# Inference and hypothesis testing

## EDS 222

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Fall 2021

# Announcements/check-in

- Assignments to slow down to give space for final project (details on the next slide)
    - Assignment 04 will cover the next two weeks of material
- Increasing divergence between course material and IMS textbook (including today's lecture)
- Some additional nuances on interactions and multiple linear regression at end of week 05 slide deck, for your reference (e.g., multicollinearity)

# Final project

## Goal:

Apply **some of** the statistical concepts you have learned in this course to **answer an environmental data science question**.[*]

## Two parts:

1. Technical blog post: see examples here, here, and here

2. 3-5 minute in-class presentation

> Please run your idea by me via email, Slack, or in person before committing so that I can help assess whether it is feasible and appropriate!

[*]: Your project *must* include concepts from the second half of the course.

# Final project

## Some example topics:

- Are political views on climate change associated with recent natural disaster exposure?

- Detecting trends in air quality for disadvantaged groups across California

- Spatial patterns of deforestation during COVID-19

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- Are there gendered health effects of wildfire smoke?

# Today

### Thinking about uncertainty

Sampling distributions

### Hypothesis testing: conceptual foundations

Null hypotheses, alternative hypotheses, rejecting the null

### Hypothesis testing: in practice

The Central Limit Theorem, standard errors, Z-scores, p-values

### Confidence

Confidence intervals

# Thinking about uncertainty

# Why does uncertainty matter?

## All our sample statistics (e.g., sample means, regression parameters) are uncertain

- We have a *randomly drawn sample* and are trying to learn about the population from our sample
- But our sample statistics would have been different had we randomly drawn a different set of observations!
- This is **natural variability** and it means that all our sample statistics are uncertain estimates of population parameters, even if they are unbiased (e.g., no convenience sampling, no systematic non-response, etc.)

# Why does uncertainty matter?

Key question: Is our estimate indicating anything more than sampling variability or "noise"?

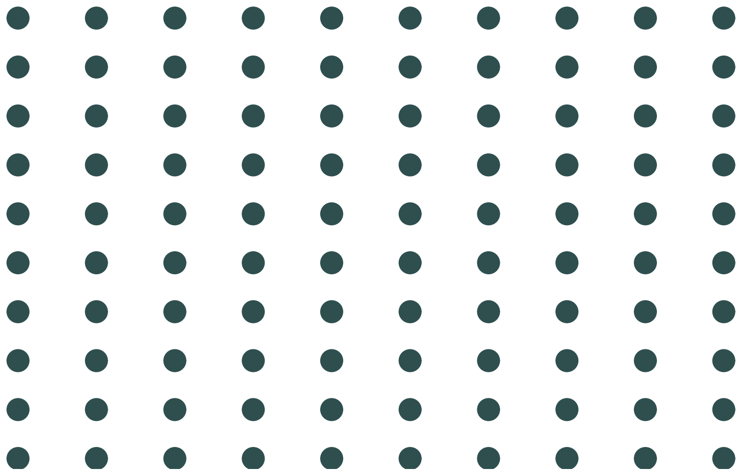
- This is the question **statistical inference** and **hypothesis testing** are trying to answer

# Why does uncertainty matter?

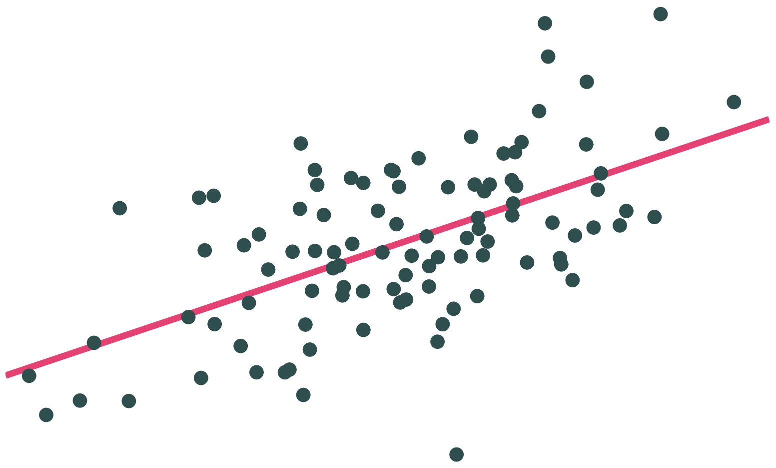
## Example: Gender wage gap

- We collect data on annual earnings and sex for 100 Bren alumni. We are interested in whether the population of all Bren alumni exhibit a gender wage gap.
- We see a mean difference between men and women in our 100-observation sample of $4,500 per year, but a wide range of earnings across both men and women
- Does this mean there is a gender wage gap, or did we just *happen* to get a few high-earning men and a few high-earning women in this group?
- If we collected another independent sample of 100, would the gap be the same?

