

## Exercises chapter 10

Andreas Slättelid

### Exercise 10.2

i)

We are asked to show that the solution to the GBM under  $Q$  is given by:

$$S(t) = S(0) \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma \widetilde{W}_t \right)$$

Now from arbitrage theory, we have that  $\widetilde{S}_t$  is a  $(Q, \mathcal{F})$ -martingale, here I just state the dynamics of  $\widetilde{S}$  under  $P$ :

$$\begin{aligned} d(e^{-rt} S_t) &= S_t e^{-rt} \sigma \left[ \frac{\mu - r}{\sigma} dt + dW_t \right] \\ &= \widetilde{S}_t \sigma [\theta dt + dW_t] \end{aligned}$$

From Girsanov's thm we have that  $Z_t$  a martingale

$$Z_t = \exp \left( \int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right)$$

yields that  $\widetilde{W}_t = W_t - \int_0^t \varphi_s ds$  is a  $(Q, \mathcal{F})$ -BM.

So combining the martingale representation theorem with a clever way of finding  $\varphi_s$ , we see that  $\varphi_t = -\theta$ , because, we then get that:

$$\widetilde{W}_t = W_t + \int_0^t \theta ds$$

is a  $(Q, \mathcal{F})$ -BM. This gives us  $W_t = \widetilde{W}_t - \theta t$ , from theory we have that the solution to a GBM under  $(P)$  is:

$$\begin{aligned} S(t) &= S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right) \quad (P) \\ &= S(0) \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma (\widetilde{W}_t - \theta t) \right) \quad (Q) \\ &= S(0) \exp \left( \left( \mu - \frac{\sigma^2}{2} - \sigma \theta \right) t + \sigma \widetilde{W}_t \right) \\ &= S(0) \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma \widetilde{W}_t \right) \end{aligned}$$

ii)

We are asked to find the price of the claim  $X = \max(NS_T, G)$ , now from mathematical theory we have that the price is given by:

$$\pi(t) = e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t]$$

The goal here is to use the explicit formula given by Black & Scholes:

$$BS(t, T, S_t, K) = e^{-r(T-t)} \mathbb{E}_Q[(S_T - K)^+ | \mathcal{F}_t]$$

where  $(x)^+ = \max(x, 0)$ , so let's do so, before we dig in, we show some useful tricks concerning the maximum:

$$\max(A, B) = \max(B - A, 0) + A$$

$$\max(A, B) = \max(A - B, 0) + B$$

$$\max(A, B) = (A - B)^+ + B$$

Hence:

$$\begin{aligned} \max(NS_T, G) &= (NS_T - G)^+ + G \\ &\Downarrow \\ \frac{\max(NS_T, G)}{N} &= \left(S_T - \frac{G}{N}\right)^+ + \frac{G}{N} \\ &= Z + \frac{G}{N} \\ &\Downarrow \\ \max(NS_T, G) &= NZ + G \end{aligned}$$

Now let's use finance theory:

$$\begin{aligned} \pi(t) &= e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t] \\ &= e^{-r(T-t)} \mathbb{E}_Q[(NZ + G) | \mathcal{F}_t] \\ &= N e^{-r(T-t)} \mathbb{E}_Q[Z | \mathcal{F}_t] + e^{-r(T-t)} G \\ &= N e^{-r(T-t)} \mathbb{E}_Q \left[ \left(S_T - \frac{G}{N}\right)^+ | \mathcal{F}_t \right] + e^{-r(T-t)} G \\ &= NBS(t, T, S_t, G/N) + e^{-r(T-t)} G \end{aligned}$$

### Exercise 10.3

We are dealing with an endowment insurance, hence  $S = \{*, \dagger\}$ , the contractual information provided is:

```
T      <- 10      #length of contract
x0     <- 60      #age of insured
pi0    <- 10000
S0     <- 1
beta   <- 0.005  #deduction charges
r      <- 0.025
sigma  <- 0.25   #volatility
```

From the text, we get:

$$\begin{aligned} C(T) &= \max((1 - 0.03)\pi_0 S_T (1 - \beta)^{T-1}, \pi_0) \\ C(T) &= \max(NS_T, \pi_0) \\ C(T) &= N \left( S_T - \frac{\pi_0}{N} \right)^+ + \pi_0 \end{aligned}$$

We also write down the policy functions, where we do not include premiums:

$$a_*(t) = \begin{cases} 0, & t < 0 \\ 0, & t \in [0, T) \\ C(T), & t \geq T \end{cases} \quad a_\dagger(t) = 0 \quad a_{*\dagger}(t) = 0 \quad \dot{a}_*(t) = 0$$

We also recall what we mean by the B&S-notation:

$$BS(t, T, S_t, K) = e^{-r(T-t)} \mathbb{E}_Q \left[ (S_T - K)^+ \middle| \mathcal{F}_t \right]$$

