Part I: Differential Calculus

1. Differential Calculus for functions of a single variable. Taylor's Formula.

Problem: Digital computing machines do not store values of nonlinear functions such as $\sin x$ but rather compute them (whenever needed) using only basic arithmetic operations: $+, -, \cdot, \div$

How is this done in practice?

Math. Insight: Polynomials

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

are constructed via a finite nr. of arithmetic operations.

(So, that's what we need!)

§1.1. Limits and Continuity (CAUCHY 1821) "ε-δ defs"

Def (limit of f at a point x^*) $f:(a,b)\to\mathbb{R}$

$$\lim_{x\to x^*} f(x) = \ell \quad \text{if}$$

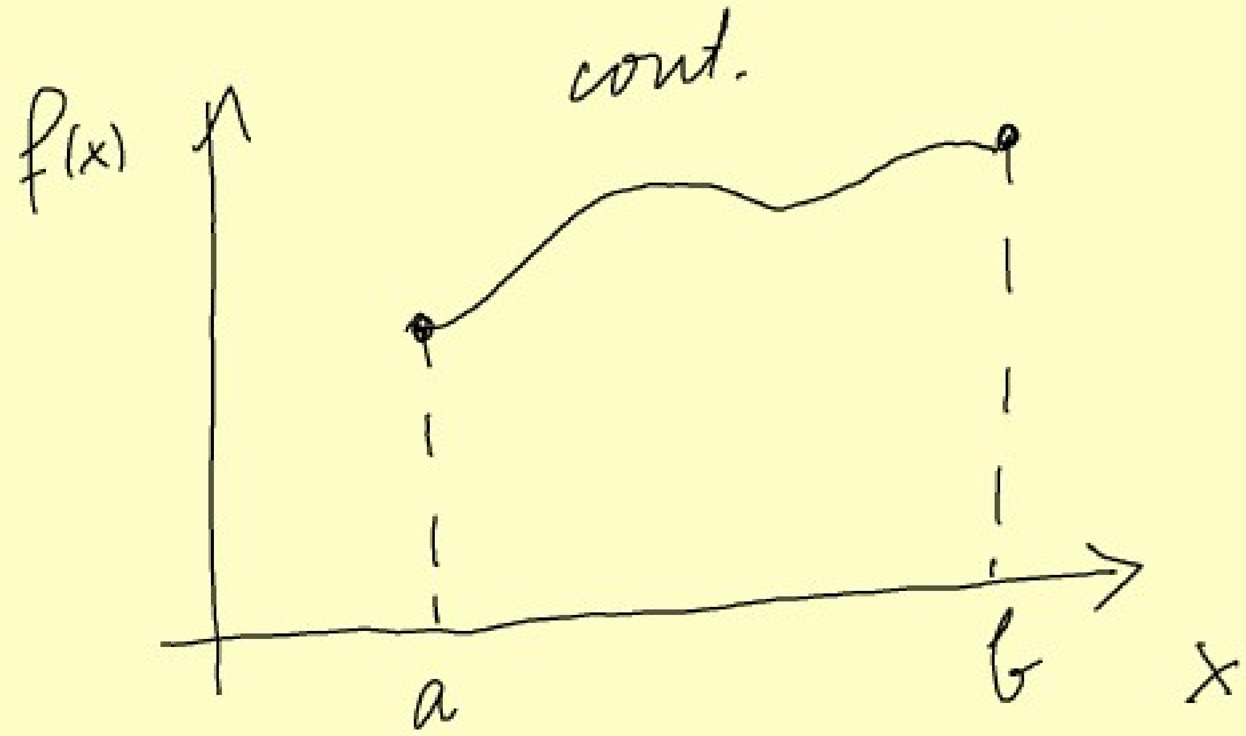
$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $|f(x)-\ell| < \varepsilon$

