



Binomial Model,
Value of Stock,
Black–Scholes

Andreas Stahel

Market
Observations

Binomial Model
for Value of
Stock

Future Value of
Stock

Value of an
Option on a
Stock

The End

From a Binomial Model for the Value of Stock to Option Pricing by Black–Scholes–Merton

Andreas Stahel

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Gesellschaft für Mathematik an Schweizer Fachhochschulen (GMFH)



Goals of this Presentation

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- Have a closer look at the daily changes of stock values.
- Use a simple binomial model to describe the evolution of stock values.
- Derive a finite difference approximation for a PDE modelling the evolution of the values of a stock.
- Give a very brief explanation of options.
- Derive the Black–Scholes PDE for the values of European put or call options.

Key point: All based on elementary algebra, some analysis and probability, i.e. no advanced mathematical tools.



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Examine historical data of stock values, in this case the daily values for the years 2015–2019. Use $S(i)$, the value on the i -th day of trading. Use relative changes (e.g. $+0.5\%$) and determine

$$\text{daily gain}(i) = \log\left(\frac{S(i)}{S(i-1)}\right)$$

The value on the N -th day of trading is given by

$$S(N) = S(0) \cdot \exp\left(\sum_{i=1}^N \text{daily gain}(i)\right)$$

Examine histograms of daily gains for a few stocks.



Market Observations II

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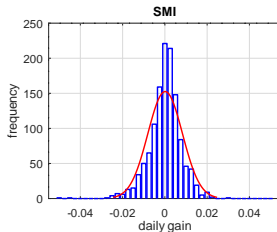
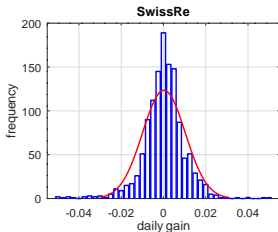
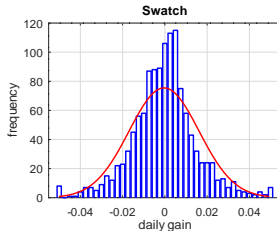
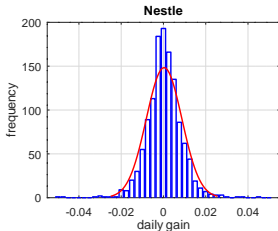
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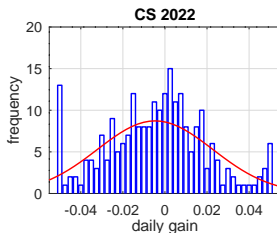
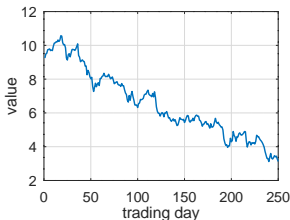
The End

Historical data based on five years from 2015 until 2019.



A normal distribution is often a very good assumption, but too many extreme events occur. As example consider the results for the Credit Swiss stock in 2022.

- annual gain -69%, -0.49% per day
- annual volatility 50.4%, 3.25% per day



Observe the wider spread of the daily gains and the large number of extreme events.



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Based on the data estimate the daily gain and its standard deviation. This allows to determine an annual gain and the corresponding standard deviation, called **volatility**, of the value of the stock.

	Nestle	Swatch	SwissRe	SMI
daily gain	+0.031%	-0.039%	+0.021%	+0.013%
annual gain r	+7.609%	-8.867%	+5.189%	+3.282%
daily stddev	0.842%	1.656%	1.012%	0.819%
volatility σ	13.05%	25.65%	15.67%	12.69%

To create a model for the development of the value $S(t)$ of a stock **assume** a known annual gain r (resp. $e^r - 1$) and a volatility σ . This will allow to determine the value V of an option based on this stock.



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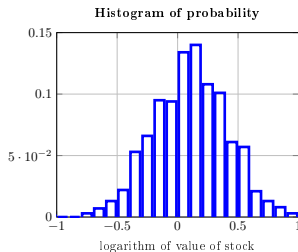
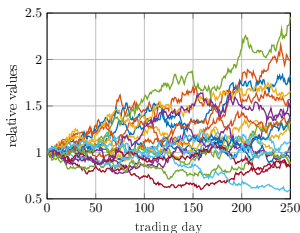
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The histograms of daily gains look (almost) like normal distributions. Use Monte Carlo simulations to model future values of a stock. Based on data from 1990 to 1999 for IBM stock estimate the daily gain at 0.0605% and its standard deviation at 1.94%.



