

Exercises chapter 9

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Exercise 9.3 Markov property of B&S

i)

We are asked to show that the solution to the GBM under Q is given by:

$$S(t) = S(0) \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma \widetilde{W}_t \right)$$

Now from arbitrage theory, we have that \widetilde{S}_t is a (Q, \mathcal{F}) -martingale, here I just state the dynamics of \widetilde{S} under P :

$$\begin{aligned} d(e^{-rt} S_t) &= S_t e^{-rt} \sigma \left[\frac{\mu - r}{\sigma} dt + dW_t \right] \\ &= \widetilde{S}_t \sigma [\theta dt + dW_t] \end{aligned}$$

From Girsanov's thm we have that Z_t a martingale

$$Z_t = \exp \left(\int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right)$$

yields that $\widetilde{W}_t = W_t - \int_0^t \varphi_s ds$ is a (Q, \mathcal{F}) -BM.

So combining the martingale representation theorem with a clever way of finding φ_s , we see that $\varphi_t = -\theta$, because, we then get that:

$$\widetilde{W}_t = W_t + \int_0^t \theta ds$$

is a (Q, \mathcal{F}) -BM. This gives us $W_t = \widetilde{W}_t - \theta t$, from theory we have that the solution to a GBM under (P) is:

$$\begin{aligned} S(t) &= S(0) \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (P) \\ &= S(0) \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma (\widetilde{W}_t - \theta t) \right) \quad (Q) \\ &= S(0) \exp \left(\left(\mu - \frac{\sigma^2}{2} - \sigma \theta \right) t + \sigma \widetilde{W}_t \right) \\ &= S(0) \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma \widetilde{W}_t \right) \end{aligned}$$

ii)

We are asked to find the price of the claim $X = \max(NS_T, G)$, now from mathematical theory we have that the price is given by:

$$\pi(t) = e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t]$$

The goal here is to use the explicit formula given by Black & Scholes:

$$BS(t, T, S_t, K) = e^{-r(T-t)} \mathbb{E}_Q[(S_T - K)^+ | \mathcal{F}_t]$$

where $(x)^+ = \max(x, 0)$, so let's do so, before we dig in, we show some useful tricks concerning the maximum:

$$\max(A, B) = \max(B - A, 0) + A$$

$$\max(A, B) = \max(A - B, 0) + B$$

$$\max(A, B) = (A - B)^+ + B$$

Hence:

$$\begin{aligned} \max(NS_T, G) &= (NS_T - G)^+ + G \\ &\Downarrow \\ \frac{\max(NS_T, G)}{N} &= \left(S_T - \frac{G}{N}\right)^+ + \frac{G}{N} \\ &= Z + \frac{G}{N} \\ &\Downarrow \\ \max(NS_T, G) &= NZ + G \end{aligned}$$

Now let's use finance theory:

$$\begin{aligned} \pi(t) &= e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t] \\ &= e^{-r(T-t)} \mathbb{E}_Q[(NZ + G) | \mathcal{F}_t] \\ &= N e^{-r(T-t)} \mathbb{E}_Q[Z | \mathcal{F}_t] + e^{-r(T-t)} G \\ &= N e^{-r(T-t)} \mathbb{E}_Q \left[\left(S_T - \frac{G}{N}\right)^+ | \mathcal{F}_t \right] + e^{-r(T-t)} G \\ &= NBS(t, T, S_t, G/N) + e^{-r(T-t)} G \end{aligned}$$

Exercise 9.4

We are dealing with an endowment insurance, hence $S = \{*, \dagger\}$, the contractual information provided is:

```
T      <- 10      #length of contract
x0     <- 60      #age of insured
pi0    <- 10000
S0     <- 1
beta   <- 0.005   #deduction charges
r      <- 0.025
sigma  <- 0.25    #volatility
```

From the text, we get:

$$\begin{aligned} C(T) &= \max((1 - 0.03)\pi_0 S_T (1 - \beta)^{T-1}, \pi_0) \\ C(T) &= \max(NS_T, \pi_0) \\ C(T) &= N \left(S_T - \frac{\pi_0}{N} \right)^+ + \pi_0 \end{aligned}$$

We also write down the policy functions, where we do not include premiums:

$$a_*(t) = \begin{cases} 0, & t < 0 \\ 0, & t \in [0, T) \\ C(T), & t \geq T \end{cases} \quad a_{\dagger}(t) = 0 \quad a_{*\dagger}(t) = 0 \quad \dot{a}_*(t) = 0$$

We also recall what we mean by the B&S-notation:

$$BS(t, T, S_t, K) = e^{-r(T-t)} \mathbb{E}_Q \left[(S_T - K)^+ \middle| \mathcal{F}_t \right]$$