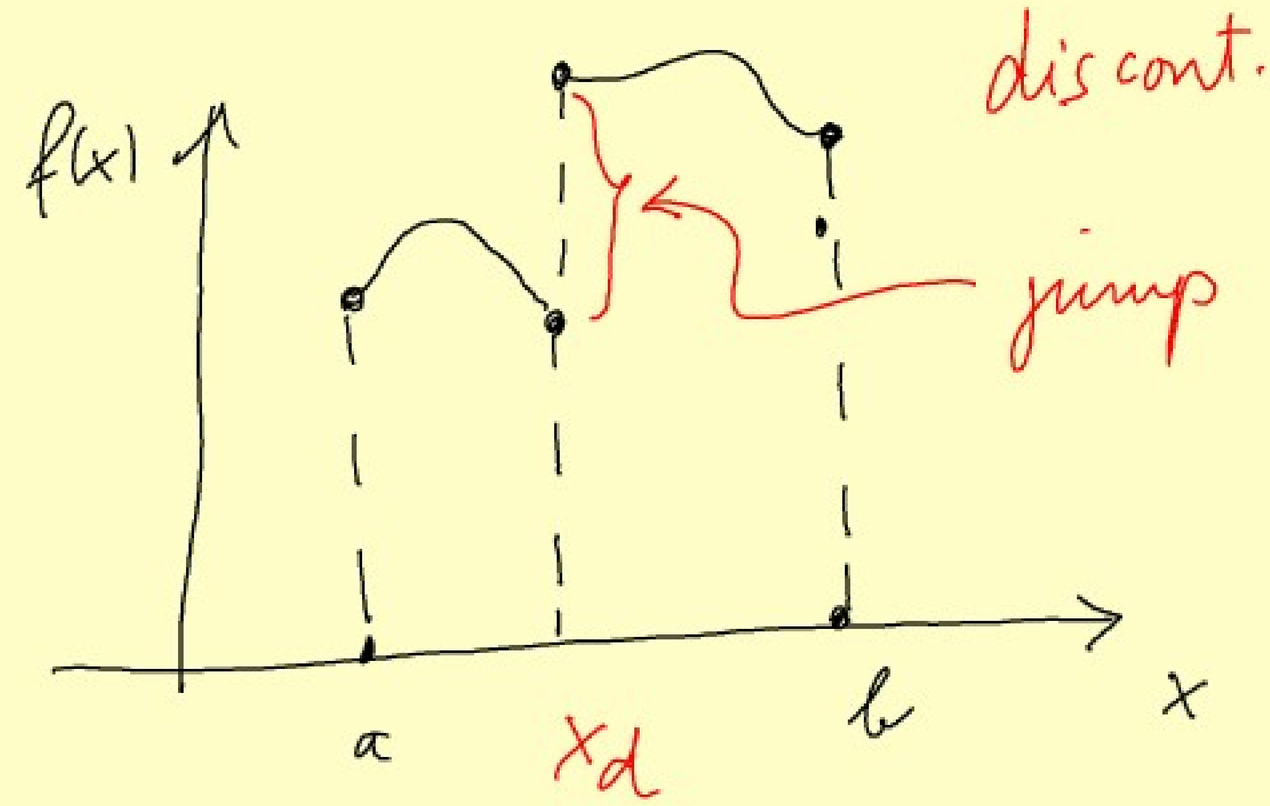
for all $x\in(a,b)$ with $|x-x^*|<\delta$.

Def $f:(a,b)\to\mathbb{R}$ is <u>cont</u> at $x^*\in(a,b)$

if $\displaystyle\lim_{x\to x^*} f(x) = f(x^*)$.

Intuitively: "f cont you can draw its graph without removing pencil from paper"





$\boxed{\mathrm{T}}$ (WEIERSTRASS ~ 1860) $f : [a, b] \to \mathbb{R}$ cont.

