13.12.2021 $13^{\underline{00}}$

Part III: Sequences & Series

11. Series of reals

Theme: Infinite sums and sums of infinitesimals

$$1 + (-1) + 1 + (-1) + \dots$$

2 groupings: $\underbrace{(1-1)}_{=0} + \underbrace{(1-1)}_{=0} + \dots = 0$

$$1 + \underbrace{(-1+1)}_{=0} + \underbrace{(-1+1)}_{=0} + \dots = 1$$

§ 11.1 Sequences of reals

What is a sequence?

A sequence is a map from the discrete set $\mathbb{N}^*$ to $\mathbb{R}$ (or some other space)

$(a_n)_{n\in\mathbb{N}^*}$ $\qquad \mathbb{N}^* \ni n \longmapsto a_n \in \mathbb{R}$

Lazy notation: (a_n) (omit $n \in \mathbb{N}^*$)

Not: $a_n \xrightarrow{n\to\infty} \ell$

Def: (a_n) converges to $\ell \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}^*$ s.t. $\forall n > N(\varepsilon)$ we have $|a_n - \ell| < \varepsilon$.

1HB3-5

Motivation: APPROXIMATION
standard example $1,4142\ldots \to \sqrt{2}$

Def: (a_n) is called fundamental (or Cauchy) sequence if

$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N}^*$ s.t.
$\forall m, n > N(\varepsilon)$ we have $|a_m - a_n| < \varepsilon$.

T 1. In $\mathbb{R}$ every Cauchy seq. converges.

Rk. Any convergent seq. is Cauchy. however, the converse does not always hold

Counterexample: $\mathbb{Q}$
$a_1 = 1,\ a_2 = 1,4,\ a_3 = 1,41,\ a_4 = 1,414\ldots$
(better and better approx of $\sqrt{2}$)

$(a_n)^{\subseteq \mathbb{Q}}$ is Cauchy but it does not converge to a limit in $\mathbb{Q}$

In this case, we say that $\mathbb{Q}$ is not complete (by contrast to $\mathbb{R}$ which is complete = contains all limits of Cauchy sequences.).

1HB3-5

Important INSIGHTS:

1. Regard sequences as infinite length versions of $(a_1, \dots, a_n) \in \mathbb{R}^n$

2. All the above defs. work also in $\mathbb{R}^d$ (not $\mathbb{R}$) or some other, more general/abstract set X provided one can measure dist. in X.

§ 11.2 Series of reals

Let $(a_n)_{n \in \mathbb{N}^*}$ seq. of reals ($a_n \in \mathbb{R}$)

and define $(s_n)_{n \in \mathbb{N}^*}$ sequence of partial sums

by $s_n := a_1 + a_2 + \dots + a_n$

Def A series is a pair $((a_n), (s_n))$

and is denoted as $\sum_{n=1}^{\infty} a_n$

Def $\sum_{n=1}^{\infty} a_n$ is (CONV) if (s_n) conv.

If $s_n \xrightarrow{n \to \infty} S \in \mathbb{R}$ we write $\sum_{n=1}^{\infty} a_n = S$

and say that S is the sum of the series.

Lazy notation: $\sum a_n$

1HB3-5

$\sum a_n$ (DIV) if $s_n \to \pm\infty$ or $\lim \nexists$

Meaning of (CONV): 1 finite sum makes sense

There are two questions: • Does $\sum a_n$ CONV?
• What is the sum?

Based on $\boxed{T}$1. (s_n) conv $\Leftrightarrow$ (s_n) Cauchy

i.e. $\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}^*$ such that
$\forall p \in \mathbb{N} \quad |a_{N(\varepsilon)+1} + a_{N(\varepsilon)+2} + \dots + a_{N(\varepsilon)+p}| < \varepsilon$

$\boxed{T}$2 Cauchy's general convergance crit.