# Population *vs.* sample



**Population**

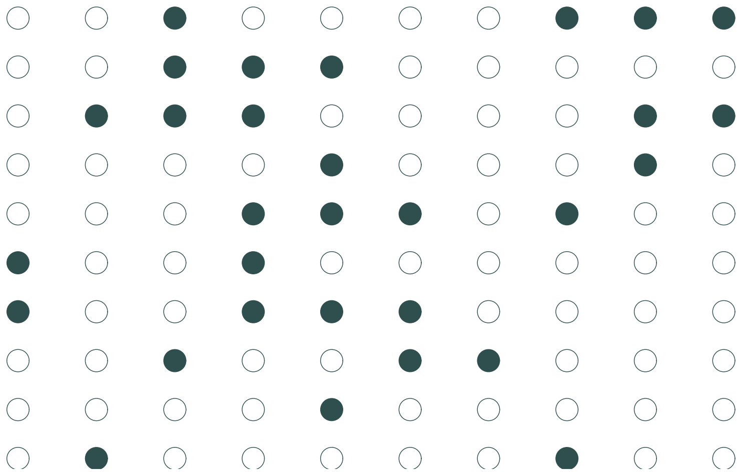


**Population relationship**

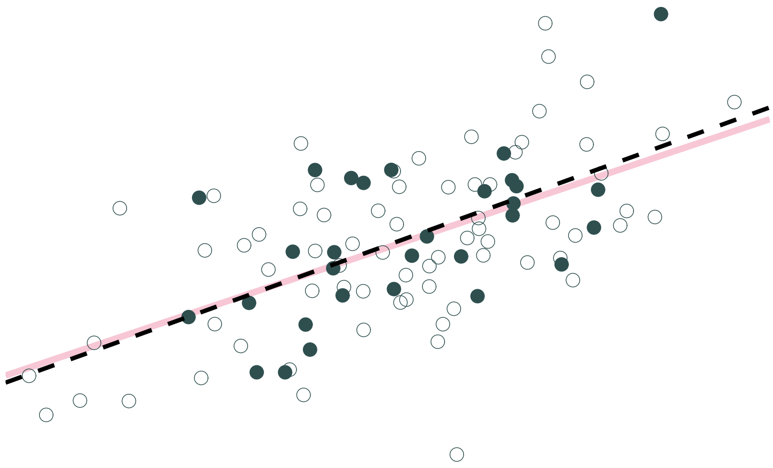
$$y_i = 2.53 + 0.57 x_i + u_i$$

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

# Population *vs.* sample



**Sample 1:** 30 random individuals



**Population relationship**

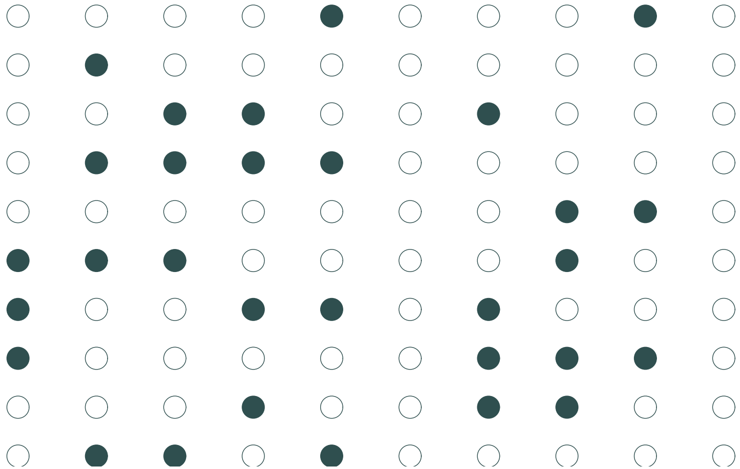
$$y_i = 2.53 + 0.57x_i + u_i$$

**Sample relationship**

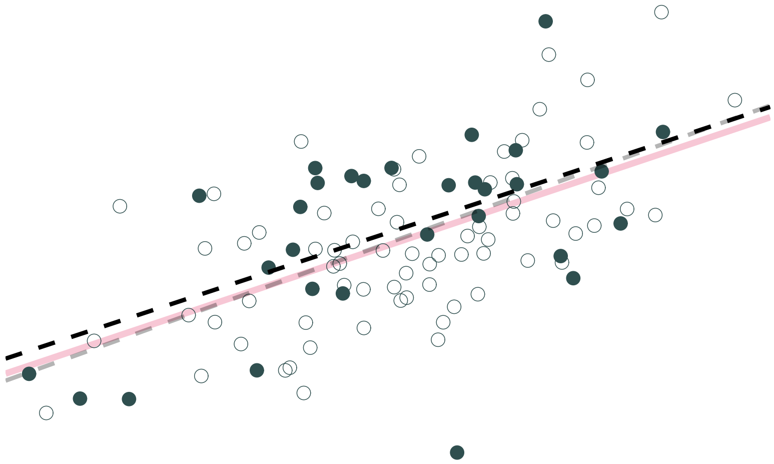
$$\hat{y}_i = 2.36 + 0.61x_i$$

# Population *vs.* sample

count: false



**Sample 2:** 30 random individuals



**Population relationship**

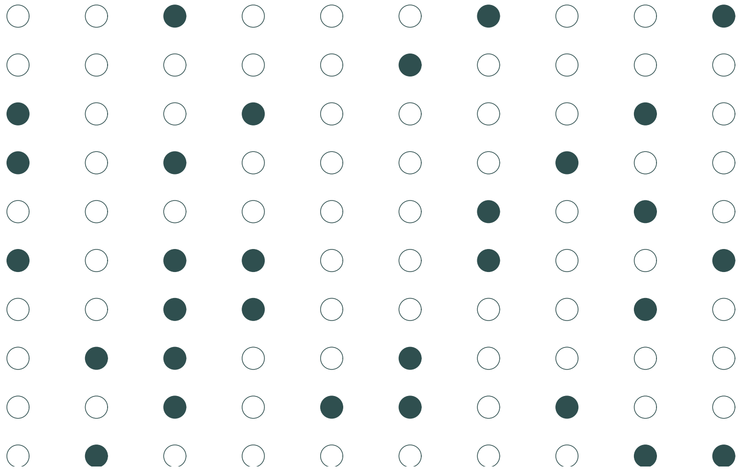
$$y_i = 2.53 + 0.57 x_i + u_i$$

**Sample relationship**

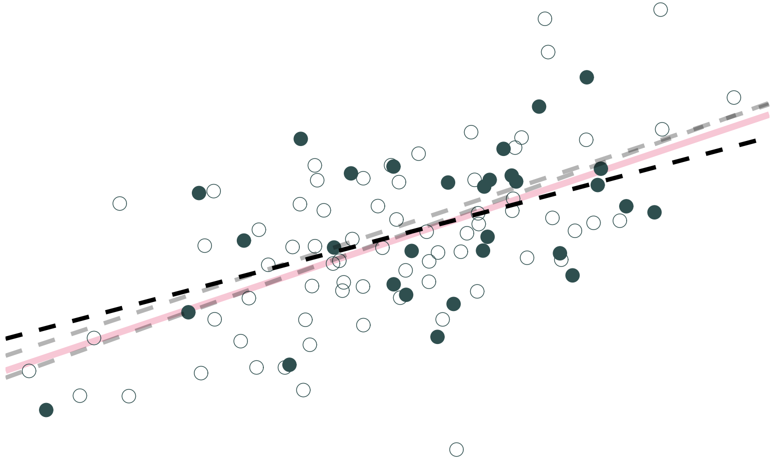
$$\hat{y}_i = 2.79 + 0.56 x_i$$

# Population *vs.* sample

count: false



**Sample 3:** 30 random individuals



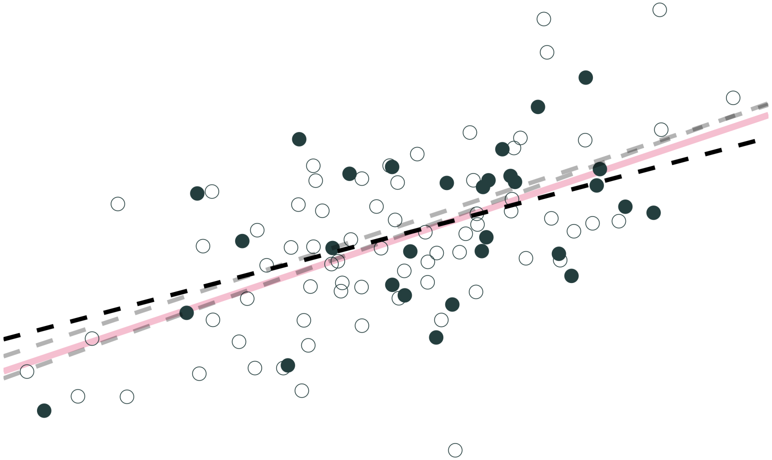
**Population relationship**

$$y_i = 2.53 + 0.57 x_i + u_i$$

**Sample relationship**

$$\hat{y}_i = 3.21 + 0.45 x_i$$

# Population *vs.* sample



- On **average**, our regression lines match the population line very nicely.
- However, **individual lines** (samples) can really miss the mark.
- Differences between individual samples and the population lead to **uncertainty** for the statistican

> Keeping track of uncertainty allows us to test hypotheses about the population using just our sample

# Hypothesis testing: conceptual foundations

# Hypothesis testing

A **hypothesis test** is a statistical method used to evaluate competing claims about population parameters *based on a sample of data*

- $H_0$: **Null hypothesis** A default hypothesis that the measured quantity (e.g., sample mean, difference in means, regression parameters) is zero. In other words: whatever I recovered in my sample is due to random chance.
- $H_A$: **Alternative hypothesis** A hypothesis that the measured quantity is nonzero. In other words: whatever I recovered in my sample is due to true population differences or effects.

# Hypothesis testing: Example

## Example: Are runners getting slower or faster over time?

Random sample of 100 runners from the 2017 Cherry Blossom 10-mile race.

- Mean 2006 finish time for all runners: 93.29 minutes
- Mean 2017 finish time: 98.78 minutes
- Standard deviation 2017 finish time: 16.59 minutes

- $H_0$: The average time was the same in the two years. That is, $\mu = \mathbf{93.29}$
- $H_A$: The average time was not the same in the two years. That is, $\mu \neq \mathbf{93.29}$

# Hypothesis testing: Example

## Example: Are runners getting slower or faster over time?

We call the calculated statistic of interest the **point estimate**

Here, the difference in mean finish time between 2017 sample and the 2006 census of runners is:

- 98.78 - 93.29 = 5.49

> Hypothesis test asks if this point estimate is actually different from zero once we account for sampling variability

# Hypothesis testing: Rejecting the null

## How do we choose between $H_0$ and $H_A$?

If the data conflict so much with $H_0$ that the null cannot be deemed reasonable we **reject the null**

For example:

- The distribution of 2017 10-mile finish times are so far from the 2006 mean that we can reject the means are the same
- Wages across a random sample of 100 Bren alumni are so strongly differentiated across men and women that we reject a gender wage gap of zero

Rejecting the null involves both a **point estimate** and a measure of **uncertainty** or spread in your data

# Hypothesis testing: in practice

# Hypothesis testing in five steps

The general framework for implementing a hypothesis test is:

1. **Define the null** and alternative hypotheses
2. Collect data and compute the **point estimate of the statistic**
3. Model the **variability of the statistic**
4. Given this variability, **quantify the probability that your sample statistic differs from the null** by the observed amount, if the null were true
5. Based on #4, either **reject** or **fail to reject** the null

# Hypothesis testing in five steps

The general framework for implementing a hypothesis test is:

1. **Define the null** and alternative hypotheses
2. Collect data and compute the **point estimate of the statistic**

We already know all about these two steps.

- Null and alternative hypotheses will depend entirely on the statistical question of interest.
- Data collection and point estimates (e.g., means, regression parameters, variances, etc.) we have studied at length in this class

# Hypothesis testing in five steps

The general framework for implementing a hypothesis test is:

1. **Define the null** and alternative hypotheses
2. Collect data and compute the **point estimate of the statistic**
3. Model the **variability of the statistic**

Ack! What is this? Something about how much noise there is in a sample statistic in any given sample...

Let's turn to some definitions.

# Sampling distribution

A **sampling distribution** is the distribution of all possible values of a sample statistic from samples of a given size from a given population.

- The sampling distribution describes how sample statistics (e.g., mean, regression parameters) vary from one sample (or study) to the next
- This is *not* the same as the **data distribution**!
    - Distribution of your data = distribution within one sample (e.g., gives you *one* sample mean)
    - Sampling distribution = distribution across samples (e.g., gives you *many* sample means)

# Sampling distribution

For example, recall our regression above, where the population model is:

$$y_i = \mathbf{2.53} + \mathbf{0.57}x_i + u_i$$

- A regression using one sample gives us *one* set of coefficients, called the **point estimates**: $\hat{\beta}_0 = \mathbf{2.36}$ and $\hat{\beta}_1 = \mathbf{0.61}$
- If we could collect 1000 samples and run that regression 1,000 times, we would recover the **sampling distribution** for each coefficient

# Sampling distribution

## Why do we need a sampling distribution?

Tells us how certain we are that our sample statistics are informative of a population parameter

- Wide sampling distribution = high uncertainty = hard to prove anything about the population

## But how do we obtain one of these?

You only have one sample of data! Where does the sampling distribution come from?

