Seminar 10

$[f]_E = [f(e_1) \mid f(e_2) \mid f(e_3)]$.

If we have the bases $B = e \cdot S$ and $B' = e' \cdot T$, then $[f]_{BB'} = T^{-1} \cdot [f]_{ee'} \cdot S$.

$[f]_E \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ for a basis in $ker(f)$.

$dim(Im(f)) = dim(f(e)) = dim(< f(e_1), f(e_2), f(e_3), f(e_4) >) =$ maximum number of linearly independent vectors in $[f]_E = rank([f]_E)$.

f is an automorphism $\iff det([f]_E) \neq 0$ and $[2f]_E = 2[f]_E$.

1. We use $[f]_E = [f(e_1) f(e_2) f(e_3)]$. So, we compute $\begin{cases} f(e_1) = f(1,0,0) = (1,0,2) \\ f(e_2) = f(0,1,0) = (1,1,1) \\ f(e_3) = f(0,0,1) = (0,-1,1) \end{cases}$

   Hence, $[f]_E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.

2. $\begin{cases} f(v_1) = f(1,1,0) = (1,-1) \\ f(v_2) = f(0,1,1) = (1,0) \\ f(v_3) = f(1,0,1) = (0,-1) \end{cases}$

   So, $[f]_{BE'} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$.

   From $f(v_1) = (1,-1)$, we get $(1,-1) = a_1 v_1' + a_2 v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{2}{3}$ and $a_1 = \frac{1}{3}$.

   From $f(v_2) = (1,0)$, we get $(1,0) = a_1 v_1' + a_2 v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{1}{3}$ and $a_1 = \frac{2}{3}$.

   From $f(v_3) = (0,-1)$, we get $(0,-1) = a_1 v_1' + a_2 v_2' = (a_1 + a_2, a_1 - 2a_2) \Rightarrow a_2 = \frac{1}{3}$ and $a_1 = -\frac{1}{3}$.

   Hence, $[f]_{BB'} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

3. (i) Let $v = (a,b,c) \in \mathbb{R}^3 \Rightarrow f(v) = f(ae_1 + be_2 + ce_3) \iff f(v) = af(e_1) + bf(e_2) + cf(e_3)$, as f is an homomorphism.

   Hence, $f(v) = a + 4b - 2c, 2a + 3b + c, 3a + 2b + 4c, 4a + b + c) \in \mathbb{R}^4$

(ii) $[f]_E = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{bmatrix}$

(iii) $[f]_e \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} x_1 + 4x_2 - 2x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ 3x_1 + 2x_2 + 4x_3 = 0 \\ 4x_1 + x_2 + x_3 = 0 \end{cases}$

$\bar{A} = \begin{bmatrix} 1 & 4 & -2 & | & 0 \\ 2 & 3 & 1 & | & 0 \\ 3 & 2 & 4 & | & 0 \\ 4 & 1 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -2 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{cases} x_3 = \alpha \in \mathbb{R} \\ x_2 = \alpha \\ x_1 = -2\alpha \end{cases} \Rightarrow$

$ker(f) =< (-2, 1, 1) >\Rightarrow dim(ker(f)) = 1$.

For the image, we have the matrix: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -2 & 1 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & -6 \end{bmatrix} \Rightarrow$

$dim(Im(f)) = 3$.

4. (i) $v = (1, 4, 1, -1) = e_1 + 4e_2 + e_3 - e_4 \Rightarrow f(v) = f(e_1) + 4f(e_2) + f(e_3) - f(e_4) = (1, -1, 2, 1) + 4(1, 1, 1, 2) + (-3, 1, -5, -4) - (2, 4, 1, 5) = (0, 0, 0, 0) \Rightarrow v \in ker(f)$.

$v' \in Im(f) \iff \exists v$ such that $f(v) = v'$. So, $v' = af(e_1) + bf(e_2) + cf(e_3) + df(e_4) \Rightarrow \begin{cases} a + b - 3c + 2d = 2 \\ -a + b + c + 4d = -2 \\ 2a + b - 5c + d = 4 \\ a + 2b - 4c + 5d = 2 \end{cases}$.

By solving the system, we get that $c, d \in \mathbb{R}$, $b = c - 3d$ and $a = 2 + 2c + d$. Hence, there is a v such that $f(v) = v'$.

(ii) We use $[f]_E \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. By solving the system here, we get that $x_4 = a$, $x_3 = b$, $x_2 = b - 3a$ and $x_1 = 2b + a \iff < (1, -3, 0, 1), (2, 1, 1, 0) >= ker(f) \Rightarrow dim = 2$.

