Mandatory assignment STK4500 1 of 1

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Exercise 1

a)

We start by showing that $\mu_{m\ m-1}(t) = m\mu(t)$ for $m \in S \setminus \{0\}$. We have that $Z = \{Z_t\}_{t\geq 0}$ represents the number of people alive at time t, with state space $S = \{0, \ldots, N\}$. We can define Z_t as:

$$Z_t = \sum_{i=1}^{N} X_t^{(i)} \quad X_t^{(i)} = \begin{cases} 1, & \text{person i is alive at time t} \\ 0, & \text{person i is dead at time t} \end{cases}$$

Since every person in the group have same age and mortality μ, we get that:

$$\mu(t) = \lim_{h\downarrow 0} \frac{P(X_{t+h}^{(i)} = 0 | X_t^{(i)} = 1)}{h}$$

Now by following the definition we get:

$$\mu_{m\ m-1}(t) = \lim_{h\downarrow 0} \frac{P(Z_{t+h} = m-1 | Z_t = m)}{h}$$

By definition of conditional probability we have:

$$P(Z_{t+h} = m-1 | Z_t = m) = \frac{P(Z_{t+h} = m-1, Z_t = m)}{P(Z_t = m)}$$

Now let's work a bit with the sets:

$$\begin{aligned} \{Z_{t+h} = m-1, Z_t = m\} &= \bigcup_{i=1}^{m} \{X_{t+h}^{(i)} = 0, X_{t+h}^{(j)} = 1, j = 1, \ldots, m, j \neq i\} \\ &= \bigcup_{i=1}^{m} \{X_{t+h}^{(i)} = 0, X_t^{(i)} = 1, X_{t+h}^{(j)} = 1, j = 1, \ldots, m, j \neq i\} \\ &= \bigcup_{i=1}^{m} \{A^{(i)}, B^{(i)}, C^{(j)}\} \end{aligned}$$

We have that $A^{(i)} \cap B^{(i)}$ is independent of $C^{(j)} = \{X^{(j)}_{t+h} = 1, j = 1, \dots, m, j \neq i\}$ furthermore $\alpha_{i,j} = \{A^{(i)}, B^{(i)}, C^{(j)}\}$ is a disjoint sequence of sets, thus:

$$
\begin{aligned}
P(\{Z_{t+h} = m-1, Z_t = m\}) &= P\left(\bigcup_{i=1}^{m} \alpha_{i,j}\right) \\
&= \sum_{i=1}^{m} P(\alpha_{i,j}) \\
&= \sum_{i=1}^{m} P(X^{(i)}_{t+h} = 0, X^{(i)}_t = 1) P(X^{(j)}_{t+h} = 1, j = 1, \dots, m, i \neq j) \\
&= \sum_{i=1}^{m} P(X^{(i)}_{t+h} = 0 | X^{(i)}_t = 1) P(X^{(i)}_t = 1) P(X^{(j)}_{t+h} = 1, j = 1, \dots, m, i \neq j)
\end{aligned}
$$

Now the last two sets are independent, meaning that we get:

$$
\begin{aligned}
P(X^{(i)}_t = 1) P(X^{(j)}_{t+h} = 1, j = 1, \dots, m, i \neq j) &= P((X^{(i)}_t = 1, X^{(j)}_{t+h} = 1, j = 1, \dots, m, i \neq j) \\
&= P(Z_t = m)
\end{aligned}
$$

This leaves us with:

$$
P(\{Z_{t+h} = m-1, Z_t = m\}) = \sum_{i=1}^{m} P(X^{(i)}_{t+h} = 0 | X^{(i)}_t = 1) P(Z_t = m)
$$

This gives us the desired result:

$$
\begin{aligned}
\mu_{m\ m-1}(t) &= \lim_{h \downarrow 0} \frac{P(\{Z_{t+h} = m-1, Z_t = m\})}{P(\{Z_t = m\})} \\
&= \sum_{i=1}^{m} \lim_{h \downarrow 0} P(X^{(i)}_{t+h} = 0 | X^{(i)}_t = 1) P(Z_t = m) \frac{1}{P(Z_t = m)} \\
&= \sum_{i=1}^{m} \mu(t) \\
&= m\mu(t)
\end{aligned}
$$

