminus A$ with $\mu(B \setminus A) = 0$, thus $C \setminus A$ is a null set, which means that $C \setminus A \in \mathcal{A}$. We also have that $C = A \cup C \setminus A \in \mathcal{A}$.

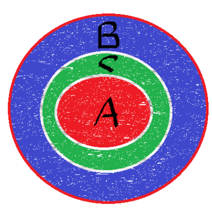


Figure 1: Included sets

When we defined $\overline{\mathcal{A}}$ we said that it was the smallest sigma-algebra containg $\mathcal{A}$ and $\mathcal{N}$, which may arise some questions with regards to sigma-algebras. For instance, we do have the following sigma algebras: $\mathcal{A}_0 = \{\emptyset, X\}$ and $\mathcal{P}(X)$, the problem with these are that in most practical cases the first one is to small, and the other one to big. So what if we want a more appropriate sigma-algebra, that just contains the "necessary" information?

Let's say we have some collection of information $\mathcal{B}$ and we want the smallest sigma-algebra containig $\mathcal{B}$, we will call this one the sigma-algebra generated by $\mathcal{B}$ and we denote it by $\sigma(\mathcal{B})$. And this is in fact the smallest sigma-algebra containg $\mathcal{B}$. Before we can interperet what $\sigma(\mathcal{B})$ means, we need a lemma:

Lemma 3 (Intersection of σ-algebras is a σ-algebra) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathcal{I}$ be a non-empty index set and let $\mathcal{F}_i$, $i \in \mathcal{I}$ be σ-algebras on X, then:*

$$\mathcal{F} = \bigcap_{i \in \mathcal{I}} \mathcal{F}_i = \{A \subseteq X : A \in \mathcal{F}_i,\ \forall i \in \mathcal{I}\}$$

is a σ-algebra on X

Proof 7 *We do have from the definition that: $\mathcal{F} = \bigcap_{i \in \mathcal{I}} \mathcal{F}_i = \{A \subseteq X : A \in \mathcal{F}_i,\ \forall i \in \mathcal{I}\}$, and since all $\mathcal{F}_i$ are sigma-algebras we have that $\emptyset \in \mathcal{F}_i,\ \forall i \in \mathcal{I}$, hence $\emptyset \in \mathcal{F}$.*
Assume that $A \in \mathcal{F}$, this means that we have $A \in \mathcal{F}_i,\ \forall i \in \mathcal{I}$, but then $A^C \in \mathcal{F}_i,\ \forall i \in \mathcal{I}$, hence $A^C \in \mathcal{F}$.
Again: assume that $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}$, which means that: $\{A_n\}_{n \in \mathbb{N}} \in \mathcal{F}_i,\ \forall i \in \mathcal{I}$, and thus $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}_i, \forall i \in \mathcal{I}$, which means that: $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{F}$

Definition 5 (generated sigma-algebra) *Let X be a non-empty set, and let $\mathcal{B}$ be a collection of subsets of X. Then the σ-algebra on X generated by $\mathcal{B}$ $\sigma(\mathcal{B})$ is defined to be the intersection of all σ-algebras $\mathcal{F}$ on X such that $\mathcal{B} \subseteq \mathcal{F}$, i.e:*

$$\sigma(\mathcal{B}) = \bigcap \{\mathcal{F} \subseteq \mathcal{P}(X) : \mathcal{F} \text{ is a } \sigma - \text{algebra on } X \text{ and } \mathcal{B} \subseteq \mathcal{F}\}$$

And this is the smallest σ-algebra on X containing $\mathcal{B}$.

So why is this the smallest σ-algebra containing $\mathcal{B}$? Well we have that we have a bunch of σ-algebras $\mathcal{F}$ on X containing $\mathcal{B}$, so these are in fact all the possible σ-algebras on X which contains $\mathcal{B}$, so some of these will be bigger and some will be smaller, but if we take the intersection, we will actually obtain the smallest one containig $\mathcal{B}$.

Example 8 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two collections of subsets of X, such that $\mathcal{A} \subseteq \sigma(\mathcal{B})$ and $\mathcal{B} \subseteq \sigma(\mathcal{A})$, then $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.*
Let $A \in \mathcal{A}$, so this means that $A \in \sigma(\mathcal{A}),\ \forall A \in \mathcal{A}$ but by assumption we have that $A \in \sigma(\mathcal{B}),\ \forall A \in \mathcal{A}$. Thus $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{B})$. Let $B \in \mathcal{B}$, so we then have: $B \in \sigma(\mathcal{B})$ and also again by assumption $B \in \sigma(\mathcal{A})$, thus $\sigma(\mathcal{B}) \subseteq \sigma(\mathcal{A})$, hence $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.

Example 9 *Let X be a metric space, let's say $X = \mathbb{R}$, and let $\mathcal{G}$ be the collection of all open sets, i.e $\mathcal{G} = \{(a, b) : a \leq b \wedge a, b \in \mathbb{R}\}$ and let $\mathcal{F}$ be the collection of all closed subsets of X, i.e:*
$\mathcal{F} = \{[a, b] : a \leq b \wedge a, b \in \mathbb{R}\}$, then $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$.
Let G_1 be an open set i.e $G_1 \in \mathcal{G}$ so that $G_1 \in \sigma(\mathcal{G})$. But since $\sigma(\mathcal{G})$ is a sigma-algebra we must have that $G_1^C \in \sigma(\mathcal{G})$, $G_1^C \in \mathcal{F}$, thus $\mathcal{F} \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$. Now let $F_1 \in \mathcal{F}$, so that $F_1 \in \sigma(\mathcal{F})$, but then $F_1^C \in \sigma(\mathcal{F})$. Again we have: F_1^C open which means that: $F_1^C \in \mathcal{G}$, thus $\mathcal{G} \subseteq \sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G})$, hence $\sigma(\mathcal{G}) = \sigma(\mathcal{F})$.

One important feature of generated sigma-algebra's is the Borel-sigma-algebra. Let's assume that X is a metric space e.g $X = \mathbb{R}$, and let $\mathcal{B}$ be the collection of all open sets.
Then $\sigma(\mathcal{B})$ is called the Borel-σ-algebra, and this is the smallest σ-algebra containg all open sets. The sets in $\sigma(\mathcal{B})$ are called Borel-sets. And any measure defined on $\sigma(\mathcal{B})$ is called a Borel-measure. We will later learn that there exists an unique complete measure μ on $\sigma(\mathcal{B})$ such that $\mu([a, b]) = b - a$ for all $a < b$. this is called the Lesbegue-measure. In general there is no guarantee for the Borle-measure being complete, but the Lesbegue is.

1.3 Measurable functions

Our aim is to define what $\int f d\mu$ means, but such integrals relies heavely on sequences of so called simple functions f_n. We have that the limit of these: $\lim_{n\to\infty} f_n(x)$ could be $\pm\infty$, we would therefore like to work with functions $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Before we begin the study of measurable functions it's good to recall some notions from set-theory.