We use $dim(Im(f)) = rank[f]_E$ and we know that $rank[f]_E = 2 \Rightarrow dim(Im(f)) = 2$ and $Im(f) =< (1, 1, -3, 2), (-1, 1, 1, 4) >$.

$$\text{(iii)} \quad \begin{cases} f(1,0,0,0) = (1,-1,2,1) = (x,-x,2x,x) \\ f(0,1,0,0) = (1,1,1,2) = (y,y,y,2y) \\ f(0,0,1,0) = (-3,1,-5,-4) = (-3z,z,-5z,-4z) \\ f(0,0,0,1) = (2,4,1,5) = (2t,4t,t,5t) \end{cases}$$

$\Rightarrow f(x,y,z,t) = (x+y-3z+2t, -x+y+z+4t, 2x+y-5z+t, x+2y-4z+5t)$.

5. $\varphi(e_1) = \varphi(1\cdot 1+0\cdot X+0\cdot X^2) = (1+0)+(0+0)X+(1+0)X^2 = 1+X^2$

$\varphi(e_2) = \varphi(0\cdot 1+1\cdot X+0\cdot X^2) = (0+1)+(1+0)X+(0+0)X^2 = 1+X$

$\varphi(e_3) = \varphi(0\cdot 1+0\cdot X+1\cdot X^2) = (0+0)+(0+1)X+(0+1)X^2 = X+X^2$

Then: $[\varphi]_E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$\varphi(b_1) = \varphi(1\cdot 1+0\cdot X+0\cdot X^2) = (1+0)+(0+0)X+(1+0)X^2 = 1+X^2$

$\varphi(b_2) = \varphi(-1\cdot 1+1\cdot X+0\cdot X^2) = (-1+1)+(1+0)X+(-1+0)X^2 = X-X^2$

$\varphi(B_3) = \varphi(1\cdot 1+0\cdot X+1\cdot X^2) = (1+0)+(0+1)X+(1+1)X^2 = 1+X+2X^2$

Then: $[\varphi]_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

6. $det[f]_B = 1 \neq 0 \Rightarrow f$ is an automorphism $\Rightarrow [2f]_B = 2[f]_B = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$.

$$\begin{cases} f(v_1) = (1,-1) \\ f(v_2) = (2,-1) \end{cases} \Rightarrow \begin{cases} a_1x+b_1y = 1 \\ a_2x+b_2y = -1 \end{cases}.$$

From $x=1, y=2 \Rightarrow \begin{cases} a_1+2b_1 = 1 \\ a_2+2b_2 = -1 \end{cases} \Rightarrow a_1 = -1$ and $a_2 = -1$.

From $x=1, y=3 \Rightarrow \begin{cases} a_1+3b_1 = 2 \\ a_2+3b_2 = -1 \end{cases} \Rightarrow b_1 = 1$ and $b_2 = 0$.

Hence, $f(x,y) = (y-x, -x)$.

$$\begin{cases} g(v'_1) = (-7,5) \\ g(v'_2) = (-13,7) \end{cases} \Rightarrow \begin{cases} a_1x+b_1y = -7 \\ a_2x+b_2y = 5 \end{cases}.$$

From $x = 1, y = 0 \Rightarrow a_1 = -7$ and $a_2 = 5$. From $x = 2, y = 1 \Rightarrow b_1 = 1$ and $b_2 = -3 \Rightarrow g(x, y) = (y - 7x, 5x - 3y)$.

Now, we compute $(f + g)(x, y) = f(x, y) + g(x, y) = (y - x, -x) + (y - 7x, 5x - 3y)$. And we apply this to the vectors v_1, v_2. So, $(f + g)(v_1) = (-4, -2)$ and $(f + g)(v_2) = (-2, -5) \Rightarrow [f + g]_B = \begin{bmatrix} -4 & -2 \\ -2 & -5 \end{bmatrix}$.

In the end, we compute $(f \circ g)(x, y) = f(g(x, y)) = (12x - 4y, -y + 7x)$ and we apply this to the vectors v'_1, v'_2. So, $(f \circ g)(v'_1) = (12, 7)$ and $(f \circ g)(v'_2) = (20, 13) \Rightarrow [f \circ g]_{B'} = \begin{bmatrix} 12 & 20 \\ 7 & 13 \end{bmatrix}$.

7. $f(e_1) = (cos(\alpha), sin(\alpha))$ and $f(e_2) = (-sin(\alpha), cos(\alpha))$.

   So, $[f]_E = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}$.

   We compute $det([f]_E) = cos^2(\alpha) + sin^2(\alpha) = 1 \neq 0 \Rightarrow f$ is an automorphism.

8. $dim_{\mathbb{Z}_2}(V) = 2 \Rightarrow \mid V \mid = 2^2 = 4$ and $\mid M_2(\mathbb{Z}_2) \mid = 2^4 = 16$.

   As $End_{\mathbb{Z}_2}(V)$ is isomorphic to $M_2(\mathbb{Z}_2) \Rightarrow \mid End_{\mathbb{Z}_2}(V) \mid = 2^4$.