We are now asked to show that $\mu_{mn}(t) = 0$ for $|m - n| \geq 2$, let's assume that $N = 2$, so that i can manage this exercise:

$$
\mu_{20}(t) = \lim_{h \downarrow 0} \frac{p_{20}(t, t+h)}{h}
$$

Let's work with the probability in this case:

$$
\begin{aligned}
p_{20}(t, t+h) &= P(Z_{t+h} = 0 | Z_t = 2) \\
&= P(X^{(2)}_{t+h} = \dagger, X^{(1)}_{t+h} = \dagger | X^{(2)}_t = *, X^{(1)}_t = *) \\
&= P(X^{(2)}_{t+h} = \dagger | X^{(2)}_t = *, X^{(1)}_t = *) \times P(X^{(1)}_{t+h} = \dagger | X^{(2)}_t = *, X^{(1)}_t = *) \\
&= P(X^{(2)}_{t+h} = \dagger | X^{(2)}_t = *) \times P(X^{(1)}_{t+h} = \dagger | X^{(1)}_t = *)
\end{aligned}
$$

Here we used the fact that lives are independet, as well as the Markov property. Now letting $h \to 0$, we get:

$$
\lim_{h \downarrow 0} P(X^{(2)}_{t+h} = \dagger | X^{(2)}_t = *) = 0
$$

This leaves us with:

$$
\mu_{20}(t) = 0
$$

b)

We are asked to argue for why $p_{mn}(t,s) = 0$ for $n \geq m+1$, this follows from the fact that if this were not the case we would let the transition $\dagger \to *$ to be allowed. Hence this needs to be zero.

We were also asked to argue for $n \leq m$ that:

$$p_{mn}(t,s) = \binom{m}{n} p(t,s)^n [1 - p(t,s)]^{m-n}$$

This follows from the fact that we have a group of friends with same age, same mortality μ and the fact that they have independent lives. We thus get a sequence of independent Bernoulli trials where we have the same probability of success(survival) and failure(death), when we then sum up the number of success in $[t,s]$ we get a Binomial-distribution, and hence the probability is as above.

c)

We assume that $\int_0^\infty \mu(u)du = \infty$, this means that

$$p(0,\infty) = \lim_{s\to\infty} p(0,s) = \lim_{s\to\infty} e^{\int_0^s \mu(u)du} = 0$$

Now, lets look at the sets again:

$$\{Z_s = 0\} = \bigcap_{i=0}^{N} \{X_s^{(i)} = \dagger\}$$

$$\Downarrow$$

$$P(\{Z_s = 0\}) = P\left(\bigcap_{i=0}^{N} \{X_s^{(i)} = \dagger\}\right) = \prod_{i=1}^{N} P(\{X_s^{(i)} = \dagger\}) = (1 - p(0,s))^N$$

Now, if we apply the limit on the sets:

$$P(\lim_{s\to\infty} Z_s = 0) = \left(1 - \lim_{s\to\infty} p(0,s)\right)^N = (1 - p(0,\infty))^N = 1$$

Exercise 2

In this exercise we are considering a socalled Tontine of Friends in Continous time. We are dealing with a regular continous markov chain X consisting of a total of N friends. We focus on one of the participants called the chosen one referred to as $C1$.

Our state space S looks like the following:

$$S = \{0, 1, \ldots, N-1\} \times \{*, \dagger\}$$

We also use the convention that $(m, *)$ means: m participants in the group except $C1$ are alive and the chosen one, $C1$ is alive. $(m, \dagger)$ means that m participants in the gruop except $C1$ are alive, while $C1$ is dead.

a) Transition Probabilities

We are asked to argue for the following:

$$\mu_{(m,*)(m-1,\dagger)}(t) = 0$$
$$\mu_{(m,*)(m-1,*)}(t) = \mu_{(m,\dagger)(m-1,\dagger)}(t) = m\mu(t)$$

$\mu_{(m,*)(m-1,\dagger)}(t) = 0$ means that the group consisting of m persons goes to $m-1$ i.e one has died, and also at the same time $C1$ have gone from $*$ to $\dagger$. This is zero as it represents instantanoius jumps of size two immideately.

$\mu_{(m,*)(m-1,*)}(t) = \mu_{(m,\dagger)(m-1,\dagger)}(t) = m\mu(t)$: $\mu_{(m,*)(m-1,*)}(t)$: the group goes from $m \to m-1$, while $* \to *$. From **Exercise 1** we have that the lives are assumed to be stochastically independent, same age and have the same force of mortality $\mu(t)$. Now: if the group goes from $m \to m-1$ and $C1$ remains in $*$ we have that there are $\binom{m}{m-1}$ ways to go from m to $m-1$, and given the fact that all have same force of mortality we get: $\mu_{(m,*)(m-1,*)}(t) = \binom{m}{m-1}\,\mu(t) = m\mu(t)$. Exact same argument applies for $\mu_{(m,\dagger)(m-1,\dagger)}(t)$.

$\mu_{(m,*)(m,\dagger)}(t)$: the group remians of m people, while $C1$ has gone from $* \to \dagger$, given the fact that all were assumed to have the same mortality rate we get: $\mu_{(m,*)(m,\dagger)}(t) = \mu(t)$

We also have that $p_{(m,j)(n,j)}(t, s) = 0$ for $m, n \in \{0, \ldots, N-1\}$ $n \geq m+1, j \in \{*, \dagger\}$, if this probability was non-zero we would allowd for the transition $\dagger \to *$, which is not reasonable, hence this probability is zero.