Definition 6 (inverse image of B under f) *Let X, Y be two non-empty sets, and let $f : X \to Y$ with $B \subseteq Y$, we then define the inverse image of B as:*

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Proposition 5 *Let $\mathcal{B}$ be a family/collection of subsets of Y, then for all functions $f : X \to Y$ we have:*

- $f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B)$
- $f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$

Proposition 6 (inverse image of complement) *Let $f : X \to Y$, and let $D \subseteq Y$, then:*

$$f^{-1}(D^C) = (f^{-1}(D))^C$$

Proposition 7 *Let $f : X \to Y$ and let $g : X \to Y$, further let $S \subseteq Y$, then:*

$$(g \circ f)^{-1}(S) = f^{-1}(g^{-1}(S))$$

Proof 8 *We start by the following:*

$$
\begin{aligned}
x &\in (g \circ f)^{-1}(S) \\
(g \circ f)(x) &\in S \\
f(x) &\in g^{-1}(x) \\
x &\in f^{-1}(g^{-1}(S))
\end{aligned}
$$

Definition 7 (Continuity and open sets) *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$, then the following are equivalent:*

- *f is continuous*
- *if $V \subseteq Y$ is open, then $f^{-1}(V)$ is open.*
- *if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed.*

Definition 8 (measurable function) *Assume that $(X, \mathcal{A}, \mu)$ is a measure space. A function $f : X \to \overline{\mathbb{R}}$ is called measurable if:*

$$
\begin{aligned}
\{x \in X : f(x) < r\} &\in \mathcal{A},\ \forall r \in \mathbb{R} \\
&\Updownarrow \\
\{x \in X : f(x) \in [-\infty, r)\} &\in \mathcal{A} \\
&\Updownarrow \\
f^{-1}([-\infty, r)] &\in \mathcal{A}
\end{aligned}
$$

We also have a really useful proposition which tells us that if we want to check measurability, then we have more then one way to do so:

Proposition 8 (equivalent definitions of measurable functions) *The following are equivivallent:*

i) $\{x : f(x) < r\} \in \mathcal{A}$ *for all* $r \in \mathbb{R}$

ii) $\{x : f(x) \leq r\} \in \mathcal{A}$ *for all* $r \in \mathbb{R}$

iii) $\{x : f(x) \geq r\} \in \mathcal{A}$ *for all* $r \in \mathbb{R}$

iv) $\{x : f(x) > r\} \in \mathcal{A}$ *for all* $r \in \mathbb{R}$

Proof 9 *$i) \implies ii)$:*
Assume that $\{x : f(x) < r\} \in \mathcal{A}$, we can rewrite $ii)$ as: $\{x : f(x) \leq r\} = \bigcap_{n \in \mathbb{N}} \{x : f(x) < r + 1/n\} = \bigcap_{n \in \mathbb{N}} A_n$, but since $A_n \in \mathcal{A}$ we have that $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{A}$, hence $i) \implies ii)$.
$ii) \implies i)$:
Assume that $\{x : f(x) \leq r\} \in \mathcal{A}$, we then get:
$\{x : f(x) < r\} = \bigcup_{n \in \mathbb{N}} \{x : f(x) \leq r - 1/n\} \in \mathcal{A}$.
$i) \iff iii)$:

Assume that i) holds, and let: $\{x : f(x) < r\} = A$*, so that* $A \in \mathcal{A}$*. But then:*
$\{x : f(x) \geq r\} = A^C \in \mathcal{A}$*, this means that:*
$\{x : f(x) < r\} = \{x : f(x) \geq r\}^C$*, i.e:* $A = (A^C)^C$
ii) $\iff$ *iv):*
Let $A = \{x : f(x) \leq r\} \in \mathcal{A}$*, but then agian:*
$\{x : f(x) \leq r\} = \{x : f(x) > r\}^C$ *so* $A = (A^C)^C$

Since we now know the useful equivalent definition of measurable functions, then we can also understand the following proposition:

Proposition 9 *Assume that* $f : X \to \overline{\mathbb{R}}$ *is measurable, then* $f^{-1}(I) \in \mathcal{A}$ *for all intervals* I*, e.g:* $I = (a, b], I = [a, b), I = (a, b), I = [a, b]$

Proof 10 *This is actually just a consequence of proposition 8: let's choose* $I = (a, b)$*, we must then show* $f^{-1}((a, b)) \in \mathcal{A}$*:*

$$\begin{aligned} f^{-1}((a,b)) &= \{x \in X : f(x) \in (a,b)\} \\ &= \{x \in X : f(x) > a\} \cap \{x \in X : f(x) < b\} \\ &= A_1 \cap A_2 \in \mathcal{A} \end{aligned}$$

Proposition 10 *Any open set* $G \subseteq \mathbb{R}$ *is a countable union of open intervals.*

Proof 11 *Let* $\mathcal{I} = \{(a, b) : a, b \in \mathbb{Q}, a < b\}$*, so* $\mathcal{I}$ *is a collection of all open rational intervals.* $\mathcal{I}$ *is countable as* $\mathbb{Q}$ *is countable. Let* $\mathcal{I}_G = \{(a, b) \in \mathcal{I} : (a, b) \subseteq G\}$*, we have that* $\mathcal{I}_G \subseteq \mathcal{I}$*, hence* $\mathcal{I}_G$ *is countable. We claim that* $G = \bigcup_{(a,b)\in\mathcal{I}_G}(a, b)$*.*
By definition we have that $\bigcup_{(a,b)\in\mathcal{I}_G}(a, b) \subseteq G$*. As this is by definition the collection of all* (a, b) *such that* $(a, b) \subseteq G$*.*
Now we need to show $G \subseteq \bigcup_{(a,b)\in\mathcal{I}_G}(a, b)$*.*
We have that G *is open, i.e* $\exists\, \epsilon > 0$ *s.t* $B_\epsilon(x) \subseteq G$ *which means that* $(x - \epsilon, x + \epsilon) \subseteq G$*. Since* $\mathbb{Q}$ *is dence we have that for* $a, b \in \mathbb{Q}$ *that* $(a, b) \in \mathcal{I}_G \subseteq B_\epsilon(x)$*, thus* $x \in \bigcup_{(a,b)\in\mathcal{I}_G}(a, b)$*, thus* $G \subseteq \bigcup_{(a,b)\in\mathcal{I}_G}(a, b)$*.*
Finally: $G = \bigcup_{(a,b)\in\mathcal{I}_G}(a, b)$

Example 10 *We have that* $f^{-1}(\{\infty\}) \in \mathcal{A}$ *,*$f^{-1}(\{-\infty\}) \in \mathcal{A}$ *and that* $f^{-1}(\{-\infty, \infty\}) \in \mathcal{A}$

$$\begin{aligned} f^{-1}(\{\infty\}) &= \{x \in X : f(x) = \infty\} \\ &= \bigcap_{n\in\mathbb{N}} \{x \in X : f(x) > n\} \in \mathcal{A} \\ f^{-1}(\{-\infty\}) &= \{x \in X : f(x) = -\infty\} \\ &= \bigcup_{n\in\mathbb{N}} \{x \in X : f(x) < -n\} \in \mathcal{A} \\ f^{-1}(\{-\infty, \infty\}) &= f^{-1}((-\infty, \infty)^C) = [f^{-1}((-\infty, \infty)]^C \in \mathcal{A} \end{aligned}$$

