9. The Riemann Integral: related topics

§ 9.1. Numerical computation of the Riemann Int.: the Trapezoidal Rule

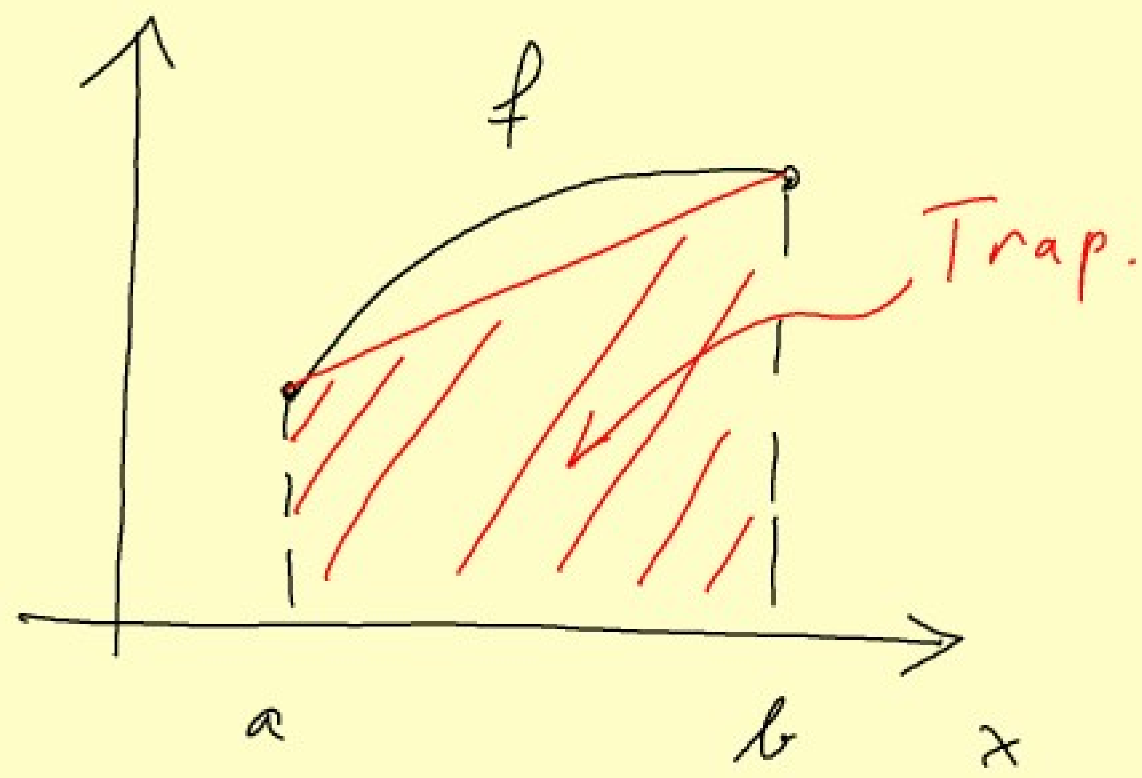
$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{when} \quad \underset{\text{antideriv.}}{F'} = f$$

Most function's don't admit simple/nice antideriv.

$\Rightarrow$ You want to approximate the Riemann Int.

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$



How good is this?

$$\text{error} = \left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left(f(a)+f(b)\right) \right| = \frac{(b-a)^3}{12} \left| f''(\xi) \right|$$

for some $\xi \in (a,b)$

roughly
- if $(b-a) \gg 1$ approx is bad
- if $(b-a) \ll 1$ —"— good!

$f'' \sim O(1)$

How can we exploit this?

