Lecture Notes on Robot to Table Calibration

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1 Introduction

We consider the setup shown in Figure 1 where a robot is located on a table. Two coordinate frames are considered: World and Robot Base. To let the tool center point (TCP) move to a specific point on the table, it is necessary to know the coordinate transformation H_W^R from World to Robot Base.

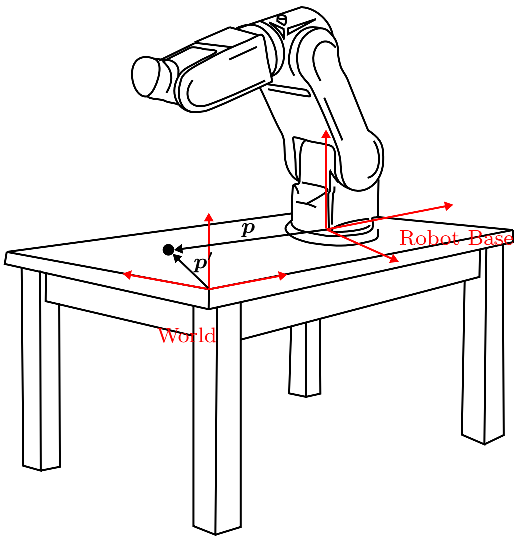

Figure 1: Robot on table, where coordinate frames are attached to the table (World frame) and robot (Robot Base frame).

If the homogeneous transformation H_W^R is known, then the point $\boldsymbol{p}$ in Robot Base frame can be transformed to the point $\boldsymbol{p}'$ in World frame by

$$\begin{bmatrix} \boldsymbol{p}' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{\mathrm{W}}^{\mathrm{R}} & \boldsymbol{T}_{\mathrm{W}}^{\mathrm{R}} \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix}}_{H_{\mathrm{W}}^{\mathrm{R}}} \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} \tag{1}$$

where $R_{\mathrm{W}}^{\mathrm{R}}$ is the rotation matrix and $\boldsymbol{T}_{\mathrm{W}}^{\mathrm{R}} \in \mathbb{R}^3$ is the translation vector.

In the following, we attempt to find a robot-table calibration given by H_W^R, by moving the TCP of the robot to different known points on the table $\boldsymbol{p}_i'$ for $i = 1, \dots, N$ (given in the World frame). When the TCP of the robot is at each point $\boldsymbol{p}_i'$, it is possible to read the TCP position of the robot $\boldsymbol{p}_i$ for $i = 1, \dots, N$ (given in the Robot Base frame). These measured positions are used for finding the homogeneous transformation $H_{\mathrm{W}}^{\mathrm{R}}$ or rather $R_{\mathrm{W}}^{\mathrm{R}}$ and $\boldsymbol{T}_{\mathrm{W}}^{\mathrm{R}} \in \mathbb{R}^3$ based on the relation (given by (1))

$$\boldsymbol{p}' = R_{\mathrm{W}}^{\mathrm{R}} \boldsymbol{p} + \boldsymbol{T}_{\mathrm{W}}^{\mathrm{R}} \tag{2}$$

2 Problem Formulation

This section poses the mathematical problem, which is solved to find the coordinate transformation $H_{\mathrm{W}}^{\mathrm{R}}$. The text is based on [1], which contains more details on the calibration. The problem is formulated as a least squares problem, i.e., we obtain the least squares solution to (2).

Problem 1. *Given two sets of points $\boldsymbol{p}_i, \boldsymbol{p}'_i \in \mathbb{R}^3$ for $i = 1, \dots, N$ that are related as*

$$\boldsymbol{p}'_i = R\boldsymbol{p}_i + \boldsymbol{T} + \boldsymbol{w}_i$$

where R is a rotation matrix, $\boldsymbol{T} \in \mathbb{R}^3$ is a translation vector, and $\boldsymbol{w}_i \in \mathbb{R}^3$ is a noise vector. Find R and $\boldsymbol{T}$ that minimizes

$$\sum_i ||\boldsymbol{p}'_i - (R\boldsymbol{p}_i + \boldsymbol{T})|| \tag{3}$$

The least squares solution provides the least average error in the relation (2), and since noise is present ($\boldsymbol{w}_i$) then the solution is an approximation.

