8. Computation of multiple integrals

Th 1. (FUBINI) $f : A \times B \to \mathbb{R}$ integrable

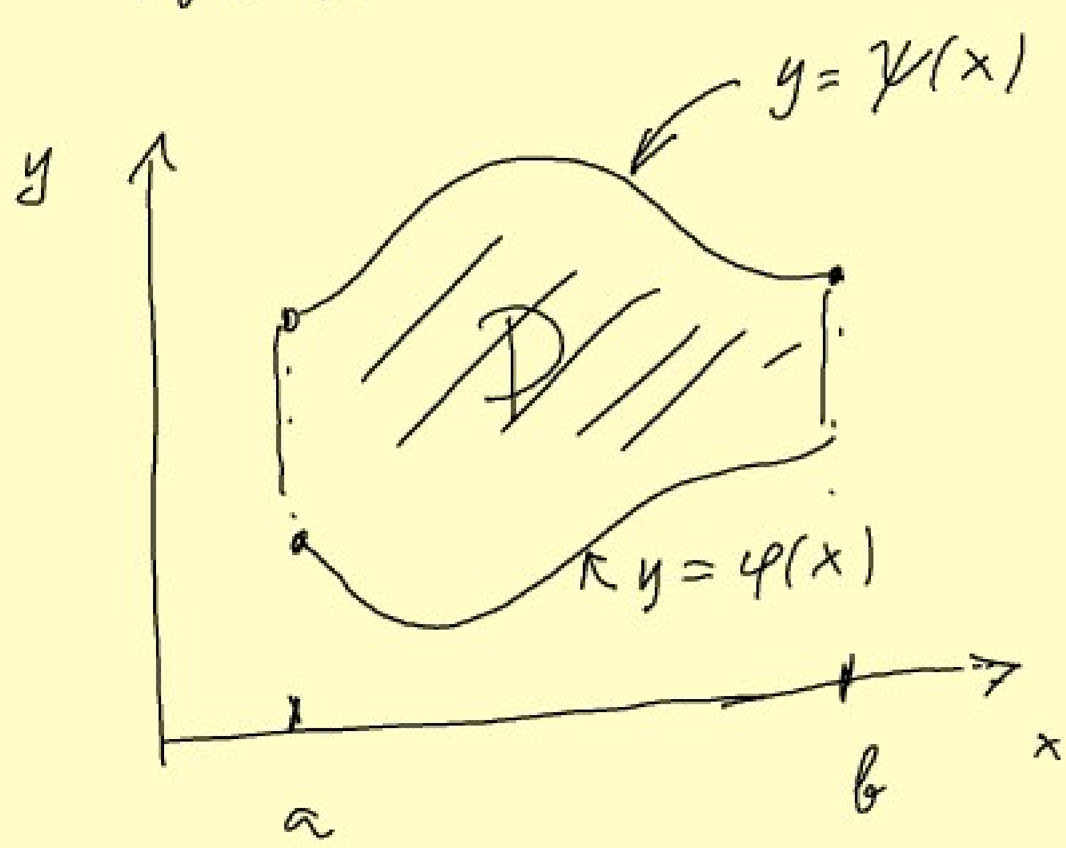
A, B bounded and Jordan measurable

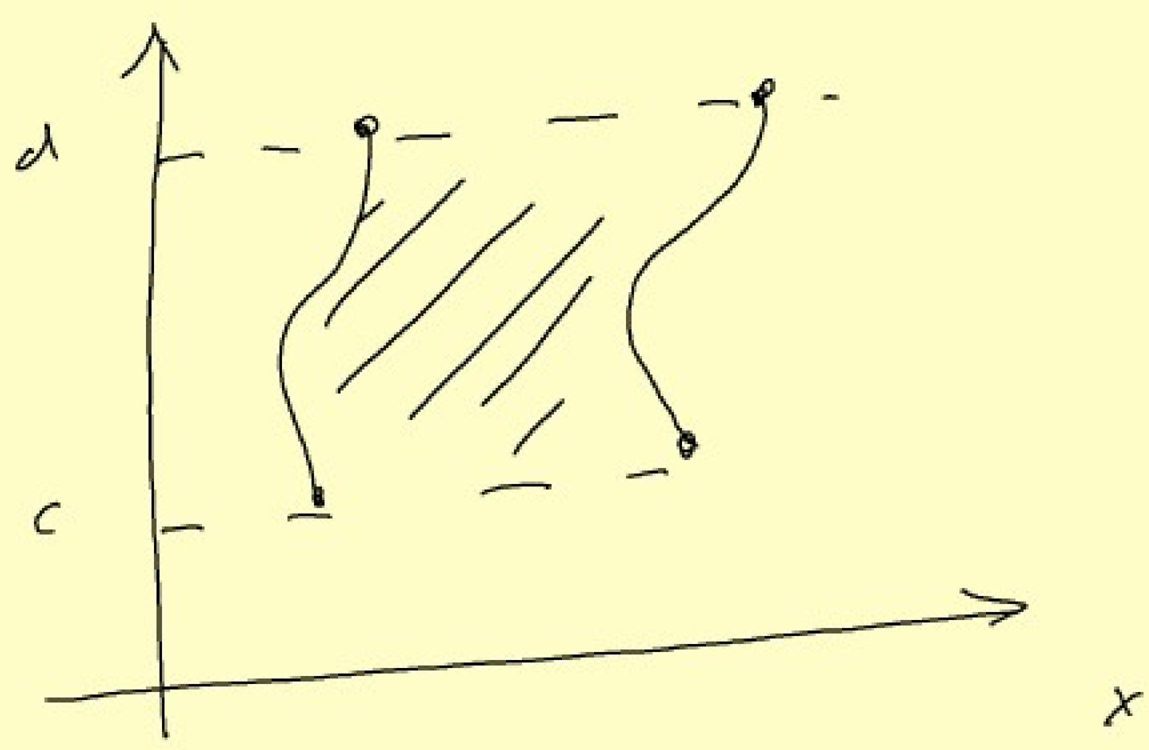
Then $$\iint_{A\times B} f(x,y)\,dx\,dy = \int_A \left(\int_B f(x,y)\,dy \right) dx = \int_B \left(\int_A f(x,y)\,dx \right) dy$$

IDEA: reduce the multiple integral to the comput of several simpler integrals.

§ 8.1. Double integrals

