6.12.2021

10. Extensions of the Riemann Integral: Improper integrals (w. a parameter).

Famous model: ideal gas
MAXWELL 1860, BOLTZMANN 1872, 1877

example $\displaystyle\int_0^{\infty} x^2 e^{-x^2}\,dx = ?$

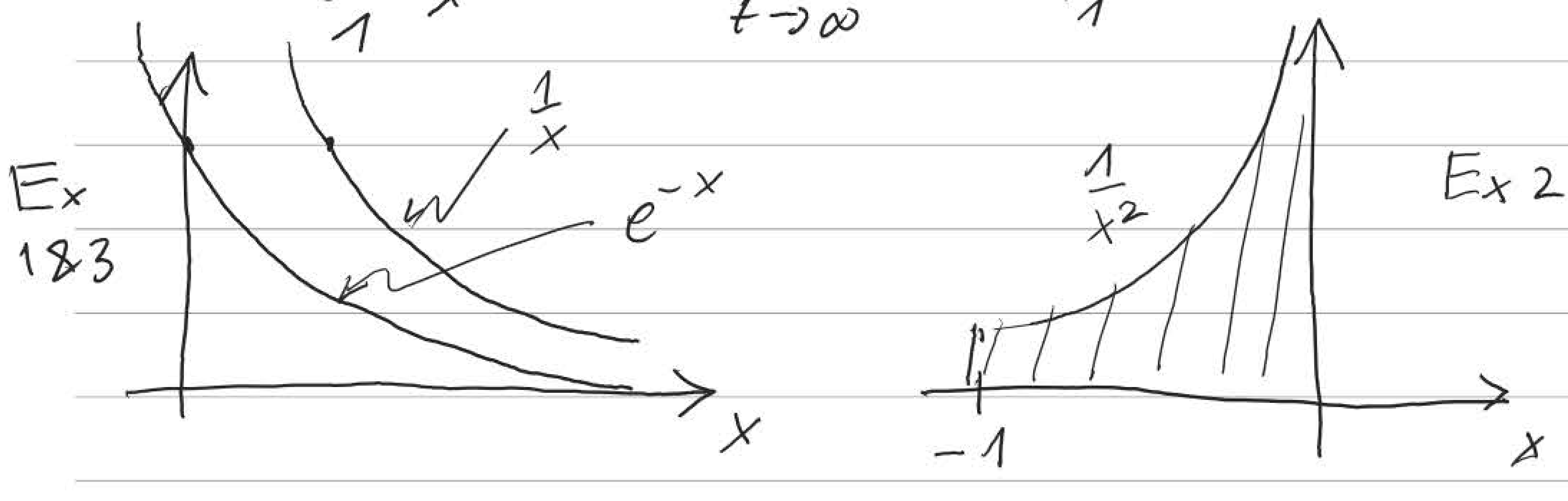
§ 10.1. Improper integrals

"Improper integrals are limits"

Ex. 1. $\displaystyle\int_0^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = 1$

Ex. 2. $\displaystyle\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{\substack{t\to 0\\ t<0}} \int_{-1}^{t} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \left(-\frac{1}{x}\Big|_{-1}^{t}\right) = \infty$

Ex. 3. $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x \Big|_1^t = \infty$



Def: $f : [a, b) \to \mathbb{R}$ ($b = \infty$ admitted!)
integrable (Riemann) on any $[a, \bar b]$, $\bar b < b$ (locally integrable)

we call $\underbrace{\int_a^b f(x)\,dx}_{\text{the improper}}$ convergent (CONV) if $\displaystyle\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx \ \exists < \infty$.

otherwise (DIV)

T 1. (CAUCHY) $f:[a,b)\to\mathbb{R}$ locally integrable

$$\int_a^b f(x)\,dx \text{ (CONV)} \iff \forall \varepsilon>0\ \exists b_\varepsilon<b$$

$$\underbrace{\text{such that}}_{\forall t\in(b_\varepsilon,b)} \left|\int_{b_\varepsilon}^t f(x)\,dx\right| < \varepsilon.$$

[Meaning: (CONV) $\iff \int_{b_\varepsilon}^t$ can be made arbitr. small.

