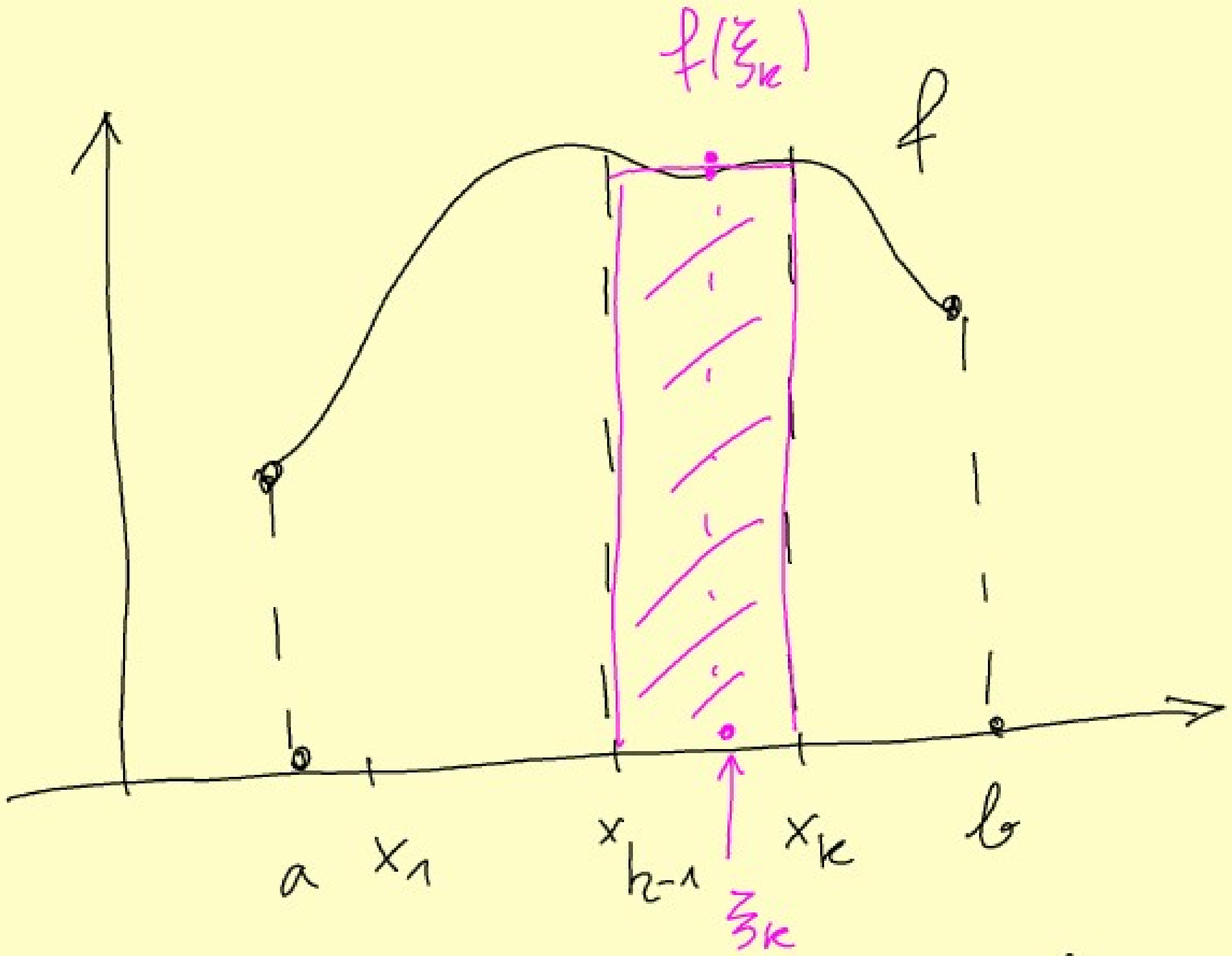
# 7. Measurable sets and the multiple integral

Short Recap: The Riemann integral

- simple functions don't have simple antiderivatives
  $\int e^{-x^2}\,dx \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{1}{\ln x}\,dx$
- RIEMANN: "Concentrate on the Riemann integral (not) antiderivatives"
- cont. functions are Riemann integrable
  (but $\exists$ integrable functions which are discont.)




area under graph $f$ = ?