Then f is bounded and attains its min (m) and max (M) in the sense that

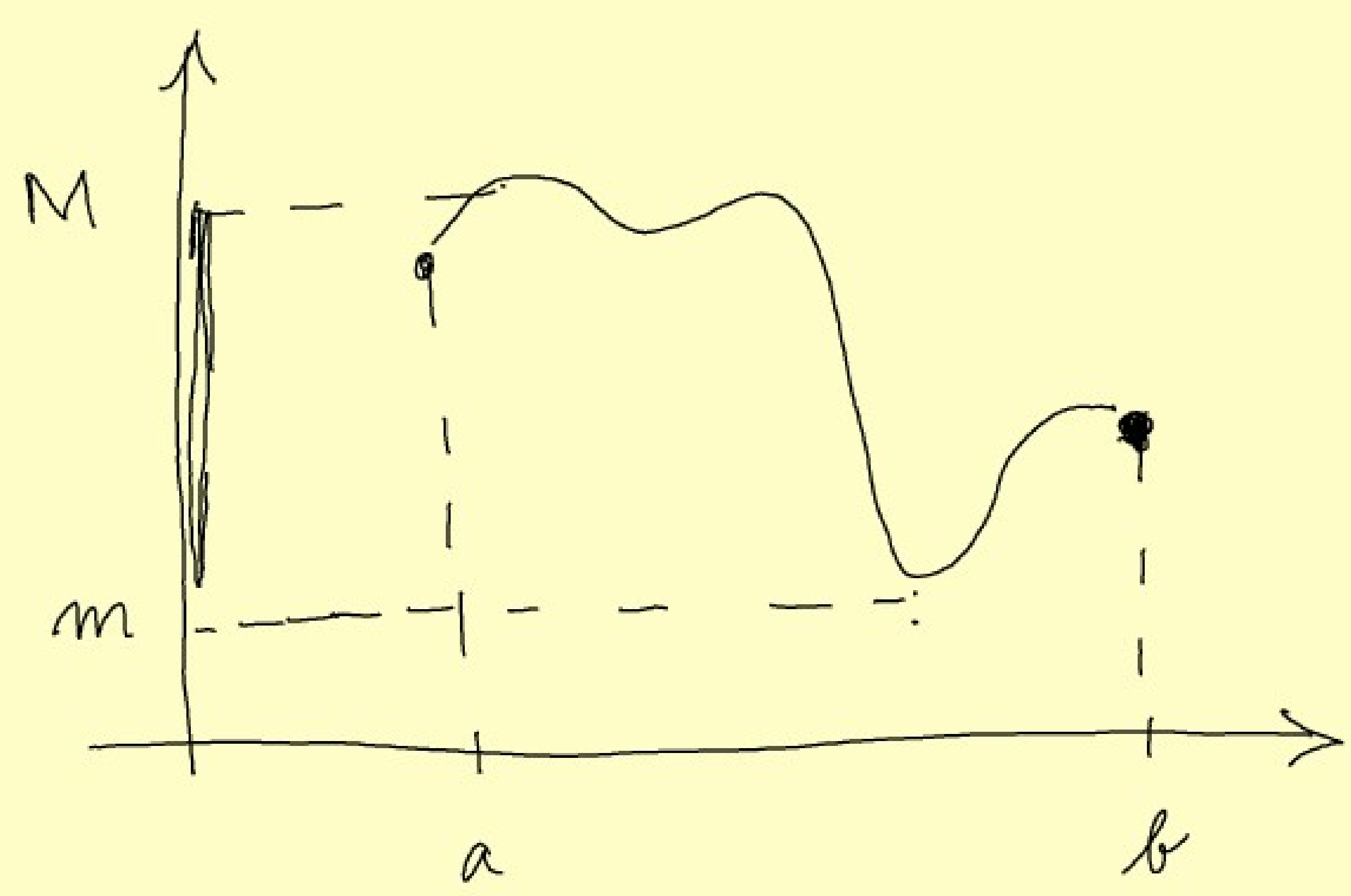
$$f([a, b]) = [m, M]$$

not. for the image of $[a, b]$ through f

$$f([a, b]) = \{ y \in \mathbb{R} : \exists x \in [a, b] \text{ s.t. } y = f(x) \}$$

Rk $\quad \forall x \in [a, b] \quad \exists y \in [m, M] \text{ s.t. } y = f(x)$

"you pass through all points between m and M"



§ 1.2. Differential Calculus and Mean-value Theorems

"Derivative" IDEA Newton (Velocity)

Def (differentiability) $f:(a,b)\to\mathbb{R}$ is diffable at $x^*\in(a,b)$ if the limit

$$\lim_{h\to 0} \frac{f(x^*+h)-f(x^*)}{h}$$ exists & is finite ($<\infty$)

Rk If the limit exists but is infinite, then we say that f has derivative at x^* (but is not diffable at x^*).

