

Regularization by Tikhonov Functionals

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1 Introduction

This note is a documentation for the Octave codes `regularization.m` and `regularization2D.m`. With these commands a Tikhonov regularization is used to approximate given data points by smooth curves or surfaces.

- With the command `regularization()`, for given data points $(x_i, y_i) \in \mathbb{R}^2$, given functions $g_1(x)$ and $g_2(x)$ on the interval $a \leq x \leq b$, using the regularization parameters $\lambda_1, \lambda_2 \geq 0$, the functional

$$F(u) = \sum_{i=1}^M (y_i - u(x_i))^2 + \lambda_1 \int_a^b (u'(x) - g_1(x))^2 dx + \lambda_2 \int_a^b (u''(x) - g_2(x))^2 dx$$

is minimized.

- With the command `regularization2D()`, for given data points $(x_i, y_i, z_i) \in \mathbb{R}^3$ and then regularization parameters $\lambda_1 \geq 0, \lambda_2 > 0$, minimize the functional

$$\begin{aligned} F(u) &:= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\ &= \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \iint_{\Omega} u_x^2 + u_y^2 dA + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 dA \end{aligned}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$.

The organization of the notes is as follows:

- In section 2 the setup and documentation for the regularization with functions of one independent variable is explained. A few examples are given and connections to splines shown.
- In section 3 the algorithm and the resulting code is explained. The corresponding Euler–Lagrange equations are derived and the jump conditions determined.
- In section 4 the setup and documentation for the regularization with functions of two independent variables is explained.
 - The code and its documentation is shown.
 - A few examples are given, illustration essential effects.
 - An elementary proof of existence of a unique solution is given and a few remarks to similar results in the literature given. The Euler–Lagrange equations and the natural boundary conditions are derived.

2 Introduction to Regularization in One Dimension

For the data points (x_i, y_i) for $1 \leq i \leq M$, positive regularization parameters $0 \leq \lambda_1$ and $0 \leq \lambda_2$ and the target functions $g_1(x)$ and $g_2(x)$ find a function $u : [a, b] \rightarrow \mathbb{R}$ such that the Tikhonov functional

$$\begin{aligned} F(u) &= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\ &= \sum_{i=1}^M (y_i - u(x_i))^2 + \lambda_1 \int_a^b (u'(x) - g_1(x))^2 dx + \lambda_2 \int_a^b (u''(x) - g_2(x))^2 dx \end{aligned} \quad (1)$$

is minimized.

- The functional F_D pushes the optimal solution towards a function going through the given data points (x_i, y_i) .
- The functional $\lambda_1 F_1$ pushes the optimal solution towards a function with derivative $g_1(x)$. For large values of λ_1 the result satisfies $u'(x) \approx g_1(x)$. For small values of $\lambda_1 > 0$ and $\lambda_2 = 0$, the data will be approximated by straight line segments between the data points.
- The functional $\lambda_2 F_2$ pushes the optimal solution towards a function with second derivative $g_2(x)$. With $g_2 = 0$ the solution is pushed towards straight line segments.

The above optimization is realized in Octave by using finite difference approximations for the derivatives and elementary numerical integration. A linear system of equations is used to determine discrete approximations $\vec{u} \in \mathbb{R}^{N+1}$ to the exact minimizer $u(x)$.

The built-in help is shown below and the next section provides a few examples on how to use the function `regularization()`.

```

help regularization
-->
-- Function File: [GRID,U] = regularization (DATA, INTERVAL, N, F1, F2)

Apply a Tikhonov regularization, the functional to be minimized is
F = FD + LAMBDA1*F1 + LAMBDA2*F2
= sum_(i=1)^M (y_i-u(x_i))^2 + LAMBDA1*int_a^b (u'(x) - G1(x))^2 dx
+ LAMBDA2*int_a^b (u''(x) - G2(x))^2 dx

With LAMBDA1 = 0 and G2(x) = 0 this leads to a smoothing spline.

Parameters:
* DATA is a M*2 matrix with the x values in the first column and
the y values in the second column.
* INTERVAL = [a,b] is the interval on which the regularization
is applied.
* N is the number of subintervals of equal length. GRID will
consist of N+1 grid points.
* F1 is a structure containing the information on the first
regularization term, integrating the square of the first
derivative.
* F1.LAMBDA is the value of the regularization parameter
LAMBDA1>=0.
* F1.G is the function handle for the function G1(X). If
not provided G1=0 is used.
* F2 is a structure containing the information on the second
regularization term, integrating the square of the second
derivative. If F2 is not provided LAMBDA2=0 is assumed.
* F2.LAMBDA is the value of the regularization parameter
LAMBDA2>=0.
* F2.G is the function handle for the function G2(X). If
not provided G2=0 is used.

Return values:
* GRID is the grid on which U is evaluated. It consists of N+1
equidistant points on the INTERVAL.
* U are the values of the regularized approximation to the DATA
evaluated at GRID.

```

See also: `csaps`, `regularization2D`, `demo regularization`.

2.1 Examples

2.1.1 A first example

As a first example consider the data points $x = y = (3.2, 4, 5, 5.2, 5.6)$ on the interval $[0, 10]$ with 101 grid points. Then two regularizations are performed, leading to the results in Figure 1.

1. Use $\lambda_1 = 10^{-2}$ and aim for a slope of 0.1 by setting $g_1(x) = 0.1$. For $\lambda_2 = 0$ this leads to the piecewise straight lines in Figure 1 with slope 0.1, as soon as outside of the domain with specified values.
2. For the second attempt change $\lambda_2 = 0$ to $\lambda_2 = 10^{-2} = 0.01$ leading to the smoother curve in Figure 1. Again the slope on the outside approaches 0.1.

Observe that the values of regularized curves do **not** coincide with the given data points, just zoom into the figure.

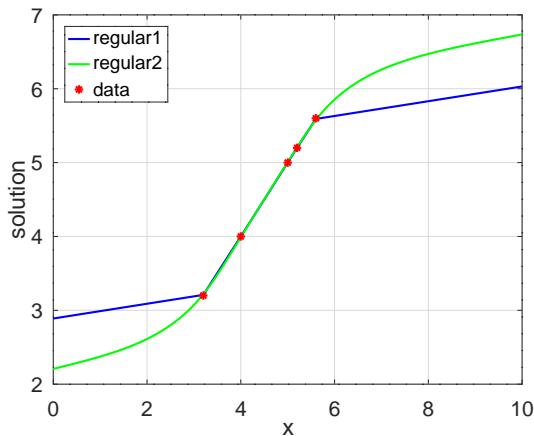


Figure 1: Results of the first example for regularization

Example1.m

```

N = 100;
interval = [0,10];
x = [3.2,4,5,5.2,5.6]'; y = x;
F1.lambda = 1e-2;
F1.g = @(x)0.1*ones(size(x));
%% regularize towards slope 0.1, no smoothing
F2.lambda = 0;
[grid,u1] = regularization([x,y],interval,N,F1);
%% regularize towards slope 0.1, with some smoothing
F2.lambda = 1e-2;
[grid,u2] = regularization([x,y],interval,N,F1,F2);

figure(1); plot(grid,u1,'b',grid,u2,'g',x,y,'*r')
xlabel('x'); ylabel('solution');
legend('regular1','regular2','data','location','northwest')

```

2.1.2 A second example

As a second example consider the data points $y = \sin(x)$ for 15 points $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$, with some random noise added. Then two regularizations on the interval $[0, \pi]$ with 1000 subintervals are performed, leading to the results in Figure 2.

1. Use $\lambda_2 = 10^{-3}$ and $\lambda_1 = 0$, leading to the blue curve in Figure 1. Observe that the curvature outside of $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$ vanishes, i.e. find straight line segments with the slope determined by the closest data points.
2. For the second attempt change to $\lambda_1 = 10^{-2}$, this causes the slope of the green curve to tend towards 0 on the outside.

Both calls use $g_1 = g_2 = 0$. Observe that the situation is asymmetric, caused by the randomness of the data points.

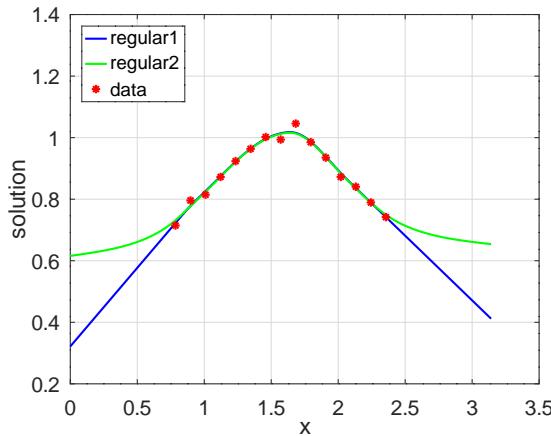


Figure 2: Results of the second example for regularization

Example2.m

```

N = 1000;
interval = [0,pi];
x = linspace( pi/4,3*pi/4,15)';
y = sin(x)+ 0.03*randn(size(x));
clear F1 F2
F1.lambda = 0;
%% regularize by smoothing only
F2.lambda = 1e-3;
[grid,u1] = regularization([x,y],interval,N,F1,F2);
%% regularize by smoothing and aim for slope 0
F1.lambda = 1*1e-2;
[grid,u2] = regularization([x,y],interval,N,F1,F2);

figure(1); plot(grid,u1,'b',grid,u2,'g',x,y,'*r')
xlabel('x'); ylabel('solution');
legend('regular1','regular2','data','location','northwest')

```

2.1.3 A third example

As third example consider 200 data points, located on four straight line segments with slope -2 , and some random noise added. Then three regularizations are performed, using F_1 only with $\lambda_1 = 0.001, 0.1$ and 3 , leading to the results in Figure 3. Thus the functional to be minimized is

$$F(u) = \sum_{i=1}^{200} (y_i - u(x_i))^2 + \lambda_1 \int_0^1 (u'(x) - g_1(x))^2 dx$$

for different values of the regularization parameter $\lambda_1 > 0$ and either $g_1 = 0$ or $g_1 = -2$.

1. The blue curve in Figure 3(a) with $\lambda_1 = 0.001$ is not very regular and has many spikes. The curve is rather close to the data points. Thus $\lambda_1 = 0.001$ is on the small side.
2. The green curve in Figure 3(a) with $\lambda_1 = 0.1$ is rather smooth and not very far away from the data points. Thus $\lambda_1 = 0.1$ might be a good choice.
3. The magenta curve in Figure 3(a) with $\lambda_1 = 3$ is very smooth, but far away from the data points. The zero slopes at the end points are visible. Thus $\lambda_1 = 3$ is on the large side.

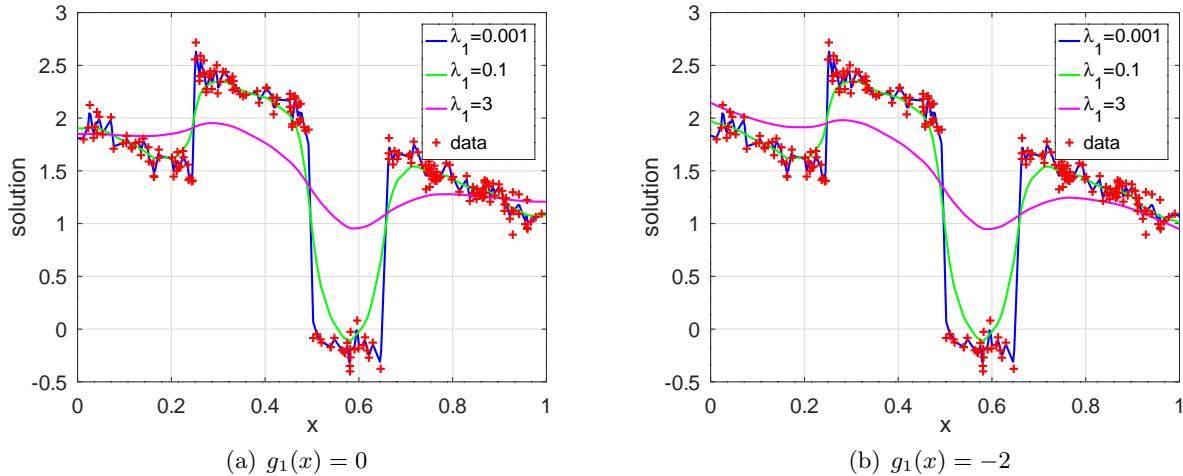


Figure 3: Results of the third example for regularization

Since it is “known” that the slope of the curve should be -2 , rerun the above regularization using $g_1(x) = -2$, i.e. minimize

$$F(u) = \sum_{i=1}^{200} (y_i - u(x_i))^2 + \lambda_1 \int_0^1 (u'(x) + 2)^2 dx.$$

This leads to Figure 3(b), where the slopes at the endpoints are closer to the desired value of -2 .

Example3.m

```

interval = [0,1]; N = 400;
x = rand(200,1);
y = 2 - 2*x + (x>0.25) - 2*(x>0.5).* (x<0.65)+ 0.1*randn(length(x),1);
clear F1
%% F1.g = @ (x)-2*ones(size(x)); %% use this when aiming for a slope -2
F1.lambda = 1e-3; [grid,u1] = regularization([x,y],interval,N,F1);
F1.lambda = 1e-1; [grid,u2] = regularization([x,y],interval,N,F1);
F1.lambda = 5e+0; [grid,u3] = regularization([x,y],interval,N,F1);

figure(1); plot(grid,u1,'b',grid,u2,'g',grid,u3,'m',x,y,'+r')
xlabel('x'); ylabel('u and f')
legend ('\lambda_1=0.001', '\lambda_1=0.1', '\lambda_1=3', 'data')

```

2.1.4 Regularization to a circle

If we assume that the data points should sit on a given circle, we can feed this information to the regularization algorithm by providing the function $g_2(x)$. The regularization is used twice, with the results in Figure 4.