Answer $= \int_a^b f(x)\,dx$

Riemann $\int$

$a = x_0 \quad x_n = b$

Idea: cut $[a,b]$ in slices ($\Delta$ "partition" $x_1, x_2, \ldots$)

choose $\xi_k$ intermediate points $\xi$ ($\xi_k \in [x_{k-1}, x_k]$)

area of one slice $= f(\xi_k)(x_k - x_{k-1})$

Riemann Sum = add up all slices

$$\sigma(f;\Delta;\xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \xrightarrow[\max_k |x_k - x_{k-1}| \to 0]{n\to\infty} \int_a^b f(x)\,dx$$

## § 7.1. A simple example

$f : [a,b] \times [c,d] \to \mathbb{R}$ cont

Then $$\int_a^b \left( \int_c^d f(x,y)\, dy \right) dx = \int_c^d \left( \int_a^b f(x,y)\, dx \right) dy$$

Idea of Proof:

Define $$H(t) = \int_a^t \left( \int_c^d f(x,y)\, dy \right) dx - \int_c^d \left( \int_a^t f(x,y)\, dx \right) dy$$

$$H : [a,b] \to \mathbb{R}$$

Aim: show that $H(b) = 0$ ✓

We compute the derivative

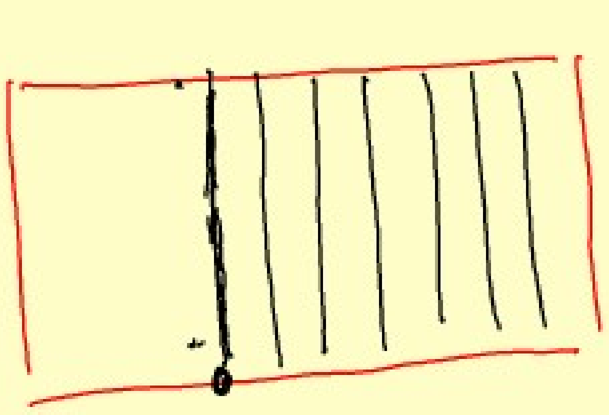
$$\underbrace{H'(t)}_{\frac{d}{dt} H(t)} = \int_c^d f(t,y)\, dy - \int_c^d \frac{d}{dt} \left( \int_a^t f(x,y)\, dx \right) dy$$

$$= 0 \implies H = \text{const}$$

$$H(b) = H(a) = 0$$

because $\int_a^a \cdots = 0$

Geometry



$[a,b] \times [c,d]$

fix $x \in [a,b]$ then integrate w.r.t. $y$

then integrate w.r.t. $x$

## §7.2. Jordan Measurability

The point is: we first have to establish which domains $D \subset \mathbb{R}^d$ are "good" for integrating $f$ (over them)

Want: for $f : D \to \mathbb{R}$ $\quad$ "$\int_D$" $f(\underbrace{x_1, \dots, x_d}_{x})$ "$dx$"

integral for function of several variables

We need a framework in which
LENGTH - AREA - VOLUME - ?

$d=1$ $\qquad$ $d=2$ $\qquad$ $d=3$ $\qquad$ $d$ arbitrary

$B = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_d, b_d]$ "box" in dim $= d$

$\operatorname{int} B = (a_1, b_1) \times \dots \times (a_d, b_d)$ interior of $B$