WHY derivatives? OPTIMIZATION

$\boxed{T}$ (FERMAT) $f:(a,b)\to\mathbb{R}$ has a local min/max at $x^*\in(a,b)$.
If f is diffable at x^* then $f'(x^*)=0$.

Rk The converse statement doesn't hold ($f'(x^*)=0 \not\Rightarrow x^*$ local extremum)

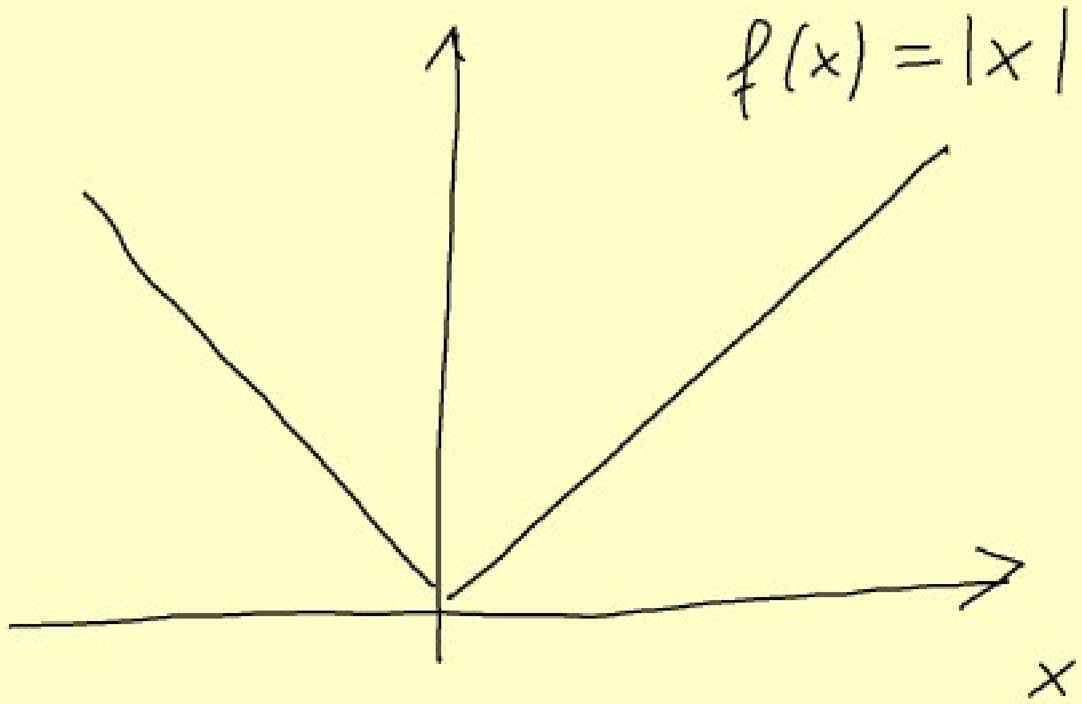
Def (local extrema) x^* is local mini/max of f

