Seminar 8

A matrix A is **invertible** if $det(A) \neq 0$.

Kronecker-Capelli: a system is compatible if $Rang(A) = Rang(\bar{A})$, where A is the matrix of the system and $\bar{A}$ is A with a column consisting of the free terms.

Rouche: a system is compatible if all the characteristic determinants are zero.

Cramer: $x_i = \frac{det(A_i)}{det(A)}$ are the solutions of a system, where A_i is the matrix A, by replacing the column i with the column of the free terms.

Gauss-Jordan: zeros under the main diagonal.

1. The matrix A is invertible if $det(A) \neq 0 \iff det(A) = -1 \neq 0$.

$$A^{-1} = \frac{1}{det(A)} A^* \iff A^{-1} = \begin{bmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{bmatrix}.$$

$$AX = B \mid A^{-1} \cdot \Rightarrow X = A^{-1}B \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = -11 \\ x_3 = 19 \end{cases}.$$

2. (i) $\bar{A} = \begin{bmatrix} 1 & 1 & 1 & -2 & | & 5 \\ 2 & 1 & -2 & 1 & | & 1 \\ 2 & -3 & 1 & 2 & | & 3 \end{bmatrix}$. So $Rang(A) = Rang(\bar{A}) = 3 \Rightarrow$ compatible system with "number of columns - $Rang(A)$" unknowns, i.e. 1 unknown. Let's say $x_4 = \alpha \in \mathbb{R}$, then the system becomes:

$$\begin{cases} x_1 = 5 - x_2 - x_3 + 2\alpha \\ 10 - 2x_2 - 2x_3 + 4\alpha + x_2 - 2x_3 + \alpha = 1 \\ 10 - 2x_2 - 2x_3 + 4\alpha - 3x_2 + x_3 + 2\alpha = 3 \end{cases}$$

By solving the system, we get that $x_1 = 2$, $x_2 = 1 + \alpha$, $x_3 = 2 + \alpha$ and $x_4 = \alpha$, with $\alpha \in \mathbb{R}$.

(ii) $\bar{A} = \begin{bmatrix} 1 & -2 & 1 & 1 & | & 1 \\ 1 & -2 & 1 & -1 & | & -1 \\ 1 & -2 & 1 & 5 & | & 5 \end{bmatrix}$. So $Rang(A) = Rang(\bar{A}) = 2$, by using the submatrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$ compatible system with 2 unknowns. Let's take $x_1 = \alpha$ and $x_2 = \beta$ both in $\mathbb{R}$, then the system becomes:

$$\begin{cases} x_3 + x_4 = 1 - \alpha + 2\beta \\ x_3 - x_4 = -1 - \alpha + 2\beta \\ x_3 + 5x_4 = 5 - \alpha + 2\beta \end{cases}$$

By solving the system, we get that $x_1 = \alpha$, $x_2 = \beta$, $x_3 = -\alpha + 2\beta$ and $x_4 = 1$, with $\alpha, \beta \in \mathbb{R}$.

(iii) $\bar{A} = \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 1 & -1 & 1 & | & 1 \\ 2 & -1 & 2 & | & 3 \\ 1 & 0 & 1 & | & 4 \end{bmatrix}$. So, $Rang(A) = 2 \neq 3 = Rang(\bar{A}) \Rightarrow$ incompatible system.

3. Similar to the previous exercise.

4. $A = \begin{bmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{bmatrix} \Rightarrow det(A) = -2abc$. Then the system is compatible determinate if $det(A) \neq 0 \iff abc \neq 0 \iff a, b, c \neq 0$.

Now, let's compute the solutions: $x = \frac{det(A_x)}{det(A)}$, where $det(A_x) = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} \Rightarrow$

$$\begin{cases} x = \frac{b^2+c^2-a^2}{2bc} \\ y = \frac{a^2+c^2-b^2}{2ac} \\ z = \frac{b^2+a^2-c^2}{2ab} \end{cases}$$

5. (i) $\begin{bmatrix} 2 & 2 & 3 & | & 3 \\ 1 & -1 & 0 & | & 1 \\ -1 & 2 & 1 & | & 2 \end{bmatrix}$. We change L_2 with $L_1 \Rightarrow \begin{bmatrix} 1 & -1 & 0 & | & 1 \\ 2 & 2 & 3 & | & 3 \\ -1 & 2 & 1 & | & 2 \end{bmatrix}$.

We do $L_2 = L_2 - 2L_1$ and $L_3 = L_3 + L_1 \Rightarrow \begin{bmatrix} 1 & -1 & 0 & | & 1 \\ 0 & 4 & 3 & | & 1 \\ 0 & 1 & 1 & | & 3 \end{bmatrix}$.