We now let $n \leq m$, we then have:

$$p_{(m,*)(n,*)}(t,s) = \binom{m}{n} p(t,s)^{n+1}[1-p(t,s)]^{m-n}$$

We here have that all individuals have the same survival-probability $p(t,s)$

$$\begin{aligned}
p_{(m,*)(n,*)}(t,s) &= P[\text{Group goes from m to n in [t,s]}, \text{C1 remains in * for [t,s]}] \\
&\overset{indep.lives}{=} P[\text{Group goes from m to n in [t,s]}] P[\text{C1 remains in * for [t,s]}] \\
&= \binom{m}{n} p(t,s)^n [1-p(t,s)]^{m-n} \times p(t,s) \\
&= \binom{m}{n} p(t,s)^{n+1}[1-p(t,s)]^{m-n}
\end{aligned}$$

We also have:

$$\begin{aligned}
p_{(m,*)(n,*)}(t,s) &= P[\text{Gruop goes from m to n in [t,s]}, \text{C1 goes from alive to dead in [t,s]}] \\
&\overset{indep.lives}{=} P[\text{Gruop goes from m to n in [t,s]}] P[\text{C1 goes from alive to dead in [t,s]}] \\
&= \binom{m}{n} p(t,s)^n [1-p(t,s)]^{m-n} \times [1-p(t,s)] \\
&= \binom{m}{n} p(t,s)^n [1-p(t,s)]^{m-n+1}
\end{aligned}$$

b)

We have that the contract starts at $t = 0$, and that all participants pay a single premium π_0 at this time. These premiums are invested into a fund, which we will regard as a stocastich process $S = \{S(t), t \geq 0\}$, which looks like

$$S(t) = S(0)e^{\rho t}$$

Furthermore the fund will not start paying out until we reach $t = T_0$, where after $t \geq T_0$ will have the value: $S(t) = S(0)e^{\rho T_0}$ with a constant interest rate $r > 0$, we are asked to derive $a_{(m,*)}(s)$, when we do not consider premiums.

$$a_{(m,*)}(t) = \begin{cases} 0 & , t \in [0, T_0) = 0 \\ \frac{\rho t S(0) e^{\rho T_0}}{m+1} & , t \geq T_0 \end{cases}$$

c)

In this exercise we are asked to calcualte the reserves, we first start off by remembering the formula applicable to our situation, and then translate it into our setting:

$$V_i^+(t, A) = \frac{1}{v(t)} \left[\sum_{j \in S} \int_t^\infty v(s) p_{ij}^x(t,s) da_j(s) \right]$$

Now translating this into our setting:

$$\begin{aligned}
V^+_{(m,*)}(t) &= \sum_{n=0}^{m} \int_t^\infty \frac{v(s)}{v(t)} p_{(m,*)(n,*)}(t,s) da_{(n,*)}(s) \\
&= \sum_{n=0}^{m} \int_t^\infty \frac{v(s)}{v(t)} p_{(m,*)(n,*)}(t,s) \mathbb{1}_{[t\geq T_0]} \frac{\rho S(0) e^{\rho T_0}}{n+1} ds \\
&= \rho S(0) e^{\rho T_0} \sum_{n=0}^{m} \int_{t \vee T_0}^\infty \binom{m}{n} \frac{p(t,s)^{n+1}[1-p(t,s)]^{m-n}}{n+1} e^{-r(s-t)} ds
\end{aligned}$$

d)

By Fubini we have:

$$V^+_{(m,*)} = \rho S(0) e^{\rho T_0} \int_{t \vee T_0}^\infty \sum_{n=0}^{m} \binom{m}{n} \frac{p(t,s)^{n+1}(1-p(t,s))^{m-n}}{n+1} e^{-r(s-t)} ds$$

Let $X \sim Bin(m,p)$, then:

$$E\left[\frac{1}{1+X}\right] = \sum_{n=0}^{m} \binom{m}{n} \frac{p^n (1-p)^{m-n}}{n+1}$$

We also have:

$$E\left[\frac{1}{1+X}\right] = \frac{1-(1-p)^{m+1}}{(m+1)p}$$

This means that we can rewrite $V^+_{(m,*)}(t)$:

$$\begin{aligned}
V^+_{(m,*)}(t) &= \rho S(0) e^{\rho T_0} \int_{t \vee T_0}^\infty pE\left[\frac{1}{1+X}\right] e^{-r(s-t)} ds \\
&= \frac{\rho S(0) e^{\rho T_0}}{m+1} \int_{t \vee T_0}^\infty (1-(1-p(t,s))^{m+1}) e^{-r(s-t)} ds
\end{aligned}$$

e)