Example 1. *Throughout this note, points $\boldsymbol{p}_i = (x_i, y_i, z_i)$ and $\boldsymbol{p}'_i = (x'_i, y'_i, z'_i)$ are considered. A scatter plot of the points is shown in Figure 2.*

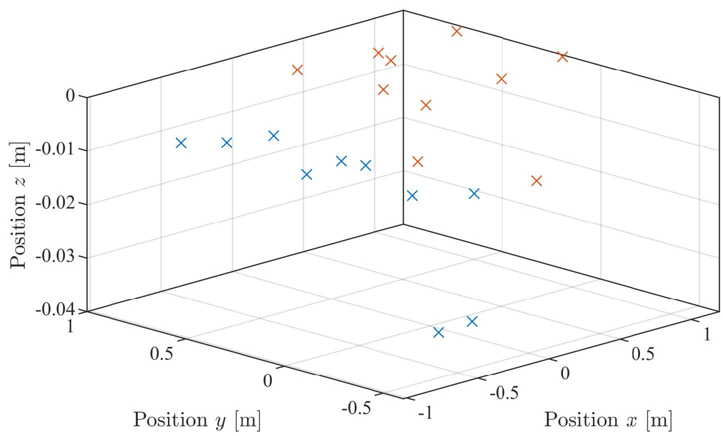

Figure 2: Scatter plot of points $\boldsymbol{p}_i$ (red) and $\boldsymbol{p}'_i$ (blue).

The red points are given in the World frame and the blue points are given in Robot Base frame.

3 Computation of Calibration

This section provides the theoretical background on solving Problem 1 and presents an algorithm for finding $H^{\mathrm{R}}_{\mathrm{W}}$. The following shows that Problem 1 can be solved by first finding the rotation matrix R and subsequently finding the translation vector $\boldsymbol{T} \in \mathbb{R}^3$.

A key to finding the calibration is the possibility to decompose the problem into two sequential steps, which is possible due to the following.

Lemma 1. *If $\{\hat{R}, \hat{\boldsymbol{T}}\}$ is a minimizer of (3) then $\{p'_i\}$ and $\{p''_i = \hat{R}p_i + \hat{T}\}$ have the same centroid.*

Recall that the centroid of the points $p_1, p_2, \dots, p_N$ is

$$C_p = \frac{1}{N}\sum_{i=1}^{N} p_i$$

If $\{p'_i\}$ and $\{p''_i\}$ have the same centroid, then it is possible to transform the points to have a centroid at the origin and then find the rotation $\hat{R}$. The transformation of the centroid is found in the following example.

Example 2. *To get a centroid of both $\boldsymbol{p}_i = (x_i, y_i, z_i)$ and $\boldsymbol{p}'_i = (x'_i, y'_i, z'_i)$ to be zero, we compute the centroid of both sets of points and subtract the centroid from each point. The centroids of the sets of points are*

$$\boldsymbol{C}_p = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{p}_i \qquad \text{and } \boldsymbol{C}_{p'} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{p}'_i$$

Two new sets of points with zero centroid are defined as

$$\boldsymbol{q}_i = \boldsymbol{p}_i - \boldsymbol{C}_p \qquad \text{and } \boldsymbol{q}'_i = \boldsymbol{p}'_i - \boldsymbol{C}_{p'}$$

A scatter plot of the points is shown in Figure 3 from which it is seen that the centroid of the two sets of points is at the origin.