The Composite Trapezoidal rule

IDEA: Sub-divide $[a,b]$ into n equal subintervals $x_i - x_{i-1} = \frac{b-a}{n}$

use Trapezoidal rule on each $[x_{i-1}, x_i]$

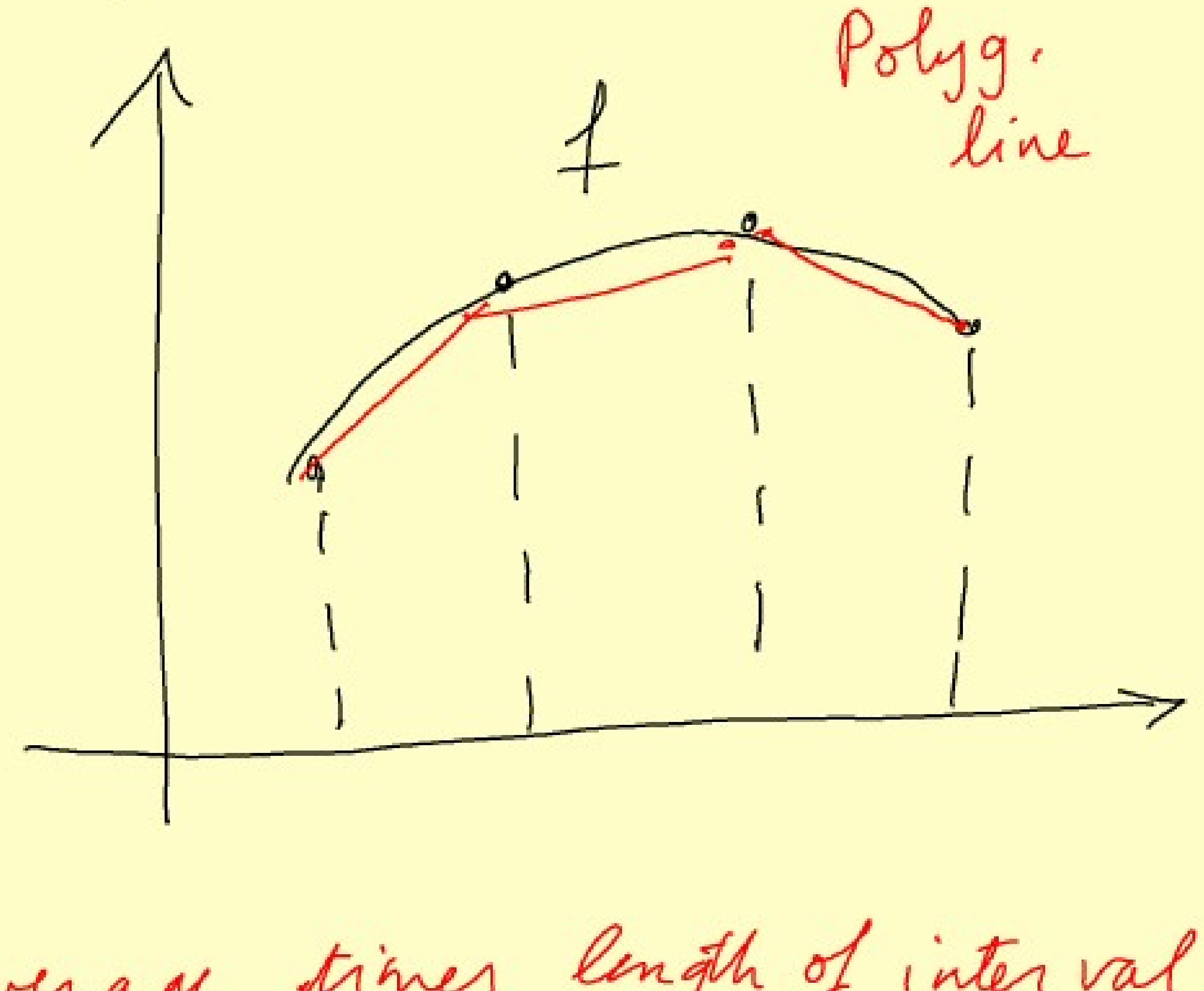
$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{n} \frac{b-a}{n}\,\frac{1}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

$$\left(\overset{\text{this rewrites as}}{=} \frac{b-a}{n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)\right)$$

$$\text{error} = \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} \frac{b-a}{n}\,\frac{1}{2}\cdots \right| = \frac{b-a}{12}\cdot\frac{(b-a)^2}{n^2} f''(\xi)$$