We start with the single premium π_0: According to the equivalence principle, we have that:

$$\pi_0 = V^+_{(N-1,*)}(0) = \frac{\rho S(0) e^{\rho T_0}}{N} \int_{T_0}^\infty (1-(1-p(0,s))^N) e^{-rs} ds$$

We also want the yearly premiums, from lecture notes we have: $V^+_*(t, A) = V^+_*(t, A_*) + V^+_*(t, \tilde{A}_*)$ Where we in the first expression do not consider premiums, and in the other we only care about premiums, i.e $a_{(m,*)}(t)$ remains, while:

$$\tilde{a}_{(m,*)}(t) = \begin{cases} -\pi t & , t \in [0, T_0) = 0 \\ -\pi T_0 & , t \geq T_0 \end{cases}$$

with $\dot{\tilde{a}}_{(m,*)}(t) = -\pi$ for $t \in (0, T_0)$ Now, we get the following:

$$\begin{aligned}
V^+_{(m,*)}(t, \tilde{A}) &= \frac{1}{v(t)} \int_t^{T_0} v(s) p_{(m,*)(n,*)}(t,s)(-\pi ds) \\
&= -\pi \int_t^{T_0} \sum_{n=0}^{m} \binom{m}{n} p(t,s)^{n+1}(1-(p(t,s))^{m-n}) ds
\end{aligned}$$

Now translating the above idea, we get:

$$V^+_{(m,*)}(t,A) = V^+_{(m,*)}(t,A_*) + V^+_{(m,*)}(t,\tilde{A}_*)$$

And by the equivalence principle we will set

$$V^+_{(m,*)}(0,A) = V^+_{(m,*)}(0,A_*) + V^+_{(m,*)}(0,\tilde{A}_*) = 0$$

and solve for π, leading to:

$$\begin{aligned}
0 &= \frac{\rho S(0)e^{\rho T_0}}{m+1}\int_{T_0}^{\infty}(1-(1-p(0,s))^{m+1})e^{-rs}ds - \pi\int_0^{T_0}\sum_{n=0}^{m}\binom{m}{n}p(0,s)^{n+1}(1-p(0,s))^{m-n}ds \\
&\Downarrow \\
\pi &= \frac{\frac{\rho S(0)e^{\rho T_0}}{m+1}\int_{T_0}^{\infty}(1-(1-p(0,s))^{m+1})e^{-rs}ds}{\int_0^{T_0}\sum_{n=0}^{m}\binom{m}{n}p(0,s)^{n+1}(1-p(0,s))^{m-n}ds}
\end{aligned}$$

f) Thiele's equation

$$\frac{d}{dt}V_i^+(t) = r(t)V_i^+(t) - \dot{a}_i(t) - \sum_{j\neq i}\mu_{ij}^x(t)(a_{ij}(t) + V_j^+(t) - V_i^+(t))$$

$$\Downarrow$$

$$\begin{aligned}
\frac{d}{dt}V^+_{(m,*)}(t) &= r(t)V^+_{(m,*)}(t) - \frac{\rho S(0)e^{\rho T_0}}{m+1} - \left(\mu^x_{(m,*)(m,\dagger)}(t)(-V^+_{(m,*)}(t)) + \mu_{(m,*)(m-1,*)}[V^+_{(m-1,*)}(t) - V^+_{(m,*)}(t)]\right) \\
&= r(t)V^+_{(m,*)}(t) - \frac{\rho S(0)e^{\rho T_0}}{m+1} + \mu^x(t)V^+_{(m,*)}(t) - m\mu^x(t)[V^+_{(m-1,*)}(t) - V^+_{(m,*)}(t)] \\
&= [r(t) + (m+1)\mu^x(t)]\,V^+_{(m,*)}(t) - \frac{\rho S(0)e^{\rho T_0}}{m+1} - m\mu^x(t)V^+_{(m-1,*)}(t)
\end{aligned}$$

With final condition: $V^+_{(m,*)}(T) = 0$

g) Numerical example:

We start off by the contractual information:

```
r   <- 0.03   #interest-rate
rho <- 0.07
S0  <- 100000
N   <- 10     #number of person
x   <- 30     #age of insured
T0  <- 40     #withdrawal
T   <- 90     #length of contract
G   <- 0      #Gender = male
Y   <- 2022   #Year
```

We were also asked to use the mortalities proposed by Finanstilsynet:

```
w <- function(G, x){
  if(G==0){
    return( min(2.671548-0.17248*x+0.001485*x^2,0) )
  }
  if(G==1){
    return( min(1.287968-0.10109*x+0.000814*x^2,0) )
  }
}
```