Exercises chapter 6

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Exercise 6.2

We have the following contractual information:

```
T <- 10     #length of contract
x <- 60     #age of insured
D <- 20000  #disability pension while disabeld
B <- 50000  #death benefit
r <- 0.05   #intensity rate
```

Where our state space is $S = \{*, \diamond, \dagger\}$, we have the following policy-functions:

$$a_{\diamond}(t) = \begin{cases} 0, & t < 0 \\ Dt, & t \in [0, T) \\ DT, & t \geq T \end{cases} \quad a_{*\dagger}(t) = \begin{cases} B, & t \in [0, T) \\ 0, & else \end{cases}$$

Since we for now do not care about the premiums in the policy-functions, we get

$$\dot{a}_{*}(t) = 0$$

and

$$\dot{a}_{\diamond}(t) = D\mathbb{1}_{[0,T)}(t)$$

We can set up thieles differential equations:

$$\frac{d}{dt}V_{*}^{+}(t) = r(t)V_{*}(t) - \mu_{*\diamond}^{x}(t)[V_{\diamond}^{+}(t) - V_{*}^{+}(t)] - \mu_{*\dagger}^{x}(t)[B\mathbb{1}_{[0,T)}(t) - V_{*}^{+}(t)]$$
$$\frac{d}{dt}V_{\diamond}^{+}(t) = r(t)V_{\diamond}(t) - D\mathbb{1}_{[0,T)}(t) - \mu_{*\diamond}^{x}(t)[V_{*}^{+}(t) - V_{\diamond}^{+}(t)] + \mu_{\diamond\dagger}^{x}(t)V_{\diamond}^{+}(t)$$

With finial conditions: $V_{*}^{+}(T) = V_{\diamond}^{+}(T) = 0$

In order to solve this numerically, we partition up $[0, T]$ in times $t_i = t_0 + ih$, with $0 = t_0 < \cdots < t_n = T$.

We also iterate backwards, giving us:

$$\frac{V_{*}^{+}(t_i) - V_{*}^{+}(t_{i-1})}{h} = rV_{*}(t_i) - \mu_{*\diamond}^{x}(t_i)[V_{\diamond}^{+}(t_i) - V_{*}^{+}(t_i)] - \mu_{*\dagger}^{x}(t_i)[B - V_{*}^{+}(t_i)]$$

We then solve for $V_{*}(t_{i-1})$, giving us:

$$V_{*}^{+}(t_{i-1}) = V_{*}^{+}(t_i) - h\left[rV_{*}^{+}(t_i) - \mu_{*\dagger}^{x}(t_i)[V_{\diamond}^{+}(t_i) - V_{*}^{+}(t_i)] - \mu_{*\dagger}^{x}(t)[B - V_{*}(t_i)]\right]$$

We do the same for $V_{\diamond}^{+}(t_{i-1})$:

$$V_{\diamond}^{+}(t_{i-1}) = V_{\diamond}^{+}(t_i) - h\left[rV_{\diamond}^{+}(t_i) - D\mathbb{1}_{[0,T)}(t) - \mu_{\diamond *}^{x}(t_i)[V_{*}^{+}(t_i) - V_{\diamond}^{+}(t_i)] + \mu_{\diamond\dagger}^{x}(t_i)V_{\diamond}^{+}(t_i)\right]$$

```r
#states 0:alive, 1:disabled, 2:dead

mu01 <- function(u){
  a1 <- 4*10^{-4}
  b1 <- 3.4674*10^{-6}
  c1 <- 0.138755
  return(a1 + b1*exp(c1*u))
}

mu02 <- function(u){
  a2 <- 5*10^{-4}
  b2 <- 7.5858*10^{-5}
  c2 <- 0.087498
  return(a2 + b2*exp(c2*u))
}

mu12 <- function(u){
  return(mu02(u))
}

mu10 <- function(u){
  return(0.1*mu01(u))
}
```

```r
#a.e. derivative of a_{\diamond}(t)
dis_dot <- function(t){
  if(t >= 0 && t<T){ return(D) }
  if(t >= T){ return(0) }
}

#PV of policy V0 and V1
h <- 1/12
N <- T/h

PV_act <- PV_dis <- rep(0,N+1)
PV_act[N+1] <- PV_dis[N+1] <- 0 #by construction of contract

for(i in N:1){
  PV_act[i] <- PV_act[i+1] - h*(r*PV_act[i+1]
                                - mu01(x+(i+1)*h)*(PV_dis[i+1]-PV_act[i+1])
                                - mu02(x+(i+1)*h)*(B-PV_act[i+1])
                                )

  PV_dis[i] <- PV_dis[i+1] - h*(r*PV_dis[i+1]- dis_dot((i+1)*h)
                                - mu10(x+(i+1)*h)*(PV_act[i+1]-PV_dis[i+1])
                                + mu12(x+(i+1)*h)*PV_dis[i+1]
                                )
}
```