We've discussed simple domains





$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx.$$

Notation & why we write "dxdy" etc.

Whenever there is no ambiguity: simplified notation

$\int_D f(x)\, dx$! this implies it's a multiple integral, that $f = f(x_1, \dots, x_d)$ and $dx = dx_1\, dx_2 \dots dx_d$

Whenever more details are helpful you write more details

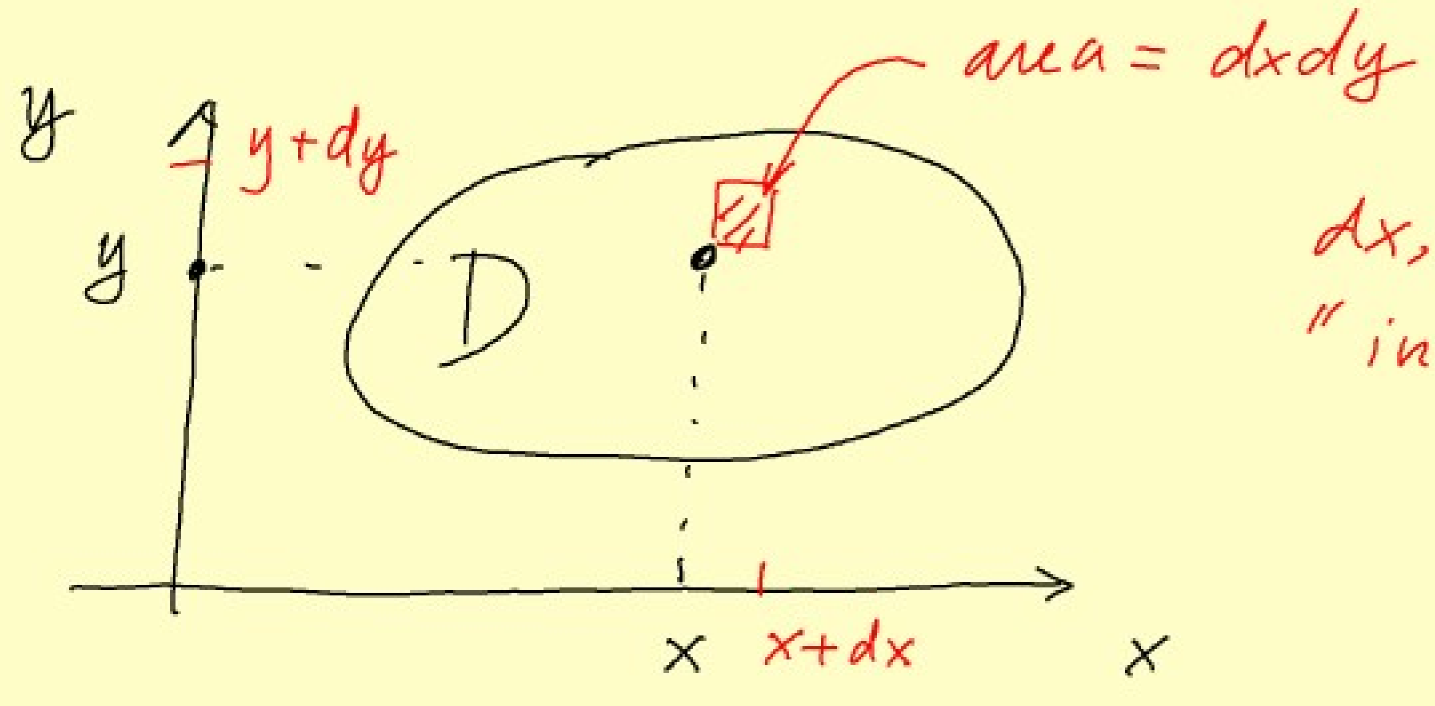
e.g. $\iint_D f(x,y)\, dx\, dy$ this is a double integral

