Inverse Problems

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1 Task a

The first row of the state transition matrix is shown in fig. 1, we not that the matrix has a smoothing property and that it is circular, i.e that the effect of position 0 is equal the effect at position 100.

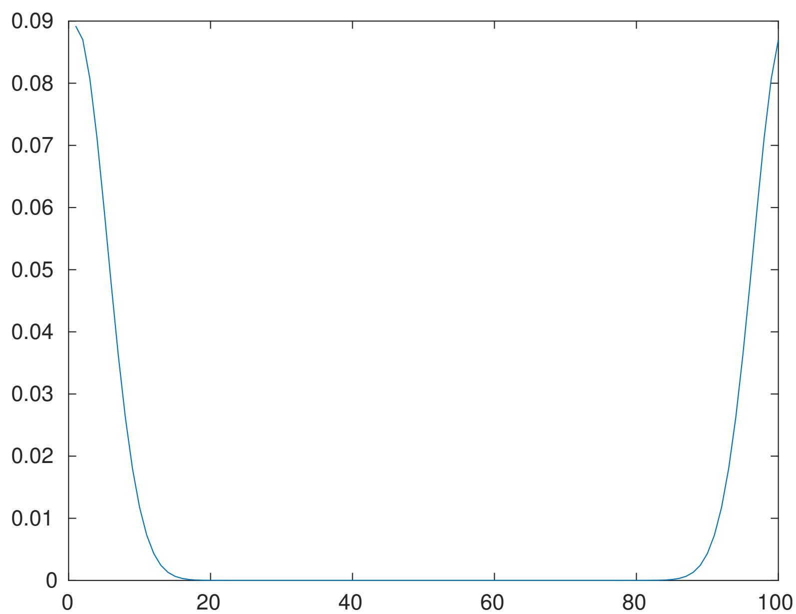

Figure 1: First row of matrix A. Illustrating the smoothing and circular nature of the system.

From the given measurements displayed in fig. 2 the naive solution would be to ignore the noise and do the inverse eq. (3) to directly calculate $\mathbf{h}_0$. This solution is displayed in fig. 3 and is not very successful. The problem is that the matrix is close to singular, and with the added noise the direct inversion becomes unstable.

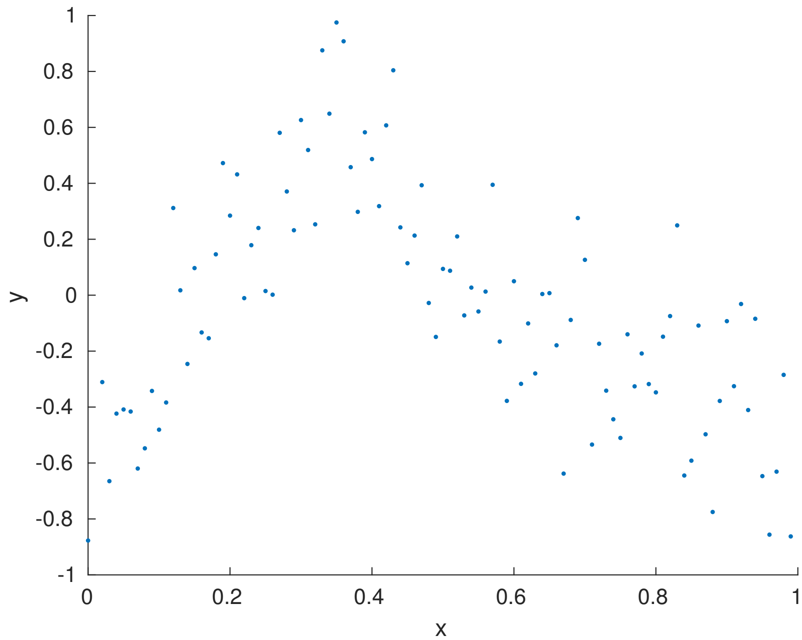

Figure 2: The given measurements of $\mathbf{h}$ at t = 1 ms.

The singular values of A can be seen in fig. 4. We can represent the matrix A using a singular value decomposition such that

$$A = USV^{\top} \tag{1}$$

where S is a diagonal matrix with the singular values of A on the diagonal. From this the inversion can be calculated as

$$\hat{\mathbf{h}_0} = \sum_{k=1}^{n} \frac{\mathbf{u}_k^{\top} y}{s_k} \mathbf{v}_k \tag{2}$$

Where s_k is the k-th singular value. The problems is that because of the noise, the upper part of the fraction will not go towards zero as it is suppose to with a well defined matrix, thus as the singular values approaches zero the fraction approaches infinity. By removing the singular values with the smallest values we avoid this problem, however this again removes some of the high frequensy components of the system. Then we have to tune the number of singualr values to include (k), this is a typical noise-to-signal ratio tuning. See fig. 5, k = 15 seems accurate.

