5. Natural Deduction

Language & Logic

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Recap

- Notation for arguments (in sequent form)
  - P1, P2 : C
  - P1, P2 $\vdash$ C (if proven to be valid)
  - where P1, P2, C are formula in propositional logic

- Proofs (of argument validity)
  - using natural deduction
  - with a given set of inference rules (syntactic transforms)
  - so far: list of premise introductions & inference rule applications
  - not necessary unique (nor easy to construct)
  - but easy to check, rigorously once, constructed
  - fixed proof notation

Exercise Sheet 2 Feedback

- Proof for Question 5 from Exercise sheet 2

$$\neg P \to \neg Q, P \to Z, \neg\neg Q \;:\; Z$$

1.	$\neg P \to \neg Q$	Premise	{1}
2.	$\neg\neg Q$	Premise	{2}
3.	$\neg\neg P$	Modus Tollens$_{1,2}$	{1,2}
4.	P	Double Negative Elimination$_{3}$	{1,2}
5.	$P \to Z$	Premise	{5}
6.	Z	Modus Ponens$_{5,4}$	{1,2,5}

- There are variants, but we will stick largely to the above
  - perhaps with abbreviations for rule annotations

Exercise Sheet 2 Feedback

- Proof for Question 5 from Exercise sheet 2

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2.	$\neg\neg Q$	Premise	$\{2\}$
3.	$\neg\neg P$	Modus Tollens$_{1,2}$	$\{1,2\}$
4.	P	Double Negative Elimination$_3$	$\{1,2\}$
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Lines of propositions used in the inference rule

Lines of all premises on which this depends

Exercise Sheet 2 Feedback

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Inference rules can be applied
to any propositions (including compounds)

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4.	P	Double Negative Elimination$_3$	$\{1,2\}$
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Don't skip steps: every line should
be the application of <u>one</u> inference rule

Proof strategies

- How do we go about constructing a proof?

- Simple examples so far
  - look at the available propositions and rules to manipulate them
  - work backwards from the conclusion
  - or forwards from the premises
  - look at the main connective of each proposition

Example

- Proof for Question 2 from Exercise sheet 2

$$P, Q, R \; : \; P \wedge (Q \wedge R)$$

1.	Q	Premise	$\{1\}$
2.	R	Premise	$\{2\}$
3.	$Q \wedge R$	$\wedge$-Introduction$_{1,2}$	$\{1,2\}$
4.	P	Premise	$\{4\}$
5.	$P \wedge (Q \wedge R)$	$\wedge$-Introduction$_{4,3}$	$\{1,2,4\}$

Today

- This lecture (and the next):

  - more systematic list of inference rules

  - more complex inference rules & proof techniques

  - and strategies to apply them

Inference rules

- We will always assume a fixed set of inference rules

- Previously, used an arbitrary set:
  - Modus Ponens & Modus Tollens
  - double negative elimination
  - $\wedge$-introduction & $\wedge$-elimination

- Time to be systematic…
  - 4 connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$)
  - each one has rules for introduction and elimination
  - which introduce and eliminate the connective, respectively
  - e.g. to form a conclusion or break up a premise

Introduction & Elimination

- Conjunction
  - $\wedge$-introduction
  - $\wedge$-elimination
- Disjunction
  - $\vee$-introduction
  - $\vee$-elimination
- Implication
  - $\rightarrow$-introduction
  - $\rightarrow$-elimination
- Negation
  - $\neg$-introduction
  - $\neg\neg$-elimination

Introduction & Elimination

- Conjunction
  - $\wedge$-introduction – already seen
  - $\wedge$-elimination – already seen
- Disjunction
  - $\vee$-introduction
  - $\vee$-elimination
- Implication
  - $\rightarrow$-introduction
  - $\rightarrow$-elimination – already seen (Modus Ponens)
- Negation
  - $\neg$-introduction
  - $\neg\neg$-elimination – already seen (double negative elimination)

But <u>no</u> Modus Tollens

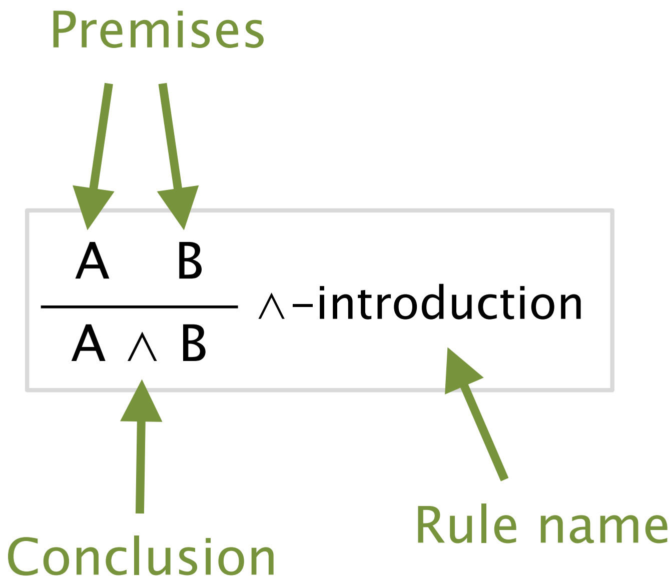
Introduction & Elimination

- Conjunction
  - $\wedge$-introduction – already seen
  - $\wedge$-elimination – already seen
- Disjunction
  - $\vee$-introduction – new
  - $\vee$-elimination – new (case analysis)
- Implication
  - $\rightarrow$-introduction – new (conditional proof)
  - $\rightarrow$-elimination – already seen (Modus Ponens)
- Negation
  - $\neg$-introduction – new (reductio ad absurdum)
  - $\neg\neg$-elimination – already seen (double negative elimination)