Proposition 11 *Assume that $f : X \to \overline{\mathbb{R}}$ is measurable. If $B \subseteq \mathbb{R}$ is open or closed, then $f^{-1}(B) \in \mathcal{A}$.*

Proof 12 *Assume that B is open, then by proposition 10, we have that B is a countable union of open sets, i.e: $B = \bigcup_{n\in\mathbb{N}} I_n$:*

$$f^{-1}(B) = f^{-1}(\bigcup_{n\in\mathbb{N}} I_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(I_n) \in \mathcal{A}$$

Assume now that B is closed, we will here use the fact that : A open $\iff$ A^C closed. But we must be a bit careful as we are working on $\overline{\mathbb{R}}$, we have that $B = \mathbb{R} \setminus G$, with G open. But accounting for $\overline{\mathbb{R}}$ we get: $B = \mathbb{R} \setminus G = \overline{\mathbb{R}} \setminus (G \cup \{-\infty, \infty\}) = (G \cup \{-\infty, \infty\})^C$, this gives us:

$$\begin{aligned} f^{-1}(B) &= f^{-1}[(G \cup \{-\infty, \infty\})^C] = \left(f^{-1}[(G \cup \{-\infty, \infty\})]\right)^C \\ &= \left(f^{-1}(G) \cup f^{-1}(\{-\infty, \infty\})\right)^C \in \mathcal{A} \end{aligned}$$

We have that $f^{-1}(G)$ is open, thus $f^{-1}(G) \in \mathcal{A}$, from example 10, we have that $f^{-1}(\{-\infty, \infty\}) \in \mathcal{A}$, thus we get the above inclusion.

Example 11 *Assume that $f : X \to \mathbb{R}$ is measurable, then $f^{-1}(B) \in \mathcal{A}$ for all Borel sets $B \in \mathcal{B}$. To see why: let $\mathcal{T} = \{S \subseteq \mathbb{R} : f^{-1}(S) \in \mathcal{A}\}$, this is a σ-algebra. $\mathcal{B}$ is the smallest sigma-algebra generated by open sets, thus by definition $\mathcal{B} \subseteq \mathcal{T}$. We must therefore have that $f^{-1}(B) \in \mathcal{A}$.*

Proposition 12 *Assume that $f : X \to \mathbb{R}$ is measurable, and that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, then: $\phi \circ f$ is measurable.*

Proof 13 *We have the following:*

$$\begin{aligned} \{x : (\phi \circ f)(x) < r\} &= (\phi \circ f)^{-1}((-\infty, r)) \\ &= f^{-1}[\phi^{-1}(-\infty, r)] \end{aligned}$$

If we let $V = (-\infty, r)$, we have that $V \subseteq \mathbb{R}$ and that V is open, we also have that ϕ is continuous, thus by definition 7, we get that $\phi^{-1}(V)$ is open. And by proposition 11, that $f^{-1}(\phi^{-1}(V)) \in \mathcal{A}$

Theorem 2 (sums and products of measurable functions) *Assume that $f, g : X \to \mathbb{R}$ are measurable functions, then:*

i) $f + g$ *is measurable.*

ii) $f - g$ *is measurable.*

iii) fg *is measurable.*

Proof 14 *We will start bu proving i):*

$$\{x : f(x) + g(x) < r\} = \{x : f(x) < r - g(x)\}$$

We have that $\mathbb{Q}$ is dense. This means that $\exists\, q \in \mathbb{Q}$ such that:
$f(x) < q < r - g(x)$ thus:

$$\{x : f(x) < r - g(x)\} = \bigcup_{q\in\mathbb{Q}} [\{x : f(x) < q\} \cap \{q < r - g(x)\}] \in \mathcal{A}$$

Here: $\{q < r - g(x)\} = \{g(x) < r - q\} \in \mathcal{A}$, hence the inclusion.
For part ii), we observe that $f(x) - g(x) = f(x) + (-g(x))$, hence we only need to show that $-g(x)$ is measurable, but this holds by definition.
Now for part iii), we want to do something clever, namely use our previous propositions etc:

$$fg = 1/2[(f + g)^2 - f^2 - g^2]$$

By proposition 12, we have that the composition is measurable for two measurable functions, hence $f^2 \in \mathcal{A}$, $g^2 \in \mathcal{A}$ and by i) $(f + g) \in \mathcal{A} \implies (f + g)^2 \in \mathcal{A}$, thus $fg \in \mathcal{A}$. A constant times a measurable function is also measurable, this holds by definition.

Example 12 (Finite sums and product of measurable functions are measurable) *Let $f_1, \dots f_n$ be measurable functions, we then have that:*
$(f_1 + f_2 + \dots + f_n)$ is measurable, and that $\prod_{i=1}^n f_i$ is measurable.

$$(f_1 + f_2 + \dots + f_{n-1} + f_n) = (f_1 + f_2) + (f_2 + f_3) + \dots + (f_{n-1} + f_n)$$

By theorem 2, we have that $(f_1 + f_2)$ is measurable, as well as all the other sums of two, hence the entire sum is measurable.
What about the product, we have again from the same theorem that f, g measurable $\implies fg$ measurable.

$$\begin{aligned} f_{n-1}f_n &= g_{n-1,n} \text{ (measurabe by theorem 2)} \\ f_{n-2}g_{n,n-1} &= g_{n-2,n} \text{ (measurabe by theorem2)} \\ &\vdots \\ f_1 g_{2,n} &= \prod_{i=1}^n f_i \end{aligned}$$

We see that all of the above are products of measurable functions, hence we end up with the entire product measurable.