This is a possible tool to determine the value of an option based on this specific stock.



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An (overly?) simple model:

Examine a stock with values $S(t)$, resp. $S(i \Delta t)$. For each time step Δt (e.g. one day) the value will

- either be multiplied by $e^{+u} > 1$ with probability $\frac{1}{2}(1 + \Delta p)$
- or be multiplied by $e^{-u} < 1$ with probability $\frac{1}{2}(1 - \Delta p)$

$$S_1 = S(1 \Delta t) = \begin{cases} S_0 e^{+u} & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ S_0 e^{-u} & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

$$S_2 = S(2 \Delta t) = \begin{cases} S_1 e^{+u} & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ S_1 e^{-u} & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

$$S_3 = S(3 \Delta t) = \begin{cases} S_2 e^{+u} & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ S_2 e^{-u} & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

\vdots



Binomial Model II

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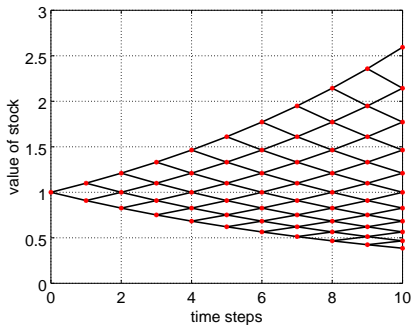
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For a starting value $S_0 = 1$ this can be visualized by a mesh with a linear scale for the values $S_i = S(i \Delta t)$.





Binomial Model III

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For easier calculations use a logarithmic scale $z(t) = \ln(S(t))$.

$$z_1 = \begin{cases} z_0 + u & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ z_0 - u & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

$$z_2 = \begin{cases} z_1 + u & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ z_1 - u & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

$$z_3 = \begin{cases} z_2 + u & \text{with probability } \frac{1}{2}(1 + \Delta p) \\ z_2 - u & \text{with probability } \frac{1}{2}(1 - \Delta p) \end{cases}$$

\vdots



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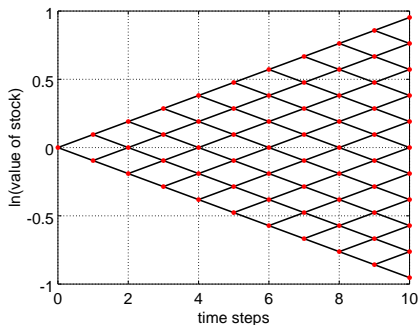
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Of the n steps k went upwards, i.e. find a binomial distribution.

$$\begin{aligned} P(z_n = z_0 + (2k - n)u) &= \binom{n}{k} \frac{1}{2^n} (1 + \Delta p)^k (1 - \Delta p)^{n-k} \\ &= \text{PDF}(k, B(n, \frac{1}{2}(1 + \Delta p))) \end{aligned}$$



E and σ^2 for a Binomial Distribution I

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Since the expected value E and the variance σ^2 of a binomial distribution will be used, here an elementary, elegant derivation:

- For the expected value E use the standard binomial formula with $q = 1 - p$. Differentiate once with respect to p , then multiply by p .

$$(p + q)^n = \sum_{i=0}^n \binom{n}{i} p^i q^{n-i}$$

$$p n (p + q)^{n-1} = p \frac{d}{dp} (p + q)^n = \sum_{i=0}^n i \binom{n}{i} p^i q^{n-i}$$

$$p n = \sum_{i=0}^n i \binom{n}{i} p^i (1-p)^{n-i} = E(X)$$



E and σ^2 for a Binomial Distribution II

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- For the variance σ^2 differentiate twice, multiply by p^2 .

$$\begin{aligned} p^2 n(n-1)(p+q)^{n-2} &= p^2 \frac{d^2}{dp^2} (p+q)^n \\ &= \sum_{i=0}^n i(i-1) \binom{n}{i} p^i q^{n-i} \\ p^2 n(n-1) &= \sum_{i=0}^n i(i-1) \binom{n}{i} p^i (1-p)^{n-i} \\ &= E(X^2) - E(X) = E(X^2) - np \\ E(X^2) &= p^2 n(n-1) + np = np(np - p + 1) \\ \sigma^2 &= E(X^2) - E(X)^2 \\ &= np(np - p + 1) - n^2 p^2 = np(1-p) \end{aligned}$$



