TFY 4345 Computational Assignment 2 (2021): The trebuchet

Introduction

This is the second (and last) compulsory computational exercise for the subject TFY4345 - Classical Mechanics (2021), which has to be returned by **November 16th at 13:00** on Blackboard. For crucial background information, please read the sections preceding the tasks. Prepare your report using either LaTeX or by writing it directly into Jupyter Notebook using markdown cells. Pay special attention on the quality of figures as these are the most important content (are the labels visible, is the line thickness appropriate, etc.). Attached source code should be zipped. You may program in any language as long as you prepare your own code from scratch. You can reuse your own code from the previous assignment. The code should be easy to read with comments and consistent naming schemes. Assessment: Accepted (1) or requires a revision (0).

The trebuchet

In this assignment you will model the trajectory of a projectile launched by a trebuchet. A trebuchet is a type of catapult used in medieval Europe before the advent of gunpowder. It was the largest and most destructive catapult design, developed and optimized over centuries to tear down the increasingly thicker castle walls. In contrast to other catapults, the trebuchet is powered by gravity and not torsion.

A trebuchet consists of a long wooden beam attached to a raised axle, such that the beam can rotate through a wide arc. A sling is attached on one side of the beam, which holds the projectile, while a counterweight is attached on the other side. The process of firing a trebuchet first begins with raising the counterweight to a certain height and loading the sling with a projectile. Then, the counterweight is released, resulting in a rapid rotation of the beam around the axle. At the same time on the other side of the beam, the sling is pulled upwards with the rotation of the lever. During this motion, the slings will begin to rotate around the point it is attached to the beam. When the beam has rotated to a certain angle the projectile is released from the sling and hurled at the target. An image of a reconstructed trebuchet is shown in Fig. 1.

Figure 1: Counterweight trebuchet at Château des Baux, France

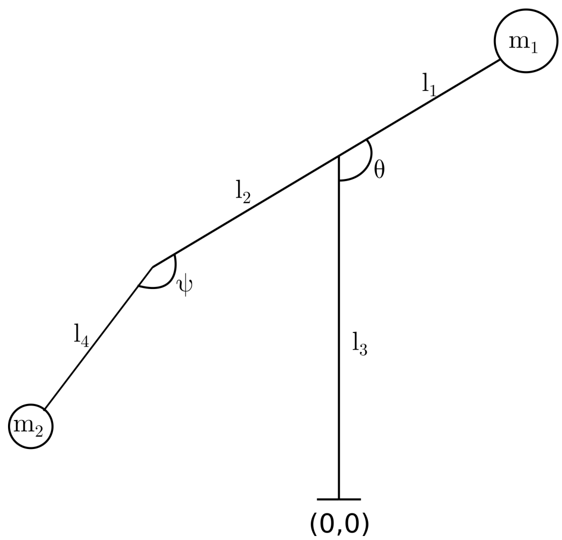

Figure 2: schematic of a our simplified trebuchet model.

There were many variations of the trebuchet, some did not have slings and worked similarly to a seesaw at the playground, others had a hinged counterweight (such as in Fig. 1) and a trough for the projectile to glide on under the trebuchet. In this assignment we assume that the counterweight is not hinged and that it is instead attached to the end of the trebuchet arm. In addition, we will assume that the the projectile swings unconstrained through the air, i.e. we ignore the ground below the trebuchet and air-resistance. Our simplified trebuchet can then be described by the schematic displayed in Fig. 2.

Here the mass of the counter weight and projectile is denoted by m_1 and m_2, respectively. We place the base of the trebuchet at the origin. The height of the axle is given by l_3, while the length of the beam is given by $l_1 + l_2$. The length between the axle and the center of mass of the counter-weight is denoted by l_1, while the length between the axle and the point where the sling is attached is denoted by l_2. Moreover, l_4 denotes the length of the sling. The angle between the beam and the sling is given by ψ, while the angle between the beam and the stand the axle rests on is denoted by θ.

