Course 4: 25.10.2021

2.3 Generated subspace

For a vector space V over K, we denote by $S(V)$ the set of all subspaces of V. Sometimes, this set is denoted by $S_K(V)$ if we like to emphasize the field K.

Theorem 2.3.1 *Let V be a vector space over K and let $(S_i)_{i\in I}$ be a family of subspaces of V. Then $\bigcap_{i\in I} S_i \in S(V)$.*

Proof. For each $i \in I$, we have $S_i \in S(V)$, hence $0 \in S_i$. Then $0 \in \bigcap_{i\in I} S_i \neq \emptyset$. Now let $k_1, k_2 \in K$ and $x, y \in \bigcap_{i\in I} S_i$. Then $x, y \in S_i$, $\forall i \in I$. But $S_i \in S(V)$, $\forall i \in I$. It follows that $k_1x + k_2y \in S_i$, $\forall i \in I$, hence $k_1x + k_2y \in \bigcap_{i\in I} S_i$. Therefore, $\bigcap_{i\in I} S_i \in S(V)$. $\square$

Remark 2.3.2 In general, the union of two subspaces of a vector space is not a subspace. For instance, $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$ are subspaces of the canonical real vector space $\mathbb{R}^2$, but $S\cup T$ is not a subspace of $\mathbb{R}^2$. Indeed, for instance, we have $(1, 0), (0, 1) \in S\cup T$, but $(1, 0)+(0, 1) = (1, 1) \notin S\cup T$.

Now we are interested in how to “complete” a given subset of a vector space to a subspace in a minimal way. This is the motivation for the following definition.

Definition 2.3.3 Let V be a vector space and let $X \subseteq V$. Then we denote

$$\langle X\rangle = \bigcap\{S \leq V \mid X \subseteq S\}$$

and we call it the *subspace generated by X* or the *subspace spanned by X*.

Here X is called the *generating set* of $\langle X\rangle$.

If $X = \{x_1, \dots, x_n\}$, we denote $\langle x_1, \dots, x_n\rangle = \langle\{x_1, \dots, x_n\}\rangle$.

Remark 2.3.4 (1) $\langle X\rangle$ is the “smallest” (with respect to inclusion) subspace of V containing X.

(2) $\langle\emptyset\rangle = \{0\}$.

(3) If $S \leq V$, then $\langle S\rangle = S$.

Definition 2.3.5 A vector space V over K is called *finitely generated* if $\exists x_1, \dots, x_n \in V$ $(n \in \mathbb{N})$ such that $V = \langle x_1, \dots, x_n\rangle$. Then the set $\{x_1, \dots, x_n\}$ is called a *system of generators for V*.

Definition 2.3.6 Let V be a vector space over K and $x_1, \dots, x_n \in V$ $(n \in \mathbb{N})$. A finite sum of the form

$$k_1x_1 + \dots + k_nx_n\,,$$

where $k_i \in K$, $x_i \in X$ $(i = 1, \dots, n)$, is called a (finite) *linear combination* of the vectors $x_1, \dots, x_n$.

Let us now determine how the elements of a generated subspace look like.

Theorem 2.3.7 *Let V be a vector space over K and let $\emptyset \neq X \subseteq V$. Then*

$$\langle X\rangle = \{k_1x_1 + \dots + k_nx_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\}\,,$$

that is, the set of all finite linear combinations of vectors of X.

Proof. We prove the result in 3 steps, by showing that

$$L = \{k_1x_1 + \dots + k_nx_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \dots, n\,,\ n \in \mathbb{N}^*\}$$

is the smallest subspace of V containing X.