Rk. $a_n \xrightarrow{n\to\infty} 0$ is a necessary (but not sufficient) cond for $\sum a_n$ (CONV)

i.e. $\sum a_n$ (CONV) $\underset{\not\Leftarrow}{\Rightarrow}$ $a_n \xrightarrow{n\to\infty} 0$

Counterexample (see later): $\sum_{n=1}^{\infty} \frac{1}{n}$ (DIV)

this is used to prove that a seq. is divergent:

if you have prove $a_n \not\to 0$ Then $\sum a_n$ (DIV)

Example: $a_n = 1 \quad \forall n = 1, 2, \dots$
const. seq. ↑ generates a (DIV) series

1HB3-5

§ 11.3. Series of positive reals

Standing assumption $a_n > 0 \ \forall n \in \mathbb{N}^*$

T 3. (CAUCHY's Integral crit.)

$f : [1, \infty) \to \mathbb{R}_+$ decreasing

Define $(f_n)_{n \in \mathbb{N}^*}$ by $f_n := f(n)$

$$\sum_{n=1}^{\infty} f_n \quad \begin{matrix} \text{(CONV)} \iff \text{(CONV)} \\ \text{(DIV)} \iff \text{(DIV)} \end{matrix} \quad \int_1^{\infty} f(x)\,dx$$

Connection between series & improper integrals!

Proof: $s_n = f_1 + \dots + f_n$, (s_n)

f decreasing $\Rightarrow f_{n+1} = f(n+1) \le f(x) \le f(n) = f_n$ for any $x \in [n, n+1]$

integrate over $[n, n+1]$ w.r.t. x

i.e. $$f_{n+1} \le f(x) \le f_n \quad \Big| \int_n^{n+1} dx$$

$$f_{n+1} \underbrace{\int_n^{n+1} dx}_{1} \le \int_n^{n+1} f(x)\,dx \le f_n \underbrace{\int_n^{n+1} dx}_{1}$$

so $$f_{n+1} \le \int_n^{n+1} f(x)\,dx \le f_n \quad \Big| \sum_{n=1}^{N}$$

$$s_{N+1} - f_1 \le \int_1^{N+1} f(x)\,dx \le s_N$$

conclusion follows from (CONV)/(DIV) of $\int$.

Applications of Cauchy's Integral Crit.

1. The <u>harmonic series</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ is (DIV)

[because $\int_1^{\infty} \frac{1}{x}\,dx$ is (DIV)]

2. The <u>gen. harmonic series</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (CONV)

[because $\int_1^{\infty} \frac{1}{x^2}\,dx$ is (CONV)]

3. The <u>geometric series</u> $\sum_{\substack{n=1 \\ \text{or } n=0}}^{\infty} q^n$

is (CONV) if $|q| < 1$

(DIV) otherwise

HW: a) Compute $\int_{1 \text{ or } 0}^{\infty} q^x\,dx$

(and discuss $|q| \lesseqgtr 1$ cases)

b) When $|q| < 1$, compute

$$\sum_{n=0}^{\infty} q^n = ? \quad \text{(sum of series)}$$

Rk: Computing the sum of a series can be very tricky! Example is $\sum \frac{1}{n^2}$ which is (CONV) but sum is not easy to compute.

(see Lecture $3.^{rd}$ Jan)

13.12.2021 cont'ed. $16^{\underline{40}}$

§11.3 Series of positive reals (Cont'ed)

→ there will be additional material (uploaded in Files) which you'll have to study on your own. (CONV) CRITERIA

