Setul 4
de probleme şi exerciţii de matematică
(cu privire la serii cu termeni oarecare, serii de funcţii şi serii de puteri)

S4.1 Folosind diverse criterii de convergenţă, să se stabilească natura seriilor:

$$\text{a) } \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{2^{n-1}}; \quad \text{b) } \sum_{n=0}^{\infty} (-1)^n \frac{\ln 2 + 3^n}{\ln 3 + 2^n}; \quad \text{c) } \sum_{n=1}^{\infty} \frac{1}{\sqrt[n+1]{\ln(n+1)}}; \quad \text{d) } \sum_{n=1}^{\infty} \frac{\sin n \cdot \cos n^2}{\sqrt{n}}, n \in \mathbb{N}^*;$$

$$\text{e) } \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)!!}{2^n \cdot n!}; \quad \text{f) } \sum_{n=0}^{\infty} 2^{-\sqrt{n^2+1}}; \quad \text{g) } \sum_{n=1}^{\infty} (-1)^{n-1} \ln\left(\frac{n^2+2}{n^2+1}\right);$$

$$\text{h) } \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^{n+1}}{n^{n+2}}; \quad \text{i) } \sum_{n=1}^{\infty} (-1)^{n-1} \ln\left(\frac{n^2+2}{n^2+1}\right); \quad \text{j) } \sum_{n=1}^{\infty} (-1)^n \frac{1}{n - \ln n};$$

$$\text{k) } \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+1}; \quad \text{l) } \sum_{n=1}^{\infty} \frac{n+1}{n} \cdot \frac{\sin \frac{n\pi}{6}}{\sqrt{n^3+1}}; \quad \text{m) } \sum_{n=0}^{\infty} \frac{a^n + \operatorname{sh} n}{3^n} \cdot b^n, a, b \in \mathbb{R};$$

$$\text{n) } \sum_{n=1}^{\infty} \operatorname{tg}^n\left(a + \frac{b}{n}\right), a, b \in \left(0, \frac{\pi}{2}\right); \quad \text{o) } \sum_{n=1}^{\infty} (-1)^{n-1} n^{\alpha} \left(\ln\left(\frac{n+2}{n}\right)\right)^{\beta}, \alpha, \beta \in \mathbb{R}.$$

S4.2 Date fiind seriile $\displaystyle\sum_{n \in \mathbb{N}^*} \frac{(-1)^n}{n^2}$ şi $\displaystyle\sum_{n \in \mathbb{N}^*} \frac{n}{2^n}$, să se găsească produsul lor Cauchy.

S4.3 Să se studieze mulţimea de convergenţă pentru următoarele serii de funcţii:

$$\text{a) } \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \left(\frac{1-x}{1-2x}\right)^n, x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}; \quad \text{b) } \sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln n} \left(\frac{1-x^2}{1+x^2}\right)^n, x \in \mathbb{R};$$

$$\text{c) } \sum_{n=1}^{\infty} \frac{\ln(1+a^n)}{n^x}, a \geq 0; \quad \text{d) } \sum_{n=1}^{\infty} \frac{\sin^n x}{n^{\alpha}}, \alpha \in \mathbb{R}.$$

S4.4 Să se studieze convergenţa punctuală şi convergenţa uniformă a seriilor de funcţii:

$$\text{a) } \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2} x^{2n+1}, x \in \mathbb{R}; \quad \text{b) } \sum_{n=1}^{\infty} \frac{1}{n! x^n}, x \in \mathbb{R}; \quad \text{c) } \sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln n} \left(\frac{1-x^2}{1+x^2}\right)^n, x \in \mathbb{R};$$

$$\text{d) } \sum_{n=1}^{\infty} \left[e - \left(1 + \frac{1}{n}\right)^n\right] \frac{\cos nx}{n+1}, x \in \mathbb{R}; \quad \text{e) } \sum_{n=1}^{\infty} \frac{\cos \frac{2n\pi}{3}}{\sqrt{x^3+n}}, x \in \mathbb{R}.$$

S4.5 Să se găsească mulţimea de convergenţă corespunzătoare fiecăreia dintre seriile de puteri ce urmează:

$$\text{a) } \sum_{n \in \mathbb{N}} \left[2 + (-1)^n\right] x^n,\ x \in \mathbb{R}; \quad \text{b) } \sum_{n \in \mathbb{N}} \frac{n+1}{\sqrt{n^4+n^3+1}} \left(\frac{x+1}{2x+3}\right)^n,\ x \in \mathbb{R} \setminus \left\{\frac{3}{2}\right\};$$

$$\text{c) } \sum_{n \in \mathbb{N}^*} \left(\cos \frac{1}{n}\right)^{\frac{n^2+2}{n+2}} \cdot x^n, x \in \mathbb{R}; \quad \text{d) } \sum_{n \in \mathbb{N}^*} (-1)^{n-1} \frac{(x-4)^n}{n \cdot 3^n}, x \in \mathbb{R}; \quad \text{e) } \sum_{n \in \mathbb{N}^*} \frac{x^n}{n^p},\ p \in \mathbb{R};$$

f) $\sum\limits_{n=2}^{\infty}\frac{x^n}{3^n\cdot n\cdot\ln n}$ $x\in\mathbb{R}$; g) $\sum\limits_{n=1}^{\infty}\frac{2^n(x+1)^{2n}}{(4n+1)^2}$, $x\in\mathbb{R}$. h) $\sum\limits_{n\in\mathbb{N}^*}\left(\sqrt{n}-1\right)^n\cdot x^n, x\in\mathbb{R}$.

i) $\sum\limits_{n\in\mathbb{N}}(-1)^n\frac{1}{3^{\frac{n}{2}}\sqrt{1+n^2}}\text{tg}^n x$, $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$; j) $\sum\limits_{n\in\mathbb{N}^*}\frac{n!}{(a+1)(a+2)\cdot\ldots\cdot(a+n)}x^n, a>0$.

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