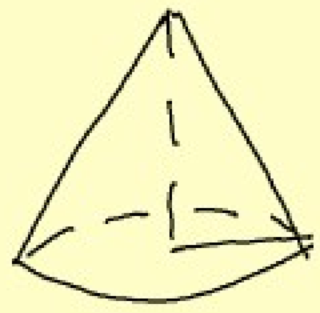
2. Calculus for functions of several variables I.

The Geometry $\mathbb{R}^d$, partial derivatives, the Gradient.

WHY functions of several variables?

Reality is complicated...

- Volume of a cone  $V = \dfrac{\pi r^2 h}{3}$

  $V = V(r, h)$

- You buy a new BMW

  Price = Price (Options) ← many: motor
  lether seats
  gear box
  color ...

Problem: (How) can we describe construct all the geometric objects that we need (in $\mathbb{R}^d$) starting with a single simple concept?

Math insight: "Yes we can!"
based on the so called inner (or dot) product.

§ 2.1. The Geometry $\mathbb{R}^d$

I will try to avoid the word "vector"

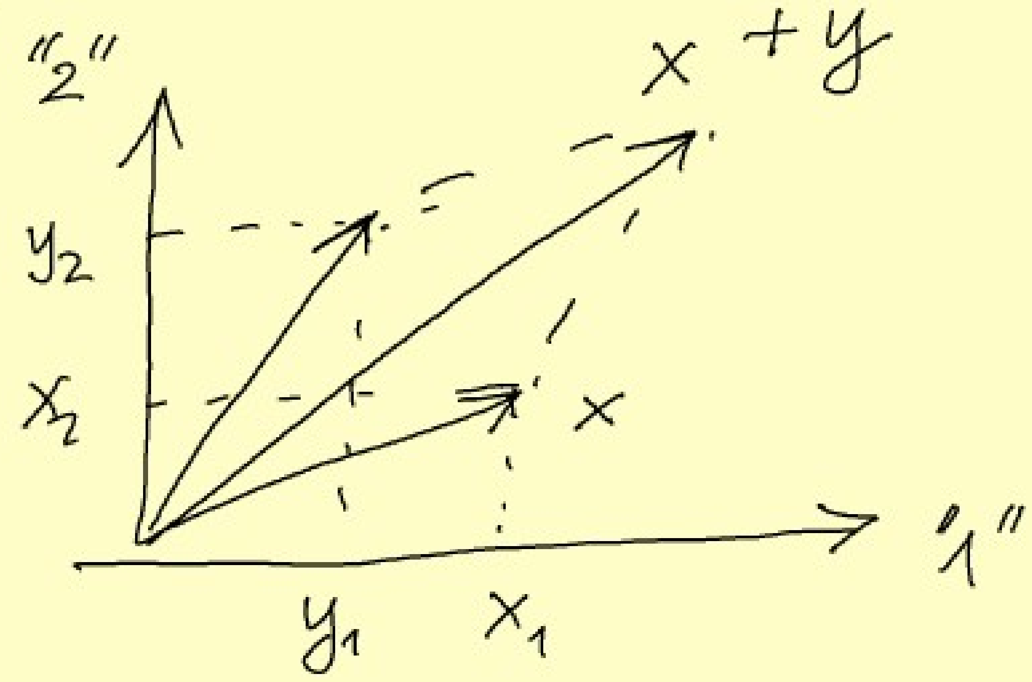
$x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ element of $\mathbb{R}^d$

Carthesian prod. $\mathbb{R} \times \mathbb{R} \times \dots$

Two op. on $\mathbb{R}^d$

- "+" addition $\quad x + y = (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d)$