Example 13 *Let $(X, \mathcal{A}, \mu)$ be a measure space, we have that the indicator function $\mathbf{1}_A(x)$ is measurable $\iff A \in \mathcal{A}$. We also get that $f(x) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(x)$ is measurable.*

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^C \end{cases}$$

We start by recalling the definition of measurability of functions:
$\mathbf{1}_A^{-1}((-\infty, r)) = \{x : \mathbf{1}_A(x) < r\}$, *we have that* $\mathbf{1}_A(x) : X \to \{0, 1\}$ *so for all* $x \in X$ *we they will either get mapped to one or zero.*

$$\begin{aligned} r > 1 : \{x : \mathbf{1}_A(x) < r\} &= X \in \mathcal{A} \\ r \leq 0 : \{x : \mathbf{1}_A(x) < r\} &= \emptyset \in \mathcal{A} \\ 0 < r \leq 1 : \{x : \mathbf{1}_A(x) < r\} &= A^C \in \mathcal{A} \end{aligned}$$

For part two, we have: $\sum_{i=1}^n a_i \mathbf{1}_{A_i}(x)$, *we have that* f *measurable* $\implies$ cf *measurable and by example 12 we get that the finite sum of measurable functions is measurable.*

Definition 9 (function being finite almost everywhere) *A function* $f : X \to \overline{\mathbb{R}}$ *is said to be finite almost everywhere if:*
$\{x : f(x) = \pm\infty\}$ *has measure zero, i.e:*

$$\mu(\{x : f(x) = \pm\infty\}) = 0$$

Definition 10 (functions being equal almost everywhere) *The measurable functions* $f, \widetilde{f} : X \to \overline{\mathbb{R}}$ *are said to be equal almost everywhere if* $\{x : f(x) \neq \widetilde{f}(x)\}$ *has measure zero, i.e:*

$$\mu(\{x : f(x) \neq \widetilde{f}(x)\}) = 0$$

Result 1 *If* $f : X \to \overline{\mathbb{R}}$ *is finite almost everywhere, then there exists a function* $\widetilde{f}(x) : X \to \mathbb{R}$ *such that* f *and* $\widetilde{f}$ *are equal almost everywhere, we just set:*

$$\widetilde{f}(x) = \begin{cases} f(x) & \textit{if } f(x) \in \mathbb{R} \\ 0 & \textit{if } f(x) \in \{-\infty, \infty\} \end{cases}$$

HERE I WANT TO INCLUDE STUFF ABOUT LIMINF, AND LIMSUP, AND A THEOREM INVOLVING THESE, BUT FOR NOW: KINDA BORING.

1.4 Integration of simple functions

We are now ready to look at $\int f d\mu$ shall mean, at least for simple functions f. We will also see that this notion of integration, will with the right assumptions lead to when we can use $\lim_{n\to\infty} \int f d\mu = \int \lim_{n\to\infty} f d\mu$.

But first we need some notion about non-negative simple functions.
In ordinary integration, we deal with upper and lower approximations of the integral, the idea here is that we represent these lower approximations by functions, and let $\mu(A_i)$ represent the area/volume on the particular set.

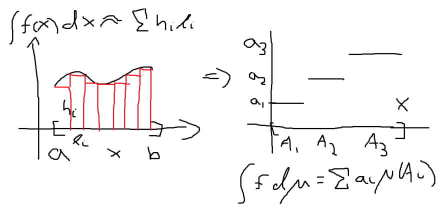


Figure 2: Analogous integration

Definition 11 (non-negative simple function on standard form) *Let $a_1, \dots a_n$ be distinct taking values in $[0, \infty)$, further let $A_i = \{x : f(x) = a_i\}$ be measurable sets, forming a partition of X, i.e: $\bigcup_{i=1}^{n} A_i = X$. Then we define the simple function $f : X \to \mathbb{R}$ by:*

$$f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$

Why are we working with non-negative simple functions? The reason is that in measure theory we have the convention $0 * \infty = 0$, since we allow the measure to take on ∞ volume, however notions of $\infty - \infty$ are still undefined.

So by letting f be a non-negative simple function, we don't get into that trouble as the a_i's are positive. One question that arises then is: but what about functions taking negative values? We solve this by decomposing the function in a smart way, so that we can integrate negative functions as well.

We also have that the requirement of the a_i's being distinct, is quite a strict requirement, and it turns out, that this does not need to be the case.

Definition 12 (Integral of non-negative simple function on standard form) *Assume that $f : X \to \mathbb{R}$ is a non-negative simple function on standard form, $f(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$, we then define:*

$$\int f d\mu = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}(x)$$

Lemma 4 *Let $b_1, \dots b_m$ be non-negative numbers, not necessarily distinct, where the B_j's are disjoint and form a partition of $X = \bigcup_{j=1}^{m} B_j$, we then get the integral of $g(x) = \sum_{j=1}^{m} b_j \mathbf{1}_{B_j}(x)$ is defined as:*

$$\int g d\mu = \sum_{j=1}^{m} b_j \mu(B_j)$$

Proof 15 *Let $a_1, \dots a_n$ be the distinct values of $b_1, \dots b_m$. We then get:*

$$\sum_{j=1}^{m} b_j \mu(B_j) = a_1[\mu(B_{1_1}) + \dots \mu(B_{1_{n_1}})]$$
$$+ a_2[\mu(B_{2_1}) + \dots \mu(B_{2_{n_2}})]$$
$$\vdots$$
$$+ a_n[\mu(B_{n_1}) + \dots \mu(B_{2_{n_n}})]$$

Here: $A_i = B_{i_1} \cup B_{i_2} \cup \dots \cup B_{i_{n_i}}$, here the B_{i_j}'s are disjoint, so that:

$$\sum_{j=1}^{m} b_j \mu(B_j) = \sum_{i=1}^{n} a_i \mu(A_i)$$

Proposition 13 (properties of the integral) *Let f, g be two non-negative simple functions, and let $c \in [0, \infty)$, then:*

i) $\int cf d\mu = c \int f d\mu$

ii) $\int (f + g) d\mu = \int f d\mu + \int g d\mu$

Until now, we have only seen how integrals over the entire space looks like, i. e: $\int f d\mu = \int_X f d\mu$, what about only parts of X, let's say we are interested in the region $B \in \mathcal{A}$, so we want $\int_B f d\mu$. We then get:

$$\int_B f d\mu = \int \mathbf{1}_B f d\mu$$

Here we have: $\mathbf{1}_B f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i} \mathbf{1}_B$, using the fact that: $\mathbf{1}_A \mathbf{1}_B = \mathbf{1}_{A \cap B}$, we get that: $\mathbf{1}_B f = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i \cap B}$

Proposition 14 *Assume that f, g are two non-negative simple functions, if $g \leq f$, then:*

$$\int g d\mu \leq \int f d\mu$$

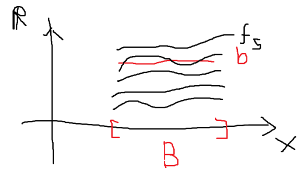
Proof 16 *Since $g \leq f$, we will have that $f - g$ is non-negative, and using proposition 13, we get:*

$$\int f d\mu = \int (g + (f - g)) d\mu = \int g d\mu + \int (f - g) d\mu$$
$$\geq \int g d\mu$$

Lemma 5 *Let B be a measurable set, and let $b \in \mathbb{R}_+$. Assume that $\{f_n\}$ is an increasing sequence of non-negative simple functions sucht that:* $\lim_{n \to \infty} f_n(x) \geq b$, $\forall x \in B$, *then:*

$$\lim_{n \to \infty} \int_B f_n(x) d\mu \geq b\mu(B)$$

This lemma is easiest understood with the help of figures, in the belowe figure, we have graphed the situation.



The idea here is that $\{f_n\}$ is an increasing sequence, so eventually we will get past our limit b, in the figure, we have that f_5 is past this limit, and thus $\lim_{n\to\infty} f_n(x) \geq b$, $\forall x \in B$.