(i) Let $x \in X$. Then $x = 1 \cdot x \in L$, hence $L \neq \emptyset$. Now let $k, k' \in K$ and $v, v' \in L$. Then $v = \sum_{i=1}^{n} k_i x_i$ and $v' = \sum_{j=1}^{m} k'_j x'_j$ for some $k_1, \ldots, k_n, k'_1, \ldots, k'_m \in K$ and $x_1, \ldots, x_n, x'_1, \ldots, x'_m \in X$. Hence

$$kv + k'v' = k \sum_{i=1}^{n} k_i x_i + k' \sum_{j=1}^{m} k'_j x'_j = \sum_{i=1}^{n} (kk_i) x_i + \sum_{j=1}^{m} (k'k'_j) x'_j \in L\,,$$

because it is a finite linear combination of vectors of X. Hence we have $L \leq V$.

(ii) Choose $n = 1$ and $k_1 = 1$ in order to see that $X \subseteq L$.

(iii) Let $S \leq V$ be such that $X \subseteq S$. Let $k_1, \ldots, k_n \in K$ and $x_1, \ldots, x_n \in X$. Since $X \subseteq S$ and $S \leq V$, it follows that $k_1 x_1 + \cdots + k_n x_n \in S$. Hence $L \subseteq S$.

Thus, we have $\langle X \rangle = L$ by the remark from the beginning of the proof. $\square$

Corollary 2.3.8 *Let V be a vector space over K and let $x_1, \ldots, x_n \in V$. Then*

$$\langle x_1, \ldots, x_n \rangle = \{k_1 x_1 + \cdots + k_n x_n \mid k_i \in K\,,\ x_i \in X\,, i = 1, \ldots, n\}\,.$$

Example 2.3.9 (a) Consider the canonical real vector space $\mathbb{R}^3$. Then

$$\begin{aligned}
\langle (1,0,0), (0,1,0), (0,0,1) \rangle &= \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\} \\
&= \{(k_1,0,0) + (0,k_2,0) + (0,0,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} \\
&= \{(k_1,k_2,k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\} = \mathbb{R}^3\,.
\end{aligned}$$

Hence $\mathbb{R}^3$ is generated by the three vectors.

(b) Consider the canonical vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$. Then

$$\begin{aligned}
\langle (\widehat{1},\widehat{0},\widehat{0}), (\widehat{0},\widehat{1},\widehat{0}) \rangle &= \{k_1(\widehat{1},\widehat{0},\widehat{0}) + k_2(\widehat{0},\widehat{1},\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} \\
&= \{(k_1,\widehat{0},\widehat{0}) + (\widehat{0},k_2,\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} = \{(k_1,k_2,\widehat{0}) \mid k_1, k_2 \in \mathbb{Z}_2\} \neq \mathbb{Z}_2^3\,.
\end{aligned}$$

Hence $\mathbb{Z}_2^3$ is not generated by the two vectors $(\widehat{1},\widehat{0},\widehat{0})$ and $(\widehat{0},\widehat{1},\widehat{0})$. But it is generated by $(\widehat{1},\widehat{0},\widehat{0})$, $(\widehat{0},\widehat{1},\widehat{0})$ and $(\widehat{0},\widehat{0},\widehat{1})$, hence it is finitely generated.

(c) Consider the subspace $S = \{(x,y,z) \in \mathbb{R}^3 \mid x - y - z = 0\}$ of the canonical real vector space $\mathbb{R}^3$. Let us write it as a generated subspace. Expressing $x = y + z$, we have:

$$\begin{aligned}
S &= \{(y+z,y,z) \mid y,z \in \mathbb{R}\} = \{(y,y,0) + (z,0,z) \mid y,z \in \mathbb{R}\} \\
&= \{y(1,1,0) + z(1,0,1) \mid y,z \in \mathbb{R}\} = \langle (1,1,0), (1,0,1) \rangle.
\end{aligned}$$

Alternatively, one may express y or z by using the other two components and get other writings of S as a generated subspace.

Definition 2.3.10 Let V be a vector space over K and let $S, T \leq V$. Then we define the *sum* of the subspaces S and T as the set $S + T = \{s + t \mid s \in S\,,\ t \in T\}$.