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Determine $E(z_n)$ and $\text{Var}(z_n)$ for the binomial distribution $B(n, \frac{1}{2}(1 + \Delta p))$ for the values of the stock. Use $p = \frac{1}{2}(1 + \Delta p)$.

$$\begin{aligned} E(z_n) &= \sum_{k=0}^n (z_0 + (2k - n)u) \text{PDF}(k, B(n, \frac{1}{2}(1 + \Delta p))) \\ &= (z_0 - nu) E(1) + 2u E(k) \\ &= z_0 - nu + un(1 + \Delta p) \\ &= z_0 + nu\Delta p \end{aligned}$$



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$$\begin{aligned}
 \text{Var}(z_n) &= E((z_n - E(z_n))^2) \\
 &= \sum_{k=0}^n ((2k - n)u - nu\Delta p)^2 \text{PDF}(k, B(n, \frac{1}{2}(1 + \Delta p))) \\
 &= u^2 \sum_{k=0}^n (4k^2 - 2kn + n^2 - 2(2k - n)n\Delta p + n^2(\Delta p)^2) \cdot \\
 &\quad \cdot \text{PDF}(k, B(n, \frac{1}{2}(1 + \Delta p))) \\
 &= u^2 (4E(k^2) - 4n(1 + \Delta p)E(k) + n^2(1 + \Delta p)^2 E(1)) \\
 &= u^2 \left(4n \frac{1}{2}(1 + \Delta p) \frac{1}{2}(1 - \Delta p + n(1 + \Delta p)) - \right. \\
 &\quad \left. - 4n(1 + \Delta p)n \frac{1}{2}(1 + \Delta p) + n^2(1 + \Delta p)^2 \right) \\
 &= \dots \\
 &= u^2 n (1 - (\Delta p)^2)
 \end{aligned}$$



Compatibility Condition and Parameter Selection I

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Take n time steps of equal length $\Delta t = \frac{1}{n}$ to reach time $t = 1$.
Based on the market observations ask that the expectation value $E(z_n)$ and the variance σ^2 satisfy

$$\begin{aligned}z_0 + r &= E(z_n) = z_0 + n u \Delta p \\ \sigma^2 &= V(z_n) = u^2 n (1 - (\Delta p)^2)\end{aligned}$$

Solve the two equations for the parameters Δp and u .

$$\begin{aligned}\Delta p = \frac{r}{n u} &\implies \sigma^2 = u^2 n \left(1 - \frac{r^2}{n^2 u^2}\right) = n u^2 - \frac{r^2}{n} \\ &\implies u = \frac{\sqrt{n \sigma^2 + r^2}}{n} \\ &\implies \Delta p = \frac{r}{n u} = \frac{r}{\sqrt{n \sigma^2 + r^2}}\end{aligned}$$

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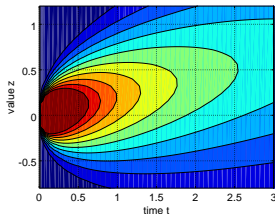
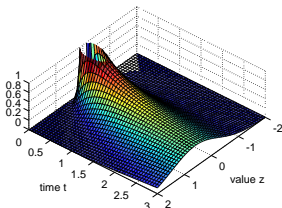
For $n \gg 1$ (with $t = 1$) find the approximations

$$\Delta p = \frac{r}{n u} \approx \frac{r}{\sigma \sqrt{n}} \quad \text{and} \quad u \approx \frac{\sigma}{\sqrt{n}}.$$

Using the De Moivre–Laplace theorem with these parameters and $n \rightarrow \infty$ leads to a normal distribution at time t , given by

$$\text{PDF}(t, z) = \frac{1}{\sigma \sqrt{2\pi} \sqrt{t}} e^{-(z - (z_0 + r t))^2 / (2 t \sigma^2)}$$

with mean $z_0 + r t$ and variance $t \sigma^2$.