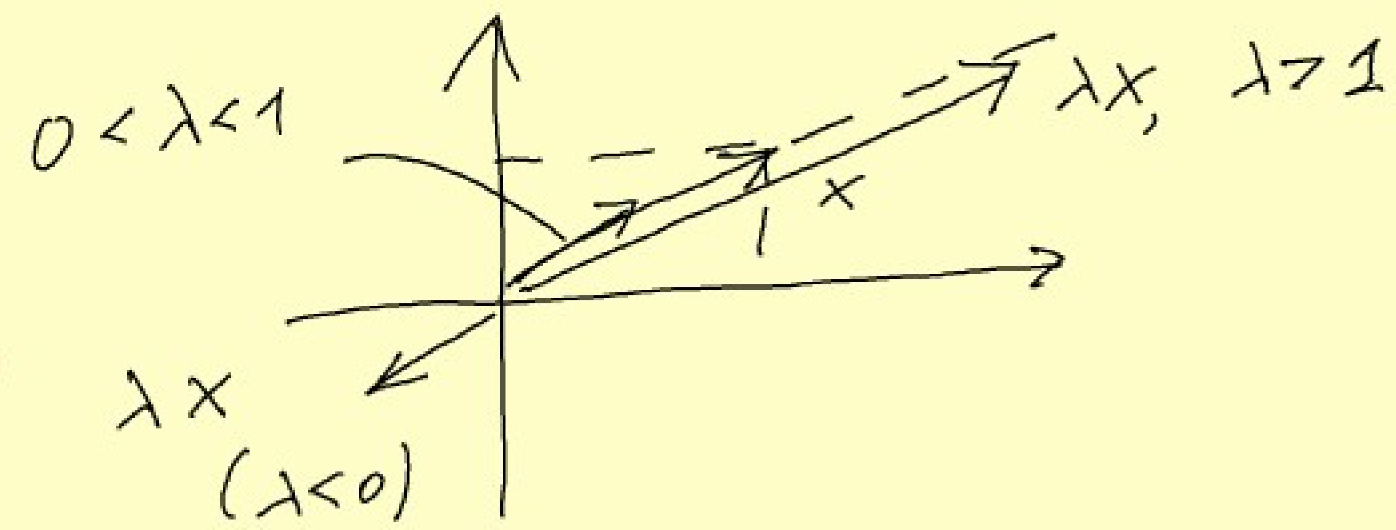
$x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$



- multiplic by a scalar $(\lambda \in \mathbb{R})$

$\lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$

geom: "scaling"



You know all these things!

These are just operations with matrices! $\quad 1\,[\cdot \ \cdot \ \cdot \ \cdot]$ d col.

$\lambda(x+y) = \lambda x + \lambda y$, $\quad (\lambda + \mu)x = \lambda x + \mu x$ etc.

Careful you have $0_{\mathbb{R}}$ (scalar) and $0_{\mathbb{R}^d} = (0, \dots, 0)$

$0_{\mathbb{R}} x = 0_{\mathbb{R}^d}$ and $\lambda 0_{\mathbb{R}^d} = 0_{\mathbb{R}^d}$ $\quad \forall \lambda \in \mathbb{R}. \quad \forall x \in \mathbb{R}^d$

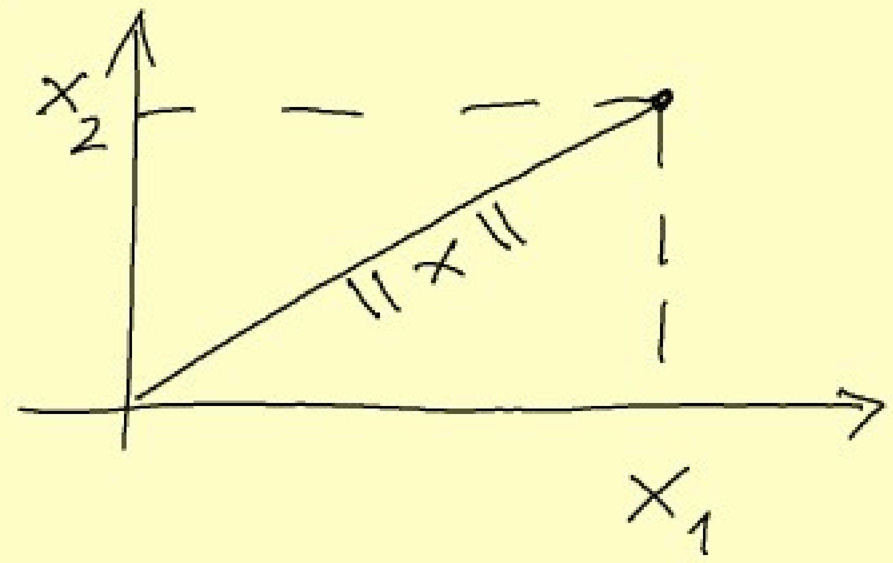
Straight line through the origin $= \{\lambda x : \lambda \in \mathbb{R}\}$
pointing in the direction of $x \neq 0_{\mathbb{R}^d}$

- the inner (dot) product (altern. not. $\langle x, y \rangle$)

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_d y_d$$

- norm of x (length)

$$\|x\| = \sqrt{x \cdot x}$$
$$= \sqrt{x_1^2 + x_2^2 + \ldots x_d^2}$$



- distance $\operatorname{dist}(x, y) = \|x - y\|$

the properties (see below) can be generalized axiomatically.

