Exercises chapter 10

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Exercise 10.1

In this exercise we are dealing with a term-insurance, meaning that $S = \{*, \dagger\}$, we have stochastic interest rates, which follows a Vasicke model, furthermore, the risk-premium is given by $\gamma = 0$, meaning that we will have the same model under P as under Q, the contractual information provided is:

```
T  <- 10
x0 <- 50
Benefit  <- 200000 #need B as function in ATS-form of Vasicek

#Vasicek parameters:
a     <- 5
b     <- 0.04
sigma <- 0.01
r0    <- 0.02
gamma <- 0

#0:alive, 1: dead
mu01 <- 0.009

p_surv <- function(t,s){
  f <- mu01
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}
```

We clarify, what the survival-probability is:

$$\begin{aligned} p(t,s) &= \exp\left(-\int_t^s \mu_{*\dagger}(u)du\right) \\ &= \exp(-0.009[s-t]) \end{aligned}$$

The desired Vasicek model is in this case given by:

$$dr_t = a(b - r_t)dt + \sigma dW_t^Q$$

Furthermore it's said that premiums are paid continiously, giving us the following policy-functions:

$$a_*(t) = \begin{cases} 0, & t < 0 \\ -\pi t, & t \in [0,T) \\ -\pi T, & t \geq T \end{cases} \quad a_{*\dagger}(t) = \begin{cases} B, & t \in [0,T) \\ 0, & else \end{cases}$$

and

$$\dot{a}_*(t) = -\pi \mathbb{1}_{[0,T)}(t)$$

We start off by stating the general formula for reserves in continous-time:

$$V_i^+(t,A) = \sum_j \frac{v(T)}{v(t)} p_{ij}^x(t,T)\Delta a_j(T) + \sum_j \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t,s) da_j(s) + \sum_{k\neq j} \int_t^T \frac{v(s)}{v(t)} p_{ij}^x(t,s)\mu_{jk}^x(s) a_{jk}(s) ds$$

Furthermore, we define:

$$V_i^+(t,r_t) = \mathbb{E}_Q[V_i^+(t,A)|\mathcal{F}_t]$$

we also have:

$$P(t,T) = \mathbb{E}_Q[e^{-\int_t^T r_u du}|\mathcal{F}_t]$$

We have $i = *$ and $j \in \{*, \dagger\}$, this gives us:

$$V_*^+(t,r_t) = -\pi \int_t^T P(t,s) p_{**}^x(t,s) ds + B \int_t^T P(t,s) p_{**}^x(t,s)\mu_{*\dagger}^x(s) ds$$

Since we are dealing with the Vasicek model, we have that it belongs to a class called: Affine term structures, which in simplicity means that the zero-coupon bond price $P(t,T)$ have an explicit form:

$$P(t,T) = e^{-A(T-t)r_t + B(T-t)}$$

Where:

$$A(h) = \frac{1-e^{-ah}}{h} \text{ and } B(h) = \left(b + \frac{\gamma\sigma}{a} - \frac{\sigma^2}{2a^2}\right)[A(h) - h] - \frac{\sigma^2}{4a} A(h)^2$$

Now from the equivalence principle, we get that the fair premium π is given by:

$$0 = V_*^+(0,A) = \mathbb{E}_Q[V_*^+(0,r_0)]$$

Hence:

$$0 = -\pi \int_0^T P(0,s) p_{**}^x(0,s) ds + B \int_0^T P(0,s) p_{**}^x(0,s)\mu_{*\dagger}^x(s) ds$$

$$\Downarrow$$

$$\pi = \frac{B \int_0^T P(0,s) p_{**}^x(0,s)\mu_{*\dagger}^x(s) ds}{\int_0^T P(0,s) p_{**}^x(0,s) ds}$$

We have been given the following:

$$\mu_{*\dagger}(t) = 0.009$$

```r
#0:alive, 1:dead
mu01 <- function(u){
  return(0.009)
}

p_surv <- function(t,s){
  f <- Vectorize(mu01)
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}
```

```r
A <- function(h){
  return((1-exp(a*h))/h)
}

B <- function(h){
  res <- (b + gamma*sigma/a - (sigma**2)/(2*a**2))*(A((h))-(h)) -
         ((sigma**2)/(4*a))*A((h))**2
  return(res)
}

P <- function(u){
  ans <- exp((-1)*A(u)*r0 + B(u))
  return(ans)
}

above <- function(u){
  return(p_surv(0,u)*P(u)*mu01(u))
}

below <- function(u){
  return(p_surv(0,u)*P(u))
}

upper <- Benefit*integrate(Vectorize(above), 0, T)$value
lower <- integrate(Vectorize(below), 0, T)$value

yearly_premium <- upper/lower
yearly_premium
```

```
## [1] 1800
```