Babeş-Bolyai University, Faculty of Mathematics and Computer Science
Computer Science Groups 911-917, Academic Year 2021-2022

Mathematical Analysis
Exercise Sheet 2

6. Study the continuity of

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point in $\mathbb{R}$ and $|f|$ is continuous on $\mathbb{R}$.

8. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that f has at least one fixed point $x^* \in [0,1]$, that is, $f(x^*) = x^*$.

9. Let $f : D \to \mathbb{R}$ be a function, defined on a nonempty set $D \subseteq \mathbb{R}$, and let S be a nonempty set of D. We say that f is *Lipschitzian on* S if there exists a real number $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in S.$$

Prove that:
a) If f is Lipschitzian on S, then f is continuous on S.
b) If f is differentiable on an interval S and $f' : S \to \mathbb{R}$ is bounded, then f is Lipschitzian on S.
c) The function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, is not Lipschitzian on $[0, +\infty)$. However, f is Lipschitzian on $[a, +\infty)$ for any $a > 0$.

Homework

HW 10. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$

How many times is this function differentiable?

HW 11. Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[3]{x}$, is not differentiable at 0 although its derivative at 0 exists.

HW 12. Let $f,\ g : [0,1] \to \mathbb{R}$ be two continuous functions, such that $f(x) = g(x)$, $\forall x \in [0,1] \cap \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in [0,1]$.