To model the mechanics of our trebuchet we first need to obtain the equations of motion (EOM), we will do this by using Lagrangian mechanics. The position vectors of the masses are shown in Fig. 3. The position of the counter-weight is described by

$$\vec{r}_1 = \begin{bmatrix} l_1 \sin(\theta) \\ -l_1 \cos(\theta) + l_3 \end{bmatrix}, \tag{1}$$

while the position of the projectile is given by

$$\vec{r}_{2b} = \vec{r}_{2a} - \begin{bmatrix} l_4 \sin(\psi - \theta) \\ l_4 \cos(\psi - \theta) \end{bmatrix} = \begin{bmatrix} -l_2 \sin(\theta) - l_4 \sin(\psi - \theta) \\ l_2 \cos(\theta) + l_3 - l_4 \cos(\psi - \theta) \end{bmatrix}. \tag{2}$$

The potential energy of each mass is given by $V_i = m_i g h$, while the kinetic

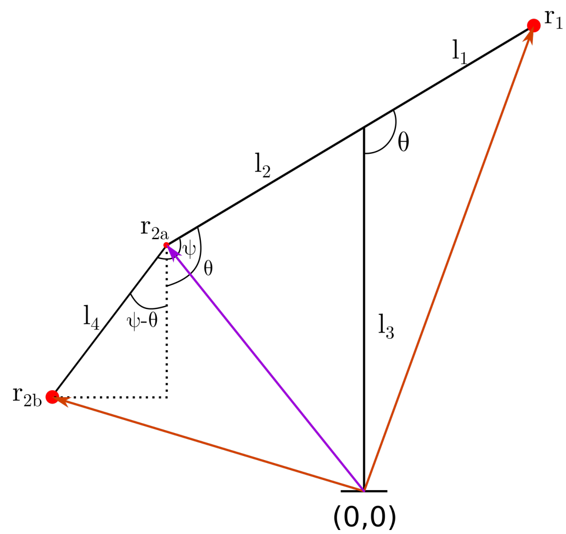

Figure 3: A schematic of a our simplified trebuchet model. The position vectors of the projectile, counterweight and sling joint are shown with all the relevant angles.

energy of each mass is given by $T_i = \frac{1}{2} m_i v_i^2$, where $\vec{v}_i = \mathrm{d}\vec{r}_i/\mathrm{d}t$. The Lagrangian is then given by

$$
\begin{aligned}
L(\theta, \dot{\theta}, \psi, \dot{\psi}) = &\sum_i T_i - V_i \\
=&\frac{1}{2}\left(m_1 l_1^2 + m_2 l_2^2\right)\dot{\theta}^2 - g l_3\left(m_1 + m_2\right) \\
&+ g\left(m_1 l_1 - m_2 l_2\right)\cos(\theta) + \frac{1}{2} m_2 l_4^2 (\dot{\psi} - \dot{\theta})^2 \\
&+ m_2 l_2 l_4 \dot{\theta}(\dot{\psi} - \dot{\theta})\cos(\psi) + g m_2 l_4 \cos(\psi - \theta).
\end{aligned} \tag{3}
$$

The equations of motion can then be obtained by applying the Euler-Lagrange equations for the two generalized coordinates θ and ψ given by

$$
\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \tag{4}
$$

and

$$
\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = 0, \tag{5}
$$

respectively. The two resulting EOM can then be organized as the following matrix equation

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{6}
$$

The matrix elements are given by

$$
\begin{aligned}
a_{11}(\psi) =& - l_1^2 m_1 - l_2^2 m_2 + 2 l_2 l_4 m_2 \cos(\psi) - l_4^2 m_2, \\
a_{12}(\psi) =& - l_4 m_2 \left(l_2 \cos(\psi) - l_4\right), \\
a_{22} =& - l_4^2 m_2,
\end{aligned} \tag{7}
$$

while the vector elements are given by

$$
\begin{aligned}
b_1(\theta, \dot{\theta}, \psi, \dot{\psi}) =& g m_1 l_1 \sin(\theta) - g m_2 \left[l_2 \sin(\theta) + l_4 \sin(\psi - \theta)\right] \\
&- l_2 l_4 m_2 (\dot{\psi} - 2\dot{\theta}) \sin(\psi)\dot{\psi},
\end{aligned} \tag{8}
$$

and

$$
b_2(\theta, \psi, \dot{\theta}) = l_4 m_2 \left[g \sin(\psi - \theta) - l_2 \sin(\psi)\dot{\theta}^2\right]. \tag{9}
$$

