CS109 assignment 2

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Warmup

1. Say in Silicon Valley, 36% of engineers program in Java and 24% of the engineers who program in Java also program in C++. Furthermore, 33% of engineers program in C++.
a. What is the probability that a randomly selected engineer programs in Java and C++?
b. What is the conditional probability that a randomly selected engineer programs in Java given that he/she programs in C++?

Solution

a. $P(J, C) = P(J) * P(C|J) = 0.36 * 0.24 = 0.0864$

b. $P(J|C) = \frac{P(C|J)P(J)}{P(C)} = \frac{0.24*0.36}{0.33} = 0.2618$

2. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let E be the event that both cards are Aces. Let F be the event that the first card chosen is the Ace of Spades. Compute $P(E|F)$.
Solution

Intuitively, we'd say that if we already know that the first card is Ace of Spades, there are 51 cards left to choose from and 3 of those would make event E happen, so the probability is $\frac{3}{51}$. We can check this result if we apply Bayes' rule: $P(E|F) = \frac{P(F|E)P(E)}{P(F)}$. We know that $P(F) = \frac{1}{52}$, as there is exactly one Ace of Spades in the deck. Also, $P(E) = P(\text{first card is Ace})P(\text{second card is Ace}) = \frac{4}{52}\frac{3}{51}$. Given that we have a pair of Aces, the probability that first one in the pair is the Ace of Spades is $\frac{1}{4} = P(F|E)$. Replacing the values in Bayes' rule above, we get the same result.

3. Five servers are located in a computer cluster. After one year, each server independently is still working with probability p, and otherwise fails (with probability 1 − p).
a. What is the probability that at least 1 server is still working after one year?
b. What is the probability that exactly 3 servers are still working after one year?
c. What is the probability that at least 3 servers are still working after one year?

Solution

a. $P(\#\text{working clusters} \geq 1) = 1 - P(\#\text{working clusters} = 0) = 1 - (1-p)^5$.
b. $P(\#\text{working clusters} = 3) = \binom{5}{3}p^3(1-p)^2 = 10p^3(1-p)^2$

c.

$$P(\#\text{working clusters} >= 3) = P(\#\text{working clusters} = 3) + P(\#\text{working clusters} = 4) + P(\#\text{working clusters} = 5) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5(1-p)^0$$

4. A website wants to detect if a visitor is a robot. They give the visitor

three CAPTCHA tests that are hard for robots, but easy for humans. If the visitor fails in one of the tests, they are flagged as a robot. The probability that a human succeeds at a single test is 0.95, while a robot only succeeds with probability 0.3. Assume all tests are independent.
a. If a robot visits the website, what is the probability they get flagged?
b. If a visitor is human, what is the probability they get flagged?
c. The fraction of visitors on the site that are robots is 1/10. Suppose a visitor gets flagged. What is the probability that visitor is a robot?

Solution

a. $P(\text{R is flagged}) = 1 - P(\text{R gets all 3 tests correct}) = 1 - 0.3^3 = 0.973$.

b. $P(\text{H is flagged}) = 1 - P(\text{H gets all 3 tests correct}) = 1 - 0.95^3 = 0.1426$.

c. $P(\text{V=R}|\text{V is flagged}) = \frac{P(\text{V is flagged}|\text{V=R})P(\text{V=R})}{P(\text{V is flagged})} = \frac{P(\text{R is flagged})P(\text{V=R})}{P(\text{R is flagged})P(\text{V=R})+P(\text{H is flagged})P(\text{V=H})} = \frac{0.973*0.1}{0.973*0.1+0.1426*0.9} = 0.4312$

5. Recall the game set-up in the "St. Petersburg's paradox" discussed in class: there is a fair coin which comes up "heads" with probability p = 0.5. The coin is flipped repeatedly until the first "tails" appears. Let N = number of coin flips before the first "tails" appears (i.e., N = the number of consecutive "heads" that appear). Given that no one really has infinite money to offer as payoff for the game, consider a variant of the game where you win $\text{MIN}(2^N, X)$, where X is the maximum amount that the game provider will pay you after playing. Compute the expected payoff of the game for the following values of X. Show your work.
a. X = \$5.
b. X = \$500.
c. X = \$4096.

Solution

In the most general way, we can write:

$$\mathbb{E}[\text{payoff}] = \sum_{i=0}^{i=\infty} (\frac{1}{2})^i \frac{1}{2} \min(2^i, X) = \sum_{i=0}^{i=\lfloor \log_2 X \rfloor} (\frac{1}{2^{i+1}})2^i + \sum_{i > \lfloor \log_2 X \rfloor} \frac{1}{2^{i+1}} X =$$
$$\frac{1}{2}(1 + \lfloor \log_2 X \rfloor) + \frac{1}{2^{\lfloor \log_2 X \rfloor + 1}} X \sum_{i>0} \frac{1}{2^i} = \frac{1}{2}(1 + \lfloor \log_2 X \rfloor) + \frac{X}{2^{\lfloor \log_2 X \rfloor}}$$

a. Using X=\$5 in the equation above, we get: $\mathbb{E}[\text{payoff}] = \frac{3}{2} + \frac{5}{4} = 2$.

b. $\mathbb{E}[\text{payoff}|X = \$500] = \frac{9}{2} + \frac{500}{256} = 5$.

c. $\mathbb{E}[\text{payoff}|X = \$4096] = \frac{13}{2} + \frac{4096}{4096} = 7$.

6. A bit string of length n is generated randomly such that each bit is generated independently with probability p that the bit is a 1 (and 0 otherwise). How large does n need to be (in terms of p) so that the probability that there is at least one 1 in the string is at least 0.7?

Solution

$P(\#_1(string) \geq 1) = 1 - P(\#_1(string) = 0) = 1 - (1-p)^n$. Thus, $P(\#_1(string) \geq 1) \geq 0.7$ is equivalent to: $1 - (1-p)^n \geq 0.7 \iff 0.3 \geq (1-p)^n \iff \log_{1-p} 0.3 \leq n$