And from this we can find the one-time premium $\pi_0 = V_*^+(0)$:

```
pi0 <- PV_act[1]
pi0
```

```
## [1] 20608.17
```

We were asked to find the yearly premium π, we then use the method, where we create an artificial policy, where one pays a premium of NOK 1 during $[0, T)$, i.e this is the only contribution of interest, thus:

$$
\begin{aligned}
&a_*^{Prem=1}(t) = \begin{cases} 0, & t < 0 \\ -t, & t \in [0, T) \\ -T, & t \geq T \end{cases} \implies \dot{a}_*^{Prem=1}(t) = -1\mathbb{1}_{[0,T)}(t) \\
&a_\diamond^{Prem=1}(t) = 0, \ \forall t \implies \dot{a}_\diamond^{Prem=1}0 \\
&a_{*\dagger}^{Prem=1}(t) = 0
\end{aligned}
$$

We can now set up the new thieles equations:

$$
\begin{aligned}
\frac{d}{dt}V_*^+(t, A^{Prem=1}) &= rV_*^+(t) - \dot{a}_*(t) - \mu_{*\diamond}^x(t)[V_\diamond^+(t) - V_*^+(t)] + \mu_{*\dagger}^x(t)V_*^+(t) \\
\frac{d}{dt}V_\diamond^+(t, A^{Prem=1}) &= rV_\diamond(t) - \mu_{*\diamond}^x(t)[V_*^+(t) - V_\diamond^+(t)] + \mu_{\diamond\dagger}^x(t)V_\diamond^+(t)
\end{aligned}
$$

And again solve for $V_*^+(t_{i-1})$ and $V_\diamond^+(t_{i-1})$:

$$
\begin{aligned}
V_*^+(t_{i-1}) &= V_*^+(t_i) - h\left[rV_*^+(t_i) - \dot{a}_*(t_i) - \mu_{*\dagger}^x(t_i)[V_\diamond^+(t_i) - V_*^+(t_i)] + \mu_{*\dagger}^x(t)V_*(t_i)\right] \\
V_\diamond^+(t_{i-1}) &= V_\diamond^+(t_i) - h\left[rV_\diamond^+(t_i) - \mu_{\diamond *}^x(t_i)[V_*^+(t_i) - V_\diamond^+(t_i)] + \mu_{\diamond\dagger}^x(t_i)V_\diamond^+(t_i)\right]
\end{aligned}
$$

```
#a.e derivative a_{*}:
a_dot_prem1 <- function(t){
  if(t >= 0 && t < T){ return(-1) }
  if(t >= T){ return(0) }
}

PV_prem1_act <- PV_prem1_dis <- rep(0,N+1)
PV_prem1_act[N+1] <- PV_prem1_dis[N+1] <- 0 #follows from contract

for(i in N:1){
  PV_prem1_act[i] <- PV_prem1_act[i+1] - h*(r*PV_prem1_act[i+1] - a_dot_prem1((i+1)*h)
                                  - mu01(x+(i+1)*h)*(PV_prem1_dis[i+1]-PV_prem1_act[i+1])
                                  + mu02(x+(i+1)*h)*(PV_prem1_act[i+1])
                                  )

  PV_prem1_dis[i] <- PV_prem1_dis[i+1] - h*(r*PV_prem1_dis[i+1]
                                  - mu10(x+(i+1)*h)*(PV_prem1_act[i+1]-PV_prem1_dis[i+1])
                                  + mu12(x+(i+1)*h)*PV_prem1_dis[i+1]
                                  )
}
PV_prem1_act[1]
```

```
## [1] -6.453883
```

