Seminar 2

Homogeneous relation $\varphi : M \to M$.

A graph of a relation φ is a set $A = \{(x, y) \mid x\varphi y\}$, i.e. all the pairs of elements, which are in relation φ with each other. A relation is also given by its graph.

An equivalence relation has to be reflexive (R), transitive (T) and symmetric (S).

R:$\forall x \in A : x\rho x$

T:$\forall x, y, z \in A$ if $x\rho y$ and $y\rho z \Rightarrow x\rho z$

S:$\forall x, y \in A : x\rho y \Rightarrow y\rho x$

We say that $h = (R, M, H)$ is a relation if $H \subseteq R \times M$. And h is a function if $\forall x \in R :\mid h < x >\mid = 1$ (i.e. injective).

We say that $(A_i)_{i\in I}$ is a partition if $\cup_{i\in I} A_i = A$ and $A_i \cap A_j = \emptyset, \forall i, j \in I, i \neq j$.

1. $x \text{ r } y \Rightarrow x < y \Rightarrow R = \{(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$

   $x \text{ s } y \Rightarrow x \mid y \Rightarrow S = \{(2,4),(2,6),(3,6),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

   $x \text{ t } y \Rightarrow gcd(x,y) = 1 \Rightarrow T = \{(2,3),(3,2),(2,5),(5,2),(3,4),(4,3),(3,5),(5,3),(4,5),$ $(5,4),(5,6),(6,5)\}$

   $x \text{ v } y \Rightarrow x \equiv y(mod 3) \Rightarrow V = \{(3,6),(6,3),(2,5),(5,2),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

2. i $\varphi : A \to B \Rightarrow$ Number of $\varphi = 2^{|A\times B|} = 2^{mn}$ Because we have m elements from A, which can form pairs with n elements from B, so mn pair in the end. But those pairs can be written in 2 different ways, like $(a, b), (b, a)$, so it gives us the number stated before.

   ii $\varphi : A \to A \Rightarrow$ Number of $\varphi = 2^{|A\times A|} = 2^{n^2}$

3. $A = \{1, 2, 3\}$

   $R = \{(1,1),(2,2),(3,3)\}$

   $T = \{(1,2),(2,3),(1,3)\}$

   $S = \{(1,2),(2,1)\}$

4. $(\mathbb{R}, \neq)$

$$R : \forall x \in \mathbb{R}, x \neq x(false)$$

$(\mathbb{N}, |)$

$$R : \forall x \in \mathbb{N}, x \mid x(true)$$

$$T : \forall x, y, z \in \mathbb{N} y \mid x, z \mid y \Rightarrow z \mid x(true)$$

$$S : \forall x, y \in \mathbb{N}, x \mid y \iff y \mid x(false)$$

The same goes for $(\mathbb{Z}, |)$.

$(V^3, \perp)$

$$R : \forall x \in V^3, x \perp x(false)$$

$(V^3, \|)$

$$R : \forall x \in V^3, x \parallel x(false)$$

$(V^2, \equiv)$

$$R : \forall x \in V^2, x \equiv x(true)$$

$$T : \forall x, y, z \in V^2, x \equiv y, y \equiv z \Rightarrow x \equiv z(true)$$

$$S : \forall x, y \in V^2, x \equiv y \iff y \equiv x(true)$$

$(V^2, \sim)$

$$R : \forall x \in V^2, x \sim x(true)$$

$$T : \forall x, y, z \in V^2, x \sim y, y \sim z \Rightarrow x \sim z(true)$$

$$S : \forall x, y \in V^2, x \sim y \iff y \sim x(true)$$

5. i $R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2), (4,4)\}$. From the pairs $(1,1), (2,2), (3,3), (4,4)$ we can say that R_1 is reflexive. From pairs like $(1,2), (2,1)$ we check that R_1 is symmetric. And from pairs like $(1,2), (2,3), (1,3)$ we check that R_1 is transitive. So r_1 is an equivalence.$\Rightarrow \pi = \{1, 2, 3, 4\}$.