Finite Difference Approximations of Derivatives

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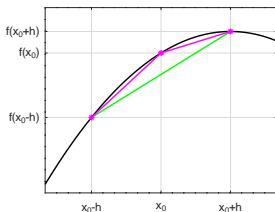
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$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

$$f''(x_0) \approx \frac{f'(x_0 + h/2) - f'(x_0 - h/2)}{h}$$

$$\approx \frac{1}{h} \left(\frac{f(x_0 + h) - f(x_0)}{h} - \frac{f(x_0) - f(x_0 - h)}{h} \right)$$

$$= \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$



From the Binomial Model to a PDE I

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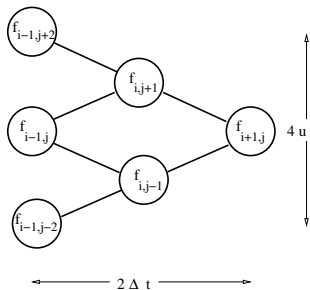
The End

The goal is to verify that the above binomial model leads to the finite difference discretization of a PDE (Partial Differential Equation).

Use

$f_{i,j}$ = value of probability at $t = i \Delta t$ and $z = j \cdot u$

and the stencil





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Applying the formula for the binomial tree

$$f_{i+1,j} = \frac{1}{2} (1 + \Delta p) f_{i,j-1} + \frac{1}{2} (1 - \Delta p) f_{i,j+1}$$

repeatedly with $u = \frac{\sigma}{\sqrt{n}}$, $\Delta p = \frac{r}{\sigma \sqrt{n}} = \frac{r}{n u}$ and elementary algebra leads to

$$2 f_{i,j-1} = (1 + \Delta p) f_{i-1,j-2} + (1 - \Delta p) f_{i-1,j}$$

$$2 f_{i,j+1} = (1 + \Delta p) f_{i-1,j} + (1 - \Delta p) f_{i-1,j+2}$$



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$$\begin{aligned}4 f_{i+1,j} &= (1 + \Delta p) 2 f_{i,j-1} + (1 - \Delta p) 2 f_{i,j+1} \\&= (1 + \Delta p) ((1 + \Delta p) f_{i-1,j-2} + (1 - \Delta p) f_{i-1,j}) \\&\quad + (1 - \Delta p) ((1 + \Delta p) f_{i-1,j} + (1 - \Delta p) f_{i-1,j+2}) \\&= (1 + \Delta p)^2 f_{i-1,j-2} + 2 (1 - (\Delta p)^2) f_{i-1,j} + (1 - \Delta p)^2 f_{i-1,j+2} \\&= \left(1 + \frac{r}{n u}\right)^2 f_{i-1,j-2} + 2 \left(1 - \frac{r^2}{n^2 u^2}\right) f_{i-1,j} + \left(1 - \frac{r}{n u}\right)^2 f_{i-1,j+2} \\&= (f_{i-1,j-2} + 2 f_{i-1,j} + f_{i-1,j+2}) + \frac{2 r (f_{i-1,j-2} - f_{i-1,j+2})}{n u} + \\&\quad + \frac{r^2 (f_{i-1,j-2} - 2 f_{i-1,j} + f_{i-1,j+2})}{n^2 u^2}\end{aligned}$$



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Use $\Delta t = \frac{1}{n} = \frac{u^2}{\sigma^2}$ and the limit as $n \rightarrow \infty$, or $\Delta t \rightarrow 0$ and $u \rightarrow 0$.

$$\begin{aligned}
 4 \frac{f_{i+1,j} - f_{i-1,j}}{\Delta t} &= \sigma^2 \frac{f_{i-1,j-2} - 2f_{i-1,j} + f_{i-1,j+2}}{u^2} + 2r \frac{f_{i-1,j-2} - f_{i-1,j+2}}{u} + \\
 &\quad + \frac{r^2}{n} \frac{f_{i-1,j-2} - 2f_{i-1,j} + f_{i-1,j+2}}{u^2} \\
 \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta t} &= \frac{\sigma^2}{2} \frac{f_{i-1,j-2} - 2f_{i-1,j} + f_{i-1,j+2}}{4u^2} - r \frac{f_{i-1,j+2} - f_{i-1,j-2}}{4u} + \\
 &\quad + \frac{r^2}{2n} \frac{f_{i-1,j-2} - 2f_{i-1,j} + f_{i-1,j+2}}{4u^2}
 \end{aligned}$$

If $f_{i,j}$ is the discretization of a continuous function $f(t, z)$, i.e. $f_{i,j} = f(i \Delta t, j \Delta z)$, take the limit $n \rightarrow \infty$ and recognize the above as an explicit finite difference approximation of the dynamic heat equation

$$\frac{\partial f(t, z)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f(t, z)}{\partial z^2} - r \frac{\partial f(t, z)}{\partial z} + 0.$$

The above function PDF(t, z) = $f(t, z)$ satisfies this PDE.



