Seminar 6

We say that $v_1, v_2, \ldots v_n$ are **linearly independent** if

$$a_1v_1 + a_2v_2 + \ldots a_nv_n = 0 \iff a_1 = a_2 = \cdots = a_n = 0$$

or if the determinant, given by the vectors written on lines, is different from zero.

We say that B is a **basis** if the vectors in B are linearly independent and the vectors in B generate the whole space.

1. (i) $a_1v_1 + a_2v_2 + a_3v_3 = 0 \iff \begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 + a_2 + 5a_3 = 0 \\ a_2 + 2a_3 = 0 \end{cases}$

   From the last equation we have $a_2 = -2a_3$ so our system becomes $\begin{cases} a_1 - 4a_3 + a_3 = 0 \\ -a_1 - 2a_3 + 5a_3 = 0 \end{cases} \Rightarrow a_1 - 3a_3 = 0 \Rightarrow a_1 = 3a_3 \Rightarrow S = \{(3a, -2a, a) \mid a \in \mathbb{R}\} \Rightarrow v_1, v_2, v_3$ are linearly dependent.

   (ii) $a_1v_1 + a_2v_2 = 0 \iff \begin{cases} a_1 + 2a_2 = 0 \\ -a_1 + a_2 = 0 \\ a_2 = 0 \end{cases} \Rightarrow a_1 = 0 \Rightarrow v_1, v_2$ are linearly independent.

2. (i) $a_1v_1 + a_2v_2 + a_3v_3 = 0 \iff \begin{cases} a_1 - a_2 + 3a_3 = 0 \\ 2a_2 + a_3 = 0 \\ 2a_1 + a_2 + a_3 = 0 \end{cases} \Rightarrow$ By simple computations we get that $a_1 = a_2 = a_3 = 0 \Rightarrow v_1, v_2, v_3$ are linearly independent.

   (ii) $\begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = -192 \neq 0 \Rightarrow v_1, v_2, v_3, v_4$ are linearly independent.

3. $\begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 1 & 0 & a \end{vmatrix} = a(\sqrt{2} - a)(\sqrt{2} + a) = 0$ (if this is 0, the vectors are dependent, if not, they are independent) $\Rightarrow a = 0$ or $a = \sqrt{2}$ or $a = -\sqrt{2} \Rightarrow$ for v_1, v_2, v_3 to be linearly independet $a \in \mathbb{R} \setminus \{-\sqrt{2}, 0, \sqrt{2}\}$.

4. $a_1v_1+a_2v_2+a_3v_3 = 0 \iff \begin{cases} a_1 + 2a_2 = 0 \\ -2a_1 + a_2 + aa_3 = 0 \\ a_2 + a_3 = 0 \\ -a_1 + 2a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = -a_3 \\ a_1 = 2a_3 \\ -4a_3 - a_3 + aa_3 = 0 \end{cases} \Rightarrow$ $(a-5)a_3 = 0 \Rightarrow a \in \mathbb{R} \setminus \{5\}$. (Here, a_3 could not be 0, as all of them would have been 0, which means that the vectors would have been linearly independent).

5. (i) $\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow (v_1, v_2, v_3)$ linealry independent.
$\forall u = (u_1, u_2, u_3) \in \mathbb{R}^3, \exists! a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1v_1 + a_2v_2 + a_3v_3 = u \Rightarrow \begin{cases} a_1 - a_2 + a_3 = u_1 \\ a_1 + a_3 = u_2 \\ 2a_2 + a_3 = u_3 \end{cases} \Rightarrow$$

By simple computations we get that $\begin{cases} a_1 = 3u_2 - u_3 - 2u1 \\ a_2 = u_2 - u_1 \\ a_3 = u_3 - 2u_2 + 2u_1 \end{cases} \Rightarrow$
(v_1, v_2, v_3) generates $\mathbb{R}^3 \Rightarrow (v_1, v_2, v_3)$ is a basis.

(ii) We have to solve all three systems $a_1v_1 + a_2v_2 + a_3v_3 = e_1$ and $a_1v_1 + a_2v_2 + a_3v_3 = e_2$ and $a_1v_1 + a_2v_2 + a_3v_3 = e_3$. In other words, to find a_1, a_2, a_3 in each case. $\Rightarrow \begin{cases} e_1 = 2v_1 + v_2 - 2v_3 \\ e_2 = 3v_1 + v_2 - 2v_3 \\ e_3 = -v_1 + v_3 \end{cases}$

(iii) In (e_1, e_2, e_3) we have the coordinates for u as $(1, -1, 2)$. So, $a_1v_1 + a_2v_2 + a_3v_3 = (1, -1, 2)$, and by solving the system, we find $a_1 = -7, a_2 = -2, a_3 = 6 \Rightarrow$ In the basis (v_1, v_2, v_3), u has the coordinates $(-7, -2, 6)$.

6.

