Seminar 9

1. We have the matrix $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{bmatrix}$.

   I shall write only the operations on lines: $\begin{cases} L3 \leftrightarrow L1 \\ L2 = L2 - 2L1, L4 = L4 - 2L1 \\ L3 = L3 - L2, L4 = L4 - L3 \end{cases}$

   We get to the matrix: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.We have one line of zeros, so the rank comes from the other three $\Rightarrow Rank = 3$.

2. Now we have the matrix $\begin{bmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{bmatrix}$.

   I shall write only the operations on lines: $\begin{cases} L2 = L2 + 2L1, L3 = L3 + L1 \\ L3 = 2L3 + L2 \\ L2 = \frac{1}{2}(L2 + L3) \\ L1 = L1 - L3 \end{cases}$

   We get the matrix: $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix}$. We see the first three columns that get our non-zero determinant, so the rank is 3.

3. The same as *Exercise 2*.

4. For this, we put near our matrix, the identity matrix.

   $\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 2 & 1 & -2 & | & 0 & 1 & 0 \\ 2 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$.

   Now, $L2 = L2 - 2L1$ and $L3 = L3 - 2L1$: $\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -3 & -6 & | & -2 & 1 & 0 \\ 0 & -6 & -3 & | & -2 & 0 & 1 \end{bmatrix}$.

From here $L3 = L3 - 2L2$ and $\begin{bmatrix} 1 & 2 & 2 & | & 1 & 0 & 0 \\ 0 & -3 & -6 & | & -2 & 1 & 0 \\ 0 & 0 & 9 & | & 2 & -2 & 1 \end{bmatrix}$.

Then $L1 = 3L1 + 2L2$, so $\begin{bmatrix} 3 & 0 & 6 & | & -1 & 2 & 0 \\ 0 & -3 & -6 & | & -2 & 1 & 0 \\ 0 & 0 & 9 & | & 2 & -2 & 1 \end{bmatrix}$.

It is obvious that we can do $L1 = \frac{1}{3}L1$, $L2 = -\frac{1}{3}L2$ and $L3 = \frac{1}{9}L3$. $\begin{bmatrix} 1 & 0 & -2 & | & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & | & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & | & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}$.

In the end, $L2 = L2 - 2L3$ and $L1 = L1 + 2L3:$ $\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & | & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & | & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{bmatrix}$.

So, we get on the first part the identity matrix and on the second part the inverse of our initial matrix.

5. The same as *Exercise 4*.

6. We build the matrix $X = \begin{bmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{bmatrix}$.

   By applying $L_2 - L_1, L_3 - L_1$ and then $L_3 + L_2$, we obtain the echelon form $X = \begin{bmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   As, the last raw is formed only of zeroes, the three vectors $v_!, v_2, v_3$ are linearly dependent.

7. We build the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{bmatrix}$.

   I shall write only the operations on lines: $\begin{cases} L4 = L4 - 2L3 \\ L2 = L2 - 2L1, L3 = L3 - L1 \\ L3 = \frac{1}{5}L3 \\ L3 = L3 - L2 \end{cases}$

We obtain the matrix: $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So the rank is $2 \Rightarrow dim < X >= 2 \Rightarrow < X >=< v1, v2 >$ (given by the vectors forming the non-zero lines).

8. The same as *Exercise 7.*

9. For S we have the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$. If we do $L3 + L3 + L1$ and after that $L3 = L3 - L2$ we get to the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, $dim(S) = 2$ and $S =< (1, 0, 4), (2, 1, 0) >$.

   For T we have the matrix $\begin{bmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{bmatrix}$. If we do $L2 = L2 + L1$ and after that $L3 = L3 + L2$ we get to the matrix $\begin{bmatrix} -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$.

   So $dim(T) = 2$ and $T =< (-3, -2, 4), (5, 2, 4) >$.

   Remember: $dim(S+T) = dim < S \cup T >$ and $dim(S+T) = dim(S) + dim(T) - dim(S \cap T)$.

   For $S + T$ we have the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{bmatrix}$.

   The operations on lines are: $\begin{cases} L3 = L3 + L1, L5 = L5 + L4 \\ L3 = L3 - L2, L6 = L6 + L5 \\ L5 = L5 - 2L1 \end{cases}$.

And we get to the matrix $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, $dim(S+T) = 3$ and $< S+T > = < (1,0,4), (2,1,0), (-3,-2,4) >$.

So, the dimension for $S \cap T$ is obtained from the equality above, hence $dim(S \cap T) = 1$.

10. The same as *Exercise 9*.