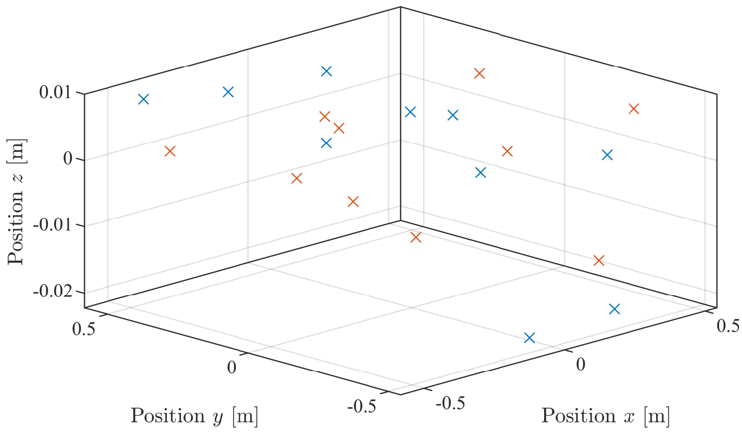

Figure 3: Scatter plot of points $\boldsymbol{q}_i$ (red) and $\boldsymbol{q}'_i$ (blue).

It is seen from Figure 3 that both sets of points have zero mean value.

When the centroid of the sets of points has been translated to the origin it is possible to find the rotation $\hat{R}$ from the following result.

Proposition 1. *Let $\{R, T\}$ be a minimizer of* (3)*. Then R is also a minimizer of*

$$\sum_i ||\boldsymbol{q}'_i - R\boldsymbol{q}_i|| \tag{4}$$

where

$$\boldsymbol{q}_i = \boldsymbol{p}_i - \boldsymbol{C}_p \qquad \text{and } \boldsymbol{q}'_i = \boldsymbol{p}'_i - \boldsymbol{C}_{p'}$$

Finding the minimizer of (4) can be accomplished via a singular value decomposition (SVD), which is often used in least squares optimization. A singular value decomposition of a matrix $A \in \mathbb{R}^{m \times n}$ with $m \leq n$ is

$$A = U\Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix with nonnegative elements.

The SVD is used to finding the rotation matrix $\hat{R}$ as explained in the following proposition.

Proposition 2. *Let*

$$U\Sigma V^T = H = \sum_{i=1}^{N} \boldsymbol{q}_i(\boldsymbol{q}'_i)^T$$

be a SVD of H. Then $R = VU^T$ is a minimizer of (4)*.*

The following example uses Proposition 2 to find the rotation matrix $\hat{R}$

Example 3. *To compute the rotation that relates points $\boldsymbol{q}_i$ to points $\boldsymbol{q}'_i$, we define the matrix*

$$H = \sum_{i=1}^{N} \boldsymbol{q}_i(\boldsymbol{q}'_i)^T = \begin{bmatrix} 0.1057 & 0.6680 & -0.0057 \\ -1.9019 & -0.1082 & 0.0227 \\ -0.0225 & -0.0098 & 0.0010 \end{bmatrix}$$

A singular value decomposition of H (computed in MATLAB with command `svd`*) gives*

$$U = \begin{bmatrix} -0.0853 & -0.9963 & 0.0129 \\ 0.9963 & -0.0854 & -0.0111 \\ 0.0122 & 0.0119 & 0.9999 \end{bmatrix}, V = \begin{bmatrix} -0.9962 & 0.0864 & 0.0116 \\ -0.0863 & -0.9962 & 0.0067 \\ 0.0121 & 0.0057 & 0.9999 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1.9113 & 0 & 0 \\ 0 & 0.6588 & 0 \\ 0 & 0 & 0.0007 \end{bmatrix}$$

From this data, the rotation matrix is

$$R = VU^T = \begin{bmatrix} -0.0009 & 1.0000 & 0.0062 \\ -1.0000 & -0.0009 & 0.0004 \\ 0.0005 & -0.0062 & 1.0000 \end{bmatrix}$$

Figure 4 shows Rq_i and q'_i and it is seen that the two sets of points are aligned by the rotation as expected.