This yields:

$$\begin{aligned} V_*^+(t, S_t) &= p_{**}^{x_0}(t, T) \frac{v(T)}{v(t)} \mathbb{E}_Q[C(T) | \mathcal{F}_t] \\ &= p_{**}^{x_0}(t, T) e^{-r(T-t)} \mathbb{E}_Q \left[N \left(S_T - \frac{\pi_0}{N} \right)^+ + \pi_0 \middle| \mathcal{F}_t \right] \\ &= p_{**}^{x_0}(t, T) N e^{-r(T-t)} \mathbb{E}_Q \left[N \left(S_T - \frac{\pi_0}{N} \right)^+ \middle| \mathcal{F}_t \right] + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0 \\ &= p_{**}^{x_0}(t, T) N BS(t, T, S_t, \pi_0/N) + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0 \\ &= p_{**}^{x_0}(t, T) \left[N BS(t, T, S_t, \pi_0/N) + e^{-r(T-t)} \pi_0 \right] \end{aligned}$$

```

#0: alive, 1:dead
mu01 <- function(t){
  A <- 0.0001
  B <- 0.00035
  c <- 1.075
  return(A + B*c^{t})
}

#survival function:
p_surv <- function(t,s){
  f <- mu01
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}

#Black&Scholes for call-option:
BS <- function(t,T,St, K){
  d1 <- (log(St/K) + (r + sigma^2/2)*(T-t)) / (sigma*sqrt(T-t))
  d2 <- d1 - sigma*sqrt(T-t)

  value <- St*pnorm(d1) - K*exp(-r*(T-t))*pnorm(d2)
  return(value)
}

V_reserve <- function(t, T,St){
  N <- (1-0.03)*pi0*(1-beta)^{(T-1)}

  reserve <- p_surv(t+x0, T+x0)*(N*BS(t,T,St,pi0/N) + exp(-r*(T-t))*pi0)
  return(reserve)
}

#reserve at t=0 with S0 = 1
V_reserve(t = 0, T = 10, St = 1)

## [1] 7556.453

```

Exercise 9.5

In this exercise we are asked to see what happens to the reserve at time $t = 6$, if:

- i) Fund increases by 45%
- ii) Fund increases by 5%

```

#fund has increased by 45%:
V_reserve(t = 6, T = 10, St = 1.45)

## [1] 11635.45

#fund has increased by 5%:
V_reserve(t = 6, T = 10, St = 1.05)

## [1] 9293.374

```

Exercise 9.6 B&S PDE

We are considering a call option: $X = (S_T - K)^+$, furthermore the stock price follows a GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (P)$$

We are asked to show that the claim value at time t denoted $v(t, S_t)$, satisfies the following PDE:

$$-\partial_t v + rv = \frac{1}{2} \sigma^2 x^2 \partial_{xx} v + rx \partial_x v$$

First of, by claim value at time t , we mean:

$$v(t, S_t) = e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t]$$

This is an object we can trade, and from the fundamental theorem of asset-pricing we have that all tradable assets are (Q, \mathcal{F}) -martingales after discounting. This means that:

$$\frac{v(t, S_t)}{B_t}$$

should be a martingale, here $B_t = \exp\left(\int_0^t r_u du\right)$, in our case we have a constant interest rate, meaning that $B_t = e^{rt}$. Our strategy is to use the martingale representation theorem, from which we can conclude that the dt -terms should be zero.

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) + d\left[\frac{1}{B_t}\right] dv(t, S_t) \\ &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) \end{aligned}$$

Since we work in (Q) -framework, we state the dynamics of S_t under Q first:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t \quad (Q)$$

We start with the easy one first:

$$d(e^{-rt}) = -re^{-rt} dt$$

We will use Ito's formula on $v(t, x)$:

$$\begin{aligned} dv(t, x) &= \partial_t v dt + \partial_x v dS_t + \frac{1}{2} \partial_{xx} v (dS_t)^2 \\ &= \partial_t v dt + \partial_x v [rx dt + \sigma x d\widetilde{W}_t] + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 dt \\ &= \left[\partial_t v + \partial_x vx + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 \right] dt + \partial_x v x d\widetilde{W}_t \end{aligned}$$

This leaves us with:

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= -re^{-rt} v dt + e^{-rt} \left(\left[\partial_t v + \partial_x vx + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 \right] dt + \partial_x v x d\widetilde{W}_t \right) \\ &= e^{-rt} \left[-rv + \partial_t v + rx \partial_x v + \sigma^2 x^2 \frac{1}{2} \partial_{xx} v \right] dt + e^{-rt} x \partial_x v d\widetilde{W}_t \end{aligned}$$

And then, combining the fundamental theorem of asset pricing with martingale representation theorem, we get that the dt -part must equal to zero, hence:

$$-\partial_t v + rv = \frac{1}{2} \sigma^2 x^2 \partial_{xx} v + rx \partial_x v$$