

Exercises chapter 6

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Exercise 6.1

In this exercise we are asked to calculate transition rates using an Euler scheme, we have four states, i.e $S = \{*, \diamond, \times, \dagger\}$, where \times represents critically ill. From the text, we have the following transition rate matrix:

```
#required libraries:
```

```
library(tidyverse)
```

```
library(scales)
```

```
#0: alive, 1: disabeld, 2: critcally ill, 3: dead
```

```
lambda <- function(t){
```

```
#constants given in exercise:
```

```
a1 <- 4*10**(-4)
```

```
b1 <- 3.4674*10**(-6)
```

```
c1 <- 0.138155
```

```
a2 <- 5*10**(-4)
```

```
b2 <- 7.5858*10**(-5)
```

```
c2 <- 0.08749
```

```
#alive
```

```
m01 <- a1 + b1*exp(c1*t)
```

```
m02 <- 0.05*m01
```

```
m03 <- a2 + b2*exp(c2*t)
```

```
m00 <- -(m01 + m02 + m03)
```

```
#disabeld
```

```
m10 <- 0.1*m01
```

```
m12 <- m02
```

```
m13 <- m03
```

```
m11 <- -(m10 + m12 + m13)
```

```
#critically ill
```

```
m23 <- 1.2*m13
```

```
m21 <- 0
```

```
m20 <- 0
```

```
m22 <- -(m23 + m21 + m20)
```

```
#deceased
```

```
m30 <- m31 <- m32 <- m33 <- 0
```

```
transition_rates <- c(m00, m01, m02, m03,
```

```
                      m10, m11, m12, m13,
```

```
                      m20, m21, m22, m23,
```

```
                      m30, m31, m32, m33)
```

```

L <- matrix(transition_rates, nrow=4, byrow=TRUE)

return(L)
}

field <- function(t,M){
  return(M%*%lambda(t))
}

#field(t0, P0)

Euler <- function(t0,P0,h,tn){
  if(t0==tn){ return(P0)}
  N <- (tn-t0)/h
  D <- dim(P0)[1] #gives dimension of max matrix dim = c(m,n)
  #Initial condition at s
  P <- array(diag(D*(N+1)), dim=c(D,D,N+1))
  #First iteration
  P[, ,1] <- P0 + h*field(t0,P0)

  for(n in 1:N){
    P[, ,n+1] <- P[, ,n]+h*field(t0+n*h,P[, ,n])
  }
  return(P) #returns array, array(data , dim= c(3,3,2)), this stores 2 3x3 matrices
}

t0 <- 30 #start age
P0 <- diag(4) #initial start with P(s,s) = I
tn <- 110 #end age
h <- 1/12 #step size
N <- (tn-t0)/h #number of steps

sol <- Euler(t0,P0,h,tn) #contains ages transition probs from 30 to 110
#p_surv(30,75):
sol[1,1, ] [12*35]

## [1] 0.6189569

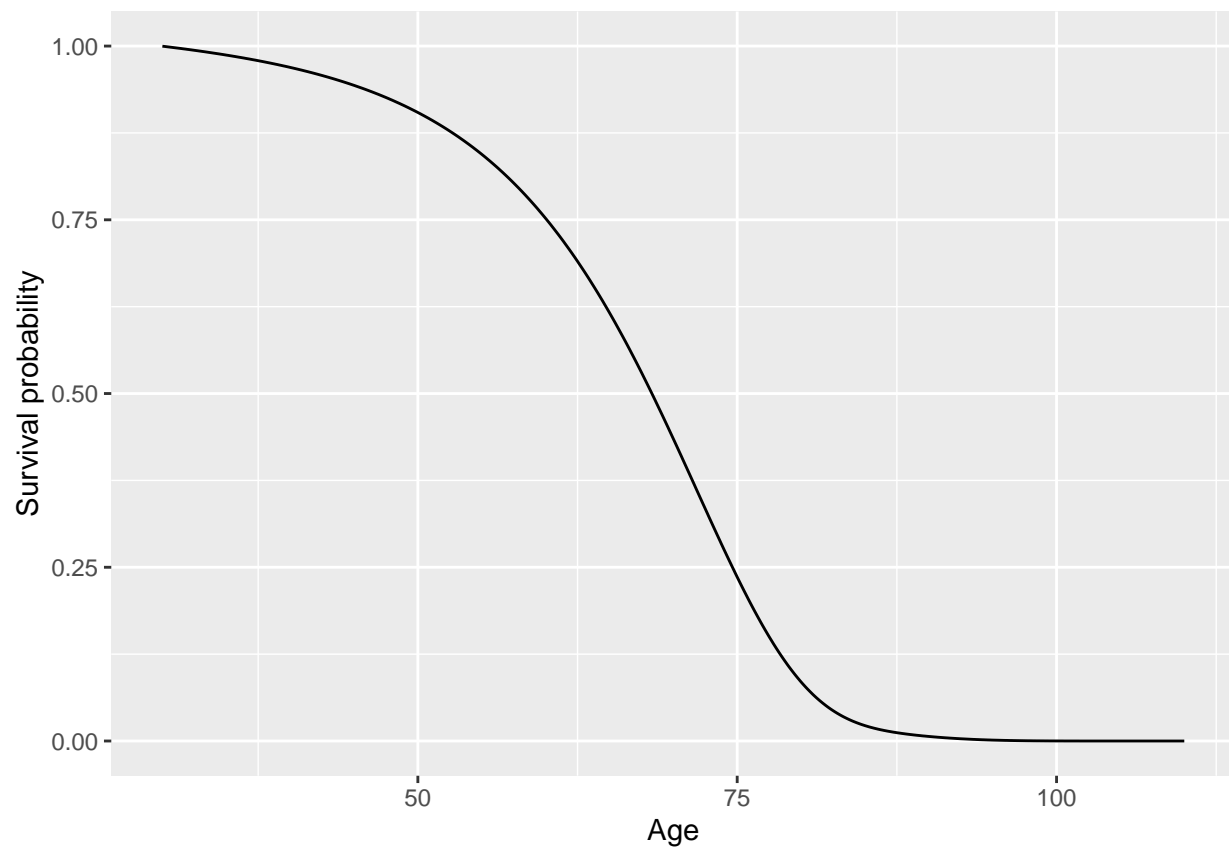
```

```
a <- seq(30, 110, length = length(sol[1,1, ]))  
b <- sol[1,1, ]
```

```
df <- data.frame(a,b)  
colnames(df) <- c("age", "survival_prob")
```

```
fig <- df %>%  
  ggplot(aes(x = age, y = survival_prob)) +  
  geom_line() +  
  scale_y_continuous() +  
  xlab("Age") +  
  ylab("Survival probability")
```