From the Binomial Model to a PDE V

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Thus the probability distribution of the value of a stock satisfies the same PDE as a heat equation with some drift contribution

$$\frac{\partial \text{PDF}(t, z)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \text{PDF}(t, z)}{\partial z^2} - r \frac{\partial \text{PDF}(t, z)}{\partial z}$$

with time t and $z = \ln(S)$, where S is the value of the stock. This is a PDE with constant coefficients. Rewriting this with the stock value $S = e^z$ as independent variable with $V(t, S) = \text{PDF}(t, \ln(S))$ leads to

$$\frac{\partial V(t, S)}{\partial t} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V(t, S)}{\partial S^2} + \left(\frac{\sigma^2}{2} - r\right) S \frac{\partial V(t, S)}{\partial S}.$$

This is a PDE with variable coefficients. Often this form is used in the literature!



What is an Option?

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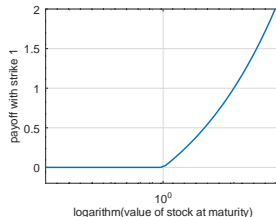
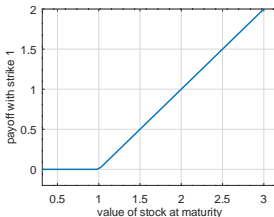
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As holder of an **European call option** you have the right, but not the obligation, to buy the underlying asset (e.g. stock, currency) at a given maturity date T for the given strike price K . For this right you have to pay a price V , the value of the option.

On the maturity date T there are different outcomes possible:

- If at time T the value S of the stock is below the strike price K , do nothing.
- If at time T the value S of the stock is above the strike K , call your option, and sell the stock on the market, you gain $S - K$.





Different Types of Options

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There are many different types of options.

- European call option: right to buy at maturity date
- European put option: right to sell at maturity date
- American call option: right to buy at any time before the maturity date
- American put option: right to sell at any time before the maturity date
- Different payoffs are possible, e.g. binary
- many, many more, some incredibly complicated



Options: Insurance or Speculation? I

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A put option with a stock as underlying asset:

- **Insurance:** you own a large chunk of Swisscom stock and want to use it a year from now to pay off your mortgage, since you retire. To make sure that you obtain a minimal price for your Swisscom stock, you buy a put option with your minimal price as strike. You have to buy this insurance at a fair price.
- **Speculation:** your gut feeling (or insider knowledge) tells you that the value of the Swisscom stock will fall drastically within a year. With a put option you assure that you can sell at the strike price. If a year from now the value is considerably lower than your strike, you buy on the market and make a (hopefully large) profit. You risk to lose the price of this option, bought at a fair price.



Options: Insurance or Speculation? II

Binomial Model,
Value of Stock,
Black-Scholes

Andreas Stahel

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Binomial Model
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The End

A call option with a currency exchange as underlying asset:

- **Insurance:** as a small business owner you have a large contract in the USA and your work will be paid a year from now with 1 million USD. With the help of a call option for CHF (payed in USD) with a strike of $0.90 \frac{\text{CHF}}{\text{USD}}$ you are assured to have at least this “exchange rate”, even if the value of the USD would fall drastically. You have to buy this insurance at a fair price.
- **Speculation:** your gut feeling (or insider knowledge) tells you that the value of USD will fall drastically within a year. Thus you buy a call option for CHF at a strike of $0.90 \frac{\text{CHF}}{\text{USD}}$ for 1 million USD at a fair price. If the exchange rate on the market a year from now would be $0.80 \frac{\text{CHF}}{\text{USD}}$ you buy 1'000'000 USD for 800'000 CHF on the market and then obtain 900'000 CHF by calling the option. It would be a nice gain of 100'000 CHF, but you risk to loose the price of the option.



Fair Value of an Option

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The key point is to determine the fair value V of an option, as function of time and the current value S of the underlying asset:

The fair value equals the expected payoff

Assume that there is a save interest rate r_0 (currently rather small). For a call option with payoff $\max\{0, S - K\}$ this leads to

$$\begin{aligned} V(t, S) e^{+r_0(T-t)} &= \int_0^\infty \max\{0, \tilde{S} - K\} \cdot \text{PDF}(T, \tilde{S}) d\tilde{S} \\ &= \int_K^\infty (\tilde{S} - K) \cdot \text{PDF}(T, \tilde{S}) d\tilde{S} \end{aligned}$$

and based on this generate explicit formulas for the value $V(t, S)$ for European call or put options at time t with current value S .