1. First with $\lambda_1 = 0$, $\lambda_2 = 10^{-1}$ and $g_1(x) = g_2(x) = 0$.
2. Then we provide the exact second derivative of the circle function as $g_2(x)$.

Observe that outside of the domain with data points, the second result deviates considerably less from the circle.

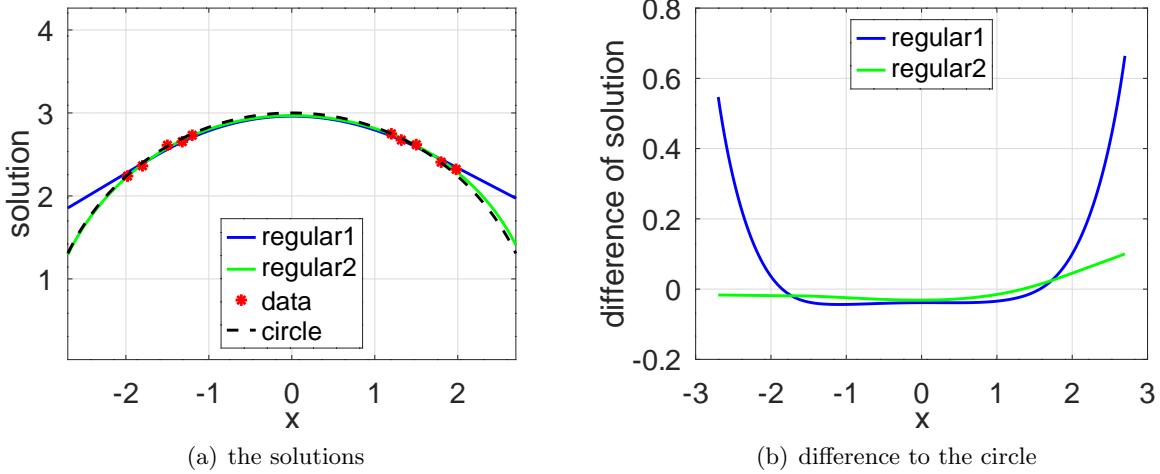


Figure 4: Regularization towards a given circle

CircleRegularization.m

```
R = 3;
x = R*[ 0.4, 0.44, 0.6, 0.66, 0.5];
x = [-fliplr(x),x]';
N = 200;
interval = 0.9*[-R,R];
y = sqrt(R^2-x.^2) + 0.03*randn(size(x));
clear F1 F2
F1 = 0;
F2.lambda = 1e-1;
[grid,u1] = regularization([x,y],interval,N,F1,F2);
F2.g = @(x)-R^2./((R^2-x.^2).^(3/2));
[grid,u2] = regularization([x,y],interval,N,F1,F2);

circle = sqrt(R^2-grid.^2);

figure(1); plot(grid,u1,'b',grid,u2,'g',x,y,'*r',grid,circle,'k')
xlabel('x'); ylabel('solution')
axis equal
legend('regular1','regular2','data','circle')

figure(2); plot(grid,u1-circle,'b',grid,u2-circle,'g')
xlabel('x'); ylabel('difference of solution')
legend('regular1','regular2')
```

2.2 Regularization and splines

In this subsection it is pointed out that smoothing splines are a special case of the presented regularization algorithm.

2.2.1 Smoothing splines

Using `regularization()` with $\lambda_1 = g_1 = g_2 = 0$ leads to results coinciding with a smoothing spline, computed by the command `csaps()` from the splines package of *Octave*. As an example Figure 5 shows the result of a smoothing spline (with parameter $p = 0.99$) and the regularization result with $\lambda_2 = 10^{-2}$. For `csaps()` use the parameter $p = \frac{1}{1+\lambda_2} = \frac{1}{1.01} \approx 0.99$.

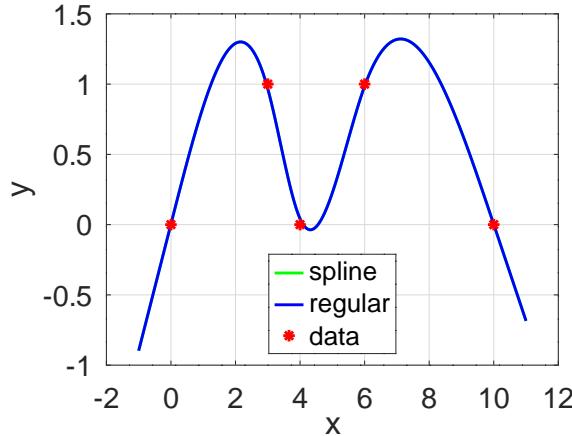


Figure 5: Comparison of regularization and a smoothing spline

SmoothingSpline.m

```

pkg load splines
N = 1000;           interval = [-1,11];
x = [0 3 4 6 10]'; y = [0 1 0 1 0]';
clear F1 F2
F1 = 0; F2.lambda = 1e-2; p = 1/(1+F2.lambda)

[grid,u] = regularization([x,y],interval,N,F1,F2);
yspline = csaps(x,y,p,grid); % smoothing spline

figure(1); plot(grid,yspline,'g',grid,u,'b',x,y,'*r')
legend('spline','regular','data','location','south')

```

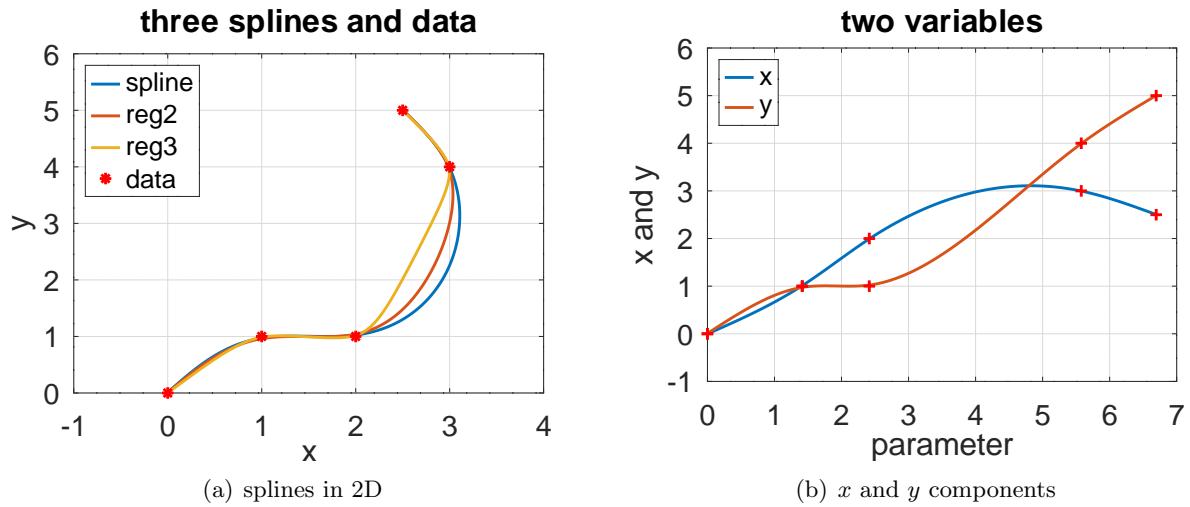
2.2.2 A 2D spline and variations by using `regularization()`

The command `regularization()` can be used to generate 2D splines, connecting points in the plane \mathbb{R}^2 . The example below uses the points

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 2.5 \\ 5 \end{pmatrix}.$$

To obtain a uniform parametrization the distance between these points is used to construct the splines. The code uses three sets of parameters and the results are shown in Figure 6.

1. With $\lambda_1 = 0$ and $\lambda_2 = 10^{-2}$ a smoothing spline is generated, i.e. only the integral based on the second derivatives is controlled.
2. With $\lambda_1 = \lambda_2 = 10^{-2}$ the first and second derivatives are used in the Tikhonov functional.
3. With $\lambda_1 = 10^{-2}$ and $\lambda_2 = 10^{-3}$ the integral of the second derivatives is allowed to grow, which leads to sharper corners in Figure 6.

Figure 6: 2D splines generated by `regularization()`

```

Spline2D.m

Points = [0 0; 1 1; 2 1; 3 4; 2.5 5]; % the given data points

% determine distance between points
dPoints = diff(Points);
Dist = sqrt(dPoints(:,1).^2+dPoints(:,2).^2);
Positions = [0;cumsum(Dist)];
N = length(Points)*10+1; %% number of points on curve to be created

F1.lambda = 0e-2; F2.lambda = 1e-2; %% smoothing spline
[tx,x] = regularization([Positions,Points(:,1)], [0,max(Positions)], N,F1,F2);
[ty,y] = regularization([Positions,Points(:,2)], [0,max(Positions)], N,F1,F2);

figure(2);
plot(tx,x,ty,y,Positions,Points(:,1),'+r',Positions,Points(:,2),'+r')
xlabel('parameter'); ylabel('x and y');
legend('x','y'); title('the two variables')

F1.lambda = 1e-2; F2.lambda = 1e-2; %% keep fits derivative small
[tx,x2] = regularization([Positions,Points(:,1)], [0,max(Positions)], N,F1,F2);
[ty,y2] = regularization([Positions,Points(:,2)], [0,max(Positions)], N,F1,F2);

F1.lambda = 1e-2; F2.lambda = 1e-3; %% allow large second derivative
[tx,x3] = regularization([Positions,Points(:,1)], [0,max(Positions)], N,F1,F2);
[ty,y3] = regularization([Positions,Points(:,2)], [0,max(Positions)], N,F1,F2);

figure(1); plot(x,y,x2,y2,x3,y3,Points(:,1),Points(:,2),'*r')
xlabel('x'); ylabel('y');
legend('spline','reg2','reg3','data')
title('three splines and data')

```

3 The Algorithm and the Code

3.1 The computational grid

The interval $[a, b]$ is divided in N segments of equal length $\Delta x = \frac{b-a}{N}$ and $\bar{x}_j = a + (j - 1) \Delta x$ for $j = 1, 2, 3, \dots, N + 1$. This leads to a vector $\vec{u} \in \mathbb{R}^{N+1}$ with $u_j = u(\bar{x}_j)$.

3.2 The data functional and its interpolation matrix

For $\bar{x}_j \leq x < \bar{x}_{j+1}$ use a piecewise linear interpolation of the function $u(x)$ to determine an approximate value at x .

$$u(x) \approx (1 - \theta) u(\bar{x}_j) + \theta u(\bar{x}_{j+1}) \quad \text{where } \theta = \frac{x - \bar{x}_j}{\Delta x}.$$

This leads to an interpolation matrix $\mathbf{I} \in \mathbb{M}^{M \times (N+1)}$ with the nonzero entries $I_{i,j} = 1 - \theta$ and $I_{i,j+1} = \theta$. For the algorithm the matrix $\mathbf{I}^T \mathbf{I} \in \mathbb{M}^{(N+1) \times (N+1)}$ will be used. The above leads to a contribution of

$$\begin{bmatrix} (1 - \theta)^2 & \theta(1 - \theta) \\ \theta(1 - \theta) & \theta^2 \end{bmatrix}$$

in rows/columns j and $j + 1$ of $\mathbf{I}^T \mathbf{I}$. This leads to

$$\begin{aligned} F_D(u) &= \sum_{i=1}^M (y_i - u(x_i))^2 \approx \langle \vec{y} - \mathbf{I} \vec{u}, \vec{y} - \mathbf{I} \vec{u} \rangle \\ \frac{\partial F_D}{\partial \vec{u}} &\approx 2 \mathbf{I}^T (\mathbf{I} \vec{u} - \vec{y}) \end{aligned}$$

If a data point satisfies $x_i = \bar{x}_j$, then $\theta = 0$ and the j -th component of the above gradient equals $2(u_j - y_i)$. This is the only nonzero component.

3.3 Integration of first derivatives

3.3.1 The continuous formulation

The contribution

$$F_1(u) = \int_a^b (u'(x) - g_1(x))^2 dx$$

to the Euler–Lagrange equation is given by $\frac{d}{dx}(u'(x) - g_1(x)) = 0$ and the natural boundary conditions are $u'(a) - g_1(a) = u'(b) - g_1(b) = 0$.