We will now use the equvalence principle, which tells us:

$$\pi V_*^+(0, A^{Prem=1}) + V_*^+(0) = 0$$
$$\Downarrow$$
$$\pi = -\frac{V_*(0)}{V_*^+(0, A^{Prem=1})}$$

This gives us:

```
pi0
```

```
## [1] 20608.17
```

```
yearly_premium <- (-1)*pi0/PV_prem1_act[1]
yearly_premium
```

```
## [1] 3193.142
```

Exercise 6.4

We are dealing with an endownment insurance, meaning that $S = \{*, \dagger\}$, we are asked to calculate the second moment of the benefit payments V_t^+ for $t = 10, \dots, 20$. The contractual information provided is:

```
x <- 40
T <- 20
E <- 115000
B <- 220000
r <- 0.03
```

We start by writing up the policy functions:

$$a_*^{Pre}(n) = \begin{cases} 0 & , n = 0, \dots, T-1 \\ E & , else \end{cases} \quad a_{*\dagger}^{Post}(n) = \begin{cases} B & , n = 0, \dots, T-1 \\ 0 & , else \end{cases}$$

From chapter.5 we are given the following formula:

$$\mathbb{E}[(V_t^+)^p | X_t = i] = (v_t)^p \sum_j p_{ij}^x(t, t+1) \sum_{k=0}^{p} \binom{p}{k} (a_{ij}^{Post}(t))^{p-k} \mathbb{E}[(V_{t+1}^+)^k | X_{t+1} = j]$$

Where v_t is the one-step discount factor in $[t, t+1]$ i.e:

$$v(t) = v_0 v_1 \dots v_{t-1}$$

Let's denote $V_2(t) := \mathbb{E}[(V_t^+)^2 | X_t = *]$, and then translate the equation above into our situation:

$$\begin{aligned} V_2(t) &= (v_t)^2 \sum_j p_{*j}^x(t, t+1) \sum_{k=0}^{2} \binom{2}{k} (a_{*j}^{Post}(t))^{2-k} \mathbb{E}[(V_{t+1}^+)^k | X_{t+1} = j] \\ &= (v_t)^2 p_{**}^x(t, t+1) \mathbb{E}[(V_{t+1}^+)^2 | X_{t+1} = *] + (v_t)^2 p_{*\dagger}^x(t, t+1)(a_{*\dagger}^{Post}(t))^2 \\ &= (v_t)^2 p_{**}^x(t, t+1) V_2(t+1) + (v_t)^2 p_{*\dagger}^x(t, t+1)(a_{*\dagger}^{Post}(t))^2 \\ &= e^{-2r}[p_{**}^x(t, t+1) V_2(t+1) + p_{*\dagger}^x(t, t+1)(a_{*\dagger}^{Post}(t))^2] \end{aligned}$$

This has now become an difference equation, with final condition:

$$V_2(T) = (\Delta a_*^{Pre}(T))^2 = E^2$$

From the exercise it's also given that:

$$\mu^{x}_{*\dagger}(t) = 0.0018 + 0.0004t$$

```
#0:alive, 1:dead

mu01_x <- function(t){
  return(0.0018 + 0.0004*t)
}

p_surv_x <- function(t,s){
  f <- mu01_x
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}
```

```
V2 <- rep(0,T+1)
V2[T+1] <- E^{2}

for(n in (T-1):0){
  V2[n+1] <- exp(-2*r)*(p_surv_x(n,n+1)*V2[n+2] + B^{2}*(1-p_surv_x(n,n+1)))
}
V2
```

```
##  [1]  6167825284  6465436986  6765434135  7068218098  7374219626  7683901471
##  [7]  7997761234  8316334482  8640198138  8969974190  9306333731  9650001385
## [13] 10001760131 10362456580 10733006746 11114402361 11507717773 11914117511
## [19] 12334864556 12771329405 13225000000
```