If $S \cap T = \{0\}$, then $S + T$ is denoted by $S \oplus T$ and is called the *direct sum* of the subspaces S and T.

Theorem 2.3.11 *Let V be a vector space over K and let $S, T \leq V$. Then $S + T = \langle S \cup T \rangle$.*

Proof. First, let $v = s + t \in S + T$, for some $s \in S$ and $t \in T$. Then $v = 1 \cdot s + 1 \cdot t$ is a linear combination of the vectors $s, t \in S \cup T$, hence $v \in \langle S \cup T \rangle$. Thus, $S + T \subseteq \langle S \cup T \rangle$.

Now let $v \in \langle S \cup T \rangle$. Then

$$v = \sum_{i=1}^{n} k_i v_i = \sum_{i \in I} k_i v_i + \sum_{j \in J} k_j v_j\,,$$

where $I = \{i \in \{1, \ldots, n\} \mid v_i \in S\}$ and $J = \{j \in \{1, \ldots, n\} \mid v_j \in T \setminus S\}$. But the first sum is a linear combination of vectors of S, hence it belongs to S, whereas the second sum is a linear combination of vectors of T, hence it belongs to T. Thus, $v \in S + T$ and consequently $\langle S \cup T \rangle \subseteq S + T$.

Therefore, $S + T = \langle S \cup T \rangle$. $\square$

Corollary 2.3.12 *Let V be a vector space over K and let $S, T \leq V$. Then $S+T \leq V$.*

Proof. By Theorem 2.3.11. □

Theorem 2.3.13 *Let V be a vector space over K and let $S, T \leq V$. Then*

$$V = S \oplus T \Longleftrightarrow \forall v \in V\,, \exists! s \in S\,, t \in T\ : v = s + t\,.$$

Proof. $\Longrightarrow$. Assume that $V = S \oplus T$. Let $v \in V$. Then $\exists s \in S\,, t \in T$ such that $v = s + t$. Now suppose that $\exists s' \in S\,, t' \in T$ such that $v = s' + t'$. Then $s + t = s' + t'$, whence $s - s' = t' - t \in S \cap T = \{0\}$. Hence $s = s'$ and $t = t'$, that show the uniqueness.

$\Longleftarrow$. Assume that $\forall v \in V$, $\exists! s \in S\,, t \in T$ such that $v = s + t$. Then $V \subseteq S + T$. Clearly, we have $S + T \subseteq V$ and consequently $V = S + T$. Now suppose that $0 \neq v \in S \cap T$. Then $v = v + 0 = 0 + v$. But this is a contradiction, since we have the uniqueness of writing of v as a sum of an element of S and an element of T. Therefore, $S \cap T = \{0\}$ and thus, $V = S \oplus T$. □

Example 2.3.14 Consider the canonical real vector space $\mathbb{R}^2$. Then $\mathbb{R}^2 = S \oplus T$, where $S = \{(x, 0) \mid x \in \mathbb{R}\}$ and $T = \{(0, y) \mid y \in \mathbb{R}\}$.

2.4 Linear maps

Definition 2.4.1 Let V and V' be vector spaces over K. A map $f : V \to V'$ is called:

(1) *(K-)linear map* (or *(vector space) homomorphism* or *linear transformation*) if

$$f(v_1 + v_2) = f(v_1) + f(v_2)\,, \quad \forall v_1, v_2 \in V,$$

$$f(kv) = kf(v)\,, \quad \forall k \in K, \forall v \in V.$$

(2) *isomorphism* if it is a bijective K-linear map;
(3) *endomorphism* if it is a K-linear map and $V = V'$;
(4) *automorphism* if it is a bijective K-linear map and $V = V'$.

Remark 2.4.2 (1) When defining a K-linear map, we consider vector spaces over the same field K.

(2) If $f : V \to V'$ is a K-linear map, then the first condition from its definition tells us that f is a group homomorphism between $(V, +)$ and $(V', +)$. Then we have $f(0) = 0'$ and $f(-v) = -f(v)$, $\forall v \in V$.