But <u>no</u> Modus Tollens

Inference rules

- From now now, we now switch to a more common style for presenting inference rules:



Inference rules seen so far

- Conjunction ($\wedge$)

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-introduction} \qquad \frac{A \wedge B}{A}\ \wedge\text{-elimination} \qquad \frac{A \wedge B}{B}\ \wedge\text{-elimination}$$

- $\rightarrow$-elimination
  - (sometimes called “Modus Ponens”)

$$\frac{A \rightarrow B \quad A}{B}\ \rightarrow\text{-elimination}$$

- $\neg\neg$-elimination(sometimes abbreviated to DNE)

$$\frac{\neg\neg A}{A}\ \neg\neg\text{-elimination}$$

$\vee$–Introduction

- $\vee$ –introduction

$$\frac{A}{A \vee B}\ \vee\text{–introduction} \qquad \frac{A}{B \vee A}\ \vee\text{–introduction}$$

for any formula B

- Simple, but slightly counterintuitive
  - we'll illustrate some uses of it later

- Proof notation
  - same as inference rules seen so far
  - (single) proof line index is added to rule annotation
  - and premise dependencies are copied from that line

Example ($\vee$–Introduction)

- $P, P \rightarrow Q \ : \ (P \wedge Q) \vee R$

$\rightarrow$-introduction

- Allows us to prove an implication $A \rightarrow B$
  - also known as conditional proof
  - A is the antecedent
  - B is the consequent

Recall: $\vdash$ means "if we assume the LHS, we can prove the RHS"

$$\frac{A \vdash B}{A \rightarrow B} \;\rightarrow\text{-introduction}$$

- New proof technique:
  - make a hypothesis A (also called an assumption)
  - temporarily assume that A is true, and prove B (a sub-proof)
  - infer $A \rightarrow B$ and discharge the hypothesis

1.	A	Hypothesis
	…	
n.	B	
n+1.	$A \rightarrow B$	$\rightarrow$-introduction$_{1,n}$

"sub-proof"

Example (conditional proof)

- $P \rightarrow Q, Q \rightarrow R \;:\; P \rightarrow (Q \wedge R)$

1.	$P \rightarrow Q$	Premise	{1}
2.	$Q \rightarrow R$	Premise	{2}
3.	P	Hypothesis	{3}
4.	Q	$\rightarrow$-elimination$_{1,3}$	{1,3}
5.	R	$\rightarrow$-elimination$_{2,4}$	{1,2,3}
6.	$Q \wedge R$	$\wedge$-introduction$_{4,5}$	{1,2,3}
7.	$P \rightarrow (Q \wedge R)$	$\rightarrow$-introduction$_{3,6}$	{1,2}

Dependencies for hypotheses done the same as for premises

Dependency on hypothesis removed when "discharged"

- Notes:
  - lines around hypothesis + sub-proof
  - sub-proof can use propositions proved outside, but not vice-versa
  - cannot complete the proof while inside the sub-proof!

Example (nested conditional)

- $(P \wedge Q) \rightarrow R \ : \ P \rightarrow (Q \rightarrow R)$

$\vee$–Elimination

- Allows us to use a premise whose main connective is $\vee$

$$\frac{A \vee B \quad A \vdash C \quad B \vdash C}{C} \; \vee\text{–elimination}$$

- Proof technique
  - two separate sub-proofs, one assuming A, one assuming B
  - show that the same proposition C can be deduced from both
  - "case analysis"

1.	$A \vee B$	
2.	A	Hypothesis
	…	
n.	C	
n+1.	B	Hypothesis
	…	
m.	C	
m+1.	C	$\vee$–elimination$_{1,2,n,n+1,m}$

Example (∨–elimination)

- $P \vee Q$: $Q \vee P$

1.	$P \vee Q$	Premise	{1}
2.	P	Hypothesis	{2}
3.	$Q \vee P$	∨–introduction	{2}
4.	Q	Hypothesis	{4}
5.	$Q \vee P$	∨–introduction	{4}
6.	$Q \vee P$	∨–elimination$_{1,2,3,4,5}$	{1}

- Note:
  - there are 5 line number indices for the rule annotation
  - the dependencies for line 6 are derived by combining the dependencies for lines 1, 3 and 5, then removing the dependencies for lines 2 and 4