fig



Exercise 6.2

We have the following contractual information:

```
T <- 10      #length of contract
x <- 60      #age of insured
D <- 20000   #disability pension while disabeld
B <- 50000   #death benefit
r <- 0.05    #intensity rate
```

Where our state space is $S = \{*, \diamond, \dagger\}$, we have the following policy-functions:

$$a_{\diamond}(t) = \begin{cases} 0, & t < 0 \\ Dt, & t \in [0, T) \\ DT, & t \geq T \end{cases} \quad a_{*\dagger}(t) = \begin{cases} B, & t \in [0, T) \\ 0, & else \end{cases}$$

Since we for now do not care about the premiums in the policy-functions, we get

$$\dot{a}_*(t) = 0$$

and

$$\dot{a}_{\diamond}(t) = D\mathbb{1}_{[0, T)}(t)$$

We can set up thieles differential equations:

$$\begin{aligned} \frac{d}{dt} V_*^+(t) &= r(t)V_*(t) - \mu_{*\diamond}^x(t)[V_{\diamond}^+(t) - V_*^+(t)] - \mu_{*\dagger}^x(t)[B\mathbb{1}_{[0, T)}(t) - V_*^+(t)] \\ \frac{d}{dt} V_{\diamond}^+(t) &= r(t)V_{\diamond}(t) - D\mathbb{1}_{[0, T)}(t) - \mu_{*\diamond}^x(t)[V_*^+(t) - V_{\diamond}^+(t)] + \mu_{\diamond\dagger}^x(t)V_{\diamond}^+(t) \end{aligned}$$

With finial conditions: $V_*^+(T) = V_{\diamond}^+(T) = 0$

In order to solve this numerically, we partition up $[0, T]$ in times $t_i = t_0 + ih$, with $0 = t_0 < \dots < t_n = T$.