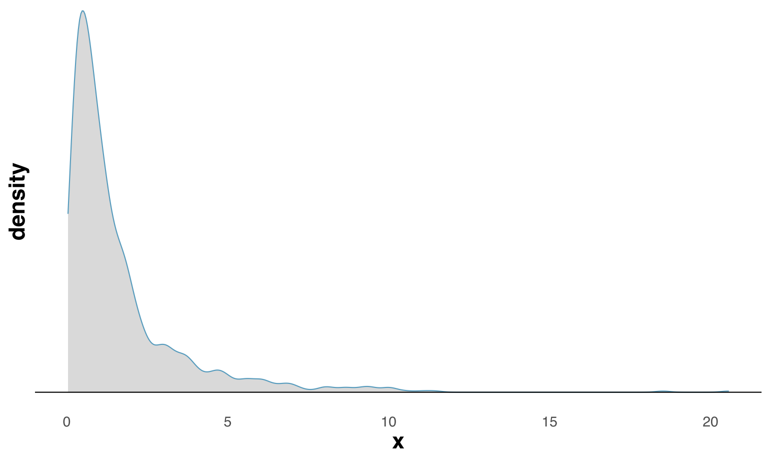
--

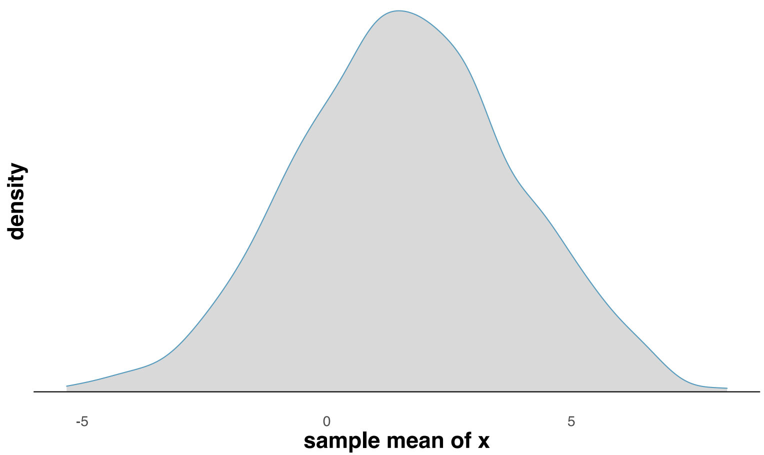
**We derive the sampling distribution from applying the Central Limit Theorem**

# Central Limit Theorem

The **Central Limit Theorem (CLT)** establishes that the sampling distribution of a population parameter is **normal** if the sample size *n* is sufficiently large and observations are drawn randomly and independently.

This is true *even if* the underlying data are not normally distributed!



# Central Limit Theorem

The **Central Limit Theorem (CLT)** tells us our sampling distribution is normal, but only if $n$ is big enough.

## Question: How big does our sample need to be?

## Answer: Rule of thumb is $n \geq 30$

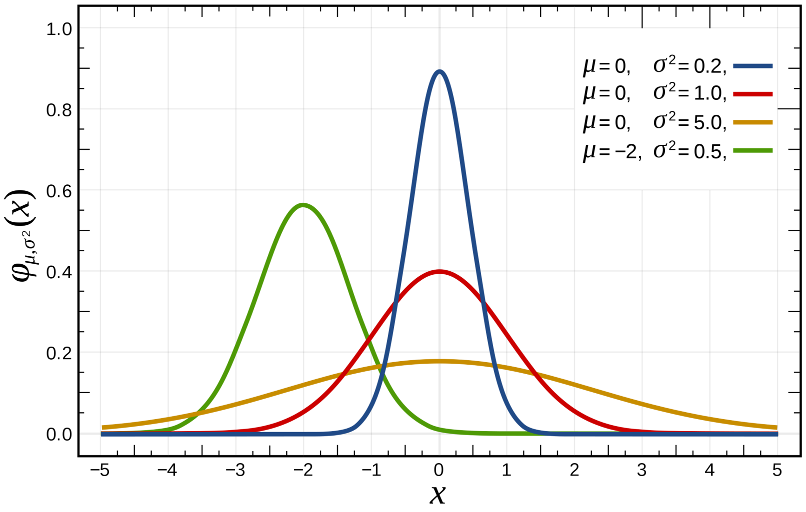
But this is not a hard and fast rule! Be cautious about hypothesis testing and inference with small sample sizes.

# Standard errors

So we know the sample statistic is drawn from a normal distribution...

but there are so many normal distributions!

**We need to know the $\mu$ and $\sigma$ of our sampling distribution** in order to fully model the variability of our statistic.



# Standard errors

## We are testing the likelihood that the null is true

Therefore, the mean of our sampling distribution is defined by the null.

For example:

- $H_0$: 10-mile finish times in 2017 have the same mean as 2006. $\mu = \mathbf{93.29}$
- $H_0$: Male and female wages have a mean *difference* of zero. $\mu = \mathbf{0}$.
- $H_0$: There is *no effect* of neonicotinoid use on colony collapse disorder. $\mu = \beta_1 = \mathbf{0}$.

# Standard error of the sample mean

The standard deviation of your sampling distribution is called the **standard error**

How you calculate the standard error depends on the research question.

For example, if we are interested in a sample mean, our friend the **Central Limit Theorem** tells us that:

$$SE = \frac{s^2}{\sqrt{n}}$$

> Q: What happens to the standard error as sample size increases? Why?

# Standard error for regression slope

The standard deviation of your sampling distribution is called the **standard error**

How you calculate the standard error depends on the research question.

For example, if we are interested in a regression slope, the CLT plus some algebra tell us that:

$$SE = \sqrt{var(\hat{\beta}_1)} = \sqrt{\frac{s^2}{\sum_i (x_i - \bar{x})^2}}$$

> Q: What happens to the standard error as sample size increases? Why?

# SE for comparing two means

The standard deviation of your sampling distribution is called the **standard error**

How you calculate the standard error depends on the research question.

For example, if we are interested in the difference between two means, the CLT plus some algebra tell us that:

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

> In all these cases, the SE is the standard deviation of the sampling distribution!