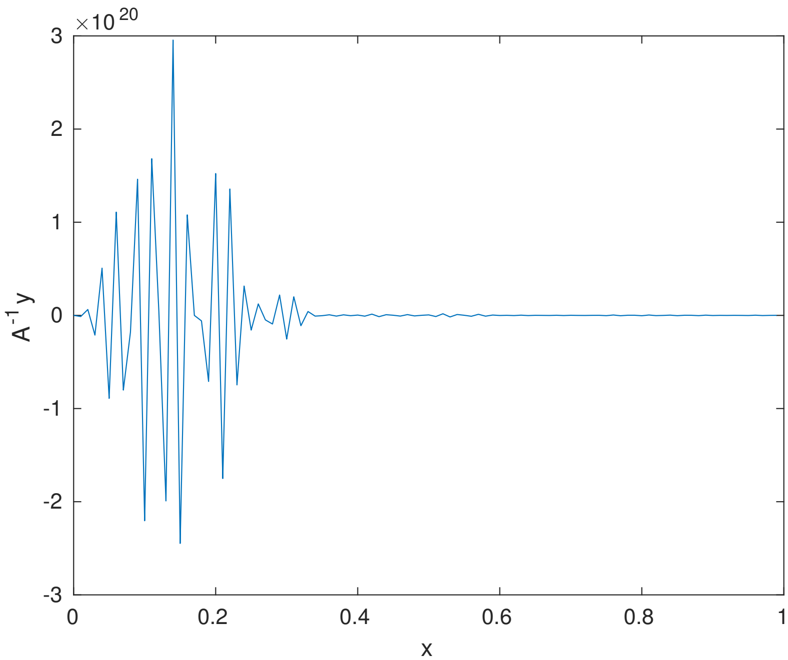

Figure 3: Naive solution of the problem: $\hat{\mathbf{h}}_0 = A^{-1}\mathbf{y}$.

2 Task b

The measurement equation is given in eq. (3).

$$\mathbf{y} = A\mathbf{h}_0 + \epsilon, \quad \epsilon \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon) \tag{3}$$

Where the measurement noise are assumed to be individually independent, thus $\Sigma_\epsilon = \text{diag}(0.25^2)$. The measurement likelihood is given in eq. (4).

$$p(\mathbf{y}|\mathbf{h}_0) \sim \mathcal{N}(A\mathbf{h}_0, \Sigma_\epsilon) \tag{4}$$

Assuming the prior given in eq. (5).

$$p(\mathbf{h}_0) \sim \mathcal{N}(0, \Sigma_{\mathbf{h}_0}) \tag{5}$$

Using Bayes rule we can estimate the posterior model:

$$p(\mathbf{h}_0|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{h}_0)p(\mathbf{h}_0)}{p(\mathbf{y})} = \text{const} \cdot p(\mathbf{y}|\mathbf{h}_0)p(\mathbf{h}_0) \propto \mathcal{N}(\mu, \Sigma) \tag{6}$$

Because the likelihood model is linear the covariance and mean can be calculated as[1]

$$\Sigma = (\Sigma_{\mathbf{h}_0}^{-1} + A^\top \Sigma_\epsilon^{-1} A)^{-1} \tag{7}$$

[1] https://www.cs.ubc.ca/ murphyk/Teaching/CS340-Fall07/reading/gauss.pdf

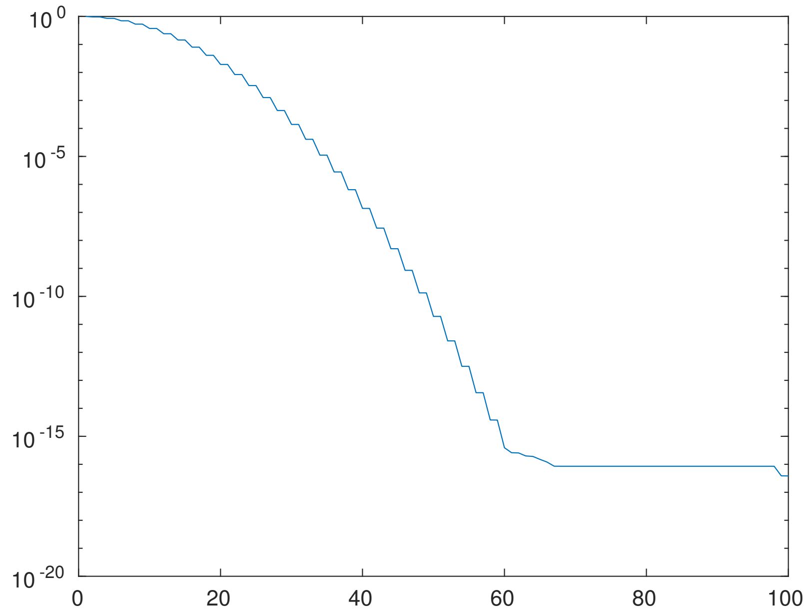

Figure 4: Singular values of A.

$$\mu = \Sigma A^\top \Sigma_\epsilon^{-1} \mathbf{y} \tag{8}$$

Because the posteriori is also a gaussian the MAP estimator will be equal to the mean given in eq. (8). From fig. 6 we can see that the solution is similar to the SVD truncated solution for k = 20 in fig. 5 even though we do not use truncate the corresponding SVD solution in task b. Instead the bias act as a regulation, adding small values to the singular values, thus preventing the inversion from becoming unstable.