12. Sequences & series of functions

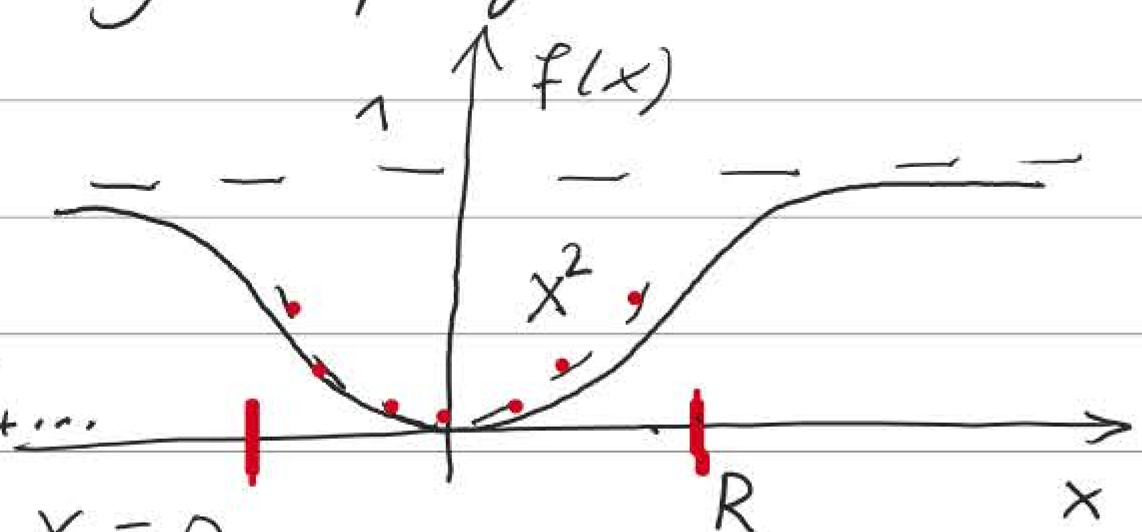
Motivation: Approx. again!

Taylor $f(x) = T_n(x) + R_n(x)$

small reminder

Taylor polynomial

Ex. $f(x) = \dfrac{x^2}{1+x^2}$



$f(x) = x^2 + a_4 x^4 + a_6 x^6 + \dots$

Taylor around $x_0 = 0$

define a sequence of functions
$f_0(x) = 0$, $f_1(x) = 0$, $f_2(x) = x^2$, $f_3(x) = 0$
$f_4(x) = a_4 x^4$, etc.

and $\sum_{n=0}^{N} f_n(x) = \sum_{n=0}^{N} a_n x^n \xrightarrow{N\to\infty} f(x)$

§12.1. Sequences of functions

$f, f_n : [a,b] \to \mathbb{R} \quad n = 1, 2, \dots$

Def [pointwise convergence] $f_n \xrightarrow[n\to\infty]{p.w.} f$

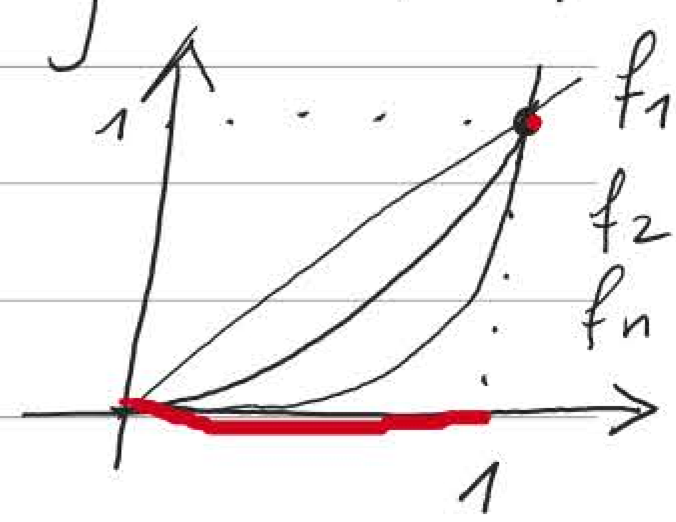
For any fixed $x \in [a,b]$ we have $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

(i.e., $\forall \varepsilon > 0 \ \exists N = N(\varepsilon, x) \in \mathbb{N}^*$ s.t. $\forall n > N(\varepsilon, x)$ we have $|f_n(x) - f(x)| < \varepsilon$

"For every x you could have different N")

Ex. $f, f_n : [0,1] \to \mathbb{R}$ $\quad f_n(x) = x^n$ ← are all cont.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} =: f(x)$$



but notice that $f(x)$ is discont

This motivates:

Def [unif. conv.] $f_n \xrightarrow[n\to\infty]{u} f$ if

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ s.t.