$v(B) = (b_1 - a_1)(b_2 - a_2) \dots (b_d - a_d)$ "volume" of the box

$v(\phi) = 0$ the empty set has zero volume

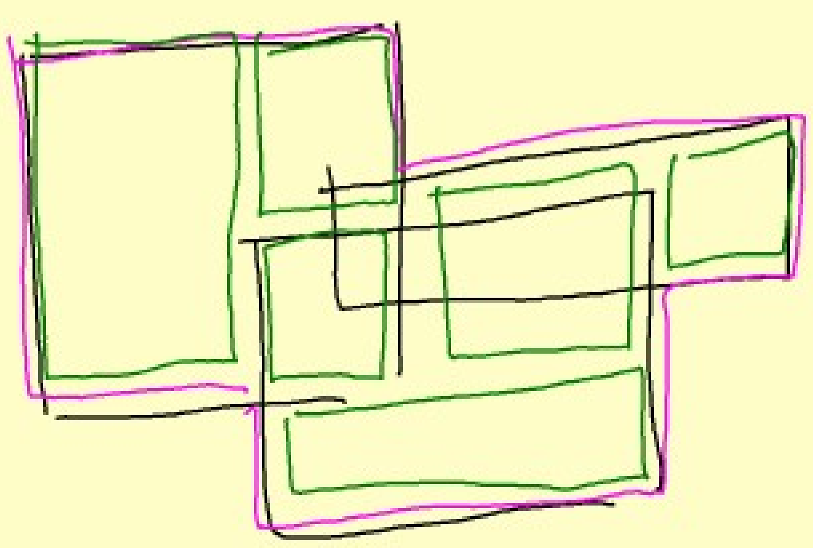
**Def** we call a set $A \subset \mathbb{R}^d$ elementary if it is a finite reunion of (nonoverlapping) boxes

$$A = \bigcup_{i=1}^{N} B_i, \quad B_i \text{ box}, \quad \operatorname{int} B_i \cap \operatorname{int} B_j = \phi \quad \forall i \neq j$$

