

Exercises chapter 10

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Exercise 10.1

In this exercise we are dealing with a term-insurance, meaning that $S = \{*, \dagger\}$, we have stochastic interest rates, which follows a Vasicek model, furthermore, the risk-premium is given by $\gamma = 0$, meaning that we will have the same model under P as under Q , the contractual information provided is:

```
T <- 10
x0 <- 50
B <- 200000

#Vasicek parameters:
a <- 5
b <- 0.04
sigma <- 0.01
r0 <- 0.02

#0:alive, 1: dead
mu01 <- 0.009

p_surv <- function(t,s){
  f <- mu01
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}
```

We clarify, what the survival-probability is:

$$\begin{aligned} p(t, s) &= \exp\left(-\int_t^s \mu_{*\dagger}(u) du\right) \\ &= \exp(-0.009[s - t]) \end{aligned}$$

The desired Vasicek model is in this case given by:

$$dr_t = a(b - r_t)dt + \sigma dW_t^Q$$

Furthermore it's said that premiums are paid continuously, giving us the following policy-functions:

$$a_*(t) = \begin{cases} 0, & t < 0 \\ -\pi t, & t \in [0, T) \\ -\pi T, & t \geq T \end{cases} \quad a_{*\dagger}(t) = \begin{cases} B, & t \in [0, T) \\ 0, & \text{else} \end{cases}$$

and

$$\dot{a}_*(t) = -\pi \mathbb{1}_{[0, T)}(t)$$

We start off by stating the general formula for reserves in continuous-time:

$$V_i^+(t, A) = \sum_j \frac{v(T)}{v(t)} p_{ij}^x(t, T) \Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t, s) da_j(s) + \sum_{k \neq j} \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t, s) \mu_{jk}^x(s) a_{jk}(s) ds$$

Furthermore, we define:

$$V_i^+(t, r_t) = \mathbb{E}_Q[V_i^+(t, A) | \mathcal{F}_t]$$

we also have:

$$P(t, T) = \mathbb{E}_Q[e^{-\int_t^T r_u du} | \mathcal{F}_t]$$

We have $i = *$ and $j \in \{*, \dagger\}$, this gives us:

$$V_*^+(t, r_t) = -\pi \int_t^T P(t, s) p_{**}^x(t, s) ds + B \int_t^T P(t, s) p_{**}^x(t, s) \mu_{*\dagger}^x(s) ds$$

Since we are dealing with the Vasicek model, we have that it belongs to a class called: Affine term structures, which in simplicity means that the zero-coupon bond price $P(t, T)$ have an explicit form:

$$P(t, T) = e^{-A(T-t)r_t + B(T-t)}$$

Where:

$$A(h) = \frac{1 - e^{-ah}}{h} \quad \text{and} \quad B(h) = \left(b + \frac{\gamma\sigma}{a} - \frac{\sigma^2}{2a^2} \right) [A(h) - h] - \frac{\sigma^2}{4a} A(h)^2$$

Now from the equivalence principle, we get that the fair premium π is given by:

$$0 = V_*^+(0, A) = \mathbb{E}_Q[V_*^+(0, r_0)]$$

Hence:

$$\begin{aligned} 0 &= -\pi \int_0^T P(0, s) p_{**}^x(0, s) ds + B \int_0^T P(0, s) p_{**}^x(0, s) \mu_{*\dagger}^x(s) ds \\ &\Downarrow \\ \pi &= \frac{B \int_0^T P(0, s) p_{**}^x(0, s) \mu_{*\dagger}^x(s) ds}{\int_0^T P(0, s) p_{**}^x(0, s) ds} \end{aligned}$$