(N1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0_{\mathbb{R}^d}$

(N2) $\|\lambda x\| = |\lambda| \, \|x\|, \quad \forall \lambda \in \mathbb{R}, x \in \mathbb{R}^d$

(N3) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathbb{R}^d$

HW: "axioms" of dist.

$\{ z \in \mathbb{R} : |x - z| \leq \varepsilon \}$

- generalize "$[x - \varepsilon, x + \varepsilon] \subset \mathbb{R}$"

$$B_r(x) = \{ y \in \mathbb{R}^d : \|x - y\| < r \}, \quad r > 0$$

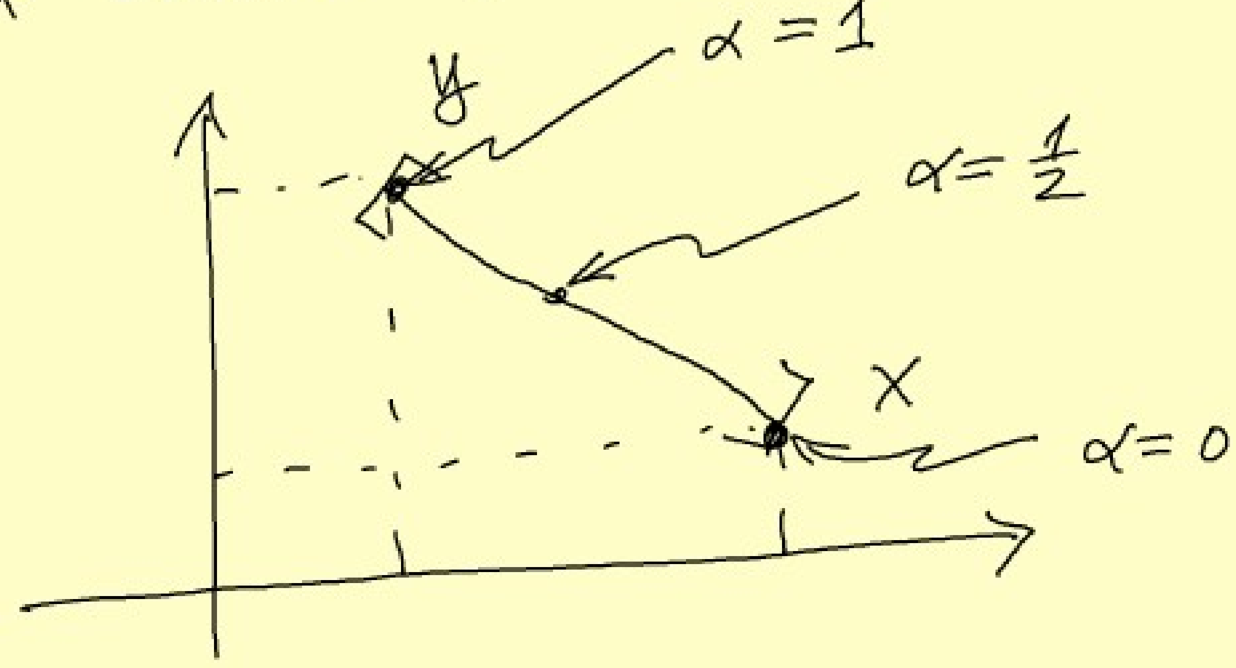
open ball centered at x and of radius r.

