03.01.2022!

13. Fourier Series

What & Why?

What are the most important ideas of univ. level Math?

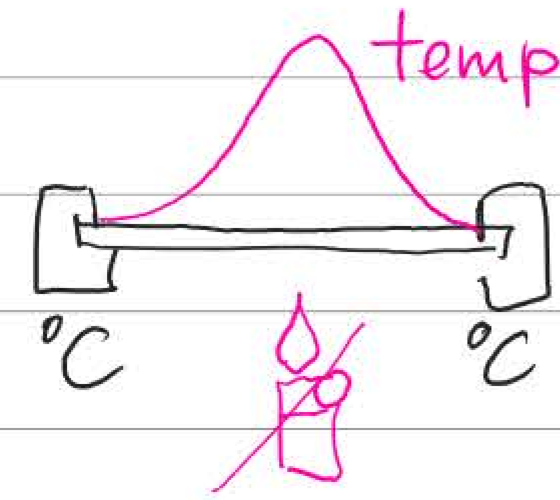
Some very bright guy said: Optimization, FOURIER analysis

signal processing, telecom, control theory

Partial Differential Equns.

It all started with J. FOURIER 1807, 1822 who studied the HEAT equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x)$$



§13.1. Small Recap: From Power Series to Fourier Series

What is a (numerical) series?
(= "infinite sum that makes sense")

$(a_n)_{n\in\mathbb{N}^*}$ sequence of reals, $s_n = a_1 + a_2 + \dots + a_n$

seq. of partial sums

$$\sum_{n=1}^{\infty} a_n \ \text{(CONV)} \iff s_n \xrightarrow{n\to\infty} S < \infty$$

the sum of the series is a number

Series of functions $f_n, f : [a,b] \to \mathbb{R}$

$$\sum_{n=1}^{\infty} f_n$$

(p CONV) "pointwise" w.r.t. x

(u CONV) unif. w.r.t. $x \in [a,b]$

The sum of the series is a function

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

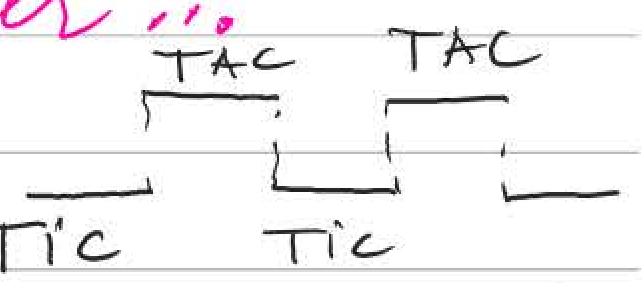
Why series of functions? (Approximation)

Taylor approx: $f(x) \approx T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

want to let $n \to \infty$ (perfect approx)

$\to$ Power series $(x_0 = 0)$ $\quad f(x) = \sum_{n=1}^{\infty} a_n x^n$ $\quad a_n \in \mathbb{R}$, $\quad f_n(x) = x^n$

FOURIER: We can do better!!!



- ask for less regularity
  (Taylor: $n+1$ cont. derivatives, $n \to \infty$)

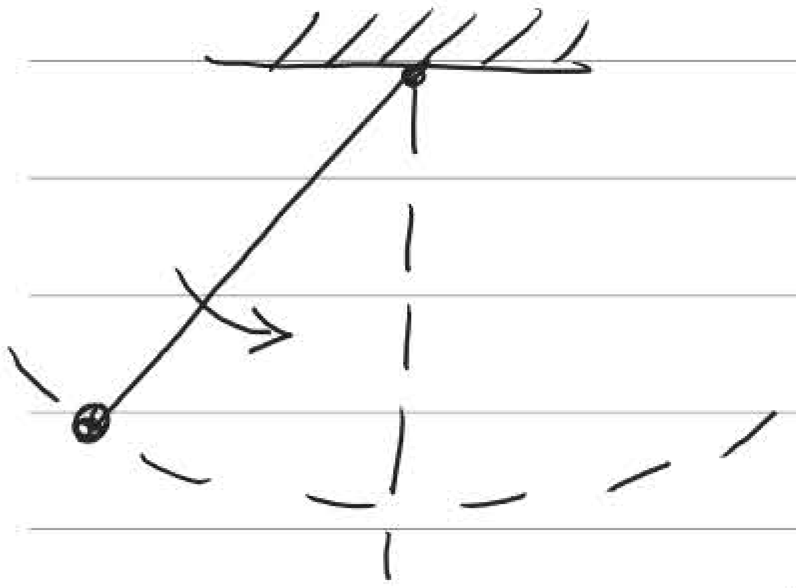
- global approx (not just x_0-dependent)

The Price: give up Polynomials, replace them by "trigonometric" polynomials

$\to$ replace Power Series by Trigonometric Series

§13.2. Fourier Series ($\mathcal{F}$-series)