3.3.2 The discrete implementation

To evaluate $\int_a^b (u'(x) - g(x))^2 dx$ use the finite difference approximation

$$u'(\frac{\bar{x}_j + \bar{x}_{j+1}}{2}) \approx \frac{u(\bar{x}_{j+1}) - u(\bar{x}_j)}{\Delta x} = \frac{u_{j+1} - u_j}{\Delta x}$$

Thus multiplying the vector \vec{u} by the matrix

$$\mathbf{A}_1 = \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \end{bmatrix} \in \mathbb{M}^{N \times (N+1)}$$

will evaluate the first derivatives at the midpoints of the grid. In the algorithm the matrix

$$\mathbf{A}_1^T \mathbf{A}_1 = \frac{1}{(\Delta x)^2} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix} \in \mathbb{M}^{(N+1) \times (N+1)}$$

is used. This is the matrix generated by a finite difference method to solve $u''(x) = f(x)$ with the boundary conditions $u'(a) = u'(b) = 0$. The vector $\vec{g}_1 \in \mathbb{R}^N$ contains the values of the function $g_1(x)$ at the midpoints of the grid. Then use

$$\begin{aligned} F_1 &= \int_a^b (u'(x) - g_1(x))^2 dx \approx \Delta x \sum_{j=1}^N \left(\frac{u_{j+1} - u_j}{\Delta x} - g_j \right)^2 = \Delta x \langle \mathbf{A}_1 \vec{u} - \vec{g}_1, \mathbf{A}_1 \vec{u} - \vec{g}_1 \rangle \\ \frac{\partial F_1}{\partial \vec{u}} &\approx 2 \Delta x \mathbf{A}_1^T (\mathbf{A}_1 \vec{u} - \vec{g}_1) \in \mathbb{R}^{N+1}. \end{aligned}$$

3.3.3 Boundary contributions

If the boundary conditions should be $u'(a) - g_1(a) = m_a$ and $u'(b) - g_1(b) = m_b$, then modify the functional F_1 by¹

$$F_1(u) = \int_a^b (u'(x) - g_1(x))^2 dx + u(a) 2 m_a - u(b) 2 m_b.$$

Thus the first and last component of the gradient $\frac{\partial F_1}{\partial \vec{u}}$ obtain an additional contribution of $+2 m_a$, resp. $-2 m_b$.

3.4 Integration of second derivatives

3.4.1 The continuous formulation

The contribution

$$F_2(u) = \int_a^b (u''(x) - g_2(x))^2 dx$$

to the Euler–Lagrange equation² is given by $\frac{d^2}{dx^2} (u''(x) - g_2(x)) = 0$ and the natural boundary conditions are $u'' - g_2 = u''' - g'_2 = 0$ at $x = a$ and $x = b$.

¹Minimize $F_1(u + \varepsilon \phi)$ with respect to ε for all test functions ϕ .

$$\begin{aligned} F_1(u + \varepsilon \phi) &= \int_a^b (u' + \varepsilon \phi' - g_1)^2 dx + (u(a) + \varepsilon \phi(a)) 2 m_a - (u(b) + \varepsilon \phi(b)) 2 m_b \\ &= F_1(u) + 2 \varepsilon \int_a^b (u' - g_1) \phi' dx + 2 \varepsilon \phi(a) m_a - 2 \varepsilon \phi(b) m_b + O(\varepsilon^2) \\ &= F_1(u) + 2 \varepsilon \left((u' - g_1) \phi \Big|_{x=a}^b - \int_a^b (u' - g_1)' \phi dx + \phi(a) m_a - \phi(b) m_b \right) + O(\varepsilon^2) \end{aligned}$$

This leads to the Euler–Lagrange equation $(u' - g_1)' = 0$ and the boundary conditions $u'(a) - g_1(a) - m_a = 0$ and $u'(b) - g_1(b) - m_b = 0$.

²Two integrations by part lead to

$$F_2(u + \varepsilon \phi) = \int_a^b (u'' + \varepsilon \phi'' - g_2)^2 dx = F_2(u) + 2 \varepsilon \int_a^b (u'' - g_2) \phi'' dx + O(\varepsilon^2)$$

3.4.2 The discrete implementation

To evaluate $\int_a^b (u''(x) - g_2(x))^2 dx$ use the finite difference approximation

$$u''(\bar{x}_j) \approx \frac{u(\bar{x}_{j-1}) - 2u(\bar{x}_j) + u(\bar{x}_{j+1})}{(\Delta x)^2} = \frac{u_{j-1} - 2u_j + u_{j+1}}{(\Delta x)^2}$$

Thus multiplying the vector \vec{u} by the matrix

$$\mathbf{A}_2 = \frac{1}{\Delta x^2} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ & & -1 & 2 & -1 \end{bmatrix} \in \mathbb{M}^{(N-1) \times (N+1)}$$

leads to approximations of the second derivatives at the interior grid points. In the algorithm use the matrix

$$\mathbf{A}_2^T \mathbf{A}_2 = \frac{1}{\Delta x^4} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -4 & 1 \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 & 1 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -4 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \in \mathbb{M}^{(N+1) \times (N+1)}$$

This is the matrix generated by a finite difference method to solve $u^{(4)}(x) = f(x)$ with the natural boundary conditions $u''(a) = u'''(a) = u''(b) = u'''(b) = 0$. The vector $\vec{g}_2 \in \mathbb{R}^{N-1}$ contains the values of the function $g_2(x)$ at the interior grid points. Then use

$$\begin{aligned} F_2 &= \int_a^b (u''(x) - g_2(x))^2 dx \approx \Delta x \sum_{j=2}^N \left(\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x} - g_j \right)^2 \\ &= \Delta x \langle \mathbf{A}_2 \vec{u} - \vec{g}_2, \mathbf{A}_2 \vec{u} - \vec{g}_2 \rangle \\ \frac{\partial F_2}{\partial \vec{u}} &\approx 2 \Delta x \mathbf{A}_2^T (\mathbf{A}_2 \vec{u} - \vec{g}_2) \in \mathbb{R}^{N+1}. \end{aligned}$$

3.5 The Euler–Lagrange equation, natural boundary conditions and jump conditions

3.5.1 The continuous formulation

Since the functional to be minimized is

$$\begin{aligned} F(u) &= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u), \\ &= F_2(u) + 2\varepsilon \left((u'' - g_2) \phi' \Big|_{x=a}^b - \int_a^b (u'' - g_2)' \phi' dx \right) + O(\varepsilon^2) \\ &= F_2(u) + 2\varepsilon \left((u'' - g_2) \phi' - (u'' - g_2)' \phi \Big|_{x=a}^b + \int_a^b (u'' - g_2)'' \phi dx \right) + O(\varepsilon^2). \end{aligned}$$

the Euler–Lagrange equation, based on the Lagrangian $\lambda_1 F_1 + \lambda_2 F_2$, is given by

$$\lambda_2 (u''(x) - g_2(x))'' - \lambda_1 (u'(x) - g_1(x))' = 0. \quad (2)$$

The natural boundary conditions are

$$0 = \lambda_2 (u'' - g_2)' - \lambda_1 (u' - g_1 - m_a) \quad \text{and} \quad 0 = -\lambda_2 (u'' - g_2).$$

Now take the discrete data (x_i, y_i) into account. Use the functional from equation (1) and pay special attention to the points $x = x_i$. Let $x_0 = a$ and $x_{M+1} = b$. Then use integration by parts on the sub-intervals $[x_i, x_{i+1}]$.

$$\begin{aligned} F(u + \phi) &= F_D(u + \phi) + \lambda_1 F_1(u + \phi) + \lambda_2 F_2(u + \phi) \\ &= \sum_{i=1}^M (u(x_i) + \phi(x_i) - y_i)^2 + \\ &\quad + \lambda_1 \int_a^b (u'(x) + \phi'(x) - g_1(x))^2 dx + \lambda_2 \int_a^b (u''(x) + \phi''(x) - g_2(x))^2 dx \\ &\approx F(u) + 2 \sum_{i=1}^M (u(x_i) - y_i) \phi(x_i) + \\ &\quad + 2 \sum_{i=0}^M \int_{x_i}^{x_{i+1}} \lambda_1 (u'(x) - g_1(x)) \phi'(x) + \lambda_2 (u''(x) - g_2(x)) \phi''(x) dx \\ &= F(u) + 2 \sum_{i=1}^M (u(x_i) - y_i) \phi(x_i) + \\ &\quad + 2 \lambda_1 \sum_{i=0}^M (u'(x_{i+1}) - g_1(x_{i+1})) \phi(x_{i+1}) - (u'(x_i) - g_1(x_i)) \phi(x_i) \\ &\quad + 2 \lambda_2 \sum_{i=0}^M (u''(x_{i+1}) - g_2(x_{i+1})) \phi'(x_{i+1}) - (u''(x_i) - g_2(x_i)) \phi'(x_i) \\ &\quad + 2 \sum_{i=0}^M \int_{x_i}^{x_{i+1}} -\lambda_1 (u'(x) - g_1(x))' \phi(x) - \lambda_2 (u''(x) - g_2(x))' \phi'(x) dx \\ &= F(u) + 2 \sum_{i=1}^M (u(x_i) - y_i) \phi(x_i) + \\ &\quad + 2 \lambda_1 \sum_{i=0}^M (u'(x_{i+1}) - g_1(x_{i+1})) \phi(x_{i+1}) - (u'(x_i) - g_1(x_i)) \phi(x_i) \\ &\quad + 2 \lambda_2 \sum_{i=0}^M (u''(x_{i+1}) - g_2(x_{i+1})) \phi'(x_{i+1}) - (u''(x_i) - g_2(x_i)) \phi'(x_i) \\ &\quad - 2 \lambda_2 \sum_{i=0}^M (u''(x_{i+1}) - g_2(x_{i+1}))' \phi(x_{i+1}) - (u''(x_i) - g_2(x_i))' \phi(x_i) \\ &\quad + 2 \sum_{i=0}^M \int_{x_i}^{x_{i+1}} -\lambda_1 (u'(x) - g_1(x))' \phi(x) + \lambda_2 (u''(x) - g_2(x))'' \phi(x) dx \end{aligned}$$

To start out use test functions $\phi(x)$ vanishing at the data points $x = x_i$, i.e. $\phi_i(x_i) = \phi'(x_i) = 0$ for $i = 1, 2, 3, \dots, M$. For $x \neq x_i$ this leads to the Euler–Lagrange equation (2). Since we are free to choose the values of $\phi(x_i)$ and $\phi'(x_i)$ there are boundary and jump conditions.

- At $x = x_0 = a$ find the natural boundary conditions

$$-\lambda_1(u'(a) - g_1(a)) + \lambda_2(u''(a) - g_2(a))' = 0 \quad \text{and} \quad -\lambda_2(u''(a) - g_2(a)) = 0.$$

- At $x = x_{M+1} = b$ find the natural boundary conditions

$$+\lambda_1(u'(b) - g_1(b)) - \lambda_2(u''(b) - g_2(b))' = 0 \quad \text{and} \quad +\lambda_2(u''(b) - g_2(b)) = 0.$$

- At a data point $x = x_i$ find

$$\begin{aligned} (u(x_i) - y_i) &= -\lambda_1(u'(x_i-) - g_1(x_i-)) + \lambda_2(u''(x_i-) - g_2(x_i-)) \\ &\quad + \lambda_1(u'(x_i+) - g_1(x_i+)) - \lambda_2(u''(x_i+) - g_2(x_i+)). \end{aligned}$$

This is a jump condition, i.e. the expression

$$+\lambda_1(u'(x) - g_1(x)) - \lambda_2(u''(x) - g_2(x))$$

jumps by $u(x_i) - y_i$ as x crosses x_i . Since the first derivative $u'(x)$ is the integral of $u''(x)$, conclude that u' is continuous. This leads to a jump condition for $u''(x)$. If in addition the functions $g_1(x)$ and $g_2(x)$ are continuous find

$$\begin{aligned} u(x_i) - y_i &= \lambda_2(u''(x_i+) - u''(x_i-)) \\ u''(x_i+) - u''(x_i-) &= \frac{1}{\lambda_2}(u(x_i) - y_i). \end{aligned}$$

Thus the exact minimizer $u(\cdot)$ of the functional (1) solves the Euler–Lagrange equation (2) piecewise on all sub-intervals $x_i < x < x_{i+1}$ and it satisfies the above natural boundary conditions and the jump conditions for the second derivative $u''(x)$ at $x = x_i$.