§10.2. Testing the convergence of improper integrals

T 2. (comparison I)

$f, g:[a,b)\to\mathbb{R}$ locally integrable

and $0\le f(x)\le g(x)\quad \forall x\in[a,b)$

Then (i) $\int_a^b g \text{ (CONV)} \Rightarrow \int_a^b f \text{ (CONV)}$

(ii) $\int_a^b f \text{ (DIV)} \Rightarrow \int_a^b g \text{ (DIV)}$

T 3. (comparison II) $g(x)>0$ on $[a,b)$

and $\lim\limits_{x\to b}\dfrac{f(x)}{g(x)} = L\in\mathbb{R}\ (<\infty)$

Then (i) if $L\neq 0$ $\int f$ & $\int g$ are both (DIV) or both (CONV)

(ii) if $L=0$ $\int g$ CONV $\Rightarrow \int |f|$ (CONV)

absolute (CONV)

Rk. Usually, you compare with x^α

$$\int_1^\infty x^\alpha \, dx \quad \begin{cases} \text{(CONV) for } \alpha < -1 \\ \text{(DIV) for } \alpha \geq -1 \end{cases}$$

$$\int_0^1 x^\beta \, dx \quad \begin{cases} \text{(CONV) for } \beta > -1 \\ \text{(DIV) for } \beta \leq -1 \end{cases}$$

HW: Prove these

Ex 4. $\int_1^\infty \frac{\sin x}{x^2} dx$ (CONV) because

$$\left| \frac{\sin x}{x^2} \right| \leq x^{-2} \quad \text{and} \quad \int_1^\infty x^{-2} \text{ (CONV)}.$$

Ex 5. $\int_0^\infty e^{-x} (\sin x)^3 dx$ (CONV)

because $\left| e^{-x} (\sin x)^3 \right| \leq e^{-x}$

and $\int_0^\infty e^{-x} dx = 1$ (CONV).

Rk. (should be placed after "II" Cauchy)

$$\int_a^b |f(x)| \, dx \text{ (CONV)} \Rightarrow \int_a^b f(x) \, dx \text{ (CONV)}.$$

[absolute CONV $\Rightarrow$ CONV].

Rk. $(a, b]$, (a, b) work the same way

HW: $\int_{-\infty}^\infty e^{-|x|} dx$

1HB3-5

Rk. Int. by parts & change of var both work if the improper integral is (CONV).

§10.3. Improper integrals with parameter

$$f : [a, b) \times [c, d] \longrightarrow \mathbb{R}$$

$$F : [c, d] \longrightarrow \mathbb{R}$$

improper   with parameter

$$F(y) = \int_a^b f(x, y)\, dx$$

So: "Improper $\int$ with parameter are functions (of y)"

Def The improper integral w. param.

$\int_a^b f(x,y)\,dx$ converges uniformly to F (w.r.t. y)

if $\forall \varepsilon > 0\ \exists b_\varepsilon < b : \forall t \in (b_\varepsilon, b)$ and $\forall y \in [c, d]$ we have

$$\left| \int_{b_\varepsilon}^{t} f(x,y)\,dx - F(y) \right| < \varepsilon \quad \text{and} \quad b_\varepsilon \neq b(y)$$

indep of y

Notation: (u CONV)

Ex. 6 (Euler Gamma function)

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx,$$

$n \in \mathbb{N}$
$\Gamma(n+1) = n!$
$\Gamma(1/2) = \sqrt{\pi}$

1HB3-5

$\boxed{T}$ 4. (Continuity) $f : [a, b) \times [c, d] \to \mathbb{R}$

If f cont (as function of two var)

and if $\int_a^b f(x, y)\, dx$ (u CONV)

then $F(y) = \int_a^b f(x, y)\, dx$ cont (in y)

$\boxed{T}$ 5. (Diff-ability) f cont, $\frac{\partial f}{\partial y}$ cont

and $\int_a^b f(x, y)\, dx$ (u CONV), $\int_a^b \frac{\partial f}{\partial y}(x, y)\, dx$ (u CONV)

Then $F(y) = \int_a^b f(x, y)\, dx$ is diff-able

and $F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx$

$\boxed{T}$ 6. (Integrability)

f cont, $\int_a^b f(x, y)\, dx$ (u CONV) then

$F(y) = \int_a^b f(x, y)\, dx$ is y-integrable

and $\int_c^d F(y)\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$

$$= \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

Ex. 7. $f(x) = \begin{cases} \frac{e^x - 1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

f diffable? $f^{(n)}(0) = ?$

trick $f(x) = \int_0^1 e^{xy}\, dy$

$$f'(x) = \int_0^1 \frac{\partial}{\partial x}(e^{xy})\, dy$$

$$= \int_0^1 y\, e^{xy}\, dy = \dots$$

$$f''(x) = \int_0^1 y^2 e^{xy}\, dy = \dots$$

$$f'(0) = \int_0^1 y\, e^{0y}\, dy = \frac{1}{2}$$

$$f''(0) = \int_0^1 y^2 e^{0y}\, dy = \frac{1}{3}$$

$$\dots$$

$$f^{(n)}(0) = \int_0^1 y^n e^{0y}\, dy = \frac{1}{n+1}$$