Proof 17 *Let $a \in \mathbb{R}_+$ be any number less than b, and let $A_n = \{x \in B : f_n(x) \geq a\}$, since $a < b$ and $\lim_{n\to\infty} f_n(x) \geq b$, we have: $B = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n \subseteq A_{n+1}$, thus $\{A_n\}_{n\in\mathbb{N}}$ is increasing. This allows us to use continuity of measure: $\mu(B) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$ If $m < \mu(B)$, then: $\exists N \in \mathbb{N}$ s.t: $\mu(A_n) \geq m$, $\forall n \geq N$ (Here we use the fact that $\{A_n\}$ is increasing, thus $\mu(A_n) \leq \mu(A_{n+1})$, $\forall n \in \mathbb{N}$) And for such n, we get: $\int_B f_n(x)d\mu \geq am$ why?*

$$\int_B f_n(x)d\mu \geq \int_{A_n} f_n(x)d\mu \geq \int_{A_n} a d\mu = a\mu(A_n) \geq am$$
$$\int_B f_n(x)d\mu \geq am$$

But this holds for arbitrary $a < b$ and $m < \mu(B)$, hence we must have:

$$\lim_{n\to\infty} \int_B f_n(x)d\mu \geq b\mu(B)$$

Proposition 15 *Assume that g is a non-negative simple function and let $\{f_n\}$ be an increasing sequence of non-negative simple functions such that: $\lim_{n\to\infty} f_n(x) \geq g(x)$, $\forall x \in X$, then:*

$$\lim_{n\to\infty} \int f_n d\mu \geq \int g(x)d\mu$$

Proof 18 *We have that g is a simple function, and let us represent it on standard form: $g(x) = \sum_{i=1}^m b_i \mathbf{1}_{B_i}(x)$. This means that $\bigcup_{i=1}^m B_i = X$ (a partition)*

$$\int f_n d\mu = \int \mathbf{1}_X f_n d\mu = \int \mathbf{1}_{\bigcup_{i=1}^m B_i} f_n d\mu = \int \sum_{i=1}^m \mathbf{1}_{B_i} f_n d\mu = \sum_{i=1}^m \int \mathbf{1}_{B_i} f_n d\mu$$
$$= \sum_{i=1}^m \int_{B_i} f_n d\mu$$

And form lemma 5, we have:

$$\lim_{n\to\infty}\int_{B_i} f_n(x)d\mu \geq b_i\mu(B_i)$$

Using this property we get:

$$\lim_{n\to\infty}\sum_{i=1}^{m}\int_{B_i} f_n d\mu = \sum_{i=1}^{m}\lim_{n\to\infty}\int_{B_i} f_n d\mu \geq \sum_{i=1}^{m} b_i\mu(B_i) = \int g d\mu$$

Hence:

$$\lim_{n\to\infty}\int f_n d\mu \geq \int g d\mu$$

Example 14 *let f be a non-negative simple function on $(X,\mathcal{A},\mu)$, then:*

$$\nu(B) = \int_B f d\mu$$

is a measure on $(X,\mathcal{A})$
Before we begine we notice the following properties about indicator functions: $\mathbf{1}_{\emptyset} = 0$, $\forall x \in X$ *as well as for A, B disjoint:* $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$.
Here we also let $\{A_n\}_{n\in\mathbb{N}}$ represent a disjoint sequence

$$\begin{aligned}
\nu(\emptyset) &= 0 : \\
\nu(\emptyset) &= \int_{\emptyset} f d\mu = \int \mathbf{1}_{\emptyset} f d\mu = 0 \\
\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \sum_{n\in\mathbb{N}} \nu(A_n) : \\
\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \int_{\bigcup_{n\in\mathbb{N}} A_n} f d\mu = \int \mathbf{1}_{\{\bigcup_{n\in\mathbb{N}} A_n\}} f d\mu \\
&= \int (\mathbf{1}_{A_1} f + \mathbf{1}_{A_2} f + \dots) d\mu \\
&= \int \mathbf{1}_{A_1} f d\mu + \int \mathbf{1}_{A_2} f d\mu + \dots \\
&= \int_{A_1} f d\mu + \int_{A_2} f d\mu + \dots \\
&= \sum_{n\in\mathbb{N}} \int_{A_n} f d\mu = \sum_{n\in\mathbb{N}} \nu(A_n)
\end{aligned}$$

1.5 Integrals of non-negative functions

We have spoken about how to integrate simple functions, and how we then interpret $\int f d\mu$, but how about $\int f d\mu$ for f measurable?

Definition 13 (Integral of measurable function) *Assume that $f : X \to \overline{\mathbb{R}}_+$ is measurable, then we define:*

$$\int f d\mu = \sup \left\{ \int g d\mu : \ g \text{ is a non-negative simple function, } g \leq f \right\}$$

This means that the integral of a measurable non-negative function f is just the integral of the largest possible non-negative simple function g. Fortunately there are many theorems and propositions about how to actually calculate this integral, as it's rather theoretical.

Proposition 16 *Assume that f is a non-negative measurable function, and that $\{f_n\}$ is an increasing sequence of non-negative simple functions converging to f, then:*

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

Proof 19 *We have that f_n is a non-negative simple function, with:*
$f_n \leq f$, $\forall n \in \mathbb{N}$, then: $\int f_n d\mu \leq \int f d\mu$. As $\{f_n\}$ is increasing, so is $\{\int f_n d\mu\}$, so $\lim_{n\to\infty} f_n$ exists (could be ∞). Thus $\lim_{n\to\infty} \int f_n d\mu \leq \int f d\mu$
Now for the other direction: let $g \leq f$ be a non-negative simple function. Then: $\lim_{n\to\infty} f_n \geq g$, we have that g is any function less than or equal to f, thus choose $g = f$, hence:
$\lim_{n\to\infty} \int f d\mu \geq \int f d\mu$

We now get a really useful proposition, which makes our life integrating measurable functions easier:

Proposition 17 *Assume that $f : X \to \overline{\mathbb{R}}_+$ is measurable, then there is an increasing sequence $\{f_n\}$ converging pointwise to f. Furthermore: we have that:*

$$f(x) < 2^n :$$
$$f(x) - 2^{-n} < f_n(x) \leq f(x)$$
$$f(x) \geq 2^n :$$
$$f_n(x) = 2^n$$

Proof 20 *Let's partition the y-axis $[0, 2^n]$ into sub-intervals of length 2^{-n}, and consider the measurable sets:*

$$A_{n,k} = \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{(k+1)}{2^n} \right\}$$
$$A_n = \{x \in X : f(x) \geq 2^n\}$$

We can then define:

$$f_n(x) = \sum_{k=0}^{4^n-1} \frac{k}{2^n}\mathbf{1}_{A_{n,k}}(x) + 2^n\mathbf{1}_{A_n}(x)$$