(far better than $\frac{1}{n}$ because comput. effort $\sim n$)

Rk: • What we have done is just: approximate the function f by a piece-wise linear function (polygonal line)

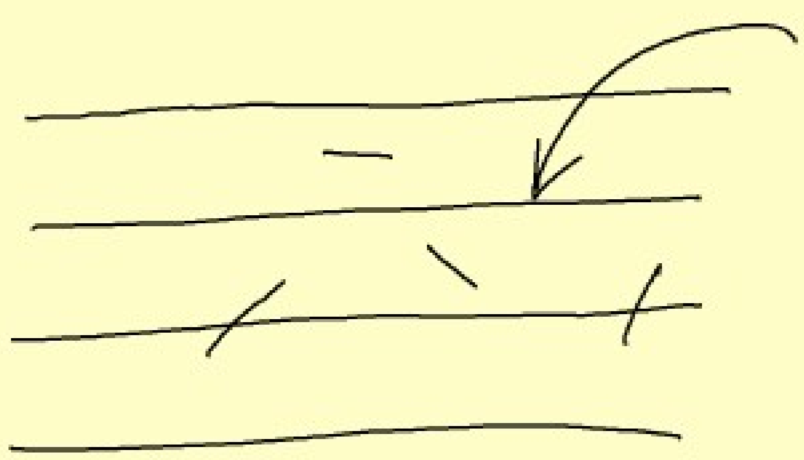


• Formula contains an average times length of interval.

§ 9.2. Integrals and Probability Theory

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total nr. of outcomes}}$$

The Buffon Needle Problem (1733)



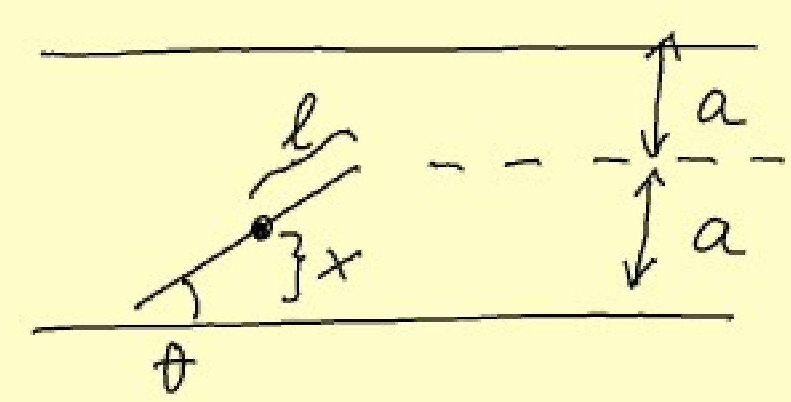
Throw Needles on the table cloth (randomly).
What is the probability (for one needle) to cross a line?

2ℓ = length of needle

$2a$ = dist. between lines ($\ell < a$)

x = dist from center of needle to closest line
$x \in [0, a]$

θ = angle needle makes with horizontal lines
$\theta \in [0, \pi]$



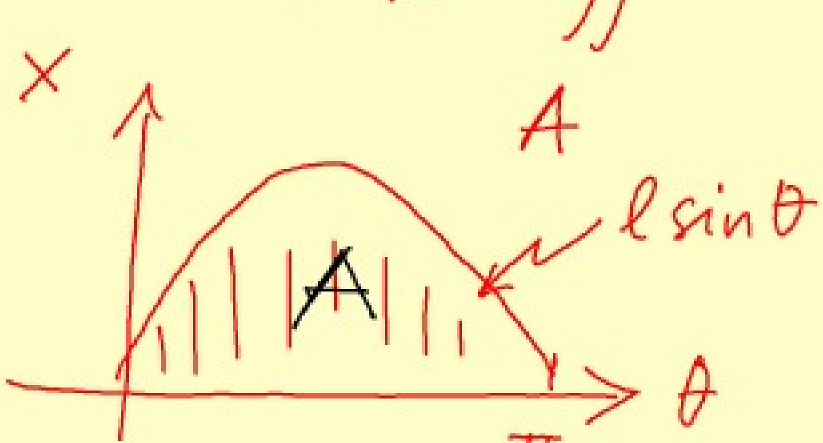
Ω = all possible outcomes
$= \{(x, \theta) \in [0, a] \times [0, \pi]\}$