3.5.2 The discrete implementation

For the functional $F(u)$ find the discretization

$$\begin{aligned} F(u) &= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\ &= \sum_{i=1}^M (y_i - u(x_i))^2 + \lambda_1 \int_a^b (u'(x) - g_1(x))^2 dx + \lambda_2 \int_a^b (u''(x) - g_2(x))^2 dx \\ &\approx \langle \vec{y} - \mathbf{I}\vec{u}, \vec{y} - \mathbf{I}\vec{u} \rangle + \lambda_1 \Delta x \langle \mathbf{A}_1 \vec{u} - \vec{g}_1, \mathbf{A}_1 \vec{u} - \vec{g}_1 \rangle + \lambda_2 \Delta x \langle \mathbf{A}_2 \vec{u} - \vec{g}_2, \mathbf{A}_2 \vec{u} - \vec{g}_2 \rangle \\ \frac{1}{2} \frac{\partial F}{\partial \vec{u}} &\approx \mathbf{I}^T (\mathbf{I}\vec{u} - \vec{y}) + \lambda_1 \Delta x \mathbf{A}_1^T (\mathbf{A}_1 \vec{u} - \vec{g}_1) + \lambda_2 \Delta x \mathbf{A}_2^T (\mathbf{A}_2 \vec{u} - \vec{g}_2) \in \mathbb{R}^{N+1}. \end{aligned}$$

This leads to a system of linear equations for $\vec{u} \in \mathbb{R}^{N+1}$.

$$(\mathbf{I}^T \mathbf{I} + \Delta x (\lambda_1 \mathbf{A}_1^T \mathbf{A}_1 + \lambda_2 \mathbf{A}_2^T \mathbf{A}_2)) \vec{u} = \mathbf{I}^T \vec{y} + \Delta x (\lambda_1 \mathbf{A}_1^T \vec{g}_1 + \lambda_2 \mathbf{A}_2^T \vec{g}_2) \in \mathbb{R}^{N+1}$$

3.6 The code for the function regularization()

The above algorithm is implemented in an *Octave* function `regularization.m`. Not shown below are the copyright, the documentation and the demos, just the implementation of the algorithm.

regularization.m

```

## Copyright (C) 2019 Andreas Stahel

function [grid,u] = regularization (data, interval, N, F1, F2)
    a = interval (1); b = interval (2);
    grid = linspace (a, b, N + 1)';
    dx = grid (2) - grid (1);
    x = data (:, 1);
    ## select points in interval only
    ind = find ((x >= a) .* (x <= b));
    x = x (ind);
    y = data (:, 2); y = y (ind);
    M = length (x);
    Interp = sparse (M, N + 1); ## interpolation matrix
    pos = floor ((x - a) / dx) + 1;
    theta = mod ((x - a) / dx, 1);
    for ii = 1:M
        if theta (ii) > 10*eps
            Interp (ii, pos (ii)) = 1 - theta (ii);
            Interp (ii, pos (ii) + 1) = theta (ii);
        else
            Interp (ii, pos (ii)) = 1;
        endif
    endfor
    mat = Interp' * Interp;
    rhs = (Interp' * y);
    if isfield (F1, 'lambda')
        A1 = spdiags ([-ones(N, 1), +ones(N, 1)], [0, 1], N, N + 1) / dx;
        mat = mat + F1.lambda * dx * A1' * A1;
        if isfield (F1, 'g')
            g1 = F1.g (grid (1:end - 1) + dx / 2);
            rhs = rhs + F1.lambda * dx * A1' * g1;
        endif
    endif
    if exist ('F2')
        A2 = spdiags (ones (N, 1) * [1, -2, 1], [0, 1, 2], N - 1, N + 1) / dx^2;
        mat = mat + F2.lambda * dx * A2' * A2;
        if isfield (F2, 'g')
            g2 = F2.g (grid (2:end - 1));
            rhs = rhs + F2.lambda * dx * A2' * g2;
        endif
    endif
    u = mat \ rhs;
endfunction

```

4 Regularization with Two Independent Variables

For a problem with two independent variables $(x, y) \in \Omega \subset \mathbb{R}^2$ and parameters $\lambda_1 \geq 0, \lambda_2 > 0$ and data points $(x_i, y_i, z_i) \in \Omega \times \mathbb{R}$ the functional to be minimized is

$$\begin{aligned}
F(u) &:= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\
&= \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \iint_{\Omega} u_x^2 + u_y^2 \, dA + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2 u_{xy}^2 \, dA.
\end{aligned}$$

- The functional F_D pushes the optimal solution towards a function going through the given data points (s_i, y_i, z_i) .
- The functional $\lambda_1 F_1$ pushes the optimal solutions towards a horizontal plane.
- The expression F_2 corresponds to the bending energy in a thin plate. Thus $\lambda_2 F_2$ pushes the optimal solutions towards surfaces with minimal curvature.

The above optimization is realized in *Octave* by using finite difference approximations for the derivatives and elementary numerical integration. A linear system of equations is used to determine discrete approximations $\vec{u} \in \mathbb{R}^N$ to the exact minimizer $u(x, y)$.

4.1 Documentation for the usage of the code

The built-in help is shown below and the next section provides a few examples on how to use the function `regularization2D()`.

```

help regularization2D
-->
-- Function File:
[GRID, U, DATA_VALID] = regularization2D(DATA, BOX, N, LAMBDA1, LAMBDA2)

Apply a Tikhonov regularization, the functional to be minimized is
F = FD + LAMBDA1 * F1 + LAMBDA2 * F2
= sum_(i=1)^M (y_i - u(x_i))^2 +
  + LAMBDA1 * dintegral (du/dx)^2 + (du/dy)^2 dA +
  + LAMBDA2 * dintegral (d^2u/dx^2)^2 + (d^2u/dy^2)^2 + 2*(d^2u/dxdy) dA

With LAMBDA1 = 0 and LAMBDA2>0 this leads to a thin plate smoothing spline.

Parameters:
* DATA is a M*3 matrix with the (x,y) values in the first two
  columns and the y values in the third column.
  Only data points strictly inside the BOX are used
* BOX = [x0,x1;y0,y1] is the rectangle x0<=x<=x1 and y0<=y<=y1 on
  which the regularization is applied.
* N = [N1,N2] determines the number of subintervals of equal length.
  GRID will consist of (N1+1)x(N2+1) grid points.
* LAMBDA1 >= 0 is the value of the first regularization parameter
* LAMBDA2 > 0 is the value of the second regularization parameter

Return values:
* GRID is the grid on which U is evaluated. It consists of
  (N1+1)x(N2+1) equidistant points on the the rectangle BOX.
* U are the values of the regularized approximation to the DATA
  evaluated on the GRID.
* DATA_VALID returns the values data points used and the values
  of the regularized function at these points

See also: tpaps, regularization, demo regularization2D.

```

4.2 Examples

4.2.1 A first example, approximating a surface with random noise

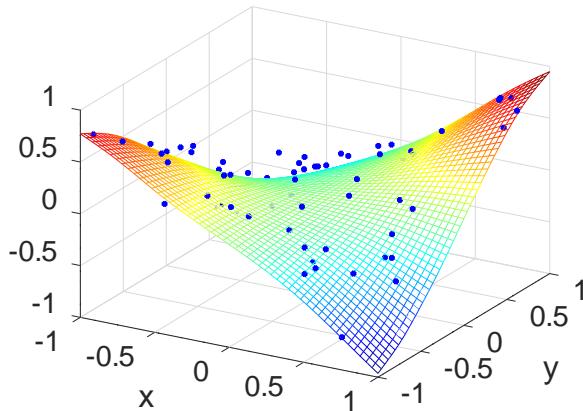
On the square $-1 < x, y < 1$ examine points given by $z_i = x_i \cdot y_i$ with some random noise added. Then a smoothing surface is fit through those points.

Octave

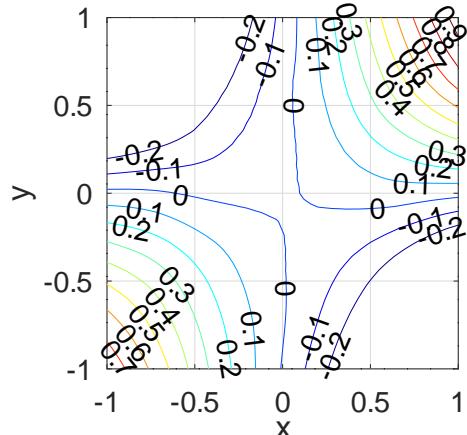
```

lambda1 = 0; lambda2 = 0.05
M = 100; x = 2*rand(M,1)-1; y = 2*rand(M,1)-1; z = x.*y + 0.1*randn(M,1);
data = [x,y,z];
[grid,u] = regularization2D(data,[-1 1;-1 1],[50 50],lambda1,lambda2);
figure(1); mesh(grid.x, grid.y,u)
xlabel('x'); ylabel('y');
hold on
plot3(data(:,1),data(:,2),data(:,3),'*b','Markersize',2)
view([30,30]); hold off

```



(a) the surface



(b) the contour lines

Figure 7: A regularized surface, determined by random data

4.2.2 A second example, determined by a few points

As a second example examine five data points, four on the x and y axis at $x, y = \pm 1$ at height 0 and at the origin at height 1. Then a regularization on the square $-1.5 \leq x, y \leq +1.5$ with parameters $\lambda_1 = 0$ and $\lambda_2 = 0.05$ is applied.

Octave

```

lambda1 = 0; lambda2 = 0.05
M = 4; angles = [1:M]/M*2*pi; data = zeros(M+1,3);
data(1:M,1) = cos(angles); data(1:M,2) = sin(angles); data(M+1,3) = 1;
[grid,u] = regularization2D(data,[-1.5 1.5;-1.5 1.5],[40 40],lambda1,lambda2);

```

Use the usual Octave commands to generate the surface plot and contour lines in Figure 8.

Octave

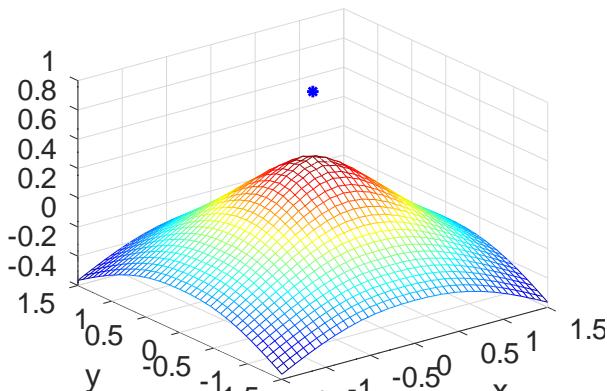
```

figure(1); mesh(grid.x, grid.y,u)
xlabel('x'); ylabel('y');
hold on
plot3(data(:,1),data(:,2),data(:,3),'*b')
hold off

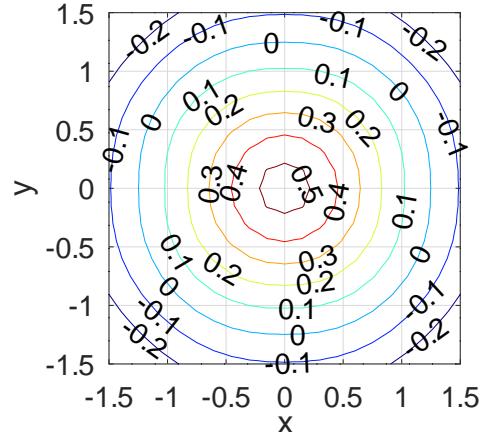
figure(2); [c,h] = contour(grid.x, grid.y,u,[-0.2:0.1:1]); clabel(c,h);
xlabel('x'); ylabel('y'); axis equal

```

The above can be recomputed with



(a) the surface

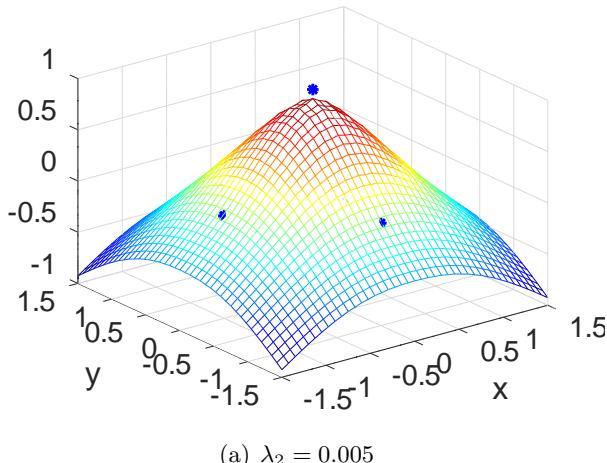


(b) contour lines

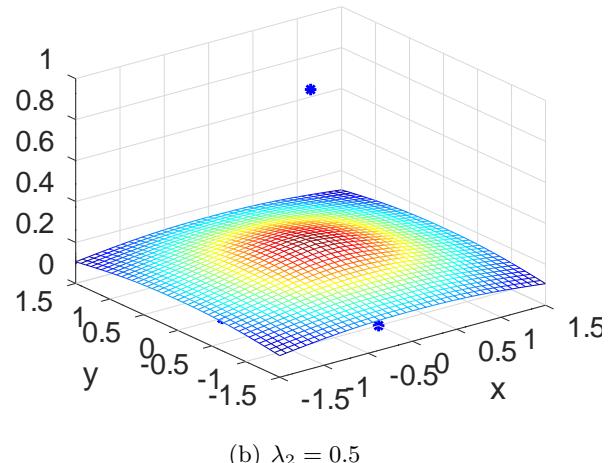
Figure 8: A surface determined by 5 data points, with regularization parameter $\lambda_2 = 0.05$

- a smaller parameter $\lambda_2 = 0.005$, leading to a surface with larger curvature, but closer to the data points.
 - a large parameter $\lambda_2 = 0.5$, leading to a surface with smaller curvature, but further away from the data points.

Find the resulting graphs in Figure 9.



(a) $\lambda_2 = 0.005$



(a) $\lambda_2 = 0.005$ (b) $\lambda_2 = 0.5$

Figure 9: A surface determined by 5 data points with different regularization parameters λ_2

4.2.3 A third example, to illustrate that $\lambda_2 > 0$ is necessary

Remark 4-4 states that the parameter $\lambda_2 > 0$ has to be positive, otherwise the problem is not well defined and the discrete solution generated depends on the grid size used. The code below uses the same data as the previous example, but with parameters $\lambda_1 = 0.1$ and $\lambda_2 = 0$. The code is used twice, once with a 50×50 grid, and once with a 100×100 grid. Figure 10 shows the results. Observe that the contour lines for the finer grid are closer together. This is consistent with the observation in Remark 4-4 and illustrates that $\lambda_2 > 0$ is necessary to obtain well defined results.