$f_n(x)$ *is by construction a simple function, as it is constants* a_i *times an indicator function.*
Now, assume that $f(x) < 2^n$*, then:* $f(x) - 2^{-n} < f_n(x) \leq f(X)$*, so that* $\lim_{n\to\infty} f_n(x) = f(x)$*. It get trapped, hence it must converge. If* $f(x) \geq 2^n$*, then* $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2^n = \infty = f(x)$*, thus* $f_n \to f$ *pointwise.* $\{f_n\}$ *is increasing, this is easiest understood by a drawing.*

Corollary 1 *Assume that* f *is a non-negative measurable function. Then there is an increasing sequence* $\{f_n\}$ *of non-negative simple functions converging pointwise to* f*, and for such a sequence we have:*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$$

Proposition 18 *Let* $f, g : X \to \overline{\mathbb{R}}_+$ *and let* $c \in [0, \infty)$*, then:*

i) $\int cf d\mu = c \int f d\mu$

ii) $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

iii) $g \leq f,\ \int g d\mu \leq \int f d\mu$

Theorem 3 (Monotone Convergence Theorem) *Assume that* $f : X \to \overline{\mathbb{R}}_+$ *is measurable, and assume that* $\{f_n\}$ *is an increasing sequence of non-negative* <u>*measurable*</u> *functions converging pointwise to* f *so that* $\lim_{n\to\infty} f_n(x) = f,\ \forall x \in X$*, then:*

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$$

Proof 21 *For each* n*, let* h_n *be the* n*-th simple function approximation to* f_n*, remember:* f_n *is measurable. This means that we have:*
$f_n(x) - 2^{-n} < h_n(x) \leq f_n(x)$*. But* $\lim_{n\to\infty} f_n(x) = f(x)$*, thus* $\lim_{n\to\infty} h_n(x) = f(x)$*.*
$\{h_n\}$ *is an increasing sequence of simple functions converging to* f*, thus by proposition 16 we have:* $\lim_{n\to\infty} \int h_n d\mu = \int f d\mu$*. Furthermore, since* $f_n \geq h_n,\ \forall n \in \mathbb{N}$*, we get:*

$$\lim_{n\to\infty} \int f_n d\mu \geq \lim_{n\to\infty} \int h_n d\mu = \int f d\mu$$

For the other inequality: we have that $f_n \leq f,\ \forall n \in \mathbb{N}$*, which means that* $\int f_n d\mu \leq \int f d\mu$ *and thus:*

$$\lim_{n\to\infty} \int f_n d\mu \leq \int f d\mu$$

Theorem 4 (Fatou's lemma) *Let $\{f_n\}$ be a sequence of non-negative measurable funtions, then:*

$$\liminf_{n\to\infty}\int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu$$

What is nice about Fato's lemma? It all boils down to the requirements of the non-negative measurable sequence $\{f_n\}$, since for once, we do not demand the sequence to be increasing, even thogh this is actually included in the lim inf, which we will see in the proof.

Proof 22 *Let $g_n = \inf_{k\geq n} f_k$, then $\{g_n\}$ is an increasing sequence of measurable functions. And by monotone convergence theorem(MCT), we have:*

$$\lim_{n\to\infty}\int g_n d\mu = \int \lim_{n\to\infty} g_n d\mu = \int \lim_{n\to\infty}\inf_{n\geq k} f_k d\mu = \int \liminf_{n\to\infty} f_n d\mu \tag{1}$$

On the other hand we have that $g_n \leq f_n$, $\forall n \in \mathbb{N}$, which means that $\int g_n d\mu \leq \int f_n d\mu$, but this means that: (Assuming that the limit exists, since then: $\lim = \liminf$*)*

$$\lim_{n\to\infty}\int g_n d\mu = \liminf_{n\to\infty}\int g_n d\mu \leq \liminf_{n\to\infty}\int f_n d\mu$$

But equation 1, tells us that: $\lim_{n\to\infty}\int g_n d\mu = \int \liminf_{n\to\infty} f_n d\mu$, thus:

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty}\int f_n d\mu$$
$$\Updownarrow$$
$$\liminf_{n\to\infty}\int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu$$

let's see some examples of how we can use this information:

Example 15 (Usage of MCT) *Let $\{u_n\}$ be a sequence of non-negative measurable functions, then we have:*

$$\sum_{n=1}^{\infty}\int u_n d\mu = \int \sum_{n=1}^{\infty} u_n d\mu$$

We have that either $\sum_{n=1}^{\infty} u_n < \infty$ or $\sum_{n=1}^{\infty} u_n = \infty$, either way, call this sum f, i.e: $\sum_{n=1}^{\infty} u_n(x) = f(x)$

$$f_N = \sum_{n=1}^{N} u_n \implies \lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} u_n(x)$$

We have that $\{f_N\}$ is a sequence of non-negative measurable functions, with $\lim_{N\to\infty} f_N(x) = f(x)$, this allows us to use MCT, so that:

$$\lim_{N\to\infty} \int f_N(x) d\mu = \int \lim_{N\to\infty} f_N(x) d\mu \tag{2}$$

$$\lim_{N\to\infty} \int f_N(x) d\mu = \lim_{N\to\infty} \int \sum_{n=1}^{N} u_n(x) d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int u_n(x) d\mu \tag{3}$$

$$= \sum_{n=1}^{\infty} \int u_n(x) d\mu \tag{4}$$

An by equation 2, we have:

$$\sum_{n=1}^{\infty} \int u_n(x) d\mu = \int \sum_{n=1}^{\infty} u_n(x) d\mu$$

1.6 Integrable functions

So far we have covered how to integrate $f : X \to [0, \infty]$, and in this section we will see how we deal with $f : X \to \overline{\mathbb{R}}$, so that we can integrate negative functions as well.

The idea is to split the integral into two positive measurable functions, $f = f_+ - f_-$, where we have:

$$f_+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad f_-(x) = \begin{cases} -f(x) & f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 14 (integrability of f) *We say that f is integrable, if it's measurable and f_+ and f_- are integrable, i.e:*

$$\int f_+ d\mu < \infty \;\; \textit{and} \;\; \int f_- d\mu < \infty$$

Definition 15 (Integral of f) *Assume that $f : X \to \overline{\mathbb{R}}$ is integrable, we then define the integral of f as:*

$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu$$

Lemma 6 (f integrable $\iff$ $|f|$ integrable) *f is integrable $\iff$ $|f|$ integrable*

Lemma 7 *Assume that $g, h : X \to [0, \infty]$ are integrable (which implies measurable) functions and that $f = g - h$, where this is defined. Then f is integrable, and:*

$$\int f d\mu = \int g d\mu - \int h d\mu$$

Proof 23 *Since g, h are integrable, we have that they are finite a.e, hence $f = g - h$ is defined for all most all $x \in X$. $B_g = \{x \in X : g(x) = \infty\}$ and $B_f = \{x \in X : f(x) = \infty\}$ both has $\mu(B_g), \mu(B_f) = 0$, so the integral over these sets are zero. Hence we can assume that $f = g - h$ is defined for all $x \in X$.*
We have that f is integrable since:

$$\int |f| d\mu = \int (|g - h|) d\mu \leq \int (|g| + |h|) d\mu$$
$$= \int |g| d\mu + \int |h| d\mu < \infty$$