or $\int_0^1 \dots \int_0^1 e^{x_1 + x_2 + \dots + x_n}\, dx_1 \dots dx_n$ n-integral(s)

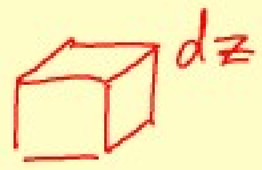
Why do we write "dx dy" (or $dx_1\, dx_2 \dots dx_d$)

two reasons:

- it indicates the numer (and name) of variables you integrate w.r.t.
- geometric interpretation: surface element or volume element



dx, dy "infinitezimals"

dim = 3 

volume = dx dy dz

§ 8.2. Change of variables in the multiple integral

We need to take a step back to differential calculus and have a look at

$f: \mathbb{R}^d \to \mathbb{R}^m$ (not just $\underset{m=1}{f: \mathbb{R}^d \to \mathbb{R}}$)

$f = (f_1, f_2, \dots, f_m)$ with $f_1, \dots, f_m : \mathbb{R}^d \to \mathbb{R}$

components of f

f is called vector field or vector-valued function

The question is: what's the appropriate notion of derivative for f?

(for $f: \mathbb{R}^d \to \mathbb{R}$ it was ∇f)

Jacobi matrix
$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_d}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \dots & \\ \dots & & & \\ \frac{\partial f_m}{\partial x_1}(x) & & \dots & \frac{\partial f_m}{\partial x_d}(x) \end{bmatrix}$$

$$= \begin{bmatrix} [\quad \nabla f_1(x) \quad] \\ [\quad \nabla f_2(x) \quad] \\ \dots \\ [\quad \nabla f_m(x) \quad] \end{bmatrix}$$

Rk. The Jacobi matrix turns out to be the Frechet differential of f ...

We are interested in <u>changes of coordinates</u>

$u : \mathbb{R}^d \to \mathbb{R}^d \qquad u = (u_1, \dots, u_d)$

u_i has cont partial derivative

Def u is called <u>regular</u> if $\det(J_u(x)) \neq 0$

T2. (Change of vars in the multiintegral)

Let $\Delta, D \subset \mathbb{R}^d$ bounded, closed, measurable.

