Exerciseses chapter 4

Andreas Slåttelid

23/2/2022

Exercise 4.3

In this exercise we are considering a disability insurance, hence $S = \{*, \diamond, \dagger\}$. The contractual information given are: as follows:

```
x <- 30      #age
T <- 35      #length of contract
D <- 20000  #disability pension
r <- 0.03    #intereset rate
premium_yearly <- 2500
```

Furthermore the transition rates are given by:

```
lambda <- function(t){
  #we do not have any t-dependency
  m12 <- 0.0279
  m13 <- 0.0229
  m11 <- -(m12+m13)
  m21 <- 0
  m23 <- 0.0229
  m22 <- -(m21+m23)
  m31 <- m32 <- m33 <- 0
  L <- matrix(c(m11,m12,m13,m21,m22,m23,m31,m32,m33), nrow=3, byrow=TRUE)

  return(L)
}
```

We also have the following policy-functions:

$$a_*(t) = \begin{cases} 0, & t < 0 \\ -\pi t, & t \in [0, T) \\ -\pi T, & t \geq T \end{cases}$$

$$a_\diamond(t) = \begin{cases} 0, & t < 0 \\ Dt, & t \in [0, T) \\ DT, & t \geq T \end{cases}$$

We were asked to calculate $V_j^+(t, A)$ for $j = *, \diamond$ and for $t = 30$. We then get:

$$\begin{aligned} V_*^+(t, A) &= \frac{1}{v(t)} \left[\int_t^T v(s) p_{**}(x+t, x+s) da_*(s) + \int_t^T v(s) p_{*\diamond}(x+t, x+s) da_\diamond(s) \right] \\ &= \frac{1}{v(t)} \left[-\pi \int_t^T v(s) p_{**}(x+t, x+s) ds + D \int_t^T v(s) p_{*\diamond}(x+t, x+s) ds \right] \end{aligned}$$

$$
\begin{aligned}
V_{\diamond}^{+}(t,A) &= \frac{1}{v(t)}\left[\int_t^T v(s)p_{\diamond\diamond}(x+t,x+s)da_{\diamond}(s)\right] \\
&= \frac{1}{v(t)}\left[D\int_t^T v(s)p_{\diamond\diamond}(x+t,x+s)ds\right]
\end{aligned}
$$

We will now use an Euler-scheme to simulate what's going on, and then use a Riemann-sum to compute the integrals:

$$
\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} f(x_i^*)\Delta x
$$

Where $\Delta x = \frac{b-a}{n}$, which in our case corresponds to: h and $x_i^* \in [x_i, x_{i+1}]$

```
field <- function(t,M){
  return(M%*%lambda(t))
}

#field(t0, P0)

Euler <- function(t0,P0,h,tn){
  if(t0==tn){ return(P0)}
  N <- (tn-t0)/h
  D <- dim(P0)[1] #gives dimension of mxn matrix dim = c(m,n)
  #Initial condition at s
  P <- array(diag(D*(N+1)), dim=c(D,D,N+1))
  #First iteration
  P[,,1] <- P0


  for(n in 1:N){
    P[,,n+1] <- P[,,n]+h*field(t0+n*h,P[,,n])
  }
  return(P) #returns array, array(data , dim= c(3,3,2)), this stores 2 3x3 matricies
}

t0 <- 60       #start age
P0 <- diag(3)  #initial start with P(s,s) = I
tn <- 110      #end age
h <- 1/12      #step size
N <- (tn-t0)/h #number of steps

sol <- Euler(t0,P0,h,tn) #contains ages transition probs from 60 to 65.

v <- function(t){
  return(exp(-r*t))
}


#v(s), but s in [30,35]
v_ages <- sapply(seq(30, 35, by = h), v) #we apply the function v to each element
```

```r
#sum1 where we do not care about the premium until afterwards, we use Riemann-sum.
s1 <- 0
for (i in 1:(length(v_ages)-1)){
  s1 <- s1 + v_ages[i]*sol[1,1, ][i]*h #survival i.e * -> *
}

#sum2: do not care about D=20.000 until afterwards
s2 <- 0
for (i in 1:(length(v_ages)-1)){
  s2 <- s2 + v_ages[i]*sol[1,2,][i]*h #disability i.e * -> dis
}

#V*(30, A):
V_30_A <- (1/v(30))*((-1)*premium_yearly*s1 + D*s2)
V_30_A
```

```
## [1] -4781.495
```

```r
#V_[disabeld](30,A):

s3 <- 0
for (i in 1:(length(v_ages)-1)){
  s3 <- s3 + v_ages[i]*sol[2,2,][i]*h #disability i.e dis -> dis
}


#V_[disabeld](30,A):
V_30_dis <- (1/v(30))*D*s3
V_30_dis
```

```
## [1] 88057.1
```

Exercise 4.6

Required-libraries

```
list.of.packages <- c("tidyverse", "scales")
new.packages <- list.of.packages[!(list.of.packages %in% installed.packages()[,"Package"])]
if(length(new.packages)) install.packages(new.packages)
library(tidyverse)
library(scales)   #nice formatting of plots
```

