Recap Part I (Diff Calculus)

Main Objective: Optimization (relevant to Machine Learning)

Lecture 1: Diff Calculus in dim = 1

Main (new) Result: TAYLOR's Thm

You can approx a smooth enough function by polinomials

Lecture 2: Geometry of $\mathbb{R}^d$ $(d > 1)$

$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_d y_d$

inner / scalar product

allows us to compute distances, angels / orthogon / etc.

Lecture 3: Derivatives for $f : \mathbb{R}^d \to \mathbb{R}$

The Gradient ∇f, Hesse matrix $\nabla^2 f$

Chain Rule (Don't try to prove the Thm / My bad!)

Lecture 4: Optimization, <u>Least</u> Squares, ML

Lecture 5: Contraint Optimization (Lagrange Multiplier Method), (Planar) Curves and the Implicit Function Thm.

Part II : Integral Calculus

6. Antiderivatives & the Riemann Integral

Def $f : I \subset \mathbb{R} \to \mathbb{R}$

F is an antideriv of f if $F' = f$

The set of all antiderivatives of f is called integral of f and by

$\int f(x)\,dx = F(x) + C$

(two antiderivs differ only by a const)

Th 1. f cont on I then f has antiderivs.

§ 6.1. Nonelementary function & antiderivatives

elementary functions = constructed via a finite nr of operations applied to x^α, a^x, $\sin x$

The Fundam. difference between differential and integral calculus is that

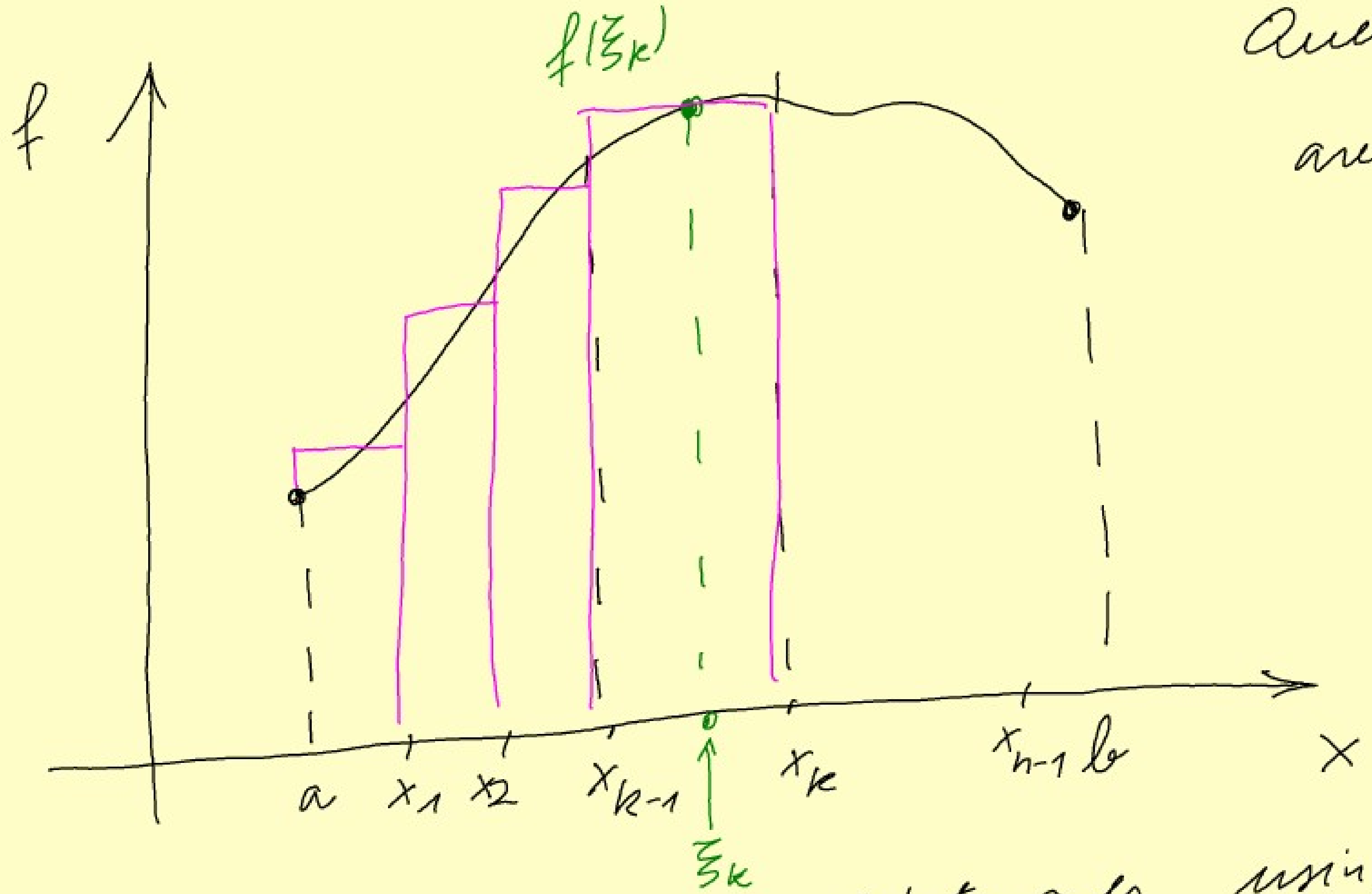
- the derivative of an elementary f is elementary
- the antideriv. of an elem. f. need not be elementary!

There no algorithm capable of deciding whether f has elementary antideriv or not!

BAD $\int e^x/x \, dx$, $\int e^{-x^2} dx$, $\int 1/\ln x \, dx$???

§ 6.2. The Riemann integral

Bernhard RIEMANN (1826 – 1866)



Question:

area under graph of $f = ?$

$\Delta = \{x_0, \dots, x_n\}$

$x_0 = a < x_1 < \dots < x_n = b$

divide $[a,b]$ in subintervals using

in each subinterval pick one intermediate point

$\xi = \{\xi_1, \dots, \xi_n\}$, $\xi_k \in [x_{k-1}, x_k]$

Area under graph (one slice) $\approx f(\xi_k)(x_k - x_{k-1})$

$$\sigma(f;\Delta;\xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

the sum of areas of all slices

take n larger & larger $\Rightarrow$ better & better approx.

Def: $f : [a,b] \to \mathbb{R}$ is called Riemann integrable if $\exists\, I \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ with the prop. that for any Δ of $[a,b]$ with $\max_k |x_k - x_{k-1}| < \delta$ and any ξ we have $\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < \varepsilon$.

T2. f cont $\Rightarrow$ f Riemann integrable

T3. (LEIBNIZ - NEWTON) Fundam Thm of Calculus

If f integrable and admits antideriv

then $$\underbrace{\int_a^b f(x)\,dx}_{\text{Riemann integral (Notation)}} = F(b) - F(a)$$

T4. If f cont. then

$$F(x) = \int_a^x f(r)\,dr \text{ is antideriv of } f.$$

Properties

- f, g integrable $\Rightarrow$ $\alpha f + \beta g$ integrable $\int(\alpha f+\beta g) = \alpha\int f + \beta\int g$
- $f(x) \le g(x)$ $\forall x \in [a,b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
- f integrable $\Rightarrow$ $|f|$ integrable $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$
- f cont, g integrable $\Rightarrow \exists c$ a.i. $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$
- integration by parts
- change variables f cont, u diffable

$$\int_a^b f(u(t))\,u'(t)\,dt = \int_{u(a)}^{u(b)} f(x)\,dx$$

Rk. Integrable functions need not be continuous