   $R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3)\}$. We check that R_2 is reflexive, transitive, but not symmetric. So r_2 is not an equivalence.

ii For $\pi_1 \Rightarrow \{1\} \cup \{2\} \cup \{3,4\} = \{1,2,3,4\} = M$, $\{1\} \cap \{2\} = \emptyset, \{1\} \cap \{3,4\} = \emptyset, \{2\} \cap \{3,4\} = \emptyset \Rightarrow \pi_1$ is a partition of $M \Rightarrow Gr = \{(1,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}$.

For $\pi_2 \Rightarrow \{1\} \cup \{1,2\} \cup \{3,4\} = \{1,2,3,4\} = M$, but $\{1\} \cap \{1,2\} = \{1\} \neq \emptyset \Rightarrow \pi_2$ is not a partition of M.

6. We check if r is reflexive, transitive and symmetric, which it is, so r is an equivalence relation. We compute $\mathbb{C}/r = \{r(z) \mid z \in \mathbb{C}\} = \{zrz \mid z \in \mathbb{C}\} = \{r(z) \mid\mid z \mid = \mid \overline{z} \mid, z \in \mathbb{C}\} = \{0\} \cup \{C(0, \mid z \mid)\}$.

   We now check the same for s and by simple computations, we get that s is also an equivalence relation. And we compute $\mathbb{C}/s = \{s(z) \mid z \in \mathbb{C}\} = \{zsz \mid arg(z) = arg(\overline{z}), z \in \mathbb{C}\} = \{$ the line starting from $O \mid$ which has the angle $arg(z)$ with $Ox\} \cup \{0\}$.

7.
$$R : \forall x \in \mathbb{Z} : x\rho_n y \Rightarrow n \mid (x-x), (true)$$
$$T : \forall x,y,z \in \mathbb{Z} : x\rho_n y, y\rho_n z \Rightarrow n \mid (x-y), n \mid (y-z) \Rightarrow n \mid [(x-y)+(y-z)] \Rightarrow n \mid (x-z), (true)$$
$$S : \forall x,y \in \mathbb{Z} : x\rho_n y \Rightarrow n \mid (x-y) \iff n \mid (y-x) \Rightarrow y\rho_n x, (true)$$

   So, ρ_n is an equivalence relation.

   $\mathbb{Z}/\rho_0 = \{\{x\} \mid x \in \mathbb{Z}\} \iff 0 \mid x-y \iff x = y$

   $\mathbb{Z}/\rho_1 = \{\mathbb{Z}\} \iff 1 \mid x-y$

   $\mathbb{Z}/\rho_n = \{\hat{0}, \hat{1}, \ldots, \widehat{n-1}\}$

8. From the set $M = \{1,2,3\}$ we can get the partitions: $\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{2,3\},\{1\}\}, M$. With each partition, we get the graph of a relation. For example, for the first partition, we get $\{(1,1),(2,2),(3,3)\}$. So this can be the equality relation, which is an equivalence relation. And it goes like this for every partition.

9. For h to be a function: $\forall x \in \mathbb{Z}$, we have $\mid h < x > \mid = 1$.

   We know that $h < x > = \{y \in M \mid (x,y) \in \mathbb{Z} \times M\} = \{y \in M \mid \exists z \in \mathbb{Z} : x = 4x + y\}$. So, $y \in M$ is the residue of x divided by 4, which is uniquelly determined.

   Hence, $h < x >$ has exactly one element for each $x \in \mathbb{Z} \Rightarrow h$ is a function.

10. First, we take the relation r:

    We try to see if it is symmetric: $mrn \iff \exists a \in \mathbb{N}$ with $m = 2^a n$. And $nrm \iff \exists b \in \mathbb{N}$ with $n = 2^b m$. This means $m = 2^a 2^b m \Rightarrow 2^{a+b} = 1 \Rightarrow a + b = 0$, where $a, b \in \mathbb{N} \Rightarrow a = b = 0 \Rightarrow m$ must be equal to n. So, it is not symmetric $\Rightarrow$ it is not an equivalence relation.

    Now, we take the relation s.

    It is reflexive, as $msm \iff m = m$.

    It is transitive. From msn and nsq, we get $m = n^2 q$ and $n^2 = q \Rightarrow m = q$ (one of the three happens).

    It is symmetric, as msn and $nsm \Rightarrow$ we get all three to be true. So, it is an equivalence relation.

    One shall verify all cases.