$m(\Omega) = a\pi$ (Ω is a box)

A = favorable outcomes
$= \{(x, \theta) : \underline{0 \le x \le \ell \sin\theta},\ \theta \in [0, \pi]\}$

if $\ell \sin\theta > x$ you don't cross (geometry)

$$m(A) = \iint dx\, d\theta = \int_0^{\pi} \left(\int_0^{\ell \sin\theta} dx \right) d\theta = \ldots = 2\ell$$



$$\text{Probab} = \frac{m(A)}{m(\Omega)} = \frac{2\ell}{a\pi}$$

§ 9.3. Monte Carlo integration

IDEA (Stan. ULAM) : use randomness to solve a deterministic probl.

Name ("Monte Carlo") comes from Ulam's uncle who used to lose family money in the MC Casinos.

This idea was famously used in computations for the bombs.

How does it work (Naive ...)

$$I = \int_\Omega f(x)\,dx\,, \quad x = (x_1, \dots, x_N) \quad N \text{ very large}$$

Riemann : subdivide Ω in n subdomains of equal measure $m(\Omega_i) = \frac{1}{n}\, m(\Omega)$

$$\left(m(\Omega) = \int_\Omega dx \right)$$

$$I \approx \sum_{i=1}^{n} f(\xi_i)\, \underbrace{m(\Omega_i)}_{\frac{m(\Omega)}{n}}$$

this rewrites as an average

$$= m(\Omega)\, \frac{1}{n} \sum_{i=1}^{n} f(\xi_i)$$

Ulam: choose these randomly

Law of Large Numbers ; average obtained from a large nr. of samples should be close to the avg. over entire possibility space.

<u>Manhattan Project.</u>

research Project of US (military) for development of Atomic bomb.

Jan 1939 FERMI lands in NY and at the same time Niels BOHR was giving a Lecture in Princeton
↳ about splitting Nuclei

August 2. 1939 Einstein (– Szilárd) <u>Letter</u>

Leó Szilárd, Edward Teller, E. Wigner convince Einstein to sign a letter written by L. Sz. to president F.D. Roosevelt.

History Facts : 1930-32 Hitler's Rise to Power
1932 Einstein → US (Princeton)
1930 IAS (Institute of Advanced Study)
Einstein, J.v. Neumann, H. Weyl, K. Gödel.
... Robert Oppenheimer.
Dec. 6. 1941 : Pearl Harbour
(US join the war!)

1941 Frisch - Peierls compute explosive power of Uranium

1942 Oppenheimer: <u>Chain Reaction</u>

1942 June: Oppenheimer's Research Summer
Edward Teller: <u>Critical Mass</u>

Manhattan Proj
R. Oppenheimer = Director (together w. gen. L. Groves)
H. Bethe = Director of "T"
Theoretical Department

1943 Site Y = Los Alamos Lab (New Mexico)

1943 (Dec.) British Mission Arrives
(incl. Klaus Fuchs = Rus. Spy)

1943 Bethe <u>Calls</u> Ulam to Los Alamos
→ resposible for "hydrodyn. computations"
working with "female computers"

1944 Oppenh. convinces Fermi to join
Sept. 1944. Fermi = assoc. Director

1945 July 16th Trinity Test
August 6 & 9 Hiroshima & Nagasaki