$M \subset \Delta$ of measure zero $(m(M) = 0)$ and

$u : \Delta \to D$ with cont. diffable components and

(i) u is injective on $\Delta \setminus M$

(ii) u is regular on $\Delta \setminus M$

Then, if $f : D \to \mathbb{R}$ is integrable over D we have

$$\int_D f(x)\,dx = \int_\Delta f(u(z)) \underbrace{|\det J_f(z)|\, dz}_{\text{the "new" infinitezimal surface/volum element.}}$$

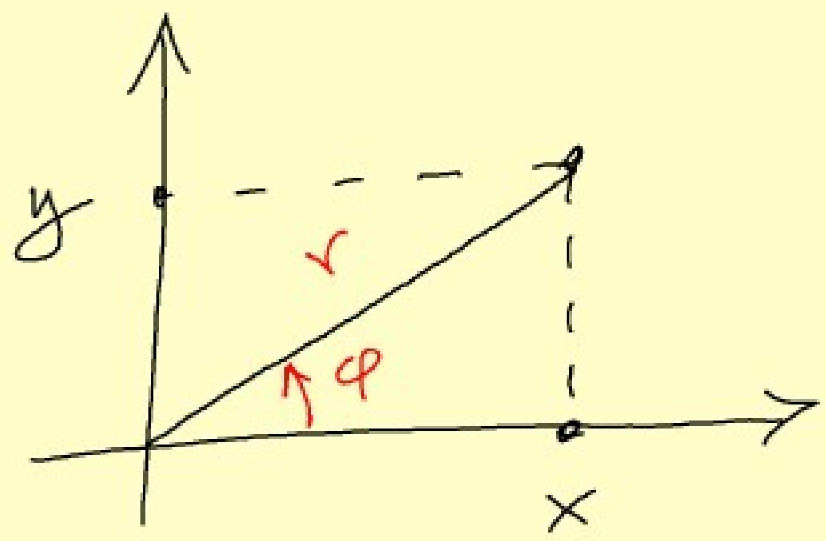
Standard changes of coordinates

2D : polar coordinates

3D : spherical coordinates, cylindric, etc.

Polar coords:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \operatorname{arctg}\frac{y}{x} \end{cases}$$



$r \geq 0$

$\varphi \in [0, 2\pi)$

$$u(r,\varphi) = (\overset{x}{r\cos\varphi}, \overset{y}{r\sin\varphi})$$

$u : \Delta \to D$ ← change of coords

$$r, \varphi \longrightarrow x, y$$

$$J_u(r,\varphi) = \begin{bmatrix} \frac{\partial}{\partial r}(r\cos\varphi) & \frac{\partial}{\partial \varphi}(r\cos\varphi) \\ \frac{\partial}{\partial r}(r\sin\varphi) & \frac{\partial}{\partial \varphi}(r\sin\varphi) \end{bmatrix}$$

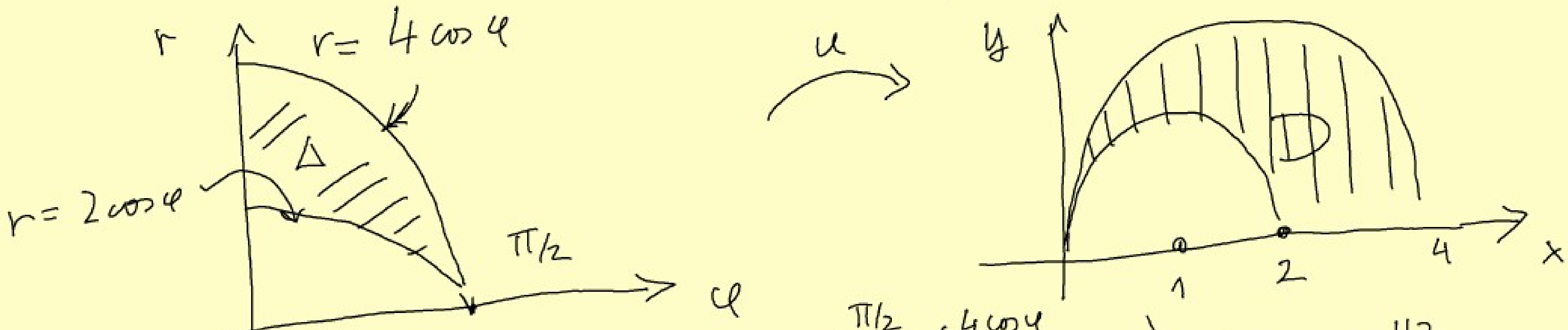
$$= \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

$$\det J_u(r,\varphi) = r\cos\varphi^2 + r\sin\varphi^2 = r \neq 0 \quad (\text{for } r \neq 0)$$