For such $A$: $v(A) = \sum_{i=1}^{N} v(B_i)$

you can define / compute volume

Overlapping vs. ~~non~~overlapping



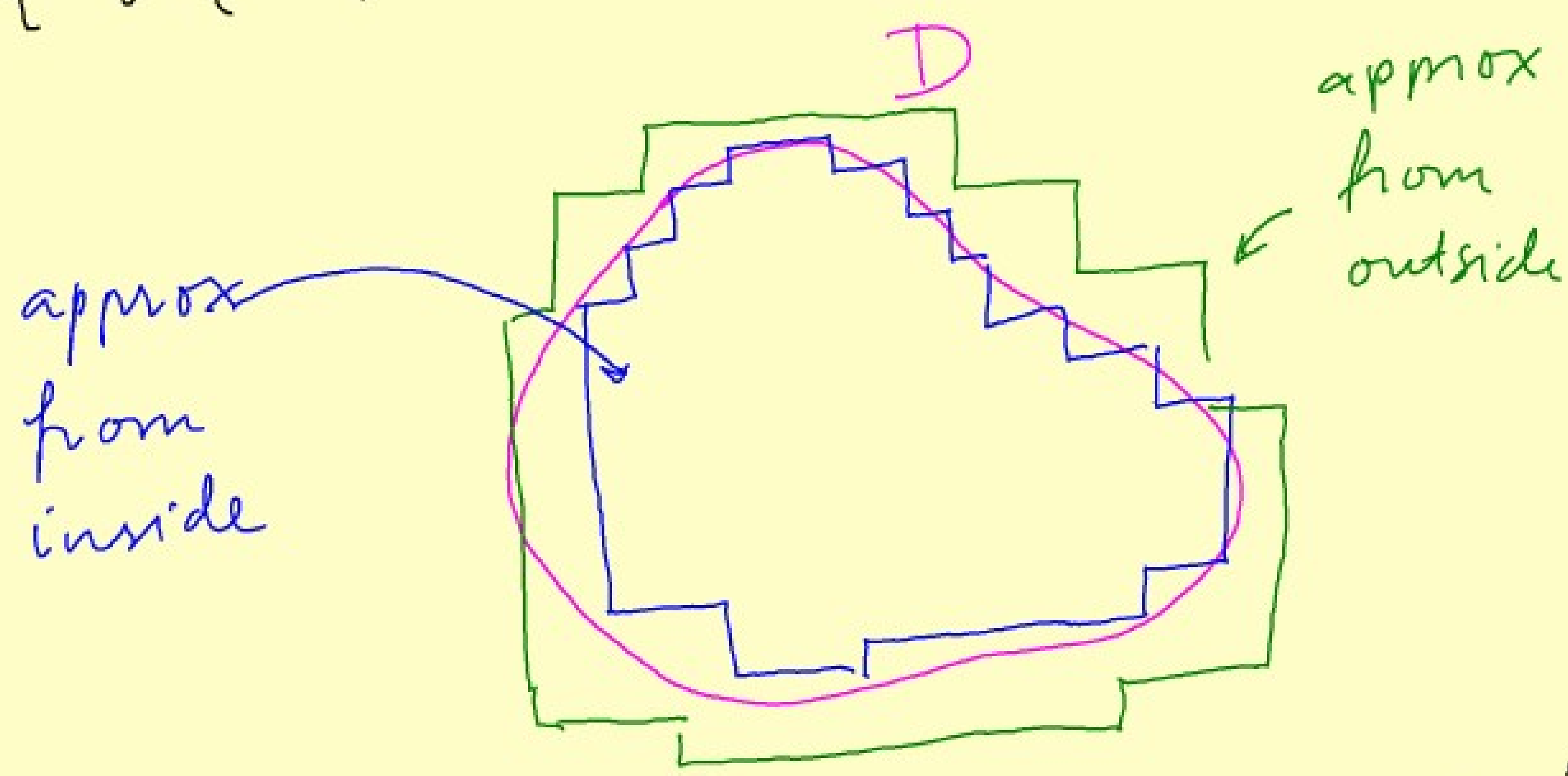
$A = \bigcup B_i = \bigcup \tilde{B}_j$

overlapping — nonoverlapping

Now consider a bounded set $D \subset \mathbb{R}^n$

$$m_i(D) = \sup\{ v(A) : A \text{ elementary and } \cancel{D \subseteq A} \; A \subseteq D \}$$

$$m_o(D) = \inf\{ v(A) : A \text{ ---''--- } D \subseteq A \}$$




**Def** A bounded set $D \subset \mathbb{R}^d$ is <u>Jordan measurable</u> if $m_i(D) = m_o(D)$ (inner & outer approx coincide). The common value will be denoted by $m(D)$ and is called the Jordan <u>measure</u> of $D$.

| LENGTH | — | AREA | — | VOLUME | — | MEASURE |
|---|---|---|---|---|---|---|
| $d=1$ | — | $d=2$ | | $d=3$ | | $d$ arbitrary |

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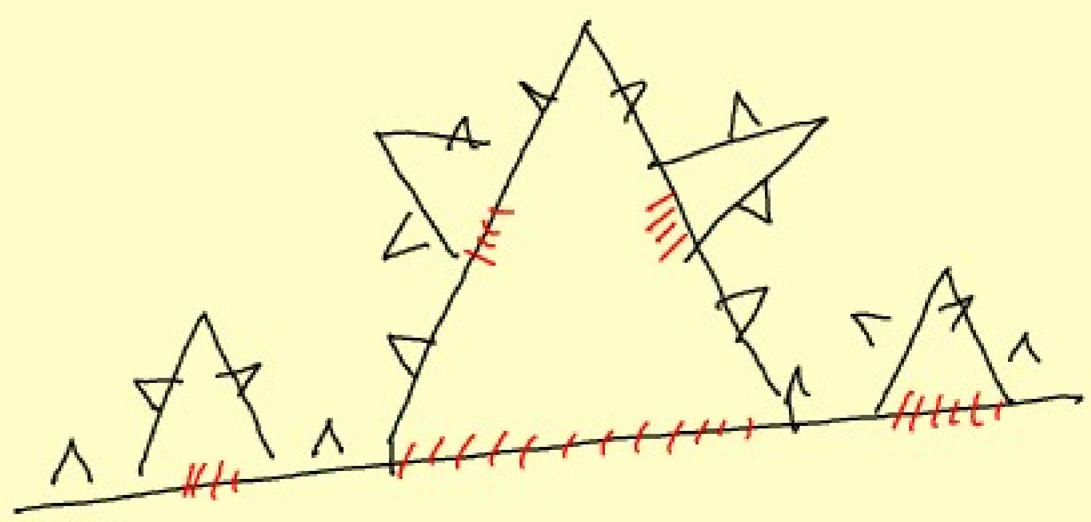
Examples and counterexamples (dim = 2)

- Any set $D$ with piecewise smooth boundary is Jordan measurable (in $\mathbb{R}^2$).

etc.