Again $L_1 = L_1 + L_3$ and $L_2 = L_2 - 3L_3 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & -8 \\ 0 & 1 & 1 & | & 3 \end{bmatrix}$.

Then $L_3 = L_3 - L_2 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & -8 \\ 0 & 0 & 1 & | & 11 \end{bmatrix}$. In the end $L_1 = L_1 - L_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -7 \\ 0 & 1 & 0 & | & -8 \\ 0 & 0 & 1 & | & 11 \end{bmatrix} \Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$.

(ii) I shall write only the operations on lines, so: $\begin{cases} L_3 \leftrightarrow L_1 \\ L_2 = L_2 - L_1, L_3 = L_3 - 2L_1 \\ L_3 = L_3 - 3L_2 \\ L_1 = L_1 - L_2 \end{cases} \Rightarrow$
$\begin{bmatrix} 1 & 0 & -7 & | & 1 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. As the last line is all zeros, then the third unknown is $z = \alpha \in \mathbb{R} \Rightarrow x = 1 + 7\alpha$ and $y = 1 - 3\alpha$.

(iii) I shall write only the operations on lines, so: $\begin{cases} L_2 = L_2 - L_1, L_3 = L_3 - 2L_1, L_4 = L_4 - L_1 \\ L_2 = L_2 - 2L_4, L_3 = L_3 - 3L_4 \end{cases}$
$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & | & -4 \\ 0 & 0 & 0 & | & -6 \\ 0 & -1 & 0 & | & 1 \end{bmatrix} \Rightarrow \begin{cases} x + y + z = 3 \\ -y = 1 \end{cases} \Rightarrow y = -1,\ z = \alpha \in \mathbb{R}$
and $x = 4 - \alpha$.

6. I shall write only the operations on lines, so: $\begin{cases} L_2 \leftrightarrow L_1 \\ L_2 = L_2 - 2L_1, L_3 = L_3 - L_1 \\ L_3 = L_3 + L_2 \end{cases} \Rightarrow$
$\begin{bmatrix} 1 & 2 & -1 & 4 & | & 2 \\ 0 & -3 & 3 & -7 & | & -3 \\ 0 & 0 & 0 & 0 & | & \lambda - 2 - 3 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ -3x_2 + 3x_3 - 7x_4 = -3 \\ 0 = \lambda - 2 - 3 \\ x_1 = \alpha \\ x_4 = \beta \end{cases}$. By solving this system we get that $\lambda = 5$, $x_2 = 1 - \alpha - \frac{5}{3}\beta$ and $x_3 = -\alpha + \frac{2}{3}\beta$.

7. I shall write only the operations on lines, so: $\begin{cases} L_3 \leftrightarrow L_1 \\ L_2 = L_2 - L_1 \\ L_3 = L_3 - aL_1 \\ L_3 = L_3 + L_2 \end{cases} \Rightarrow$

$$\begin{bmatrix} 1 & 1 & a & | & a^2 \\ 0 & a-1 & 1-a & | & a-a^2 \\ 0 & 0 & 2-a-a^2 & | & 1-a^3+a-a^2 \end{bmatrix} \Rightarrow \begin{cases} x+y+az = a^2 \\ (a-1)y+(1-a)z = a(1-a) \\ (2-a-a^2)z = 1-a^2+a-a^3 \end{cases}.$$

By solving the system we get: $\begin{cases} x = -1-a \\ y = 1 \\ z = 1+a \end{cases}$

8. We have two ways to solve it.

(a) $\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \frac{z}{xy} = 81 \end{cases} \Rightarrow \begin{cases} z = \frac{1}{xy} \\ z^2 = \frac{27}{x} \cdot \frac{1}{x^2y^2} \\ z = 81xy \end{cases} \Rightarrow \frac{27}{x} = 1 \Rightarrow x = 3^3.$

Now, $\frac{1}{xy} = 81xy \iff \frac{1}{3^3y} = 3^4 \cdot 3^3y \iff y^2 = \frac{1}{3^{10}}$, with $0 \leq y \Rightarrow y = \frac{1}{3^5}$. And $z = 81 \cdot 3^3 \cdot \frac{1}{3^5} \Rightarrow z = 3^2$.

(b) What if we apply log_3 for each equation?!

We will get: $\begin{cases} log_3(x+y+z) = log_3(3^0) \\ log_3(x^3y^2z^2) = log_3(3^3) \\ log_3(\frac{z}{xy}) = log_3(3^4) \end{cases}$.

And from here, we get to the same solution. (Try this at home!)