This yields:

$$\begin{aligned} V_*^+(t, S_t) &= p_{**}^{x_0}(t, T) \frac{v(T)}{v(t)} \mathbb{E}_Q[C(T) | \mathcal{F}_t] \\ &= p_{**}^{x_0}(t, T) e^{-r(T-t)} \mathbb{E}_Q \left[ N \left( S_T - \frac{\pi_0}{N} \right)^+ + \pi_0 \middle| \mathcal{F}_t \right] \\ &= p_{**}^{x_0}(t, T) N e^{-r(T-t)} \mathbb{E}_Q \left[ N \left( S_T - \frac{\pi_0}{N} \right)^+ \middle| \mathcal{F}_t \right] + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0 \\ &= p_{**}^{x_0}(t, T) N BS(t, T, S_t, \pi_0/N) + p_{**}^{x_0}(t, T) e^{-r(T-t)} \pi_0 \\ &= p_{**}^{x_0}(t, T) \left[ N BS(t, T, S_t, \pi_0/N) + e^{-r(T-t)} \pi_0 \right] \end{aligned}$$

```

#0: alive, 1:dead
mu01 <- function(t){
  A <- 0.0001
  B <- 0.00035
  c <- 1.075
  return(A + B*c^{t})
}

#survival function:
p_surv <- function(t,s){
  f <- mu01
  integral <- integrate(f, lower = t, upper = s)$value
  return(exp((-1)*integral))
}

#Black&Scholes for call-option:
BS <- function(t,T,St, K){
  d1 <- (log(St/K) + (r + sigma^2/2)*(T-t)) / (sigma*sqrt(T-t))
  d2 <- d1 - sigma*sqrt(T-t)

  value <- St*pnorm(d1) - K*exp(-r*(T-t))*pnorm(d2)
  return(value)
}

V_reserve <- function(t, T,St){
  N <- (1-0.03)*pi0*(1-beta)^{(T-1)}

  reserve <- p_surv(t+x0, T+x0)*(N*BS(t,T,St,pi0/N) + exp(-r*(T-t))*pi0)
  return(reserve)
}

#reserve at t=0 with S0 = 1
V_reserve(t = 0, T = 10, St = 1)

```

```
## [1] 7556.453
```

## Exercise 10.4

In this exercise we are asked to see what happens to the reserve at time  $t = 6$ , if:

- i) Fund increases by 45%
- ii) Fund increases by 5%

```

#fund has increased by 45%:
V_reserve(t = 6, T = 10, St = 1.45)

```

```
## [1] 11635.45
```

```

#fund has increased by 5%:
V_reserve(t = 6, T = 10, St = 1.05)

```

```
## [1] 9293.374
```

## Exercise 10.5

We are considering a call option:  $X = (S_T - K)^+$ , furthermore the stock price follows a GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (P)$$

We are asked to show that the claim value at time  $t$  denoted  $v(t, S_t)$ , satisfies the following PDE:

$$-\partial_t v + rv = \frac{1}{2} \sigma^2 x^2 \partial_{xx} v + rx \partial_x v$$

First of, by claim value at time  $t$ , we mean:

$$v(t, S_t) = e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t]$$

This is an object we can trade, and from the fundamental theorem of asset-pricing we have that all tradable assets are  $(Q, \mathcal{F})$ -martingales after discounting. This means that:

$$\frac{v(t, S_t)}{B_t}$$

should be a martingale, here  $B_t = \exp\left(\int_0^t r_u du\right)$ , in our case we have a constant interest rate, meaning that  $B_t = e^{rt}$ . Our strategy is to use the martingale representation theorem, from which we can conclude that the  $dt$ -terms should be zero.

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) + d\left[\frac{1}{B_t}\right] dv(t, S_t) \\ &= d\left[\frac{1}{B_t}\right] v(t, S_t) + \frac{1}{B_t} dv(t, S_t) \end{aligned}$$

Since we work in  $(Q)$ -framework, we state the dynamics of  $S_t$  under  $Q$  first:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t \quad (Q)$$

We start with the easy one first:

$$d(e^{-rt}) = -re^{-rt} dt$$

We will use Ito's formula on  $v(t, x)$ :

$$\begin{aligned} dv(t, x) &= \partial_t v dt + \partial_x v dS_t + \frac{1}{2} \partial_{xx} v (dS_t)^2 \\ &= \partial_t v dt + \partial_x v [rx dt + \sigma x d\widetilde{W}_t] + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 dt \\ &= \left[ \partial_t v + \partial_x vx + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 \right] dt + \partial_x v x d\widetilde{W}_t \end{aligned}$$

This leaves us with:

$$\begin{aligned} d\left[\frac{v(t, S_t)}{B_t}\right] &= -re^{-rt} v dt + e^{-rt} \left( \left[ \partial_t v + \partial_x vx + \frac{1}{2} \partial_{xx} v \sigma^2 x^2 \right] dt + \partial_x v x d\widetilde{W}_t \right) \\ &= e^{-rt} \left[ -rv + \partial_t v + rx \partial_x v + \sigma^2 x^2 \frac{1}{2} \partial_{xx} v \right] dt + e^{-rt} x \partial_x v d\widetilde{W}_t \end{aligned}$$

And then, combining the fundamental theorem of asset pricing with martingale representation theorem, we get that the  $dt$ -part must equal to zero, hence:

$$-\partial_t v + rv = \frac{1}{2} \sigma^2 x^2 \partial_{xx} v + rx \partial_x v$$