- A famous counterexample: the von KOCH snowflake

Fractal & NOT Jordan meas.



Rk: An important extension of Jordan meas. is due to LEBESGUE (1901, 1902, 1904 book)

The only difference is that L. considers elementary sets which are not <u>finite</u> but <u>countable</u> reunions of boxes.

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## § 7.3. The multiple integral (in the sense of Riemann)

- $D \subset \mathbb{R}^d$ bounded and Jordan measurable

$\Delta = \{ D_1, \dots, D_n \}$ a partition of $D$

Jordan measurable

$\text{int}\, D_i \cap \text{int}\, D_j = \emptyset \quad i \neq j$

$D = D_1 \cup \dots \cup D_n$

$\left( \|\Delta\| := \right) \max\limits_i \delta(D_i) = \max\limits_i \left( \sup\{ \|x-y\| : x, y \in D_i \} \right)$

diameter of $D_i$

$\xi = \{\xi_i\}, \quad \xi_i \in D_i$ intermediate points

- $f : D \to \mathbb{R}$

Jordan measure

$$\sigma(f; \Delta; \xi) = \sum_{i=1}^{n} f(\xi_i)\, m(D_i)$$

Riemann sum.

some value.

Def "$f$ is Riemann integrable if the Riemann sum converges to $I \in \mathbb{R}$"
(as $n \to \infty$ and $\max \delta(D_i) \to 0$)

The limit $I =: \int_D f(x)\, dx$ or $= \iint \dots \int f(x_1, \dots, x_d)\, dx_1 \dots dx_d$

Notations.

## 8. Computation of multiple integrals

IDEA: reduce the multiple integral to the computation of (several) simple integrals.

T 1. (FUBINI) $f : A \times B \to \mathbb{R}$ integrable

$A, B$ bounded and Jordan measurable

Then $$\iint_{A\times B} f(x,y)\,dx\,dy = \int_A \left( \int_B f(x,y)\,dy \right) dx$$