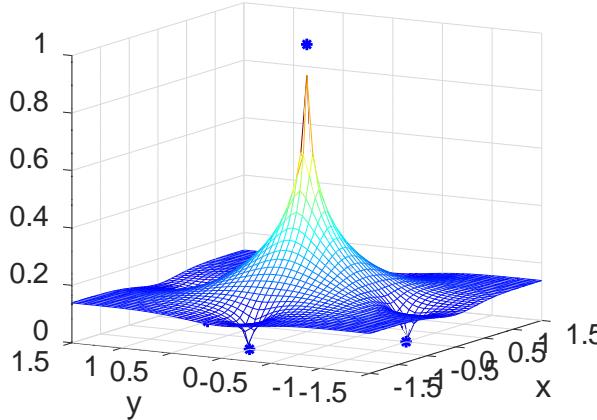
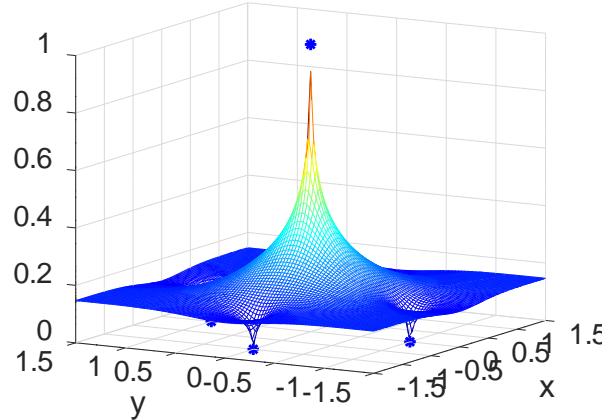
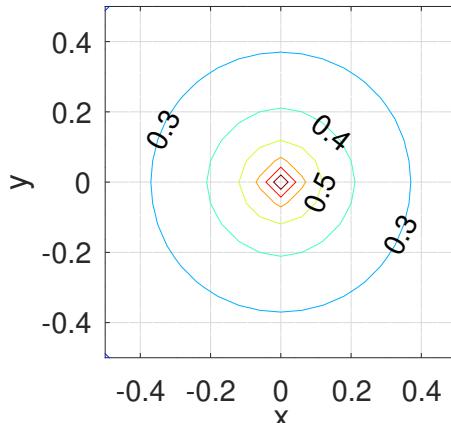
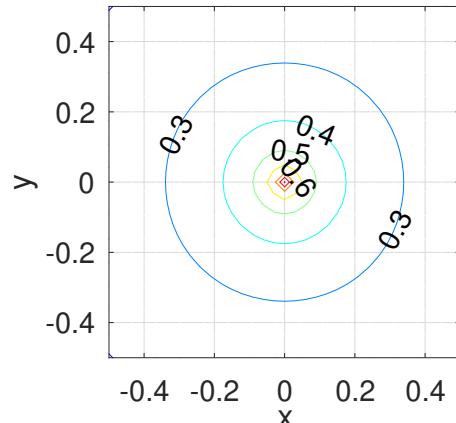
Octave

```

lambda1 = 0.1; lambda2 = 0;
M = 4; angles = [1:M]/M*2*pi; data = zeros(M+1,3);
data(1:M,1) = cos(angles); data(1:M,2) = sin(angles); data(M+1,3)=1;

[grid,u] = regularization2D(data,[-1.5 1.5;-1.5 1.5],[50 50],lambda1,lambda2)

```

(a) Graph with a 50×50 grid(b) Graph with a 100×100 grid(c) Contour with a 50×50 grid(d) Contour with a 100×100 gridFigure 10: Regularization with $\lambda_1 = 0.1$ and $\lambda_2 = 0$ on different grids

4.2.4 A fourth example, approximating a given function

Generate data on a 21×21 grid on the unit square $-1 \leq x, y \leq +1$ of the function $f(x, y) = \exp(-(x^2 + y^2))$. Then a regularization with $\lambda_1 = 0$ and $\lambda_2 = 0.05$ is applied on the larger rectangle $(x, y) \in [-1, +1] \times [-1, +2]$. Find the resulting surface and the difference between the regularized function and the original function $f(x, y) = \exp(-(x^2 + y^2))$ in Figure 11. Observe that the difference on the unit square is rather small and for $y > 1$ the regularized surface shows very little curvature, but a clear slope.

Octave

```

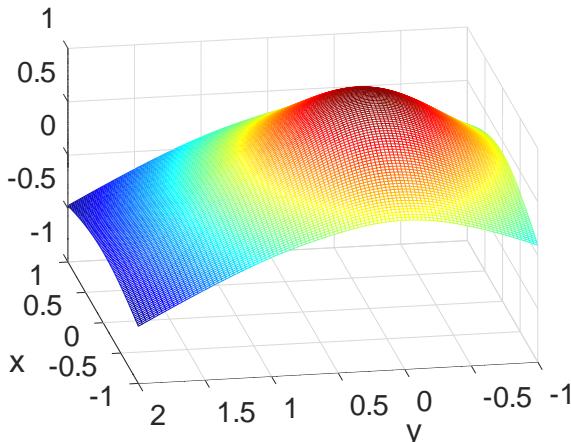
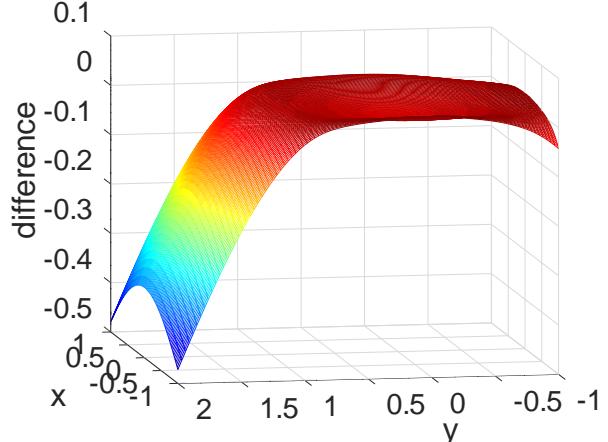
lambda1 = 0; lambda2 = 0.05;
N = 21; [xx,yy] = meshgrid(linspace(-1,1,N));

```

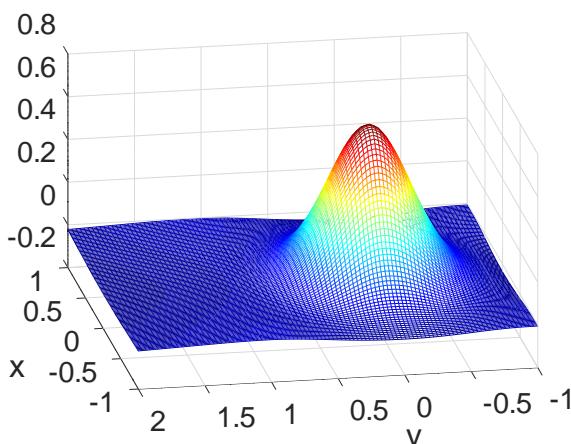
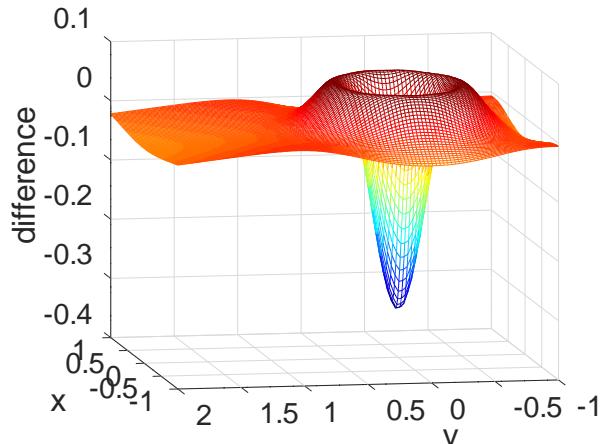
```

function z = ff(x,y)
    z = exp(-(x.^2+y.^2));
endfunction
zz = ff(xx,yy);
data = [xx(:),yy(:),zz(:)];
Nreg = 100;
[grid,u] = regularization2D(data,[-1,1;-1 2],[Nreg,Nreg],lambda1,lambda2);

```

(a) the surface $u(x,y)$ (b) the difference $u(x,y) - f(x,y)$ Figure 11: A regularized surface determined by $\exp(-(x^2 + y^2))$ on the unit square

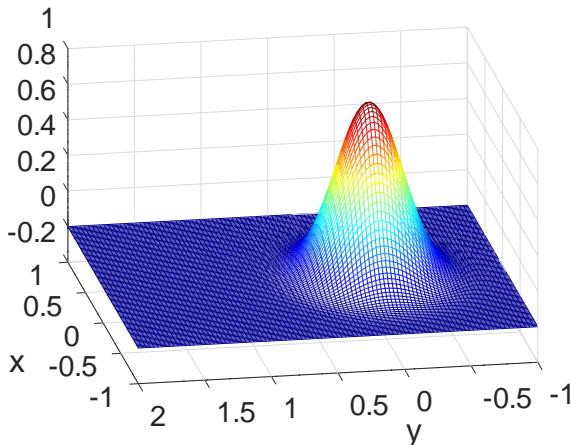
The above is repeated with the function $f(x,y) = \exp(-9(x^2 + y^2))$, leading to Figure 12. The graph of this function is almost horizontal on the boundary of the unit square, thus the extension of the regularized surface for $y > 1$ is almost horizontal. The curvature of $f(x,y)$ close to the origin $(0,0)$ is large and as a consequence the regularized surface deviates from $f(x,y)$.

(a) the surface $u(x,y)$ (b) the difference $u(x,y) - f(x,y)$ Figure 12: A regularized surface determined by $\exp(-9(x^2 + y^2))$ on the unit square

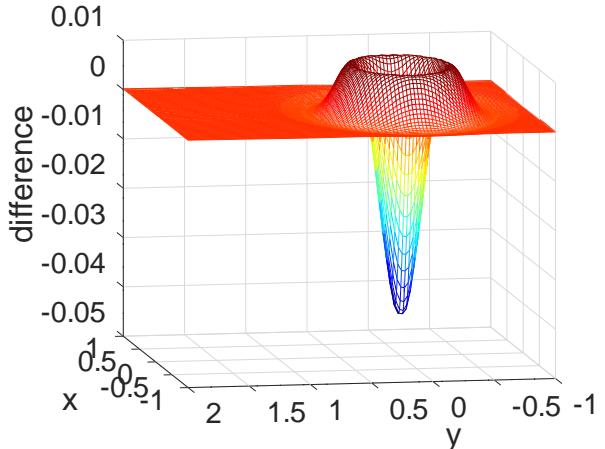
The same setup can be examined using the function `tpaps()` from the package `splines`, leading to Figure 13. Observe that shape of the regularization is similar and the construction by

`tpaps()` leads to a smaller difference to the original function $\exp(-9(x^2 + y^2))$.

```
pkg load splines
p = 1/(1+lambda2);
[u2,p] = tpaps(data(:,1:2),data(:,3),p,[grid.x(:),grid.y(:)]);
u2 = reshape(u2,Nreg+1,Nreg+1);
```



(a) the surface $u(x, y)$



(b) the difference $u(x, y) - f(x, y)$

Figure 13: A regularized surface determined by $\exp(-9(x^2 + y^2))$ on the unit square, using `tpaps()`

4.2.5 A fifth example, comparing `regularization2d()` and `tpaps()`

The function `tpaps()` is using smoothing splines, minimizing the bending energy of the thin plate over all of \mathbb{R}^2 , and not the rectangle used by `regularization2D()`. Thus the functional

$$F(u) = F_D(u) + \lambda_2 F_2(u) = \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2 u_{xy}^2 \, dA$$

is minimized

- with the rectangle $\Omega = [x_0, x_1] \times [y_0, y_1]$ for `regularization2D()`.
- with $\Omega = \mathbb{R}^2$ for `tpaps()`.

In Figure 14 find the bending plate energy $F_2(u)$ (integration over the rectangle) and the data difference $F_D(u)$ displayed as function of the parameter λ_2 . Figure 15 shows that `regularization2D()` leads to smaller thin plate energies on the rectangle, compared to `tpaps()`, as is to be expected since `regularization2D()` minimizes over the rectangle.