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & {\color{red}1} \\ 1 & 1 & \dots & 1 & 1 & {\color{red}2} \\ 1 & 1 & \dots & 1 & 2 & {\color{red}3} \\ \dots & \dots & \dots & \dots & \dots & {\color{red}\dots} \\ 1 & 2 & \dots & n-2 & n-1 & {\color{red}n} \end{vmatrix}$$

$$= (-1)^{n+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 2 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 \end{vmatrix} + (-1)^{n+2} \cdot 2 \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 2 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 \end{vmatrix}$$

$$+ (-1)^{n+3} \cdot 3 \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 \end{vmatrix} + \dots + (-1)^{n+n} \cdot n \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-2 \end{vmatrix}$$

All of them are zero, as two lines are equal, except the first two determinants.

$$(-1)^{n+1}(1-2) \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 2 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 \end{vmatrix} = (-1)^{n+2} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 2 \\ \dots & \dots & \dots & \dots \\ 1 & 2 & \dots & n-1 \end{vmatrix}$$

By induction, we get that

$$\Delta = (-1)^{(n+2)+(n+1)+\dots+2} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (-1)^{\frac{(n+1)(n+4)}{2}} \neq 0$$

So, they form a basis in $\mathbb{R}^n$.

For a vector $(x_1, x_2, \dots, x_n) = \alpha_1 v_1 + \dots \alpha_n v_n$

$$\begin{cases} \alpha_1 + \alpha_2 + \dots + \alpha_{n-1} + \alpha_n = x_1 \\ \alpha_1 + \alpha_2 + \dots + \alpha_{n-1} + 2\alpha_n = x_2 \\ \alpha_1 + \alpha_2 + \dots + 2\alpha_{n-1} + 3\alpha_n = x_3 \\ \vdots \end{cases}$$

If we compute $L_2 - L_1$ we get that $\alpha_n = x_2 - x_1$. Then $L_3 - L_2$, we get $\alpha_{n-1} = x_3 - 2x_2 + x_1$. So on, by induction, we get that:

$$\begin{cases} \alpha_{n-p} = x_{p+2} - 2x_{p+1} + x_p, \text{ if } p \geq 1 \\ \alpha_n = x_2 - x_1, \text{ if } p = 0 \end{cases}$$

7. We know that (E_1, E_2, E_3, E_4) is a basis in $M_2(\mathbb{R})$ and so, the coordinates of B in the basis are $(2, 1, 1, 0)$.

   For the second one we have to solve the system $a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = 0 \Rightarrow a_1 = a_2 = a_3 = a_4 = 0 \Rightarrow A_1, A_2, A_3, A_4$ are linearly independent. Then $\forall A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}), \exists! a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = A \Rightarrow \begin{cases} a_1 = a - b \\ a_2 = b - c \\ a_3 = c - d \\ a_4 = d \end{cases}$

   $\Rightarrow < A_1, A_2, A_3, A_4 > = M_2(\mathbb{R})$ so it is a basis of $M_2(\mathbb{R})$. Then, the coordinates of B is this basis are $(1, 0, 1, 0)$.

8. We know that E is a basis in $\mathbb{R}_2[X]$ and so, the coordinates of f in E are (a_0, a_1, a_2).

   For the second one, we have

$$\alpha_1 \cdot 1 + \alpha_2 \cdot (X - a) + \alpha_3 \cdot (X - a^2) = 0 \iff \begin{cases} \alpha_1 - a\alpha_2 + a^2\alpha_3 = 0 \\ \alpha_2 - 2\alpha_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

   $\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow B$ has linearly independent vectors.

   $\forall f = b_0 + b_1X + b_2X^2 \in \mathbb{R}_2[X], \exists! \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$f = \alpha_1 \cdot 1 + \alpha_2 \cdot (X - a) + \alpha_3 \cdot (X - a^2) \Rightarrow \begin{cases} \alpha_1 = b_0 + ab_1 - a^2b_2 \\ \alpha_2 = b_1 + 2ab_2 \\ \alpha_3 = b_2 \end{cases}$$

   $\Rightarrow < B > = \mathbb{R}_2[X] \Rightarrow B$ is a basis of $\mathbb{R}_2[X]$. And so, the coordinates of B in this basis are $(a_0 + aa_1 - a^2a_2, a_1 + 2aa_2, a_2)$.

9. $\mathbb{Z}_2^3 = \{(\hat{0}, \hat{0}, \hat{0}), (\hat{1}, \hat{0}, \hat{0}), (\hat{0}, \hat{1}, \hat{0}), (\hat{0}, \hat{0}, \hat{1}), (\hat{1}, \hat{1}, \hat{0}), (\hat{1}, \hat{0}, \hat{1}), (\hat{0}, \hat{1}, \hat{1}), (\hat{1}, \hat{1}, \hat{1})\}$. So, $\mid \mathbb{Z}_2^3 \mid = 2^3$.

   A pair $(z_1, z_2, z_3) \in \mathbb{Z}_2^3$ is a base $\iff z_1, z_2, z_3$ are linearly independent.

   Take $z_1 \in \mathbb{Z}_2^3 \setminus \{(\hat{0}, \hat{0}, \hat{0})\} \Rightarrow z_1$ is a part of the base $\Rightarrow z_1$ can be chosen in $2^3 - 1$ ways. If $z_2, z_3 \in \mathbb{Z}_2^3 \Rightarrow z_1, z_2, z_3$ linearly independent $\iff z_2 \in \mathbb{Z}_2^3 \setminus < z_1 >$ and $z_3 \in \mathbb{Z}_2^3 \setminus < z_1, z_2 >$. So, z_2 can be chosen in

$(2^3-1)-1$ ways and z_3 in $((2^3-2)-1)-1$ ways. Hence, the number of basis of $\mathbb{Z}_2^3$ is $(2^3-1)(2^3-2)(2^3-4)=168$.

10. It is the same thing as finding how many basis are in $\mathbb{Z}_2^3$.