$$= \int_B \left( \int_A f(x,y)\,dx \right) dy$$

Ex. $D = \underbrace{[0,1]}_{A} \times \underbrace{[1,2]}_{B}$

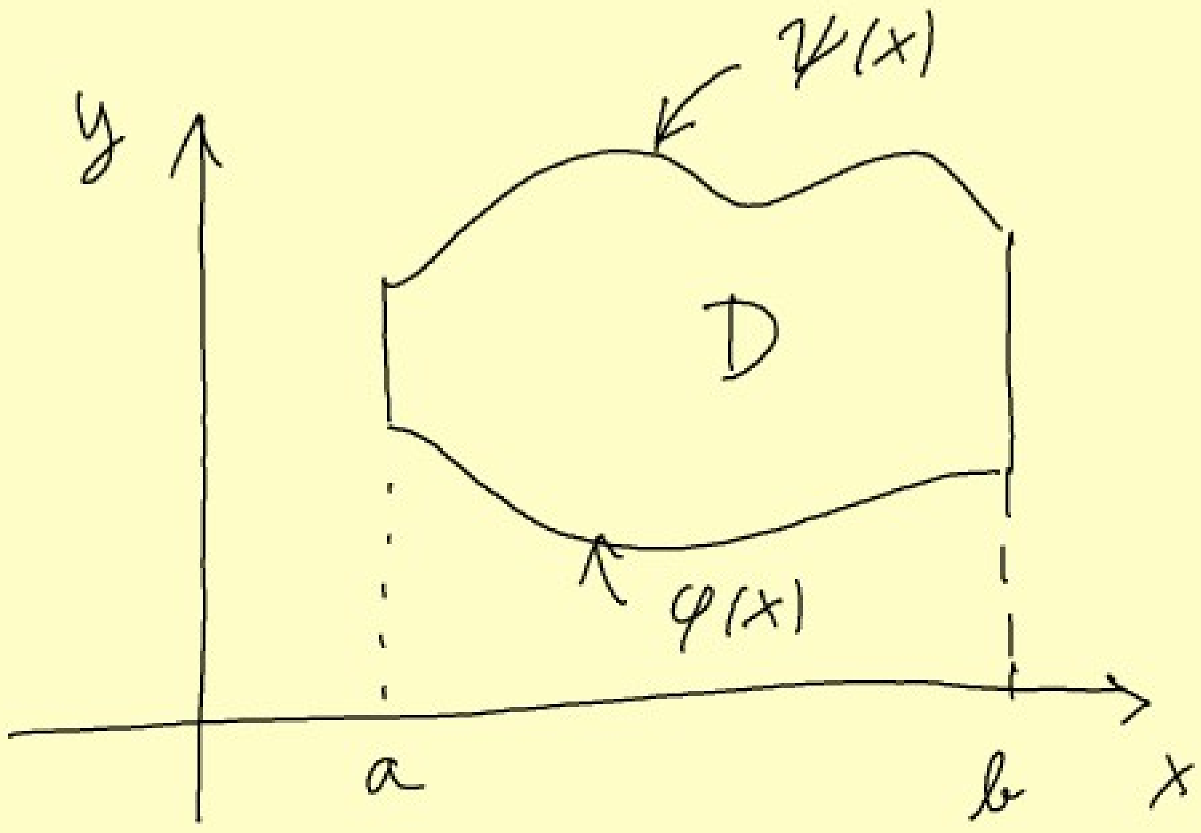
$$\iint_D xy\,dx\,dy = \int_0^1 \left( \int_1^2 xy\,dy \right) dx$$

$$= \left( \int_0^1 x\,dx \right) \left( \int_1^2 y\,dy \right)$$

$$= \frac{x^2}{2}\Big|_0^1 \cdot \frac{y^2}{2}\Big|_1^2 = \frac{1}{2} \cdot \left(2 - \frac{1}{2}\right.$$