We also iterate backwards, giving us:

$$\frac{V_*^+(t_i) - V_*^+(t_{i-1})}{h} = rV_*(t_i) - \mu_{*\diamond}^x(t_i)[V_{\diamond}^+(t_i) - V_*^+(t_i)] - \mu_{*\dagger}^x(t_i)[B - V_*^+(t_i)]$$

We then solve for $V_*(t_{i-1})$, giving us:

$$V_*^+(t_{i-1}) = V_*^+(t_i) - h [rV_*(t_i) - \mu_{*\diamond}^x(t_i)[V_{\diamond}^+(t_i) - V_*^+(t_i)] - \mu_{*\dagger}^x(t_i)[B - V_*(t_i)]]$$

We do the same for $V_{\diamond}^+(t_{i-1})$:

$$V_{\diamond}^+(t_{i-1}) = V_{\diamond}^+(t_i) - h [rV_{\diamond}^+(t_i) - D\mathbb{1}_{[0, T)}(t) - \mu_{*\diamond}^x(t_i)[V_*^+(t_i) - V_{\diamond}^+(t_i)] + \mu_{\diamond\dagger}^x(t_i)V_{\diamond}^+(t_i)]$$

```
#states 0:alive, 1:disabled, 2:dead
```

```
mu01 <- function(u){
  a1 <- 4*10^{-4}
  b1 <- 3.4674*10^{-6}
  c1 <- 0.138755
  return(a1 + b1*exp(c1*u))
}
```

```
mu02 <- function(u){
```

```

a2 <- 5*10^{-4}
b2 <- 7.5858*10^{-5}
c2 <- 0.087498
return(a2 + b2*exp(c2*u))
}

mu12 <- function(u){
  return(mu02(u))
}

mu10 <- function(u){
  return(0.1*mu01(u))
}

#a.e. derivative of a_{\diamond}(t)
dis_dot <- function(t){
  if(t >= 0 && t<T){ return(D) }
  if(t >= T){ return(0) }
}

#PV of policy V0 and V1
h <- 1/12
N <- T/h

PV_act <- PV_dis <- rep(0,N+1)
PV_act[N+1] <- PV_dis[N+1] <- 0 #by construction of contract

for(i in N:1){
  PV_act[i] <- PV_act[i+1] - h*(r*PV_act[i+1]
    - mu01(x+(i+1)*h)*(PV_dis[i+1]-PV_act[i+1])
    - mu02(x+(i+1)*h)*(B-PV_act[i+1])
  )

  PV_dis[i] <- PV_dis[i+1] - h*(r*PV_dis[i+1]- dis_dot((i+1)*h)
    - mu10(x+(i+1)*h)*(PV_act[i+1]-PV_dis[i+1])
    + mu12(x+(i+1)*h)*PV_dis[i+1]
  )
}

```

And from this we can find the one-time premium $\pi_0 = V_*^+(0)$:

```
pi0 <- PV_act[1]
pi0
```

```
## [1] 20608.17
```

We were asked to find the yearly premium π , we then use the method, where we create an artificial policy, where one pays a premium of NOK 1 during $[0, T)$, i.e this is the only contribution of interest, thus:

$$a_*^{Prem=1}(t) = \begin{cases} 0, & t < 0 \\ -t, & t \in [0, T) \\ -T, & t \geq T \end{cases} \implies \dot{a}_*^{Prem=1}(t) = -1\mathbb{1}_{[0, T)}(t)$$

$$a_\diamond^{Prem=1}(t) = 0, \forall t \implies \dot{a}_\diamond^{Prem=1}(t) = 0$$

$$a_{*\dagger}^{Prem=1}(t) = 0$$

We can now set up the new thieles equations:

$$\frac{d}{dt}V_*^+(t, A^{Prem=1}) = rV_*^+(t) - \dot{a}_*(t) - \mu_{*\diamond}^x(t)[V_\diamond^+(t) - V_*^+(t)] + \mu_{*\dagger}^x(t)V_*^+(t)$$

$$\frac{d}{dt}V_\diamond^+(t, A^{Prem=1}) = rV_\diamond^+(t) - \mu_{*\diamond}^x(t)[V_*^+(t) - V_\diamond^+(t)] + \mu_{\diamond\dagger}^x(t)V_\diamond^+(t)$$