# Summary: Standard errors

If we could collect many samples from the population, and we computed our statistic for each sample (e.g., mean, slope coefficient), we could construct the **sampling distribution.**

The **standard error** is our estimate of the the standard deviation of the sampling distribution. We can never actually collect hundreds of independent samples, so we use our single sample to approximate the true sampling distribution standard deviation, leveraging the **Central Limit Theorem**

**Standard error** measures how dispersed our sample statistic is around the population parameter of interest (highly dispersed = large SE = a lot of uncertainty about the population parameter from our one sample)

# Hypothesis testing in five steps

The general framework for implementing a hypothesis test is:

1. **Define the null** and alternative hypotheses
2. Collect data and compute the **point estimate of the statistic**
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4. Given this variability, **quantify the probability that your sample statistic differs from the null** by the observed amount, if the null were true

# Hypothesis testing in five steps

Step 4: quantify the probability that your sample statistic differs from the null by the observed amount, if the null were true

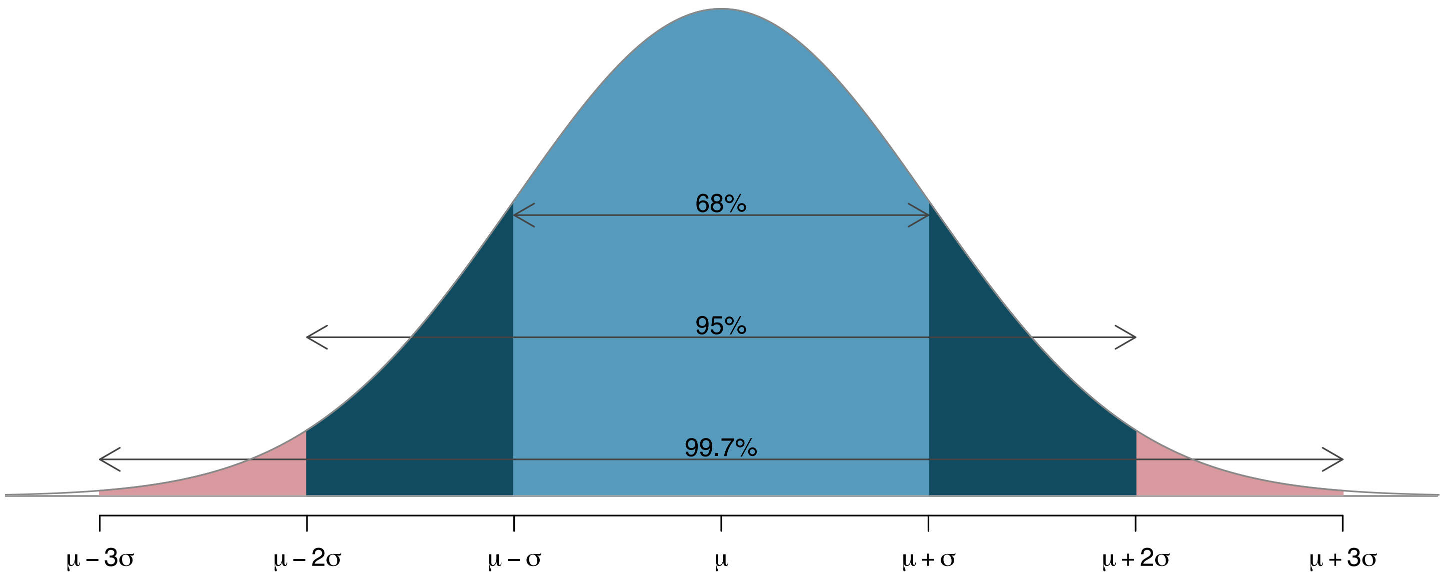
- I know how that my sample statistic is drawn from a normal distribution with mean $\mu$ and an estimated standard deviation given by $SE$.

- This should tell me something about **how unlikely it was** that I happened to draw my point estimate if the null were true, right?

- Yep! But we need a couple more definitions to get all the way there.

# The 68-95-99.7 rule

For a normal distribution:




Probabilities for falling within 1, 2, and 3 standard deviations of the mean in a normal distribution.

# Z-score

- The 68-95-99.7 rule is helpful if your point estimate (sample statistic) is exactly 1, 2, or 3 standard deviations from the mean.
- But what about all the other values?

## Z-score: How many standard deviations is a value from the mean?

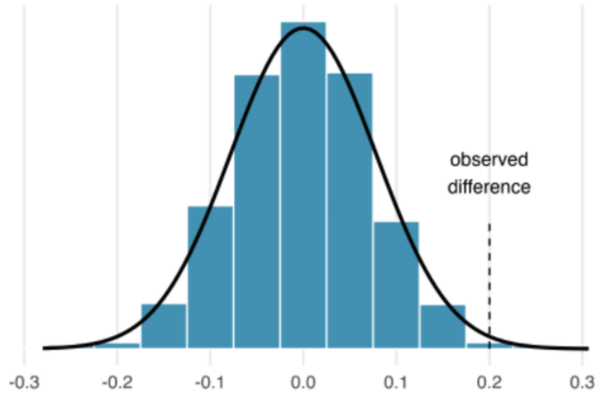
$$z = \frac{x_i - \mu}{\sigma}$$

# Z-score for hypothesis testing

- When testing hypotheses, we care about how far our point estimate is from the null.

Z-score for hypothesis testing: How many standard deviations is a point estimate from the null?

$$z = \frac{\text{point estimate} - \text{null value}}{SE}$$



# Quantifying probabilities: *p*-value

- The Z-score is also called the **test statistic**
- The Z-score/test statistic allows us to compute the ***p*-value**:

***p*-value** is the probability of getting a point estimate *at least as extreme* as ours **if the null hypothesis were true.**