Furthermore we have the following decomposition:

$$f = f_+ - f_- = g - h \implies f_+ + h = g + f_-$$
$$\int (f_+ + h) d\mu = \int (g + f_-) d\mu$$
$$\int f_+ d\mu + \int h d\mu = \int g d\mu + \int f_- d\mu$$
$$\int f d\mu = \int f_+ d\mu - \int f_- d\mu = \int g d\mu - \int h d\mu$$

Theorem 5 (properties of the integral) *Let $f, g : X \to \overline{\mathbb{R}}$, and let $c \in \mathbb{R}$, then:*

i) $\int cf d\mu = c \int f d\mu$

ii) $\int (f + g) d\mu = \int f d\mu + \int g d\mu$

iii) If $g \leq f$, then: $\int g d\mu \leq \int f d\mu$

From the previous section we defined MCT, which stated that for $\{f_n\}$ measurable, non-negative and increasing with $\lim_{n\to\infty} f_n(x) = f(x)$, then: $\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$. We would also like a theorem like this for f_n only measurable, i.e $f_n : X \to \overline{\mathbb{R}}$, and fortunately we have such a theorem for more general measurable functions, where we don't require $\{f_n\}$ increasing.

Theorem 6 (Lesbegue's Dominated Convergence Theorem) *Assume that $\{f_n\}$ is a sequence of measurable functions converging pointwise to f, i.e: $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in X$, furthermore assume that there is an integrable function $g : X \to \overline{\mathbb{R}}_+$ such that: $|f_n(x)| \leq g(x)$, $\forall n \in \mathbb{N}$, $\forall x \in X$, then:*

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$$

Proof 24 *Before we dig into the proof we need to recall some notions and make some observations. First: Fatou's lemma: $f_n \geq 0$, then:*

$\liminf \int f_n d\mu \geq \int \liminf f_n d\mu$.
Secondly: $\liminf(-a_n) = -\limsup(a_n)$, *this is true since:* $\inf(-S) = -\sup(S)$

$$S = \{1, 2, 3\} \implies \sup(S) = 3$$
$$-S = \{-1, -2, -3\} \implies \inf(-S) = -3$$
$$\inf(-S) = -\sup(S)$$

Using this we get that: $\liminf(c - a_n) = c - \limsup(a_n)$
Now we want to use Fatou's lemma, observe that $\{g + f_n\}$ *is a sequence of non-negative measurable functions. This is true, since* $g \geq |f_n|$ *for all* n, *hence it must be non-negative. Fatou gives us:*

$$\liminf \left[\int (g + f_n) d\mu\right] \geq \int \liminf(g + f_n) d\mu$$

Now working with RHS:

$$\int \liminf(g + f_n) d\mu = \int (g + f) d\mu = \int g d\mu + \int f d\mu$$

Working with LHS:

$$\liminf \left[\int (g + f_n) d\mu\right] = \liminf \left[\int g d\mu + \int f_n d\mu\right]$$
$$= \int g d\mu + \liminf \int f_n d\mu$$

LHS $\geq$ *RHS:*

$$\int g d\mu + \liminf \int f_n d\mu \geq \int g d\mu + \int f d\mu$$
$$\liminf \int f_n d\mu \geq \int f d\mu$$

$\{g - f_n\}$ *is a non-negative sequence of measurable functions, again use Fatou:*

$$\liminf \left[\int (g - f_n) d\mu\right] \geq \int \liminf(g - f_n) d\mu$$

RHS:

$$\int \liminf(g - f_n) d\mu = \int (g - f) d\mu = \int g d\mu - \int f d\mu$$

LHS:

$$\liminf \left[\int (g - f_n) d\mu\right] = \liminf \left[\int g d\mu - \int f_n d\mu\right]$$
$$= \int g d\mu + \liminf \left(-\int f_n d\mu\right)$$
$$= \int g d\mu - \limsup \left(\int f_n d\mu\right)$$

Finally, LHS $\geq$ RHS:

$$\int g d\mu - \limsup\left(\int f_n d\mu\right) \geq \int g d\mu - \int f d\mu$$
$$-\limsup\left(\int f_n d\mu\right) \geq -\int f d\mu$$
$$\Updownarrow$$
$$\limsup\left(\int f_n d\mu\right) \leq \int f d\mu$$

Summarizing our findings:

$$\limsup\left(\int f_n d\mu\right) \leq \int f d\mu \leq \liminf\left(\int f_n d\mu\right)$$

And this can only happen when:

$$\limsup\left(\int f_n d\mu\right) = \int f d\mu = \liminf\left(\int f_n d\mu\right)$$

Meaing that:

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$
$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$$

We also have some useful applications of DCT:

Theorem 7 *Let $f : \mathbb{R} \times X \to \mathbb{R}$ be a function which is:*

i) continuous in the first variable, i.e for each $y \in X$, the function $x \mapsto f(x, y)$ is continous.

ii) for each $x \in X$, the function $y \mapsto f(x, y)$ is measurable.

iii) Assume also that there is an integrable function $g : X \to \mathbb{R}_+$ such that: $|f(x, y)| \leq g(y)$, $\forall x, y \in X$, then:

$$h(x) = \int f(x, y) d\mu(y)$$

is continuous

Proposition 19 (Sequential continuity) *Let $f : X \to Y$ and let $x \in X$ be fixed, then the following are equivavlent:*

i) f is continuous at x

ii) if $\{x_n\}$ is any sequence in X converging to x, then:

$$\lim_{n\to\infty} f(x_n) = f(x)$$

Proof 25 *From proposition 19 it's suffices to show that h is continuous by showing that if $\{a_n\} \to a$, then $\lim_{n\to\infty} h(a_n) = h(a)$*

$$\lim_{n\to\infty} h(a_n) = \lim_{n\to\infty} \int f(a_n, y) d\mu(y)$$

Now let $k_n(y) = f(a_n, y)$, we have that $|k_n(y)| \leq g(y),\ \forall y \in Y$, which also means that $k_n(y) \leq g(y),\ \forall y \in Y$. Furthermore by the assumptions we have that $k_n(y)$ is measurable, for all $n \in \mathbb{N}$, thus $\{k_n\}$ is a sequence of measurable functions, which then allows us to use DCT.

$$\lim_{n\to\infty} \int f(a_n, y) d\mu(y) = \int \lim_{n\to\infty} f(a_n, y) d\mu(y) = \int f(a, y) d\mu(y)$$

Which means:

$$\lim_{n\to\infty} h(a_n) = h(a)$$

Thus h is continuous.

1.7 Comparison between Riemann integral and Lesbegue

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We can then consider it's Riemann-integral.

