Exercises chapter 9

Andreas Slåttelid

Exercise 9.1 Unit-linked term insurance

We are dealing with a term insurance, so $S = \{*, \dagger\}$, the contractual information provided is:

```
x <- 55
T <- 10
r <- 0.05

#GBM parameters:
sigma <- 0.25
S0 <- 1

#management:
beta <- 0.006
```

From the text, we get the following payout in case of death:

$$C(t) = (1 - 0.04)\pi_0 \frac{S_t}{S_0} e^{-\beta t} 1.10$$

$$a_{*\dagger}(t) = \begin{cases} C(t), & t \in [0, T) \\ 0, & else \end{cases}$$

We recall the formula for continous reserves:

$$V_i^+(t, A) = \sum_j \frac{v(T)}{v(t)} p_{ij}^x(t, T)\Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t, s) da_j(s) + \sum_{k \neq j} \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t, s)\mu_{jk}^x(s) a_{jk}(s) ds$$

only the last part will be relevant, furthermore $V_i^+(t, S_t) = \mathbb{E}_Q[V_i^+(t, A)|\mathcal{F}_t]$, translating this into our situation:

$$V_*^+(t, S_t) = \int_t^T p_{**}^x(t, s)\mu_{jk}^x(s)\mathbb{E}_Q\left[\frac{v(s)}{v(t)} C(s) \middle| \mathcal{F}_t\right] ds$$

Idea here is to use martingale theory, furthermore, let $C(s) = N(s)S_s$, where $N(s)$ represents the collection of deterministic terms in $C(s)$. This gives:

$$\mathbb{E}_Q\left[\frac{v(s)}{v(t)} C(s) \middle| \mathcal{F}_t\right] = N(s)\mathbb{E}_Q\left[\frac{v(s)}{v(t)} S_s \middle| \mathcal{F}_t\right] = N(s)\frac{1}{v(t)}\mathbb{E}_Q[v(s)S_s|\mathcal{F}_t] = N(s)S_t$$

We were asked to calculate $V_*^+(0, S_0)$:

$$
\begin{aligned}
V_*^+(0,S_0) &= \int_0^T p_{**}^x(0,s)\mu_{*\dagger}^x(s)N(s)v(0)S_0 ds \\
&= \int_0^T p_{**}^x(0,s)\mu_{*\dagger}^x(s)(1-0.04)\pi_0 e^{-\beta s}1.10 ds \\
&= (1-0.04)\pi_0 1.10 \int_0^T p_{**}^x(0,s)\mu_{*\dagger}^x(s)e^{-\beta s} ds
\end{aligned}
$$

Exercise 9.3 Markov property of B&S

i)

We are asked to show that the solution to the GBM under Q is given by:

$$S(t) = S(0)\exp\left((r - \frac{\sigma^2}{2})t + \sigma\widetilde{W}_t\right)$$

Now from arbitrage theory, we have that $\widetilde{S}_t$ is a $(Q, \mathcal{F})$-martingale,
here I just state the dynamics of $\widetilde{S}$ under P:

$$\begin{aligned} d(e^{-rt}S_t) &= S_t e^{-rt}\sigma\left[\frac{\mu - r}{\sigma}dt + dW_t\right] \\ &= \widetilde{S}_t\sigma[\theta dt + dW_t] \end{aligned}$$

From Girsanov's thm we have that Z_t a martingale

$$Z_t = \exp\left(\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds\right)$$

yields that $\widetilde{W}_t = W_t - \int_0^t \varphi_s ds$ is a $(Q, \mathcal{F})$-BM.

So combining the martingale representation theorem with a clever way of finding φ_s, we see that $\varphi_t = -\theta$, because, we then get that:

$$\widetilde{W}_t = W_t + \int_0^t \theta ds$$

is a $(Q, \mathcal{F})$-BM. This gives us $W_t = \widetilde{W}_t - \theta t$, from theory we have that the solution to a GBM under (P) is:

$$\begin{aligned} S(t) &= S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \quad (P) \\ &= S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma(\widetilde{W}_t - \theta t)\right) \quad (Q) \\ &= S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2} - \sigma\theta\right)t + \sigma\widetilde{W}_t\right) \\ &= S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}_t\right) \end{aligned}$$

ii)