$$p - value = Pr(Z < -|z| \text{ or } Z > |z|) = 2 * Pr(Z > |z|)$$

# Quantifying probabilities: $p$-value

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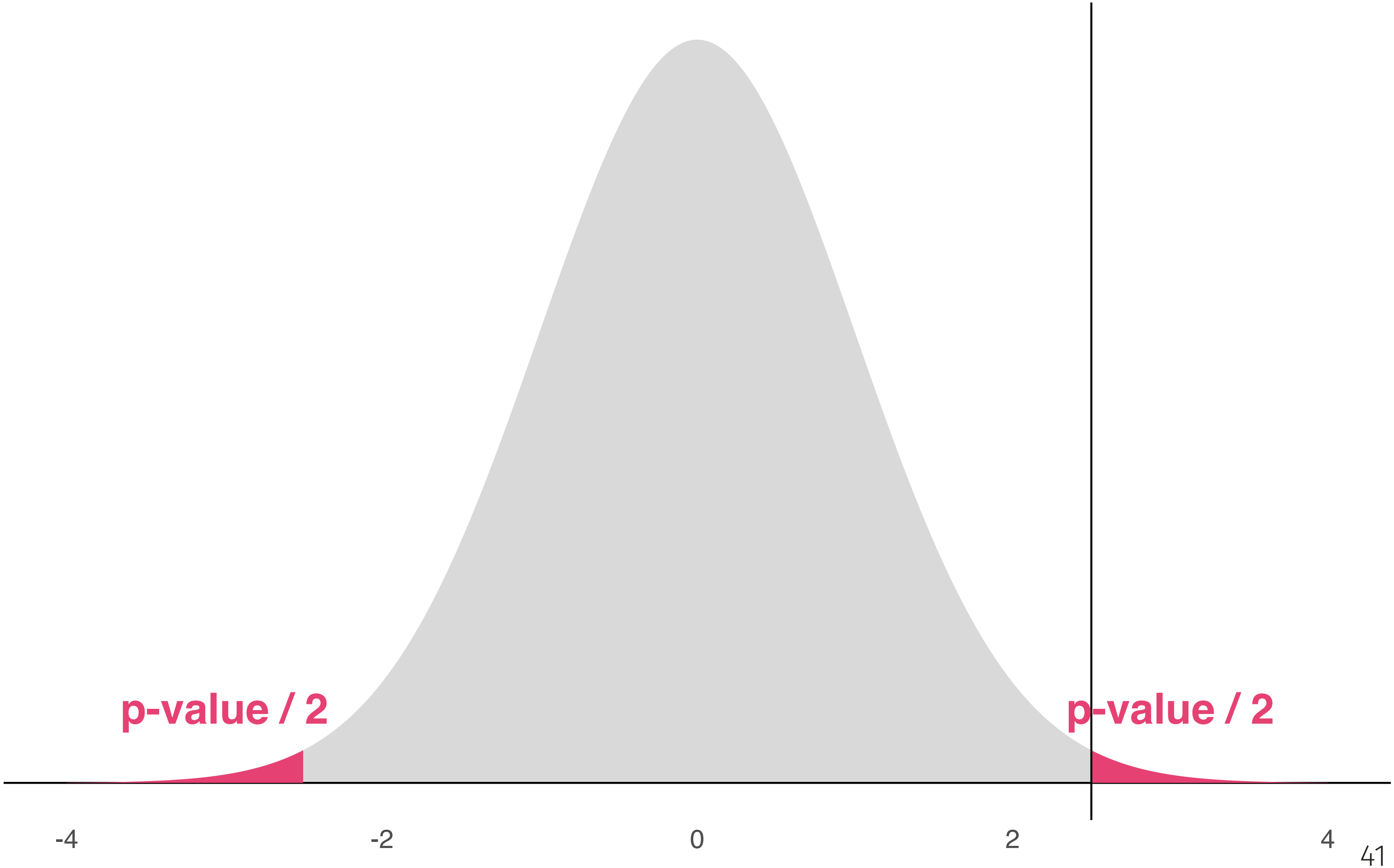

# Quantifying probabilities: $p$-value

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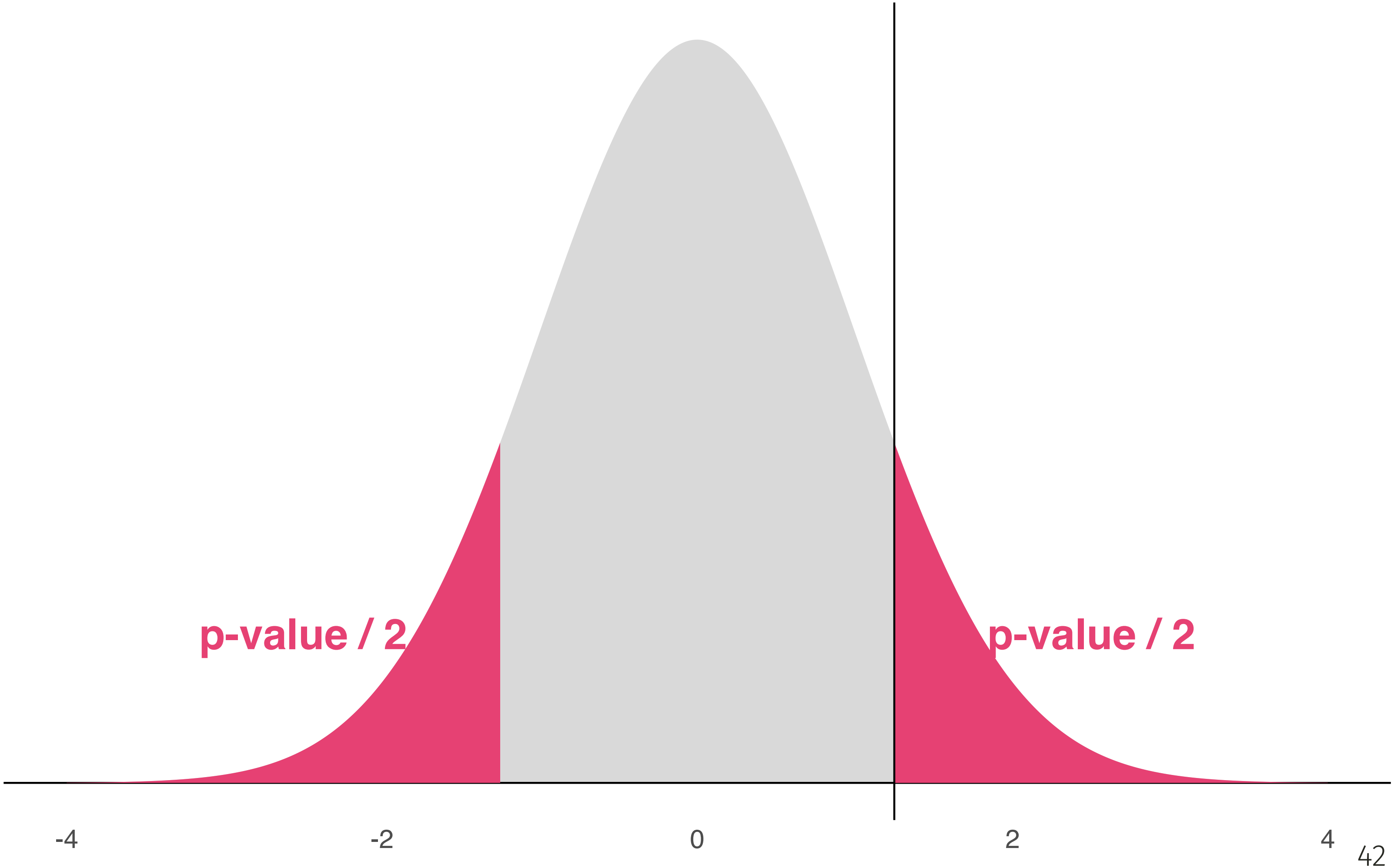