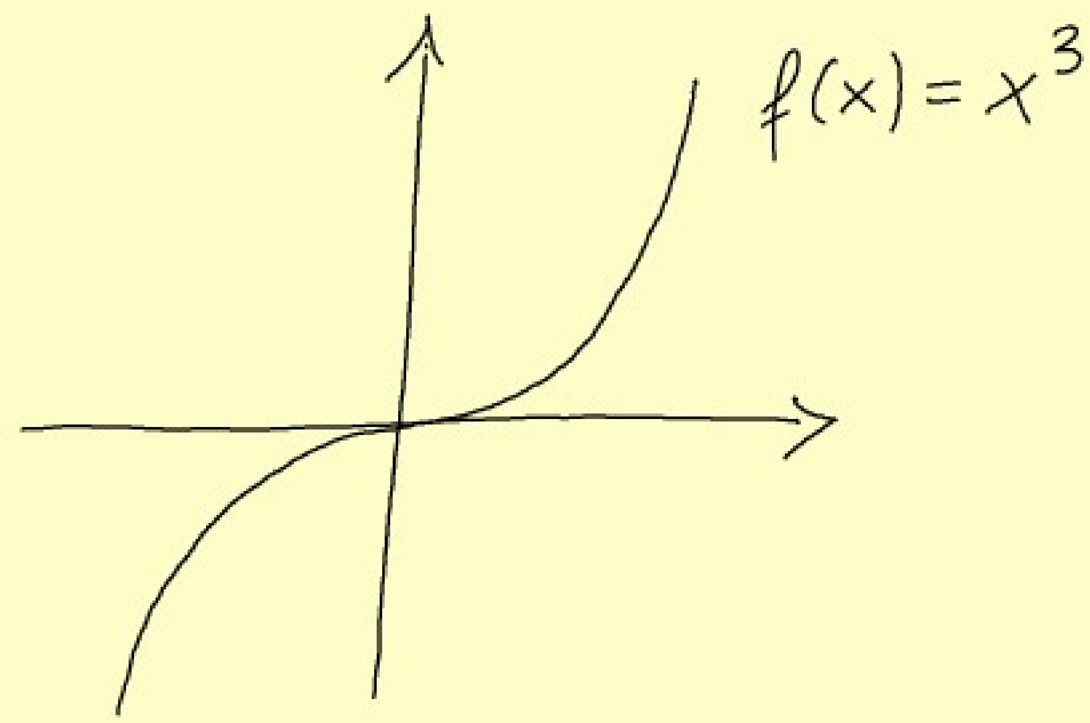
$f:(a,b)\to\mathbb{R}$ if $f(x^*) \begin{smallmatrix}\leq\\ \geq\end{smallmatrix} f(x)$

$\forall x \in [x^*-\varepsilon, x^*+\varepsilon]$, $\varepsilon > 0$.

Def (critical or stationary points)

x^* is <u>crit</u> pt. of f if $f'(x^*) = 0$.





0 is a minimum but f is <u>not</u> diffable at 0 (so 0 is not a crit.)

$f'(0) = 0$ but 0 is <u>not</u> extr.