4.3 The algorithm and the code for the function `regularizaion2D()`

The functional to be minimized is

$$\begin{aligned} F(u) &:= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\ &= \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \langle u, u \rangle_1 + \lambda_2 \langle u, u \rangle_2 \end{aligned}$$

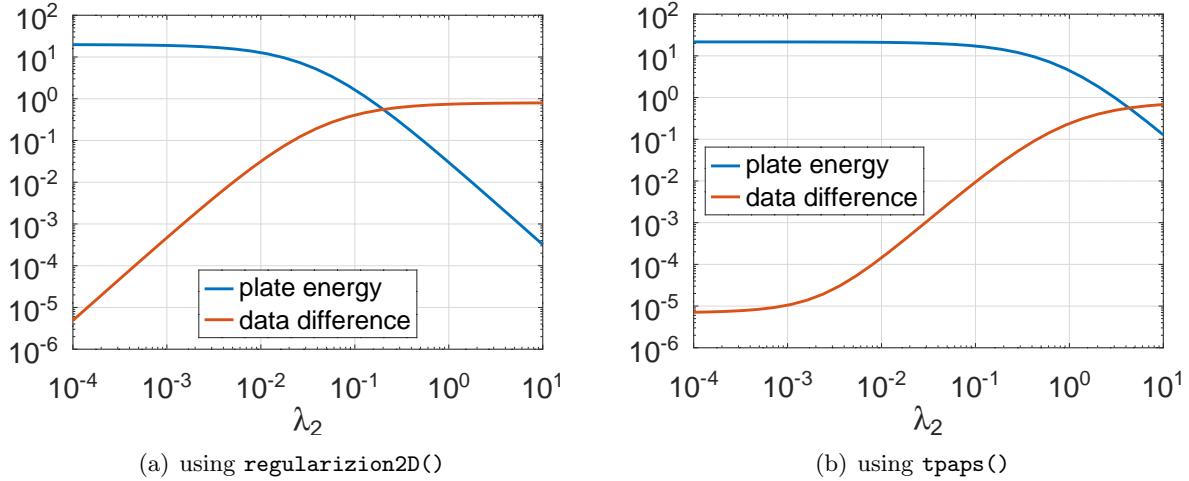


Figure 14: Comparing the contributions by the data differences F_D and the bending energy F_2 , for different values of $\lambda_2 > 0$

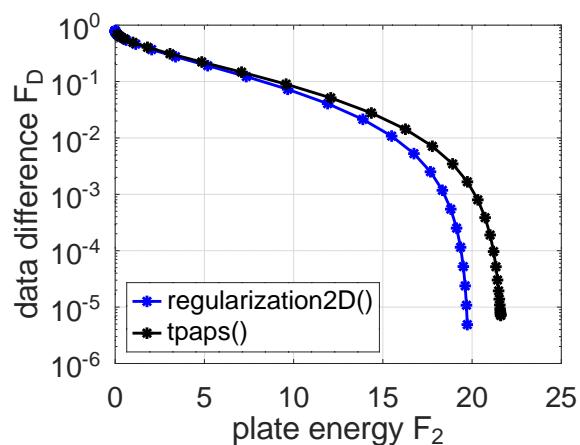


Figure 15: Comparing the energy contributions by `regularization2D()` and `tpaps()`, for different values of $\lambda_2 > 0$

$$\begin{aligned}
&= \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \iint_{\Omega} u_x^2 + u_y^2 \, dA + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 \, dA \\
&\approx \langle \text{Interp } \vec{u} - \vec{z}, \text{Interp } \vec{u} - \vec{z} \rangle + \lambda_1 (\langle \mathbf{D}_x \vec{u}, \mathbf{W}_x \mathbf{D}_x \vec{u} \rangle + \langle \mathbf{D}_y \vec{u}, \mathbf{W}_y \mathbf{D}_y \vec{u} \rangle) + \\
&\quad + \lambda_2 (\langle \mathbf{D}_{xx} \vec{u}, \mathbf{W}_x \mathbf{D}_{xx} \vec{u} \rangle + \langle \mathbf{D}_{yy} \vec{u}, \mathbf{W}_y \mathbf{D}_{yy} \vec{u} \rangle + 2 \langle \mathbf{D}_{xy} \vec{u}, \mathbf{W}_{xy} \mathbf{D}_{xy} \vec{u} \rangle) =: F(\vec{u})
\end{aligned}$$

Then the necessary analytical condition $\nabla F(\vec{u}) = \vec{0}$ leads to the discrete condition

$$\begin{aligned}
\text{Interp}^T \vec{z} &= \text{Interp}^T \text{Interp} \vec{u} + \lambda_1 (\mathbf{D}_x^T \mathbf{W}_x \mathbf{D}_x + \mathbf{D}_y^T \mathbf{W}_y \mathbf{D}_y) \vec{u} \\
&\quad + \lambda_2 (\mathbf{D}_{xx}^T \mathbf{W}_x \mathbf{D}_{xx} + \mathbf{D}_{yy}^T \mathbf{W}_y \mathbf{D}_{yy} + 2 \mathbf{D}_{xy}^T \mathbf{W}_{xy} \mathbf{D}_{xy}) \vec{u}.
\end{aligned}$$

The above algorithm is implemented in the Octave function `regularization2D.m`. Not shown below are the copyright, the documentation and the demos, just the implementation of the algorithm.

regularization2D.m

```

## Copyright (C) 2021 Andreas Stahel

function [grid,u,data_valid] = regularization2D (data, box, N, lambda1, lambda2)

%% generate the grid
N = N+1; %% now N is the number of grid points in either direction
x = linspace(box(1,1),box(1,2),N(1));
y = linspace(box(2,1),box(2,2),N(2));
[xx,yy] = meshgrid(x,y);
dx = diff(box(1,:))/(N(1)-1); dy = diff(box(2,:))/(N(2)-1);
grid.x = xx; grid.y = yy;
x = data (:,1); y = data (:,2); z = data (:,3);
## select points in box only
ind = find ((x > box(1,1)) .* (x < box(1,2)) .* (y > box(2,1)) .* (y < box(2,2)));
x = x (ind); y = y (ind); z = z (ind);
%% generate the sparse interpolation matrix
M = length (x);
x_ind = floor((x-box(1,1))/dx);
xi = mod(x,dx)/dx;
y_ind = floor((y-box(2,1))/dy);
nu = mod(y,dy)/dy;
row = ones(4,1)*[1:M];
index_base = N(2)*x_ind+y_ind+1;
index = index_base + [0,N(2),1,N(2)+1]; index = index';
coeff = [(1-xi).*(1-nu),xi.*(1-nu),(1-xi).*nu,xi.*nu]; coeff = coeff';
Interp = sparse(row(:,index(:,coeff(:,M,N(1)*N(2));
mat = Interp' * Interp;
rhs = (Interp' * z);

%%% derivative with respect to x
Dx = kron(spdiags(ones(N(1),1)*[-1 1],[0 1],N(1)-1,N(1))/dx,speye(N(2)));
Wx = ones(N(2),1); Wx(1) = 1/2; Wx(N(2)) = 1/2;
Wx = kron(speye(N(1)-1),diag(Wx))*dx*dy;
%%% derivative with respect to y
Dy = kron(speye(N(1)),spdiags(ones(N(2),1)*[-1 1],[0 1],N(2)-1,N(2))/dy);
Wy = ones(N(1),1); Wy(1) = 1/2; Wy(N(1)) = 1/2;
Wy = kron(diag(Wy),speye(N(2)-1))*dx*dy;
mat += lambda1*(Dx'*Wx*Dx + Dy'*Wy*Dy);

%%% second derivative with respect to x
Dxx = spdiags(ones(N(1),1)*[1 -1 -1 1],[-1 0 1 2],N(1)-1,N(1));
Dxx(1,1:4) = [3 -7 5 -1]; Dxx(N(1)-1,N(1)-3:N(1)) = [-1 5 -7 3];

```

```

Dxx = Dxx/(2*dx^2);
Dxx = kron(Dxx, speye(N(2)));
%%% second derivative with respect to y
Dyy = spdiags(ones(N(2),1)*[1 -1 -1 1], [-1 0 1 2], N(2)-1, N(2));
Dyy(1,1:4) = [3 -7 5 -1]; Dyy(N(2)-1, N(2)-3:N(2)) = [-1 5 -7 3];
Dyy = Dyy/(2*dy^2);
Dyy = kron(speye(N(1)), Dyy);
%%% mixed second derivative
Dy2 = kron(speye(N(1)-1), spdiags(ones(N(2),1)*[-1 1], [0 1], N(2)-1, N(2))/dy);
Dxy = Dy2*Dx*sqrt(dx*dy);
mat += lambda2*(Dxx'*Wx*Dxx + Dyy'*Wy*Dyy + 2*Dxy'*Dxy);

%%% solve
u = reshape(mat \ rhs, N(2), N(1));
if nargout>2
    data_valid =[x,y,Interp*u(:)];
endif
endfunction

```

4.4 The mathematics for the result in two dimensions

4.4.1 Analytical results and the proof of unique existence

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary, satisfying the cone condition. Thus the usual Sobolev imbedding result apply ([[Adam03](#)]). Let $(x_i, y_i) \in \Omega$ and z_i for $i = 1, 2, \dots, M$ be a set of data points. Let u and v be real valued functions defined on Ω and use the notations

$$\begin{aligned}
\vec{u} &:= (u_1, u_2, \dots, u_M)^T \\
\langle \vec{u}, \vec{v} \rangle &= \sum_{i=1}^M u_i v_i \\
\langle u, v \rangle &= \iint_{\Omega} u(x, y) v(x, y) dA \\
\langle u, v \rangle_1 &= \langle u_x, v_x \rangle + \langle u_y, v_y \rangle = \iint_{\Omega} \frac{\partial u(x, y)}{\partial x} \frac{\partial v(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} \frac{\partial v(x, y)}{\partial y} dA \\
\langle u, v \rangle_2 &= \langle u_{xx}, v_{xx} \rangle + \langle u_{yy}, v_{yy} \rangle + 2 \langle u_{xy}, v_{xy} \rangle
\end{aligned}$$

This leads to the usual L_2 and Sobolev norms.

$$\begin{aligned}
\|u\|_{L_2(\Omega)}^2 &= \langle u, u \rangle \\
\|u\|_{H^1(\Omega)}^2 &= \langle u, u \rangle + \langle u, u \rangle_1 \\
\|u\|_{H^2(\Omega)}^2 &= \langle u, u \rangle + \langle u, u \rangle_1 + \langle u, u \rangle_2
\end{aligned}$$

To prove the main result use a Poincaré type inequality, based on the value of the function u at one point $x_0 \in \Omega$, instead of the usual zero values on the boundary $\partial\Omega$ or the average value of u .

4-1 Lemma : For a convex, bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary examine functions $u \in W^{1,p}(\Omega)$ with $p > 4$. For any $x_0 \in \Omega$ with $u_0 := u(x_0)$ there exists a constant c such that

$$\|u - u_0\|_{L_2(\Omega)} \leq c \|\nabla u\|_{L_p(\Omega)}, \quad (3)$$

where the constant c depends on p and Ω .

Proof : The standard Sobolev imbedding implies $u \in C^0(\bar{\Omega})$ and the evaluation at one point $x_0 \in \Omega$ is well defined. For the convex domain $\Omega \subset \mathbb{R}^2$ and a parameter $0 < t \leq 1$ use a rescaled, contracted domain

$$\Omega_t := \{w = x + t y \mid \text{for some } y \in \Omega\} \subset \Omega \subset \mathbb{R}^2.$$

Integrating the inequality

$$\begin{aligned} |u(x) - u(x_0)| &= \left| \int_0^1 \langle \nabla u(x_0 + t(x - x_0)), (x - x_0) \rangle dt \right| \leq \int_0^1 \|x - x_0\| \|\nabla u(x_0 + t(x - x_0))\| dt \\ |u(x) - u(x_0)|^2 &\leq \max_{x \in \Omega} \|x - x_0\|^2 \int_0^1 \|\nabla u(x_0 + t(x - x_0))\|^2 dt \end{aligned}$$

over the domain Ω leads to

$$\begin{aligned} \iint_{\Omega} |u(x) - u_0|^2 dA_y &\leq \max_{x \in \Omega} \|x - x_0\|^2 \left(\iint_{\Omega} \int_0^1 \|\nabla u(x_0 + t(x - x_0))\|^2 dt dA \right) \\ &= C(\Omega) \int_0^1 \left(\iint_{\Omega} \|\nabla u(x_0 + t(x - x_0))\|^2 dA \right) dt \\ &\quad \text{substitution } w = x_0 + t(x - x_0), dA_w = t^2 dA, \text{vol}(\Omega_t) = t^2 \text{vol}(\Omega) \\ &= C(\Omega) \int_0^1 \left(\iint_{\Omega_t} \frac{1}{t^2} \|\nabla u(w)\|^2 dA_w \right) dt \\ &\quad \text{Hölder's inequality on } \Omega_t \text{ with } r = \frac{p}{2}, \frac{1}{r} + \frac{1}{r'} = 1, 1 < r' < 2 \\ &\quad \text{use } \left(\iint_{\Omega_t} t^{-2r'} dA_w \right)^{1/r'} = t^{-2} \text{vol}(\Omega_t)^{1/r'} = \text{vol}(\Omega)^{1/r'} t^{2/r'-2} \\ &\leq C(\Omega) \int_0^1 c(p, \Omega) t^{\frac{2}{r'}-2} \left(\iint_{\Omega_t} \|\nabla u(w)\|^{2r} dA_w \right)^{1/r} dt \\ &\leq C(\Omega) c(p, \Omega) \left(\int_0^1 t^{\frac{2}{r'}-2} dt \right) \|\nabla u\|_{L_p(\Omega)}^2 \\ &= C(\Omega) c(p, \Omega) \left(\frac{2}{r'} - 1 \right) \|\nabla u\|_{L_p(\Omega)}^2, \text{ use } \frac{2}{r'} - 2 > -1 \\ &\leq C_1(p, \Omega) \|\nabla u\|_{L_p(\Omega)}^2 \end{aligned}$$

Thus inequality (3) is verified. \square

Examine a convex, bounded domain $\Omega \subset \mathbb{R}^2$ and data points $(x_i, y_i) \in \Omega$ for $i = 1, 2, 3, \dots, M$. At those points the values of a function u should be close to given values z_i . For a function $u \in H^2(\Omega)$ the evaluation at the points (x_i, y_i) is well defined and the values $u(x_i, y_i)$ lead to a vector $\vec{u} \in \mathbb{R}^M$. For positive regularization parameters $0 < \lambda_1$ and $0 < \lambda_2$ search for a minimizer of the functional

$$\begin{aligned} F(u) &:= F_D(u) + \lambda_1 F_1(u) + \lambda_2 F_2(u) \\ &= \langle \vec{u} - \vec{z}, \vec{u} - \vec{z} \rangle + \lambda_1 \langle u, u \rangle_1 + \lambda_2 \langle u, u \rangle_2 \\ &= \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \iint_{\Omega} u_x^2 + u_y^2 dA + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 dA. \end{aligned} \tag{4}$$

4–2 Theorem : If at least one data point is given, then the functional $F(u)$ in (4) has exactly one minimizer $u_* \in H^2(\Omega)$.