Ex. $I = \iint_D \sqrt{x^2+y^2}\,dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 : 2x \leq x^2+y^2 \leq 4x,\ y \geq 0$

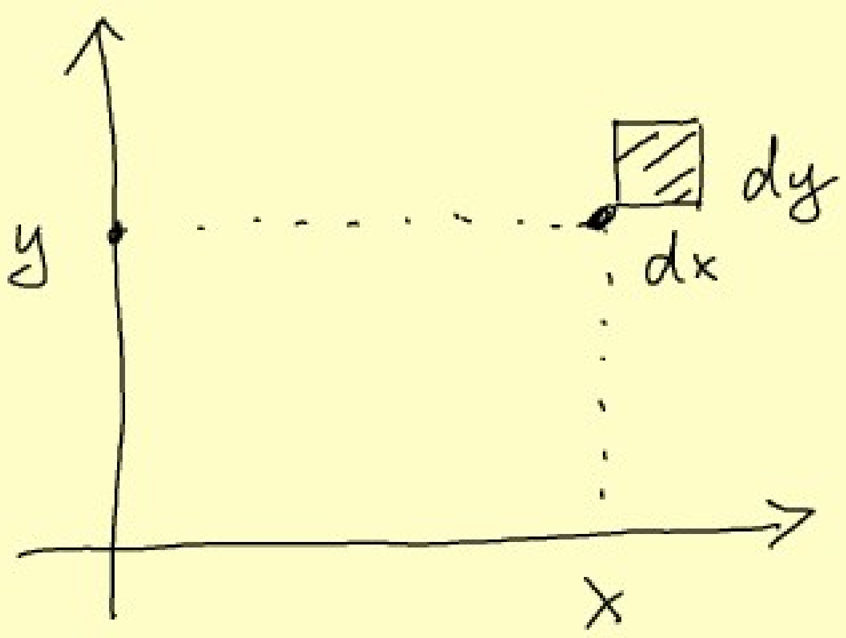
IDEA $\quad (x-x_0)^2 + (y-y_0)^2 = R^2$ eq of circle of radius R centre at $(x_0$

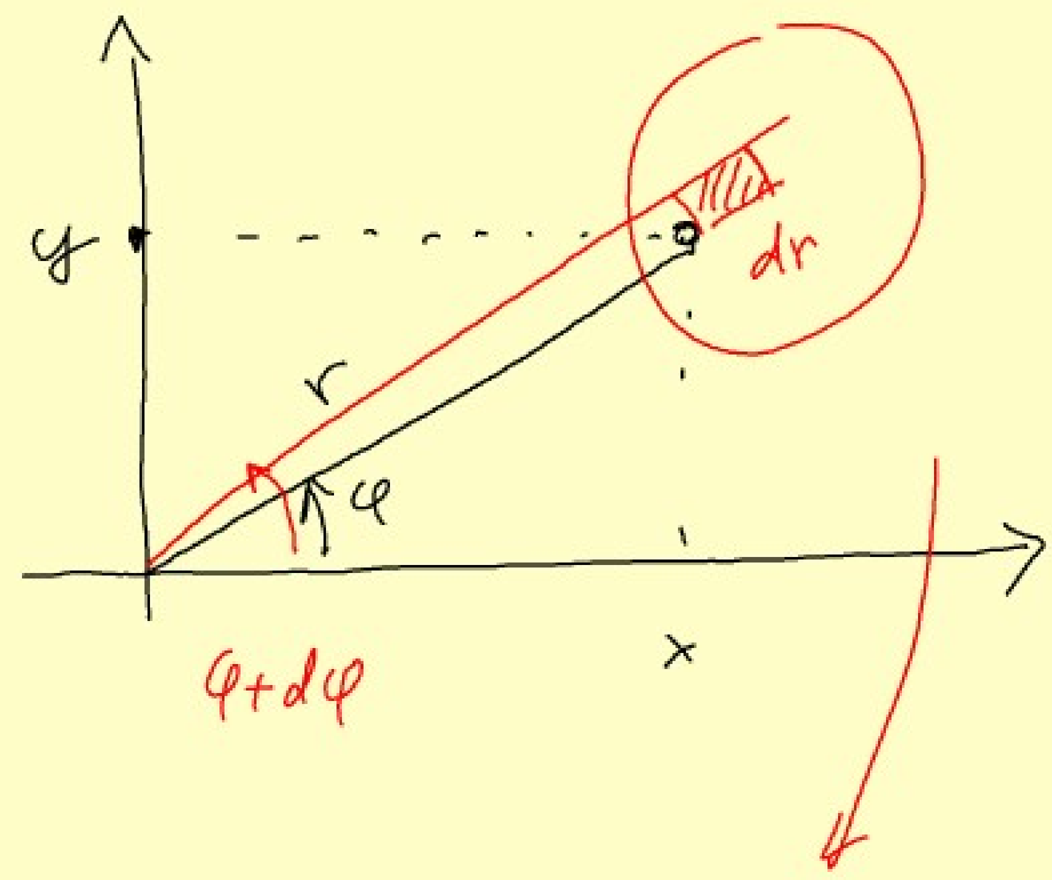
interior / ext of circle $\leq$ / $\geq$