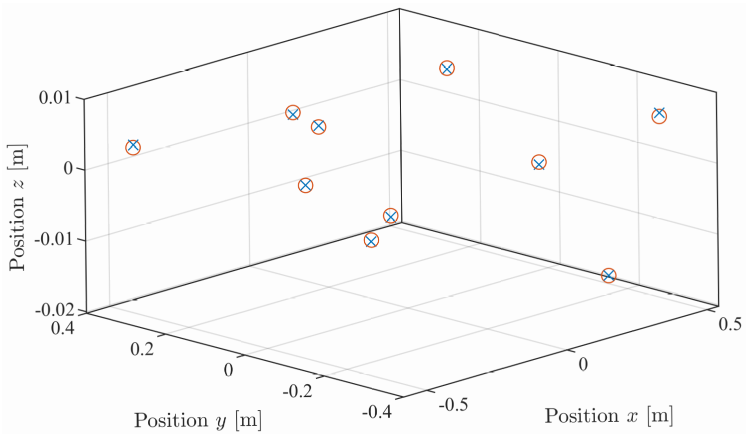

Figure 4: Scatter plot of points Rq_i (red circles) and q'_i (blue crosses).

The final step is to find the translation $\boldsymbol{T} \in \mathbb{R}^3$, which can be accomplished by finding a transformation between the centroids, i.e.

$$\hat{\boldsymbol{T}} = \boldsymbol{C}'_p - \hat{R}\boldsymbol{C}_p$$

Example 4. *The final result is illustrated in Figure 5 that shows the two original sets of points are aligned by the identified coordinate transformation.*

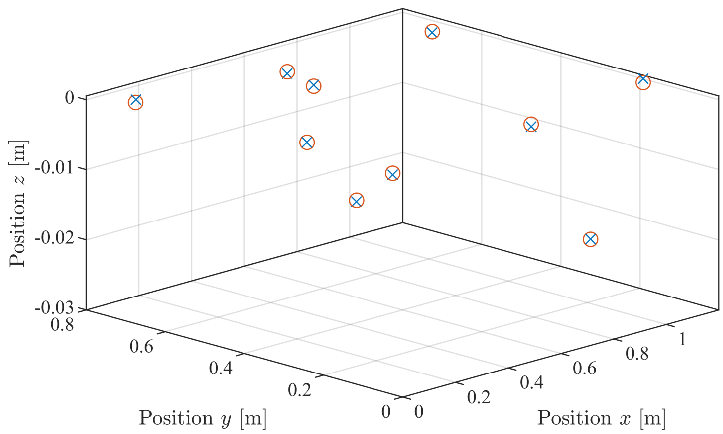

Figure 5: Scatter plot of points $Rp_i + \boldsymbol{T}$ (red circles) and p'_i (blue crosses).

To summarize, the coordinate transformation can be found via the following algorithm.

Algorithm 1 Robot-Table Calibration

Input: Points $\boldsymbol{p}_i, \boldsymbol{p}'_i \in \mathbb{R}^3$ for $i = 1, \ldots, N$.

Output: $\hat{R}, \hat{\boldsymbol{T}}$

Procedure:

$\boldsymbol{C}_p \leftarrow \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{p}_i$ ▷ Compute centroid

$\boldsymbol{C}_{p'} \leftarrow \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{p}'_i$

$\boldsymbol{q}_i \leftarrow \boldsymbol{p}_i - \boldsymbol{C}_p$

$\boldsymbol{q}'_i \leftarrow \boldsymbol{p}'_i - \boldsymbol{C}_{p'}$

$H \leftarrow \sum_{i=1}^{N} \boldsymbol{q}_i (\boldsymbol{q}'_i)^T$ ▷ Compute SVD

$[U, \Sigma, V] \leftarrow \texttt{svd}(H)$

$\hat{R} \leftarrow VU^T$ ▷ Compute rotation matrix

$\hat{\boldsymbol{T}} \leftarrow \boldsymbol{C}'_p - \hat{R}\boldsymbol{C}_p$ ▷ Compute translation vector

References

[1] K. S. Arun, T. S. Huang, and S. D. Blostein. Least-squares fitting of two 3-D point sets. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-9(5):698–700, Sep. 1987.