Proof : Proceed in three steps: first establish an a-priori estimate, then verify existence, followed by the proof of uniqueness.

A-priori estimate:

Let $\bar{z} = \frac{1}{M} \sum_{i=1}^M z_i$ be the average value and then use the variance σ^2 of the z values

$$M \sigma^2 = \sum_{i=1}^M (z_i - \bar{z})^2.$$

The constant function $u(x, y) = \bar{z}$ satisfies $F(u) = M \sigma^2$. Any function $u \in H^2(\Omega)$ with $F(u) \leq M \sigma^2$ satisfies

$$\langle u, u \rangle_1 \leq \frac{1}{\lambda_1} M \sigma^2 \quad \text{and} \quad \langle u, u \rangle_2 \leq \frac{1}{\lambda_2} M \sigma^2.$$

Define a subset $A \subset H^2(\Omega)$, based on $m = \max_i |z_i|$.

$$A := \{u \in H^2(\Omega) \mid \min_{x \in \bar{\Omega}} u(x) \leq +m, \max_{x \in \bar{\Omega}} u(x) \geq -m, \langle u, u \rangle_1 \leq \frac{1}{\lambda_1} M \sigma^2, \langle u, u \rangle_2 \leq \frac{1}{\lambda_2} M \sigma^2\}$$

This set $A \subset H^2(\Omega)$ is not empty, since the constant function $u(x, y) = \bar{z} \in A$. The Sobolev imbedding for the gradient ∇u leads to

$$\|\nabla u\|_{L_p(\Omega)} \leq c \|\nabla u\|_{H^1(\Omega)} = c \sqrt{\langle u, u \rangle_1 + \langle u, u \rangle_2} \quad \text{for some } 4 \leq p < \infty .$$

For the set A to be bounded in $H^2(\Omega)$ an a-priori bound on $\|u\|_{L_2(\Omega)}$ is needed. Since $u \in A$ use an $x_0 \in \Omega$ with $|u(x_0)| = |u_0| \leq m$. Inequality (3) in the above Lemma implies

$$\|u\|_{L_2(\Omega)} \leq \|u_0\|_{L_2(\Omega)} + \|u - u_0\|_{L_2(\Omega)} \leq m \sqrt{\text{vol}(\Omega)} + \sqrt{c} \|\nabla u\|_{L_p(\Omega)} .$$

Thus the set $A \subset H^2(\Omega)$ is uniformly bounded in the L_2 norm, and consequently in the H^2 norm.

Existence:

Use a minimizing sequence $u_n \in A$ of the functional $F(u)$ in (4). Based on the above a-priori estimate the sequence is bounded in $H^2(\Omega)$. Since $H^2(\Omega)$ is compactly imbedded in $W^{1,p}(\Omega)$ for some $p > 4$, there exists a convergent subsequence (again denoted by u_n) in $W^{1,p}(\Omega)$ and $u_n \rightharpoonup u_*$. Since $H^2(\Omega)$ is a Hilbert space there exists a weakly convergent subsequence u_n and $u_* \in H^2(\Omega)$. The weak convergence implies

$$\langle \vec{u}_n - \vec{z}, \vec{u}_* - \vec{z} \rangle + \lambda_1 \langle u_n, u_* \rangle_1 + \lambda_2 \langle u_n, u_* \rangle_2 \xrightarrow{n \rightarrow \infty} \langle \vec{u}_* - \vec{z}, \vec{u}_* - \vec{z} \rangle + \lambda_1 \langle u_*, u_* \rangle_1 + \lambda_2 \langle u_*, u_* \rangle_2 = F(u_*)$$

and $u_* \in H^2(\Omega)$ is a minimizer of the functional F .

Uniqueness:

Assume $u_1 \in H^2(\Omega)$ is a second minimizer. Let $v = u_1 - u_*$ and examine

$$\begin{aligned} g(t) &:= F(u_* + t v) \\ &= \langle \vec{u}_* + t \vec{v} - \vec{z}, \vec{u}_* + t \vec{v} - \vec{z} \rangle + \lambda_1 \langle u_* + t v, u_* + t v \rangle_1 + \lambda_2 \langle u_* + t v, u_* + t v \rangle_2 . \end{aligned}$$

Thus g is a polynomial of degree 2 with $g(0) = g(1) = F(u_*)$. Since u_* is a minimizer, conclude $\frac{d^2 g}{dt^2} = 0$, i.e. $0 = \langle \vec{v}, \vec{v} \rangle + \lambda_1 \langle v, v \rangle_1 + \lambda_2 \langle v, v \rangle_2$. Thus $\langle v, v \rangle_1 = \iint_{\Omega} \|\nabla v\|^2 dA = 0$ and the function v has to be a constant. Then $\langle \vec{v}, \vec{v} \rangle = 0$ implies that $v = 0$ and thus the minimizer u_* is unique. \square

4–3 Remark : The Euler–Lagrange Equation and Natural Boundary Conditions

To find the corresponding PDE and the natural boundary conditions use standard calculus of variations arguments. For smooth perturbations ϕ examine

$$\begin{aligned}
F(u + \phi) &= \langle \vec{u} + \vec{\phi} - \vec{z}, \vec{u} + \vec{\phi} - \vec{z} \rangle + \lambda_1 \iint_{\Omega} \|\nabla u + \nabla \phi\|^2 dA + \\
&\quad + \lambda_2 \iint_{\Omega} (u_{xx} + \phi_{xx})^2 + (u_{yy} + \phi_{yy})^2 + 2(u_{xy} + \phi_{xy})^2 dA \\
&\approx F(u) + 2\langle \vec{u} - \vec{z}, \vec{\phi} \rangle + 2\lambda_1 \iint_{\Omega} \langle \nabla u, \nabla \phi \rangle dA + \\
&\quad + 2\lambda_2 \iint_{\Omega} u_{xx}\phi_{xx} + u_{yy}\phi_{yy} + 2u_{xy}\phi_{xy} dA \\
&= F(u) + 2\langle \vec{u} - \vec{z}, \vec{\phi} \rangle + 2\lambda_1 \iint_{\Omega} \langle \nabla u, \nabla \phi \rangle dA + \\
&\quad + 2\lambda_2 \iint_{\Omega} \left\langle \begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix}, \nabla \phi_x \right\rangle + \left\langle \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix}, \nabla \phi_y \right\rangle dA \\
&= F(u) + 2\langle \vec{u} - \vec{z}, \vec{\phi} \rangle + 2\lambda_1 \oint_{\partial\Omega} \phi \cdot \langle \nabla u, \vec{n} \rangle ds - 2\lambda_1 \iint_{\Omega} \Delta u \cdot \phi dA \\
&\quad + 2\lambda_2 \oint_{\partial\Omega} \left\langle \begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix}, \vec{n} \right\rangle \phi_x + \left\langle \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix}, \vec{n} \right\rangle \phi_y ds - \\
&\quad - 2\lambda_2 \iint_{\Omega} \left\langle \begin{pmatrix} u_{xxx} + u_{xyy} \\ u_{xxy} + u_{yyy} \end{pmatrix}, \nabla \phi \right\rangle dA \\
&= F(u) + 2\langle \vec{u} - \vec{z}, \vec{\phi} \rangle + 2\lambda_1 \oint_{\partial\Omega} \phi \cdot \langle \nabla u, \vec{n} \rangle ds - 2\lambda_1 \iint_{\Omega} \Delta u \cdot \phi dA \\
&\quad + 2\lambda_2 \oint_{\partial\Omega} \left\langle \begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix}, \vec{n} \right\rangle \phi_x + \left\langle \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix}, \vec{n} \right\rangle \phi_y - \left\langle \begin{pmatrix} u_{xxx} + u_{xyy} \\ u_{xxy} + u_{yyy} \end{pmatrix}, \vec{n} \right\rangle \phi ds + \\
&\quad + 2\lambda_2 \iint_{\Omega} (u_{xxxx} + 2u_{xxyy} + u_{yyyy}) \phi dA
\end{aligned}$$

The resulting Euler–Lagrange equation is

$$-\lambda_1 \Delta u + \lambda_2 \Delta^2 u = \Delta(-\lambda_1 u + \lambda_2 \Delta u) = 0. \quad (5)$$

The effect of the data at (x_i, y_i) is not taken into account with the above Euler–Lagrange equation and the equation is valid away from (x_i, y_i) .

To derive the natural boundary conditions use [LandLifs75, §12, p.50ff]. Examine directional derivatives on the boundary $\partial\Omega$ in normal direction \vec{n} and tangential direction \vec{t} .

$$\begin{aligned}
\vec{n} &= \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \frac{\partial}{\partial n} := n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} \\
\vec{t} &= \begin{pmatrix} -n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad \frac{\partial}{\partial t} := -n_2 \frac{\partial}{\partial x} + n_1 \frac{\partial}{\partial y}
\end{aligned}$$

$$\frac{\partial}{\partial x} = n_1 \frac{\partial}{\partial n} - n_2 \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial y} = n_2 \frac{\partial}{\partial n} + n_1 \frac{\partial}{\partial t}$$

One of the above boundary contributions can be written in the form

$$\begin{aligned} \text{BC} &= \oint_{\partial\Omega} \left\langle \begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix}, \vec{n} \right\rangle \phi_x + \left\langle \begin{pmatrix} u_{xy} \\ u_{yy} \end{pmatrix}, \vec{n} \right\rangle \phi_y \, ds \\ &= \oint_{\partial\Omega} u_{xx} n_1 \phi_x + u_{xy} n_2 \phi_x + u_{xy} n_1 \phi_y + u_{yy} n_2 \phi_y \, ds \\ &= \oint_{\partial\Omega} (u_{xx} + u_{yy}) n_1 \phi_x - u_{yy} n_1 \phi_x + u_{xy} n_2 \phi_x + u_{xy} n_1 \phi_y + (u_{xx} + u_{yy}) n_2 \phi_y - u_{xx} n_2 \phi_y \, ds \\ &= \oint_{\partial\Omega} \Delta u \frac{\partial \phi}{\partial n} + (u_{xy} \phi_y - u_{yy} \phi_x) n_1 - (u_{xx} \phi_y - u_{xy} \phi_x) n_2 \, ds \\ &= \oint_{\partial\Omega} \Delta u \frac{\partial \phi}{\partial n} + (u_{xy} (n_2 \frac{\partial \phi}{\partial n} + n_1 \frac{\partial \phi}{\partial t}) - u_{yy} (n_1 \frac{\partial \phi}{\partial n} - n_2 \frac{\partial \phi}{\partial t})) n_1 - \\ &\quad - (u_{xx} (n_2 \frac{\partial \phi}{\partial n} + n_1 \frac{\partial \phi}{\partial t}) + u_{xy} (n_1 \frac{\partial \phi}{\partial n} - n_2 \frac{\partial \phi}{\partial t})) n_2 \, ds \\ &= \oint_{\partial\Omega} \Delta u \frac{\partial \phi}{\partial n} + (u_{xy} n_1 n_2 - u_{yy} n_1^2 - u_{xx} n_2^2 + u_{xy} n_1 n_2) \frac{\partial \phi}{\partial n} + \\ &\quad + (u_{xy} n_1^2 + u_{yy} n_1 n_2 - u_{xx} n_1 n_2 - u_{xy} n_2^2) \frac{\partial \phi}{\partial t} \, ds \\ &= \oint_{\partial\Omega} \Delta u \frac{\partial \phi}{\partial n} - (n_1^2 u_{yy} + n_2^2 u_{xx} - 2 n_1 n_2 u_{xy}) \frac{\partial \phi}{\partial n} + \\ &\quad + ((n_1^2 - n_2^2) u_{xy} + n_1 n_2 (u_{yy} - u_{xx})) \frac{\partial \phi}{\partial t} \, ds \end{aligned}$$

The second contribution can be integrated by parts over the closed curve $\partial\Omega$, leading to

$$\oint_{\partial\Omega} ((n_1^2 - n_2^2) u_{xy} + n_1 n_2 (u_{yy} - u_{xx})) \frac{\partial \phi}{\partial t} \, ds = - \oint_{\partial\Omega} \frac{\partial}{\partial t} ((n_1^2 - n_2^2) u_{xy} + n_1 n_2 (u_{yy} - u_{xx})) \phi \, ds.$$

For the second boundary contribution use

$$\oint_{\partial\Omega} \left\langle \begin{pmatrix} u_{xxx} + u_{xyy} \\ u_{xxy} + u_{yyy} \end{pmatrix}, \vec{n} \right\rangle \phi \, ds = \oint_{\partial\Omega} \frac{\partial (u_{xx} + u_{yy})}{\partial n} \phi \, ds.$$

The sum of all boundary integrals has to vanish for all ϕ , i.e.