Oscillations: Galileo Galilei
(first Pend. Clock!)



$$u(t) = A\cos(\omega t + \varphi)$$

amplitude, frequency, initial phase

easy trigon.
$$= a\cos\omega t + b\sin\omega t$$

The Trigonometric series ($x \in [-\pi, \pi]$)

$$(1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

is the $\mathcal{F}$-series associated to
$f : [-\pi, \pi] \to \mathbb{R}$ integrable if

$$(2)\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\[2ex] b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{cases}$$

Rk: f even then $b_n = 0$
f odd then $a_n = 0$

Rk (Counterexample) Not all trig series are $\mathcal{F}$-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx$$

NOT (u CONV)
$\nexists f$ s.t. (2) holds

F-series are trig series with assoc. "signal".

1HB3-5

T1. (BESSEL Inequality)

Let $f : [-\pi, \pi] \to \mathbb{R}$ square integrable

then $$s_n = \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx \quad (3)$$

where a_n, b_n are F-coeffs given in (2).

Idea of Proof:

Define $$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \quad (4)$$

Compute $$0 \leq \int_{-\pi}^{\pi} (f(x) - f_n(x))^2 dx =$$

$$= \int_{-\pi}^{\pi} f(x)^2 dx \qquad T_1$$

$$- 2 \int_{-\pi}^{\pi} f(x) f_n(x) dx \qquad T_2$$

$$+ \int_{-\pi}^{\pi} f_n(x)^2 dx \qquad T_3$$

using the "magic" properties of sin & cos

P2 $$\int_{-\pi}^{\pi} \cos kx \cos jx \, dx = 0 \quad \forall k, j \in \mathbb{N}^*, \ k \neq j$$

$$\int_{-\pi}^{\pi} \sin kx \sin jx \, dx = 0 \quad k \neq j$$

$$\int_{-\pi}^{\pi} \sin kx \cos jx \, dx = 0$$

$$\int_{-\pi}^{\pi} (\sin kx)^2 dx = \int_{-\pi}^{\pi} (\cos kx)^2 dx = \pi$$

HW: compute T_2 & T_3 and check (3)

1HB3-5

II 3. (PARSEVAL's Equality / Identity)

If the F-series assoc. to f (u CONV)
then (3) holds with equality as $n \to \infty$

$$(5) \quad \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx$$

What's the Deep Insight behind
this F-thing?

Square integrable functions form "space"
with good geometry (just like $\mathbb{R}^n$
but with $n \to \infty$) called Hilbert space

The operations are

"+" (sum) $\quad (f+g)(x) = f(x) + g(x)$

"αf" (scaling) $\quad (\alpha f)(x) = \alpha f(x), \quad \alpha \in \mathbb{R}$

"$\langle f, g \rangle$" $\quad \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx \in \mathbb{R}$

inner product
Scalar product (the $\mathbb{R}^d$ not was $x \cdot y$)

$\sin kx$, $\cos kx$ form an orthog base
(just like $(1, 0, 0, \dots, 0)$
$(0, 1, 0, \dots, 0)$
...
$(0, 0, \dots 0, 1, 0)$
$(0, 0, \dots 0, 0, 1)$ did in $\mathbb{R}^d$)

§13.3. The convergence of F-series

Important open question in the 19^{th} century:
When does a F-series (CONV)?

P4 (DIRICHLET'S Formula)

$$(6)\quad f_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2}\,\frac{\sin\frac{2n+1}{2}t}{\sin\frac{t}{2}}\,dt$$

f_n in (4)

see [D. Popa] for a proof.

T5 (DIRICHLET)

$f: [-\pi,\pi] \to \mathbb{R}$ piecewise-diffable ($\exists\, x_i$, $i=\overline{0,N}$, $x_0=-\pi$, $x_N=\pi$ s.t. f diff on each (x_i, x_{i+1}))

then F-series converges at any x and its sum is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \frac{f(x+0)+f(x-0)}{2}$$

left, right limits

Proof: based on P4 see [D. Popa].

§13.4. Concluding Remarks and a Remarkable Application

Rk. Gibbs' Phenomenon: there is a price to pay for discont. (in the signal); namely the F-approx will oscillate close to the discont.

1HB3-5

Application of F-series:

Compute sum of $\sum_{n=1}^{\infty} \frac{1}{n^2} = ? = \frac{\pi^2}{6}$

Idea: apply Parseval Th to $f(x) = \frac{x}{2}$

use (2) to see that $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{2} \sin nx \, dx \overset{\substack{\text{HW}\\ \downarrow}}{=} \frac{(-1)^{n+1}}{n}$$

(5) $$\underbrace{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}_{} = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx}_{}$$

reduces to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad \frac{1}{\pi} 2 \int_0^{\pi} \frac{x^2}{4} \, dx = \frac{1}{\pi} \frac{x^3}{6}\Big|_0^{\pi} = \frac{\pi^2}{6}$