$$I = \iint_\Delta r \underbrace{r}_{\det J_u} \,dr\,d\varphi = \int_0^{\pi/2} \left(\int_{2\cos\varphi}^{4\cos\varphi} r^2\,dr \right) d\varphi = \dots \frac{112}{9}$$

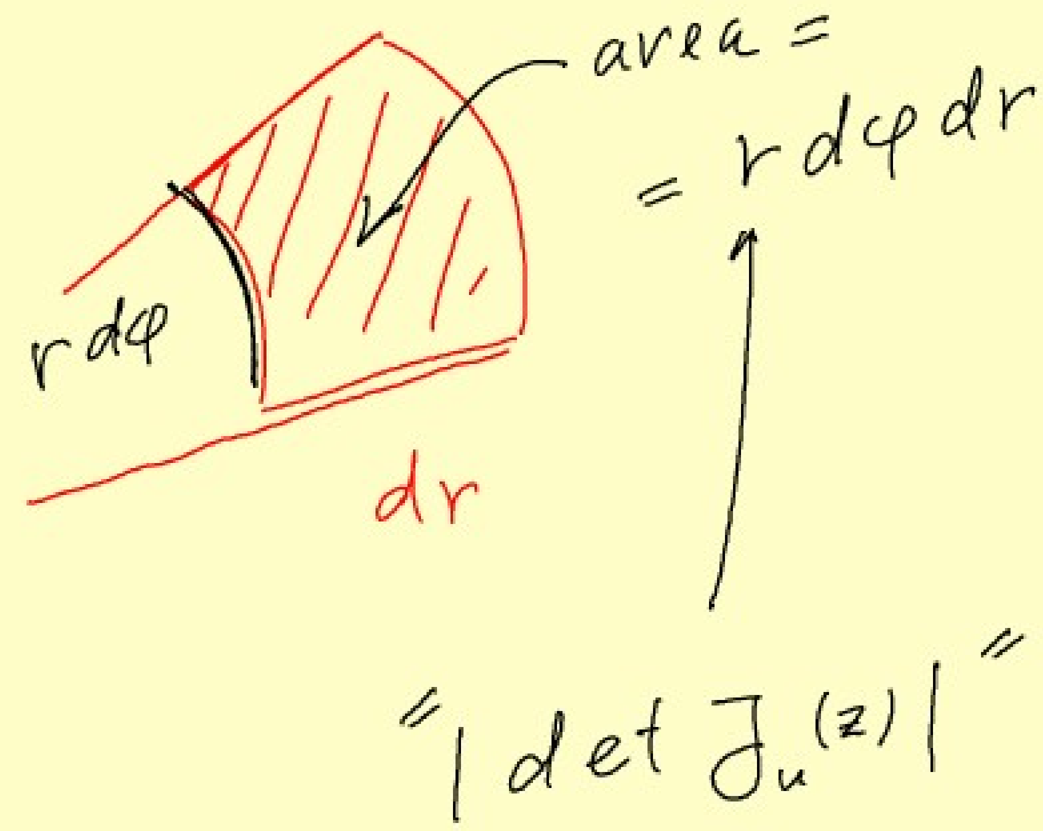
The geometric interp. of $|\det J_u^{(z)}|\, dz$





$d\varphi$ is "angle"

$r\, d\varphi$ is "length"



Spherical coords

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

$$|\det J_u(r, \varphi, \theta)| = r^2 \sin\theta$$