And again solve for $V_*^+(t_{i-1})$ and $V_\diamond^+(t_{i-1})$:

$$V_*^+(t_{i-1}) = V_*^+(t_i) - h[rV_*^+(t_i) - \dot{a}_*(t_i) - \mu_{*\diamond}^x(t_i)[V_\diamond^+(t_i) - V_*^+(t_i)] + \mu_{*\dagger}^x(t_i)V_*^+(t_i)]$$

$$V_\diamond^+(t_{i-1}) = V_\diamond^+(t_i) - h[rV_\diamond^+(t_i) - \mu_{*\diamond}^x(t_i)[V_*^+(t_i) - V_\diamond^+(t_i)] + \mu_{\diamond\dagger}^x(t_i)V_\diamond^+(t_i)]$$

```
#a.e derivative a_{*}:
a_dot_prem1 <- function(t){
  if(t >= 0 && t < T){ return(-1) }
  if(t >= T){ return(0) }
}

PV_prem1_act <- PV_prem1_dis <- rep(0, N+1)
PV_prem1_act[N+1] <- PV_prem1_dis[N+1] <- 0 #follows from contract

for(i in N:1){
  PV_prem1_act[i] <- PV_prem1_act[i+1] - h*(r*PV_prem1_act[i+1] - a_dot_prem1((i+1)*h)
    - mu01(x+(i+1)*h)*(PV_prem1_dis[i+1]-PV_prem1_act[i+1])
    + mu02(x+(i+1)*h)*(PV_prem1_act[i+1])
  )

  PV_prem1_dis[i] <- PV_prem1_dis[i+1] - h*(r*PV_prem1_dis[i+1]
    - mu10(x+(i+1)*h)*(PV_prem1_act[i+1]-PV_prem1_dis[i+1])
    + mu12(x+(i+1)*h)*PV_prem1_dis[i+1]
  )
}
PV_prem1_act[1]

## [1] -6.453883
```

We will now use the equivalence principle, which tells us:

$$\begin{aligned} \pi V_*^+(0, A^{Prem=1}) + V_*^+(0) &= 0 \\ \Downarrow \\ \pi &= -\frac{V_*(0)}{V_*^+(0, A^{Prem=1})} \end{aligned}$$

This gives us:

```
pi0
```

```
## [1] 20608.17
```

```
yearly_premium <- (-1)*pi0/PV_prem1_act[1]
yearly_premium
```

```
## [1] 3193.142
```

Exercise 6.4

We are dealing with an endowment insurance, meaning that $S = \{*, \dagger\}$, we are asked to calculate the second moment of the benefit payments V_t^+ for $t = 10, \dots, 20$. The contractual information provided is:

```
x <- 40
T <- 20
E <- 115000
B <- 220000
r <- 0.03
```

We start by writing up the policy functions:

$$a_*^{Pre}(n) = \begin{cases} 0 & , n = 0, \dots, T-1 \\ E & , else \end{cases} \quad a_{*\dagger}^{Post}(n) = \begin{cases} B & , n = 0, \dots, T-1 \\ 0 & , else \end{cases}$$

From chapter.5 we are given the following formula:

$$\mathbb{E}[(V_t^+)^p | X_t = i] = (v_t)^p \sum_j p_{ij}^x(t, t+1) \sum_{k=0}^p \binom{p}{k} (a_{ij}^{Post}(t))^{p-k} \mathbb{E}[(V_{t+1}^+)^k | X_{t+1} = j]$$

Where v_t is the one-step discount factor in $[t, t+1]$ i.e:

$$v(t) = v_0 v_1 \dots v_{t-1}$$

Let's denote $V_2(t) := \mathbb{E}[(V_t^+)^2 | X_t = *]$, and then translate the equation above into our situation:

$$\begin{aligned} V_2(t) &= (v_t)^2 \sum_j p_{*j}^x(t, t+1) \sum_{k=0}^2 \binom{2}{k} (a_{*j}^{Post}(t))^{2-k} \mathbb{E}[(V_{t+1}^+)^k | X_{t+1} = j] \\ &= (v_t)^2 p_{**}^x(t, t+1) \mathbb{E}[(V_{t+1}^+)^2 | X_{t+1} = *] + (v_t)^2 p_{*\dagger}^x(t, t+1) (a_{*\dagger}^{Post}(t))^2 \\ &= (v_t)^2 p_{**}^x(t, t+1) V_2(t+1) + (v_t)^2 p_{*\dagger}^x(t, t+1) (a_{*\dagger}^{Post}(t))^2 \\ &= e^{-2r} [p_{**}^x(t, t+1) V_2(t+1) + p_{*\dagger}^x(t, t+1) (a_{*\dagger}^{Post}(t))^2] \end{aligned}$$