From the Binomial Model to Black–Scholes I

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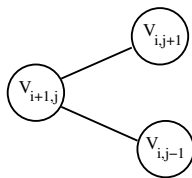
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Apply the same idea for the binomial model. Use the notations

$V_{i,j}$ = value of option at $t = T - i \Delta t$ and $z = j \cdot u$

and the risk-less interest rate r_0 and the “fair value condition”.



current value + interest = expected future value

$$e^{r_0/n} \cdot V_{i+1,j} = \frac{1}{2} (1 - \Delta p) V_{i,j-1} + \frac{1}{2} (1 + \Delta p) V_{i,j+1}$$



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With $f_{i,j} = e^{j r_0/n} \cdot V_{i,j}$ this implies

$$f_{i+1,j} = \frac{1}{2} (1 - \Delta p) f_{i,j-1} + \frac{1}{2} (1 + \Delta p) f_{i,j+1}$$

i.e. the same difference equation as for the value z of the stock, but with a change of sign in $\Delta p = \frac{r}{2} \frac{1}{nu}$. Thus the modified function $f(t, z) = e^{r_0(T-t)} \cdot V(T-t, z)$ satisfies the PDE

$$\frac{\partial f(t, z)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f(t, z)}{\partial z^2} + r \frac{\partial f(t, z)}{\partial z}$$



From the Binomial Model to Black–Scholes III

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For the original function $V(\tau, z) = e^{r_0 \tau} f(T - \tau)$ conclude

$$\begin{aligned}\frac{\partial}{\partial \tau} V(\tau, z) &= r_0 e^{r_0 \tau} f(T - \tau) - e^{r_0 \tau} \left(\frac{\sigma^2}{2} \frac{\partial^2 f(t, z)}{\partial z^2} + r \frac{\partial f(t, z)}{\partial z} \right) \\ &= r_0 V(\tau, z) - \frac{\sigma^2}{2} \frac{\partial^2}{\partial z^2} V(\tau, z) - r \frac{\partial}{\partial z} V(\tau, z)\end{aligned}$$

and consequently $V(t, z)$ solves the PDE

$$-\frac{\partial V(t, z)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 V(t, z)}{\partial z^2} + r \frac{\partial V(t, z)}{\partial z} - r_0 V(t, z)$$

This is the **Black–Scholes–Merton** PDE, leading to a Nobel Memorial Price in 1997. It is similar to a dynamic heat equation with time reversal and the final condition $V(T, z) = \text{payoff}(z)$.



Value of a Call Option on Nestle Stock

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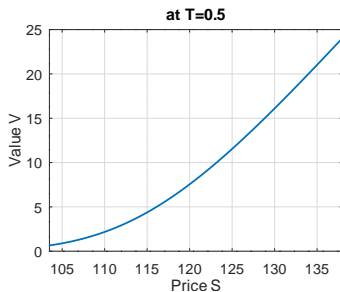
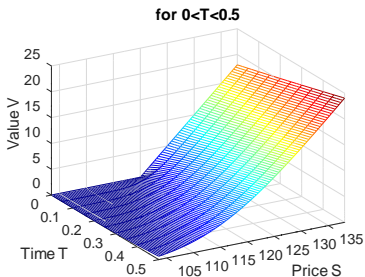
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Use the historic data for Nestle stock from 2015 until 2019 with a strike of $K = 120$. T is the time until maturity of the call option. Use *Octave* and implement a finite difference scheme to solve the PDE and determine the value V of the option as function of time T and price S of the stock.





American Options

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With an American call option you have the right to call the option at any time $t \leq T$. Thus an American option is always more valuable than a European option. The resulting PDE is again

$$-\frac{\partial V(t, z)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 V(t, z)}{\partial z^2} + r \frac{\partial V(t, z)}{\partial z} - r_0 V(t, z),$$

but with the additional obstacle

$$V(t, z) \geq \text{payoff}(z).$$

If $V(t, z) < \text{payoff}(z)$ then you would cash in immediately.

This PDE is considerably more difficult to derive and solve numerically. It is a nonlinear problem, caused by the obstacle. There are no analytical solutions.



Some Literature

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Thank You for Your Attention

That's all folks

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