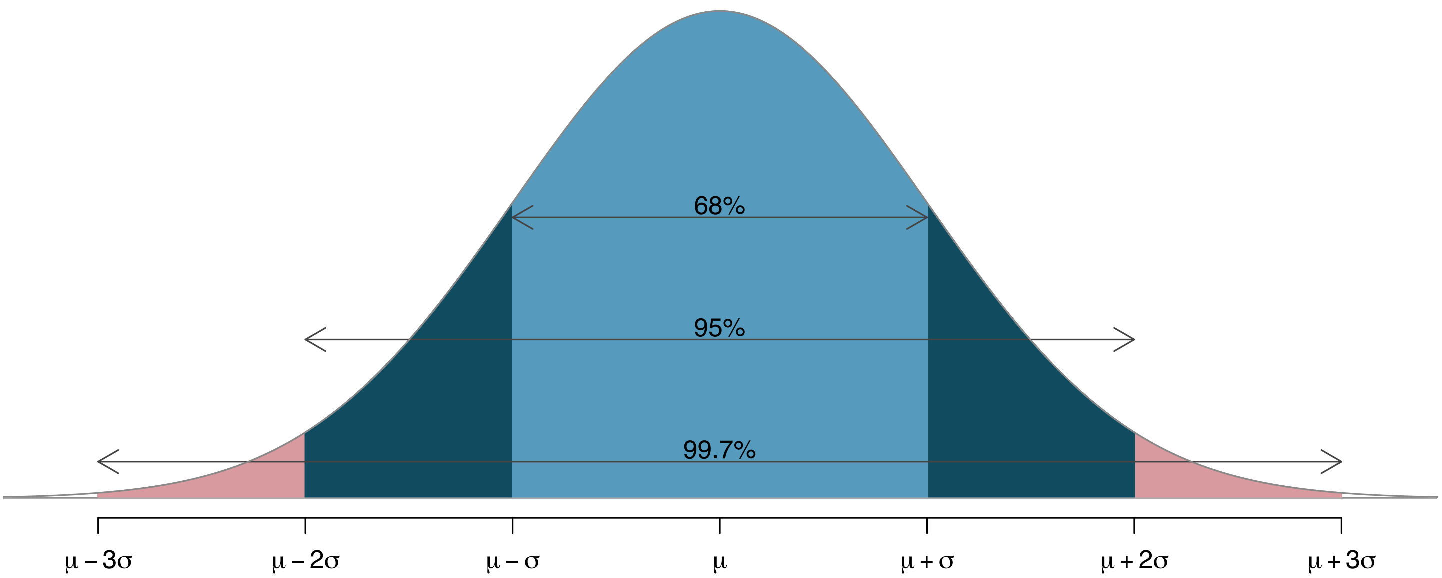
# Hypothesis testing with *p*-values

- *p*-value is the probability of observing a point estimate as extreme as yours if the null were true
- *p*-value is the area under the sampling distribution to the right and to the left of the absolute value of your test statistic (z-score, $z$)

How do I compute a *p*-value from a test-statistic?

# Hypothesis testing with $p$-values

How do I compute a $p$-value from a test-statistic?



# Hypothesis testing with *p*-values

- *p*-value is the probability of observing a point estimate as extreme as yours if the null were true

- *p*-value is the area under the sampling distribution to the right and to the left of the absolute value of your test statistic (z-score, $z$)

How do I compute a *p*-value from a test statistic?

- **In math:** Integrate the sampling distribution's probability density function between $-\infty$ and $-|z|$; multiply by 2

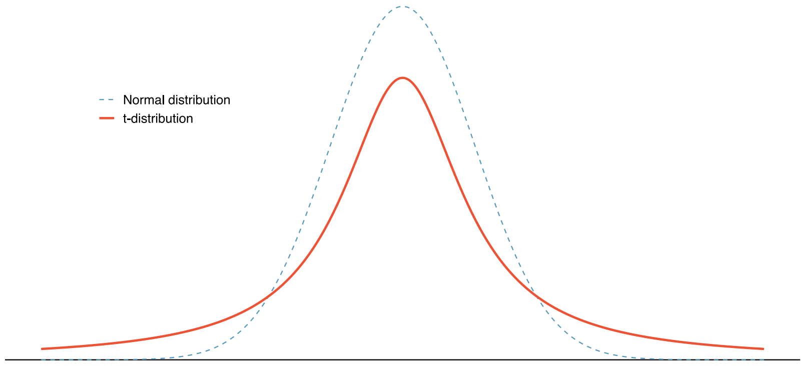
- **In `R`:** `pnorm()`, `t.test()`, `summary(lm())`, ...

# A note on the CLT

The **Central Limit Theorem** has done a lot of work for us so far. However, it only holds under the following conditions:

1. Observations in our sample are **independent**
2. We have a **large enough sample** (at the very least $n \geq 30$)

When $n$ is relatively small, we can still proceed, we just need to use a $t$-distribution (and T-score -- use `pt()` in `R`) instead of a normal distribution (and Z-score)



# Hypothesis testing in five steps

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4. Given this variability, **quantify the probability that your sample statistic differs from the null** by the observed amount, if the null were true
5. Based on #4, either **reject** or **fail to reject** the null

# Can we finally test something?

Step 5: Based on the $p$-value, either **reject** or **fail to reject** the null

- Low $p$-value $\rightarrow$ very unlikely to see your point estimate if the null were true
- High $p$-value $\rightarrow$ very likely to see your point estimate if the null were true

So...what is a low enough $p$-value to **reject the null**?

# Can we finally test something?

What is a low enough *p*-value to **reject the null**?

> This is a heavily debated question.

- Best to **report your *p*-value** alongside any conclusions you reach about your hypothesis

- Traditionally, we use a **significance level** of $\alpha = 0.05$

  - This says you have a 5% chance of observing your point estimate even if the null were true
  - Reject the null if $p < 0.05$ and $\alpha = 0.05$

- In general, reject the null if $p < \alpha$.