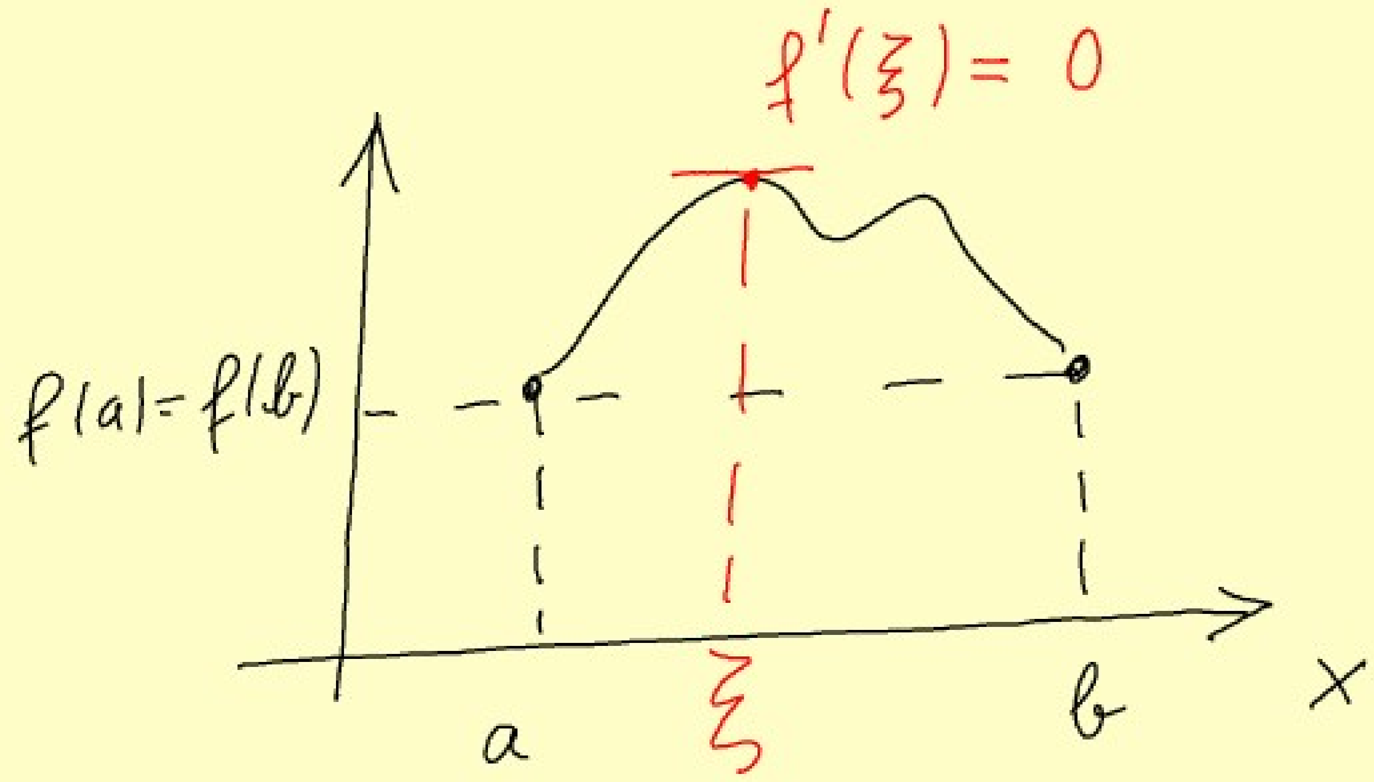
Th (ROLLE) $f : [a,b] \to \mathbb{R}$

If
- f <u>cont</u> on $[a,b]$
- f <u>diffable</u> on (a,b)
- $f(a) = f(b)$

Then $\exists\, \xi \in (a,b)$ such that

$$f'(\xi) = 0.$$

Idea of proof:
use Th Fermat
+ Th Weierstrass
[D. Popa]



! This means that f has a local min/max!

Th (LAGRANGE) $f : [a,b] \to \mathbb{R}$

If
- f <u>cont</u> on $[a,b]$
- f <u>diffable</u> on (a,b)