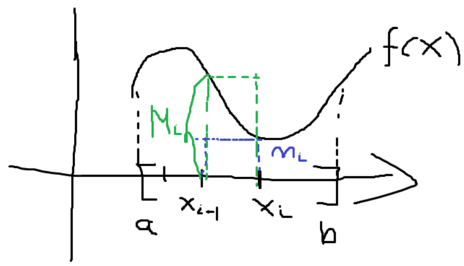


Figure 3: Riemann-integrarion

We first consider a partition Π of $[a, b]$ such that:
$\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$, further we define:
$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, we can now define the upper and lower sum as:

$$U(\Pi) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$L(\Pi) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

And then we can define the upper and lower integrals as:

$$\overline{\int_a^b} f(x)dx = \inf\{U(\Pi) : \Pi \text{ is a partition}\}$$
$$\underline{\int_a^b} f(x)dx = \sup\{L(\Pi) : \Pi \text{ is a partition}\}$$

Finally we say that f is Riemann-integrable if the upper and lower integral coincide, i.e f is Riemann-integrable if:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

Standard example of function not being Riemann-integrable, but Lebesgue-integrable:

Example 16 *Assume that f is defined by:*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \textit{otherwise} \end{cases}$$

Let's say we want to integrate over $[0,1]$, then this is not Riemann-integrable since:

$$\overline{\int_0^1} f(x)dx = 1 \neq \underline{\int_0^1} f(x)dx = 0$$

Hence f is not Riemann-integrable, but in some sence it would be logical that the integral is zero. Why is this logical? Here let $\mathbb{I}$ represent the irrational numbers, this means that: $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, and we have that $\mathbb{Q}$ is countable, and $\mathbb{R}$ uncountable, thus $\mathbb{I}$ has to be uncountable, meaing there are more irrational numbers than rational. Meaning that the integral should be zero.

If we deal with this in the Lebesgue sense: Let $f(x) = \mathbf{1}_{\hat{\mathbb{Q}}}(x)$,
here $\hat{\mathbb{Q}} = \mathbb{Q} \cap [0,1]$

$$\hat{\mathbb{Q}} = \bigcup_{i=1}^{\infty} \{q_i\} \implies \mu(\hat{\mathbb{Q}}) = \mu(\bigcup_{i=1}^{\infty} \{q_i\}) = \sum_{i=1}^{\infty} \mu(\{q_i\}) = 0$$
$$\int f d\mu = \int \mathbf{1}_{\hat{\mathbb{Q}}}(x) d\mu = \mu(\hat{\mathbb{Q}}) = 0$$

Theorem 8 (When Riemann-integral coincide with Lebesgue) *Assume that $f : [a,b] \to \mathbb{R}$ is a bounded, Riemann-integrable function on $[a,b]$. Then f is measurable and the Riemann and the Lebesgue integral coincide, i.e:*

$$\int_a^b f(x)dx = \int_{[a,b]} f d\mu$$

Proof 26 *HOPEFULLY I WILL FILL THIS OUT, BUT FOR NOW: NAH*

1.8 Different ways to converge

Assume that $\{f_n\}$ is a sequence of measurable functions, in which ways can these converge to a measurable function f?

1. Pointwise convergence:
$$\lim_{n\to\infty} f_n(x) = f(x),\ \forall x \in X$$
2. Almost everywhere convergence: There is a set $N \in \mathcal{A}$ with $\mu(N) = 0$, such that:
$$\lim_{n\to\infty} f_n(x) = f(x),\ x \notin N$$
3. Convergence in L^p:
$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0$$
4. Convergence in measure: $\forall\epsilon > 0$:
$$\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0$$

Proposition 20 *If $\{f_n\}$ converges in L^p, then $\{f_n\}$ converges in measure. Another way of formulation:*

$$f_n \xrightarrow{L^p} f \implies f_n \xrightarrow{\mu} f$$

Proof 27 *We know that:* $\lim_{n\to\infty} \int |f - f_n| d\mu = 0$, *we must show that this implies:* $\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0$

$$\begin{aligned}\epsilon^p \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) &= \int_{A_n} \epsilon^p d\mu \\ &\leq \int_{A_n} |f(x) - f_n(x)|^p d\mu \leq \int |f(x) - f_n(x)|^p d\mu \\ &\Updownarrow \\ \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) &\leq \frac{1}{\epsilon^p} \int |f(x) - f_n(x)|^p d\mu \\ &\Downarrow \\ \lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) &\leq \lim_{n\to\infty} \frac{1}{\epsilon^p} \int |f(x) - f_n(x)|^p d\mu = 0\end{aligned}$$

Which means that:

$$\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0$$

Proposition 21 *If $\{f_n\}$ is a sequence of measurable functions that converges to f in measure, then there is a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere, i.e:*

$$f_n \xrightarrow{\mu} f \implies f_{n_k} \xrightarrow{a.e} f$$

Proof 28 *By definition we have that:*
$\forall \epsilon > 0$*:* $\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0$*, so we can choose* $\epsilon = \frac{1}{k}$*. meaing that:*
$\lim_{n\to\infty} \mu(\{x \in X : |f(x) - f_n(x)| \geq \frac{1}{k}\}) = 0,\ \forall k \in \mathbb{N}$*, thus we can pick* n_k *increasing such that:*

$$\mu(\{x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k}\}) \leq \frac{1}{2^{k+1}}$$

Now let: $E_k = \{x \in X : |f(x) - f_{n_k}(x)| \geq \frac{1}{k}\}$*, and set:* $A_K = \bigcup_{k\geq K} E_k$*, so* A_k *is just the collection of all functions differing by* $1/k$ *for* $k \geq K$*, now:*

$$\mu(A_k) = \mu\left(\bigcup_{k\geq K} E_k\right) \leq \sum_{k\geq K} \mu(E_k) \leq \sum_{k\geq K} \frac{1}{2^{k+1}} = \frac{1}{2^K}$$

Where did we get the last equality from? Here we used the formula for geometric series:

$$\sum_{k=0}^{\infty} ar^k = a\left(\frac{1}{1-r}\right),\quad a = \frac{1}{2^{K+1}},\quad r = \frac{1}{2}$$
$$\sum_{k=K}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2^{K+1}}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2^K}$$

We have that $\{f_{n_k}\}$ *converges to* f *on* A_K^C*: if* $x \in A_K^C$*, well then* $x \notin A_K$*, meaning that:*
$|f(x) - f_{n_k}(x)| < \frac{1}{k},\ k \geq K$*, and this means that:* $f_{n_k}(x) \to f(x)$ *for all* $x \in A_K^C$*. As this is the case for all* $K \in \mathbb{N}$ *and* $\mu(A_K) \to 0$*, we have:*
$f_{n_k} \xrightarrow{a.e} f$

Corollary 2 (Relation between different ways to converge)

$$f_n \xrightarrow{L^p} f \implies f_n \xrightarrow{\mu} f \implies f_{n_k} \xrightarrow{a.e} f$$

References

[1] Tom L. Lindstrøm. *Spaces An introduction to Real Analysis*. American Mathematical Society, 2017. ISBN: 9781470440626.