3 Task c

Now we use a more accurate bias, thus the 95% confidence interval becomes smaller, see fig. 7.

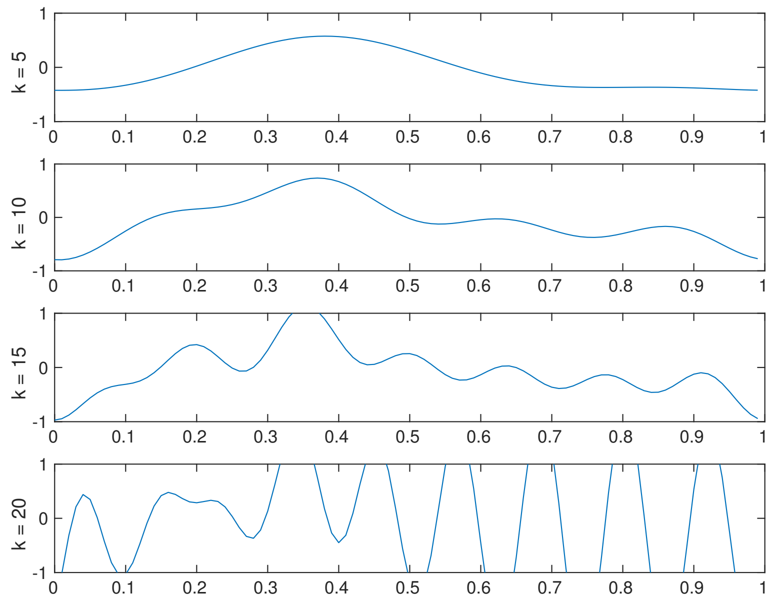

Figure 5: SVD solutions using the k largest singular values, for k = 5,10,15 and 20.

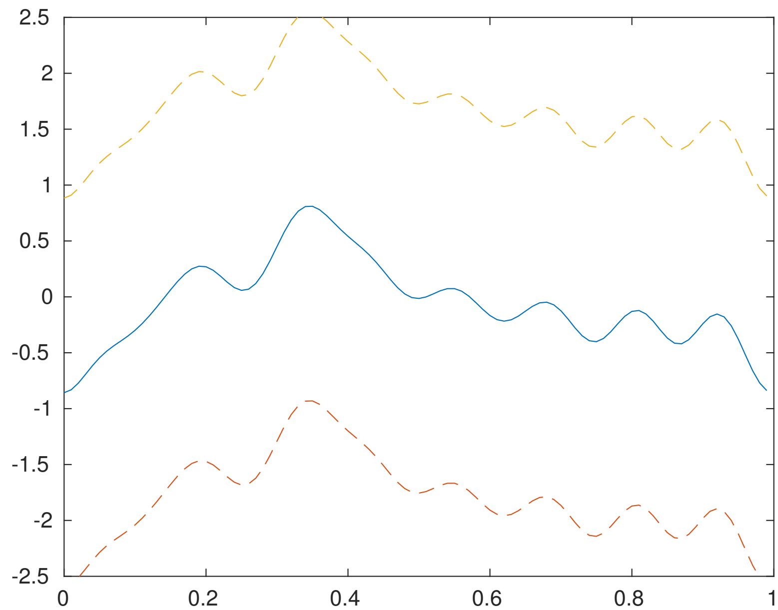

Figure 6: Bayesian inversion using a general bias. Stippled lines represents 95 % confidence interval.

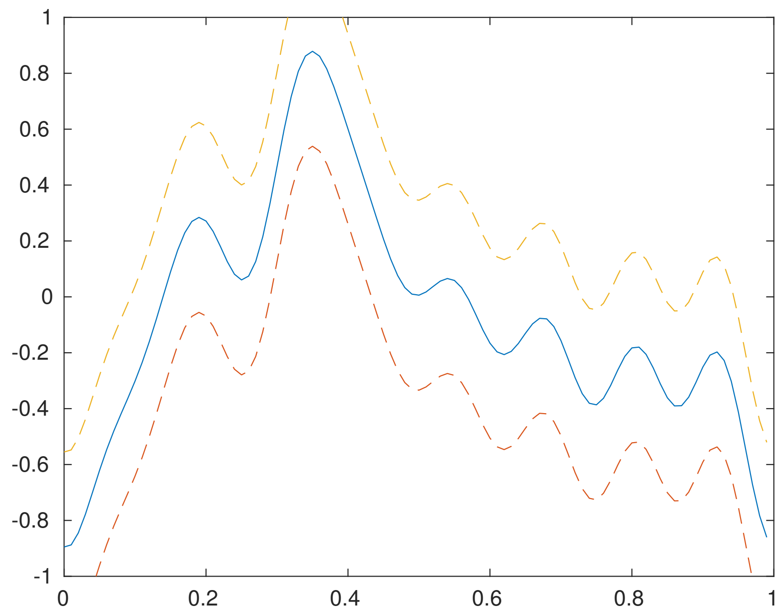

Figure 7: Bayesian inversion using a case spesific bias. Stippled lines represents 95 % confidence interval.