We denote by $V \simeq V'$ the fact that two vector spaces V and V' are isomorphic. We also denote

$$Hom_K(V, V') = \{f : V \to V' \mid f \text{ is } K\text{-linear}\}\,,$$

$$End_K(V) = \{f : V \to V \mid f \text{ is } K\text{-linear}\}\,,$$

$$Aut_K(V) = \{f : V \to V \mid f \text{ is bijective } K\text{-linear}\}\,.$$

Let us now give a characterization theorem for linear maps.

Theorem 2.4.3 *Let V and V' be vector spaces over K and $f : V \to V'$. Then*

$$f \text{ is a } K\text{-linear map} \Longleftrightarrow f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2)\,,$$

$\forall k_1, k_2 \in K$, $\forall v_1, v_2 \in V$.

Proof. $\Longrightarrow$. Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. Then

$$f(k_1v_1 + k_2v_2) = f(k_1v_1) + f(k_2v_2) = k_1f(v_1) + k_2f(v_2)\,.$$

$\Longleftarrow$. Choose $k_1 = k_2 = 1$ and then $k_2 = 0$ to get the two conditions of a K-linear map. □

Example 2.4.4 (a) Let V and V' be vector spaces over K and let $f : V \to V'$ be defined by $f(v) = 0'$, $\forall v \in V$. Then f is a K-linear map, called the *trivial linear map*.

(b) Let V be a vector space over K. Then the identity map $1_V : V \to V$ is an automorphism of V.

(c) Let V be a vector space and $S \le V$. Define $i : S \to V$ by $i(v) = v$, $\forall v \in S$. Then i is a K-linear map, called the *inclusion linear map*.

(d) Let V be a vector space over K and $a \in K$. Define $t_a : V \to V$ by $t_a(v) = av$, $\forall v \in V$. Then t_a is an endomorphism of V.

Theorem 2.4.5 *(i) Let $f : V \to V'$ be an isomorphism of vector spaces over K. Then $f^{-1} : V' \to V$ is again an isomorphism of vector spaces over K.*

(ii) Let $f : V \to V'$ and $g : V' \to V''$ be K-linear maps. Then $g \circ f : V \to V''$ is a K-linear map.

Proof. (i) Since f is an isomorphism of vector spaces over K, f is bijective, hence so is f^{-1}.

Now let $k_1, k_2 \in K$ and $v_1', v_2' \in V'$. We have to prove that

$$f^{-1}(k_1v_1' + k_2v_2') = k_1f^{-1}(v_1') + k_2f^{-1}(v_2').$$

Let us denote $v_1 = f^{-1}(v_1')$ and $v_2 = f^{-1}(v_2')$. Then $f(v_1) = v_1'$ and $f(v_2) = v_2'$, hence

$$k_1v_1' + k_2v_2' = k_1f(v_1) + k_2f(v_2) = f(k_1v_1 + k_2v_2).$$

Thus we have

$$f^{-1}(k_1v_1' + k_2v_2') = k_1v_1 + k_2v_2 = k_1f^{-1}(v_1') + k_2f^{-1}(v_2').$$

Hence f^{-1} is an isomorphism of vector spaces over K.

(ii) Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$. We have:

$$\begin{aligned}(g \circ f)(k_1v_1 + k_2v_2) &= g(f(k_1v_1 + k_2v_2)) = g(k_1f(v_1) + k_2f(v_2)) \\ &= k_1g(f(v_1)) + k_2g(f(v_2)) = k_1(g \circ f)(v_1) + k_2(g \circ f)(v_2).\end{aligned}$$

Hence $g \circ f$ is a K-linear map. □

Definition 2.4.6 Let $f : V \to V'$ be a K-linear map. Then the sets

$$\operatorname{Ker} f = \{v \in V \mid f(v) = 0'\}, \quad \operatorname{Im} f = \{f(v) \mid v \in V\}$$

are called the *kernel* and the *image* of the K-linear map f respectively.