  - Other common $\alpha$s: 0.01, 0.1

# Statistical significance

We say a point estimate is "statistically significant" when:

$p < \alpha$

For example:

> "[W]e find a **statistically-significant** effect whereby increases in surface UV intensity lowers subsequent COVID-19 growth rates...we estimate that a 1 kJm–2hr–1 increase in local UV reduces local COVID-19 growth rates by .09 (±.04, *p* =.01) percentage points over the ensuing 17 days." --- *Carleton et al., 2021*

# Hypothesis testing: Rejecting the null

We can **reject the null** hypothesis *or* **fail to reject the null** hypothesis.

We **never accept the null** hypothesis.

Why not?

- Lack of evidence is not proof! If $p > \alpha$, there is so much sampling variability that we cannot distinguish the null from the point estimate.

Think of this as innocent (null is true) until proven guilty (null is rejected).

- Failing to reject the null tells us we do not have sufficient evidence to prove there is an effect or a difference

# Constructing confidence intervals

# Why use confidence intervals?

- *p*-values are not enough for us to conclude anything meaningful about an analysis

- Effect sizes are important! We care not just about whether a treatment effects an outcome, but by *how much*

A **confidence interval** is a range of plausible values where we may find the true population value.

- It tells us something about the magnitude of the parameter of interest, as well as our uncertainty around our estimate

# Confidence intervals

When the sampling distribution of a point estimate can be modeled as normal, the point estimate we observe will be within 1.96 standard errors of the true value of interest about 95% of the time (think back to the 68-95-99.7 rule).

Thus, a 95% confidence interval for such a point estimate can be constructed:

$$\textbf{point estimate} \pm \mathbf{1.96} * \boldsymbol{SE}$$

We can be 95% confident this interval captures the true value.

> Also can see this as: `2*pnorm(-1.96) = .05`

# Confidence intervals

You can build a confidence interval for any level of $\alpha$:

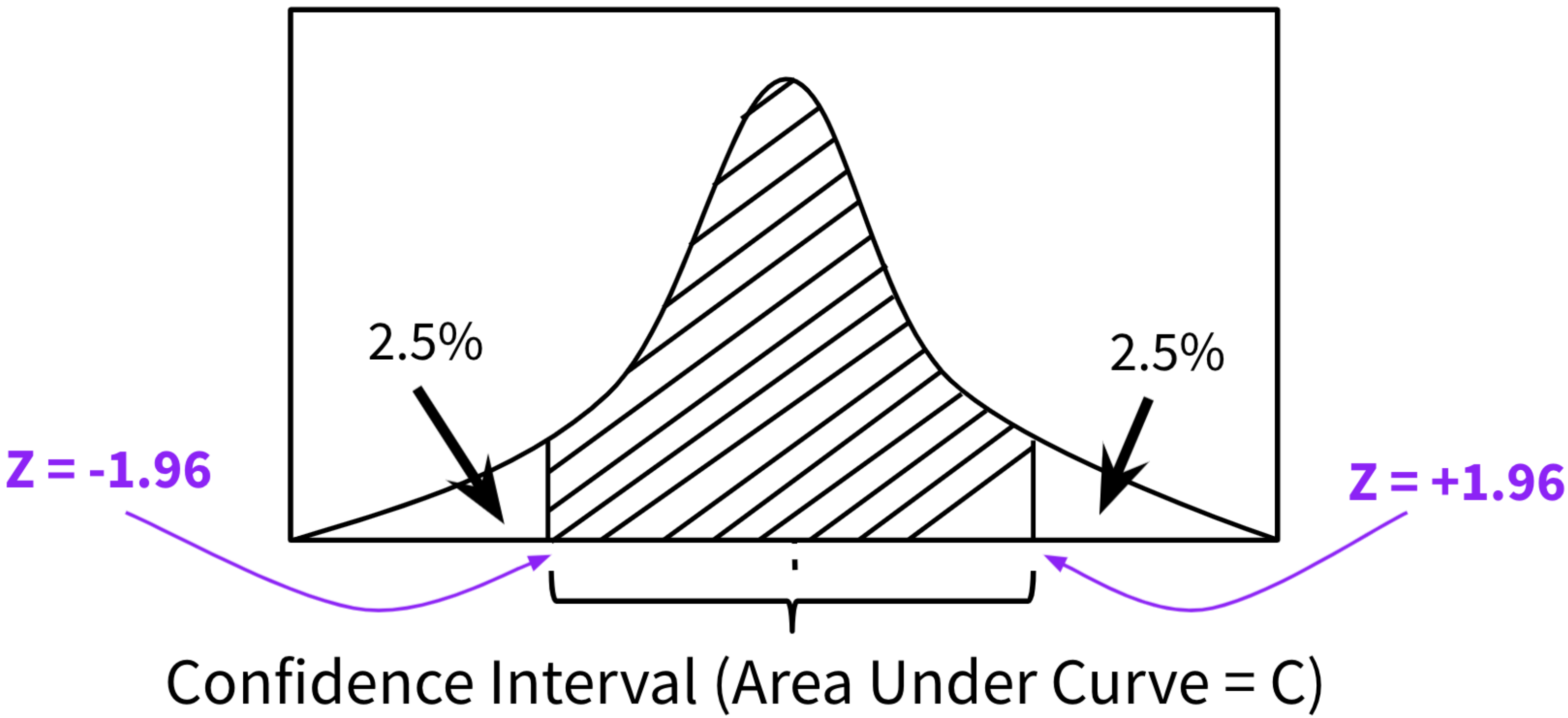
$$\textbf{point estimate} \pm z_{\alpha/2} * SE$$

For example:

- $\boldsymbol{\alpha = 0.1}$ = 90% confidence interval: $\textbf{point\_estimate} \pm \mathbf{1.64} * SE$
- $\boldsymbol{\alpha = 0.01}$ = 99% confidence interval: $\textbf{point\_estimate} \pm \mathbf{2.57} * SE$

> Why do the z-scores get larger for higher confidence intervals?

# A 95% confidence interval visual



# Confidence intervals: Interpretation

A 95% confidence interval tells you there is a **95% chance that your interval includes the true population parameter**.

## A very common misinterpretation:

There is a 95% chance the true population parameter falls inside my confidence interval.

## Why is this a big deal?

The population parameter *is not random*. So it either **is or is not** inside your CI.

Slides created via the R package **xaringan**.

Some slide components were borrowed from Ed Rubin's awesome course materials.