Using the Euler method, the EOM can be solved numerically in the following way. First, we rewrite the equation by using the finite differences approximation for the second time derivatives $\ddot{\theta}$ and $\ddot{\psi}$, which yields

$$
\boldsymbol{A}(\psi_i)\frac{1}{\Delta t}\begin{bmatrix} \dot{\theta}_{i+1} - \dot{\theta}_i \\ \dot{\psi}_{i+1} - \dot{\psi}_i \end{bmatrix} = \vec{b}(\theta_i, \dot{\theta}_i, \psi_i, \dot{\psi}_i), \tag{10}
$$

where the notation $f_i = f(t = i\Delta t)$ is used. Then, we obtain

$$
\begin{bmatrix} \dot{\theta}_{i+1} \\ \dot{\psi}_{i+1} \end{bmatrix} = \begin{bmatrix} \dot{\theta}_i \\ \dot{\psi}_i \end{bmatrix} + \Delta t (\boldsymbol{A}^{-1}(\psi_i)\vec{b}(\theta_i, \dot{\theta}_i, \psi_i, \dot{\psi}_i)). \tag{11}
$$

m_1	mass of the counterweight	2000 kg
m_2	mass of the projectile	15 kg
l_1	short part of the main beam	1.2 m
l_2	long part of the main beam,	5.7 m
l_3	stand	3.2 m
l_4	projectile sling	5 m
g	gravitation constant	9.81 m/s^2
θ_0	θ at $t = 0$	0.7π rad
$\dot{\theta}_0$	$\dot{\theta}$ at $t = 0$	0rad/s
ψ_0	ψ at $t = 0$	$\theta_0 - 0.5\pi$rad
$\dot{\psi}_0$	$\dot{\psi}$ at $t = 0$	0rad/s

Table 1: Parameters based on a trebuchet reconstruction in Denmark.

The angles are given by

$$\begin{aligned} \theta_{i+1} =& \theta_i + \dot{\theta}_i \Delta t, \\ \psi_{i+1} =& \psi_i + \dot{\psi}_i \Delta t. \end{aligned} \tag{12}$$

Eqs. (11) and (12) are solved iteratively for a given set of initial conditions $(\theta_0, \dot{\theta}_0, \psi_0, \dot{\psi}_0)$. The motion of the two masses can then be calculated numerically until the release angle $\psi = \psi_r$ is achieved. After the projectile is released we can use the usual EOM for a projectile to calculate the trajectory, the initial conditions are the vertical and horizontal coordinates and velocities of the projectile at the time of release.

Tasks

Read the background info above before you begin working on the tasks. You can reuse code from previous assignments as you see fit.

1. Express the velocities of the projectile and counterweight as functions of the angles ψ, θ and their time derivatives $\dot{\psi}, \dot{\theta}$. Then, show that the Lagrangian is given by Eq. (3). Apply the Euler-Lagrange equations for ψ and θ on the Lagrangian and show that the resulting equations of motion can be expressed by Eq. (6).

2. Use the Runge-Kutta 4th order algorithm to solve the equations of motion given by Eqs. (11) and (12). Use the parameters and initial conditions from table 1. Plot $\theta, \dot{\theta}, \psi, \dot{\psi}$ as functions of t for increasingly longer time intervals T. Are the functions periodic? Plot the trajectory of the projectile without releasing it from the sling for several seconds. Describe the motion, is it as expected?

3. Perform the calculation until θ reaches the release angle $\theta_r = 0.1\pi$. Then, use the velocity and firing angle at the moment of release as the initial

conditions for the equations of motion without air resistance from assignment 1. Calculate several trajectories for different release angles between $0°$ and $40°$. Within what range of release angles θ_r does the trebuchet fire in the forward direction $(+\hat{x})$? Change the initial throwing arm angle to $\theta_0 = 0.9\pi$ and calculate the trajectories again. Then do the same with the initial sling angle $\psi_0 = 0$. Describe what happens to the trajectories and why.

4. Plot the range and impact energy (the energy of the projectile at the moment of impact) as functions of the release angle θ_r, determine the optimal release angles for maximum range and for maximum impact, respectively. Describe and explain the reason for the shapes of the graphs, why do the release angles of the maximum values differ?