We are dealing with an endownment policy, in discrete time, this means that $S = \{*, \dagger\}$, the contractual information provided is:

```
x <- 35     #age
T <- 25     #length of contract
r <- 0.035  #interest rate
E <- 125000 #endownment
B <- 250000 #death-benfit
```

futhermore the force of mortality is given by:

```
mu <- function(t){
  return(0.0015 + 0.0004*(t-35))
}
```

discount-factor:

```
v <- function(t){
  return(exp(-r*t))
}
```

We can also calulate the survival probability:

$$p_{**}(s,t) = \exp\left(-\int_s^t \mu_{*\dagger}(u)du\right)$$

```
p_surv <- function(s,t){
  integral <- 0.0015*(t-s) + 0.0004*((1/2)*(t^2 -s^2) + 35*(s-t))
  return(exp(-integral))
}
p_surv(35,36)
```

```
## [1] 0.9983014
```

We are asked to calculate the yearly premium π, but first we start by defining the policy functions specified in the contract.

$$a_*^{Pre}(n) = \begin{cases} -\pi & ,n = 0,\dots,T-1 \\ E & ,n = T \end{cases} \quad a_{*\dagger}^{Post}(n) = \begin{cases} B & ,n = 0,\dots,T-1 \\ 0 & ,else \end{cases}$$

We can now start to calculate the reserves, but we must be aware of what happens at $n = T$:

$$\begin{aligned} V_*^+(t,A) &= \frac{1}{v(t)}\left[\sum_{n=t}^{T} v(n)p_{**}(x+t,x+n)a_*^{Pre}(n) + \sum_{n=t}^{T-1} v(n+1)p_{**}(x+t,x+n)p_{*\dagger}(x+n,x+n+1)a_{*\dagger}^{Post}(n)\right] \\ &= \frac{1}{v(t)}\left[\alpha+\beta\right] \end{aligned}$$

Here α corresponds to the first sum, and β corresponds to the second sum:

$$\alpha = \sum_{n=t}^{T-1} v(n)p_{**}(x+t, x+n)(-\pi) + v(T)p_{**}(x+t, x+T)E$$
$$\beta = \sum_{n=t}^{T-1} v(n+1)p_{**}(x+t, x+n)p_{*\dagger}(x+n, x+n+1)B$$

We will now use the equivalence principle, which states that the fair premium π should be such that $V_*(0, A) = 0$, and using this method we end up with:

$$\pi = \frac{v(T)p_{**}(x, x+T)E + B\sum_{n=0}^{T-1} v(n+1)p_{**}(x, x+n)p_{*\dagger}(x+n, x+n+1)}{\sum_{n=0}^{T-1} v(n)p_{**}(x, x+n)}$$

```
res1 <- 0
for (n in 0:(T-1)){
  res1 <- res1 + v(n+1)*p_surv(x, x+n)*(1-p_surv(x+n, x+n+1))
}

res2 <- 0
for (n in 0:(T-1)){
  res2 <- res2 + v(n)*p_surv(x, x+n)
}

above <- B*res1 + v(T)*p_surv(x,x+T)*E #whats above in the premium expression
below <- res2             #whats below in the premium expression

premium_yearly <- above/below
premium_yearly
```

```
## [1] 4095.413
```

We now also want the prospective reserves: i.e $V_*^+(t, A)$ for $t = 0, \ldots, T$:

```
#Present value of reserve when including premiums.
V_reserve_1 <- function(x=35, E = 125000 , T=25){
  #x: age of insured
  #B: benefit specified in contract
  #T: length of contract

  V_star_pre <- NULL
  for (t in 0:(T-1)){
    s <- 0
    for (n in t:(T-1)){
      s <- s + (1/v(t))*(-1)*premium_yearly*v(n)*p_surv(t+x, n+x)
    }
    V_star_pre[t+1] <- s  + (1/v(t))*v(T)*p_surv(x+t, x+T)*E
  }
  V_star_pre[T+1] <- E #specified by contract
  return(V_star_pre)
}
```

```
V_reserve_2 <- function(x=35, B=250000, T=25){
  #x: age of insured
  #B: benefit specified in contract
  #T: length of contract

  V_star_post <- NULL
  for (t in 0:(T-1)){
    s <- 0
    for (n in t:(T-1)){
      s <- s + (1/v(t))*B*v(n+1)*p_surv(x+t, x+n)*(1-p_surv(x+n, x+n+1))
    }
    V_star_post[t+1] <- s
  }
  V_star_post[T+1] <- 0 #specified by contract
  return(V_star_post)
}

reserve_total <- V_reserve_1() + V_reserve_2()
reserve_total
```

```
##  [1] 1.091394e-11 3.823145e+03 7.692302e+03 1.161236e+04 1.558847e+04
##  [6] 1.962604e+04 2.373075e+04 2.790862e+04 3.216595e+04 3.650944e+04
## [11] 4.094613e+04 4.548348e+04 5.012938e+04 5.489218e+04 5.978071e+04
## [16] 6.480437e+04 6.997309e+04 7.529742e+04 8.078858e+04 8.645848e+04
## [21] 9.231977e+04 9.838591e+04 1.046712e+05 1.111910e+05 1.179615e+05
## [26] 1.250000e+05
```