$$\begin{aligned} 0 &= \oint_{\partial\Omega} \left(\lambda_1 \frac{\partial u}{\partial n} - \lambda_2 \frac{\partial \Delta u}{\partial n} - \lambda_2 \frac{\partial}{\partial t} ((n_1^2 - n_2^2) u_{xy} + n_1 n_2 (u_{yy} - u_{xx})) \right) \phi \, ds + \\ &\quad + \lambda_2 \oint_{\partial\Omega} (\Delta u - (n_1^2 u_{yy} + n_2^2 u_{xx} - 2 n_1 n_2 u_{xy})) \frac{\partial \phi}{\partial n} \, ds. \end{aligned}$$

As consequence find the two natural boundary conditions

$$\lambda_2 \left(\frac{\partial \Delta u}{\partial n} + \frac{\partial}{\partial t} ((n_1^2 - n_2^2) u_{xy} + n_1 n_2 (u_{yy} - u_{xx})) \right) = \lambda_1 \frac{\partial u}{\partial n} \quad (6)$$

$$\Delta u - (n_1^2 u_{yy} + n_2^2 u_{xx} - 2 n_1 n_2 u_{xy}) = 0. \quad (7)$$

4–4 Remark : Second Order Derivatives are Necessary

One might be attempted to use first order derivatives only in the regularizing functional, i.e.

$$F(u) = \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_1 \iint_{\Omega} u_x^2 + u_y^2 \, dA.$$

But this does not work at all, caused by $H^1(\Omega)$ not being imbedded in $C^0(\bar{\Omega})$. To illustrate this use radial functions depending on $r = \sqrt{x^2 + y^2}$ and examine for $p > 0$

$$v_{\varepsilon}(x, y) = \begin{cases} 1 - \frac{r^p}{\varepsilon^p} & \text{for } r < \varepsilon \\ 0 & \text{for } r \geq \varepsilon \end{cases}.$$

Then $v_{\varepsilon}(0, 0) = 1$ for all $\varepsilon > 0$ and $\lim_{\varepsilon \rightarrow 0+} v_{\varepsilon}(x, y) = 0$ for $(x, y) \neq (0, 0)$. Use $\nabla r = \frac{1}{r}(x, y)$ to calculate $\|\nabla v_{\varepsilon}\| = \frac{p}{\varepsilon^p} r^{p-1}$ for $r \leq \varepsilon$ and an integration in polar coordinates to conclude

$$\begin{aligned} \|v_{\varepsilon}\|_{L_2(\mathbb{R}^2)}^2 &\leq \pi \varepsilon^2 \\ \|\nabla v_{\varepsilon}\|_{L_2(\mathbb{R}^2)}^2 &= \frac{2\pi p^2}{\varepsilon^{2p}} \int_0^{\varepsilon} r^{2p-2} r \, dr = \frac{2\pi p^2}{\varepsilon^{2p} 2p} \varepsilon^{2p} = p\pi. \end{aligned}$$

For separated data points (x_i, y_i) the function

$$u_{\varepsilon}(x, y) := \sum_{i=1}^M z_i v_{\varepsilon}(x - x_i, y - y_i)$$

satisfies (for $\varepsilon > 0$ small enough) $u_{\varepsilon}(x_i, y_i) = z_i$ and $\lim_{\varepsilon \rightarrow 0+} u_{\varepsilon}(x, y) = 0$ for all $(x, y) \neq (x_i, y_i)$. All functions satisfy $u_{\varepsilon} \in H^1(\Omega)$ since $\iint_{\Omega} \|\nabla u_{\varepsilon}\|^2 \, dA = M p\pi$. Using $\varepsilon = p$ find

$$\lim_{\varepsilon \rightarrow 0+} F(u_{\varepsilon}) = \lim_{\varepsilon \rightarrow 0+} \lambda_1 \langle u_{\varepsilon}, u_{\varepsilon} \rangle_1 = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0+} \langle u_{\varepsilon}, u_{\varepsilon} \rangle = 0.$$

The above functions u_{ε} (with $p = \varepsilon$) converge to zero, but a similar construction can lead to convergence towards any constant h by using

$$u_{\varepsilon}(x, y) := h + \sum_{i=1}^M (z_i - h) v_{\varepsilon}(x - x_i, y - y_i).$$

Thus a regularization with $\lambda_2 = 0$ is not well defined and consequently a numerical implementation can not be stable. This is illustrated by an example in Section 4.2.3 on page 19.

4–5 Remark : First Order Derivatives are Necessary

One might be attempted to set $\lambda_1 = 0$ and use second order derivatives only in the regularizing functional, i.e.

$$F(u) = \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_2 \iint_{\Omega} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 \, dA$$

- In most real world cases this will work just fine and the result in [Arca04] states just that.
- The proof of the Lemma with inequality (3) requires an estimate for $\|\nabla u\|_{L_2(\Omega)}$ and without additional assumptions on the data points (x_i, y_i) and z_i this is not available. Examine the situation when all points (x_i, y_i) are on a straight line. Then one can add a linear function with arbitrary slope without changing the value of the functional $F(u)$. Thus there is no unique solution and no bound on $\|\nabla u\|_{L_2(\Omega)}$.

- In [Arca04] find a proof for the result 4–2 with $\lambda_1 = 0$, based in a result by Necas in [Neca12]. The essential step is shown in Theorem 4–6 below.
- With the keyword smoothing D splines in [Arca04, Theorem 3.1, p.66] a proof is given, using $H^2(\Omega)$ ellipticity and the Lax–Milgram Theorem with the norm

$$\|u\|_{A,2,\Omega}^2 := \sum_{i=1}^M |u(x_i, y_i)|^2 + \langle u, u \rangle_2 .$$

Proposition 1.1 verifies that this is an equivalent norm to the usual Sobolev norm on $H^2(\Omega)$. The key argument is that $\langle u, u \rangle_2 = 0$ implies that u is a polynomial of degree 1 and [Neca12, Theorem 2.7.1] to show the equivalence of the norms. This replaces the a-priori estimate in Lemma 4–1 and is the key point of the proof. Below find the proof of the key result in [Neca12]. Observe that the proof is very elegant, but not constructive.

- In [GreeSilv94, Lemma 7.1, p.149] a proof is given with domain \mathbb{R}^2 that for non colinear data a unique solution exists.

4–6 Theorem : [Neca12, Theorem 2.7.1] Assuming that the data points (x_i, y_i) are not colinear there exists a constant c such that

$$\|u\|_{H^1(\Omega)}^2 \leq c \left(\langle u, u \rangle_2 + \sum_{i=1}^M u^2(x_i, y_i) \right) \quad \text{for all } u \in H^2(\Omega) . \quad (8)$$

This leads easily to

$$\|u\|_{H^2(\Omega)}^2 \leq c_1 \left(\langle u, u \rangle_2 + \sum_{i=1}^M u^2(x_i, y_i) \right) \quad \text{for all } u \in H^2(\Omega) . \quad (9)$$

Proof : Proof by contradiction. If no such constant exists then there exists a sequence $u_n \in H^2(\Omega)$ with $\|u_n\|_{H^1(\Omega)} = 1$ and

$$\langle u_n, u_n \rangle_2 + \sum_{i=1}^M u_n^2(x_i, y_i) \leq \frac{1}{n} .$$

This implies $\|u_n\|_{H^2(\Omega)}^2 = \|u_n\|_{H^1(\Omega)}^2 + \langle u_n, u_n \rangle_2 \leq 1 + \frac{1}{n}$. Since the imbedding $H^2(\Omega)$ in $H^1(\Omega)$ is compact use a subsequence (again denoted by u_n), such that for all $|\alpha| = 2$

$$\begin{aligned} D^\alpha u_n &\rightarrow 0 \quad \text{in } L_2(\Omega) \\ u_n &\rightarrow u \quad \text{in } H^1(\Omega) \end{aligned}$$

Thus $u_n \rightarrow u$ in $H^2(\Omega)$ and $D^\alpha u = 0$. This implies (use [Neca12, Theorem 1.1.6, p.10]) that u is a polynomial of degree 1. Since the data points are not colinear $\sum_{i=1}^M u_n^2(x_i, y_i) \leq \frac{1}{n}$ implies $u = 0$. This is in contradition to $\|u\|_{H^1(\Omega)} = 1$. \square

4-7 Remark : Find a solution of the problem with $\lambda_1 = 0$ in a document by David Eberly [Eber96], i.e. `ThinPlateSplines.pdf`, Geometric Tools, Redmond WA 98052, <https://www.geometrictools.com/> or more precise <https://www.geometrictools.com/Documentation/Documentation.html>.

The expression to be minimized is

$$F(u) = \sum_{i=1}^M (u(x_i, y_i) - z_i)^2 + \lambda_2 \iint_{\mathbb{R}^2} u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 dA,$$

i.e. integration over the plane \mathbb{R}^2 and not a bounded domain $\Omega \subset \mathbb{R}^2$. The main source seems to be [Wahb90], see also [Duch77],[GreeSilv94] (on my HD).

As Green's function use $G(r) = \frac{1}{6\pi} r^2 \ln(r)$, where $r = \sqrt{x^2 + y^2}$. Then the ansatz for the optimizer is

$$u(x, y) = c_0 + c_1 x + c_2 y + \sum_{i=1}^M \alpha_i G(\|(x - x_i, y - y_i)\|).$$

This approach is very specific to the domain $\Omega = \mathbb{R}^2$.

4.4.2 Numerical differentiation and integration on rectangles

Examine the partial derivatives of functions defined on rectangles $[a, b] \times [c, d] \subset \mathbb{R}^2$, using uniform grids. Use

$$\begin{aligned} x_i &= a + i \Delta x \quad \text{for } i = 0, 1, 2, 3, \dots, N_x \text{ with } \Delta x = \frac{b-a}{N_x} \\ y_j &= c + j \Delta y \quad \text{for } j = 0, 1, 2, 3, \dots, N_y \text{ with } \Delta y = \frac{d-c}{N_y} \end{aligned}$$

and the usual matrices to determine partial derivatives with finite difference approximations. With these the centered difference approximation is used determine the values of the derivatives at the mid points between the grid points, e.g. at $(x, y) = (x_i + \frac{\Delta x}{2}, y_j)$ for $\frac{\partial u}{\partial x}$ or at $(x, y) = (x_i, y_j + \frac{\Delta y}{2})$ for $\frac{\partial u}{\partial y}$.

$$\begin{aligned} \mathbf{D}_x &= \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \end{bmatrix} \in \mathbb{M}^{N_x \times (N_x+1)} \\ \mathbf{D}_y &= \frac{1}{\Delta y} \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \end{bmatrix} \in \mathbb{M}^{N_y \times (N_y+1)} \end{aligned}$$

With these matrices and a Kronecker product with an identity matrix the first partial derivatives can be estimated. To perform the integration use a midpoint rule in x direction and a trapezoidal rule in y direction.

$$\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \rangle = \int_c^d \left(\int_a^b \left(\frac{\partial u(x, y)}{\partial x} \right)^2 dx \right) dy \approx \int_c^d \left(\sum_{i=0}^{N_x-1} \left(\frac{\partial u(x_i + \Delta x/2, y)}{\partial x} \right)^2 \Delta x \right) dy$$

$$\approx \frac{\Delta y}{2} \left(\sum_{i=0}^{N_x-1} \left(\frac{\partial u(x_i + \Delta x/2, y_0)}{\partial x} \right)^2 \Delta x \right) + \\ + \Delta y \sum_{j=1}^{N_y-1} \left(\sum_{i=0}^{N_x-1} \left(\frac{\partial u(x_i + \Delta x/2, y_j)}{\partial x} \right)^2 \Delta x \right) + \\ + \frac{\Delta y}{2} \left(\sum_{i=0}^{N_x-1} \left(\frac{\partial u(x_i + \Delta x/2, y_{N_y})}{\partial x} \right)^2 \Delta x \right)$$

For the second order derivatives at the midpoints between the grid points use the matrix

$$\mathbf{D}_{xx} = \frac{1}{2(\Delta x)^2} \begin{bmatrix} +3 & -7 & +5 & -1 \\ +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ \ddots & \ddots & \ddots & \ddots \\ +1 & -1 & -1 & +1 \\ +1 & -1 & -1 & +1 \\ -1 & +5 & -7 & +3 \end{bmatrix} \in \mathbb{M}^{N_x \times (N_x+1)}$$

and a similar matrix \mathbf{D}_{yy} for $\frac{\partial^2 u}{\partial y^2}$. A Kronecker product is used to generate the matrix approximating the second order derivatives on the grid. These finite difference approximations are consistent of order 2. The integration over the rectangle is very similar to the first order derivatives. Similar algorithms are used to integrate $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x \partial y}$, resp. their squares.

4.4.3 Bilinear interpolation on rectangular grids

If a function u is defined on the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ the **bilinear interpolation** based on the four corners is given by

$$\begin{aligned} u(\xi, \eta) &= (1 - \xi)(1 - \eta)u_{i,j} + \xi(1 - \eta)u_{i+1,j} + (1 - \xi)\eta u_{i,j+1} + \xi\eta u_{i+1,j+1} \\ &= \langle \begin{pmatrix} (1 - \xi)(1 - \eta) \\ \xi(1 - \eta) \\ (1 - \xi)\eta \\ \xi\eta \end{pmatrix}, \begin{pmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{pmatrix} \rangle. \end{aligned}$$

This can be used to define the (sparse) interpolation matrix $\mathbf{Interp} \in \mathbb{M}^{(N_x \cdot N_y) \times M}$ to determine the interpolated values at the data points (x_i, y_i) . It is best constructed with the command `sparse()` in *Octave*.

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