# § 8.1. Double integrals

Domain $D$ is simple w.r.t. $Ox$ or $Oy$

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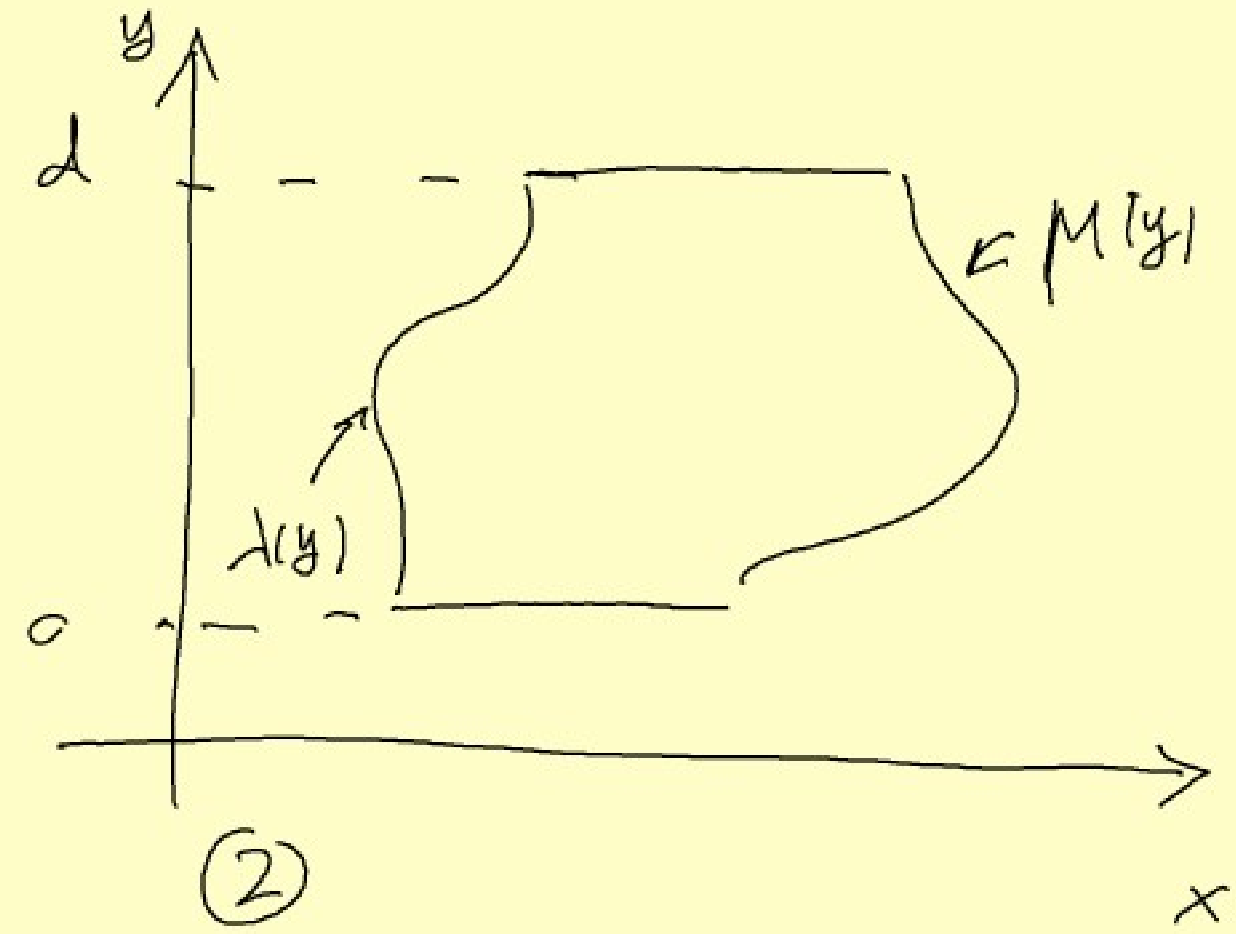


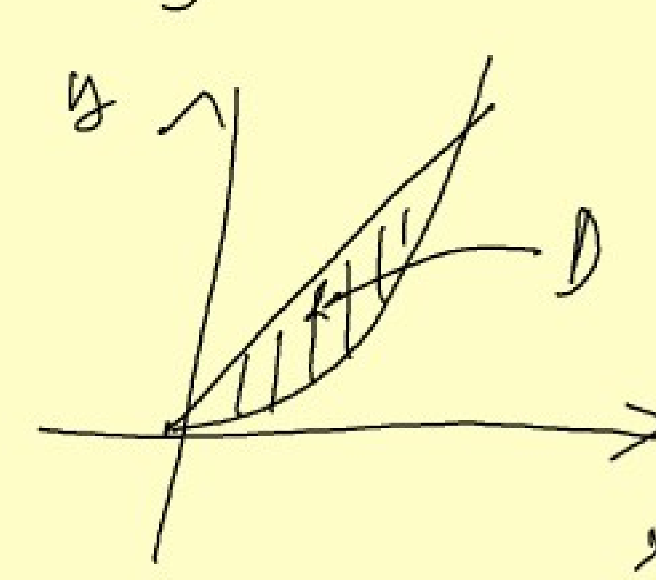

②

① $$\iint_D f(x,y)\,dx\,dy = \int_a^b \left( \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx$$

② $$\iint_D f(x,y)\,dx\,dy = \int_c^d \left( \int_{\lambda(y)}^{\mu(y)} f(x,y)\,dx \right) dy$$

Ex. $D$ is bdd by $y=x$ and $y=x^2$

$\varphi(x) = x^2, \quad \psi(x) = x$




$$I = \iint_D (x+y)\,dx\,dy = \int_0^1 \left( \int_{x^2}^{x} (x+y)\,dy \right) dx$$

$$= \int_0^1 \left[ xy + \frac{y^2}{2} \right]\Bigg|_{y=x^2}^{y=x} dx = \ldots = \frac{3}{20}$$