- orthogonality in $\mathbb{R}^d$

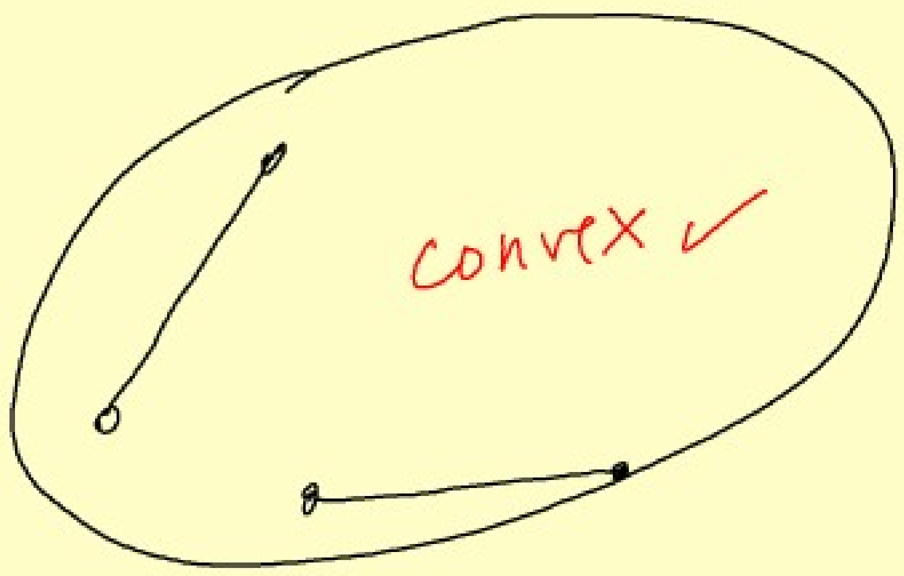
$x \perp y \iff x \cdot y = 0$ HW: (Check this in $\mathbb{R}^2$)

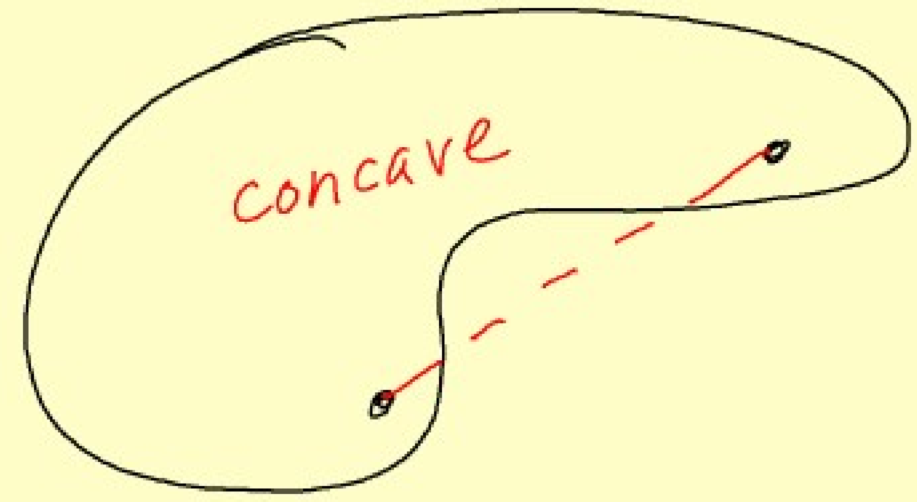
- segments in $\mathbb{R}^d$

$$\underbrace{[x,y]}_{\text{segment}} = \{ \underbrace{(1-\alpha)x + \alpha y}_{\text{convex comb. (or average)}} : \alpha \in [0,1] \subset \mathbb{R} \} \subset \mathbb{R}^d$$



- C convex set if $\forall x, y \in C$ we have $[x,y] \subset C$





Rk: the Ball is a convex set.

Rk: $\mathbb{R}^d$ can <u>not</u> be ordered (completely) !!!

$\exists\, x, y \in \mathbb{R}^d \quad x \nleq y$

§ 2.2. Partial derivatives and the Gradient.

Def $f : \mathbb{R}^d \to \mathbb{R}$ has partial derivative w.r.t. x_k at a point $x = (x_1, \dots, x_d)$ if

$$\lim_{h \to 0} \frac{f(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots) - f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots)}{h}$$

if the limit exists. Notation $\frac{\partial f}{\partial x_k}(x)$

"keep all other vars. const. and take deriv w.r.t. x_k"

Partial derivative $\to$ only partial information ("in direction of x_k")

That's why we need: The Gradient (full info)

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right) \quad x \in \mathbb{R}^d$$