Theorem 2.4.7 *Let $f : V \to V'$ be a K-linear map. Then* $\operatorname{Ker} f \le V$ *and* $\operatorname{Im} f \le V'$.

Proof. First, note that $f(0) = 0'$, hence $0 \in \operatorname{Ker} f \ne \emptyset$. Let $k_1, k_2 \in K$ and $v_1, v_2 \in \operatorname{Ker} f$. We prove that $k_1v_1 + k_2v_2 \in \operatorname{Ker} f$. Indeed, we have:

$$f(k_1v_1 + k_2v_2) = k_1f(v_1) + k_2f(v_2) = 0',$$

and so $k_1v_1 + k_2v_2 \in \operatorname{Ker} f$. Hence $\operatorname{Ker} f \le V$.

Now note that $0' = f(0) \in \operatorname{Im} f \ne \emptyset$. Let $k_1, k_2 \in K$ and $v_1', v_2' \in \operatorname{Im} f$. We prove that $k_1v_1' + k_2v_2' \in \operatorname{Im} f$. We have $v_1' = f(v_1)$ and $v_2' = f(v_2)$ for some $v_1, v_2 \in V$. It follows that

$$k_1v_1' + k_2v_2' = k_1f(v_1) + k_2f(v_2) = f(k_1v_1 + k_2v_2) \in \operatorname{Im} f.$$

Hence $\operatorname{Im} f \le V'$. □

Theorem 2.4.8 *Let $f : V \to V'$ be a K-linear map and let $X \subseteq V$. Then $f(\langle X \rangle) = \langle f(X) \rangle$.*

Proof. If $X = \emptyset$, then we have $f(\langle \emptyset \rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \langle \emptyset \rangle = \langle f(\emptyset) \rangle$.

Now assume that $X \ne \emptyset$. By Theorem 2.3.7 we have

$$\langle X \rangle = \{k_1x_1 + \dots + k_nx_n \mid k_i \in K,\ x_i \in X, i = 1, \dots, n,\ n \in \mathbb{N}^*\}.$$

Since f is a K-linear map, it follows by Theorem 2.4.3 that

$$\begin{aligned}f(\langle X \rangle) &= \{f(k_1x_1 + \dots + k_nx_n) \mid k_i \in K,\ x_i \in X, i = 1, \dots, n,\ n \in \mathbb{N}^*\} \\ &= \{k_1f(x_1) + \dots + k_nf(x_n) \mid k_i \in K,\ x_i \in X, i = 1, \dots, n,\ n \in \mathbb{N}^*\} \\ &= \langle f(X) \rangle.\end{aligned}$$

□

Extra: Image crossfade

A black-and-white image of (say) $n = 1024 \times 768$ pixels can be viewed as a vector in the real canonical vector space $\mathbb{R}^n$, where each component of the vector is the intensity of the corresponding pixel.

Let us consider two vectors representing images:

$v_1 =$ , $v_2 =$ .

Now consider the following intermediate images:

The vectors corresponding to the above images are the following linear combinations of the vectors v_1 and v_2:

$$v_1, \quad \frac{8}{9}v_1+\frac{1}{9}v_2, \quad \frac{7}{9}v_1+\frac{2}{9}v_2, \quad \frac{6}{9}v_1+\frac{3}{9}v_2, \quad \frac{5}{9}v_1+\frac{4}{9}v_2, \quad \frac{4}{9}v_1+\frac{5}{9}v_2, \quad \frac{3}{9}v_1+\frac{6}{9}v_2, \quad \frac{2}{9}v_1+\frac{7}{9}v_2, \quad \frac{1}{9}v_1+\frac{8}{9}v_2, \quad v_2.$$

One may use these images as frames in a video in order to get a crossfade effect.

Reference: P.N. Klein, Coding the Matrix. Linear Algebra through Applications to Computer Science, *Newtonian Press, 2013.*