$\forall n > N(\varepsilon)$ we have $|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [a,b]$

uniform w.r.t. x

[T] 1. (continuity) $f_n : [a,b] \to \mathbb{R}$ all cont. and $f_n \xrightarrow[n\to\infty]{u} f$ Then f is cont.

[T] 2. (integrability) f_n all cont, $f_n \xrightarrow{u} f$ then f integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

[T]3. (diff-ability) f_n all diff-able

and (i) $f_n \xrightarrow{p.w} f$

(ii) $f_n' \xrightarrow{u} g$

then f is diffable and $f' = g$.

[D. Popa, Calculus]

§12.2. Power series

Series of functions $\sum_{n=1}^{\infty} f_n$ (notation)

$(f_n)_{n \in \mathbb{N}^*}$ a seq. of functions

$s_n(x) = f_1(x) + \dots + f_n(x)$, $(s_n)_{n \in \mathbb{N}^*}$ seq. of partial sums (again functions)

Def $\sum_{n=1}^{\infty} f_n$ (pw CONV) $:\Leftrightarrow s_n \xrightarrow{p.w} f$

(u CONV) $:\Leftrightarrow s_n \xrightarrow{u} f$

Recall: Approx (motivates notation)

Notation $\sum f_n \xrightarrow{p.w} f$ or $\sum f_n \xrightarrow{u} f$

(think of Taylor)

[T]4. (WEIERSTRASS)

(f_n) seq. of functions, $f_n : [a,b] \to \mathbb{R}$

(a_n) seq. of positive reals

and

(i) $\sum a_n$ (CONV) (ii) $|f_n(x)| \le a_n$ $\forall n \ge n_0$ $\forall x \in [a,b]$

Then $\sum f_n$ (CONV). HW: Proof?

1HB3-5

Power series $\sum_{n=0}^{\infty} a_n x^n$

special interesting case of $\sum f_n$

"LT" 5 (ABEL I) $\sum a_n x^n$ power s.

$\exists_1 R \in [0, \infty]$ s.t. power series (u CONV) on $[0, R]$

(radius of conv

Proof (idea): If $R = 0$ nothing to prove

if $R > 0$ s.t. $\sum a_n R^n$ (CONV) if it holds, this implies $|a_n R^n| \le M$ $\underbrace{\quad}_{\text{then } \in \mathbb{N}}$

For $|x| < R$ rewrite (forcing $\frac{1}{R}$)

$$\sum_{n=0}^{\infty} a_n x^n = \sum a_n R^n \left(\frac{x}{R}\right)^n$$

$$\le \sum \underbrace{|a_n R^n|}_{\le M} \left|\frac{x}{R}\right|^n$$

$$\le M \sum_{n=0}^{\infty} \left|\frac{x}{R}\right|^n \quad \left(\left|\frac{x}{R}\right| < 1\right)$$

and the desired result follows from (CONV) of the geometric series.

So, a radius of convergence exist —
How can we compute it?

1HB3-5

'T' 6. (J. HADAMARD) $\sum a_n x^n$

If $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ exists, then

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} \quad \left(\text{with } \frac{1}{0} = \infty,\ \frac{1}{\infty} = 0\right)$$

Proof based on Cauchy's root crit.
(which is part of the Additional material to Sec §11.3)

Rk if $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}}$ exists, then $R = \lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}}$

'T' 7. (ABEL II) continuity of funct. approx. by power series

$f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is cont. at $x = R$
if $\sum a_n R^n$ (CONV)

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generizes the result in the 1930s

'T' ((STONE)-WEIERSTRASS 1885)
Any cont $f : [a, b] \to \mathbb{R}$
can be arbitrarily well approx
(unif) by polynomial functions.

Weierstrass' Proof = NOT constructive

1912 S.N. BERNSTEIN constructive
proof based on Bernstein Polynomials
probability inspired
requires knowledge of values $f(x_i)$ at
many points x_i

1HB3-5