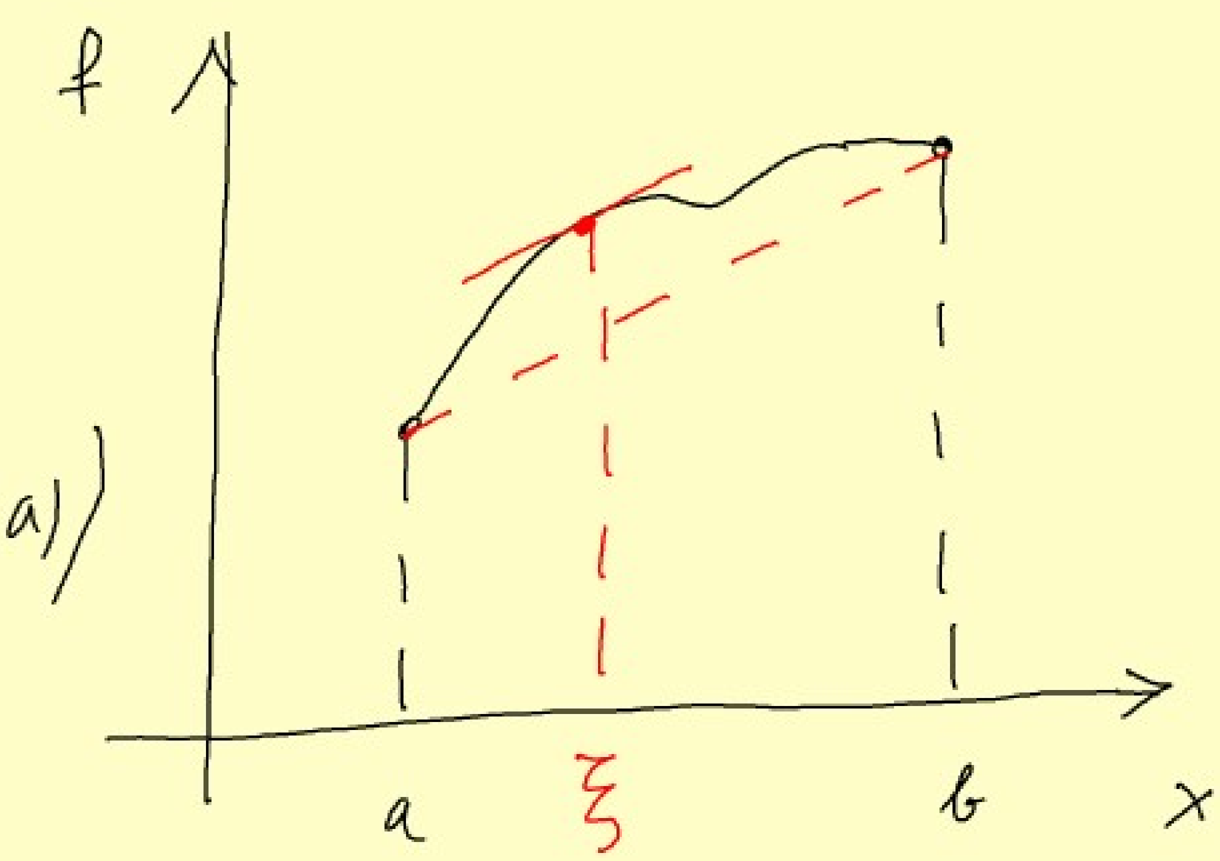
Then $\exists\, \xi \in (a,b)$ s.t.

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Idea of proof
Apply Th Rolle to

$$F(x) = (b-a) f(x) - x\,(f(b) - f(a))$$

HW: complete the proof.



Rewrite the Lagrange formula: $f(b) = f(a) + f'(\xi)(b-a)$

§ 1.3. Taylor's Formula

IDEA: approximate (nonlinear) function by polynomial

Th (Taylor) $f:(a,b)\to\mathbb{R}$ diffable $n+1$ times, $x_0 \in (a,b)$. Then, $\forall x \in (a,b)$ $\exists \xi$ between x and x_0 such that

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x) \text{ Taylor polyn.}} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\substack{R_n(x) \\ \text{Reminder}}}$$

$$f(x) = T_n(x) + R_n(x)$$

$$R_n(x) \sim \frac{1}{(n+1)!} \xrightarrow{n\to\infty} 0$$

Examples: usually $x_0 = 0$ (MacLaurin)

$$e^x \approx e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \ldots$$

$$\approx 1 + x + \frac{1}{2}x^2 + \ldots + \frac{1}{n!}x^n$$

$\approx$ approx (because we drop the reminder)

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Idea of the proof:

Step 1. Prove Taylor formula w. "integral reminder"
use integration by parts

$$(*) \quad f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \int_{x_0}^{x} f^{(n+1)}(r)\frac{(x-r)^n}{n!}\,dr$$

derivatives w.r.t. r!

$$f(x) = f(x_0) + \int_{x_0}^{x} 1 \cdot f'(r)\,dr$$

integr. by parts

$$= f(x_0) + \int_{x_0}^{x} \left(-(x-r)\right)' f'(r)\,dr$$

$$= f(x_0) - 0 + (x-x_0)f'(x_0) + \int_{x_0}^{x} \underbrace{(x-r)}_{=\frac{1}{2}\left(-(x-r)^2\right)'} f''(r)\,dr$$

and repeat integr. by parts to get (*)

Step 2. Prove that integral reminder is equiv to Lagrange reminder

$$\int_{x_0}^{x} f^{(n+1)}(r)\frac{(x-r)^n}{n!}\,dr \quad \text{vs.} \quad \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

HW: Find ξ s.t. $=$ holds HW

Hint: use 'T' Weierstrass