This has now become an difference equation, with final condition:

$$V_2(T) = (\Delta a_*^{Pre}(T))^2 = E^2$$

From the exercise it's also given that:

$$\mu_{* \dagger}^x(t) = 0.0018 + 0.0004t$$

```
#0:alive, 1:dead
```

```
mu01_x <- function(t){  
  return(0.0018 + 0.0004*t)  
}
```

```
p_surv_x <- function(t,s){  
  f <- mu01_x  
  integral <- integrate(f, lower = t, upper = s)$value  
  return(exp((-1)*integral))  
}
```

```
V2 <- rep(0,T+1)  
V2[T+1] <- E^{2}
```

```
for(n in (T-1):0){  
  V2[n+1] <- exp(-2*r)*(p_surv_x(n,n+1)*V2[n+2] + B^{2}*(1-p_surv_x(n,n+1)))  
}  
V2
```

```
## [1] 6167825284 6465436986 6765434135 7068218098 7374219626 7683901471  
## [7] 7997761234 8316334482 8640198138 8969974190 9306333731 9650001385  
## [13] 10001760131 10362456580 10733006746 11114402361 11507717773 11914117511  
## [19] 12334864556 12771329405 13225000000
```


Exercise 6.5

We are considering a permanent disability insurance, so $S = \{*, \diamond, \dagger\}$, with $\mu_{\diamond*}(t) = 0$. Contractual information:

```
x <- 60
T <- 10
D <- 30000
B <- 50000
r <- 0.05
u <- 60000
```

We are asked to find:

$$P_{\diamond}(t, u) := P[V_t^+ < u | X_t = \diamond]$$

for $t = 7$ and $u = \$60k$

Before we start, we write up the desired policy functions:

$$a_{\diamond}^{Pre}(n) = \begin{cases} D & , n = 0, \dots, T-1 \\ 0 & , else \end{cases} \quad a_{*\dagger}^{Post}(n) = \begin{cases} B & , n = 0, \dots, T-1 \\ 0 & , else \end{cases} \quad a_{\diamond\dagger}^{Post}(n) = \begin{cases} B & , n = 0, \dots, T-1 \\ 0 & , else \end{cases}$$

We will use Thiele's difference equation for distributions of mathematical reserves:

$$P_i(t, u) = \sum_k p_{ik}^x(t, t+1) P_k(t+1, v_t^{-1}[u - a_i^{Pre}(t) - a_{ik}^{Post}(t)])$$

Translating this into our situation:

$$\begin{aligned} P_{\diamond}(t, u) &= p_{\diamond\diamond}^x(t, t+1) P_{\diamond}(t+1, e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\diamond}^{Post}(t)]) \\ &\quad + p_{\diamond\dagger}^x(t, t+1) P_{\dagger}(t+1, e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)]) \\ &= p_{\diamond\diamond}^x(t, t+1) P_{\diamond}(t+1, e^r[u - a_{\diamond}^{Pre}(t)]) \\ &\quad + p_{\diamond\dagger}^x(t, t+1) P_{\dagger}(t+1, e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)]) \end{aligned}$$

We start by exploring P_{\dagger} :

$$\begin{aligned} P_{\dagger}(t+1, e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)]) &= P[V_{t+1}^+ < e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)] | X_{t+1} = \dagger] \\ &= P[0 < e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)]] \\ &= \begin{cases} 1 & , t > T \\ 0 & , t \leq T \end{cases} \end{aligned}$$

Since we are interested in $t \leq T$, we get:

$$P_{\dagger}(t+1, e^r[u - a_{\diamond}^{Pre}(t) - a_{\diamond\dagger}^{Post}(t)]) = 0$$

Giving us:

$$P_{\diamond}(t, u) = p_{\diamond\diamond}^x(t, t+1) P_{\diamond}(t+1, e^r[u - a_{\diamond}^{Pre}(t)])$$

We iterate backwards, given that we know $V_{10}^+ = 0$, as we do not need to set aside money for someone entering the contract at the end.

$$\begin{aligned} P_{\diamond}(9, 60) &= p_{\diamond\diamond}^x(9, 10) P_{\diamond}(10, e^r[60 - a_{\diamond}^{Pre}(9)]) \\ &= p_{\diamond\diamond}^x(9, 10) P(0 < e^r[60 - 30]) \\ &= p_{\diamond\diamond}^x(9, 10) \end{aligned}$$