Seminar 13

For a (n, k) code, the message has k digits and the coded message has n digits. Also, the number of check digits is $n-k$, the information rate is $\frac{k}{n}$ and the number of different syndromes is 2^{n-k}.

For a vector u and a coset $u+V$, where $V = Im(\gamma)$, a **coset leader** (the most likely error pattern) is $e = u - v = u + v \in u + V$.

$[symdrome] = H \cdot [vector]$, where H is the parity check matrix.

For similar symdromes, we choose the ones with fewer 1 digits or with the 1's bunched together.

1. (i) This is $k = 56$.
   (ii) This is $n - k = 63 - 56 = 7$.
   (iii) This is $\frac{k}{n} = \frac{9}{8}$.
   (iv) This is $2^{n-k} = 2^7$.

2. Let's start by naming the words we need to decode by u_i.

   Now, for u_1 we need to multiply this with the parity check matrix to find it's syndrome $\Rightarrow H \cdot u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. From the table, we see that for the syndrome 000 we have the coset leader 000000, which is denoted with e_1.

   Now, the most likely code vector is $v_1 = u_1 + e_1 = 101110$. So, the most likely message is 110.

   For $u_2 \Rightarrow H \cdot u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow e_2 = 000010 \Rightarrow v_2 = 011010 \Rightarrow 010.$

   For $u_3 \Rightarrow H \cdot u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow e_3 = 000110 \Rightarrow v_3 = 001101 \Rightarrow 101.$

   For $u_4 \Rightarrow H \cdot u_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow e_4 = 000110 \Rightarrow v_4 = 111001 \Rightarrow 001.$

   For $u_5 \Rightarrow H \cdot u_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow e_5 = 010000 \Rightarrow v_5 = 100011 \Rightarrow 011.$

3. **Remember:** $G = \begin{bmatrix} P \\ I_k \end{bmatrix}$ and $H = \begin{bmatrix} I_{n-k} & P \end{bmatrix}$.

   Our equations can be rewritten as: $\begin{cases} u_1 + u_4 + u_5 + u_7 = 0 \\ u_2 + u_4 + u_6 + u_7 = 0 \\ u_3 + u_4 + u_5 + u_6 = 0 \end{cases}$.

   From here we get $H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$. So, $P = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow$
   $G = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Now, for the first word 0000111, first we verify if it is a code word, which is true. To decode it, we have to compute $H \cdot u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So, the decoded word is 0111.

   For the second word 0001111, we verify if it is a code word, which is false. If we compute $H \cdot u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So, the code word would be $v = u + e$, where e is a coset leader which we don't have.

4. Received words: $\{000, 001, 010, 100, 011, 101, 110, 111\}$, which we denote with u_i. And $G = [\gamma]_{EE'}$.

   Take $(3,2)$-code, then $E = (e_1, e_2)$, $E' = (e_1, e_2, e_3)$. For $e_1 \Rightarrow 10 \Rightarrow (1+0)10 = 110 = e'_1 + e'_2$. And for $e_2 \Rightarrow 01 \Rightarrow (0+1)01 = 101 = e'_1 + e'_3$.

   So, $G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

   We use $[syndrome] = H \cdot [u_i]$ and we get the syndromes:

   $$u_1 \to 0$$

   $$u_2 \to 1$$

$$u_3 \to 1$$

$$u_4 \to 1$$

$$u_5 \to 0$$

$$u_6 \to 0$$

$$u_7 \to 0$$

$$u_8 \to 1$$

For the $(3,1)$-code, we have $G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

So the syndromes are:

$$u_1 \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_2 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_3 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_4 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_5 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_6 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_7 \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_8 \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. First we compute how many syndromes we have, i.e. $2^{n-k} = 2^3 = 8$.

   Now, we write all the possible syndromes that we may get and for each one we compute the coset leader. So, the syndromes may be $\{000, 001, 010, 100, 011, 101, 110, 111\}$.

   Now, we try to see for some possible coset leaders, what syndromes we get. And for that, we choose the coset leaders with as many 0 as possible and as fewer 1 as possible.

   Take the coset leader $u = 0000000 \Rightarrow H \cdot u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   Take $u = 1000000 \Rightarrow H \cdot u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   And so on, such that in the end we have:

$$\textit{syndromes} \rightarrow \textit{coset leader}$$
$$000 \rightarrow 0000000$$
$$001 \rightarrow 0010000$$
$$010 \rightarrow 0100000$$
$$100 \rightarrow 1000000$$
$$011 \rightarrow 0000001$$
$$101 \rightarrow 0000010$$
$$110 \rightarrow 0000100$$
$$111 \rightarrow 0001000$$

6. The parity check matrix is $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

   We have $2^{n-k} = 2^2 = 4$ syndromes.

   With the same reasoning as before, we get the table:

$$\textit{syndromes} \rightarrow \textit{coset leaders}$$
$$00 \rightarrow 00000$$
$$01 \rightarrow 01000$$
$$10 \rightarrow 10000$$
$$11 \rightarrow 00010$$

7. We know how to construct H from the previous seminar.

   $G = [\gamma]_{EE'}$, where $E = e_1$ and $E' = (e_1', e_2', e_3')$.

   For $e_1 = 1 \Rightarrow m = 1 \Rightarrow mX^{n-k} = X^2 \Rightarrow r = X + 1 \Rightarrow v = 1 + X + X^2 \Rightarrow 111$.

   So, $G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

   We have $2^{n-k} = 2^2 = 4$ syndromes.

   Now, for a coset leader $u = 000$, we get the syndrome: $H \cdot u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   In the end, we have the table:

   $$syndromes \to coset\ leaders$$

   $$00 \to 000$$

   $$01 \to 010$$

   $$10 \to 100$$

   $$11 \to 001$$

8. The same as the previous exercise.

   We have $H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ and $2^4 = 16$ syndromes.

   So, our table may look like this:

   $$syndromes \to coset\ leaders$$

   $$0000 \to 0000000$$

   $$0001 \to 0001000$$

   $$0010 \to 0010000$$

   $$0100 \to 0100000$$

   $$1000 \to 1000000$$

   $$0011 \to 0011000$$

   $$0101 \to 0000110$$

$$0110 \rightarrow 0110000$$
$$1010 \rightarrow 0001100$$
$$1100 \rightarrow 1100000$$
$$1001 \rightarrow 0000011$$
$$0111 \rightarrow 0000001$$
$$1011 \rightarrow 0000100$$
$$1101 \rightarrow 0000101$$
$$1110 \rightarrow 0000010$$
$$1111 \rightarrow 1000001$$