We are asked to find the price of the claim $X = \max(NS_T, G)$, now from mathematical theory we have that the price is given by:

$$\pi(t) = e^{-r(T-t)}\mathbb{E}_Q[X|\mathcal{F}_t]$$

The goal here is to use the explicit formula given by Black & Scholes:

$$BS(t, T, S_t, K) = e^{-r(T-t)}\mathbb{E}_Q[(S_T - K)^+|\mathcal{F}_t]$$

where $(x)^+ = \max(x, 0)$, so let's do so, before we dig in, we show some useful tricks concerning the maximum:

$$\begin{aligned}
\max(A, B) &= \max(B - A, 0) + A \\
\max(A, B) &= \max(A - B, 0) + B \\
\max(A, B) &= (A - B)^+ + B
\end{aligned}$$

Hence:

$$\begin{aligned}
\max(NS_T, G) &= (NS_T - G)^+ + G \\
&\Downarrow \\
\frac{\max(NS_T, G)}{N} &= \left(S_T - \frac{G}{N}\right)^+ + \frac{G}{N} \\
&= Z + \frac{G}{N} \\
&\Downarrow \\
\max(NS_T, G) &= NZ + G
\end{aligned}$$

Now let's use finance theory:

$$\begin{aligned}
\pi(t) &= e^{-r(T-t)}\mathbb{E}_Q[X|\mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}_Q[(NZ + G)|\mathcal{F}_t] \\
&= Ne^{-r(T-t)}\mathbb{E}_Q[Z|\mathcal{F}_t] + e^{-r(T-t)}G \\
&= Ne^{-r(T-t)}\mathbb{E}_Q\left[\left(S_T - \frac{G}{N}\right)^+ |\mathcal{F}_t\right] + e^{-r(T-t)}G \\
&= NBS(t, T, S_t, G/N) + e^{-r(T-t)}G
\end{aligned}$$

Exercise 9.4

We are dealing with an endownment insurance, hence $S = \{*, \dagger\}$, the contractual information provided is:

```
T     <- 10    #length of contract
x0    <- 60    #age of insured
pi0   <- 10000
S0    <- 1
beta  <- 0.005 #deduction charges
r     <- 0.025
sigma <- 0.25  #volatility
```

From the text, we get:

$$
\begin{aligned}
C(T) &= \max((1-0.03)\pi_0 S_T (1-\beta)^{T-1}, \pi_0)\\
C(T) &= \max(N S_T, \pi_0)\\
C(T) &= N\left(S_T - \frac{\pi_0}{N}\right)^+ + \pi_0
\end{aligned}
$$

We also write down the policy functions, where we do not include premiums:

$$
a_*(t) = \begin{cases} 0, & t < 0 \\ 0, & t \in [0, T) \\ C(T), & t \geq T \end{cases} \quad a_\dagger(t) = 0 \;\; a_{*\dagger}(t) = 0 \;\; \dot{a}_*(t) = 0
$$

We also recall what we mean by the B&S-notation:

$$
BS(t, T, S_t, K) = e^{-r(T-t)} \mathbb{E}_Q \left[(S_T - K)^+ \middle| \mathcal{F}_t \right]
$$

This yields:

$$
\begin{aligned}
V_*^+(t, S_t) &= p_{**}^{x_0}(t, T) \frac{v(T)}{v(t)} \mathbb{E}_Q[C(T)|\mathcal{F}_t]\\
&= p_{**}^{x_0}(t, T) e^{-r(T-t)} \mathbb{E}_Q[N\left(S_T - \frac{\pi_0}{N}\right)^+ + \pi_0 | \mathcal{F}_t]\\
&= p_{**}^{x_0}(t, T) N e^{-r(T-t)} \mathbb{E}_Q\left[N\left(S_T - \frac{\pi_0}{N}\right)^+ |\mathcal{F}_t\right] + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0\\
&= p_{**}^{x_0}(t, T) N BS(t, T, S_t, \pi_0/N) + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0\\
&= p_{**}^{x_0}(t, T) \left[N BS(t, T, S_t, \pi_0/N) + e^{-r(T-t)} \pi_0\right]
\end{aligned}
$$

```r
#0: alive, 1:dead
mu01 <- function(t){
  A <- 0.0001
  B <- 0.00035
  c <- 1.075
  return(A + B*c^{t})
}

#survival function:
p_surv <- function(t,s){
  f <- mu01
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}

#Black&Scholes for call-option:
BS <- function(t,T,St, K){
   d1 <- (log(St/K) + (r + sigma^2/2)*(T-t)) / (sigma*sqrt(T-t))
   d2 <- d1 - sigma*sqrt(T-t)

   value <- St*pnorm(d1) - K*exp(-r*(T-t))*pnorm(d2)
   return(value)
}

V_reserve <- function(t, T,St){

  N <- (1-0.03)*pi0*(1-beta)^{(T-1)}

  reserve <- p_surv(t+x0, T+x0)*(N*BS(t,T,St,pi0/N) + exp(-r*(T-t))*pi0)
  return(reserve)
}
#reserve at t=0 with S0 = 1
V_reserve(t = 0, T = 10, St = 1)
```

```
## [1] 7556.453
```

Exercise 9.5

In this exercise we are asked to see what happens to the reserve at time $t = 6$, if:

i) Fund increases by 45%

ii) Fund increases by 5%

```r
#fund has increased by 45%:
V_reserve(t = 6, T = 10, St = 1.45)
```

```
## [1] 11635.45
```

```r
#fund has increased by 5%:
V_reserve(t = 6, T = 10, St = 1.05)
```

```
## [1] 9293.374
```

Exercise 9.6 B&S PDE

We are considering a call option: $X = (S_T - K)^+$, furthermore the stock price follows a GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \ \ (P)$$

We are asked to show that the claim value at time t denoted $v(t, S_t)$, satisfies the following PDE:

$$-\partial_t v + rv = \frac{1}{2}\sigma^2 x^2 \partial_{xx} v + rx\partial_x v$$

First of, by claim value at time t, we mean:

$$v(t, S_t) = e^{-r(T-t)}\mathbb{E}_Q[X|\mathcal{F}_t]$$

This is an object we can trade, and form the fundamental theorem of asset-pricing we have that all tradable assets are $(Q, \mathcal{F})$-martingales after discounting. This means that:

$$\frac{v(t, S_t)}{B_t}$$

should be a martingale, here $B_t = \exp\left(\int_0^t r_u du\right)$, in our case we have a constant interest rate, meaning that $B_t = e^{rt}$. Our strategy is to use the martigale representation theorem, from which we can conclude that the dt-terms should be zero.

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) + d\left[\frac{1}{B_t}\right] dv(t, S_t) \\ &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) \end{aligned}$$

Since we work in (Q)-framework, we stat the dynamics of S_t under Q first:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t \ \ (Q)$$

We start with the easy one first:

$$d(e^{-rt}) = -re^{-rt}dt$$

We will use Ito's formula on $v(t, x)$:

$$\begin{aligned} dv(t, x) &= \partial_t v dt + \partial_x v dS_t + \frac{1}{2}\partial_{xx} v (dS_t)^2 \\ &= \partial_t v dt + \partial_x v[rxdt + \sigma x d\widetilde{W}_t] + \frac{1}{2}\partial_{xx} v \sigma^2 x^2 dt \\ &= \left[\partial_t v + \partial_x v x + \frac{1}{2}\partial_{xx} v \sigma^2 x^2\right] dt + \partial_x v x d\widetilde{W}_t \end{aligned}$$

This leaves us with:

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= -re^{-rt} v dt + e^{-rt}\left(\left[\partial_t v + \partial_x v x + \frac{1}{2}\partial_{xx} v \sigma^2 x^2\right] dt + \partial_x v x d\widetilde{W}_t\right) \\ &= e^{-rt}\left[-rv + \partial_t v + rx\partial_x v + \sigma^2 x^2 \frac{1}{2}\partial_{xx} v\right] dt + e^{-rt} x \partial_x v d\widetilde{W}_t \end{aligned}$$

And then, combining the fundamental theorme of asset pricing with martingale representation theorem, we get that the dt-part must equal to zero, hence:

$$-\partial_t v + rv = \frac{1}{2}\sigma^2 x^2 \partial_{xx} v + rx\partial_x v$$